PRECALCULUS

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This text was compiled on 03/29/2026

TABLE OF CONTENTS

Licensing

1: Functions and Graphs

- 1.1: 1.1 Function Families
- 1.2: 1.2 Graphical Transformations
- 1.3: 1.3 Point Notation and Function Notation
- 1.4: 1.4 Domain and Range
- 1.5: 1.5 Maximums and Minimums
- 1.6: 1.6 Symmetry
- 1.7: 1.7 Increasing and Decreasing
- 1.8: 1.8 Zeroes and Intercepts of Functions
- 1.9: 1.9 Asymptotes and End Behavior
- 1.10: 1.10 Continuity and Discontinuity
- 1.11: 1.11 Function Composition
- 1.12: 1.12 Inverses of Functions

2: Polynomials and Rational Functions

- 2.1: 2.1 Factoring Review
- 2.2: Advanced Factoring
- 2.3: Polynomial Expansion and Pascal's Triangle
- 2.4: Rational Expressions
- 2.5: Polynomial Long Division and Synthetic Division
- 2.6: Solving Rational Equations
- 2.10 Horizontal Asymptotes
- 2.11 Oblique Asymptotes
- 2.12 Sign Test for Rational Function Graphs
- 2.13 Graphs of Rational Functions by Hand
- 2.7 Holes in Rational Functions
- 2.8 Zeroes of Rational Functions
- 2.9 Vertical Asymptotes

3: Logs and Exponents

- 3.1: Exponential Functions
- 3.2: Properties of Exponents
- 3.3: Scientific Notation
- 3.4: Properties of Logs
- 3.5: Change of Base
- 3.6: Exponential Equations
- 3.7: Logistic Functions

4: Basic Triangle Trigonometry

- 4.1: Angles in Radians and Degrees
- 4.2: Circular Motion and Dimensional Analysis
- 4.3: Special Right Triangles

- 4.4: Right Triangle Trigonometry
- 4.5: Law of Cosines
- 4.6: Law of Sines
- 4.7: Area of a Triangle
- 4.8: Applications of Basic Triangle Trigonometry

5: Trigonometric Functions

- 5.1: The Unit Circle
- 5.2: The Sinusoidal Function Family
- 5.3: Amplitude of Sinusoidal Functions
- 5.4: Vertical Shift of Sinusoidal Functions
- 5.5: Frequency and Period of Sinusoidal Functions
- 5.6: Phase Shift of Sinusoidal Functions
- 5.7: Graphs of Other Trigonometric Functions
- 5.8: Graphs of Inverse Trigonometric Functions

6: Analytic Trigonometry

- 6.1: Basic Trigonometric Identities
- 6.2: Pythagorean Identities
- 6.3: Sum and Difference Identities
- 6.4: Double, Half, and Power Reducing Identities
- 6.5: Trigonometric Equations

7: Vectors

- 7.1: Basic Properties of Vectors
- 7.2: Operations with Vectors
- 7.3: Resolution of Vectors into Components
- 7.4: Dot Product and Angle Between Two Vectors
- 7.5: Vector Projection

8: Systems and Matrices

- 8.1 Systems of Two Equations and Two Unknowns
- 8.2 Systems of Three Equations and Three Unknowns
- 8.3 Matrices to Represent Data
- 8.4 Matrix Operations
- 8.5 Row Operations and Row Echelon Forms
- 8.6 Augmented Matrices
- 8.10 Partial Fractions
- 8.7 Determinant of Matrices
- 8.8 Cramer's Rule
- 8.9 Inverse Matrices

9: Conics

- 9.1 General Form of a Conic
- 9.2 Parabolas
- 9.3 Circles
- 9.4 Ellipses
- 9.5 Hyperbolas

- 9.6 Degenerate Conics

10: Polar and Parametric Equations

- 10.1 Polar and Rectangular Coordinates
- 10.2 Polar Equations of Conics
- 10.3 Parameters and Parameter Elimination
- 10.4 Parametric Inverses
- 10.5 Applications of Parametric Equations

11: Complex Numbers

- 11.1 Fundamental Theorem of Algebra
- 11.2 Arithmetic with Complex Numbers
- 11.3 Trigonometric Polar Form of Complex Numbers
- 11.4 De Moivre's Theorem and nth Roots

12: Discrete Math

- 12.1: Section 1-
- 12.2: Section 2-
- 12.3: Section 3-
- 12.4: Section 4-
- 12.5: Section 5-
- 12.6: Section 6-

13: Finance

- 13.1: Section 1-
- 13.2: Section 2-
- 13.3: Section 3-
- 13.4: Section 4-
- 13.5: Section 5-
- 13.6: Section 6-

14: Concepts of Calculus

- 14.1: Section 1-
- 14.2: Section 2-
- 14.3: Section 3-
- 14.4: Section 4-
- 14.5: Section 5-
- 14.6: Section 6-

15: Concepts of Statistics

- 15.1: Section 1-
- 15.2: Section 2-
- 15.3: Section 3-
- 15.4: Section 4-
- 15.5: Section 5-
- 15.6: Section 6-

16: Logic and Set Theory

- 16.1: Section 1-
- 16.2: Section 2-
- 16.3: Section 3-
- 16.4: Section 4-
- 16.5: Section 5-
- 16.6: Section 6-

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Functions and Graphs

1.1: 1.1 Function Families
1.2: 1.2 Graphical Transformations
1.3: 1.3 Point Notation and Function Notation
1.4: 1.4 Domain and Range
1.5: 1.5 Maximums and Minimums
1.6: 1.6 Symmetry
1.7: 1.7 Increasing and Decreasing
1.8: 1.8 Zeroes and Intercepts of Functions
1.9: 1.9 Asymptotes and End Behavior
1.10: 1.10 Continuity and Discontinuity
1.11: 1.11 Function Composition
1.12: 1.12 Inverses of Functions

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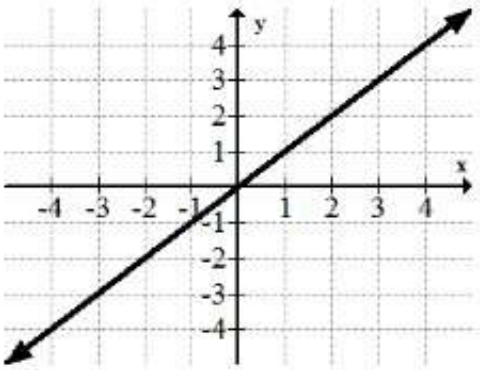
1.1: 1.1 Function Families

Functions come in all different shapes. A few are very closely related and others are very different, but often confused. For example, what is the difference between x^2 and 2^x? They both have an x and a 2 and they both equal 4 when $x = 2$, but one eventually becomes much bigger than the other.

Families of Functions

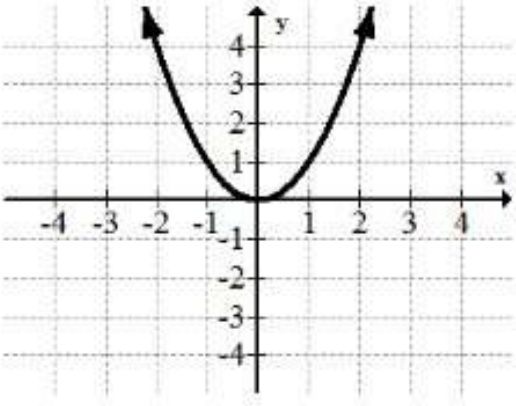
If mathematicians are cooks, then families of functions are their ingredients. Each family of functions has its own flavor and personality. Before you learn to combine functions to create an infinite number of potential models, you need to get a clear idea of the name of each function family and how it acts.

The Identity Function: $f(x) = x$



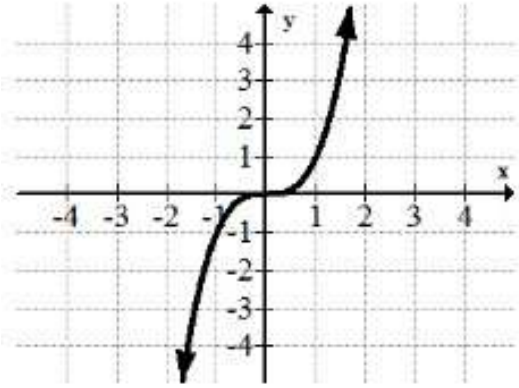
The **identity function** is the simplest function and all straight lines are transformations of the identity function family.

The Squaring Function: $f(x) = x^2$



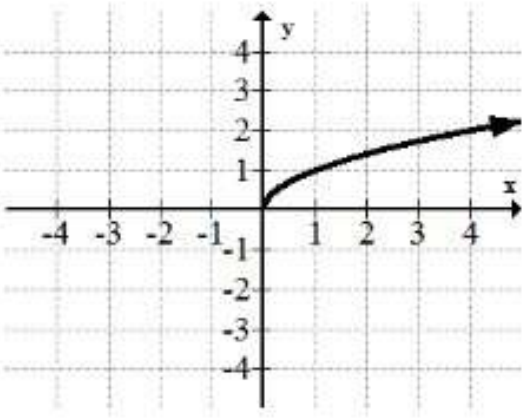
The **squaring function(quadratic function)** is commonly called a parabola and is useful for modeling the motion of falling objects. All parabolas are transformations of this squaring function.

The Cubing Function: $f(x) = x^3$



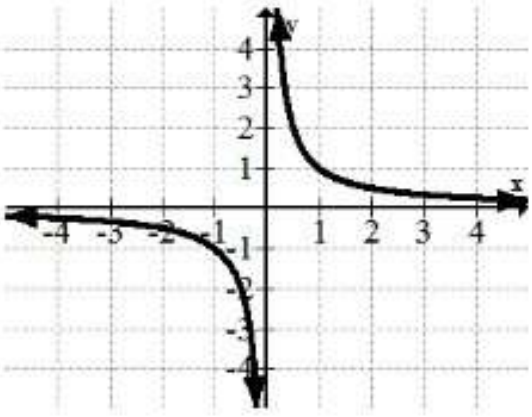
The **cubing function** has a different kind of symmetry than the squaring function. Since volume is measured in cubic units, many physics applications use the cubic function.

The Square Root Function: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$



The **square root function** is not defined over all real numbers. It introduces the possibility of complex numbers and is also closely related to the squaring function.

The Reciprocal Function: $f(x) = x^{-1} = \frac{1}{x}$



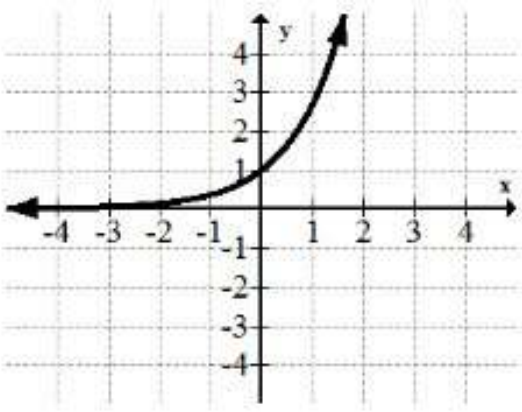
The **reciprocal function** is also known as a hyperbola and a rational function. It has two parts that are disconnected and is not defined at zero. Simple electric circuits are modeled with the reciprocal function.

So far all the functions can be grouped together into an even larger function family called the power function family.

The Power Function Family: $f(x) = cx^a$

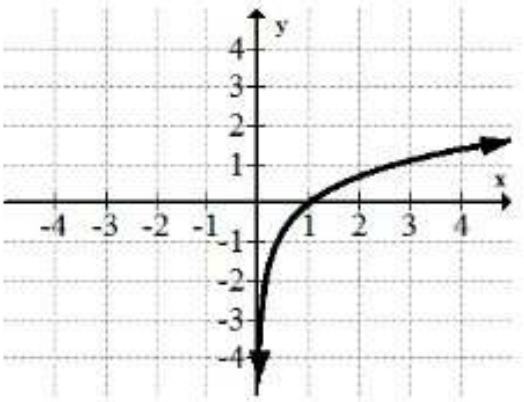
The **power function** family has two parameters. The parameter c is a vertical scale factor. The parameter a controls everything about the shape. The reason why all the functions so far are subsets of the larger power function family is because they only differ in their value of a. The power function family also shows you that there are an infinite number of other functions like quartics $(f(x) = x^4)$ and quintics $(f(x) = x^5)$ that don't really need a whole category of their own. The power function family can be extended to create polynomials and rational functions.

The Exponential Function Family: $f(x) = e^x$



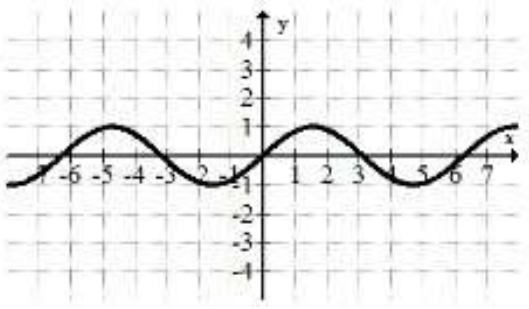
The **exponential function** family is one of the first functions you see where x is not the base of the exponent. This function eventually grows much faster than any power function. $f(x) = 2^x$ is a very common exponential function as well. Many applications like biology and finance require the use of exponential growth.

The Logarithm Function: $f(x) = \ln x$



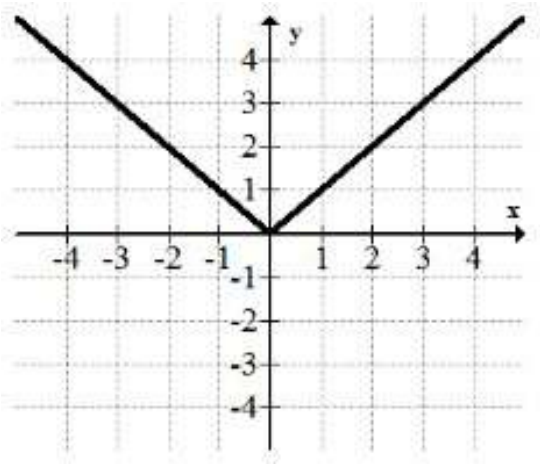
The **logarithmic function** is closely related to the exponential function family. Many people confuse the graph of the log function with the square root function. Careful analysis will show several important differences. The log function is the basis for the Richter Scale which is how earthquakes are measured.

The Periodic Function Family: $f(x) = \sin x$



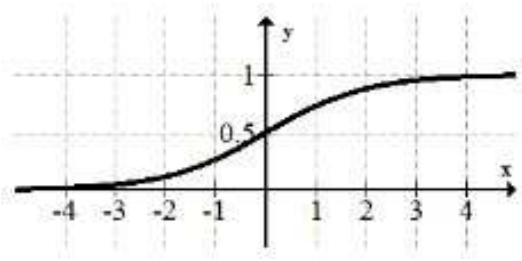
The **sine graph** is one of many periodic functions. Periodic refers to the fact that the sine wave repeats a cycle every period of time. Periodic functions are extremely important for modeling tides and other real world phenomena.

The Absolute Value Function: $f(x) = |x|$



The **absolute value function** is one of the few basic functions that is not totally smooth.

The Logistic Function: $f(x) = \frac{1}{1+e^{-x}}$



The **logistic function** is a combination of the exponential function and the reciprocal function. This curve is very powerful because it models population growths where the maximum population is limited by environmental resources.

Examples

Example 1

Earlier, you were given a problem about comparing x^2 and 2^x. While x^2 and 2^x have similar ingredients, they have very different graphical features. The squaring function is symmetric about the line $x = 0$ while the exponential function is not. When $x = 0$, the squaring function has a height of zero and the exponential function has a height of one. The squaring function has a slope that becomes steeper as x gets further from the origin while the exponential function flattens as x gets very small. All of these differences are important and not obvious at first glance.

Example 2

$y = x$, $y = e^x$, $y = \frac{1}{1+e^{-x}}$. Some functions that are close but not quite: $y = x^3$, $y = \sqrt{x}$

Example 3

Compare and contrast the graphs of the two functions:

$f(x) = \ln x$ and $h(x) = \sqrt{x}$

Similarities: Both functions increase without bound as x gets larger. Both functions are not defined for negative numbers.

Differences: The log function approaches negative infinity as x approaches 0. The square root function, on the other hand, just ends at the point (0,0).

Example 4

Describe the symmetry among the function families discussed in this concept. Consider both reflection symmetry and rotational symmetry.

Some function families have reflective symmetry with themselves:

$y = x$, $y = x^2$, $y = \frac{1}{x}$, $y = |x|$

Some function families are rotationally symmetric:

$y = x$, $y = x^3$, $y = \frac{1}{x}$, $y = \sin x$, $y = \frac{1}{1+e^{-x}}$

Some pairs of function families are full or partial reflections of other function families:

$y = x^2$, $y = \sqrt{x}$

$y = e^x$, $y = \ln x$

Review

For 1-10, sketch a graph of the function from memory.

1. $y = e^x$
2. $y = \ln(x)$
3. $y = \sin(x)$
4. $y = x^2$
5. $y = |x|$
6. $y = \frac{1}{x}$
7. $y = \frac{1}{1+e^{-x}}$
8. $y = \sqrt{x}$
9. $y = x^3$
10. $y = x$
11. Which function is not defined at 0? Why?
12. Which functions are bounded below but not above??
13. What are the differences between $y = x^2$ and $y = x^3$?
14. What is a similarity between $y = e^x$ and $y = \ln(x)$?
15. Explain why $y = \sqrt{x}$ is not defined for all values of x.

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1.2: 1.2 Graphical Transformations

The basic functions are powerful, but they are extremely limited until you can change them to match any given situation. Transformation means that you can change the equation of a basic function by adding, subtracting, and/or multiplying by constants and thus cause a corresponding change in the graph. What are the effects of the following transformations?

1. $f(x) \rightarrow f(x+3)$
2. $h(x) \rightarrow h(x)-5$
3. $g(x) \rightarrow -g(2x)$
4. $j(x) \rightarrow j\left(-\frac{x}{2}\right)$

Transforming Functions

A **function** is a rule that takes any input x and gives a specific output. When you use letters like f, g, h, or j to describe the rule, this is called function notation. In order to interpret what effect the algebraic change in the equation will have on the graph, it is important to be able to read those changes in general function notation and then apply them to specific cases.

When **transforming** a function, you can transform the **argument** (the part inside the parentheses with the x), or the function itself. There are two ways to linearly transform the argument. You can multiply the x by a constant and/or add a constant to the x as shown below:

$f(x) \rightarrow f(bx+c)$

The function itself can also be linearly transformed in the same ways:

$f(x) \rightarrow af(x)+d$

Each of the letters a,b,c, and d corresponds to a very specific change. Some of these changes are straightforward, while others may be the opposite of what you might expect.

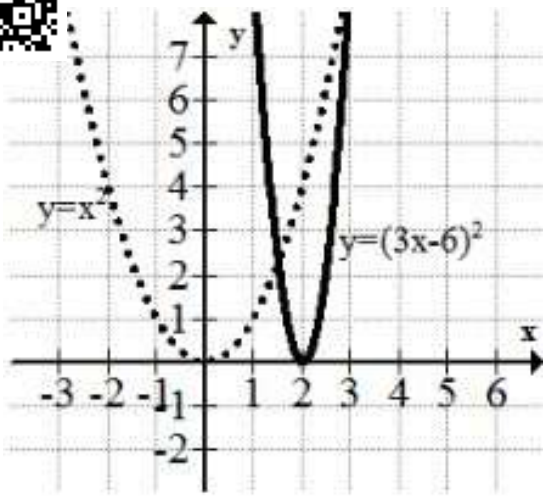
- a is a vertical stretch. If a is negative, there is also a reflection across the x axis.
- d is a vertical shift. If d is positive, then the shift is up. If d is negative, then the shift is down.

When transforming the argument of the function things are more complicated.

- $\frac{1}{b}$ is a horizontal stretch. If b is negative, there is also a reflection across the y axis.
- c is a horizontal shift. If c is positive, then the shift is to the left. If c is negative, then the shift is to the right. Notice that this is the opposite of what most people think at first.

The trickiest part with tra ment of a function is the order in which you carry out the transformations. Often it makes sense to apply the t pecific function that is known and then describe the transformation that you see.

The graph below shows th $(x) \rightarrow f(3x-6)$ applied to a simple parabola:



Clearly the graph is narrower and to the right, but in order to be specific you must look closer. First, notice that the transformation is entirely within the argument of the function. This affects only the horizontal values. This means while the graph seems like it was stretched vertically, you must keep your perspective focused on a horizontal compression.

Look carefully at the vertex of the parabola. It has moved to the right two units. This is because first the entire graph was shifted entirely to the right 6 units. Then the function was horizontally compressed by a factor of 3 which means the point (6, 0) became (2, 0) and the x value of every other point was also compressed by a factor of 3 towards the line $x = 0$. This method is counter-intuitive because it requires reading the transformations backwards (the opposite of the way the order of operations tells you to).

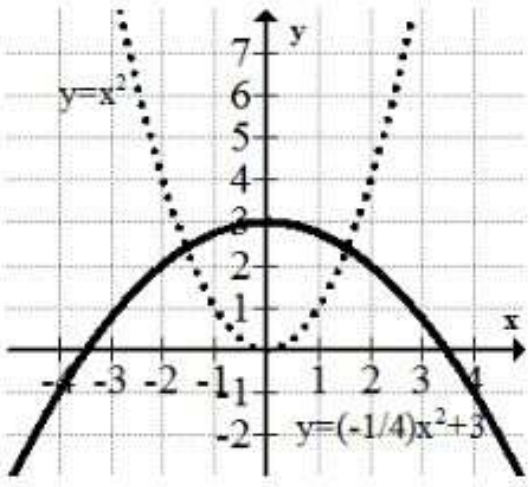
Alternatively, the argument can be factored and each component of the transformation will present itself.

$f(3(x-2))$

This time the stretch occurs from the center of the transformed graph, not the origin. This method is ultimately the preferred method.

Either way, this is a horizontal compression by a factor of 3 and a horizontal shift to the right by 2 units.

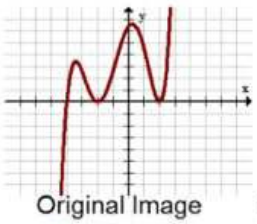
Now take the transformation $f(x) \rightarrow -\frac{1}{4}f(x) + 3$. It describes a vertical stretch by a factor of $\frac{1}{4}$, a reflection over the x axis, and a vertical shift 3 units up. As opposed to what you saw above, the order of the transformations for anything outside of the argument is directly what the order of operations dictates.

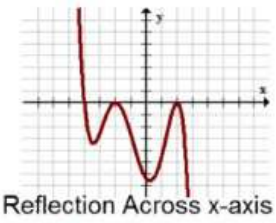


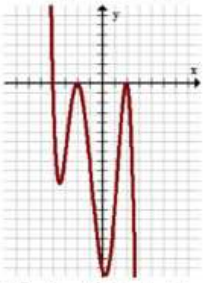
First, the parabola is reflected over the x axis and compressed vertically so it appears wider. Then, every point is moved up 3 units.

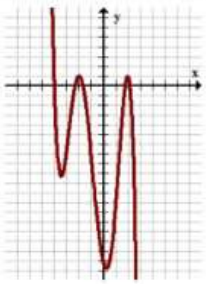
The transformation $f(x) \rightarrow -3f\left(-\frac{1}{2}x - 1\right) + 1$ contains every possible transformation. The horizontal and the vertical components do not interact with each other and so your description of the transformation can begin with either component. Here, start by describing the vertical components of the transformation:

First, there is reflection across the x axis and a vertical stretch by a factor of 3. Then, there is a vertical shift up 1 unit. Below is an image of a non-specific function going through the vertical transformations.







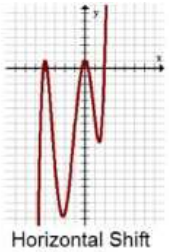


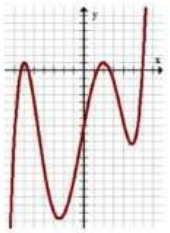
In order to figure out the horizontal components of the transformation, start by factoring the inside of the parentheses (the argument):

$f\left(-\frac{1}{2}x-1\right)=f\left(-\frac{1}{2}(x+2)\right)$

Factoring reveals a reflection across the y axis, a horizontal shift left 2 units and a horizontal stretch by a factor of 2. Below is an image of the same function going through the horizontal transformations.







Examples

Example 1

Earlier, you were given a problem about the effects of the following transformations:

$f(x)\rightarrow f(x+3)$

This transformation shifts the entire graph left 3 units. A common misconception is to shift right because the three is positive.

$h(x)\rightarrow h(x)-5$

This transformation shifts the entire graph down 5 units.

$g(x)\rightarrow -g(2x)$

This transformation is a vertical reflection across the x axis and a horizontal compression by a factor of 2.

$j(x)=j\left(-\frac{x}{2}\right)$

This transformation is a horizontal reflection across the y axis and a horizontal stretch by a factor of 2. A common misconception is to see the $\frac{1}{2}$ and believe that the x values will be half as big which is a horizontal compression. However, the \x\) values need to be twice as big to counteract this factor of $\frac{1}{2}$

Example 2

Describe the following transformation in words: $g(x) \rightarrow 2g(-x)$

Vertical stretch by a factor of 2 and a reflection across the y axis.

Example 3

Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 3.
- Vertical shift down 4 units.
- Horizontal shift right 5 units.

$\frac{1}{3}h(x-5)-4$

Example 4

Describe the transformation that would change $f(x)$ in the following ways:

- Horizontal stretch by a factor of 4 and a horizontal shift 3 units to the right.
- Vertical reflection across the x axis and a shift down 2 units.

$-f\left(\frac{1}{4}(x-3)\right)-2$ or $-f\left(\frac{1}{4}x-\frac{3}{4}\right)-2$

Review

Describe the following transformations in words.

1. $g(x) \rightarrow -g(-x)$
2. $f(x) \rightarrow -f(x+3)$
3. $h(x) \rightarrow h(x+1)-2$
4. $j(x) \rightarrow j(-x+3)$
5. $k(x) \rightarrow -k(2x)$
6. $f(x) \rightarrow 4f\left(\frac{1}{2}x+1\right)$
7. $g(x) \rightarrow -3g(x-2)-2$
8. $h(x) \rightarrow 5h(x+1)$

9. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical stretch by a factor of 2
- Vertical shift up 3 units.
- Horizontal shift right 2 units.

10. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical reflection across the x axis.
- Vertical shift down 1 unit.
- Horizontal shift left 2 units.

11. Describe the transformation that would change $g(x)$ in the following ways:

- Vertical compression by a factor of 4.
- Reflection across the y axis.

12. Describe the transformation that would change $j(x)$ in the following ways:

- Horizontal compression by a factor of 3.
- Vertical shift up 3 units.
- Horizontal shift right 2 units.

13. Describe the transformation that would change $k(x)$ in the following ways:

- Horizontal stretch by a factor of 4.
- Vertical shift up 3 units.
- Horizontal shift left 1 unit.

14. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 2.
- Horizontal shift right 3 units.
- Reflection across the y axis.

15. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical stretch by a factor of 5.
- Reflection across the x axis

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1.3: 1.3 Point Notation and Function Notation

When performing multiple transformations, it is very easy to make a small error. This is especially true when you try to do every step mentally. Point notation is a useful tool for concentrating your efforts on a single point and helps you to avoid making small mistakes.

What would $f(3x)+7$ look like in point notation and why is it useful?

Using Function Notation and Point Notation

A transformation can be written in function notation and in point notation. Function notation is very common and practical because it allows you to graph any function using the same basic thought process it takes to graph a parabola in vertex form.

Another way to graph a function is to transform each point one at a time. This method works well when a table of x, y values is available or easily identified from the graph.

Essentially, it takes each coordinate (x, y) and assigns a new coordinate based on the transformation.

$$(x, y) \longrightarrow (\text{ new } x, \text{ new } y)$$

This notation is called **point notation**. The new y coordinate is straightforward and is directly from what takes place outside $f(x)$ because $f(x)$ is just another way to write y. For example, $f(x) \rightarrow 2f(x)-1$ would have a new y coordinate of $2y-1$.

The new x coordinate is trickier. It comes from undoing the operations that affect x. For example, $f(x) \rightarrow f(2x-1)$ would have a new x coordinate of $\frac{x+1}{2}$

The function notation and  esentations of the transformation "Horizontal shift right three units, vertical shift up 4 units" are

$f(x) \rightarrow f(x-3)+4$

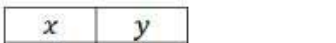
$(x, y) \rightarrow (x+3, y+4)$

Notice that the operations with the x are different.

Apply the transformation above to the following table of points.

x	y
1	3
2	5
8	−11

x	y
1	3
2	5
8	-11



x	y
4	7
5	9
11	-7

Notice that point notation greatly reduces the mental visualization required to keep all the transformations straight at once.

Examples

Example 1

Earlier, you were asked what the function $f(3x)+7$ would be when written in point notation. When written in point notation, it would be written as $(x,y)\rightarrow\left(\frac{x}{3},y+7\right)$. This is useful because it becomes obvious that the x values are all divided by three and the y values all increase by 7 .

Example 2

Convert the following function in point notation to words and then function notation.

$(x,y)\rightarrow(3x+1,-y+7)$

Horizontal stretch by a factor of 3 and then a horizontal shift right one unit. Vertical reflection over the x axis and then a vertical shift 7 units up.

$f(x)\rightarrow -f\left(\frac{1}{3}x-\frac{1}{3}\right)+7$

Example 3

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

$f(x) \rightarrow -2f(x-1)+4$

Vertical reflection across the x axis. Vertical stretch by a factor of 2. Vertical shift 4 units. Horizontal shift right one unit.

$(x, y) \rightarrow (x+1, -2y+4)$

x	y
0	5
1	6
2	7

→

x	y
1	-6
2	-8
3	-10

Example 4

Convert the following function notation into point notation and apply it to the included table of points

$f(x) \rightarrow \frac{1}{4}f(-x-3)-1$

x	y
1	4
2	8

The y component can be directly observed. For the x component you need to undo the argument. $(x, y) \rightarrow (-x-3, \frac{1}{4}y-1)$

x	y
0	0
1	4
2	8

→

x	y
-3	-1
-4	0
-5	1

Example 5

Convert the following point notation to words and to function notation and then apply the transformation to the included table of points.

$(x+3, y-1) \rightarrow (2x+6, -y)$

x	y
10	8
12	7
14	6

This problem is different because it seems like there is a transformation happening to the original left point. This is an added layer of challenge because the transformation of interest is just the difference between the two points. Notice that the x coordinate has simply doubled and the y coordinate has gotten bigger by one and turned negative. This problem can be rewritten as:

$(x, y) \rightarrow (2x, -(y+1)) = (2x, -y-1)$

$f(x) \rightarrow -f\left(\frac{x}{2}\right) - 1$

x	y
10	8
12	7
14	6

→

x	y
5	-9
6	-8
7	-7

Review

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

x	y
	5
1	6
2	7

1. $9f(x) \rightarrow -\frac{1}{2}f(x+1)$
2. $g(x) \rightarrow 2g(3x)+2$
3. $h(x) \rightarrow -h(x-4)-3$
4. $j(x) \rightarrow 3j(2x-4)+1$
5. $k(x) \rightarrow -k(x-3)$

Convert the following functions in point notation to function notation.

6. $(x,y) \rightarrow \left(\frac{1}{2}x+3, y-4\right)$
7. $(x,y) \rightarrow (2x+4, -y+1)$
8. $(x,y) \rightarrow (4x, 3y-5)$
9. $(2x,y) \rightarrow (4x, -y+1)$
10. $(x+1, y-2) \rightarrow (3x+3, -y+3)$

Convert the following functions in function notation to point notation.

11. $f(x) \rightarrow 3f(x-2)+1$
12. $g(x) \rightarrow -4g(x-1)+3$
13. $h(x) \rightarrow \frac{1}{2}h(2x+2)-5$
14. $j(x) \rightarrow 5j\left(\frac{1}{2}x-2\right)-1$
15. $k(x) \rightarrow \frac{1}{4}k(2x-4)$

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1.4: 1.4 Domain and Range

Analyze means to examine methodically and in detail. One way to analyze functions is by looking at possible inputs (domain) and possible outputs (range). Which of the basic functions have limited domains and why?

Domain and Range

Notation

Domain and range are described in interval notation. Parentheses, ( ), mean non-inclusive. Brackets, [ ], mean inclusive. The following descriptions of numbers into interval notation have been converted to interval notation.

1. All numbers.
$(-\infty, \infty)$ Note: Parentheses are always used with infinity.

2. All negative numbers not including 0.

$(-\infty, 0)$

3. All positive numbers including 0.
$[0, \infty)$

4. Every number between 1 and 4 including 1 and 4.
[1,4]

5. Every number between 5 and 6 not including 5 or 6.

(5, 6)

6. The numbers 1 through 2 including 1 but not including 2 and the numbers
10 through 25 including both 10 and 25

$[1,2) \cup [10, 25]$

Note: The $\cup$ symbol means Union and refers to the fact that if some number x is in this union, then it is either in the first group or it is in the second group. This symbol is associated with the OR statement. While it is true that the Union symbol seems to bring one group and another group together, the symbol for AND is $\cap$ which means intersection. Intersection is different from union because intersection means all numbers that are in both the first group and second group at the same time.

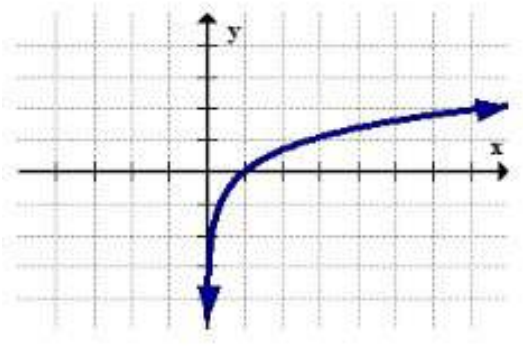
Domain and Restricted Domain

Domain is the possible inputs to a function. Many functions allow any kind of numbers to be inputted. This includes numbers that are positive, negative, zero, fractions or decimals. The squaring function $y = x^2$ is an example that has a domain of all possible real numbers. Three functions have very specific restrictions:

The square root function: $y = \sqrt{x}$
Domain restriction: $x \geq 0$

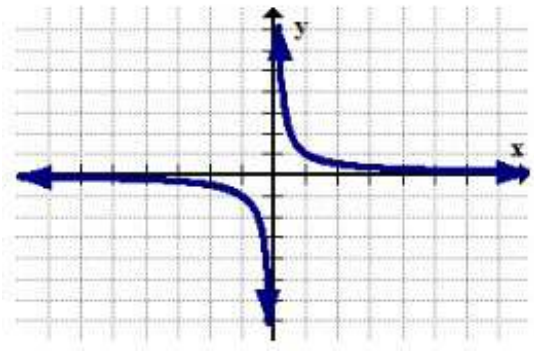
This is because the square root of a negative number is not a real number. This restriction can be observed in the graph because the curve ends at the point (0,0) and is not defined anywhere where x is negative.

The log function is only defined on numbers that are strictly bigger than zero. This is because the logarithmic function is a different way of writing exponents. One property of exponents is that any positive number raised to any power will never produce a negative number or zero. The restriction can be observed in the graph by the way the log function approaches the vertical line $x = 0$ and shoots down to infinity.



The reciprocal function: $y = \frac{1}{x}$
Domain Restriction: $x \neq 0$

The reciprocal function is restricted because you cannot divide numbers byzero. Any x values that make the denominator of a function zero are outside of the domain. This restriction can be observed in the graph by the way the reciprocal function never touches the vertical line $x = 0$.

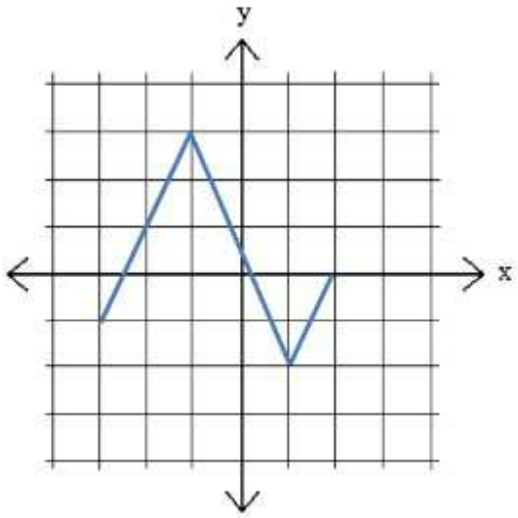


Range

Range is the possible outputs of a function. Just about any function can produce any output through the use of transformations and so determining the range of a function is significantly less procedural than determining the domain. Use what you know about the shape of each function and their equations to decide which y values are possible to produce and which y values are impossible to produce.

Finding Domain and Range





The domain and range for the graph above are:

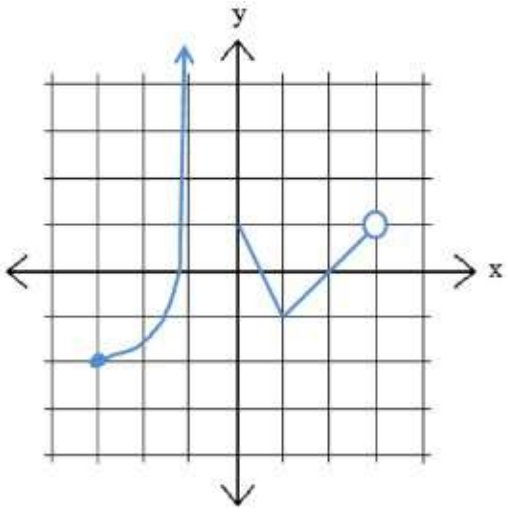
Domain: $x \in [-3, 2]$

Range: $y \in [-2, 3]$

Note that the $\in$ symbol means "is an element of" and means that the x or the y is in that interval and the numbers in the interval are always written in increasing order. [3,-2] is considered improper.

Note that even though the [-3,2] may look similar to the ordered pair that represents the point where $x = -3$ and $y = 2$, this is not the case. Both the -3 and the 2 are x values. This misconception is why you should always write $x \in$ because it reminds you of this fact. Many people get very confused when they see something like $x \in (-2, 1)$ because they see the parenthesis and immediately see a point when they should see an interval on the x axis.

In the graph below, there are two different pieces of the function.



The domain and range for the graph above are:

Domain: $x \in [-3, -1) \cup [0, 3)$

Range: $y \in [-2, \infty)$

The function seems to approach the vertical line $x = -1$ without actually reaching it $s0$ an open bracket is used. Also, the empty hole at the point $(3,$ 1) which is why the domain excludes the x value of 3 .

Examples

Example 1

Earlier, you were asked which basic functions have limited domains. The three functions that have limited domains are the square root function, the log function and the reciprocal function. The square root function has a restricted domain because you cannot take square roots of negative numbers and produce real numbers. The log function is restricted because the log function is not defined to operate on non-positive numbers. The reciprocal function is restricted because numbers that are divided by zero are not defined.

Example 2

Identify the domain and range of the following function written in a table:

x	y
0	5
1	6
2	7
$\frac{1}{2}$	6
π	$\frac{\pi}{2}$

The specific equation of the function may be hidden, but from the table you can determine the domain and range directly from the x and y values. It may be tempting to guess that other values could potentially work in the table, especially if the pattern is obvious, but this is not the kind of question that asks what the function could be. Instead this question just asks what is the stated domain and range.

Domain: $x \in \{0, 1, 2, \frac{1}{2}, \pi\}$

Range: $y \in \{5, 6, 7, \frac{\pi}{2}\}$

Notice that the two 6's that appear in the table do not need to be written twice in the range.

Example 3

Identify the domain of the following three transformed functions.

1. $y = 10\sqrt{2-x} - 3$

The argument of the function must be greater than or equal to 0.

$$2 - x \geq 0$$
$$-x \geq -2$$
$$x \leq 2$$

Domain: $x \in (-\infty, 2]$

2. $y = \frac{3x}{x^2+7x+12}$

The denominator cannot be equal to 0. First find what values of x would make it equal to zero and then you can exclude those values.

$$x^2 + 7x + 12 = 0$$
$$(x+4)(x+3) = 0$$
$$x = -4, -3$$

Domain: $x \in (-\infty, -4) \cup (-4, -3) \cup (-3, \infty)$

3. $y = -4\ln(3x - 9) + 11$

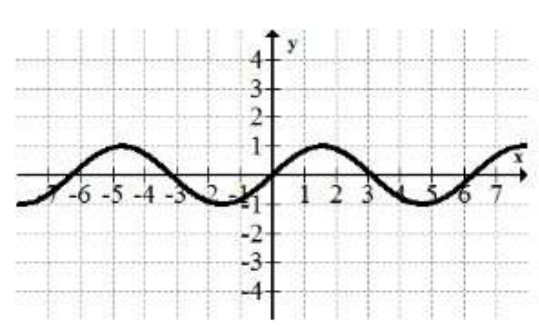
The argument must be strictly greater than 0 .

$$3x - 9 > 0$$
$$3x > 9$$
$$x > 3$$

Domain: $x \in (3, \infty)$

Example 4

What is the domain and range of the sine wave?



Domain: $x \in (-\infty, \infty)$

Range: $y \in [-1, 1]$

Review

Convert the following descriptions of numbers into interval notation.

1. All positive numbers not including 0.

2. Every number between -1 and 1 including -1 but not 1.

3. Every number between 1 and 5 not including 2 or 3, but including 1 and 5.

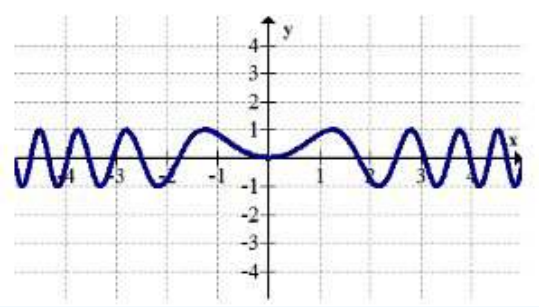
4. Every number greater than 5, not including 5.

5. All real numbers except 1.

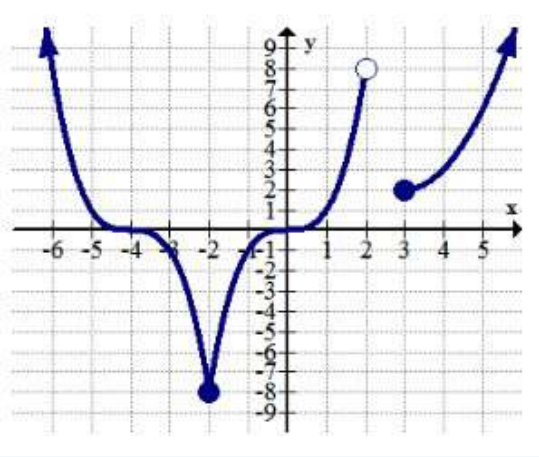
Translate the following inequalities into interval notation.

6. $-4 < x \leq 5$
7. $x > 0$
8. $-\infty < x \leq 4$ or $5 < x < \infty$

9.



10.



Given the stated domain and range, draw a possible graph.

11. Domain: $x \in [0, \infty)$ Range: $y \in (-2, 2]$

12. Domain: $x \in [-4, 1) \cup (1, \infty)$ Range: $y \in (-\infty, \infty)$

13. Given the table, find the domain and range.

x	y
-2	7
3	7
2	1
$\frac{3}{4}$	5
$\frac{\pi}{2}$	π

Find the domain for the following functions.

14. $y = -3\sqrt{x+4} - 1$

$y = \frac{7}{x+6} - 1$

16. $y = 5\ln(x^2 - 1) + 4$

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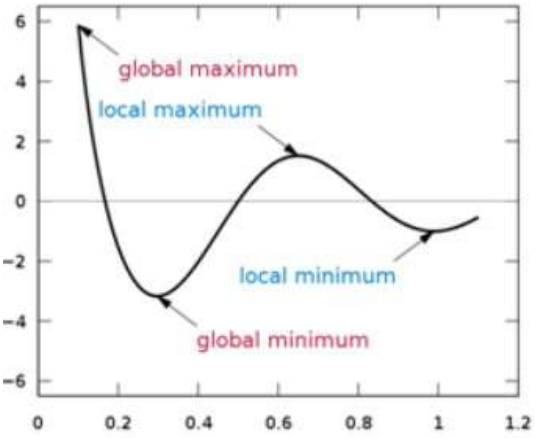
1.5: 1.5 Maximums and Minimums

When riding a roller coaster there is always one point that is the absolute highest off the ground. There are usually many other places that reach fairly high, just not as high as the first. There are also places where the roller coaster dips with one being the absolute lowest the roller coaster can go. How do you identify and distinguish between these different peaks and valleys in a precise way?

Finding Maximums and Minimums

A **global maximum** refers to the point with the largest y value possible on a function. A **global minimum** refers to the point with the smallest y value possible. Together these two values are referred to as **global extrema**. There can only be one global maximum and only one global minimum. Global refers to entire space where the function is defined. Global extrema are also called **absolute extrema**.

In addition to global maximums and global minimums, there are also **local extrema** or **relative maximums** and **relative minimums**. The word relative is used because in relation to some neighborhood, these values stand out as being the highest or the lowest.



Calculus uses advanced analytic tools to compute extreme values, but for the purposes of PreCalculus it is sufficient to be able to identify and categorize extreme values graphically or through the use of technology. For example, the TI-84 has a maximum finder when you select <2nd> then .

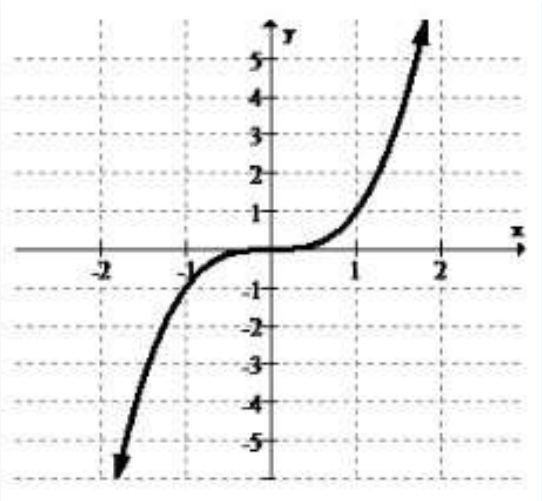
Examples

Example 1

Earlier, you were asked inguish between different peaks and valleys on a graph. Maximums and minimums should be intuitive bec dentify the highest points and the lowest points, or the peaks and the valleys, in a graph. There is a formal distinction about whether a maximum is the highest on some local open interval (does not matter how small), or whether it is simply the highest overall.

Example 2

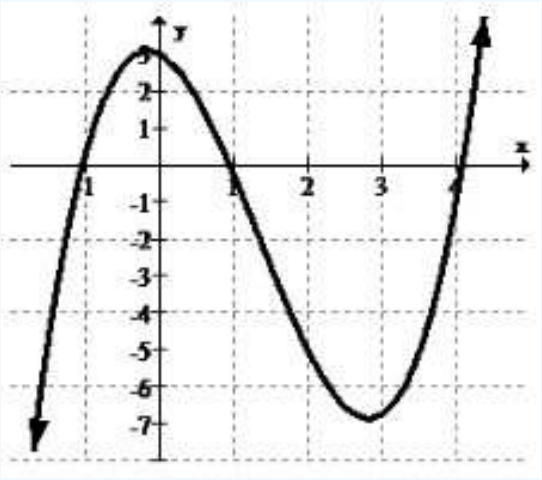
Identify and categorize all extrema.



There are no global or local maximums or minimums. The function flattens, but does not actually reach a peak or a valley.

Example 3

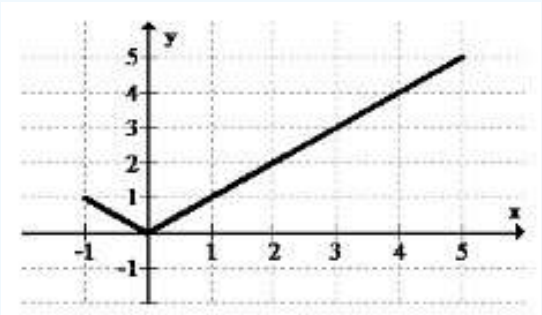
Identify and categorize all the extrema in the graph below:



Since the function appears from the arrows to increase and decrease beyond the display, there are no global extrema. There is a local maximum at approximately (0, 3) and a local minimum at approximately (2.8, -7).

Example 4

Identify and categorize all the extrema in the graph below:

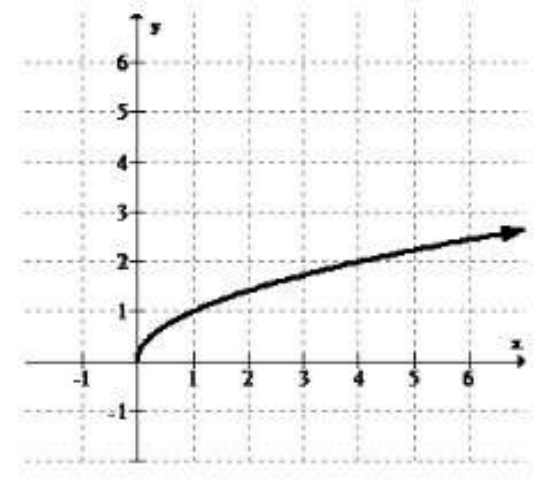


Since the function seems to abruptly end at the end points and does not go beyond the display, the endpoints are important.

There is a global minimum at (0, 0). There is a local maximum at (-1, 1) and a global maximum at (5, 5).

Example 5

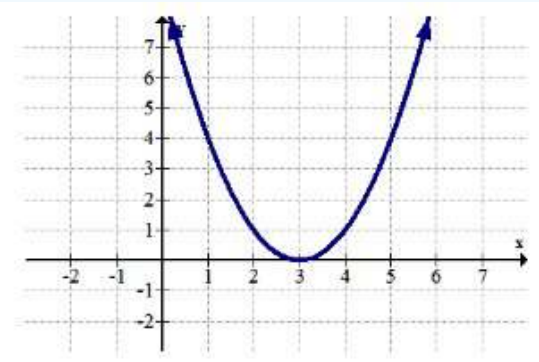
Identify and categorize all the extrema in the graph below:



Since this function appears to increase to the right as indicated by the arrow there is no global maximum. There are not any other high points either, so there are no local maximums. There is only the end point at (0, 0) which is a global minimum.

Review

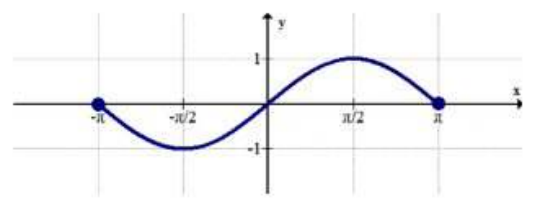
Use the graph below for 1-2.



1. Identify any global extrema.

2. Identify any local extrema.

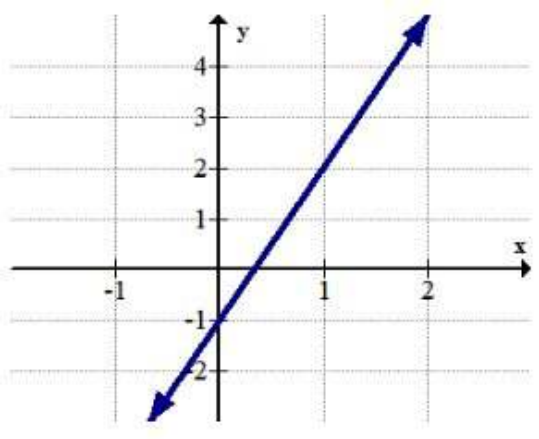
Use the graph below for 3-4.



3. Identify any global extrema.

4. Identify any local extrema.

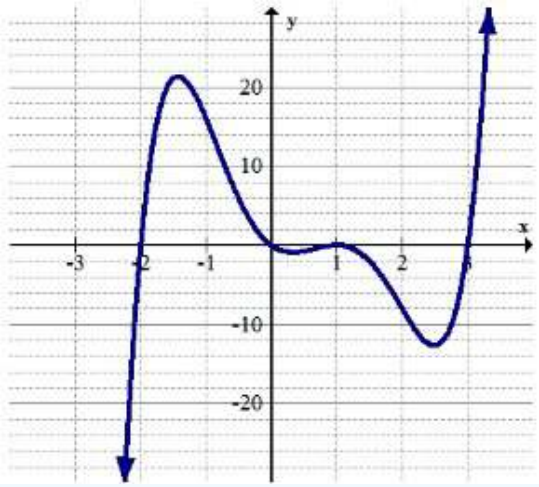
Use the graph below for 5-6.



5. Identify any global extrema.

6. Identify any local extrema.

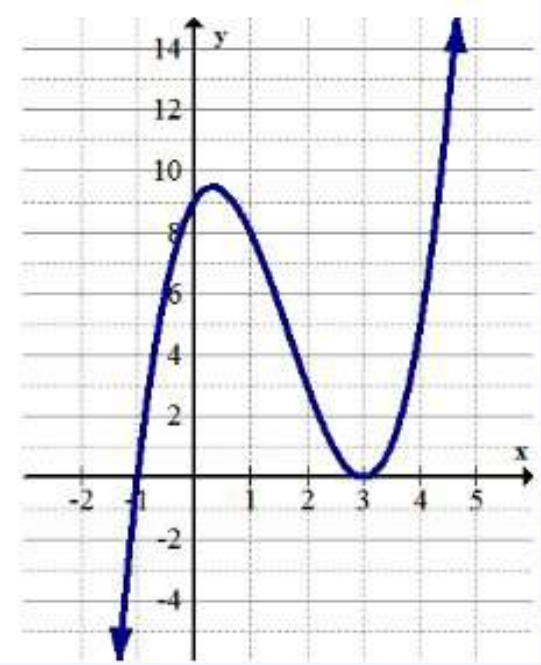
Use the graph below for 7-8.



7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10.



9. Identify any global extrema.

10. Identify any local extrema.

11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use your graphing calculator to identify and categorize the extrema of:

$f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24$

15. Use your graphing calculator to identify and categorize the extrema of:

$g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3$

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1.6: 1.6 Symmetry

Some functions, like the sine function, the absolute value function and the squaring function, have reflection symmetry across the line $x = 0$. Other functions like the cubing function and the reciprocal function have rotational symmetry about the origin.

Why is the first group categorized as even functions while the second group is categorized as odd functions?

Even and Odd Functions

Even Functions

Functions symmetrical across the line $x = 0$ (the y axis) are called even. Even functions have the property that when a negative value is substituted for x, it produces the same value as when the positive value is substituted for the x. In other words, the equation $f(-x) = f(x)$ holds true for even functions.

To show that the function $f(x) = 3x^4 - 5x^2 + 1$ is even, show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= 3(-x)^4 - 5(-x)^2 + 1 \\ &= 3x^4 - 5x^2 + 1 \\ &= f(x) \end{aligned}$$

The property that both positive and negative numbers raised to an even power are always positive is the reason why the term even is used. It does not matter that the coefficients are even or odd, just the exponents.

Odd Functions

Functions that have rotational symmetry about the origin are called odd functions. Odd functions have the property that when a negative x value is substituted into the function, it produces a negative version of the function evaluated at a positive value. In other words, the equation $f(-x) = -f(x)$ holds true for odd functions.

This property becomes increasingly important in problems and proofs of Calculus and beyond, but for now it is sufficient to identify functions that are even, odd or neither and show why.

To show that $f(x) = 4x^3 - x$ is odd, show that $f(-x) = -f(x)$.

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 + x \\ &= -\left(4x^3 + x\right) \\ &= -f(x) \end{aligned}$$

Just like even functions are named, odd functions are named because negative signs don't disappear and can always be factored out of odd functions.

Even and odd functions describe different types of symmetry, but both derive their name from the properties of exponents. A negative number raised to an even number will always be positive. A negative number raised to an odd number will always be negative.

Examples

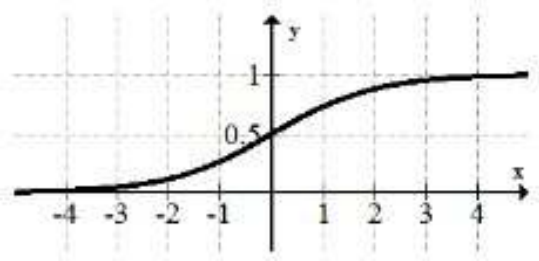
Example 1

Which of the basic functions are even, which are odd and which are neither?

Even Functions: The squaring function and the absolute value function.

Odd Functions: The identity function, the cubing function, the reciprocal function, the sine function.

Neither: The square root function, the exponential function and the log function. The logistic function is also neither because it is rotationally symmetric about the point $\left(0, \frac{1}{2}\right)$ as opposed to the origin.



Example 2

Suppose $h(x)$ is an even function and $g(x)$ is an odd function. $f(x) = h(x) + g(x)$. Is $f(x)$ even or odd? If $h(x)$ is even then $h(-x) = h(x)$. If $g(x)$ is odd then $g(-x) = -g(x)$

Therefore: $f(-x) = h(-x) + g(-x) = h(x) - g(x)$

This does not match $f(x) = h(x) + g(x)$ nor does it match $-f(x) = -h(x) - g(x)$

This is a proof that shows the sum of an even function and an odd function will never itself be even or odd.

Example 3

Determine whether the following function is even, odd, or neither.

$f(x) = x\left(x^2 - 1\right)\left(x^4 + 1\right)$

Identify whether the function is even, odd or neither and explain why.

$$\begin{aligned} f(x) &= x\left(x^2 - 1\right)\left(x^4 + 1\right) \\ f(-x) &= (-x)\left((-x)^2 - 1\right)\left((-x)^4 + 1\right) \\ &= -x\left(x^2 - 1\right)\left(x^4 + 1\right) \\ &= -f(x) \end{aligned}$$

The function is odd because $f(-x) = -f(x)$ holds true.

$f(x) = 4x^3 - |x|]$

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 - x \end{aligned}$$

This does not seem to match either $f(x) = 4x^3 - |x|$ or $-f(x) = -4x^3 + |x|$. Therefore, this function is neither even nor odd.

Note: This function is a difference of an odd function and an even function. This should be a clue that the resulting function is neither even nor odd.

Review

Determine whether the following functions are even, odd, or neither.

1. $f(x) = -4x^2 + 1$
2. $g(x) = 5x^3 - 3x$
3. $h(x) = 2x^2 - x$
4. $j(x) = (x - 4)(x - 3)^3$
5. $k(x) = x\left(x^2 - 1\right)^2$
6. $f(x) = 2x^3 - 5x^2 - 2x + 1$
7. $g(x) = 2x^2 - 4x + 2$
8. $h(x) = -5x^4 + x^2 + 2$
9. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) - g(x)$ is neither even nor odd.

10. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = \frac{h(x)}{g(x)}$ is odd.
11. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) \cdot g(x)$ is odd.
12. Is the sum of two even functions always an even function? Explain.
13. Is the sum of two odd functions always an odd function? Explain.
14. Why are some functions neither even nor odd?
15. If you know that a function is even or odd, what does that tell you about the symmetry of the function?

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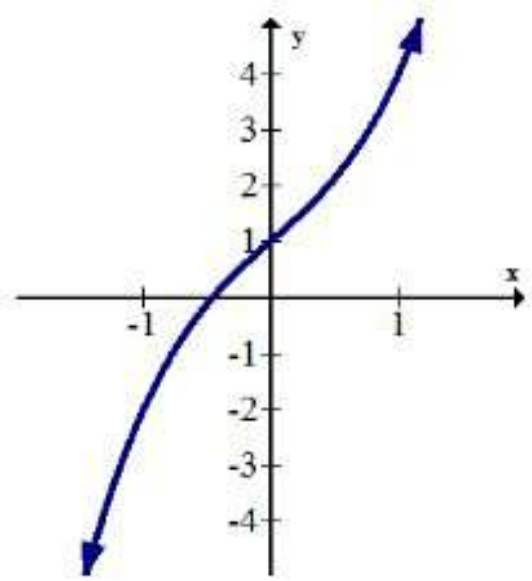
1.7: 1.7 Increasing and Decreasing

It is important to be able to distinguish between when functions are increasing and when they are decreasing. In business this could mean the difference between making money and losing money. In physics it could mean the difference between speeding up and slowing down.

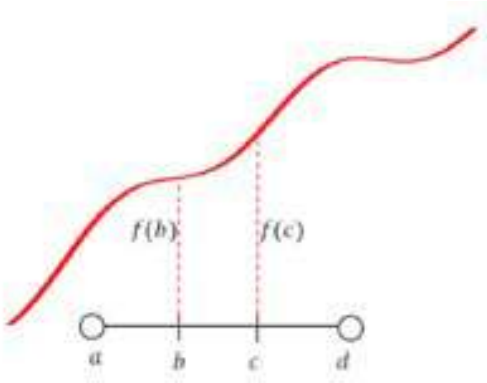
How do you decide when a function is increasing or decreasing?

Increasing and Decreasing Functions

Increasing means places on the graph where the slope is positive.

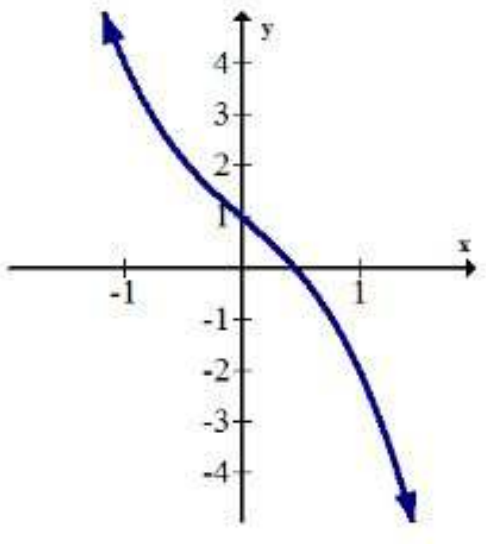


The formal definition of an increasing interval is: an open interval on the x axis of (a, d) where every $b, c \in (a, d)$ with $b < c$ has $f(b) \leq f(c)$



A interval is said to be strictly increasing if $f(b) < f(c)$ is substituted into the definition.

Decreasing means places on the graph where the slope is negative. The formal definition of decreasing and strictly decreasing are identical to the definition of increasing with the inequality sign reversed.



A function is called **monotonic** if the function only goes in one direction and never switches between increasing and decreasing.

Out of the basic functions, the monotonically increasing functions are:

$f(x) = x, f(x) = x^3, f(x) = \sqrt{x}, f(x) = e^x, f(x) = \ln x, f(x) = \frac{1}{1+e^{-x}}$

The only basic functions that are not monotonically increasing are:

$f(x) = x^2, f(x) = |x|, f(x) = \frac{1}{x}, f(x) = \sin x$

Identifying analytically where functions are increasing and decreasing often requires Calculus. For PreCalculus, it will be sufficient to be able to identify intervals graphically and through your knowledge of what the parent functions look like.

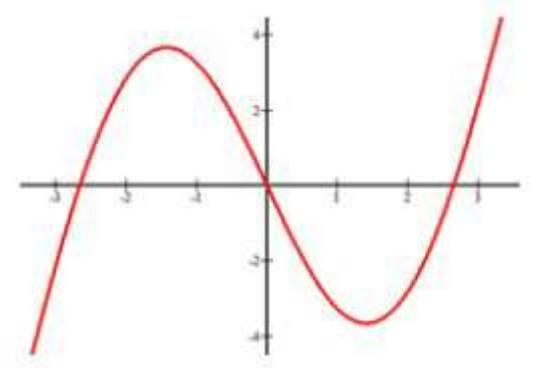
Examples

Example 1

Earlier, you were asked if a function is increasing or decreasing. Increasing is where the function has a positive slope and decreasing is where the function has a negative slope. A common misconception is to look at the squaring function and see two curves that symmetrically increase away from zero. Instead, you should always read functions from left to right and draw slope lines and decide if they are positive or negative.

Example 2

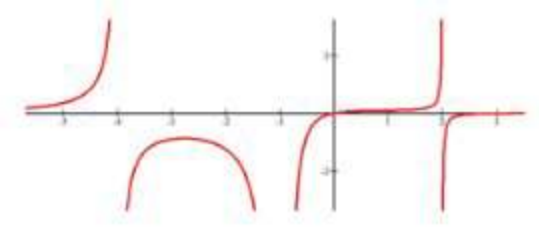
Estimate where the following function is increasing and decreasing.



Increasing: $x \in (-\infty, -1.5) \cup (1.5, \infty)$.
Decreasing: $x \in (-1.5, 1.5)$

Example 3

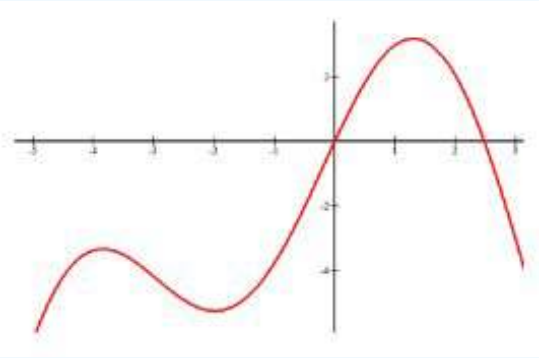
Estimate where the following function is increasing and decreasing.



Increasing $x \in (-\infty, -4) \cup (-4, -2.7) \cup (-1, 2) \cup (2, \infty)$.
Decreasing $x \in (-2.7, -1)$

Example 4

Estimate the intervals where the function is increasing and decreasing.



Increasing: $x \in (-\infty, -4) \cup (-2, 1.5)$
Decreasing: $x \in (-4, -2) \cup (1.5, \infty)$

Notice that open intervals are used because at $x = -4, -2, 1.5$ the slope of the function is zero. This is here the slope transitions from being positive to negative. The reason why open parentheses are used is because the function is not actually increasing or decreasing at those specific points.

Example 5

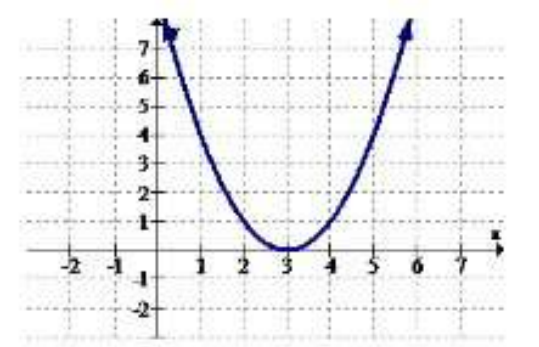
A continuous function has a global maximum at the point (3, 2), a global minimum at (5, -12) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

Increasing $x \in (-\infty, 3) \cup (5, \infty)$.
Decreasing $x \in (3, 5)$

Note: The y coordinates are not used in the intervals. A common mistake is to want to use the y coordinates.

Review

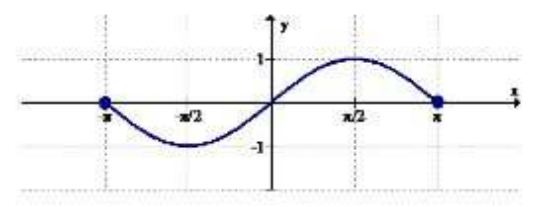
Use the graph below for 1-2.



1. Identify the intervals (if any) where the function is increasing.

2. Identify the intervals (if any) where the function is decreasing.

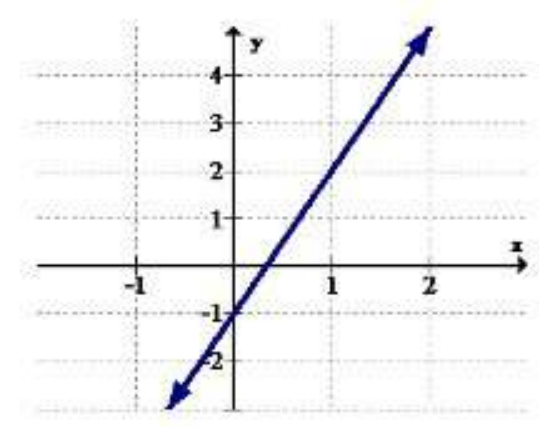
Use the graph below for 3-4.



3. Identify the intervals (if any) where the function is increasing.

4. Identify the intervals (if any) where the function is decreasing.

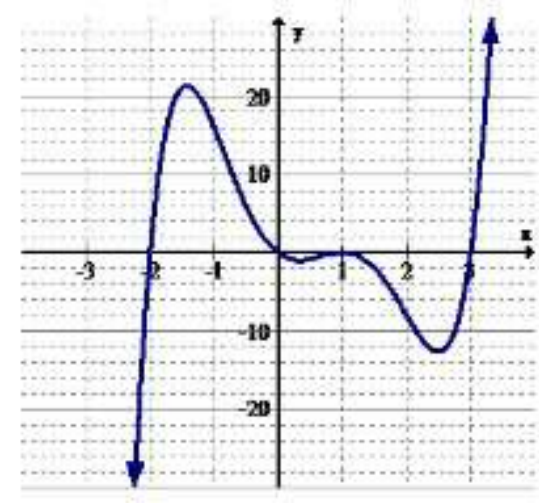
Use the graph below for 5-6.



5. Identify the intervals (if any) where the function is increasing.

6. Identify the intervals (if any) where the function is decreasing.

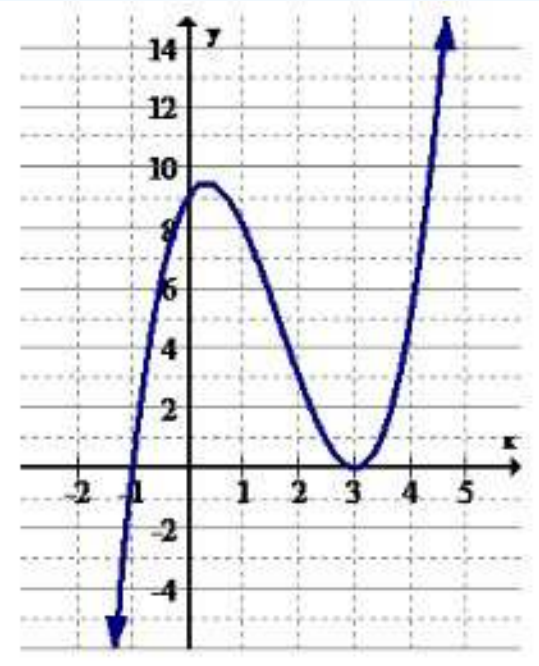
Use the graph below for 7-8.



7. Identify the intervals (if any) where the function is increasing.

8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10.



9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A continuous function has a global maximum at the point (1, 4), a global minimum at (3, -6) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

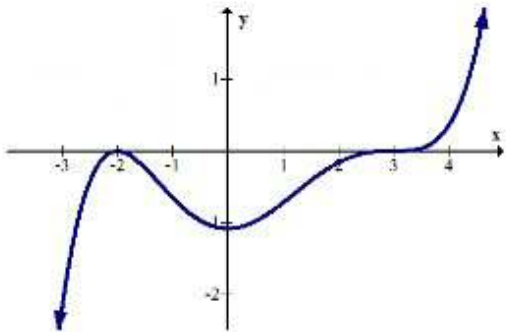
14. A continuous function has a global maximum at the point (1, 1) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

15. A continuous function has a global minimum at the point (5, -15) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

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1.8: 1.8 Zeroes and Intercepts of Functions

An **intercept** in mathematics is where a function crosses the x or y axis. What are the intercepts of this function?



X and Y Intercepts

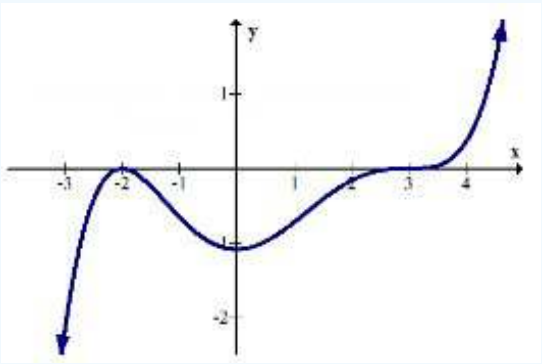
The first type of intercept you may have learned is the y -intercept when you learned the slope intercept form of a line: $y = mx + b$. A y **-intercept** is the unique point where a function crosses the y axis. It can be found algebraically by setting $x = 0$ and solving for y.

x **-intercepts** are where functions cross the x axis and where the height of thefunction is zero. They are also called roots, solutions and zeroes of a function. They are found algebraically by setting $y = 0$ and solving for x. Watch the videos below for practice:

Examples

Example 1

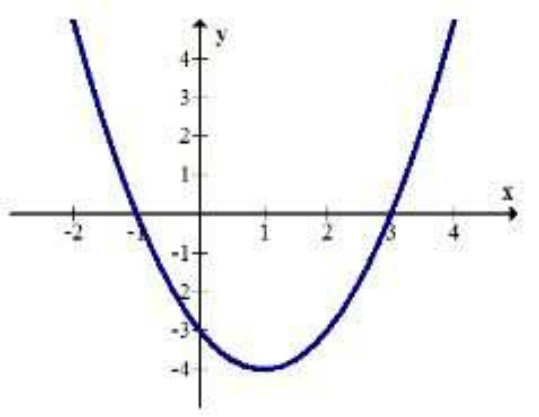
Earlier, you were asked of the graph below are.



Graphically the function has zeroes at -2 and 3 with a y intercept at about -1.1.
Note: In order for a function to pass the vertical line test, it must only have one y -intercept, but it may have multiple x -intercepts.

Example 2

What are the zeroes and y -intercepts of the parabola $y = x^2 - 2x - 3$?
Using a graph:



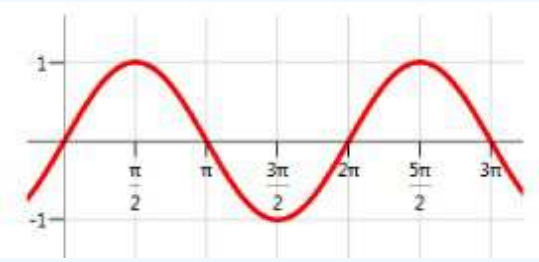
The zeroes are at (-1,0) and (3,0) . The y -intercept is at (0,-3) .
Using Algebra:

Substitute 0 for y to find zeroes.
$0 = x^2 - 2x - 3 = (x-3)(x+1)$
$y = 0, x = 3, -1$

Substitute 0 for x to find the y -intercept.
$y = (0)^2 - 2(0) - 3 = -3$
$x = 0, y = -3$

Example 3

Identify the zeroes and y -intercepts for the sine function.



The y -intercept is (0,0) . There are four zeroes visible on this portion of the graph. One thing you know about the sine graph is that it is periodic and repeats forever in both directions. In order to capture every x -intercept, you must identify a pattern instead of trying to write out every single one.

The visible x -intercepts are $0, \pi, 2\pi, 3\pi$. The pattern is that there is an x -intercept every multiple of π including negative multiples. In order to describe all of these values you should write:

The x -intercepts are $\pm n\pi$ where n is an integer $\{0, \pm 1, \pm 2, ...\}$.

Example 4

Identify the intercepts and zeroes of the function: $f(x) = \frac{1}{100}(x-3)^3(x+2)^2$

To find the y -intercept, substitute o for x :

$y = \frac{1}{100}(0-3)^3(0+2)^2 = \frac{1}{100}(-27)(4) = -\frac{108}{100} = -1.08$

To find the x -intercepts, substitute 0 for y :

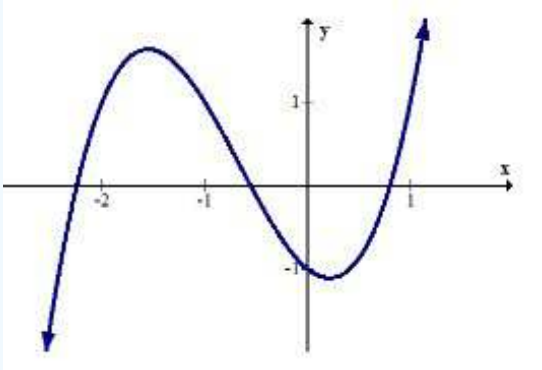
$0 = \frac{1}{100}(x-3)^3(x+2)^2$

$x = 3, -2$

Thus the y -intercept is (0,-1.08) and the x -intercepts are (3,0) and (-2,0)

Example 5

Determine the intercepts of the following function graphically.



The y -intercept is approximately (0,-1) . The x -intercepts are approximately (-2.3,0), (-0.4,0) and $(0.7, 0)$. When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

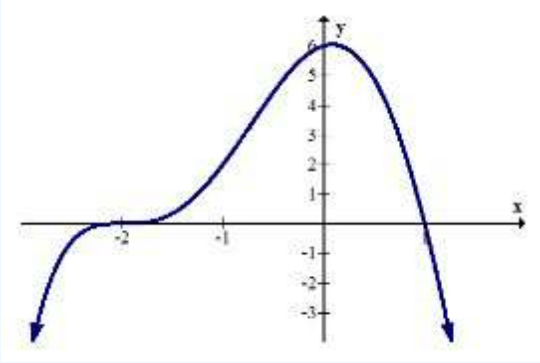
Review

1. Determine the zeroes and y -intercept of the following function using algebra:

$f(x) = (x+1)^3(x-4)$

2. Determine the roots and y -intercept of the following function using algebra or a graph:

3. Determine the intercepts of the following function graphically:



Find the intercepts for each of the following functions.

4. $y = x^2$
5. $y = x^3$
6. $y = \ln(x)$
7. $y = \frac{1}{x}$
8. $y = e^x$
9. $y = \sqrt{x}$
10. Are there any functions without a y -intercept? Explain.
11. Are there any functions without an x -intercept? Explain.
12. Explain why it makes sense that an x -intercept of a function is also called a "zero" of the function.

Determine the intercepts of the following functions using algebra or a graph.

13. $h(x) = x^3 - 6x^2 + 3x + 10$
14. $j(x) = x^2 - 6x - 7$
15. $k(x) = 4x^4 - 20x^3 - 3x^2 + 14x + 5$

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1.9: 1.9 Asymptotes and End Behavior

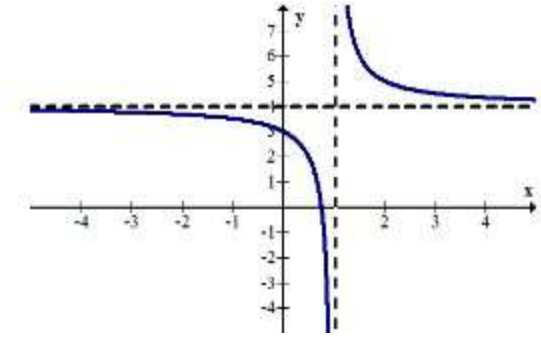
Most functions continue beyond the viewing window in our calculator or computer. People often draw an arrow next to a dotted line to indicate the pattern specifically. How can you recognize these asymptotes?

Asymptotes and End Behavior of Functions

A vertical asymptote is a vertical line such as $x = 1$ that indicates where a function is not defined and yet gets infinitely close to.

A horizontal asymptote is a horizontal line such as $y = 4$ that indicates where a function flattens out as x gets very large or very small. A function may touch or pass through a horizontal asymptote.

The reciprocal function has two asymptotes, one vertical and one horizontal. Most computers and calculators do not draw the asymptotes and so they must be inserted by hand as dotted lines.

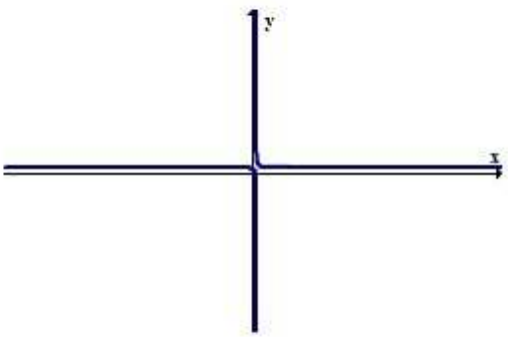


Many students have the misconception that an asymptote is a line that a function gets infinitely close to but does not touch. This is not true. Take the following function:

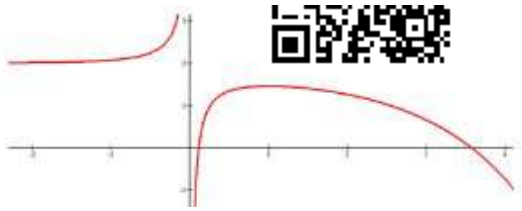


The graph appears to flatten as x grows larger. Thus, the horizontal asymptote is $y = 0$ even though the function clearly passes through this line an infinite number of times.

The reason why asymptotes are important is because when your perspective is zoomed way out, the asymptotes essentially become the graph.



To find the asymptotes and e function below, examine what happens to x and y as they each increase or decrease.



The function has a horizontal asymptote $y = 2$ as x approaches negative infinity. There is a vertical asymptote at $x = 0$. The right hand side seems to decrease forever and has no asymptote.

Note that slant asymptotes do exist and are called oblique asymptotes.

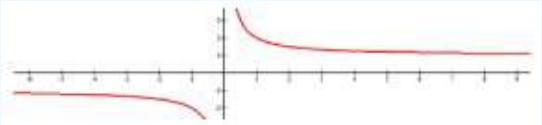
Examples

Example 1

Earlier, you were asked how to identify asymptotes on a graph. Asymptotes written by hand are usually identified with dotted lines next to the function that indicate how the function will behave outside the viewing window. The equations of these vertical and horizontal dotted lines are of the form x=___ and y=____. When problems ask you to find the asymptotes of a function, they are asking for the equations of these horizontal and vertical lines.

Example 2

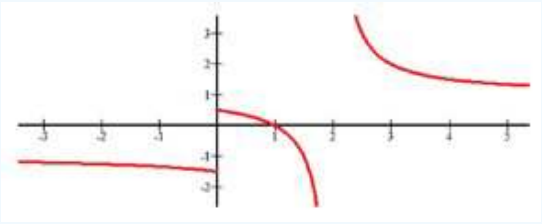
Identify the horizontal and vertical asymptotes of the following function.



There is a vertical asymptote at $x = 0$. As x gets infinitely small, there is a horizontal asymptote at $y = -1$. As x gets infinitely large, there is another horizontal asymptote at $y = 1$.

Example 3

Identify the horizontal and vertical asymptotes of the following function.

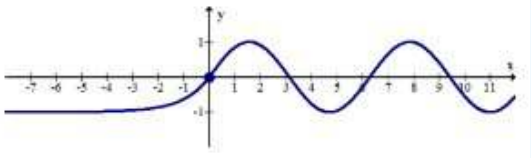


There is a vertical asymptote at $x = 2$. As x gets infinitely small there is a horizontal asymptote at $y = -1$. As x gets infinitely large, there is a horizontal asymptote at $y = 1$.

Example 4

Identify the horizontal and vertical asymptotes of the following piecewise function:

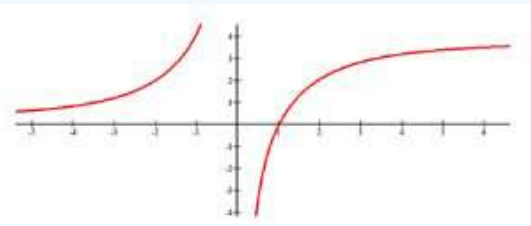
$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$



There is a horizontal asymptote at $y = -1$ as x gets infinitely small. This is because e raised to the power of a very small number becomes $0.000000\ldots$ and basically becomes zero.

Example 5

Identify the asymptotes and end behavior of the following function.

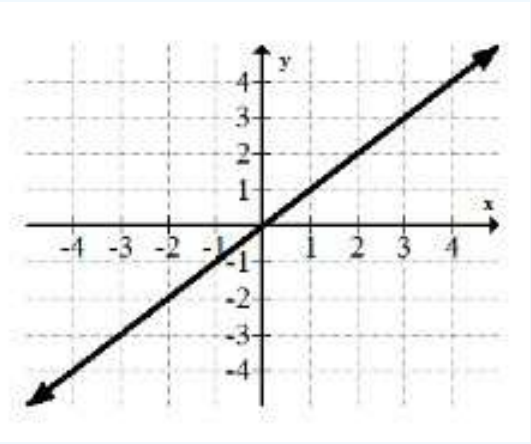


There is a vertical asymptote at $x = 0$. The end behavior of the right and left side of this function does not match. The horizontal asymptote as x approaches negative infinity is $y = 0$ and the horizontal asymptote as x approaches positive infinity is $y = 4$. At this point you can only estimate these heights because you were not given the function or the tools to find these values analytically.

Review

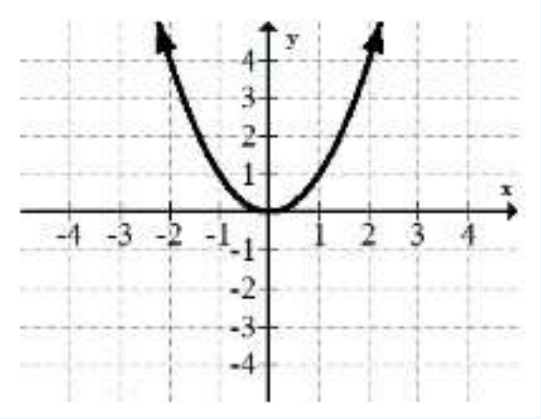
Identify the asymptotes and end behavior of the following functions.]

1. $y = x$

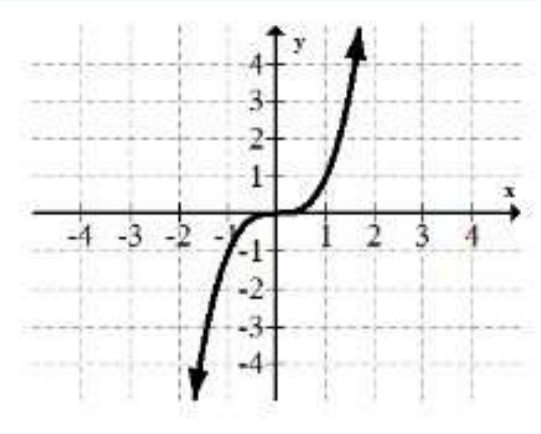


2. $y = x^2$

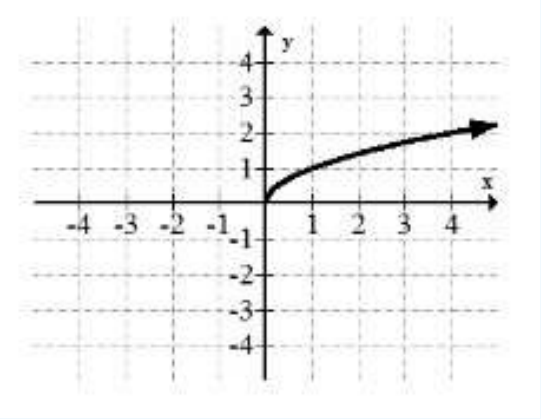
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3. $y = x^3$

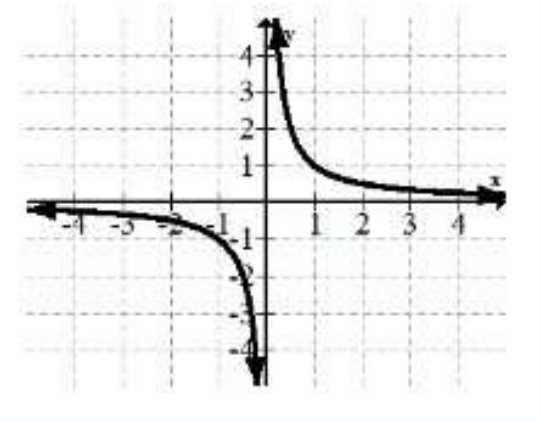


4. $y = \sqrt{x}$

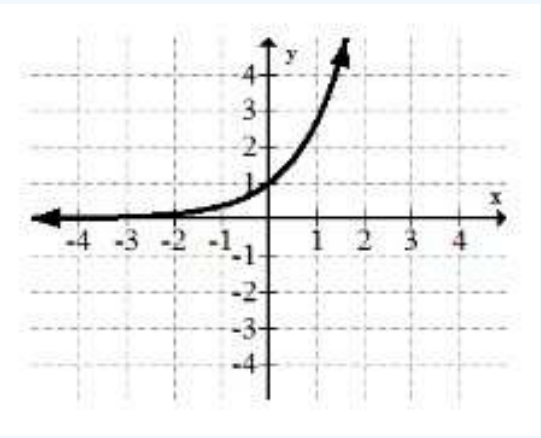


5. $y = \frac{1}{x}$

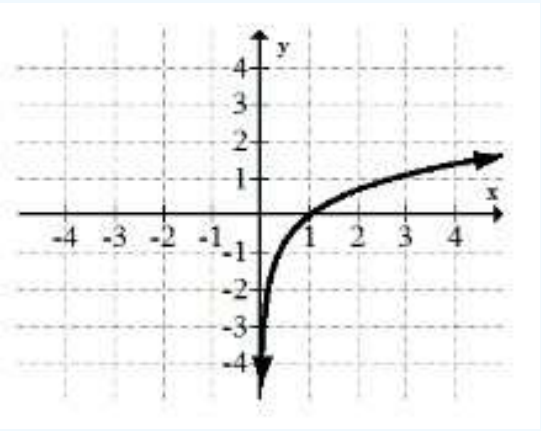
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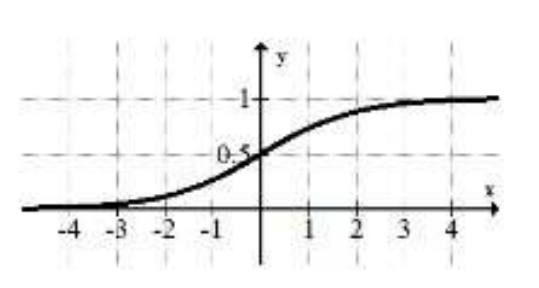
6. $y = e^x$



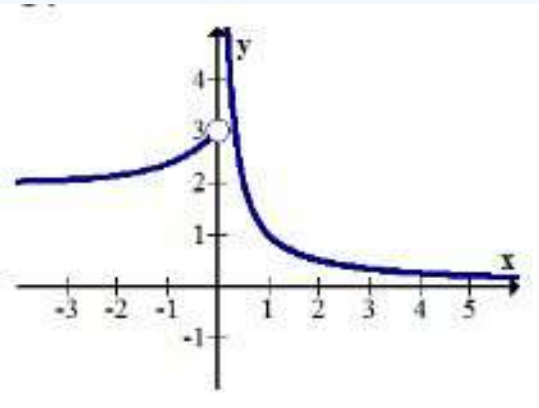
7. $y = \ln(x)$



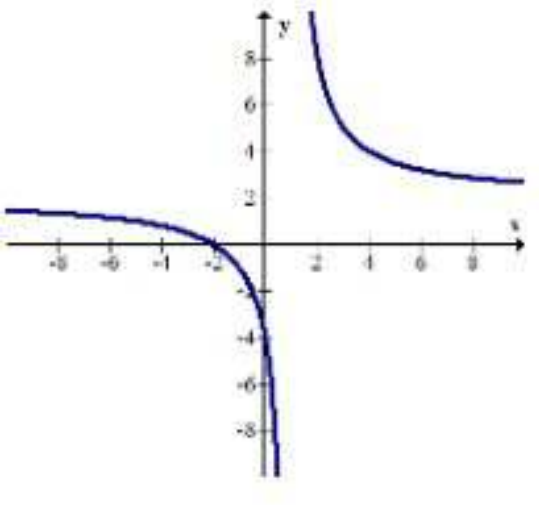
8. $y = \frac{1}{1+e^{-x}}$



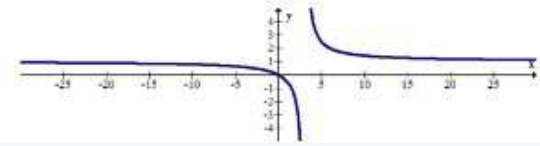
9.



10.



11.



12. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x}$ has a vertical asymptote at $x = 0$.

13. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x+3}$ has a vertical asymptote at $x = -3$.

14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = \frac{1}{x-2}$

15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = \frac{2}{x+4}$

c

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1.10: 1.10 Continuity and Discontinuity

Continuity is a property of functions that can be drawn without lifting your pencil. Some functions, like the reciprocal functions, have two distinct parts that are unconnected. Functions that are unconnected are discontinuous. What are the three ways functions can be discontinuous and how do they come about?

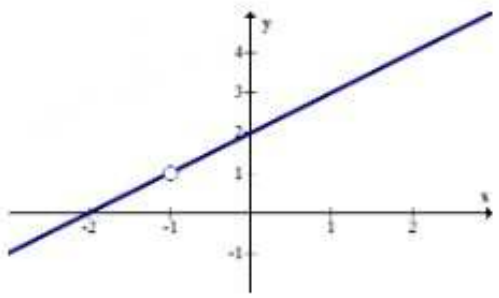
Continuity and Discontinuity of Functions

Functions that can be drawn without lifting up your pencil are called **continuous functions**. You will define continuous in a more mathematically rigorous way after you study limits.

There are three types of discontinuities: Removable, Jump and Infinite.

Removable Discontinuities

Removable discontinuities occur when a rational function has a factor with an x that exists in both the numerator and the denominator. Removable discontinuities are shown in a graph by a hollow circle that is also known as a hole. Below is the graph for $f(x) = \frac{(x+2)(x+1)}{x+1}$. Notice that it looks just like $y = x + 2$ except for the hole at $x = -1$. When graphing function, you should cancel the removable factor, graph like usual and then insert a hole in the appropriate spot at the end. There is a hole at $x = -1$ because when $x = -1, f(x) = \frac{0}{0}$.

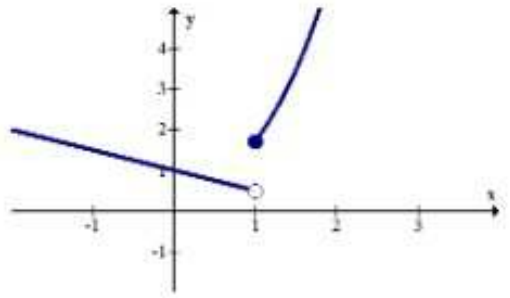


Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make $f(x)$ continuous you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

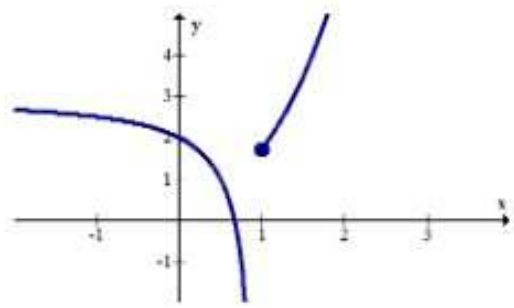
Jump Discontinuities

Jump discontinuities occur when a function has two ends that don't meet even if the hole is filled in. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity.



Infinite Discontinuities

Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below $x = 1$.



Examples

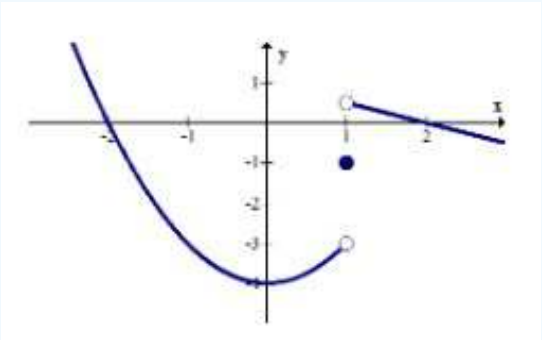
Example 1

Earlier you were asked how functions can be discontinuous. There are three ways that functions can be discontinuous. When a rational function has a vertical asymptote as a result of the denominator being equal to zero at some point, it will have an infinite discontinuity at that point. When the numerator and denominator of a rational function have one or more of the same factors, there will be removable discontinuities corresponding to each of these factors. Finally, when the different parts of a piecewise function don't "match", there will be a jump discontinuity.

Example 2

Identify the discontinuity of the piecewise function graphically.

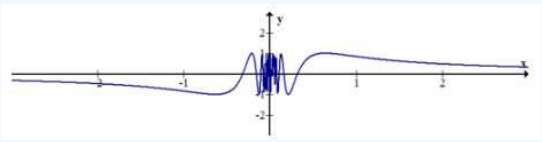
$$f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$$



There is a jump discontinuity at $x = 1$. The piecewise function describes a function in three parts; a parabola on the left, a single point in the middle and a line on the right.

Example 3

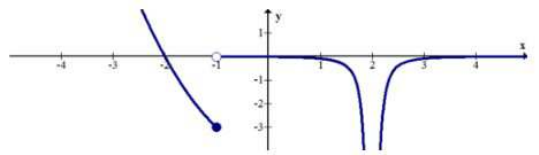
Describe the continuity or discontinuity of the function $f(x) = \sin\left(\frac{1}{x}\right)$.



The function seems to oscillate infinitely as x approaches zero. One thing that the graph fails to show is that 0 is clearly not in the domain. The graph does not shoot to infinity, nor does it have a simple hole or jump discontinuity. Calculus and Real Analysis are required to state more precisely what is going on.

Example 4

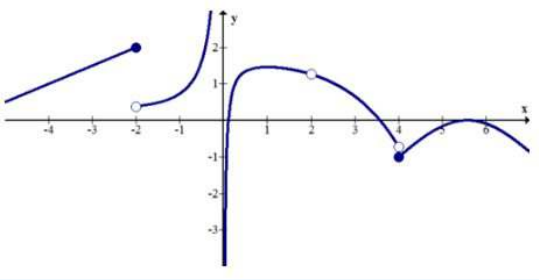
Describe the discontinuities of the function below.



There is a jump discontinuity at $x = -1$ and an infinite discontinuity at $x = 2$.

Example 5

Describe the discontinuities of the function below.

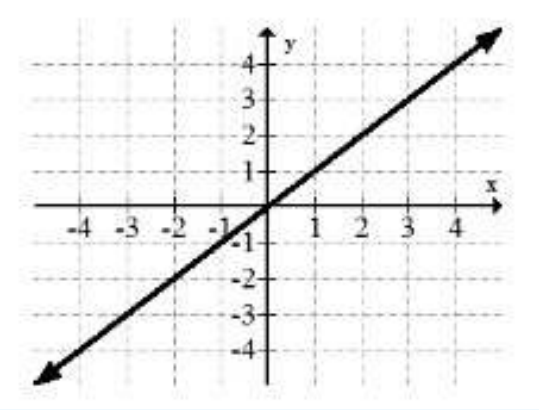


There are jump discontinuities at $x = -2$ and $x = 4$. There is a removable discontinuity at $x = 2$. There is an infinite discontinuity at $x = 0$.

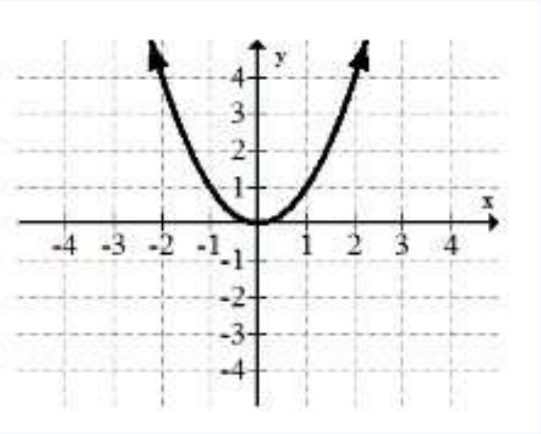
Review

Describe any discontinuities in the functions below:

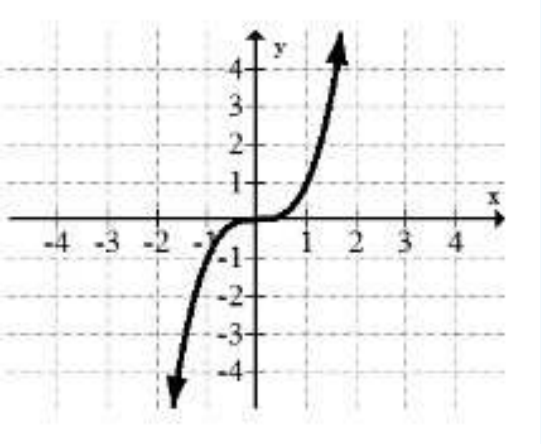
1. $y = x$



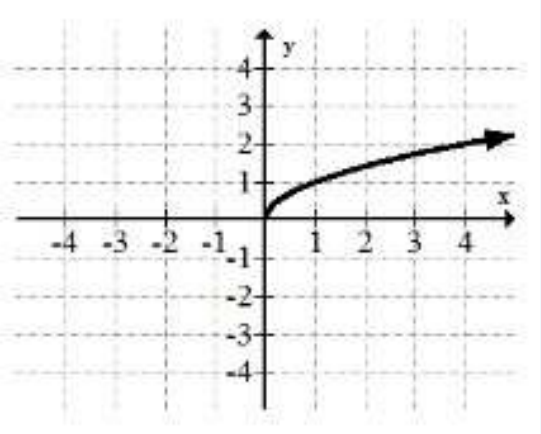
2. $y = x^2$



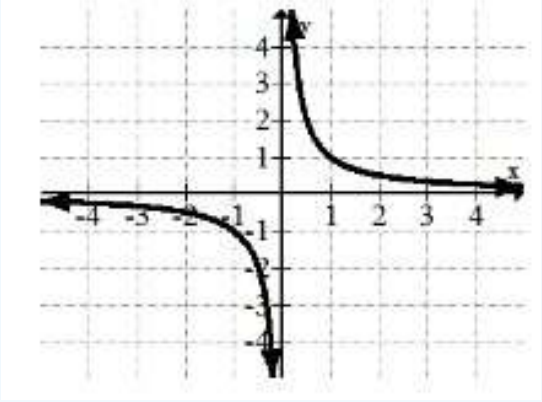
3. $y = x^3$



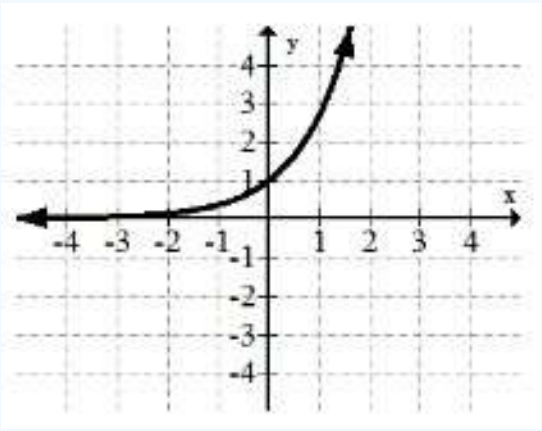
4. $y = \sqrt{x}$



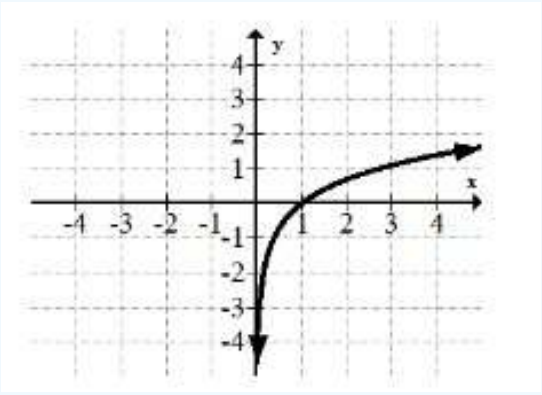
5. $y = \frac{1}{x}$



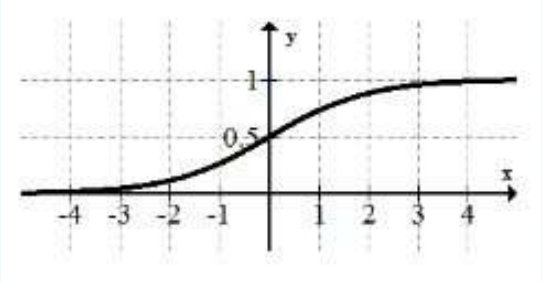
6. $y = e^x$



7. $y = \ln(x)$

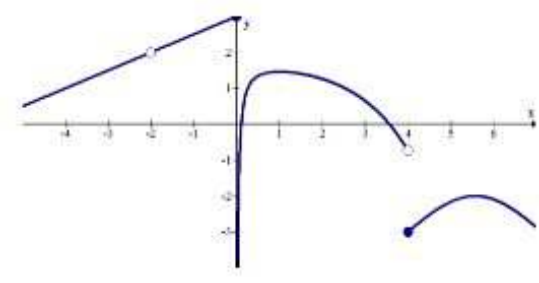


8. $y = \frac{1}{1+e^{-x}}$

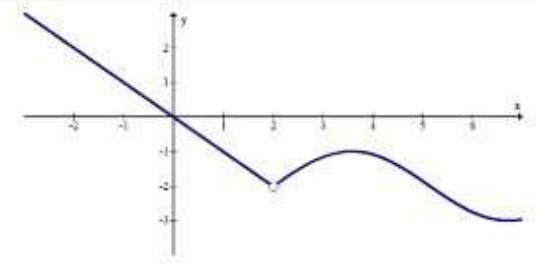


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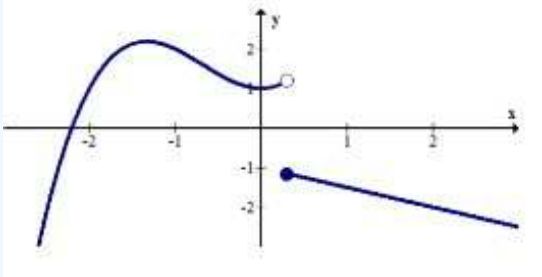
9.



10.



11.



12. $f(x)$ has a jump discontinuity at $x = 3$, a removable discontinuity at $x = 5$, and another jump discontinuity at $x = 6$. Draw a picture of a graph that could be $f(x)$.

13. $g(x)$ has a jump discontinuity at $x = -2$, an infinite discontinuity at $x = 1$, and another jump discontinuity at $x = 3$. Draw a picture of a graph that could be $g(x)$.

14. $h(x)$ has a removable discontinuity at $x = -4$, a jump discontinuity at $x = 1$, and another jump discontinuity at $x = 7$. Draw a picture of a graph that could be $h(x)$.

15. $j(x)$ has an infinite discontinuity at $x = 0$, a removable discontinuity at $x = 1$, and a jump discontinuity at $x = 4$. Draw a picture of a graph that could be $j(x)$.

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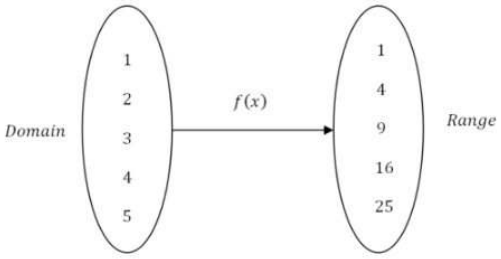
1.11: 1.11 Function Composition

Functions can be added, subtracted, multiplied and divided creating new functions and graphs that are complicated combinations of the various original functions. One important way to transform functions is through function composition. Function composition allows you to line up two or more functions that act on an input in tandem.

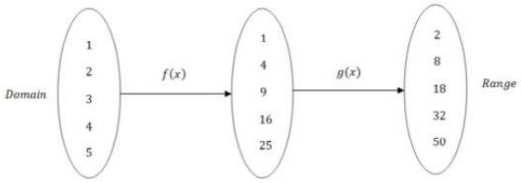
Is function composition essentially the same as multiplying the two functions together?

Composition of Functions

A common way to describe functions is a mapping from the domain space to the range space:



Function composition means that you have two or more functions and the range of the first function becomes the domain of the second function.



There are two notations used to describe function composition. In each case the order of the functions matters because arithmetically the outcomes will be different. Squaring a number and then doubling the result will be different from doubling a number and then squaring the result. In the diagram above, $f(x)$ occurs first and $g(x)$ occurs second. This can be written as:

$g(f(x))$ or $(g \circ f)(x)$

You should read this "g of f of x." In both cases notice that the f is closer to the x and operates on the x values first.

Examples

Example 1

Earlier, you were asked ition is the same as multiplying two functions together. Function composition is not the same as multiplying ther. With function composition there is an outside function and an inside function. Suppose the two functions were doubling and squaring. It is clear just by looking at the example input of the number 5 that 50 (squaring then doubling) is different from 100 (doubling then squaring). Both 50 and 100 are examples of function composition, while 250 (five doubled multiplied by five squared) is an example of the product of two separate functions happening simultaneously.

For the next two examples, use the functions below:

$f(x) = x^2 - 1$
$h(x) = \frac{x-1}{x+5}$
$g(x) = 3e^x - x$
$j(x) = \sqrt{x+1}$

Example 2

Show $f(h(x)) \neq h(f(x))$

$f(h(x)) = f\left(\frac{x-1}{x+5}\right) = \left(\frac{x-1}{x+5}\right)^2 - 1$

$h(f(x)) = h\left(x^2-1\right) = \frac{(x^2-1)-1}{(x^2-1)+5} = \frac{x^2-2}{x^2+4}$

In order to truly show they are not equal it is best to find a specific counter example of a number where they differ. Sometimes algebraic expressions may look different, but are actually the same. You should notice that $f(h(x))$ is undefined when $x = -5$ because then there would be zero in the denominator. $h(f(x))$ on the other hand is defined at $x = -5$. since the two function compositions differ, you can conclude:

$f(h(x)) \neq h(f(x))$

Example 3

What is $f(j(h(g(x))))$?

These functions are nested within the arguments of the other functions. Sometimes functions simplify significantly when composed together, as f and j do in this case. It makes sense to evaluate those two functions first together and keep them on the outside of the argument.

$f(x) = x^2 - 1; h(x) = \frac{x-1}{x+5}; g(x) = 3e^x - x; j(x) = \sqrt{x+1}$

$f(j(y)) = f(\sqrt{y+1}) = (\sqrt{y+1})^2 - 1 = y + 1 - 1 = y$

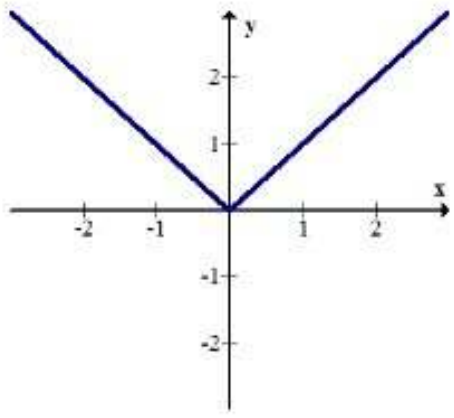
Notice how the composition of f and j produced just the argument itself?

Thus,

$$f(j(h(g(x)))) = h(g(x)) = h\left(3e^x - x\right)$$
$$= \frac{(3e^x - x) - 1}{(3e^x - x) + 5}$$
$$= \frac{3e^x - x - 1}{3e^x - x + 5}$$

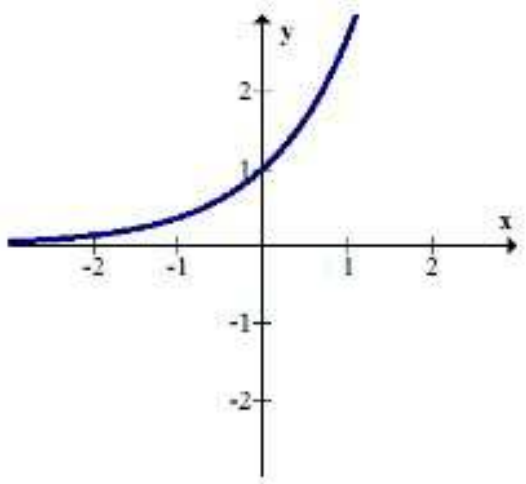
For the next two examples, use the graphs shown below:

$f(x) = |x|$

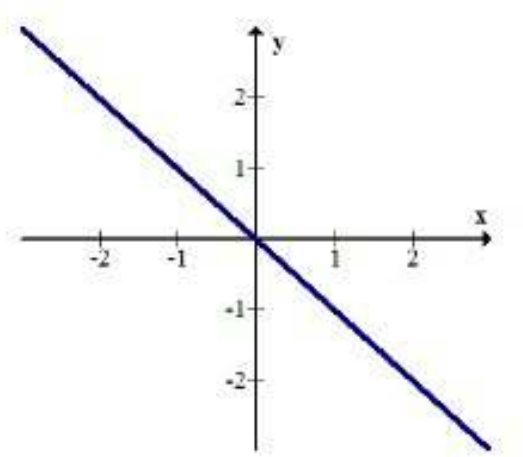


$g(x) = e^x$

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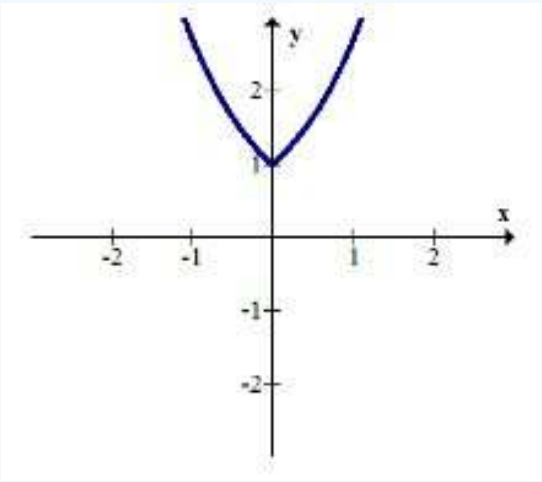
$h(x) = -x$



Example 4

Compose $g(f(x))$ and graph the result. Describe the transformation.

$g(f(x)) = g(|x|) = e^{|x|}$



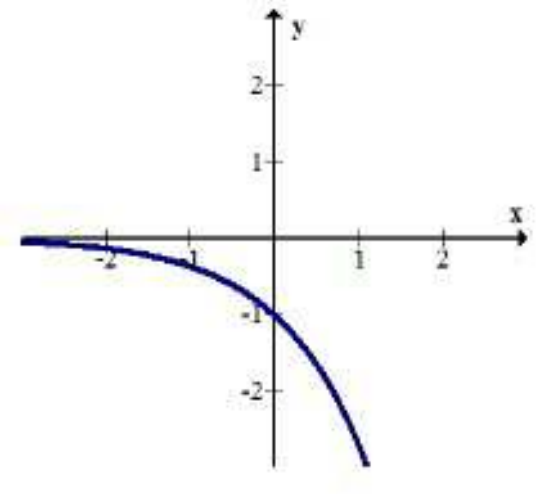
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The positive portion of the exponential graph has been mirrored over the y axis and the negative portion of the exponential graph has been entirely truncated.

Example 5

Compose $h(g(x))$ and graph the result. Describe the transformation.

$h(g(x)) = h\left(e^{x}\right) = -e^{x}$



The exponential graph has been reflected over the x -axis.

Review

For questions 1-9, use the following three functions: $f(x) = |x|, h(x) = -x, g(x) = (x-2)^2 - 3$

1. Graph $f(x), h(x)$ and $g(x)$.
2. Find $f(g(x))$ algebraically.
3. Graph $f(g(x))$ and describe the transformation.
4. Find $g(f(x))$ algebraically.
5. Graph $g(f(x))$ and describe the transformation.
6. Find $h(g(x))$ algebraically.
7. Graph $h(g(x))$ and describe the transformation.
8. Find $g(h(x))$ algebraically.
9. Graph $g(h(x))$ and describe the transformation.

For 10-16, use the following three functions: $j(x) = x^2, k(x) = |x|, m(x) = \sqrt{x}$

10. Graph $j(x), k(x)$ and $m(x)$.
11. Find $j(k(x))$ algebraically.
12. Graph $j(k(x))$ and describe the transformation.
13. Find $k(m(x))$ algebraically.
14. Graph $k(m(x))$ and describe the transformation.
15. Find $m(k(x))$ algebraically.
16. Graph $m(k(x))$ and describe the transformation.

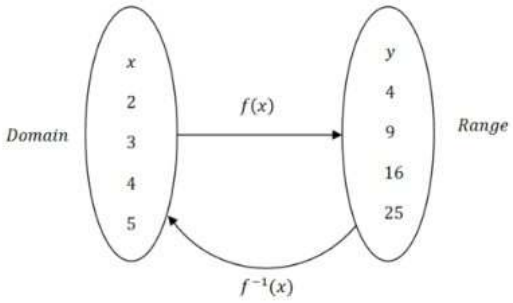
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1.12: 1.12 Inverses of Functions

Functions are commonly known as rules that take inputs and produce outputs. An i**nverse function** does exactly the reverse, undoing what the original function does. How can you tell if two functions are inverses?

Finding Inverses of Functions

A function is written as $f(x)$ and its inverse is written as $f^{-1}(x)$. A common misconception is to see the -1 and interpret it as an exponent and write $\frac{1}{f(x)}$, but this is not correct. Instead, $f^{-1}(x)$ should be viewed as a new function from the range of $f(x)$ back to the domain.

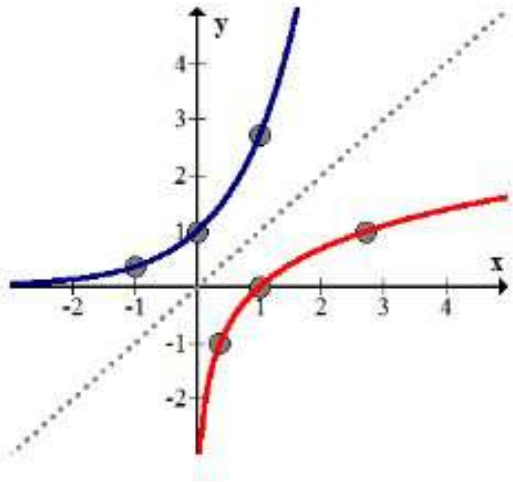


It is important to see the cycle that starts with x, becomes y and then goes back to x. In order for two functions to truly be inverses of each other, this cycle must hold algebraically.

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

When given a function there are two steps to take to find its inverse. In the original function, first switch the variables x and y. Next, solve the function for y. This will give you the inverse function. After finding the inverse, it is important to check both directions of compositions to make sure that together the function and its inverse produce the value x. In other words, verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Graphically, inverses are reflections across the line $y = x$. Below you see inverses $y = e^x$ and $y = \ln x$. Notice how the (x, y) coordinates in one graph become (y, x) coordinates in the other graph.



In order to decide whether an inverse function is also actually a function you can use the vertical line test on the inverse function like usual. You can also use the horizontal line test on the original function. The horizontal line test is exactly like the vertical line test except the lines simply travel horizontally.

Examples

Example 1

Earlier, you were asked how you can tell that two functions are inverses. You can tell that two functions are inverses if each undoes the other, always leaving the original x.

Example 2

Find the inverse, then verify the inverse algebraically. $f(x) = y = (x+1)^2 + 4$

To find the inverse, switch x and y then solve for y.

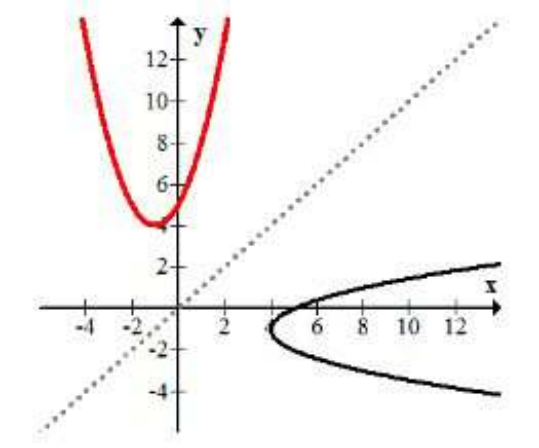
$$\begin{aligned} x &= (y+1)^2 + 4 \\ x-4 &= (y+1)^2 \\ \pm\sqrt{x-4} &= y+1 \\ -1 \pm \sqrt{x-4} &= y = f^{-1}(x) \end{aligned}$$

To verify algebraically, you must show $x = f\left(f^{-1}(x)\right) = f^{-1}(f(x))$

$$\begin{aligned} f\left(f^{-1}(x)\right) &= f(-1 \pm \sqrt{x-4}) \\ &= ((-1 \pm \sqrt{x-4}) + 1)^2 + 4 \\ &= (\pm\sqrt{x-4})^2 + 4 \\ &= x - 4 + 4 = x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left((x+1)^2 + 4\right) \\ &= -1 \pm \sqrt{((x+1)^2 + 4) - 4} \\ &= -1 \pm \sqrt{(x+1)^2} \\ &= -1 + x + 1 = x \end{aligned}$$

As you can see from the graph, the $\pm$ causes the inverse to be a relation instead of a function. This can be observed in the graph because the original function does not pass the horizontal line test and the inverse does not pass the vertical line test.



Example 3

Find the inverse of the function and then verify that $x = f\left(f^{-1}(x)\right) = f^{-1}(f(x))$.

$f(x) = y = \frac{x+1}{x-1}$

Sometimes it is quite challenging to switch x and y and then solve for y. You must be careful with your algebra.

$$x = \frac{y+1}{y-1}$$
$$x(y-1) = y+1$$
$$xy - x = y+1$$
$$xy - y = x+1$$
$$y(x-1) = x+1$$
$$y = \frac{x+1}{x-1}$$

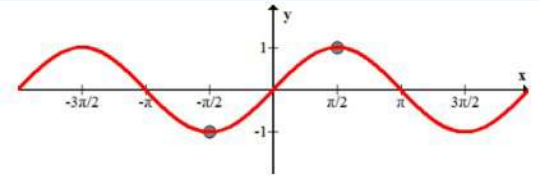
This function turns out to be its own inverse. since they are identical, you only need to show that $x = f\left(f^{-1}(x)\right)$

$f\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{x+1+x-1}{x+1-(x-1)} = \frac{2x}{2} = x$

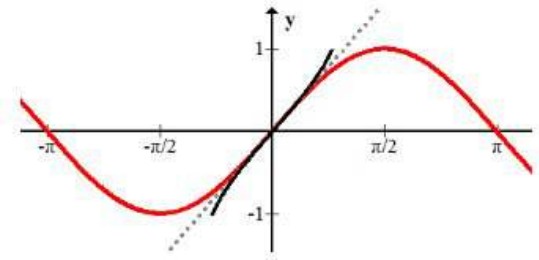
Example 4

What is the inverse of $f(x) = y = \sin x$?

The sine function does not pass the horizontal line test and so its true inverse is not a function.



However, if you restrict the domain to just the part of the x -axis between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ then it will pass the horizontal line test and the inverse will be a function.



The inverse of the sine function is called the arcsine function, $f(x) = \sin^{-1}(x)$, and is shown in black. It is truncated so that it only inverts a part of the whole sine wave. You will study periodic functions and their inverses in more detail later.

Example 5

Determine if $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ are inverses of one another.

Even though $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ have some inverted pieces, they are not inverses of each other. In order to show this, you must show that the composition does not simplify to x.

$\frac{3}{7}\left(\frac{7}{3}x + 21\right) - 21 = x + 9 - 21 = x - 12 \neq x$

Review

Consider $f(x) = x^3$

1. Sketch $f(x)$ and $f^{-1}(x)$.

2. Find $f^{-1}(x)$ algebraically. It is actually a function?

3. Verify algebraically that $f(x)$ and $f^{-1}(x)$ are inverses.

Consider $g(x) = \sqrt{x}$.

4. Sketch $g(x)$ and $g^{-1}(x)$.

5. Find $g^{-1}(x)$ algebraically. It is actually a function?

6. Verify algebraically that $g(x)$ and $g^{-1}(x)$ are inverses.

Consider $h(x) = |x|$

7. Sketch $h(x)$ and $h^{-1}(x)$.

8. Find $h^{-1}(x)$ algebraically. It is actually a function?

9. Verify graphically that $h(x)$ and $h^{-1}(x)$ are inverses.

Consider $j(x) = 2x - 5$

10. Sketch $j(x)$ and $j^{-1}(x)$.

11. Find $j^{-1}(x)$ algebraically. It is actually a function?

12. Verify algebraically that $j(x)$ and $j^{-1}(x)$ are inverses.

13. Use the horizontal line test to determine whether or not the inverse of $f(x) = x^3 - 2x^2 + 1$ is also a function.

14. Are $g(x) = \ln(x + 1)$ and $h(x) = e^{x-1}$ inverses? Explain.

15. If you were given a table of values for a function, how could you create a table of values for the inverse of the function?

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CHAPTER OVERVIEW

2: Polynomials and Rational Functions

2.1: 2.1 Factoring Review
2.2: Advanced Factoring
2.3: Polynomial Expansion and Pascal's Triangle
2.4: Rational Expressions
2.5: Polynomial Long Division and Synthetic Division
2.6: Solving Rational Equations
2.10 Horizontal Asymptotes
2.11 Oblique Asymptotes
2.12 Sign Test for Rational Function Graphs
2.13 Graphs of Rational Functions by Hand
2.7 Holes in Rational Functions
2.8 Zeroes of Rational Functions
2.9 Vertical Asymptotes

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2.1: 2.1 Factoring Review

To **factor** means to write an expression as a product instead of a sum. Factoring is particularly useful when solving equations set equal to zero because then logically at least one factor must be equal to zero. In PreCalculus, you should be able to factor even when there is no obvious greatest common factor or the difference is not between two perfect squares.

How do you use the difference of perfect squares factoring technique on polynomials that don't contain perfect squares and why would this be useful?

Factoring Functions

A polynomial is a sum of a finite number of terms. Each term consists of a constant that multiplies a variable. The variable may only be raised to a non-negative exponent. The letters $a, b, c...$ in the following general polynomial expression stand for regular numbers like $0, 5, -\frac{1}{4}, \sqrt{2}$ and the x represents the variable.

$$ax^n + bx^{n-1} + ... + fx^2 + gx + h$$

You have already learned many properties of polynomials. For example, you know the commutative property which states that terms of a polynomial can be rearranged to create an equivalent polynomial. When two polynomials are added, subtracted or multiplied the result is always a polynomial. This means polynomials are closed under addition, and is one of the properties that makes the factoring of polynomials possible. Polynomials are not closed under division because dividing two polynomials could result in a variable in the denominator, which is a rational expression (not a polynomial).

There are three methods for factoring that are essential to master.

Greatest Common Factor Method

The first method you should always try is to factor out the greatest common factor (GCF) of the expression.

To factor the following ex  the GCF method:

$-\frac{1}{2}x^4 + \frac{7}{2}x^2 - 6$

To find the GCF, it is com out the a value. In this case, try factoring out $-\frac{1}{2}$

$-\frac{1}{2}x^4 + \frac{7}{2}x^2 - 6 = -\frac{1}{2}\left(x^4 - 7x^2 + 12\right)$

In order to check to see that this is an equivalent expression, you need to distribute the $-\frac{1}{2}$. When you distribute, the first coefficient matches because it just gets multiplied by 1 , the second term becomes $\frac{7}{2}$ and the third term becomes - **6.** Note that this expression is not completely factored yet but it is simplified as much as it can be with just the GCF method.

Factoring Into Binomials Method

The second method you should try is to see if you can factor the expression into the product of two binomials.

To continue factoring the  Greatest Common Factor section, factor the following expression into the product of two binomials and a const

$-\frac{1}{2}\left(x^4 - 7x^2 + 12\right)$

Many students familiar w may be initially stuck on a problem like this. However, you should recognize that beneath the 4^{th} degree and the $-\frac{1}{2}$ the problem boils down to being able to factor $u^2 - 7u + 12$ which is just $(u-3)(u-4)$.

Start by rewriting the problem: $-\frac{1}{2}\left(x^4 - 7x^2 + 12\right)$

Then choose a temporary substitution: Let $u = x^2$.

Then substitute and factor away. Remember to substitute back at the end.

$$-\frac{1}{2}\left(u^2 - 7u + 12\right) = -\frac{1}{2}(u-3)(u-4)$$
$$= -\frac{1}{2}\left(x^2 - 3\right)\left(x^2 - 4\right)$$

This type of temporary substitution that enables you to see the underlying structure of an expression is very common in calculus. The expression is still not completely factored and since there are no more trinomials, you must apply the last method.

Difference of Squares Method

The third method of basic factoring is the difference of squares. It is recognizable as one square monomial being subtracted from another square monomial.

To finish factoring the resu  om the Factoring Into Binomials section, factor the expression into four linear factors and a constant:

$-\frac{1}{2}\left(x^2-3\right)\left(x^2-4\right)$

Many students may recog nmediately factors by the difference of squares method to be $(x-2)(x+2)$. This problem asks for more because sometimes the difference of squares method can be applied to expressions like x^2-3 where each term is not a perfect square. The number 3 actually is a square.

$3=(\sqrt{3})^2$

So the fully factored expression would be:

$-\frac{1}{2}(x-\sqrt{3})(x+\sqrt{3})(x-2)(x+2)$

Examples

Example 1

Earlier, you were asked how you use the difference of perfect squares factoring technique on polynomials that don't contain perfect squares and why it would be useful. One reason why it might be useful to completely factor an expression like $-\frac{1}{2}\left(x^4-7x^2+12\right)$ into linear factor is if you wanted to find the roots of the function $f(x)=-\frac{1}{2}\left(x^4-7x^2+12\right)$. The roots are $x=\pm\sqrt{3},\pm 2$

You should recognize that x^2-3 can still be thought of as the difference of perfect squares because the number 3 can be expressed as $(\sqrt{3})^2$. Rewriting the number 3 to fit a factoring pattern that you already know is an example of using the basic factoring techniques at a PreCalculus level.

Example 2

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$$\begin{aligned}\frac{x^5}{3}-\frac{11x^3}{3}+6x&=\frac{1}{3}x\left(x^4-11x^2+18\right)\\&=\frac{1}{3}x\left(x^2-2\right)\left(x^2-9\right)\\&=\frac{1}{3}x(x+\sqrt{2})(x-\sqrt{2})(x+3)(x-3)\end{aligned}$$

Example 3

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$-\frac{2}{7}x^4+\frac{74}{63}x^2-\frac{8}{63}$

For $-\frac{2}{7}x^4+\frac{74}{63}x^2-\frac{8}{63}$, let $u=x^2$.

$=-\frac{2}{7}u^2+\frac{74}{63}u-\frac{8}{63}$
$=-\frac{2}{7}\left(u^2-\frac{37}{9}u+\frac{4}{9}\right)$

Factoring through fractions like this can be extremely tricky. You must recognize that $-\frac{1}{9}$ and -4 sum to $-\frac{37}{9}$ and multiply to $\frac{4}{9}$.

$=-\frac{2}{7}\left(u-\frac{1}{9}\right)(u-4)$
$=-\frac{2}{7}\left(x^2-\frac{1}{9}\right)\left(x^2-4\right)$

$= -\frac{2}{7}\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)(x - 2)(x + 2)$

Example 4

Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$x^4 + x^2 - 72$

$x^4 + x^2 - 72 = (x^2 - 8)(x^2 + 9)$

$x^4 + x^2 - 72 = (x^2 - 8)(x^2 + 9)$

Notice that $(x^2 - 8)$ can be written as the difference of perfect squares because $8 = (\sqrt{8})^2 = (2\sqrt{2})^2$. On the other hand, $x^2 + 9$ cannot be written as the difference between squares because the x^2 and the 9 are being added not subtracted. This polynomial cannot be factored into strictly linear factors.

$x^4 + x^2 - 72 = (x - 2\sqrt{2})(x + 2\sqrt{2})(x^2 + 9)$

Review

Factor each polynomial into strictly linear factors if possible. If not possible, explain why not.

1. $x^2 + 5x + 6$
2. $x^4 + 5x^2 + 6$
3. $x^4 - 16$
4. $2x^2 - 20$
5. $3x^2 + 9x + 6$
6. $\frac{x^4}{2} - 5x^2 + \frac{9}{2}$
7. $\frac{2x^4}{3} - \frac{34x^2}{3} + \frac{32}{3}$
8. $x^2 - \frac{1}{4}$
9. $x^4 - \frac{37x^2}{4} + \frac{9}{4}$
10. $\frac{3}{4}x^4 - \frac{87}{4}x^2 + 75$
11. $\frac{1}{2}x^4 - \frac{29}{2}x^2 + 50$
12. $\frac{x^4}{2} - \frac{5x^2}{9} + \frac{1}{18}$
13. $x^4 - \frac{13}{36}x^2 + \frac{1}{36}$
14. How does the degree of a polynomial relate to the number of linear factors?
15. If a polynomial does not have strictly linear factors, what does this imply about the type of roots that the polynomial has?

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2.2: Advanced Factoring

The difference of perfect squares can be generalized as a factoring technique. By extension, any difference between terms that are raised to an even power like $a^6 - b^6$ can be factored using the difference of perfect squares technique. This is because even powers can always be written as perfect squares:

$$a^6 - b^6 = \left(a^3\right)^2 - \left(b^3\right)^2.$$

What about the sum or difference of terms with matching odd powers? How can those be factored?

More Factoring Techniques

Factoring a trinomial of the form $ax^2 + bx + c$ is much more difficult when $a \neq 1$. There are four techniques that can be used to factor such expressions.

Guess and Check

The educated guess and check method can be time consuming but if the first and last coefficient only have a few factors, there are a finite number of possibilities. Take the expression:

$6x^2 - 13x - 28$

The 6 can be factored into the following four pairs:

1, 6

2, 3

-1, -6

-2, -3

The -28 can be factored into the following twelve pairs:

1, -28 or -28, 1

-1, 28 or 28, -1

2, -14 or -14, 2

-2, 14 or 14, -2

4, -7 or -7, 4

-4, -7 or -7, -4

The correctly factored expression will need a pair from the top list and a pair from the bottom list. This is 48 possible combinations to try.

If you try the first pair from each list and multiply out you will see that the first and the last coefficients are correct but the b coefficient does not.

$(1x + 1)(6x - 28) = 6x^2 - 28x + 6x - 28$

A systematic approach to every one of the 48 possible combinations is the best way to avoid missing the correct pair. In this case it is:

$(2x - 7)(3x + 4) = 6x^2 + 8x - 21x - 28 = 6x^2 - 13x - 28$

This method can be extremely long and rely heavily on good guessing which is why other methods are preferable.

Factoring by Grouping

The next factoring technique is factoring by grouping. Suppose you start with an expression already in factored form:

$12x^2 + 4xz + 3xy + yz$

Notice that the first two terms are divisible by both 4 and x and the last two terms are divisible by y. First, factor out these common factors and then notice that there emerges a second layer of common factors. The binomial $(3x + z)$ is now common to both terms and can be factored out just as before.

$$\begin{aligned} 12x^2 + 4xz + 3xy + yz &= 4x(3x + z) + y(3x + z) \\ &= (3x + z)(4x + y) \end{aligned}$$

To verify your work, multiply the binomials and compare it with the original expression.

$$(4x + y)(3x + z) = 12x^2 + 4xz + 3xy + yz$$

Usually when you multiply the factored form of a polynomial, two terms can be combined because they are like terms. In this case, there are no like terms that can be combined.

Quadratic Formula

An alternative method to  uses the quadratic formula as a clue even though this is an expression and not an equation set equal to zero.

$$6x^2 - 13x - 28$$

$$a = 6, b = -13, c = -28$$

$$\begin{aligned} x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{13 \pm \sqrt{169 - 4 \cdot 6 \cdot -28}}{2 \cdot 6} &= \frac{13 \pm 29}{12} \\ = \frac{13 + 29}{12} = \frac{42}{12} &= \frac{7}{2} \\ = \frac{13 - 29}{12} = -\frac{16}{12} &= -\frac{4}{3} \end{aligned}$$

This means that when set equal to zero, this expression is equivalent to

$$\left(x - \tfrac{7}{2}\right)\left(x + \tfrac{4}{3}\right) = 0$$

Multiplying by 2 and multiplying by 3 only changes the left hand side of the equation because the right hand side will remain 0. This has the effect of shifting the coefficient from the denominator of the fraction to be in front of the x.

$$6x^2 - 13x - 28 = \left(x - \tfrac{7}{2}\right)\left(x + \tfrac{4}{3}\right) = (2x - 7)(3x + 4)$$

Factoring Algorithm

Another useful and efficient technique is the procedural factoring algorithm. The proof of the algorithm is beyond the scope of this book, but is a reliable technique for getting a handle on tricky factoring questions of the form: $6x^2 - 13x - 28$

We will factor $6x^2 - 13x - 28$ using the factoring algorithm to introduce it to you.

First, multiply the first coefficient (6) with the last coefficient (28) and set the first coefficient to 1:

$$x^2 - 13x - 168$$

Second, factor as you normally would with $a = 1$:

$$(x - 21)(x + 8)$$

Third, divide the second half of each binomial by the coefficient that was multiplied in step 1:

$$\left(x - \tfrac{21}{6}\right)\left(x + \tfrac{8}{6}\right)$$

Fourth, simplify each fraction completely:

$$\left(x - \tfrac{7}{2}\right)\left(x + \tfrac{4}{3}\right)$$

Lastly, move the denominator of each fraction to become the coefficient of x:

$$(2x - 7)(3x + 4)$$

Sum or Difference of Matching Odd Powers

The last method of advanced factoring does not involve expressions of the form $ax^2 + bx + c$ Instead, it involves the patterns that arise from factoring the sum or difference of terms with matching odd powers. The patterns are:

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

This method is shown in the examples below and the pattern is fully explored in the Review.

Examples

Example 1

Earlier, you were asked how the sum and difference of terms with matching odd powers can be factored. The sum or difference of terms with matching odd powers can be factored in a precise pattern because when multiplied out, all intermediate terms cancel each other out.

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

When a is distributed: $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$

When b is distributed: $+a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5$

Notice all the inside terms cancel: $a^5 + b^5$

Example 2

Show that $a^3 - b^3$ factors into the result given in the Sum or Difference of Matching Odd Powers section.

Factoring,

$$\begin{aligned} a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\ &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3 \end{aligned}$$

Example 3

Show that $a^3 + b^3$ factors into the result given in the Sum or Difference of Matching Odd Powers section.

Factoring,

$$\begin{aligned} a^3 + b^3 &= (a + b)(a^2 - ab + b^2) \\ &= a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3 \\ &= a^3 + b^3 \end{aligned}$$

Example 4

Factor the following expression without using the quadratic formula or trial and error:

$8x^2 + 30x + 27$

Using the factoring algorithm:

$$\begin{aligned} 8x^2 + 30x + 27 &\rightarrow x^2 + 30x + 216 \\ &\rightarrow (x + 12)(x + 18) \\ &\rightarrow \left(x + \frac{12}{8}\right)\left(x + \frac{18}{8}\right) \\ &\rightarrow \left(x + \frac{3}{2}\right)\left(x + \frac{9}{4}\right) \\ &\rightarrow (2x + 3)(4x + 9) \end{aligned}$$

Review

Factor each expression completely.

1. $2x^2 - 5x - 12$
2. $12x^2 + 5x - 3$
3. $10x^2 + 13x - 3$
4. $18x^2 + 9x - 2$
5. $6x^2 + 7x + 2$
6. $8x^2 + 34x + 35$
7. $5x^2 + 23x + 12$
8. $12x^2 - 11x + 2$

Expand the following expressions. What do you notice?

9. $(a+b)\left(a^8 - a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6 - ab^7 + b^8\right)$
10. $(a-b)\left(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6\right)$
11. Describe in words the pattern of the signs for factoring the difference of two terms with matching odd powers.
12. Describe in words the pattern of the signs for factoring the sum of two terms with matching odd powers.

Factor each expression completely.

13. $27x^3 - 64$
14. $x^5 - y^5$
15. $32a^5 - b^5$
16. $32x^5 + y^5$
17. $8x^3 + 27$
18. $2x^2 + 2xy + x + y$
19. $8x^3 + 12x^2 + 2x + 3$
20. $3x^2 + 3xy - 4x - 4y$

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2.3: Polynomial Expansion and Pascal's Triangle

The expression $(2x+3)^5$ would take a while to multiply out. Is there a pattern you can use?

Expansion with Pascal's Triangle

Expanding a Binomial

Pascal was a French mathematician in the 17^{th} century, but the triangle now named Pascal's Triangle was studied long before Pascal used it. The pattern was used around the 10^{th} century in Persia, India and China as well as many other places.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

The primary purpose for using this triangle is to introduce how to expand binomials.

$(x+y)^0 = 1$
$(x+y)^1 = x+y$
$(x+y)^2 = x^2+2y+y^2$
$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

Notice that the coefficients for the x and y terms on the right hand side line up exactly with the numbers from Pascal's triangle. This means that given $(x+y)^n$ for any power n you can write out the expansion using the coefficients from the triangle. Note that to write the coefficients for any power n, you need to look at row $n+1$ to find the coefficients.

If you were asked to expand $(3x-2)^4$ using Pascal's Triangle, you would look at the 5 th row to find the coefficients. The coefficients will be 1,4,6,4,1; however, since there are already coefficients with the x and the constant term you must be particularly careful.

$1 \cdot (3x)^4 + 4 \cdot (3x)^3 \cdot (-2) + 6 \cdot (3x)^2 \cdot (-2)^2 + 4 \cdot (3x) \cdot (-2)^3 + 1 \cdot (-2)^4$

Then it is only a matter of multiplying out and keeping track of negative signs.

$81x^4 - 216x^3 + 216x^2 - 96x + 16$

When you study how to count with combinations then you will be able to calculate the value of any coefficient without writing out the whole triangle.

Patterns and Pascal's T

There are many patterns in Pascal's Triangle, here are just a few.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

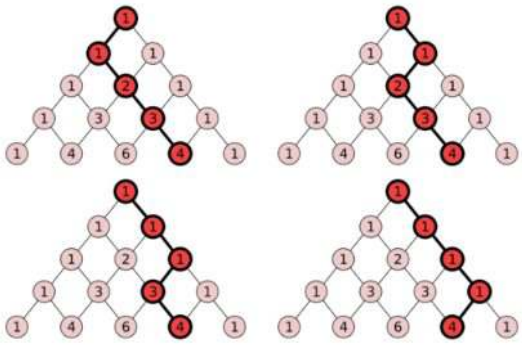
1 5 10 10 5 1

1 6 15 20 15 6 1

⋮

1. Notice the way each number is created by summing the two numbers above on the left and right hand side.
2. As you go further down the triangle the values in a row approach a bell curve. This is closely related to the normal distribution in statistics.
3. For any row that has a second term that is prime, all the numbers besides 1 in that row are divisible by that prime number.

Real Life Use: In the game Plinko where an object is dropped through a triangular array of pegs, the probability (which corresponds proportionally to the values in the triangle) of landing towards the center is greater than landing towards the edge. This is because every number in the triangle indicates the number of ways a falling object can get to that space through the preceding numbers.



Examples

Example 1

Earlier, you were asked to multiply out $(2x+3)^5$. Pascal's triangle allows you to identify that the coefficients of $(2x+3)^5$ will be $1, 5, 10, 10, 5, 1$. By carefully substituting, the expansion will be:

$1 \cdot (2x)^5 + 5 \cdot (2x)^4 \cdot 3 + 10 \cdot (2x)^3 \cdot 3^2 + 10 \cdot (2x^2) \cdot 3^3 + 5(2x)^1 \cdot 3^4 + 3^5$

Simplifying is a matter of arithmetic, but most of the work is done thanks to the patterns of Pascal's Triangle.

Example 2

Factor the following polynomial by recognizing the coefficients.

$x^4 + 4x^3 + 6x^2 + 4x + 1$

The coefficients are 1, 4, 6, 4, and 1 and those coefficients are on the 5th row. The first row of Pascal's Triangle shows the coefficients for the 0th power so the 5th row shows the coefficients for the 4th power. Thus, the factored form is:

$(x+1)^4$

Example 3

Factor the following polynomial by recognizing the coefficients.

$8x^3-12x^2+6x-1$

since the first coefficient is not 1 , you need to take the appropriate root of the first term of the expression to find the first term of the binomial. In this case, the first term of the binomial must be $2x$. Also, the last term must be -1 and the power must be 3. Now all that remains is to check.

$(2x-1)^3=(2x)^3+3(2x)^2\cdot(-1)+3(2x)^1(-1)^2+(-1)^3=8x^3-12x^2+6x-1$

Example 4

Expand the following binomial: $\left(\frac{1}{2}x-3\right)^5$

You know that the coefficients will be 1, 5, 10, 10, 5, 1.

$1\left(\frac{1}{2}x\right)^5+5\left(\frac{1}{2}x\right)^4(-3)+10\left(\frac{1}{2}x\right)^3(-3)^2+10\left(\frac{1}{2}x\right)^2(-3)^3+5\left(\frac{1}{2}x\right)(-3)^4+1\cdot(-3)^5$

$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{90x^3}{8}-\frac{270x^2}{4}+\frac{405x}{2}-243$

Remember to simplify fractions.

$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{45x^3}{4}-\frac{135x^2}{2}+\frac{405x}{2}-243$

Example 5

Expand the following trinomial: $(x+y+z)^4$

Unfortunately, Pascal's triangle does not apply to trinomials. Instead of thinking of a two dimensional triangle, you would ned to calculate a three dimensional pyramid which is called Pascal's Pyramid. The sum of all five terms below is your answer.

$1x^4+4x^3z+6x^2z^2+4xz^3+1z^4$
$4x^3y+12x^2yz+12xyz^2+4yz^3$
$6x^2y^2+12xy^2z+6y^2z^2$
$4xy^3+4y^3z$
$1y^4$

Notice how many patterns exist in the coefficients of this layer of the pyramid.

Review

Factor the following polynomials by recognizing the coefficients.

1. $x^2+2xy+y^2$
2. x^3+3x^2+3x+1
3. $x^5+5x^4+10x^3+10x^2+5x+1$
4. $27x^3-27x^2+9x-1$
5. $x^3+12x^2+48x+64$

Expand the following binomials using Pascal's Triangle.

6. $(2x-3)^3$
7. $(3x+4)^4$
8. $(x-y)^7$

9. $(a+b)^{10}$
10. $(2x+5)^5$
11. $(4x-1)^4$
12. $(5x+2)^3$
13. $(x+y)^6$
14. $(3x+2y)^3$
15. $(5x-2y)^4$

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2.4: Rational Expressions

A rational expression is a ratio, just like a fraction. However, instead of a ratio between numbers, a **rational expression** is a ratio between two expressions.

One driving question to ask is: Are the rules for simplifying and operating on rational expressions the same as the rules for simplifying and operating on fractions?

Working with Rational Expressions

When simplifying or operating on rational expressions, it is vital that each polynomial be fully factored. Once all expressions are factored, identical factors in the numerator and denominator may be canceled or removed. The reason they can be removed is that any expression divided by itself is equal to 1. An identical expression in the numerator and denominator is just an expression being divided by itself, and so equals 1.

Adding and Subtracting sions

To add or subtract rationa essential to first find a common denominator. While any common denominator will work, using the least com is a means of keeping the number of additional factors under control. Look at each rational expression you a and identify your desired common denominator. Multiply each expression by an appropriate form of 1, suc you should have your common denominator. In addition and subtraction problems, the numerator must be multiplied, combined, and re-factored to be considered simplified.

Multiplying and Dividing sions

To multiply rational expre write the product of all the numerator factors over the product of all the denominator factors and then cancel, o factors.To divide rational expressions, you should rewrite the division problem as a multiplication problem. M onal expression by the reciprocal of the second rational expression. Follow the steps above for multiplying. I on and division problems answers are most commonly left entirely factored to demonstrate everything has been reduced appropriately.

Examples

Example 1

Earlier, you were aske simplifying and operating on rational expressions are the same as the rules for simplifying and operati rational expressions are an extension of fractions and the operations of simplifying, adding, subtracting, multiplying and dividing work in exactly the same way.

Example 2

Subtract the following rational expressions.

$\frac{x-2}{x+3} - \frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$

Being able to factor effectively is of paramount importance.

$= \frac{x-2}{x+3} - \frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$
$= \frac{(x-2)}{(x+3)} - \frac{x^2(x-3)+8(x-3)}{2(x+2)(x^2-9)}$
$= \frac{(x-2)}{(x+3)} - \frac{(x-3)(x^2+8)}{2(x+2)(x+3)(x-3)}$

Before subtracting, simplify where possible so you don't contribute to unnecessarily complicated denominators.

$= \frac{(x-2)}{(x+3)} - \frac{x^2+8}{2(x+2)(x+3)}$

The left expression lacks $2(x+2)$, so multiply both its numerator and denominator by $2(x+2)$

$= \frac{2(x+2)(x-2)}{2(x+2)(x+3)} - \frac{(x^2+8)}{2(x+2)(x+3)}$
$= \frac{2(x^2-4)-x^2-8}{2(x+2)(x+3)}$
$= \frac{x^2-16}{2(x+2)(x+3)}$

Example 3

Simplify the following rational expression.

$\frac{x^2+7x+12}{x^2+4x+3} \cdot \frac{x^2+9x+8}{2x^2-128} \div \frac{x+4}{x-8} \cdot \frac{14}{1}$

First factor everything. Second, turn division into multiplication (only one term). Third, cancel appropriately which will leave the answer.

$= \frac{(x+3)(x+4)}{(x+3)(x+1)} \cdot \frac{(x+8)(x+1)}{2(x+8)(x-8)} \cdot \frac{(x-8)}{(x+4)} \cdot \frac{14}{1}$
$= \frac{(x+3)(x+4)}{(x+3)(x+1)} \cdot \frac{(x+8)(x+1)}{2(x+8)(x-8)} \cdot \frac{(x-8)}{(x+4)} \cdot \frac{14}{1}$
$= \frac{14}{2}$
$= 7$

In this example, the strike through is shown. You should use this technique to match up factors in the numerator and the denominator when simplifying.

Example 4

Combine the following rational expressions.

$\frac{1}{x^2+5x+6} - \frac{1}{x^2-4} + \frac{(x-7)(x+5)+5}{(x+2)(x-2)(x+3)(x-4)}$

First factor everything and decide on a common denominator. While the numerators do not really need to be factored, it is sometimes helpful in simplifying individual expressions before combining them. Note that the numerator of the expression on the right hand seems factored but it really is not. Since the 5 is not connected to the rest of the numerator through multiplication, that part of the expression needs to be multiplied out and like terms need to be combined.

$= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+2)(x-2)} + \frac{x^2-2x-35+5}{(x+2)(x-2)(x+3)(x-4)}$
$= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+2)(x-2)} + \frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$

Note that the right expression has 4 factors in the denominator while each of the left expressions have two that match and two that are missing from those four factors. This tells you what you need to multiply each expression by in order to have the denominators match up.

$= \frac{(x-2)(x-4)}{(x+2)(x-2)(x+3)(x-4)} - \frac{(x+3)(x-4)}{(x+2)(x-2)(x+3)(x-4)} + \frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$

Now since the rational expressions have a common denominator, the numerators may be multiplied out and combined. Sometimes instead of rewriting an expression repeatedly in mathematics you can use an abbreviation. In this case, you can replace the denominator with the letter D and then replace it at the end.

$= \frac{(x-2)(x-4)-(x+3)(x-4)+x^2-2x-30}{D}$
$= \frac{x^2-6x+8-\left[x^2-x-12\right]+x^2-2x-30}{D}$

Notice how it is extremely important to use brackets to indicate that the subtraction applies to all the terms of the middle expression not just x^2. This is one of the most common mistakes.

$= \frac{x^2-6x+8-x^2+x+12+x^2-2x-30}{D}$
$= \frac{x^2-7x-10}{D}$

After the numerator has been entirely simplified try to factor the remaining expression and see if anything cancels with the denominator which you now need to replace.

$= \frac{(x+2)(x-5)}{(x+2)(x-2)(x+3)(x-4)}$
$= \frac{(x-5)}{(x-2)(x+3)(x-4)}$

Example 5

Simplify the following expression.

$$\frac{\frac{1}{x+1}-\frac{1}{x+2}}{\frac{1}{x-2}-\frac{1}{x+1}}$$

The expression itself does not look like a rational expression, but it can be rewritten so it is more recognizable. Also, working with fractions within fractions is an important skill.

$$\begin{aligned}
&= \left(\frac{1}{x+1}-\frac{1}{x+2}\right) \div \left(\frac{1}{x-2}-\frac{1}{x+1}\right) \\
&= \left[\frac{(x+2)}{(x+1)(x+2)}-\frac{(x+1)}{(x+1)(x+2)}\right] \div \left[\frac{(x+1)}{(x+1)(x-2)}-\frac{(x-2)}{(x+1)(x-2)}\right] \\
&= \left[\frac{1}{(x+1)(x+2)}\right] \div \left[\frac{3}{(x+1)(x-2)}\right] \\
&= \frac{1}{(x+1)(x+2)} \cdot \frac{(x+1)(x-2)}{3} \\
&= \frac{(x-2)}{3(x+2)}
\end{aligned}$$

Bonus Example

Simplify the following expression which has an infinite number of fractions nested within other fractions.

$$\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\ldots}}}}}}$$

It would be an exercise in futility to attempt to try to compute this expression directly. Instead, notice that the repeating nature of the expression lends itself to an extremely nice substitution.

$$\text{Let } \cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\cfrac{-1}{2+\ldots}}}}}} = x$$

Notice that the red portion of the fraction is exactly the same as the rest of the fraction and so x may be substituted there and solved.

$$\begin{aligned}
\frac{-1}{2+x} &= x \\
-1 &= x(2+x) \\
-1 &= x^2+2x \\
0 &= x^2+2x+1 \\
0 &= (x+1)^2 \\
x &= -1
\end{aligned}$$

The reason why this expression is included in this concept is because it highlights one problem solving aspect of simplifying expressions that distinguishes PreCalculus from Algebra 1 and Algebra 2.

Review

Perform the indicated operation and simplify as much as possible.

1. $\frac{x^2+5x+4}{x^2+4x+3} \cdot \frac{5x^2+15x}{x+4}$
2. $\frac{x^2-4}{x^2+4x+4} \cdot \frac{7}{x-2}$
3. $\frac{4x^2-12x}{5x+10} \cdot \frac{x+2}{x} \div \frac{x-3}{1}$
4. $\frac{4x^3-4x}{x} \div \frac{2x-2}{x-4}$
5. $\frac{2x^3+8x}{x+1} \div \frac{x}{2x^2-2}$
6. $\frac{3x-1}{x^2+2x-15} - \frac{2}{x+5}$
7. $\frac{x^2-8x+7}{x^2-4x-21} \cdot \frac{x^2-9}{1-x^2}$
8. $\frac{2}{x+7} + \frac{1}{x-7}$
9. $\frac{6}{x-7} - \frac{6}{x+7}$
10. $\frac{3x+35}{x^2-25} + \frac{2}{x+5}$
11. $\frac{2x+20}{x^2-4x-12} + \frac{2}{x+2}$
12. $\frac{2}{x+6} - \frac{x-9}{x^2-3x-18}$
13. $-\frac{5x+30}{x^2+11x+30} + \frac{2}{x+5}$
14. $\frac{x+3}{x+2} + \frac{x^3+4x^2+5x+20}{2x^4+2x^2-40}$
15. $\cfrac{-4}{2+\cfrac{-4}{2+\cfrac{-4}{2+\cfrac{-4}{2+\cfrac{-4}{2+\cdots}}}}}$

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2.5: Polynomial Long Division and Synthetic Division

While you may be experienced in factoring, there will always be polynomials that do not readily factor using basic or advanced techniques. How can you identify the roots of these polynomials?

Rational Roots and Dividing Polynomials

There are numerous theorems that point out relationships between polynomials and their factors. For example there is a theorem that a polynomial of degree n must have exactly n solutions/factors that may or may not be real numbers. The Rational Root Theorem and the Remainder Theorem are two theorems that are particularly useful starting places when manipulating polynomials.

The Rational Root Theorem

The Rational Root Theorem states that in a polynomial, every rational solution can be written as a reduced fraction $\left(x=\frac{p}{q}\right)$, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient.

Let's identify all the possible rational solutions of the following polynomial using the Rational Root Theorem.

$$12x^{18}-91x^{17}+x^{16}+\cdots+2x^2-14x+5=0$$

The integer factors of 5 are 1,5 . The integer factors of 12 are 1,2,3,4,6 and $\mathbf{12}$. since pairs of factors could both be negative, remember to include $\pm$.

$$\pm\frac{p}{q}=\frac{1}{1},\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{6},\frac{1}{12},\frac{5}{1},\frac{5}{2},\frac{5}{3},\frac{5}{4},\frac{5}{6},\frac{5}{12}$$

The possible solutions can be found from these 24 possible rational answers. If this question required you to find a solution, then the Rational Root Theorem would give you a great starting place. Once you have one root, you can use either polynomial long division or synthetic to divide the factor out and to keep reducing the expression. We will use the Rational Root Theorem in Example 3.

Polynomial Long Division and the Remainder Theorem

Polynomial long division is identical to regular long division except the dividend and divisor are both polynomials instead of numbers.

The Remainder Theorem states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division.

Let's put this knowledge to use and use polynomial long division to divide:

$$\frac{x^3+2x^2-5x+7}{x-3}$$

First note that it is clear that 3 is not a root of the polynomial because of the Rational Root Theorem and so there will definitely be a remainder. Start a polynomial long division question by writing the problem like a long division problem with regular numbers:

$$-x-3\longdivx^3+2x^2-5x+7$$

First note that it is clear that 3 is not a root of the polynomial because of the Rational Root Theorem and so there will definitely be a remainder. Start a polynomial long division question by writing the problem like a long division problem with regular numbers:

$$-x-3\longdivx^3+2x^2-5x+7$$

Just like with regular numbers ask yourself "how many times does x go into x^3 ?" which in this case is x^2.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \end{array}$$

Now multiply the x^2 by $x-3$ and copy below. Remember to subtract the entire quantity.

Unknown environment 'tabular'

Combine the rows, bring down the next number and repeat.

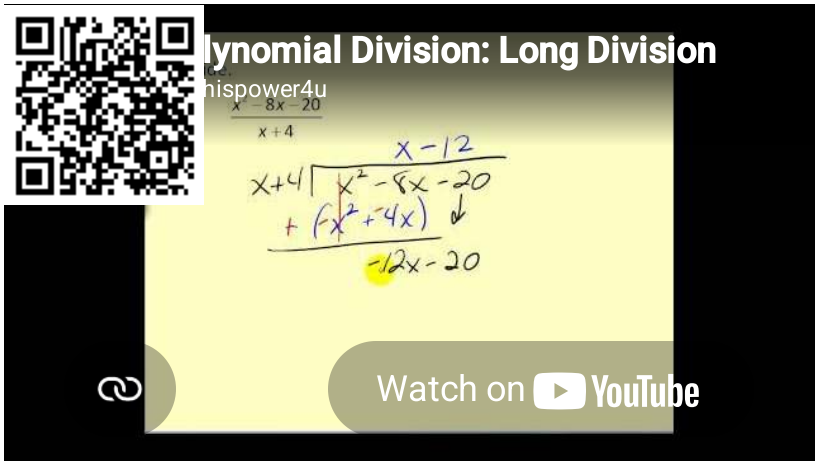
$$\begin{array}{r} x^2+5x+10 \\ 3)\overline{x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)} \\ 5x^2-5x \\ \underline{-(5x^2-15x)} \\ 10x+7 \\ \underline{-(10x-30)} \\ 37 \end{array}$$

The number 37 is the remainder. There are two things to think about at this point. First, interpret in an equation:

$$\frac{x^3+2x^2-5x+7}{x-3} = (x^2+5x+10)+\frac{37}{x-3}$$

Second, check your result with the Remainder Theorem which states that the original function evaluated at 3 must be 37 . Notice the notation indicating to substitute 3 in for x.

$$(x^3+2x^2-5x+7)\big|_{x=3} = 3^3+2\cdot 3^2-5\cdot 3+7 = 27+18-15+7 = 37$$



Synthetic Division

Synthetic division is an abbreviated version of polynomial long division where only the coefficients are used. Synthetic division is mostly used when the leading coefficients of the numerator and denominator are equal to 1 and the divisor is a first degree binomial. Let's use synthetic division to divide the same expression that we divided above with polynomial long division:

$$\frac{x^3+2x^2-5x+7}{x-3}$$

Instead of continually writing and rewriting the x symbols, synthetic division relies on an ordered spacing.

$$\pm 3 \,|\; 1 \quad 2 \quad -5 \quad 7$$

Notice how only the coefficients for the denominator are used and the divisor includes a positive three rather than a negative three. The first coefficient is brought down and then multiplied by the three to produce the value which goes beneath the 2.

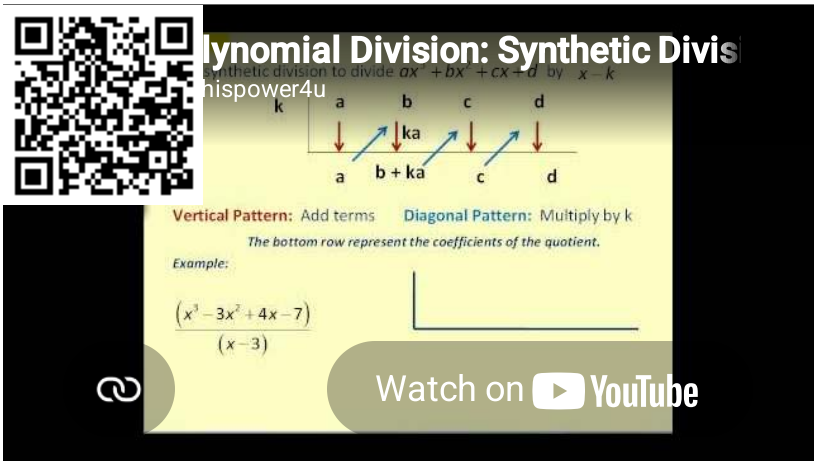
$$\begin{array}{r|llll} +3 & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & & \\ \hline & 1 & & & \end{array}$$

Next the new column is added. $2+3=5$, which goes beneath the 2^{nd} column. Now, multiply $5\cdot +3=15$, which goes underneath the -5 in the 3^{rd} column. And the process repeats...

$$\begin{array}{r|llll} +3 & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & 15 & 30 \\ \hline & 1 & 5 & 10 & 37 \end{array}$$

The last number, 37, is the remainder. When writing out the resulting expression, you will put this remainder over the divisor. The three other numbers represent the quadratic that is identical to the solution to the result from dividing the expression using polynomial long division. Note that when writing out the expression, you decrease the exponent of the leading coefficient of the original by 1.

$(1x^2 + 5x + 10) + \frac{37}{x-3}$



Examples

Example 1

Earlier, you were asked about identifying roots of polynomials that do not readily factor using the techniques you have learned so far. Identifying roots of polynomials by hand can be tricky business. The best way to identify roots is to use the rational root theorem to quickly identify likely candidates for solutions and then use synthetic or polynomial long division to quickly and effectively test them to see if their remainders are truly zero.

Example 2

Divide the following polynomials.

$\frac{x^3+2x^2-4x+8}{x-2}$

Since the leading coefficients of the numerator and denominator are both 1 and the denominator is a binomial, synthetic division is a good method to use here.

$\frac{x^3+2x^2-4x+8}{x-2} = x^2 + 4x + 4 + \frac{16}{x-2}$

Example 3

Completely factor the following polynomial.

$x^4 + 6x^3 + 3x^2 - 26x - 24$

Notice that possible roots are $\pm 1, 2, 3, 4, 6, 8, 24$. Of these 14 possibilities, four will yield a remainder of zero. When you find one, use long division or synthetic division to factor out the root that you found. Then find another zero and repeat the process.

$x^4 + 6x^3 + 3x^2 - 26x - 24$
$= (x+1)(x^3 + 5x^2 - 2x - 4)$
$= (x+1)(x-2)(x^2 + 7x + 12)$
$= (x+1)(x-2)(x+3)(x+4)$

The first zero found was -1. It was divided out of the original expression to find the remaining non-factored portion of the expression. The second zero found was 2 from the remaining piece and was divided out. Once you get down to a quadratic expression, you can use the other factoring techniques you know to factor the rest of the expression.

Example 4

Divide the following polynomials.

$\frac{3x^5-2x^2+10x-5}{x-1}$

Since the first coefficient of the numerator is not 1, polynomial long division is a good method to use here.

$\frac{3x^5-2x^2+10x-5}{x-1} = 3x^4+3x^3+3x^2+x+11+\frac{6}{x-1}$

Review

Identify all possible rational solutions of the following polynomials using the Rational Root Theorem.

1. $15x^{14}-12x^{13}+x^{12}+\cdots+2x^2-5x+5=0$
2. $18x^{11}+42x^{10}+x^9+\cdots+x^2-3x+7=0$
3. $12x^{16}+11x^{15}+3x^{14}+\cdots+6x^2-2x+11=0$
4. $14x^7-7x^6+x^5+\cdots+x^2+6x+3=0$
5. $9x^9-10x^8+3x^7+\cdots+4x^2-2x+2=0$

Completely factor the following polynomials.

6. $2x^4-x^3-21x^2-26x-8$
7. $x^4+7x^3+5x^2-31x-30$
8. $x^4+3x^3-8x^2-12x+16$
9. $4x^4+19x^3-48x^2-117x-54$
10. $2x^4+17x^3-8x^2-173x+210$

Divide the following polynomials.

11. $\frac{x^4+7x^3+5x^2-31x-30}{x+4}$
12. $\frac{x^4+7x^3+5x^2-31x-30}{x+2}$
13. $\frac{x^4+3x^3-8x^2-12x+16}{x+3}$
14. $\frac{2x^4-x^3-21x^2-26x-8}{x^3-x^2-10x-8}$
15. $\frac{x^4+8x^3+3x^2-32x-28}{x^3+10x^2+23x+14}$

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2.6: Solving Rational Equations

The techniques for solving rational equations are extensions of techniques you already know. Recall that when there are fractions in an equation you can multiply through by the denominator to clear the fraction. The same technique helps turn rational expressions into polynomials that you already know how to solve. When you multiply by a constant there is no problem, but when you multiply through by a value that varies and could possibly be zero interesting things happen.

Since every equation is trivially true when both sides are multiplied by zero, how do you account for this when solving rational equations?

Finding Solutions to Rational Equations

The first step in solving rational equations is to transform the equation into a polynomial equation. This is accomplished by clearing the fraction which means multiplying the entire equation by the common denominator of all the rational expressions. Then you should solve using what you already know. The last thing to check once you have the solutions is that they do not make the denominators of any part of the equation equal to zero when substituted back into the original equation. If so, that solution is called **extraneous** and is a "fake" solution that was introduced when both sides of the equation were multiplied by a number that happened to be zero.

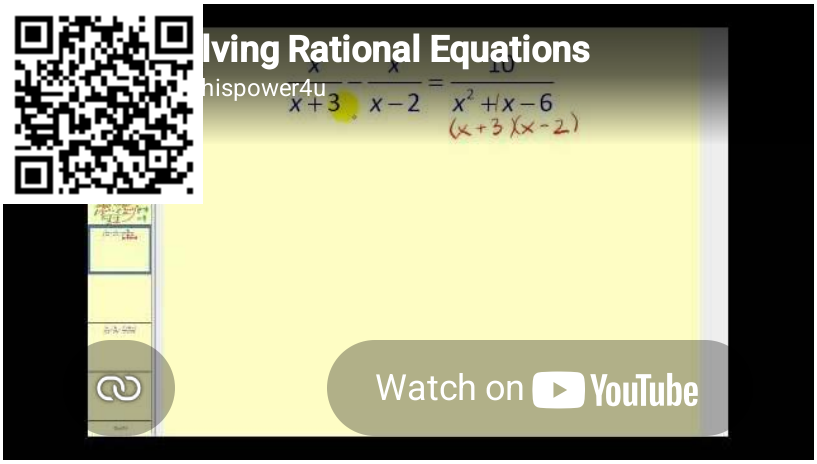
Take the following rational equation:

$x - \frac{5}{x+3} = 12$

To find the solutions of the equation, first multiply all parts of the equation by $(x+3)$, the common denominator, and then simplify.

$$\begin{aligned} x(x+3) - 5 &= 12(x+3) \\ x^2 + 3x - 5 - 12x - 36 &= 0 \\ x^2 - 9x - 41 &= 0 \\ x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 1 \cdot (-41)}}{2 \cdot 1} \\ x &= \frac{9 \pm 7\sqrt{5}}{2} \end{aligned}$$

The only potential extraneous solution would have been -3 since that is the number that makes the denominator of the original equation zero. Therefore, both answers are possible.



Examples

Example 1

Earlier, you were asked to account for the extra solutions introduced when both sides of an equation are multiplied by a variable. In order to deal with the possible extra solutions, you must check each solution to see if it makes the denominator of any fraction in the original equation zero. If it does, it is called an extraneous solution.

Example 2

Solve the following rational equation

$\frac{3x}{x+4} - \frac{1}{x+2} = -\frac{2}{x^2+6x+8}$

Multiply each part of the equation by the common denominator of $x^2 + 6x + 8 = (x + 2)(x + 4)$

$$(x+2)(x+4)\left[\frac{3x}{x+4} - \frac{1}{x+2}\right] = \left[\frac{-2}{(x+2)(x+4)}\right](x+2)(x+4)$$
$$3x(x+2) - (x+4) = -2$$
$$3x^2 + 6x - x - 2 = 0$$
$$3x^2 + 5x - 2 = 0$$
$$(3x-1)(x+2) = 0$$
$$x = \frac{1}{3}, -2$$

Note that -2 is an extraneous solution. The only actual solution is $x = \frac{1}{3}$.

Example 3

Solve the following rational equation for y.

$x = 2 + \frac{1}{2+\frac{1}{y+1}}$

This question can be done multiple ways. You can use the clearing fractions technique twice.

$$\left(2 + \frac{1}{y+1}\right)x = \left[2 + \frac{1}{2+\frac{1}{y+1}}\right]\left(2 + \frac{1}{y+1}\right)$$
$$2x + \frac{x}{y+1} = 2\left(2 + \frac{1}{y+1}\right) + 1$$
$$2x + \frac{x}{y+1} = 4 + \frac{2}{y+1} + 1$$
$$(y+1)\left[2x + \frac{x}{y+1}\right] = \left[5 + \frac{2}{y+1}\right](y+1)$$
$$2x(y+1) + x = 5(y+1) + 2$$
$$2xy + 2x + x = 5y + 5 + 2$$

Now just get the y variable to one side of the equation and everything else to the other side.

$$2xy - 5y = -3x + 7$$
$$y(2x-5) = -3x + 7$$
$$y = \frac{-3x+7}{2x-5}$$

Example 4

Solve the following rational equation.

$\frac{3x}{x-5} + 4 = x$

$\frac{3x}{x-5} + 4 = x$

$$3x + 4x - 20 = x^2 - 5x$$
$$0 = x^2 - 12x + 20$$
$$0 = (x-2)(x-10)$$
$$x = 2, 10$$

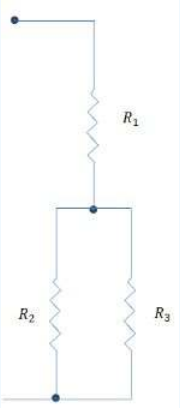
Neither solution is extraneous.

Example 5

In electrical circuits, resistance can be solved for using rational expressions. This is an electric circuit diagram with three resistors. The first resistor R_1 is run in series to the other two resistors R_2 and R_3 which are run in parallel. If the total resistance R is 100 ohms and R_1 and R_3 are each 22 ohms, what is the resistance of R_2 ?

The equation of value is:

$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$



$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$
$$100 = 22 + \frac{x \cdot 22}{x + 22}$$
$$78(x + 22) = 22x$$
$$78x + 1716 = 22x$$
$$56x = -1716$$
$$x = -30.65$$

A follow up question would be to ask whether or not ohms can be negative which is beyond the scope of this text.

Review

Solve the following rational equations. Identify any extraneous solutions.

1. $\frac{2x-4}{x} = \frac{16}{x}$
2. $\frac{4}{x+1} - \frac{x}{x+1} = 2$
3. $\frac{5}{x+3} + \frac{2}{x-3} = 1$
4. $\frac{3}{x-4} - \frac{5}{x+4} = 6$
5. $\frac{x}{x+1} - \frac{6}{x+2} = 4$
6. $\frac{x}{x-4} - \frac{4}{x-4} = 8$
7. $\frac{4x}{x-2} + 3 = 1$
8. $\frac{-2x}{x+1} + 6 = -x$
9. $\frac{1}{x+2} + 1 = -2x$
10. $\frac{-6x-3}{x+1} - 3 = -4x$
11. $\frac{x+3}{x} - \frac{3}{x+3} = \frac{6}{x^2+3x}$
12. $\frac{x-4}{x} - \frac{2}{x-4} = \frac{8}{x^2-4x}$
13. $\frac{x+6}{x} - \frac{2}{x+6} = \frac{12}{x^2+6x}$
14. $\frac{x+5}{x} - \frac{3}{x+5} = \frac{15}{x^2+5x}$

15. Explain what it means for a solution to be extraneous.

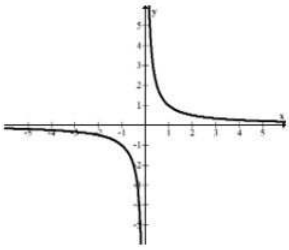
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2.10 Horizontal Asymptotes

Vertical asymptotes describe the behavior of a function as the values of x approach a specific number. **Horizontal asymptotes** describe the behavior of a function as the values of x become infinitely large and infinitely small. since functions cannot touch vertical asymptotes, are they not allowed to touch horizontal asymptotes either?

Finding Horizontal Asymptotes

Horizontal asymptotes are a means of describing end behavior of a function. End behavior essentially is a description of what happens on either side of the graph as the function continues to the right and left infinitely. When you are determining the horizontal asymptotes, it is important to consider both the right and the left hand sides, because the horizontal asymptotes will not necessarily be the same in both places. Consider the reciprocal function and note how as x goes to the right and left it flattens to the line $y = 0$.



Sometimes functions flatten out and other times functions increase or decrease without bound. There are basically three cases.

Case 1: Degree of Numerator is Less than Degree of Denominator

The first case is the function flattens out to 0 as x gets infinitely large or infinitely small. This happens when the degree of the numerator is less than the degree of the denominator. The degree is determined by the greatest exponent of x.

$f(x) = \frac{2x^8+3x^2+100}{x^9-12}$

One way to reason through why this makes sense is because when x is a ridiculously large number then most parts of the function hardly make any impact. The 100 for example is nothing in comparison and neither is the $3x^2$. The two important terms to compare are x^8 and x^9. The 2 isn't even important now because if x is even just a million than the x^9 will be a million times bigger than the x^8 and the 2 hardly matters again. Essentially, when x gets big enough, this function acts like $\frac{1}{x}$ which has a horizontal asymptote of 0.

Case 2: Degree of Numerator is Equal to the Degree of Denominator

If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is equal to the ratio of the leading coefficients.

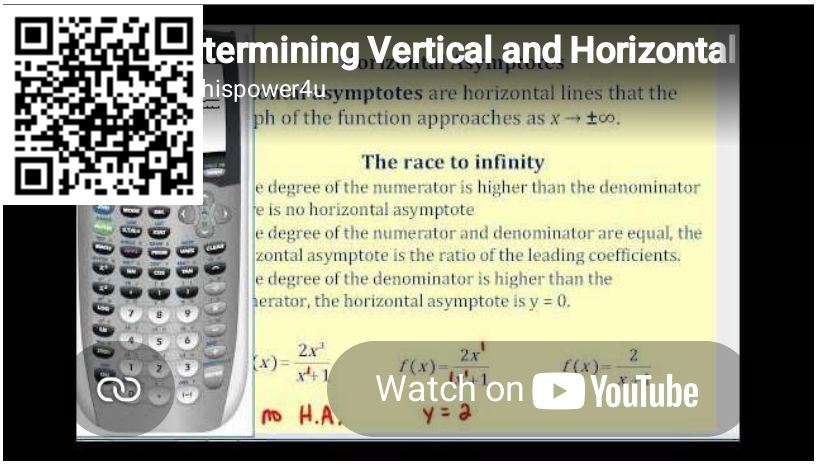
$f(x) = \frac{6x^4-3x^3+12x^2-9}{3x^4+144x-0.001}$

Notice how the degree of both the numerator and the denominator is 4. This means that the horizontal asymptote is $y = \frac{6}{3} = 2$. One way to reason through why this makes sense is because when x gets to be a very large number all the smaller powers will not really make much of an impact. The biggest contributors are only the biggest powers. Then the value of the numerator will be about twice that of the denominator. As x gets even bigger, then the function will get even closer to 2.

Case 3: Degree of Numerator is Greater than the Degree of Denominator

If the degree of the numerator is greater than the degree of the denominator, there does not exist a horizontal asymptote. You must determine if the function increases or decreases without bound in both the left and right directions.

Watch the following video, focusing on the parts about horizontal asymptotes.



Examples

Example 1

Earlier, you were asked if functions are allowed to touch their horizontal asymptotes. Functions may touch and pass through horizontal asymptotes without limit. This is a difference between vertical and horizontal asymptotes. In calculus, there are rigorous proofs to show that functions like the one in Example C do become arbitrarily close to the asymptote.

Example 2

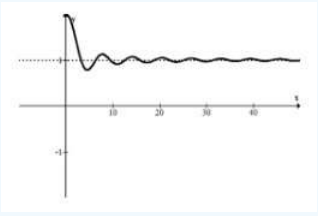
Identify the vertical and horizontal asymptotes of the following rational function.

$f(x) = \frac{(x-2)(4x+3)(x-4)}{(x-1)(4x+3)(x-6)}$

There is factor that cancels that is neither a horizontal or vertical asymptote. The vertical asymptotes occur at $x = 1$ and $x = 6$. To obtain the horizontal asymptote you could methodically multiply out each binomial, however since most of those terms do not matter, it is more efficient to first determine the relative powers of the numerator and the denominator. In this case they both happen to be 3 . Next determine the coefficient of the cubic terms only. The numerator will have $4x^3$ and the denominator will have $4x^3$ and so the horizontal asymptote will occur at $y = \frac{4}{4} = 1$

Example 3

Describe the right hand end behavior of the following function.



Notice how quickly this dampening wave function settles down. There seems to be an obvious horizontal axis on the right at $y = 1$

Example 4

Identify the horizontal asymptotes of the following function.

$f(x) = \frac{(x-3)(x+2)}{|(x-5)|\cdot(x-1)}$

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First notice the absolute value surrounding one of the terms in the denominator. The degrees of both the numerator and the denominator will be 2 which means that the horizontal asymptote will occur at a number. As x gets infinitely large, the function is approximately:

$f(x) = \frac{x^2}{x^2}$

So the horizontal asymptote is $y = -1$ as x gets infinitely large.

On the other hand, as x gets infinitely small the function is approximately:

$f(x) = \frac{x^2}{-x^2}$

So the horizontal asymptote is $y = -1$ as x gets infinitely small.

In this case, you cannot blindly use the leading coefficient rule because the absolute value changes the sign.

Example 5

Identify the horizontal asymptotes if they exist for the following 3 functions.

1. $f(x) = \frac{3x^6 - 72x}{x^6 + 999}$

The degrees of the numerator and the denominatro are equal so the horizontal asymptote is $y = 3$.

2. $h(x) = \frac{ax^4 + bx^3 + cx^2 + dx + e}{fx^4 + gx^3 + hx^2}$

The degrees of the numerator and the denominator are equal again so the horizontal asymptote is $y = \frac{a}{f}$

3. $g(x) = \frac{f(x)}{h(x)}$

As x gets infinitely large,

$$g(x) = \frac{f(x)}{h(x)} = \frac{\frac{3x^6 - 72x}{x^6 + 999}}{\frac{ax^4 + bx^3 + cx^2 + dx + e}{fx^4 + gx^3 + hx^2}} \approx \frac{3}{\frac{a}{f}} = \frac{3f}{a}$$

When you study calculus, you will learn the rigorous techniques that enable you to feel more confident about results like this.

Review

Identify the horizontal asymptotes, if they exist, for the following functions.

1. $f(x) = \frac{5x^4 - 2x}{x^4 + 32}$
2. $g(x) = \frac{3x^4 - 2x^6}{-x^4 + 2}$
3. $h(x) = \frac{3x^4 - 5x}{8x^3 + 3x^4}$
4. $j(x) = \frac{2x^3 - 15x}{-x^4 + 3}$
5. $k(x) = \frac{2x^5 - 3x}{5x^2 + 3x^4 + 2x - 7x^5}$
6. $f(x) = \frac{ax^{14} + bx^{23} + cx^{12} + dx + e}{fx^{24} + gx^{23} + hx^{21}}$
7. $g(x) = \frac{(x-1)(x+4)}{|(x-2)| \cdot (x-1)}$
8. Write a function that fits the following criteria:
   - Vertical asymptotes at $x = 1$ and $x = 4$
   - Zeroes at 3 and 5
   - Hole when $x = 6$
   - Horizontal asymptote at $y = \frac{2}{3}$
9. Write a function that fits the following criteria:
   - Vertical asymptotes at $x = -2$ and $x = 2$
   - Zeroes at 1 and 5
   - Hole when $x = 3$

- Horizontal asymptote at $y = 1$

10. Write a function that fits the following criteria:

- Vertical asymptotes at $x = 0$ and $x = 3$
- Zeroes at 1 and 2
- Hole when $x = 8$
- Horizontal asymptote at $y = 2$

11. Write a function that fits the following criteria:

- Vertical asymptotes at 2 and 6
- Zero at 5
- Hole when $x = 4$
- Horizontal asymptote at $y = 0$

12. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at when $x = 5$
- No horizontal asymptotes

Identify the vertical and horizontal asymptotes of the following rational functions.

13. $f(x) = \frac{(x-5)(2x+1)(x-3)}{(x-3)(4x+5)(x-6)}$
14. $g(x) = \frac{x(x-1)(x+3)(x-5)}{3x(x-1)(4x+3)}$
15. $h(x) = \frac{(x-2)(x+3)(x-6)}{(x-4)(x+3)^2(x+2)}$

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2.11 Oblique Asymptotes

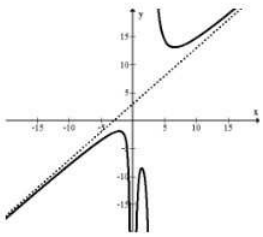
When the degree of the numerator of a rational function exceeds the degree of the denominator by one then the function has **oblique asymptotes**. In order to find these asymptotes, you need to use polynomial long division and the non-remainder portion of the function becomes the oblique asymptote. A natural question to ask is: what happens when the degree of the numerator exceeds that of the denominator by more than one?

Oblique Asymptotes

The following function is shown before and after polynomial long division is performed.

$$f(x) = \frac{x^4+3x^2+2x+14}{x^3-3x^2} = x + 3 + \frac{12x^2+2x+14}{x^3-3x^2}$$

Notice that the remainder portion will go to zero when x gets extremely large or extremely small because the power of the numerator is smaller than the power of the denominator. This means that while this function might go haywire with small absolute values of x, large absolute values of x are extremely close to the line $y = x + 3$.



Oblique asymptotes are these slanted asymptotes that show exactly how a function increases or decreases without bound. Oblique asymptotes are also called slant asymptotes.

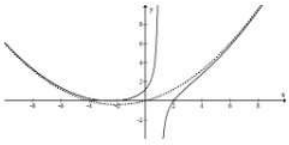
Sometimes a function will have an asymptote that does not look like a line. Take a look at the following function:

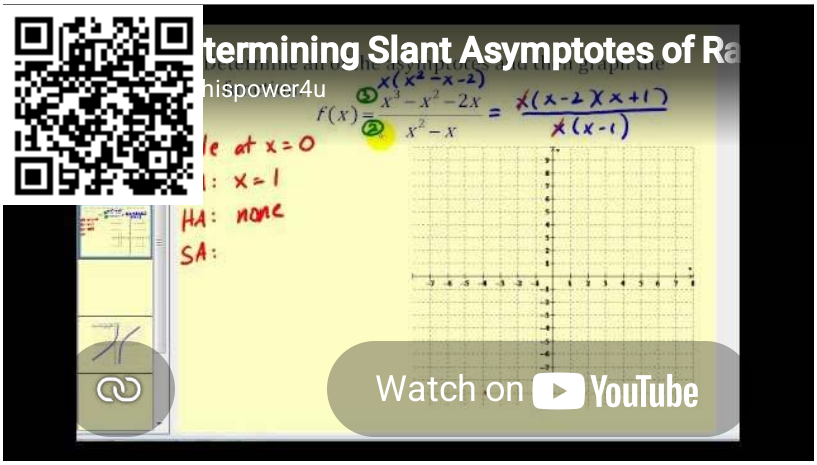
$$f(x) = \frac{(x^2-4)(x+3)}{10(x-1)}$$

The degree of the numerator is 3 while the degree of the denominator is 1 so the slant asymptote will not be a line. However when the graph is observed, there is still a clear pattern as to how this function increases without bound as x approaches very large and very small numbers.

$$f(x) = \frac{1}{10}(x^2 + 4x) - \frac{12}{10(x-1)}$$

As you can see, this looks like a parabola with a remainder. This rational function has a parabola backbone. A backbone is a function that a graph tends towards. This is not technically an oblique asymptote because it is not a line.





Examples

Example 1

Earlier, you were asked what happens when the power of the numerator exceeds the power of the denominator by more than one. As seen above, when the power of numerator exceeds the power of the denominator by more than one, the function develops a backbone that be shaped like any polynomial. Oblique asymptotes are always lines.

Example 2

Find the asymptotes and intercepts of the function:

$f(x) = \frac{x^3}{x^2-4}$

The function has vertical asymptotes at $x = \pm 2$.

After long division, the function becomes:

$f(x) = x + \frac{4}{x^2-4}$

This makes the oblique asymptote at $y = x$

Example 3

Identify the vertical and oblique asymptotes of the following rational function.

$$f(x) = \frac{x^3 - x^2 - x - 1}{(x-3)(x+4)} \tag{1}$$

After using polynomial long division and rewriting the function with a remainder and non-remainder portion it looks like this:

$$f(x) = x - 2 + \frac{13x - 25}{x^2 + x - 12} = x - 2 + \frac{13x - 25}{(x-3)(x+4)} \tag{2}$$

The oblique asymptote is $y = x - 2$. The vertical asymptotes are at $x = 3$ and $x = -4$ which are easier to observe in last form of the function because they clearly don't cancel to become holes.

Example 4

Create a function with an oblique asymptote at $y = 3x - 1$, vertical asymptotes at $x = 2, -4$ and includes a hole where x is 7 .

While there are an infinite number of functions that match these criteria, one example is:

$$f(x) = 3x - 1 + \frac{(x-7)}{(x-2)(x+4)(x-7)} \tag{3}$$

Example 5

Identify the backbone of the following function and explain why the function does not have an oblique asymptote.

$$f(x) = \frac{5x^5 + 27}{x^3} \tag{4}$$

While polynomial long division is possible, it is also possible to just divide each term by x^3.

$$f(x) = \frac{5x^5 + 27}{x^3} = \frac{5x^5}{x^3} + \frac{27}{x^3} = 5x^2 + \frac{27}{x^3} \tag{5}$$

The backbone of this function is the parabola $y = 5x^2$. This is not an oblique asymptote because it is not a line.

Review

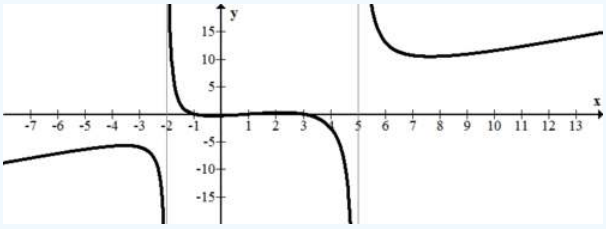
1. What is an oblique asymptote?

2. How can you tell by looking at the equation of a function if it will have an oblique asymptote or not?

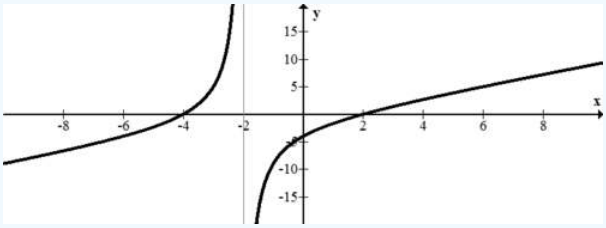
3. Can a function have both an oblique asymptote and a horizontal asymptote? Explain.

For each of the following graphs, sketch the graph and then sketch in the oblique asymptote if it exists. If it doesn't exist, explain why not.

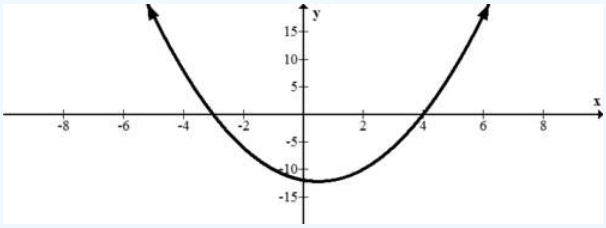
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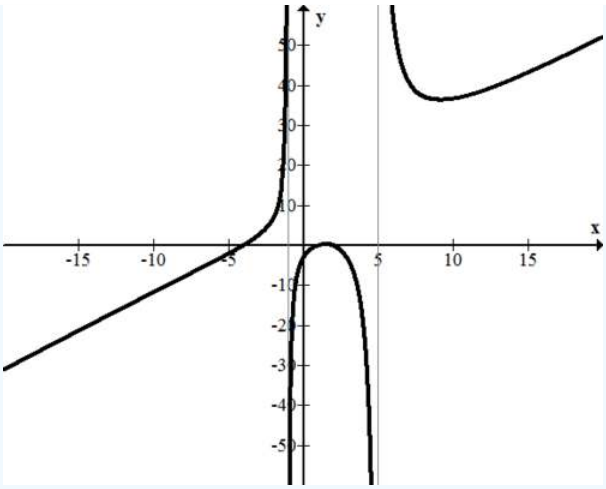


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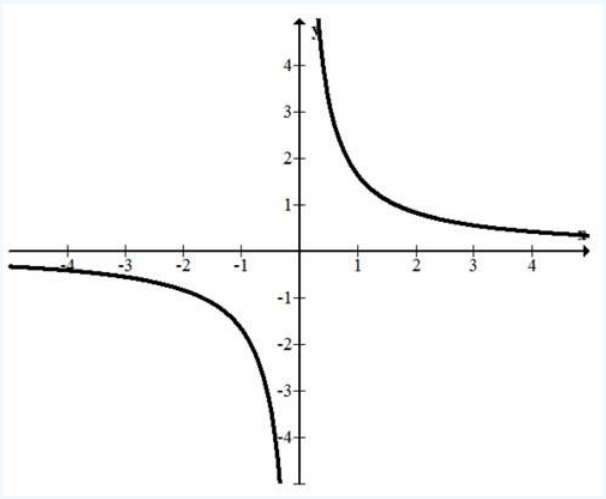


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8.



Find the equation of the oblique asymptote for each of the following rational functions. If there is not an oblique asymptote, explain why not and give an equation of the backbone of the function if one exists.

9. $f(x) = \frac{x^3-7x-6}{x^2-2x-15}$

10. $g(x) = \frac{x^3-7x-6}{x^4-3x^2-10}$

11. $h(x) = \frac{x^2+5x+6}{x^2+2x+1}$

12. $k(x) = \frac{x^4+9x^3+21x^2-x-30}{x^2+2x+1}$

13. Create a function with an oblique asymptotes at $y = 2x - 1$, a vertical asymptote at $x = 3$ and a hole where x is 7 .

14. Create a function with an oblique asymptote at $y = x$, vertical asymptotes at $x = 1, -3$ and no holes.

15. Does a parabola have an oblique asymptote? What about a cubic function?

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2.12 Sign Test for Rational Function Graphs

The asymptotes of a rational function provide a very rigid structure in which the function must live. Once the asymptotes are known you must use the sign testing procedure to see if the function becomes increasingly positive or increasingly negative near the asymptotes. A driving question then becomes how close does near need to be in order for the sign test to work?

Sign Test for Rational Functions

Consider mentally substituting the number 2.99999 into the following rational expression.

$f(x) = \frac{(x-1)(x+3)(x-5)(x+10)}{(x+2)(x-4)(x-3)}$

Without doing any of the arithmetic, simply note the sign of each term:

$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(-)}$

The only term where the value is close to zero is $(x-3)$ but careful subtraction still indicates a negative sign. The product of all of these signs is negative. This is strong evidence that this function approaches negative infinity as x approaches 3 from the left.

Next consider mentally substituting 3.00001 and going through the same process.

$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(+)}$

The product of all of these signs is positive which means that from the right this function approaches positive infinity instead. This technique is called the sign test. The **sign test** is a procedure for determining only whether a function is above or below the axis at a particular value.

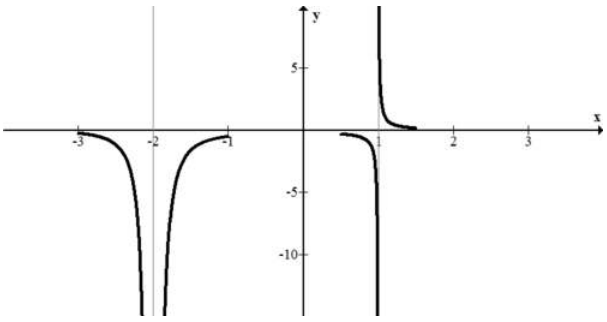
The sign test helps you sketch and graph a function. Look at the following function:

$f(x) = \frac{1}{(x+2)^2\cdot(x-1)}$

Your first step for sketching this is to identify the vertical asymptotes. The vertical asymptotes occur at $x = -2$ and $x = 1$.Then, you use the asymptotes to perform a sign test. The points to use the sign testing procedure with are $-2.001, -1.9999, 0.9999, 1.00001$. The number of decimals does matter so Iong as the number is sufficiently close to the asymptote. Note that any real number squared is positive.

$$f(-2.001) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(0.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(1.0001) = \frac{(+)}{(+)\cdot(+)} = +$$

Later when you sketch everything you will use your knowledge of zeroes and intercepts. For now, focus on just the portions of the graph near the asymptotes. Note that the graph below is NOT complete.



Examples

Example 1

Earlier, you were asked how close the numbers need to be to perform the sign test. In order to truly answer the question about how close the numbers need to be, calculus should be used. For the purposes of PreCalculus, the testing number should be closer to the vertical asymptote than any other number in the problem. If the vertical asymptote occurs at 3 and 3.01 is in the problem elsewhere, do not choose 3.1 as a sign test number.

Example 2

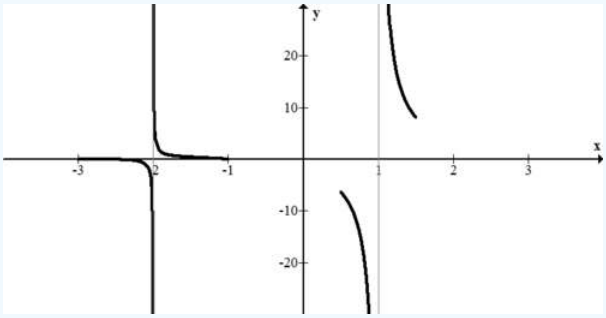
Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$f(x) = \frac{(x+1)(x-4)^2(x-1)(x+3)^3}{100(x-1)^2(x+2)}$

Note that $x = -2$ is clearly an asymptote. It may be initially unclear whether $x = 1$ is an asymptote or a hole. Just like holes have priority over zeroes, asymptotes have priority over holes. The four values to use the sign testing procedure are $-2.001, -1.9999, 0.9999, 1.00001$.

$$\begin{aligned} f(-2.001) &= \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)} = - \\ f(-1.9999) &= \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = + \\ f(0.9999) &= \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = - \\ f(1.0001) &= \frac{(+)\cdot(+)\cdot(+)\cdot(+)}{(+)\cdot(+)} = + \end{aligned} \tag{1}$$

A sketch of the behavior of this function near the asymptotes is:

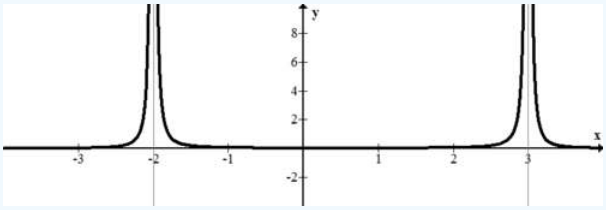


Example 3

Create a function with two vertical asymptotes at 3 and -2 such that the function approaches positive infinity from both directions at both vertical asymptotes.

Earlier, there was a function that approached negative infinity from both sides of the asymptote. This occurred because the term was squared in the denominator. An even powered term will always produce a positive term.

$f(x) = \frac{1}{(x-3)^2(x+2)^2}$



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Example 4

Create a function with three vertical asymptotes such that the function approaches negative infinity for large and small values of x and has an oblique asymptote.

There are an infinite number of possible solutions. The key is to create a function that may work and then use the sign testing procedure to check. Here is one possibility.

$f(x) = \frac{-x^7}{10(x-1)^2(x-2)^2(x-4)^2}$

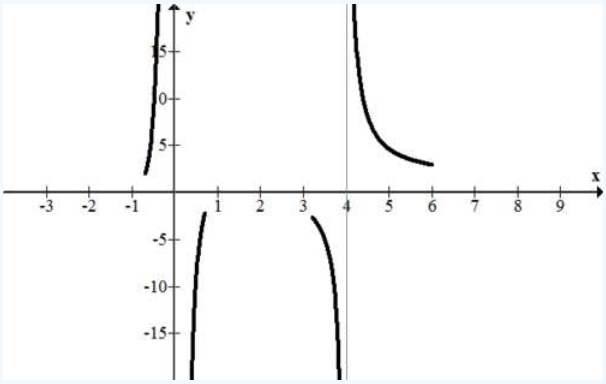
Example 5

Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$f(x) = \frac{(x-2)^3(x-1)^2(x+1)(x+3)}{x^3(x+\frac{1}{2})(x-1)(x-2)^2}$

The vertical asymptotes occur at $x = 0, -\frac{1}{2}$. Therefore the x values to sign test are -.001,0.001 3.999,4.0001

$f(-0.001) = +$
$f(0.001) = -$
$f(3.999) = -$
$f(4.0001) = +$



Review

Consider the function below for questions 1-4.

$f(x) = \frac{(x-2)^4(x+1)(x+3)}{x^3(x+3)(x-4)}$

1. Identify the vertical asymptotes.

2. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

3. What values will you need to use the sign test with in order to help you make a sketch of the graph?

4. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 5-8.

$g(x) = \frac{3(x-2)^2(x-1)^2(x+1)(x+3)}{15x^2(x+5)(x+1)(x-3)^2}$

5. Identify the vertical asymptotes.

6. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

7. What values will you need to use the sign test with in order to help you make a sketch of the graph?

8. Use the sign test and sketch the graph near the vertical asymptote(s).

Consider the function below for questions 9-12.

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$h(x) = \frac{9x^4-102x^3+349x^2-340x+100}{x^3-9x^2+24x-16}$

9. Identify the vertical asymptotes.

10. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

11. What values will you need to use the sign test with in order to help you make a sketch of the graph?

12. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 13-16.

$k(x) = \frac{2x^3-5x^2-11x-4}{3x^3+11x^2+5x-3}$

13. Identify the vertical asymptotes.

14. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

15. What values will you need to use the sign test with in order to help you make a sketch of the graph?

16. Use the sign test and sketch the graph near the vertical asymptotes.

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2.13 Graphs of Rational Functions by Hand

Sketching rational functions by hand is a mental workout because it combines so many different specific skills to produce a single coherent image. It will require you to closely examine the equation of the function in a variety of different ways in order to find clues as to the shape of the overall function. Since computers can graph these complicated functions much more accurately than people can, why is sketching by hand important?

Graphing Rational Functions

While there is no strict procedure for graphing rational functions by hand there is a flow of clues to look for in the function. In general, it will make sense to identify different pieces of information in this order and record them on a sketch.

Steps for Graphing Rational Functions By Hand

1. Examine the denominator of the rational function to determine the domain of the function. Distinguish between holes which are factors that can be canceled and vertical asymptotes that cannot. Plot the vertical asymptotes.
2. Identify the end behavior of the function by comparing the degrees of the numerator and denominator and determine if there exists a horizontal or oblique asymptote. Plot the horizontal or oblique asymptotes.
3. Identify the holes of the function and plot them.
4. Identify the zeroes and intercepts of the function and plot them.
5. Use the sign test to determine the behavior of the function near the vertical asymptotes.
6. Connect everything as best you can.

Now, apply those steps in graphing the following function:

$f(x) = \frac{4x^3-2x^2+3x-1}{8(x-1)^2(x+2)}$

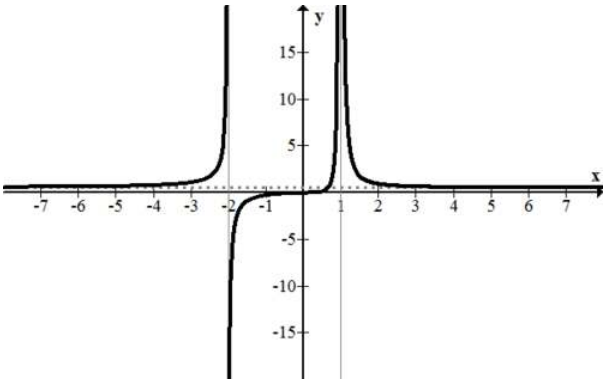
After attempting to factor the numerator you may realize that both $x = 1$ and $x = -2$ are vertical asymptotes rather than holes. The horizontal asymptote is $y = \frac{1}{2}$. There are no holes. The y intercept is:

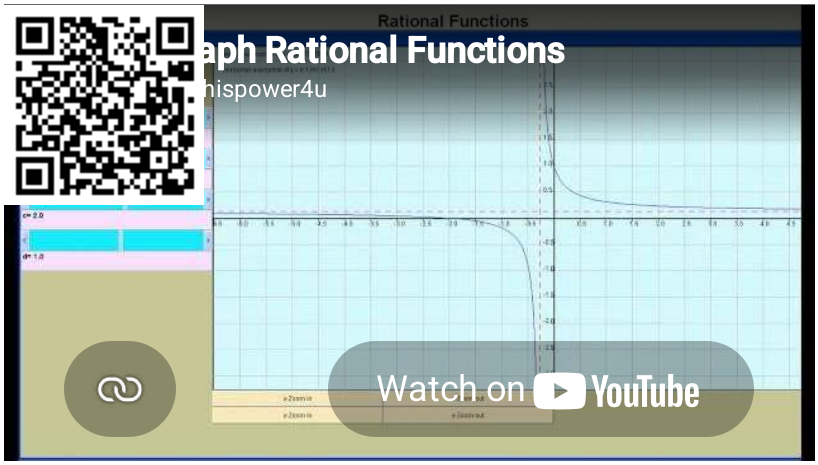
$f(0) = -\frac{1}{8 \cdot 2} = \frac{1}{16}$

The numerator is not factorable, but there is a zero between 0 and 1. You know this because there are no holes or asymptotes between 0 and 1 and the function switches from negative to positive in this region.

$4(0)^3 - 2(0)^2 + 3(0) - 1 = -1$
$4(1)^3 - 2(1)^2 + 3(1) - 1 = 4$

Putting all of this together in a sketch:





Examples

Example 1

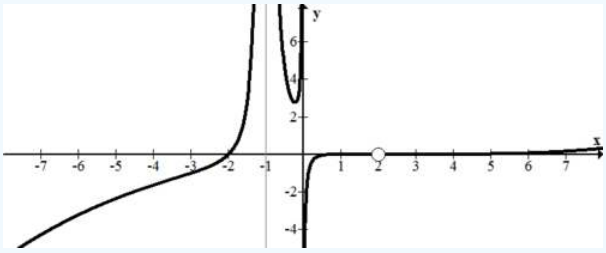
Earlier, you were asked why sketching graphs by hand is important. Computers can graph rational functions more accurately than people. However, computers may not be able to explain why a function behaves in certain ways. By being a detective and looking for clues in the equation of a function, you are applying high level analytical skills and powers of deduction. These analytical skills are vastly more important and transferable than the specific techniques involved with rational functions.

Example 2

Completely plot the following rational function.

$f(x) = \frac{(x-3)^2(x-2)^3(x-1)(x+2)}{300(x+1)^2(x-2)x}$

since this function is already factored, much of the work is already done. There is a hole at (2,0) . There are two vertical asymptotes at $x = -1, 0$. There are no horizontal or oblique asymptotes because the degree of the numerator is much bigger than the degree of the denominator. As x gets large this function grows without bound. As x get very small, this function decreases without bound. The function has no y intercept because that is where a vertical asymptote is. Besides the hole at (2 , 0), there are zeroes at (-2,0),(1,0) and $(3, 0)$. This is what the graph ends up looking like.

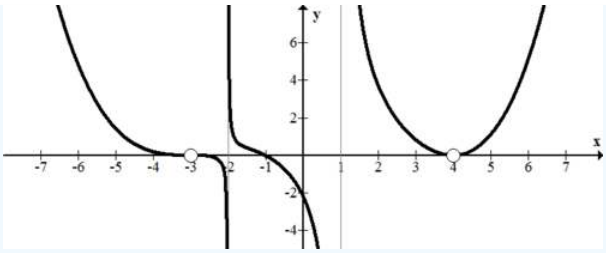


Notice on the right portion of the graph the curve seems to stay on the x-axis. In fact it does go slightly above and below x-axis, crossing through it at (1, 0), (2, 0) and (3, 0) before starting to increase.

Example 3

Estimate a function that would have the following graphical characteristics:

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First think about the vertical asymptotes and how they affect the equation of the function. Then consider zeroes and holes, and the way the graph looks at these places. Finally, use the y -intercept to refine your equation.

- The function has two vertical asymptotes at $x = -2, 1$ so the denominator must have the factors $(x+2)(x-1)$
- There is one zero at $x = -1$, so the numerator must have a factor of $(x+1)$
- There are two holes that appear to override zeroes which means the numerator and denominator must have the factors $(x+3)$ and $(x-4)$
- Because the graph goes from above the x -axis to below the x -axis at $x = -3$, the degree of the exponent of the $(x+3)$ factor must be ultimately odd.
- Because the graph stays above the x -axis before and after $x = 4$, the degree of the $(x-4)$ factor must be ultimately even.

A good estimate for the function is:

$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{(x+2)(x-1)(x+3)(x-4)}$

This function has all the basic characteristics, however it isn't scaled properly. When $x = 0$ this function has a y -intercept of -216 when it should be about -2. Thus you must divide by 108 so that the y intercept matches. Here is a better estimate for the function:

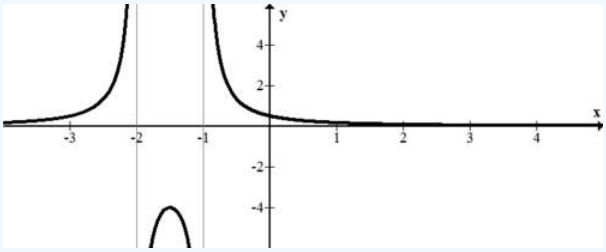
$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{108(x+2)(x-1)(x+3)(x-4)}$

Example 4

Graph the following rational function:

$f(x) = \frac{1}{x^2+3x+2}$

Here is the graph:

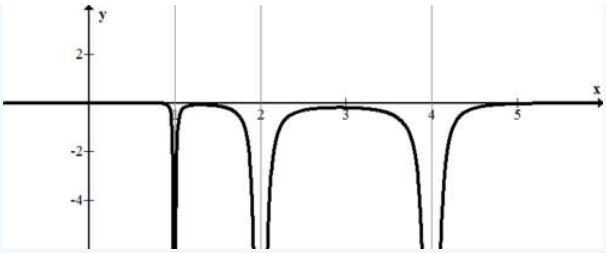


Example 5

Graph the following rational function:

$f(x) = \frac{-x^4}{10(x-1)^2(x-2)^2(x-4)^2}$

Here is the graph:



Review

Use the function below for 1-7.

$f(x) = \frac{2(x+4)(x-3)(x+1)}{8(x-1)^2(x+2)}$

1. Identify the vertical asymptotes and holes for the function.

2. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.

3. Identify any horizontal or oblique asymptotes for the function.

4. Describe the end behavior of the function.

5. Find the zeroes of the function.

6. Find the y -intercept of the function.

7. Use the information from $1-6$ to sketch the function.

Use the function below for $8-14$.

$g(x) = \frac{(x^2-9)(x^2-4)}{5(x-2)^2(x+1)^2}$

8. Identify the vertical asymptotes and holes for the function.

9. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.

10. Identify any horizontal or oblique asymptotes for the function.

11. Describe the end behavior of the function.

12. Find the zeroes of the function.

13. Find the y -intercept of the function.

14. Use the information from 8-13 to sketch the function.

15. Graph the function below by hand.

$h(x) = \frac{x^3+5x^2+2x-8}{x^2-3x-10}$

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2.7 Holes in Rational Functions

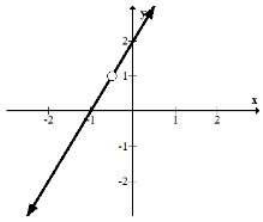
In a function like $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$, you should note that the factor $(x-1)$ clearly cancels leaving only $3x-1$. This appears to be a regular line. What happens to this line at $x = 1$?

Holes and Rational Functions

A hole on a graph looks like a hollow circle. It represents the fact that the function approaches the point, but is not actually defined on that precise x value.

Take a look at the graph of the following equation:

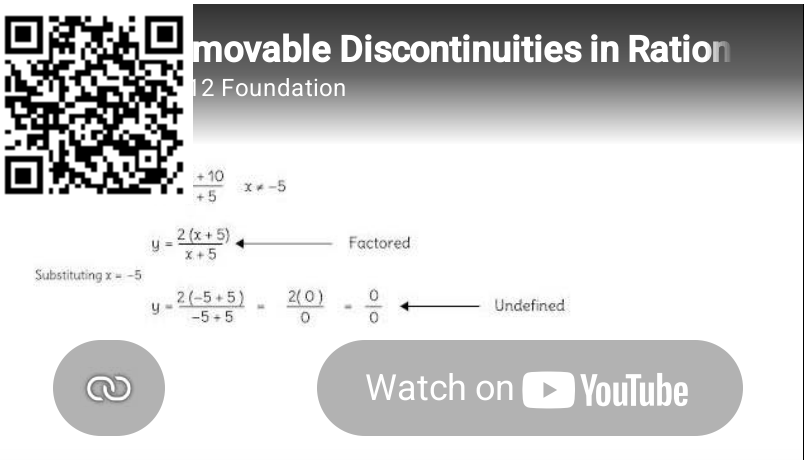
$f(x) = (2x+2) \cdot \frac{(x+\frac{1}{2})}{(x+\frac{1}{2})}$



The reason why this function is not defined at $-\frac{1}{2}$ is because $-\frac{1}{2}$ is not in the domain of the function. As you can see, $f\left(-\frac{1}{2}\right)$ is undefined because it makes the denominator of the rational part of the function zero which makes the whole function undefined. Also notice that once the factors are canceled/removed then you are left with a normal function which in this case is $2x+2$. The hole in this situation is at $\left(-\frac{1}{2}, 1\right)$ because after removing the factors that cancel, $f\left(-\frac{1}{2}\right) = 1$

This is the essence of dealing with holes in rational functions. You should cancel what you can and graph the function like normal making sure to note what x values make the function undefined. Once the function is graphed without holes go back and insert the hollow circles indicating what x values are removed from the domain. This is why holes are called removable discontinuities.

Watch the first part of this video and focus on holes in rational equations.



Examples

Example 1

Earlier, you were asked what happens to the equation $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$ at $x = 1$. since this function that is not defined at $x = 1$ there is a removable discontinuity that is represented as a hollow circle on the graph. Otherwise the function behaves precisely as $3x+1$.

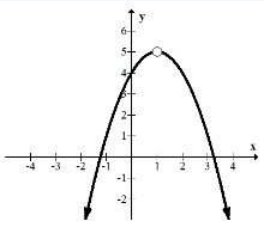
Example 2

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{-x^3+3x^2+2x-4}{x-1}$

This function requires some algebra to change it so that the removable factors become obvious. You should suspect that $(x-1)$ is a factor of the numerator and try polynomial or synthetic division to factor. When you do, the function becomes:

$f(x) = \frac{(-x^2+2x+4)(x-1)}{(x-1)}$



The removable discontinuity occurs at (1, 5).

Example 3

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{x^3-7x-6}$

This is probably one of the most challenging rational expressions with only holes that people ever try to graph by hand. There are multiple ways to start, but a good habit to get into is to factor everything you possibly can initially. The denominator seems less complicated with possible factors $(x \pm 1), (x \pm 2), (x \pm 3), (x \pm 6)$. Using polynomial division, you will find the denominator becomes:

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{(x+1)(x+2)(x-3)}$

The factors of the denominator are strong hints as to the factors of the numerator so use polynomial division and try each. When you fully factor the numerator you will have:

$f(x) = \frac{(x^3-6x^2+12x-9)(x+1)(x+2)(x-3)}{(x+1)(x+2)(x-3)}$

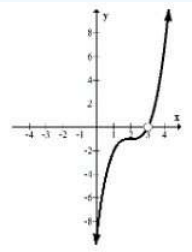
Note the factors that cancel $(x = -1, -2, 3)$ and then work with the cubic function that remains.

$f(x) = x^3 - 6x^2 + 12x - 9$

At this point it is probably reasonable to make a table and plot points to get a sense of where this cubic function lives. You also could notice that the coefficients are almost of the pattern 1 3 3 1 which is the binomial expansion. By separating the -9 into -8 -1 you can factor the first four terms.

$f(x) = x^3 - 6x^2 + 12x - 8 - 1 = (x-2)^3 - 1$

This is a cubic function that has been shifted right by two units and down one unit.



Note that there are two holes that do not fit in the graph window. When this happens you still need to note where they would appear given a properly sized window. To do this, substitute the invalid x values: $x = -1, -2, 3$ into the factored cubic that

remained after canceling.

f(x)=(x-2)^{3}-1

Holes: (3, 0); (-1, -28); (-2, -65)

Example 4

Without graphing, identify the location of the holes of the following function.

$f(x) = \frac{x^3+4x^2+x-6}{x^2+5x+6}$

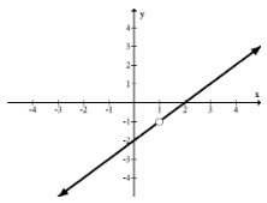
First factor everything. Then, identify the x values that make the denominator zero and use those values to find the exact location of the holes.

$f(x) = \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)}$

Holes: (-3, -4); (-2, -3)

Example 5

What is a possible equation for the following rational function?



The function seems to be a line with a removable discontinuity at $(1, -1)$. The line is has slope 1 and y -intercept of -2 and so has the equation:

$f(x) = x - 2$

The removable discontinuity must not allow the x to be 1 which implies that it is of the form $\frac{x-1}{x-1}$. Therefore, the function is:

$f(x) = \frac{(x-2)(x-1)}{x-1}$

Review

1. How do you find the holes of a rational function?

2. What's the difference between a hole and a removable discontinuity?

3. If you see a hollow circle on a graph, what does that mean?

Without graphing, identify the location of the holes of the following functions.

4. $f(x) = \frac{x^2+3x-4}{x-1}$

5. $g(x) = \frac{x^2+8x+15}{x+3}$

6. $h(x) = \frac{x^3+6x^2+2x-8}{x^2+x-2}$

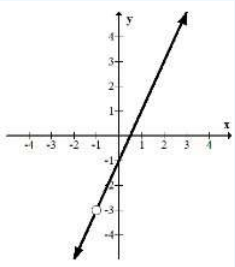
7. $k(x) = \frac{x^3+6x^2+2x-8}{x^2-3x+2}$

8. $j(x) = \frac{x^3+4x^2-17x-60}{x^2-9}$

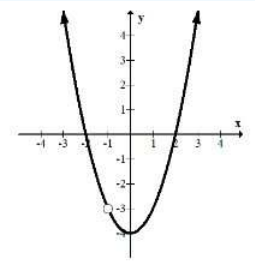
9. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-5x+4}$

10. $g(x) = \frac{x^3-4x^2-19x-14}{x^2-8x+7}$

11. What is a possible equation for the following rational function?



12. What is a possible equation for the following rational function?



Sketch the following rational functions.

13. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-x-12}$

14. $g(x) = \frac{x^3+4x^2-17x-60}{x^2+8x+15}$

15. $h(x) = \frac{x^3-4x^2-19x-14}{x^2-6x-7}$

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2.8 Zeroes of Rational Functions

The zeroes of a function are the collection of x values where the height of the function is zero. How do you find these values for a rational function and what happens if the zero turns out to be a hole?

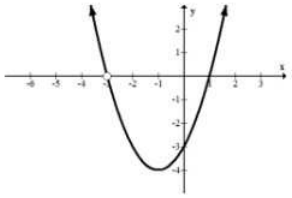
Finding Zeroes of Rational Functions

Zeroes are also known as x -intercepts, solutions or roots of functions. They are the x values where the height of the function is zero. For rational functions, you need to set the numerator of the function equal to zero and solve for the possible x values. If a hole occurs on the x value, then it is not considered a zero because the function is not truly defined at that point.

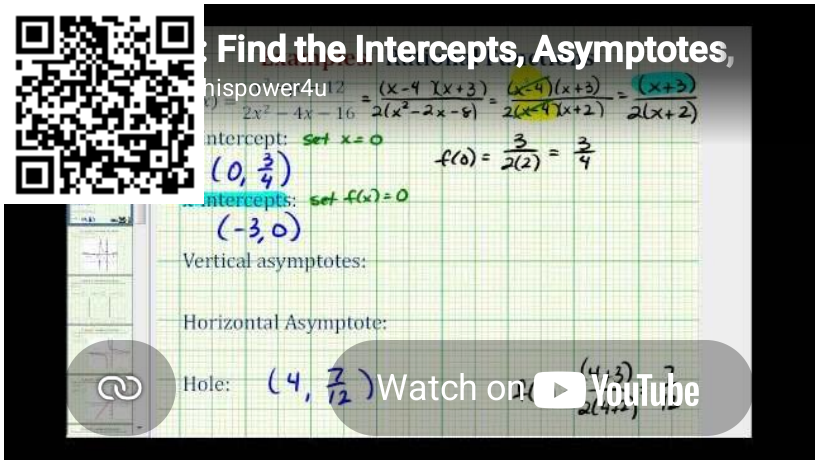
Take the following rational function:

$f(x) = \frac{(x-1)(x+3)(x+3)}{x+3}$

Notice how one of the $x+3$ factors seems to cancel and indicate a removable discontinuity. Even though there are two $x+3$ factors, the only zero occurs at $x=1$ and the hole occurs at (-3,0).



Watch the video below and focus on the portion of this video discussing holes and x -intercepts.



Examples

Example 1

Earlier, you were asked how to find the zeroes of a rational function and what happens if the zero is a hole. To find the zeroes of a rational function, set the numerator equal to zero and solve for the x values. When a hole and a zero occur at the same point, the hole wins and there is no zero at that point.

Example 2

Create a function with zeroes at $x=1,2,3$ and holes at $x=0,4$.

There are an infinite number of possible functions that fit this description because the function can be multiplied by any constant. One possible function could be:

$f(x) = \frac{(x-1)(x-2)(x-3)x(x-4)}{x(x-4)}$

Note that 0 and 4 are holes because they cancel out.

Example 3

Identify the zeroes, holes and y intercepts of the following rational function without graphing.

$f(x) = \frac{x(x-2)(x-1)(x+1)(x+2)}{(x-1)(x+1)}$

The holes occur at $x = -1, 1$. To get the exact points, these values must be substituted into the function with the factors canceled.

$f(x) = x(x-2)(x+1)(x+2)$
$f(-1) = 0, f(1) = -6$

The holes are (-1,0); $(1, 6)$. The zeroes occur at $x = 0, 2, -2$. The zero that is supposed to occur at $x = -1$ has already been demonstrated to be a hole instead.

Example 4

Identify the y intercepts, holes, and zeroes of the following rational function.

$f(x) = \frac{6x^3-7x^2-x+2}{x-1}$

After noticing that a possible hole occurs at $x = 1$ and using polynomial long division on the numerator you should get:

$f(x) = (6x^2 - x - 2) \cdot \frac{x-1}{x-1}$

A hole occurs at $x = 1$ which turns out to be the point (1,3) because $6 \cdot 1^2 - 1 - 2 = 3$.

The y -intercept always occurs where $x = 0$ which turns out to be the point (0,-2) because $f(0) = -2$

To find the x -intercepts you need to factor the remaining part of the function:

$(2x + 1)(3x - 2)$

Thus the zeroes (x -intercepts) are $x = -\frac{1}{2}, \frac{2}{3}$.

Example 5

Identify the zeroes and holes of the following rational function.

$f(x) = \frac{2(x+1)(x+1)(x+1)}{2(x+1)}$

The hole occurs at $x = -1$ which turns out to be a double zero. The hole still wins so the point (-1,0) is a hole. There are no zeroes. The constant 2 in front of the numerator and the denominator serves to illustrate the fact that constant scalars do not impact the x values of either the zeroes or holes of a function.

Review

Identify the intercepts and holes of each of the following rational functions.

1. $f(x) = \frac{x^3+x^2-10x+8}{x-2}$
2. $g(x) = \frac{6x^3-17x^2-5x+6}{x-3}$
3. $h(x) = \frac{(x+2)(1-x)}{x-1}$
4. $j(x) = \frac{(x-4)(x+2)(x+2)}{x+2}$
5. $k(x) = \frac{x(x-3)(x-4)(x+4)(x+2)}{(x-3)(x+4)}$
6. $f(x) = \frac{x(x+1)(x+1)(x-1)}{(x-1)(x+1)}$
7. $g(x) = \frac{x^3-x^2-x+1}{x^2-1}$
8. $h(x) = \frac{4-x^2}{x-2}$

9. Create a function with holes at $x = 3, 5, 9$ and zeroes at $x = 1, 2$.
10. Create a function with holes at $x = -1, 4$ and zeroes at $x = 1$.
11. Create a function with holes at $x = 0, 5$ and zeroes at $x = 2, 3$.
12. Create a function with holes at $x = -3, 5$ and zeroes at $x = 4$.
13. Create a function with holes at $x = -2, 6$ and zeroes at $x = 0, 3$.
14. Create a function with holes at $x = 1, 5$ and zeroes at $x = 0, 6$.
15. Create a function with holes at $x = 2, 7$ and zeroes at $x = 3$.

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2.9 Vertical Asymptotes

The basic rational function $f(x) = \frac{1}{x}$ is a hyperbola with a vertical asymptote at $x = 0$. More complicated rational functions may have multiple vertical asymptotes. These asymptotes are very important characteristics of the function just like holes. Both holes and vertical asymptotes occur at x values that make the denominator of the function zero. A driving question is: what makes vertical asymptotes different from holes?

Finding Vertical Asymptotes

Vertical asymptotes occur when a factor of the denominator of a rational expression does not cancel with a factor from the numerator. When you have a factor that does not cancel, instead of making a hole at that x value, there exists a vertical asymptote. The vertical asymptote is represented by a dotted vertical line. Most calculators will not identify vertical asymptotes and some will incorrectly draw a steep line as part of a function where the asymptote actually exists.

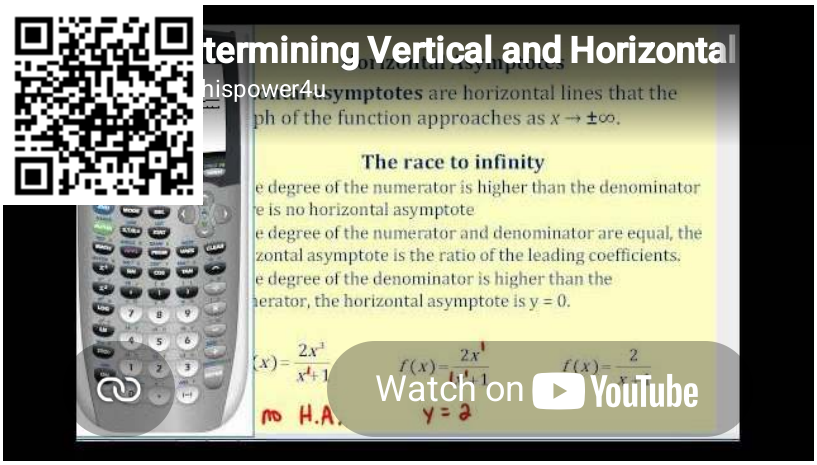
Your job is to be able to identify vertical asymptotes from a function and describe each asymptote using the equation of a vertical line.

Take the following rational function:

$f(x) = \frac{(2x-3)(x+1)(x-2)}{(x+2)(x+1)}$

To identify the holes and the equations of the vertical asymptotes, first decide what factors cancel out. The factor that cancels represents the removable discontinuity. There is a hole at (-1,15) . The vertical asymptote occurs at $x = -2$ because the factor $x + 2$ does not cancel.

Watch the following video, focusing on the parts about vertical asymptotes.



Example 1

Earlier, you were asked how asymptotes are different than holes. Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the domain of a rational function.

Example 2

Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 2 and 5
- Hole at (4, 2)

Each criteria helps build the function. The vertical asymptotes imply that the denominator has two factors that do not cancel with the numerator:

$\frac{1}{x\cdot(x-3)}$

The zeroes at 2 and 5 imply the numerator has two factors that do not cancel.

$\frac{(x-2)(x-5)}{x\cdot(x-3)}$

The hole at (4,2) implies that there is a factor $x-4$ that cancels on the numerator and the denominator.

$\frac{(x-2)(x-5)(x-4)}{x\cdot(x-3)(x-4)}$

The tricky part is that the height of the function must be 2 after the $x-4$ factor has been canceled and the 4 is substituted in. Currently it is $-\frac{1}{2}$

$\frac{(4-2)(4-5)}{4\cdot(4-3)}=-\frac{1}{2}$

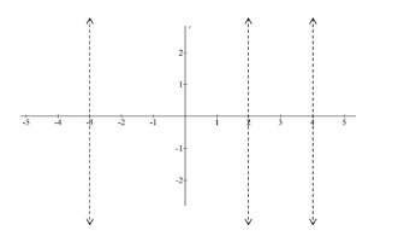
In order to make the hole exist at a height of 2, you need to multiply the function by a scalar of -4.

$f(x)=\frac{-4(x-2)(x-5)(x-4)}{x\cdot(x-3)(x-4)}$

Example 3

Draw the vertical asymptotes for the following function.

$f(x)=\frac{1}{(x-4)(x-2)(x+3)}$



Note that you may not know the characteristics of what the function does inside these vertical lines. You will soon learn how to use sign tests as well as techniques you've already learned to fill in the four sections that this function is divided into.

Example 4

Identify the holes and equations of the vertical asymptotes of the following rational function.

$f(x)=\frac{3(x-1)(x+2)(x-3)(x+4)}{5(x+\frac{1}{2})(2+x)(3-x)(x-8)}$

The vertical asymptotes occur at $x=-\frac{1}{2}, x=8$. Holes occur when x is -2 and 3. To get the height of the holes at these points, remember to cancel what can be canceled and then substitute the values. A very common mistake is to forget to cancel $\frac{x-3}{3-x}=-1$

$$g(x)=\frac{-3(x-1)(x+4)}{5\left(x+\frac{1}{2}\right)(x-8)}$$

$$g(-2)=\frac{6}{25}$$

$$g(3)=\frac{12}{25}$$

The holes are at $\left(-2,\frac{6}{25}\right),\left(3,\frac{12}{25}\right)$

Example 5

Identify the domain of the following function and then identify the holes and vertical asymptotes.

$f(x)=\frac{(3x-4)(1-x)(x^2+4)}{(3x-2)(x-1)}$

The domain of the function written in interval notation is: $\left(-\infty,\frac{2}{3}\right)\cup\left(\frac{2}{3},1\right)\cup(1,\infty)$. Note that the domain is all real numbers except for where the denominator is zero.

There are two discontinuities: one is a hole and one is a vertical asymptote. The hole occurs at (1,5) The vertical asymptote occurs at $x = \frac{2}{3}$

Notice that holes are identified as points while vertical asymptotes are identified as lines of the form $x = a$ where a is some constant.

Review

1. Write a function that fits the following criteria:

- Vertical asymptotes at 1 and 4
- Zeroes at 3 and 5
- Hole at (6, 3)

2. Write a function that fits the following criteria:

- Vertical asymptotes at -2 and 2
- Zeroes at 1 and 5
- Hole at (3, -4)

3. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 1 and 2
- Hole at (8, 21)

4. Write a function that fits the following criteria:

- Vertical asymptotes at 2 and 6
- Zero at 5
- Hole at (4, 1)

5. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at (5, 10)

Give the equations of the vertical asymptotes for the following functions.

6. $f(x) = \frac{(2-x)}{(x-2)(x-4)}$

7. $g(x) = \frac{-x}{(x+1)(x-3)}$

8. $h(x) = 6 - \frac{x+2}{(x+1)(x-5)}$

9. $j(x) = \frac{10}{x-3} - \frac{x}{(x+2)(x-3)}$

10. $k(x) = 2 - \frac{(4-x)}{(x+3)(x-4)}$

Identify the holes and equations of the vertical asymptotes of the following rational functions.

11. $f(x) = \frac{3(x-1)(x+1)(x-4)(x+4)}{4(x+4)(2+x)(4-x)(x+1)}$

12. $g(x) = \frac{x(x-3)(x-8)(x-3)(x+4)}{7(x+1)(1+x)(3-x)(x-8)}$

State the domain of the following rational functions.

13. $h(x) = \frac{x(x+1)(x-3)(x+4)}{x(3-x)(x-1)}$

14. $j(x) = \frac{x^2+3x-4}{x^2-6x-16}$

15. $k(x) = \frac{2x-10}{x^3+4x^2+3x}$

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CHAPTER OVERVIEW

3: Logs and Exponents

3.1: Exponential Functions
3.2: Properties of Exponents
3.3: Scientific Notation
3.4: Properties of Logs
3.5: Change of Base
3.6: Exponential Equations
3.7: Logistic Functions

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3.1: Exponential Functions

Exponential growth is one of the most powerful forces in nature. A famous legend states that the inventor of chess was asked to state his own reward from the king. The man asked for a single grain of rice for the first square of the chessboard, two grains of rice for the second square and four grains of rice for the third. He asked for the entire 64 squares to be filled in this way and that would be his reward. Did the man ask for too little, or too much?

Exponential Functions

Exponential functions take the form $f(x) = a \cdot b^x$ where a and b are constants. a is the starting amount when $x = 0$.b tells the story about the growth. If the growth is doubling then b is 2 . If the growth is halving (which would be decay), then b is $\frac{1}{2}$. If the growth is increasing by 6% then b is 1.06

Exponential growth is everywhere. Money grows exponentially in banks. Populations of people, bacteria and animals grow exponentially when their food and space aren't limited.

Radioactive isotopes like Carbon 14 have something called a **half-life** that indicates how long it takes for half of the molecules present to decay into other more stable molecules. It takes about 5,730 years for this process to occur which is how scientists can date artifacts of ancient humans.

Let's say a mummified animal is found preserved on the slopes of an ice covered mountain. After testing, you see that exactly one fourth of the carbon-14 has yet to decay and you want to find out how long ago was this animal alive. How would you do that?

Since this problem does not give specific amounts of carbon, it can be inferred that the time will not depend on the specific amounts. One technique that makes the problem easier to work with could be to create an example scenario that fits the one fourth ratio. Suppose 60 units were present when the animal was alive at time zero. This means that 15 units must be present today.

$15 = a \cdot \left(\frac{1}{2}\right)^x$

$60 = a \cdot \left(\frac{1}{2}\right)^0$

The second equation yields $a = 60$ and then the first equation becomes:

$15 = 60 \cdot \left(\frac{1}{2}\right)^x$

Although you may not yet have the algebraic tools to solve for x, you should still be able to see that x is 2. This does not mean that two years ago the animal was alive, it means that two half life cycles ago the animal was alive. The half life cycle for carbon 14 is 5,730 years so this animal was alive over 11,000 years ago.

Now, suppose you invested $100 the day you were born and it grew by 6% every year until you were 100 years old. How would you use an exponential function to determine how much would this investment be worth then?

The starting amount is 100 and the growth is 1.06 because it grows by 6% each year. This is enough information to write an exponential function. The x stands for time in years and the $f(x)$ stands for the amount of money in the account. Plugging in to the formula for an exponential function, you will get the equation:

$f(x) = 100 \cdot 1.06^x$

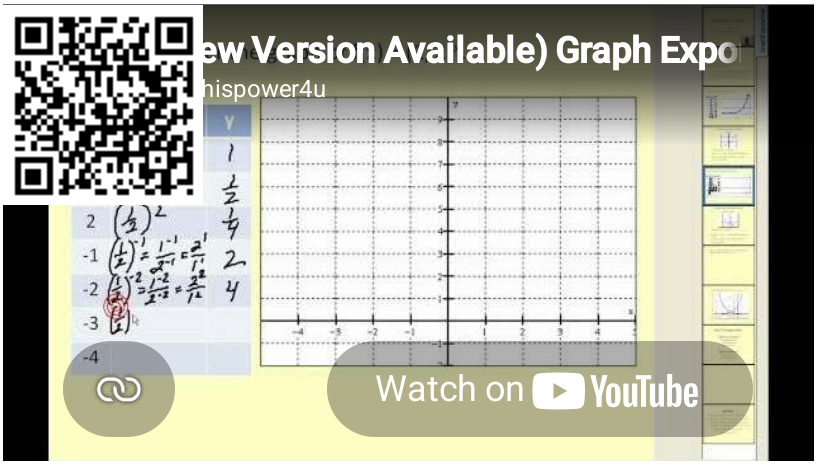
Then, you need to substitute in 100 years for x.

$f(x) = 100 \cdot 1.06^x$

$f(x) = 100 \cdot 1.06^{100}$

$f(x) \approx 33,930.21$

After a century, there will be almost $34,000 in the account. Interest has greatly increased the $100 initial investment.



Examples

Example 1

Earlier you were asked if the man asked for too little or too much rice if he gets one grain for the first square, two grains for the second, four grains for the third and so on. The number of grains of rice on the last square, the 64th square, would be almost ten quintillion (million million million). That is more rice than is produced in the world in an entire year.

$2^{63} = 9,223,372,036,854,775,808$

Example 2

Suppose forty rabbits are released on an island. The rabbits mate once every four months and produce up to 4 offspring who also produce more offspring four months later. Estimate the number of rabbits on the island in 3 years if their population grows exponentially. Assume half the population is female.

Even though parts of this problem are unrealistic, it serves to illustrate how quickly exponential growth works. Forty is the initial amount so $a = 40$. At the end of the first 4 month period 20 female rabbits could have their litters and up to 80 newborn rabbits could be born. The population has grown from 40 to 120 which means tripled. Thus, $b = 3$. The last thing to remember is that the time period is in 4 month periods. Three years must be 9 periods.

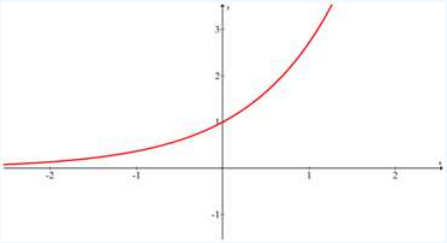
$f(x) = 40 \cdot 3^9 = 787,320$

So after three years, there could be up to 787,320 rabbits!

Example 3

Completely analyze the following exponential function.

$f(x) = e^x$



Analyze in this context means to define all the characteristics of a function.

Domain: $x \in (-\infty, \infty)$

Range: $y \in (0, \infty)$

Increasing: $x \in (-\infty, \infty)$

Decreasing: NA

Zeroes: None

Intercepts: (0,1)

Maximums: None

Minimums: None

Asymptotes: $y = 0$ as x gets infinitely small (Horizontal asymptote)

Holes: None

Example 4

Identify which of the following functions are exponential functions and which are not.

1. $y = x^6$
2. $y = 5^x$
3. $y = 1^x$
4. $y = x^x$
5. $y = x^{\frac{1}{2}}$

Exponential functions are of the form $y = a \cdot b^x$

a. $y = x^6$ is not an exponential function because x is not in the exponent.

b. $y = 5^x$ Exponential function.

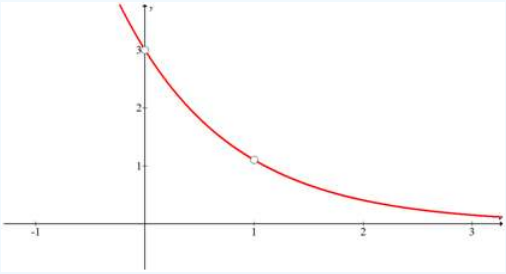
c. $y = 1^x$ Not a true exponential function because y is always 1 which is a constant function.

d. $y = x^x$ Not an exponential function because x is both the base and power of the exponent.

e. $y = x^{\frac{1}{2}}$ Not an exponential function.

Example 5

Write the exponential function that passes through the following points: $(0, 3), \left(1, \frac{3}{e}\right)$



The starting number is $a = 3$. This number is changed by a factor of $\frac{1}{e}$ which is b.

$f(x) = 3\left(\frac{1}{e}\right)^x = 3e^{-x}$

Review

1. Explain what makes a function an exponential function. What does its equation look like?

2. Is the domain for all exponential functions all real numbers?

3. How can you tell from its equation whether or not the graph of an exponential function will be increasing?

4. How can you tell from its equation whether or not the graph of an exponential function will be decreasing?

5. What type of asymptotes do exponential functions have? Explain.

6. Suppose you invested $4,500 and it grew by 4% every year for 30 years. How much would this investment be worth after 30 years?

7. Suppose you invested $10,000 and it grew by 12% every year for 40 years. How much would this investment be worth after 40 years?

Write the exponential function that passes through the following points.

8. (0, 5) and (1, 25)

9. (0, 2) and (1, 8)

10. (0, 16) and (2, 144)

11. (1, 4) and (3, 36)

12. (0, 16) and (3, 2)

13. (0, 81) and (2, 9)

14. (1, 144) and (3, 12)

15. Explain why for exponential functions of the form $y = a \cdot b^x$ the y -intercept is always the value of a

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3.2: Properties of Exponents

It is important to quickly and effectively manipulate algebraic expressions involving exponents. One simplification that comes up often is that expressions and numbers raised to the 0 power are always equal to 1. Why is this true and is it always true?

Exponent Properties

Consider the following exponential expressions with the same base and what happens through the algebraic operations. You should feel comfortable with all of these types of manipulations. Let b^y, b^x be exponential terms.

Addition and Subtraction

$b^x \pm b^y = b^x \pm b^y$

Only in the special case when $x = y$ can the terms be combined. This is a basic property of combining like terms.

Multiplication

$b^x \cdot b^y = b^{x+y}$

When the bases are the same then exponents can be added.

Division

$\frac{b^x}{b^y} = b^{x-y}$

The division rule is an extension of the multiplication rule with the possibility of a negative in the exponent.

Negative exponent

$b^{-x} = \frac{1}{b^x}$

A negative exponent means reciprocal.

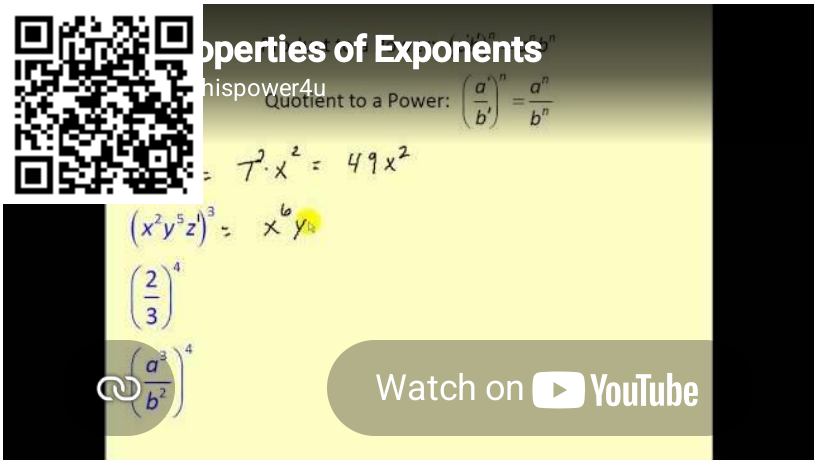
Fractional exponent

$(b)^{\frac{1}{x}} = \sqrt[x]{b}$

Square roots are what most people think of when they think of roots, but roots can be taken with any real number using fractional exponents.

Powers of Powers

$(b^x)^y = b^{x \cdot y}$



Examples

Example 1

Earlier, you were asked why expressions and numbers raised to the 0 power are always equal to 1. Consider the following pattern and decide what the next term in the sequence should be:

16, 8, 4, 2, ___

It makes sense that the next term is 1 because each successive term is half that of the previous term. These numbers correspond to powers of 2.

$2^4, 2^3, 2^2, 2^1$

In this case you could decide that the next term must be 2^0. This is a useful technique for remembering what happens when a number is raised to the 0 power.

One question that extends this idea is what is the value of 0^0 ? People have argued about this for centuries. Euler argued that it should be 1 and many other mathematicians like Cauchy and Möbius argued as well. If you search today you will still find people discussing what makes sense. In practice, many mathematicians note this value as undefined.

Example 2

Simplify the following expression until all exponents are positive.

$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0}$

$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0} = \frac{a^6b^{-9}}{ab^2 \cdot 1} = \frac{a^5}{b^{11}}$

Example 3

Simplify the following expression until all exponents are positive.

$(2x)^5 \cdot \frac{4^2}{2^{-3}}$

$(2x)^5 \cdot \frac{4^2}{2^{-3}} = \frac{2^5x^5 2^4}{2^{-3}} = \frac{2^9x^5}{2^{-3}} = 2^{12}x^5$

Example 4

Simplify the following expression using positive exponents.

$\frac{(2^6 \cdot 8^3)^{-3}}{4^2(\frac{1}{2})^4 64^{\frac{1}{3}}}$

Rewrite every exponent as a power of 2.

For example $8^3 = (2^3)^3 = 2^9$ and $64^{\frac{1}{3}} = (2^6)^{\frac{1}{3}} = 2^2$

$\frac{(2^6 \cdot 8^3)^{-3}}{4^2(\frac{1}{2})^4 64^{\frac{1}{3}}} = \frac{(2^6 \cdot 2^9)^{-3}}{2^4 2^{-4} 2^2} = \frac{(2^{15})^{-3}}{2^2} = \frac{2^{-45}}{2^2} = \frac{1}{2^{47}}$

Example 5

Solve the following equation using properties of exponents.

$(32^{0.6})^2 = x^3$

First work with the left hand side of the equation.

$$(32^{0.6})^2 = \left((2^5)^{\frac{3}{5}}\right)^2 = 2^6$$
$$2^6 = x^3$$
$$(2^6)^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$$
$$2^2 = x$$
$$4 = x$$

Review

Simplify each expression using positive exponents.

1. $81^{-\frac{1}{4}}$
2. $64^{\frac{2}{3}}$
3. $\left(\frac{1}{32}\right)^{-\frac{2}{5}}$
4. $(-125)^{\frac{1}{3}}$
5. $(4x^3y)\,(3x^5y^2)^4$
6. $(5x^3y^2)^2\,(7x^3y)^2$
7. $\frac{8a^3b^{-2}}{(-4a^2b^4)^{-2}}$
8. $\frac{5x^2y^{-3}}{(-2x^3y^2)^{-4}}$
9. $\left(\frac{3m^3n^{-4}}{2m^{-5}n^{-2}}\right)^{-4}$
10. $\left(\frac{4m^{-3}n^{-4}}{5m^5n^{-4}}\right)^{-3}$
11. $\left(\frac{a^{-1}b}{a^5b^4}\right)^{-3}$
12. $\frac{15c^{-2}d^{-6}}{3c^{-4}d^{-2}}$
13. $\frac{12e^5f}{(-2ef^3)^{-2}}$

Solve the following equations using properties of exponents.

14. $\left(81^{0.75}\right)^2 = x^3$
15. $\left(64^{\frac{1}{6}}\right)^{-3} = x^3$

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3.3: Scientific Notation

In science, measurements are often extremely small or extremely large. It is inefficient to write the many zeroes in very small numbers like 0.00000000000000523. Usually, the order of magnitude and the first few digits of the number are what people are interested in. How should you represent these extreme numbers?

Scientific Notation

Scientific notation is a means of representing very large and very small numbers in a more efficient way. The general form of scientific notation is $a \cdot 10^b$

The a is a number between 1 and 10 and most often includes a decimal. The integer b is called the order of magnitude and is a measure of the general size of the number. If b is negative then the number is small and if b is positive then the number is large.

$1,240,000 = 1.24 \cdot 10^6$
$0.0000354 = 3.54 \cdot 10^{-5}$

Note that when switching to and from scientific notation the sign of b indicates which direction and how many places to move the decimal point.

Take the number 0.000 000 000 000 000 000 000 000 000 000 910 938 22 kg. This is about the mass of an electron. The number is too long to write out all the time so it is best to write it in scientific notation.

To write this number in scientific notation, count the number of decimal places that you have to move to determine the exponent. Since the decimal is moving to the right, it should be a negative exponent. An electron's mass written in scientific notation is:

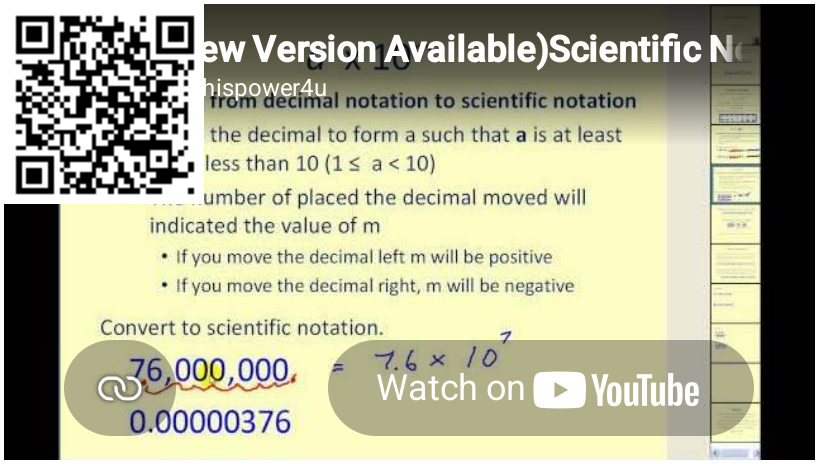
$9.1093822 \cdot 10^{-31}$

Multiplying and dividing numbers that are in scientific notation is just an exercise in exponent rules:

$(a \cdot 10^x) \cdot (b \cdot 10^y) = a \cdot b \cdot 10^{x+y}$
$(a \cdot 10^x) \div (b \cdot 10^y) = \frac{a}{b} \cdot 10^{x-y}$

Addition and subtraction require the numbers to have identical order of magnitudes.

$1.2 \cdot 10^6 - 5.5 \cdot 10^5 = 12 \cdot 10^5 - 5.5 \cdot 10^5 = 6.5 \cdot 10^5$



Examples

Example 1

Earlier, you were asked how to represent numbers with a large number of zeroes either before or after the decimal point. In order to represent an extremely large or small number you should count the number of moves necessary for the decimal point

to be directly after the first non-zero digit. This count will be the order of magnitude and will be used as the exponent of 10 as a means of representing how large or small the number is.

Example 2

The Earth's circumference is approximately 40,000,000 meters. What is the radius of the earth in scientific notation?

The relationship between circumference and radius is $C = 2\pi r$.

$$4.0 \cdot 10^7 = 2\pi r$$
$$r = \frac{4.0}{2\pi} \cdot 10^7 \approx 0.6366 \ldots 10^7 = 6.366 \cdot 10^6$$

Note that the number of significant digits required depends on the number of significant digits (or significant figures) in the original measurements. This example is an approximation, therefore the number of significant digits aren't necessarily accurate.

Example 3

Order the following numbers from least to greatest.

$$5.411 \cdot 10^{-3} \quad 7.837 \cdot 10^{-4} \quad 9.999 \cdot 10^3 \quad 9.5983 \cdot 10^{-7} \quad 8.0984 \cdot 10^3$$

First consider the order of magnitude of each number. Small numbers have negative exponents. If two numbers have the same order of magnitude, then compare the actual digits.

$$9.5983 \cdot 10^{-7} < 7.837 \cdot 10^{-4} < 5.411 \cdot 10^{-3} < 8.0984 \cdot 10^3 < 9.999 \cdot 10^3$$

Example 4

Compute the following number and use scientific notation.

$$2,000,000^3 \cdot 3,000^4$$

First convert each number to scientific notation individually, then process the exponent and multiplication.

$$\begin{aligned} 2,000,000^3 \cdot 3,000^4 &= \left(2 \cdot 10^6\right)^3 \cdot \left(3 \cdot 10^3\right)^4 \\ &= 8 \cdot 10^{18} \cdot 81 \cdot 10^{12} \\ &= 648 \cdot 10^{30} \\ &= 6.48 \cdot 10^{32} \end{aligned}$$

Example 5

Simplify the following expression.

$$\left(4.713 \cdot 10^7\right) + \left(8.985 \cdot 10^5\right) - \left(4.987 \cdot 10^2\right) \cdot \left(7.3 \cdot 10^{-6}\right) \div \left(6.74 \cdot 10^{-9}\right)$$

Resolve in order of standard order of operations

$$\begin{aligned} &\left(4.713 \cdot 10^7\right) + \left(8.985 \cdot 10^5\right) - \left(4.987 \cdot 10^2\right) \cdot \left(7.3 \cdot 10^{-6}\right) \div \left(6.74 \cdot 10^{-9}\right) \\ &= \left(4.713 \cdot 10^7\right) + \left(8.985 \cdot 10^5\right) - \left(5.40135 \cdot 10^5\right) \\ &= \left(471.3 \cdot 10^5\right) + \left(8.985 \cdot 10^5\right) - \left(5.40135 \cdot 10^5\right) \\ &= 474.8836499 \cdot 10^5 \\ &= 4.748836499 \cdot 10^7 \end{aligned}$$

Review

Write the following numbers in scientific notation.

1. 152,780

2. 0.00003256

3. 56, 320

4. 0.0821

5. 1, 000, 000, 000, 000, 000, 000, 000

6. 7.32

7. If the federal budget is $1.5 trillion, how much does it cost each individual, on average, if there are 300,000,000 people?

8. The Library of Congress has about 60,000,000 items. How could you express this number in scientific notation?

9. The sun develops 5×10^{23} horsepower per second. How much horsepower is developed in a day? In a year with 365 days?

10. A light-year is about 5,869,713,600 miles. A spacecraft travels 8.23×10^{4} miles per hour. How long will it take the spacecraft to travel a light-year?

11. Compute the following number and use scientific notation: $324,000 \cdot 30,000^{3}$.

12. Compute the following number and use scientific notation: $14,300 \cdot 20,200^{2}$.

Simplify the following expressions.

13. $\left(3.29 \cdot 10^{4}\right) - \left(3.295 \cdot 10^{5}\right) + \left(1.25 \cdot 10^{2}\right) \cdot \left(3.97 \cdot 10^{15}\right) \cdot \left(5.8 \cdot 10^{-6}\right)$

14. $\left(1.95 \cdot 10^{2}\right) + \left(6.798 \cdot 10^{6}\right) + \left(2.896 \cdot 10^{3}\right) \cdot \left(5.6 \cdot 10^{-3}\right) \div \left(2.89 \cdot 10^{4}\right)$

15. $\left(2.158 \cdot 10^{7}\right) \cdot \left(1.679 \cdot 10^{6}\right) - \left(9.98 \cdot 10^{4}\right) \cdot \left(3.4 \cdot 10^{-2}\right)$

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3.4: Properties of Logs

Log functions are inverses of exponential functions. This means the domain of one is the range of the other. This is extremely helpful when solving an equation and the unknown is in an exponent. Before solving equations, you must be able to simplify expressions containing logs. The rules of exponents are applied, but in non-obvious ways. In order to get a conceptual handle on the properties of logs, it may be helpful to continually ask, what does a log expression represent? For example, what does $\log_{10} 1,000$ represent?

Log Properties

Exponential and **logarithmic** expressions have the same 3 components. They are each written in a different way so that a different variable is isolated. The following two equations are equivalent to one another.

$b^x = a \leftrightarrow \log_b a = x$

The exponential equation on the left is read " b to the power x is a." The logarithmic equation on the right is read "log base b of a is x ".

The two most common bases for logs are 10 and e. At the PreCalculus level log by itself implies log base 10 and In implies base e.In is called the natural log. One important restriction for all log functions is that they must have strictly positive numbers in their arguments. So, if you press log -2 or log 0 on your calculator, it will give an error.

There are three basic properties of logs that correlate to properties of exponents.

Addition/Multiplication

$\log_b x + \log_b y = \log_b (x \cdot y)$
$b^{w+z} = b^w \cdot b^z$

Subtraction/Division

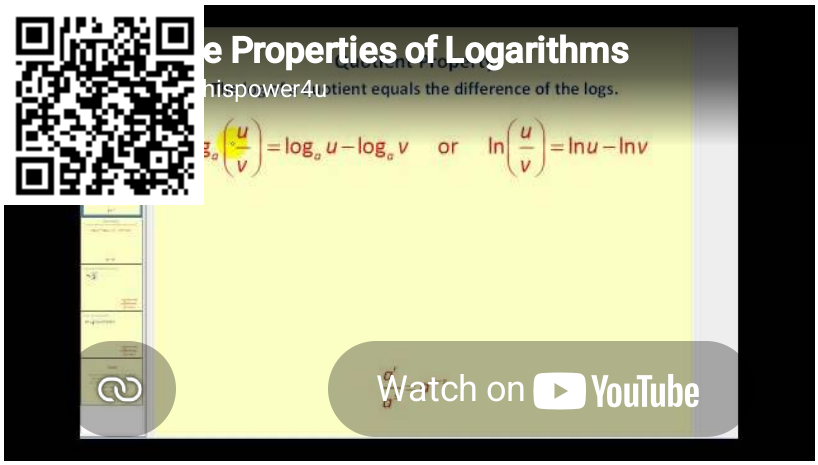
$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$
$b^{w-z} = \frac{b^w}{b^z}$

Exponentiation

$\log_b (x^n) = n \cdot \log_b x$
$(b^w)^n = b^{w \cdot n}$

There are also a few standard results that should be memorized and should serve as baseline reference tools.

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b (b^x) = x$
- $b^{\log_b x} = x$



Example

Example 1

Earlier you were asked what $\log_{10} 1,000$ represents. A log expression represents an exponent. The expression $\log_{10} 1,000$ represents the number 3 .

$\log_{10} 1000 = \log_{10} 10^3 = 3$

The reason to keep this in mind is that it can solidify the properties of logs. For example, adding exponents implies bases are multiplied. Thus adding logs means the bases of the exponents are multiplied.

Example 2

Write the expression as a logarithm of a single argument.

$\log_2 12 + \log_4 6 - \log_2 24$

Note that the center expression is of a different base. First change it to base 2 by switching back to exponential form.

$$\log_4 6 = x \leftrightarrow 4^x = 6$$
$$2^{2x} = 6 \leftrightarrow \log_2 6 = 2x$$
$$x = \frac{1}{2}\log_2 6 = \log_2 6^{\frac{1}{2}}$$

Thus the expression with the same base is:

$$\log_2 12 + \log_2 6^{\frac{1}{2}} - \log_2 24 = \log_2 \left(\frac{12 \cdot \sqrt{6}}{24}\right)$$
$$= \log_2 \left(\frac{\sqrt{6}}{2}\right)$$

Example 3

Prove the following log identity:

$\log_a b = \frac{1}{\log_b a}$

Start by letting the left side of the equation be equal to x. Then, rewrite in exponential form, manipulate, and rewrite back in logarithmic form until you get the expression from the left side of the equation.

$$\log_a b = x$$
$$a^x = b$$
$$a = b^{\frac{1}{x}}$$
$$\log_b a = \log_b b^{\frac{1}{x}} = \frac{1}{x}$$
$$x = \frac{1}{\log_b a}$$

Therefore, $\frac{1}{\log_b a} = \log_a b$ because both expressions are equal to x.

Example 4

Rewrite the following expression under a single log.

$$\ln e - \ln 4x + 2\left(e^{\ln x} \cdot \ln 5\right) = \ln\left(\frac{e}{4x}\right) + 2x \cdot \ln 5$$
$$= \ln\left(\frac{e}{4x}\right) + \ln\left(5^{2x}\right)$$
$$= \ln\left(\frac{e \cdot 5^{2x}}{4x}\right)$$

Example 5

True or false:

$(\log_3 4x) \cdot (\log_3 5y) = \log_3(4x + 5y)$

False. It is true that the log of a product is the sum of logs. It is not true that the product of logs is the log of a sum.

Review

Decide whether each of the following statements is true or false. Explain.

1. $\frac{\log x}{\log y} = \log\left(\frac{x}{y}\right)$
2. $(\log x)^n = n \log x$
3. $\log x + \log y = \log xy$

Rewrite each of the following expressions under a single log and simplify.

4. $\log 4x + \log(2x + 4)$
5. $5 \log x + \log x$
6. $4 \log_2 x + \frac{1}{2} \log_2 9 - \log_2 y$
7. $6 \log_3 z^2 + \frac{1}{4} \log_3 y^8 - 2 \log_3 z^4 y$

Expand the expression as much as possible.

8. $\log_4\left(\frac{2x^3}{5}\right)$
9. $\ln\left(\frac{4xy^2}{15}\right)$
10. $\log\left(\frac{x^2(yz)^3}{3}\right)$

Translate from exponential form to logarithmic form.

11. $2^{x+1} + 4 = 14$

Translate from logarithmic form to exponential form.

12. $\log_2(x - 1) = 12$

Prove the following properties of logarithms.

13. $\log_{b^n} x = \frac{1}{n} \log_b x$
14. $\log_{b^n} x^n = \log_b x$
15. $\log_{\frac{1}{b}} \frac{1}{x} = \log_b x$

...

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3.5: Change of Base

While it is possible to change bases by always going back to exponential form, it is more efficient to find out how to change the base of logarithms in general. since there are only base e and base 10 logarithms on most calculators, how would you evaluate an expression like $\log_3 12$?

Changing the Base of Logarithms

The **change of base property** states:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

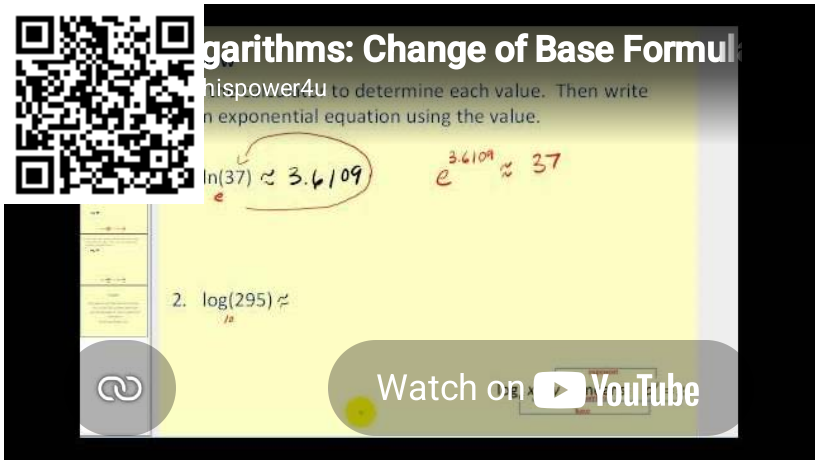
You can derive this formula by converting $\log_b x$ to exponential form and then taking the log base x of both sides. This is shown below.

$$\begin{aligned} \log_b x &= y \\ b^y &= x \\ \log_a b^y &= \log_a x \\ y \log_a b &= \log_a x \\ y &= \frac{\log_a x}{\log_a b} \end{aligned}$$

Therefore, $\log_b x = \frac{\log_a x}{\log_a b}$

If you were to evaluate $\log_3 4$ using your calculator, you may need to use the change of base formula since some calculators only have base 10 or base e. The result would be:

$$\log_3 4 = \frac{\log_{10} 4}{\log_{10} 3} = \frac{\ln 4}{\ln 3} \approx 1.262$$



Examples

Example 1

Earlier, you were asked how to use a calculator to evaluate an expression like $\log_3 12$. In order to evaluate an expression like $\log_3 12$ you have some options on your calculator:

$$\frac{\ln 12}{\ln 3} = \frac{\log 12}{\log 3} \approx 2.26$$

Some graphing calculators also have another option. Press the **MATH** followed by the **A** buttons and enter $\log_3 12$

Example 2

Prove the following log identity.

$$\log_a b = \frac{1}{\log_b a}$$

$\log_a b = \frac{\log_x b}{\log_x a} = \frac{1}{\frac{\log_x a}{\log_x b}} = \frac{1}{\log_b a}$

Example 3

Simplify to an exact result: $(\log_4 5) \cdot (\log_3 4) \cdot (\log_5 81) \cdot (\log_5 25)$

$\frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 3^4}{\log 5} \cdot \frac{\log 5^2}{\log 5} = \frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{4 \cdot \log 3}{\log 5} \cdot \frac{2 \cdot \log 5}{\log 5} = 4 \cdot 2 = 8$

Example 4

Evaluate: $\log_2 48 - \log_4 36$

$$\begin{aligned}
\log_2 48 - \log_4 36 &= \frac{\log 48}{\log 2} - \frac{\log 36}{\log 4} \\
&= \frac{\log 48}{\log 2} - \frac{\log 6^2}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{2 \cdot \log 6}{2 \cdot \log 2} \\
&= \frac{\log 48 - \log 6}{\log 2} \\
&= \frac{\log\left(\frac{48}{6}\right)}{\log 2} \\
&= \frac{\log 8}{\log 2} \\
&= \frac{\log 2^3}{\log 2} \\
&= \frac{3 \cdot \log 2}{\log 2} \\
&= 3
\end{aligned}$$

Example 5

Given $\log_3 5 \approx 1.465$ find $\log_{25} 27$ without using a log button on the calculator.

$\log_{25} 27 = \frac{\log 3^3}{\log 5^2} = \frac{3}{2} \cdot \frac{1}{\left(\frac{\log 5}{\log 3}\right)} = \frac{3}{2} \cdot \frac{1}{\log_3 5} \approx \frac{3}{2} \cdot \frac{1}{1.465} = 1.0239$

Review

Evaluate each expression by changing the base and using your calculator.

1. $\log_6 15$
2. $\log_9 12$
3. $\log_5 25$

Evaluate each expression.

4. $\log_8 (\log_4 (\log_3 81))$
5. $\log_2 3 \cdot \log_3 4 \cdot \log_6 16 \cdot \log_4 6$
6. $\log 125 \cdot \log_9 4 \cdot \log_4 81 \cdot \log_5 10$
7. $\log_5 (5^{\log_5 125})$
8. $\log (\log_6 (\log_2 64))$
9. $10^{\log_{100} 9}$
10. $(\log_4 x)(\log_x 16)$
11. $\log_{49} 49^5$

12. $3 \log_{24} 24^8$

13. $4^{\log_2 3}$

Prove the following properties of logarithms.

14. $(\log_a b)(\log_b c) = \log_a c$

15. $(\log_a b)(\log_b c)(\log_c d) = \log_a d$

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3.6: Exponential Equations

When you were first learning equations, you learned the rule that whatever you do to one side of an equation, you must also do to the other side so that the equation stays in balance. The basic techniques of adding, subtracting, multiplying and dividing both sides of an equation worked to solve almost all equations up until now. With logarithms, you have more tools to isolate a variable. Consider the following equation and ask yourself: why is $x = 3$? Logically it makes sense that if the bases match, then the exponents must match as well, but how can it be shown for examples like this one?

$$1.79898^{2x} = 1.79898^{6}$$

Solving Exponential Equations

A common technique for solving equations with unknown variables in exponents is to take the log of the desired base of both sides of the equation. Then, you can use properties of logs to simplify and solve the equation.

Take the following equation. To solve for t, you should first simplify the expression as much as possible and then take the natural log of both sides.

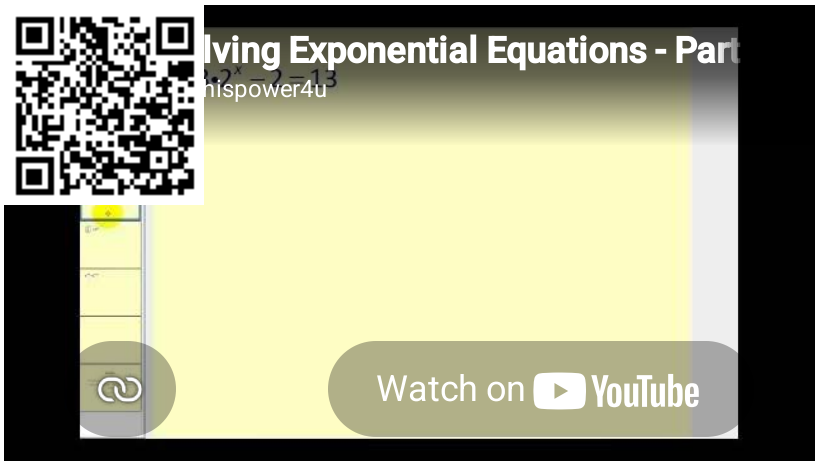
$$\begin{aligned} 9,000 &= 300 \cdot \tfrac{(1.06)^t - 1}{0.06} \\ 30 &= \frac{(1.06)^t - 1}{0.06} \\ 1.8 &= (1.06)^t - 1 \\ 2.8 &= 1.06^t \\ \ln 2.8 &= \ln\left(1.06^t\right) = t \cdot \ln(1.06) \\ t &= \frac{\ln 2.8}{\ln 1.06} \approx 17.67 \text{ years} \end{aligned}$$

It does not matter what base you use in this situation as long as you use the same base on both sides. Choosing natural log allows you to use a calculator to finish the problem.

Note that this type of equation is common in financial mathematics. The equation above represents the unknown amount of time it will take you to save $9,000 in a savings account if you save $300 at the end of each year in an account that earns 6% annual compound interest.

The other good base to use is base **10.** When solving the following equation for x : $16^x = 25$, you will need to use a calculator to get the final answer and your calculator can handle base 10 as well. First take the log of both sides. Then, use log properties and your calculator to help.

$$\begin{aligned} 16^x &= 25 \\ \log 16^x &= \log 25 \\ x \log 16 &= \log 25 \\ x &= \frac{\log 25}{\log 16} \\ x &= 1.16 \end{aligned}$$



Examples

Example 1

Earlier, you were asked how to show that if the bases match in an equation, the exponents should match. In the equation, logs can be used to reduce the equation to $2x = 6$.

$$1.79898^{2x} = 1.79898^{6}$$

Take the log of both sides and use the property of exponentiation of logs to bring the exponent out front.

$$\begin{aligned} \log 1.79898^{2x} &= \log 1.79898^{6} \\ 2x \cdot \log 1.79898 &= 6 \cdot \log 1.79898 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Example 2

Solve the following equation for all possible values of x : $(x+1)^{x-4} - 1 = 0$

$$(x+1)^{x-4} - 1 = 0$$

$$(x+1)^{x-4} = 1$$

Case 1 is that $x+1$ is positive in which case you can take the log of both sides.

$$\begin{aligned} \log (x+1)^{(x-4)} &= \log 1 \\ (x-4) \cdot \log(x+1) &= 0 \\ x-4 = 0 \text{ or } \log(x+1) &= 0 \\ x = 4 \text{ or } (x+1) &= 10^{0} = 1 \\ x &= 4, 0 \end{aligned}$$

Note that $\log 1 = 0$

Case 2 is that $(x+1)$ is negative 1 and raised to an even power. This happens when $x = -2$.

$$\begin{aligned} (x+1)^{(x-4)} &= 1 \\ (-2+1)^{(-2-4)} - 1 &= (-1)^{-6} - 1 \\ &= \frac{1}{(-1)^{6}} - 1 \\ &= 0 \end{aligned}$$

The reason why this exercise is included is because you should not fall into the habit of assuming that you can take the log of both sides of an equation. It is only valid when the argument is strictly positive. For example, $\log (-2+1)^{(-2-4)} = \log(-1)$ is not possible.

Example 3

Light intensity, measured in lumens, can be described by the relationship between i for intensity and d for depth in feet as it travels at specific depths of water in a swimming pool. What is the intensity of light at 10 feet?

$$\log \left(\frac{i}{12}\right) = -0.0145 \cdot d$$

Given $d = 10$, solve for i measured in lumens.

$$\begin{aligned} \log \left(\frac{i}{12}\right) &= -0.0145 \cdot d \\ \log \left(\frac{i}{12}\right) &= -0.0145 \cdot 10 \\ \log \left(\frac{i}{12}\right) &= -0.145 \\ \left(\frac{i}{12}\right) &= 10^{-0.145} \\ i &= 12 \cdot 10^{-0.145} \approx 8.594 \end{aligned}$$

Example 4

Solve the following equation for all possible values of x.

$\frac{e^x - e^{-x}}{3} = 14$

First solve for e^x,

$$\begin{aligned} \frac{e^x - e^{-x}}{3} &= 14 \\ e^x\left(e^x - e^{-x}\right) &= (42)e^x \\ e^{2x} - 1 &= 42e^x \\ (e^x)^2 - 42e^x - 1 &= 0 \end{aligned}$$

Let $u = e^x$

$u^2 - 42u - 1 = 0$

$u = \frac{-(-42) \pm \sqrt{(-42)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{42 \pm \sqrt{1768}}{2} \approx 42.023796, -0.0237960$

Note that the negative result is extraneous because e^x must be greater than zero, so you only proceed in solving for x for one result.

$e^x \approx 42.023796$
$x \approx \ln 42.023796 \approx 3.738$

Example 5

Solve the following equation for all possible values of x : $(\log_2 x)^2 - \log_2 x^7 = -12$.

In calculus it is common to use a small substitution to simplify the problem and then substitute back later. In this case let $u = \log_2 x$. Notice that this is a quadratic problem.

$$\begin{aligned} (\log_2 x)^2 - 7\log_2 x + 12 &= 0 \\ u^2 - 7u + 12 &= 0 \\ (u-3)(u-4) &= 0 \\ u &= 3, 4 \end{aligned}$$

Now substitute back and solve for x in each case.

$\log_2 x = 3 \leftrightarrow 2^3 = x = 8$

$\log_2 x = 4 \leftrightarrow 2^4 = x = 16$

Review

Solve each equation for x. Round each answer to three decimal places.

1. $4^x = 6$
2. $5^x = 2$
3. $12^{4x} = 1020$
4. $7^{3x} = 2400$
5. $2^{x+1} - 5 = 22$
6. $5x + 12^x = 5x + 7$
7. $2^{x+1} = 2^{2x+3}$
8. $3^{x+3} = 9^{x+1}$
9. $2^{x+4} = 5^x$
10. $13 \cdot 8^{0.2x} = 546$

11. $b^x = c + a$

12. $32^x = 0.94 - .12$

Solve each log equation by using log properties and rewriting as an exponential equation.

13. $\log_3 x + \log_3 5 = 2$

14. $2 \log x = \log 8 + \log 5 - \log 10$

15. $\log_9 x = \frac{3}{2}$

...

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3.7: Logistic Functions

Exponential growth increases without bound. This is reasonable for some situations; however, for populations there is usually some type of upper bound. This can be caused by limitations on food, space or other scarce resources. The effect of this limiting upper bound is a curve that grows exponentially at first and then slows down and hardly grows at all. This type of growth is called **logistic growth**. What are some other situations in which logistic growth would be an appropriate model?

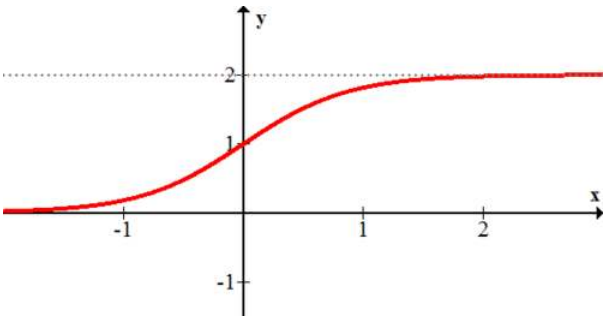
Logistic Functions

Logistic growth can be described with a logistic equation. The **logistic equation** is of the form: $f(x) = \frac{c}{1+a \cdot b^x}$

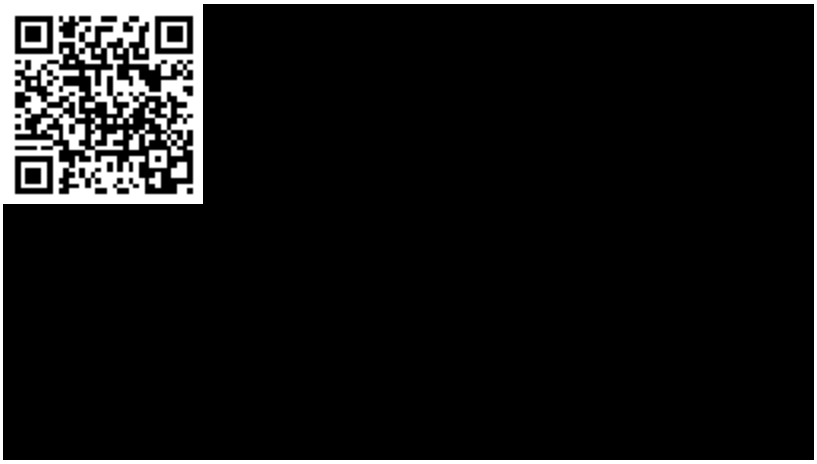
The letters a, b and c are constants that can be changed to match the situation being modeled. You will have to solve for a and b with the information that is given to you in each problem. The constant c is particularly important because it is the limit to growth. This is also known as the carrying capacity.

The following logistic function has a carrying capacity of 2 which can be directly observed from its graph.

$f(x) = \frac{2}{1+0.1^x}$



An important note about the logistic function is that it has an **inflection point**. From the previous graph you can observe that at the point (0,1) the graph transitions from curving up (concave up) to curving down (concave down). This change in curvature will be studied more in calculus, but for now it is important to know that the inflection point occurs halfway between the carrying capacity and the x axis.



Examples

Example 1

Earlier, you were asked what situations the logistic model is appropriate for. The logistic model is appropriate whenever the total count has an upper limit and the initial growth is exponential. Examples are the spread of rumors and disease in a limited population and the growth of bacteria or human population when resources are limited.

Example 2

A rumor is spreading at a school that has a total student population of 1200. Four people know the rumor when it starts and three days later three hundred people know the rumor. About how many people at the school know the rumor by the fourth day?

In a limited population, the count of people who know a rumor is an example of a situation that can be modeled using the logistic function. The population is 1200 so this will be the carrying capacity.

Identifying information: $c = 1200; (0,4); (3,300)$. First, use the point (0,4) to solve for a.

$$\frac{1200}{1+a\cdot b^0} = 4$$
$$\frac{1200}{1+a} = 4$$
$$\frac{1200}{4} = 1+a$$
$$a = 299$$

Next, use the point (3,300) to solve for b.

$$\frac{1200}{1+299\cdot b^3} = 300$$
$$4 = 1+299b^3$$
$$\frac{3}{299} = b^3$$
$$0.21568 \approx b$$

The modeling equation at $x = 4$:

$f(x) = \frac{1200}{1+299\cdot 0.21568^x} \rightarrow f(4) \approx 729$ people

A similar growth pattern will exist with any kind of infectious disease that spreads quickly and can only infect a person or animal once.

Example 3

A special kind of algae is grown in giant clear plastic tanks and can be harvested to make biofuel. The algae are given plenty of food, water and sunlight to grow rapidly and the only limiting resource is space in the tank. The algae are harvested when 95% of the tank is full leaving the tank 5% full of algae to reproduce and refill the tank. Currently the time between harvests is twenty days and the payoff is 90% harvest. Would you recommend a more optimal harvest schedule?

Identify known quantities and model the growth of the algae.

Known quantities: (0,0.05); $(20, 0.95); c = 1$ or 100%

$$0.05 = \frac{1}{1+a\cdot b^0}$$
$$1+a = \frac{1}{0.05}$$
$$a = 19$$
$$0.95 = \frac{1}{1+19\cdot b^{20}}$$
$$1+19\cdot b^{20} = \frac{1}{0.95}$$
$$b^{20} = \frac{\left(\frac{1}{0.95}-1\right)}{19}$$
$$b \approx 0.74495$$

The model for the algae growth is:

$f(x) = \frac{1}{1+19\cdot(0.74495)^x}$

The question asks about optimal harvest schedule. Currently the harvest is 90% per 20 day or a unit rate of 4.5% per day. If you shorten the time between harvests where the algae are growing the most efficiently, then potentially this unit rate might be

higher. Suppose you leave 15% of the algae in the tank and harvest when it reaches 85%. How much time will that take to yield 70%?

$$0.15 = \frac{1}{1 + 19 \cdot (0.74495)^x}$$
$$x_1 \approx 4.10897$$
$$0.85 = \frac{1}{1 + 19 \cdot (0.74495)^x}$$
$$x_2 \approx 15.8914$$
$$x_2 - x_1 \approx 15.8914 - 4.10897 \approx 11.78$$

It takes about 12 days for the batches to yield 70% harvest which is a unit rate of about 6% per day. This is a significant increase in efficiency. A harvest schedule that maximizes the time where the logistic curve is steepest creates the fastest overall algae growth.

Example 4

Determine the logistic model given $c = 12$ and the points (0,9) and (1,11)

The two points give two equations, and the logistic model has two variables. Use these points to solve for a and b.

$$9 = \frac{12}{1 + a \cdot b^0}$$
$$1 + a = \frac{12}{9}$$
$$a = \frac{1}{3}$$
$$11 = \frac{12}{1 + \left(\frac{1}{3}\right) \cdot b^1}$$
$$1 + \left(\frac{1}{3}\right) \cdot b = \frac{12}{11}$$
$$b = 0.\overline{27} = \frac{3}{11}$$

Thus the approximate model is:

$$f(x) = \frac{12}{1+\left(\frac{1}{3}\right)\cdot\left(\frac{3}{11}\right)^x}$$

Example 5

Determine the logistic model given $c = 7$ and the points (0,2) and (3,5)

The two points give two equations, and the logistic model has two variables. Use these two points to solve for a and b.

$$2 = \frac{7}{1 + a}$$
$$1 + a = \frac{7}{2}$$
$$a = 2.5$$
$$5 = \frac{7}{1 + (2.5) \cdot b^3}$$
$$1 + (2.5) \cdot b^3 = \frac{7}{5}$$
$$b^3 = 0.16$$
$$b \approx 0.5429$$

Thus the approximate model is:

$$f(x) = \frac{7}{1+(2.5)\cdot(0.5429)^x}$$

Review

For 1-5, determine the logistic model given the carrying capacity and two points.

1. $c = 12; (0, 5); (1, 7)$

2. $c = 200; (0, 150); (5, 180)$

3. $c = 1500; (0, 150); (10, 1000)$

4. $c = 1000000; (0, 100000); (-40, 20000)$

5. $c = 30000000; (-60, 10000); (0, 8000000)$

For $6 - 8$, use the logistic function $f(x) = \frac{32}{1+3e^{-x}}$

6. What is the carrying capacity of the function?

7. What is the y -intercept of the function?

8. Use your answers to 6 and 7 along with at least two points on the graph to make a sketch of the function.

For $9 - 11$, use the logistic function $g(x) = \frac{25}{1+4 \cdot 0.2^x}$

9. What is the carrying capacity of the function?

10. What is the y -intercept of the function?

11. Use your answers to 9 and 10 along with at least two points on the graph to make a sketch of the function.

For $12 - 14$, use the logistic function $h(x) = \frac{4}{1+2 \cdot 0.68^x}$

12. What is the carrying capacity of the function?

13. What is the y -intercept of the function?

14. Use your answers to 12 and 13 along with at least two points on the graph to make a sketch of the function.

15. Give an example of a logistic function that is decreasing (models decay). In general, how can you tell from the equation if the logistic function is increasing or decreasing?

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CHAPTER OVERVIEW

4: Basic Triangle Trigonometry

4.1: Angles in Radians and Degrees
4.2: Circular Motion and Dimensional Analysis
4.3: Special Right Triangles
4.4: Right Triangle Trigonometry
4.5: Law of Cosines
4.6: Law of Sines
4.7: Area of a Triangle
4.8: Applications of Basic Triangle Trigonometry

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4.1: Angles in Radians and Degrees

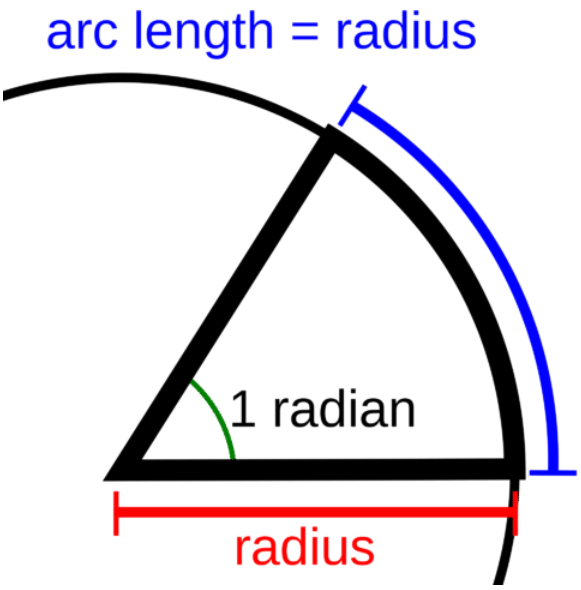
Most people are familiar with measuring angles in degrees. It is easy to picture angles like $30°, 45°$ or $90°$ and the fact that $360°$ makes up an entire circle. Over 2000 years ago the Babylonians used a base 60 number system and divided up a circle into 360 equal parts. This became the standard and it is how most people think of angles today.

However, there are many units with which to measure angles. For example, the gradian was invented along with the metric system and it divides a circle into 400 equal parts. The sizes of these different units are very arbitrary.

A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

Radians and Degrees

A **radian** is defined to be the central angle where the subtended **arc length** is the same length as the radius.



Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius r is $2\pi r$. Just over six radii (exactly 2π radii) would stretch around any circle.

To define a radian in terms of degrees, equate a circle measured in degrees to a circle measured in radians.

360 degrees $= 2\pi$ radians, so $\frac{180}{\pi}$ degrees $= 1$ radian

Alternatively, 360 degrees $= 2\pi$ radians, so 1 degree $= \frac{\pi}{180}$ radians

The conversion factor to convert degrees to radians is: $\frac{\pi}{180°}$

The conversion factor to convert radians to degrees is: $\frac{180°}{\pi}$

If an angle has no units, it is assumed to be in radians.

If you were to convert $150°$ into radians, you would multiply $150°$ by the correct conversion factor. You would get:

$150° \cdot \frac{\pi}{180°} = \frac{15\pi}{18} = \frac{5\pi}{6}$ radians

You can check your work by making sure the degree units appear on both the numerator and denominator.

If you were to convert $\frac{\pi}{6}$ radians into degrees, you would multiply $\frac{\pi}{6}$ by the correct conversion factor. You would get $\frac{\pi}{6} \cdot \frac{180°}{\pi} = \frac{180°}{6} = 30°$

Notice π appears in both the numerator and denominator and $\frac{\pi}{\pi} = 1$.



Examples

Example 1

Earlier, you were asked how many radians make up a circle. Exactly 2π radians describe a circular arc. This is because 2π radii wrap around the circumference of any circle.

Example 2

Convert $(6\pi)^\circ$ into radians.

Don't be fooled just because this has π. This number is about 19°

$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{3}$

It is very unusual to ever have a π^2 term, but it can happen.

Example 3

Convert $\frac{5\pi}{6}$ into degrees.

$\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{5 \cdot 30^\circ}{1} = 150^\circ$

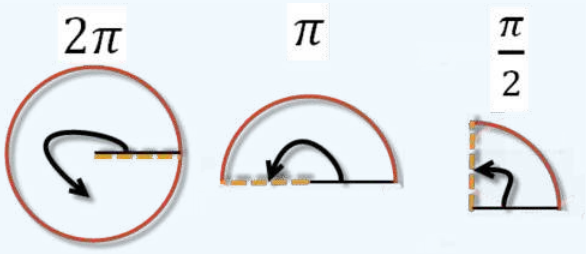
Example 4

Convert 210° into radians.

$210^\circ \cdot \frac{\pi}{180^\circ} = \frac{7 \cdot 30 \cdot \pi}{6 \cdot 30} = \frac{7\pi}{6}$

Example 5

Draw a $\frac{\pi}{2}$ angle by first drawing a 2π angle, halving it and halving the result. Recall that $\frac{\pi}{2} = 90^\circ$.



Review

Find the radian measure of each angle.

1. 120°
2. 300°
3. 90°
4. 330°
5. 270°
6. 45°
7. $(5\pi)^\circ$

Find the degree measure of each angle.

8. $\frac{7\pi}{6}$
9. $\frac{5\pi}{4}$
10. $\frac{3\pi}{2}$
11. $\frac{5\pi}{3}$
12. π
13. $\frac{\pi}{6}$
14. 3
15. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.

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4.2: Circular Motion and Dimensional Analysis

Converting between units is essential for mathematics and science in general. Radians are very powerful because they provide a link between linear and angular speed. One radian is an angle that always corresponds to an arc length of one radius. This will allow you to convert the rate at which you pedal a bike to the actual speed you can travel.

1 revolution $= 2\pi r$

The gear near the pedals on a bike has a radius of 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

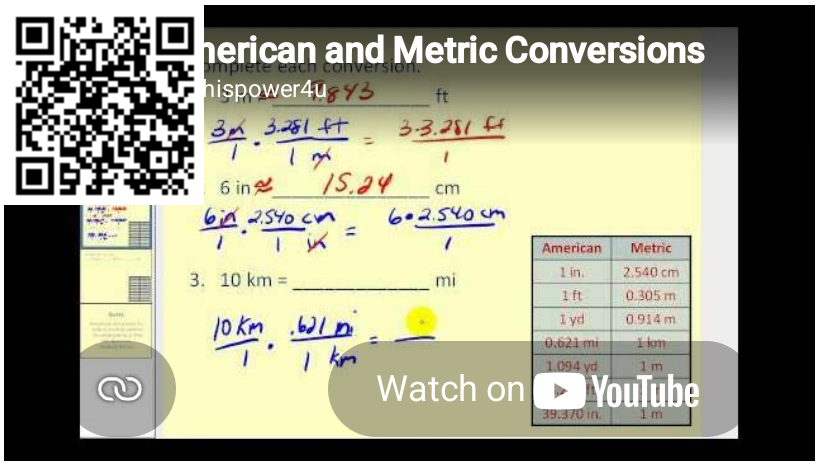
Circular Motion and Dimensional Analysis

Dimensional analysis just means converting from one unit to another. Sometimes it must be done in several steps in which case it is best to write the original amount on the left and then multiply it by all the different required conversions. To convert 3 miles to inches you write:

$$\frac{3 \text{ mile}}{1} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} = \frac{3 \cdot 5280 \cdot 12 \text{ inches}}{1} = 190080 \text{ inches}$$

Notice how miles and feet/foot can be used to convert miles to the desired unit of inches. Units, like numbers, can resolve to one when they exist in both the numerator and denominator $\left(\frac{3}{3} = 1 \text{ and } \frac{feet}{feet} = 1\right)$. Also notice each conversion factor is the same distance on the numerator and denominator, just written with different units.

Circular motion refers to the fact that on a spinning wheel points close to the center of the wheel actually travel very slowly and points near the edge of the wheel actually travel much quicker. **Angular speed** is the ratio of revolutions that occur per unit of time and linear speed is the ratio of distance per unit of time. While the two points have the same angular speed, their linear speed is very different.



Examples

Example 1

Earlier, you were given the following question: The gear near the pedals on the bike has radius 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

A bike has pedals that rotate a gear at a circular speed. The gear translates this speed to a linear speed on the chain. The chain then moves a second gear, which is a conversion to angular speed for the rear tire. This tire then converts the angular speed back to linear speed which is how fast you are moving. Instead of doing all these calculations in one step, it is easier to do each conversion in small pieces.

First convert the original gear into the linear speed of the chain.

$\frac{1 \text{ revolution}}{1 \text{ second}} \cdot \frac{2\pi \cdot 5 \text{ inches}}{1 \text{ revolution}} = 10\pi \frac{in}{\text{sec}}$

Then convert the speed of the chain into angular speed of the back gear which is the same as the angular speed of the rear tire.

$\frac{10\pi \text{ inches}}{1 \text{ second}} \cdot \frac{1 \text{ revolution}}{2\pi \cdot 3 \text{ inches}} = \frac{10}{6} \frac{\text{rev}}{\text{sec}}$

Lastly convert the angular speed of the rear tire to the linear speed of the tire in miles per hour.

$\frac{10\text{rev}}{6\text{sec}} \cdot \frac{2\pi \cdot 17\text{in}}{1\text{rev}} \cdot \frac{1\text{ft}}{12\text{in}} \cdot \frac{1\text{mile}}{5280\text{ft}} \cdot \frac{60\text{sec}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{hour}}$
$= \frac{10 \cdot 2 \cdot \pi \cdot 17 \cdot 60 \cdot 60}{6 \cdot 12 \cdot 5280} \frac{\text{miles}}{\text{hour}}$
$\approx 10.1 \frac{\text{miles}}{\text{hour}}$

Example 2

Suppose Summit High School has a circular track with two lanes for running. The interior lane is 30 meters from the center of the circle and the lane towards the exterior is 32 meters from the center of the circle. If two people run 4 laps together, how much further does the person on the outside go?

Calculate the distance each person ran separately using 1 lap to be 1 circumference and find the difference.

$\frac{4 \text{ laps}}{1} \cdot \frac{2\pi \cdot 30 \text{ meters}}{1 \text{ lap}} \approx 754$ meters
$\frac{4 \text{ laps}}{1} \cdot \frac{2\pi \cdot 32 \text{ meters}}{1 \text{ lap}} \approx 804$ meters

The person running on the outside of the track ran about 50 more meters.

Example 3

Andres races on a bicycle with tires that have a 17 inch radius. When he is traveling at a speed of 30 feet per second, how fast are the wheels spinning in revolutions per minute?

Look for ways to convert feet to revolutions and seconds to minutes.

$\frac{30 \text{ feet}}{1 \text{ second}} \cdot \frac{60 \text{ seconds}}{1 \text{ minute}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{1 \text{ revolution}}{2\pi \cdot 17 \text{ inches}} = \frac{30 \cdot 60 \cdot 12 \text{ revolutions}}{2\pi \cdot 17 \text{ minute}} \approx 202.2 \frac{\text{rev}}{\text{min}}$

Example 4

When a car travels at 60 miles per hour, how fast are the tires spinning if they have 30 inch diameters?

$\frac{60 \text{ miles}}{1 \text{ hour}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} \cdot \frac{12 \text{ inches}}{1 \text{ foot}} \cdot \frac{1 \text{ revolution}}{2\pi \cdot 15 \text{ inches}} \cdot \frac{1 \text{ hour}}{60 \text{ minute}}$
$= \frac{60 \cdot 5280 \cdot 12 \text{ revolutions}}{2\pi \cdot 15 \cdot 60 \text{ minute}} \approx 672.3 \frac{\text{rev}}{\text{min}}$

Example 5

Mike rides a bike with tires that have a radius of 15 inches. How many revolutions must Mike make to ride a mile?

$\frac{1\text{rev}}{2\pi \cdot 15\text{in}} \cdot \frac{12\text{in}}{1\text{ft}} \cdot \frac{5280\text{ft}}{1\text{mi}} \approx 672.3 \frac{\text{rev}}{\text{mile}}$

Review

For $1-10$, use the given values in each row to find the unknown value (x) in the specified units in the row.

Problem Number	Radius	Angular Speed	Linear Speed
1.	5 inches	60 rpm	$x \frac{in}{\min}$
2.	x feet	20 rpm	$2 \frac{in}{sec}$
3.	15 cm	x rpm	$12 \frac{cm}{sec}$
4.	x feet	40 rpm	$8 \frac{ft}{\text{sec}}$
5.	12 inches	32 rpm	$x \frac{in}{\text{sec}}$
6.	8 cm	x rpm	$12 \frac{cm}{\min}$
7.	18 feet	4 rpm	$x \frac{mi}{hr}$
8.	x feet	800 rpm	$60 \frac{\text{mi}}{\text{hr}}$
9.	15 in	x rpm	$60 \frac{mi}{hr}$
10	2 in	x rpm	$13 \frac{in}{\text{sec}}$

11. An engine spins a wheel with radius 5 inches at 800 rpm. How fast is this wheel spinning in miles per hour?

12. A bike has tires with a radius of 10 inches. How many revolutions must the tire make to ride a mile?

13. An engine spins a wheel with radius 6 inches at 600 rpm. How fast is this wheel spinning in inches per second?

14. Bob has a car with tires that have a 15 inch radius. When he is traveling at a speed of 30 miles per hour, how fast are the wheels spinning in revolutions per minute?

15. A circular track has two lanes. The interior lane is 25 feet from the center of the circle and the lane towards the exterior is 30 feet from the center of the circle. If you jog 6 laps, how much further will you jog in the exterior lane as opposed to the interior lane?

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4.3: Special Right Triangles

The Pythagorean Theorem is great for finding the third side of a right triangle when you already know two other sides. There are some triangles like 30-60-90 and 45-45-90 triangles that are so common that it is useful to know the side ratios without doing the Pythagorean Theorem each time. Using these patterns also allows you to totally solve for the missing sides of these special triangles when you only know one side length.

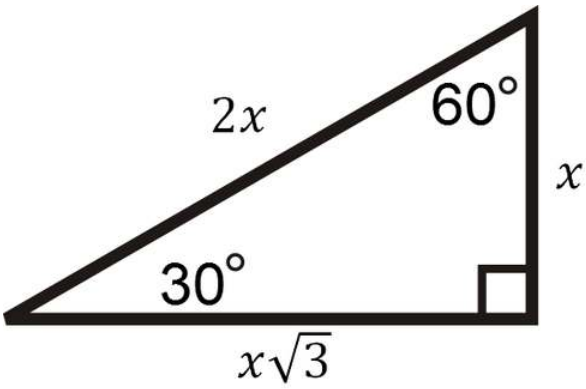
Given a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches, what is the value of x ?

Special Right Triangles

There are three types of special right triangles, 30-60-90 triangles, 45-45-90 triangles, and Pythagorean triple triangles.

30-60-90 Triangles

A 30-60-90 right triangle has side ratios $x, x\sqrt{3}, 2x$.

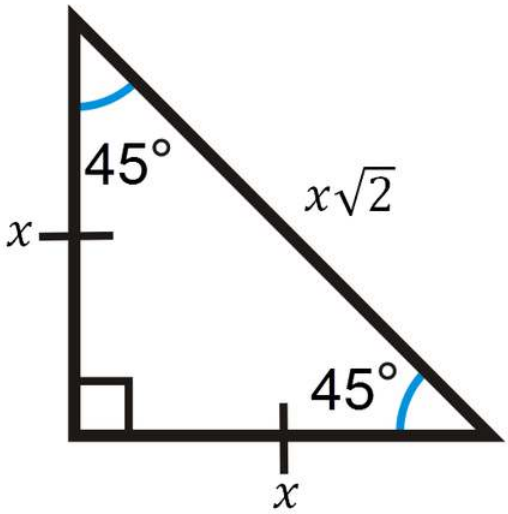


Confirm with Pythagorean Theorem:

$$x^2 + (x\sqrt{3})^2 = (2x)^2$$
$$x^2 + 3x^2 = 4x^2$$
$$4x^2 = 4x^2$$

45-45-90 Triangles

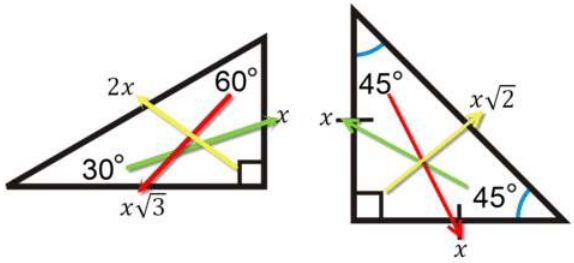
A 45 - 45 -90 right triangle has side ratios $x, x, x\sqrt{2}$.



Confirm with Pythagorean Theorem:

$$x^2 + x^2 = (x\sqrt{2})^2$$
$$2x^2 = 2x^2$$

Note that the order of the side ratios $x, x\sqrt{3}, 2x$ and $x, x, x\sqrt{2}$ is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to smallest angles and largest sides always correspond to the largest angles.



Pythagorean Triple Triangles

Pythagorean number triples are special right triangles with integer sides. While the angles are not integers, the side ratios are very useful to know because they show up everywhere. Knowing these number triples also saves a lot of time from doing the Pythagorean Theorem repeatedly. Here are some examples of Pythagorean number triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

More Pythagorean number triples can be found by scaling any other Pythagorean number triple. For example:

$3, 4, 5 \rightarrow 6, 8, 10$ (scaled by a factor of 2)

Even more Pythagorean number triples can be found by taking any odd integer like 11, squaring it to get 121, halving the result to get 60.5. The original number 11 and the two numbers that are 0.5 above and below (60 and 61) will always be a Pythagorean number triple.

$11^2 + 60^2 = 61^2$

Examples

Example 1

Earlier you were asked about a 45-45-90 right triangle with sides 6 inches, 6 inches and x inches. If you can recognize the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches $x = 6\sqrt{2}$.

Example 2

A 30-60-90 right triangle has hypotenuse of length 10. What are the lengths of the other two sides?

The hypotenuse is the side opposite 90. Sometimes it is helpful to draw a picture or make a table.

30	60	90
x	$x\sqrt{3}$	$2x$
		10

From the table you can write very small subsequent equations to solve for the missing sides.

$10 = 2x$
$x = 5$
$x\sqrt{3} = 5\sqrt{3}$

The other sides are 5 and $5\sqrt{3}$.

Example 3

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths.

30	60	90
x	$x\sqrt{3}$	$2x$
	18	

$18 = x\sqrt{3}$

$\frac{18}{\sqrt{3}} = x$

$x = \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$

$x = 6\sqrt{3}$

Note that you need to rationalize denominators.

Now use the calculated x value to solve for $2x$.

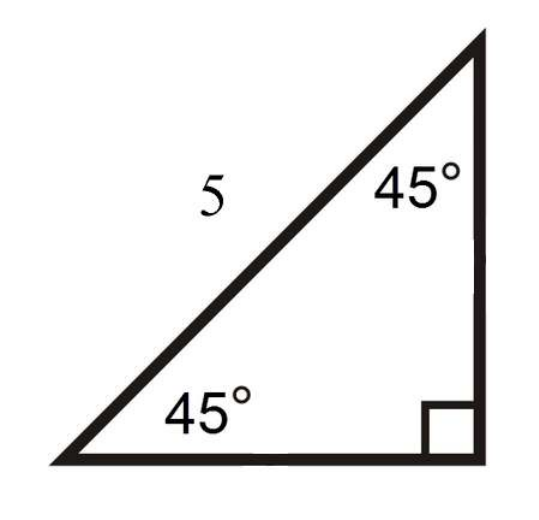
$2x = 2(6\sqrt{3})$

$2x = 12\sqrt{3}$

The other sides are $6\sqrt{3}$ and $12\sqrt{3}$.

Example 4

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.



The other sides are each $\frac{5\sqrt{2}}{2}$.

45	45	90
x	x	$x\sqrt{2}$
		5

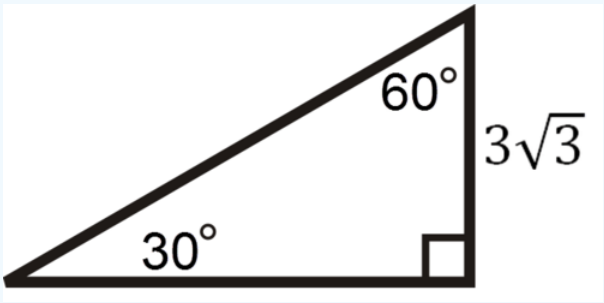
$x\sqrt{2} = 5$

$$x = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

The other sides are each $\frac{5\sqrt{2}}{2}$

Example 5

Using your knowledge of special right triangle ratios, solve for the missing sides of the right triangle.



The other sides are 9 and $6\sqrt{3}$.

30	60	90
x	$x\sqrt{3}$	$2x$
$3\sqrt{3}$		

$x = 3\sqrt{3}$

$2x = 6\sqrt{3}$

$x\sqrt{3} = 3\sqrt{3} \cdot \sqrt{3} = 9$

The other sides are 9 and $6\sqrt{3}$.

Review

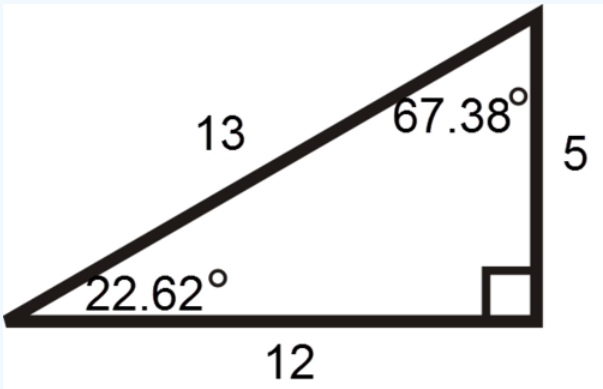
For 1-4, find the missing sides of the 45-45-90 triangle based on the information given in each row.

Problem Number	Side Opposite $45°$	Side Opposite $45°$	Side Opposite $90°$
1.	3		
2.		7.2	
3.			16
4.	$5\sqrt{2}$		

For 5-8, find the missing sides of the 30-60-90 triangle based on the information given in each row.

Problem Number	Side Opposite $30°$	Side Opposite $60°$	Side Opposite $90°$
1.	$3\sqrt{2}$		
2.		4	
3.			15
4.			$12\sqrt{3}$

Use the picture below for 9-11.



9. Which angle corresponds to the side that is 12 units?

10. Which side corresponds to the right angle?

11. Which angle corresponds to the side that is 5 units?

12. A right triangle has an angle of $\frac{\pi}{6}$ radians and a hypotenuse of 20 inches. What are the lengths of the other two sides of the triangle?

13. A triangle has two angles that measure $\frac{\pi}{4}$ radians. The longest side is 3 inches long. What are the lengths of the other two sides?

For 14-19, verify the Pythagorean Number Triple using the Pythagorean Theorem.

14. 3, 4, 5

15. 5, 12, 13

16. 7, 24, 25

17. 8, 15, 17

18. 9, 40, 41

19. 6, 8, 10

20. Find another Pythagorean Number Triple using the method explained for finding “11, 60, 61”.

...

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4.4: Right Triangle Trigonometry

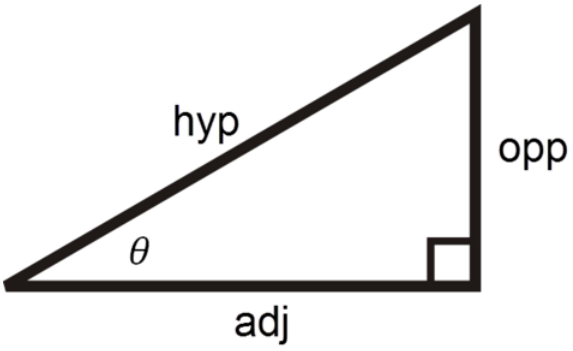
Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

Imagine trying to measure the height of a flag pole. It would be very difficult to measure vertically because it could be several stories tall. Instead walk 10 feet away and notice that the flag pole makes a 65 degree angle with your feet. Using this information, what is the height of the flag pole?

Trigonometric Functions

The six trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. **Opp** stands for the side opposite of the angle θ, **hyp** stands for hypotenuse and **adj** stands for side adjacent to the angle θ.

$$\sin\theta = \frac{opp}{hyp}$$
$$\cos\theta = \frac{adj}{hyp}$$
$$\tan\theta = \frac{opp}{adj}$$
$$\cot\theta = \frac{adj}{opp}$$
$$\sec\theta = \frac{hyp}{adj}$$
$$\csc\theta = \frac{hyp}{opp}$$

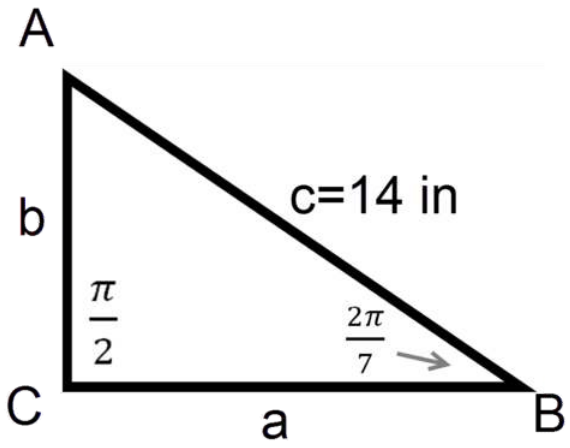


The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. Trigonometric functions are used by identifying two known pieces of information on a triangle and one unknown, setting up and solving for the unknown. Calculators are important because the operations of sin, cos and tan are already programmed in. The other three (cot, sec and csc) are not usually in calculators because there is a **reciprocal relationship** between them and tan, cos and sec.

$$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta}$$
$$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$$
$$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$$

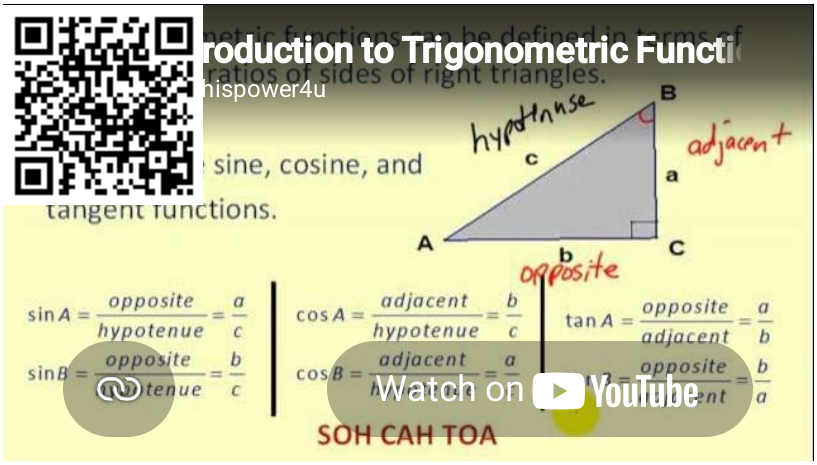
Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

Note that the images throughout this concept are not drawn to scale. If you were given the following triangle and asked to solve for side b, you would use sine to find b.



$$\sin\left(\frac{2\pi}{7}\right) = \frac{b}{14}$$

$$b = 14 \cdot \sin\left(\frac{2\pi}{7}\right) \approx 10.9\text{in}$$

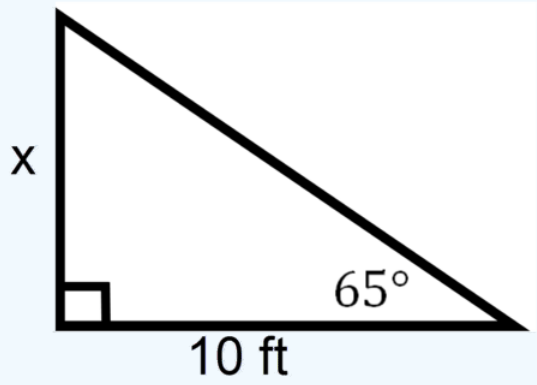


Examples

Example 1

Earlier, you were asked about the height of a flagpole that you are 10 feet away from. You notice that the flag pole makes a $65°$ angle with your feet.

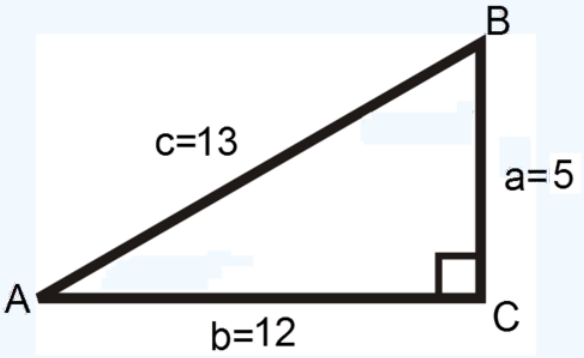
If you are 10 feet from the base of a flagpole and assume that the flagpole makes a $90°$ angle with the ground, you can use the following triangle to model the situation.



$$\tan 65^\circ = \frac{x}{10}$$
$$x = 10 \tan 65^\circ \approx 21.4 ft$$

Example 2

Solve for angle A.



This problem can be solved using sin, cos or tan because the opposite, adjacent and hypotenuselengths are all given.

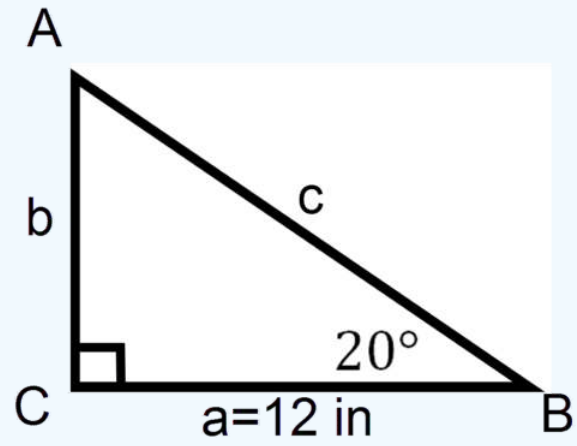
The argument, or input, of a sin function is always an angle. The arcsin, or $\sin^{-1}\theta$, function on the calculator has an argument that is a ratio of the triangle sides.

$$\sin A = \frac{5}{13}$$
$$\sin^{-1}(\sin A) = \sin^{-1}\left(\frac{5}{13}\right)$$
$$A = \sin^{-1}\left(\frac{5}{13}\right) \approx 0.39 \text{ radian } \approx 22.6^\circ$$

Example 3

Given a right triangle with $a = 12$ in , $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.

It is helpful to draw a diagram to represent the data given in a question.

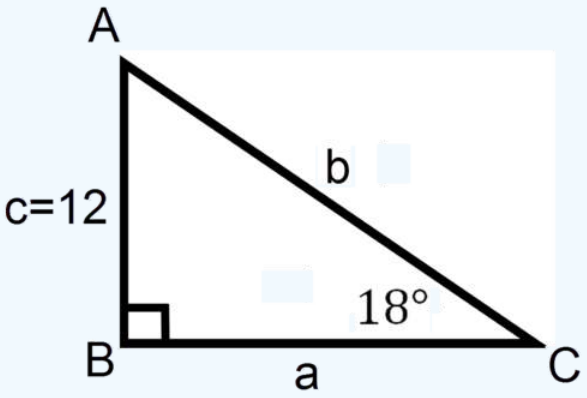


$$\cos 20^\circ = \frac{12}{c}$$
$$c = \frac{12}{\cos 20^\circ} \approx 12.77\text{in}$$

Example 4

Given $\triangle ABC$ where B is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is a ?

Drawing out this triangle, it looks like:

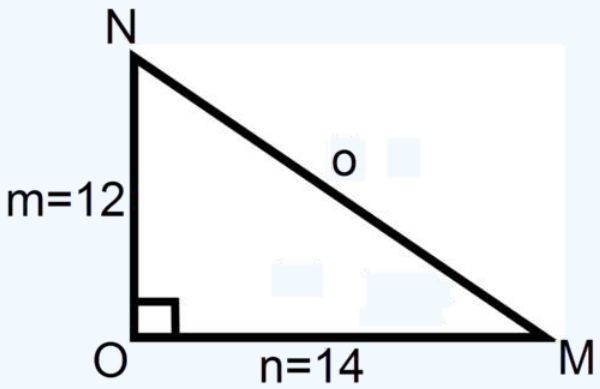


$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

Example 5

Given $\triangle MNO$ where O is a right angle, $m = 12$, and $n = 14$. What is the measure of angle M ?

Drawing out the triangle, it looks like:



$$\tan M = \frac{12}{14}$$

$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 0.7 \text{ radian } \approx 40.6^\circ$$

Review

For $1-15$, information about the sides and/or angles of right triangle ABC is given. Completely solve the triangle (find all missing sides and angles) to 1 decimal place.

Problem Number	A	B	C	a	b	c
1.	90°				4	7
2.	90°		37°	18		
3.		90°	15°		32	
4.			90°	6		11
5.	90°	12°		19		
6.		90°			17	10
7.	90°	10°			2	
8.	4°	90°		0.3		
9.	$\frac{\pi}{2}$ radian		1 radian			15
10.			$\frac{\pi}{2}$ radian	12	15	
11.			$\frac{\pi}{2}$ radian		9	14
12.	$\frac{\pi}{4}$ radian	$\frac{\pi}{4}$ radian			5	
13.	$\frac{\pi}{2}$ radian			26	13	
14.		$\frac{\pi}{2}$ radian			19	16
15.			$\frac{\pi}{2}$ radian	10		$10\sqrt{2}$

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4.5: Law of Cosines

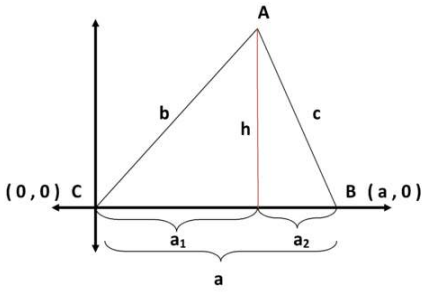
The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

The Law of Cosines

The **Law of Cosines** is:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

It is important to understand the proof:



You know four facts from the picture:

$$a = a_1 + a_2 \tag{4.5.1}$$

$$b^2 = a_1^2 + h^2 \tag{4.5.2}$$

$$c^2 = a_2^2 + h^2 \tag{4.5.3}$$

$$\cos C = \frac{a_1}{b} \tag{4.5.4}$$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$$a_2 = a - a_1 \tag{4.5.5}$$

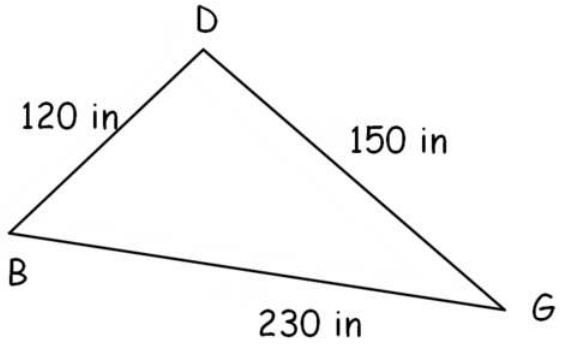
$$a_1 = b \cdot \cos C \tag{4.5.6}$$

Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

$c^2 = a_2^2 + h^2$ (3 again)
$c^2 = (a - a_1)^2 + h^2$ (substitute using 5)
$c^2 = a^2 - 2a \cdot a_1 + a_1^2 + h^2$ (FOIL)
$c^2 = a^2 - 2a \cdot b \cdot \cos C + a_1^2 + h^2$ (substitute using 6)
$c^2 = a^2 - 2a \cdot b \cdot \cos C + b^2$ (substitute using 2)
$c^2 = a^2 + b^2 - 2ab \cdot \cos C$ (rearrange terms)

There are only two types of problems in which it is appropriate to use the Law of Cosines. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called SSS (side-side-side) in geometry. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called SAS (side-angle-side).

Take the following triangle.



The measure of angle D is missing and can be found using the Law of cosines. It is necessary to set up the Law of cosines equation very carefully with D corresponding to the opposite side of **230.** The letters are not ABC like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\ 230^2 &= 120^2 + 150^2 - 2 \cdot 120 \cdot 150 \cdot \cos D \\ 230^2 - 120^2 - 150^2 &= -2 \cdot 120 \cdot 150 \cdot \cos D \\ \frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150} &= \cos D \\ D &= \cos^{-1}\left(\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150}\right) \approx 116.4^\circ \approx 2.03 \text{ radians} \end{aligned}$$

Examples

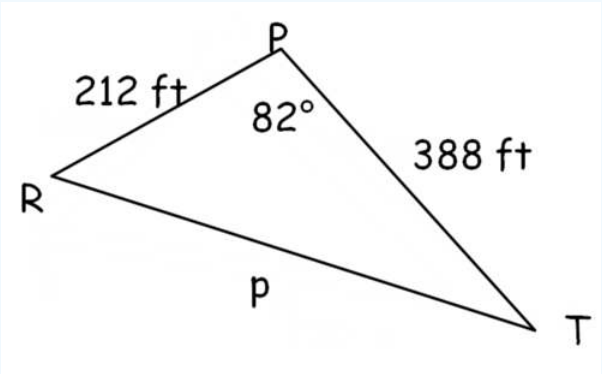
Example 1

Earlier, you were given a triangle with sides 11, 12, and 13 and asked what the measure of the angle opposite 11 is. A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\ 11^2 &= 12^2 + (13)^2 - 2 \cdot 12 \cdot 13 \cdot \cos C \\ C &= \cos^{-1}\left(\frac{11^2 - 12^2 - 13^2}{-2 \cdot 12 \cdot 13}\right) \approx 52.02^\circ \end{aligned}$$

Example 2

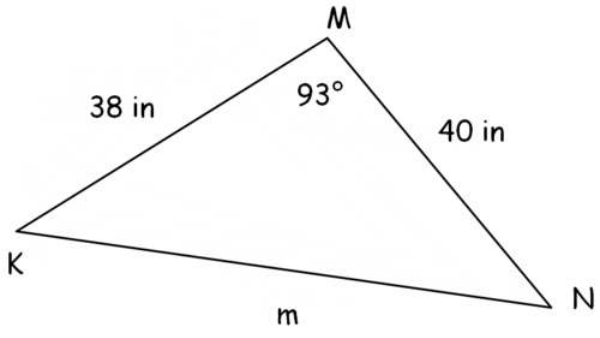
Determine the length of side p.



$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$
$$p^2 = 212^2 + 388^2 - 2 \cdot 212 \cdot 388 \cdot \cos 82^\circ$$
$$p^2 \approx 172592.354815$$
$$p \approx 415.44$$

Example 3

Determine the degree measure of angle N.



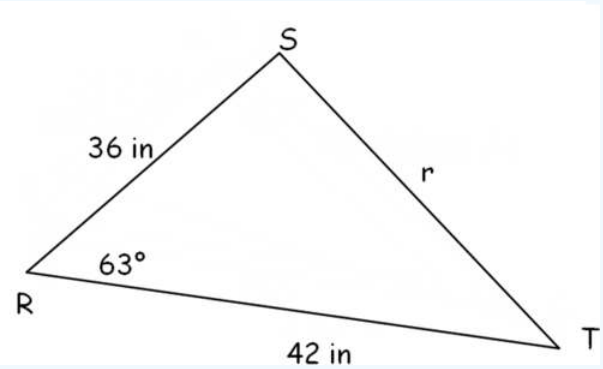
This problem must be done in two parts. First apply the Law of cosines to determine the length of side m. This is a SAS situation like Example B. Once you have all three sides you will be in the SSS situation like in Example A and can apply the Law of cosines again to find the unknown angle N.

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$
$$m^2 = 38^2 + 40^2 - 2 \cdot 38 \cdot 40 \cdot \cos 93^\circ$$
$$m^2 \approx 3203.1$$
$$m \approx 56.59$$

Now that you have all three sides you can apply the Law of cosines again to find the unknown angle N. Remember to match angle N with the corresponding side length of 38 inches. It is also best to store m into your calculator and use the unrounded number in your future calculations.

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$
$$38^2 = 40^2 + (56.59)^2 - 2 \cdot 40 \cdot (56.59) \cdot \cos N$$
$$38^2 - 40^2 - (56.59)^2 = -2 \cdot 40 \cdot (56.59) \cdot \cos N$$
$$\frac{38^2 - 40^2 - (56.59)^2}{-2 \cdot 40 \cdot (56.59)} = \cos N$$
$$N = \cos^{-1}\left(\frac{38^2 - 40^2 - (56.59)^2}{-2 \cdot 40 \cdot (56.59)}\right) \approx 42.1^\circ$$

For the next two examples, use the triangle below.



Example 4

Determine the length of side r.

$r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r \approx 41.07$

Example 5

Determine the length of side r.

$r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r \approx 41.07$

Review

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$

1. Find the measure of angle A.

2. Find the measure of angle B.

3. Find the measure of angle C.

4. Find the measure of angle C in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$

5. Find the measure of angle D.

6. Find the measure of angle E.

7. Find the measure of angle F.

In $\Delta GHI, g = 19, \angle H = 55°$, and $i = 12$

8. Find the length of h.

9. Find the measure of angle G.

10. Find the measure of angle I.

11. Explain why the Law of cosines is connected to the Pythagorean Theorem.

12. What are the two types of problems where you might use the Law of cosines?

Use the Law of cosines to determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$

14. $a = 1, b = 5, c = 4$

15. $a = 5, b = 6, c = 10$

...

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4.6: Law of Sines

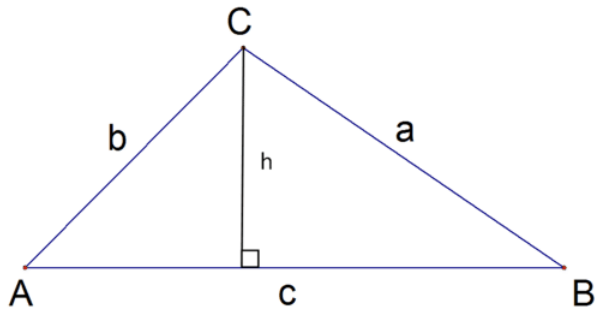
When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but SSA is not one of them. Why not?

The Law of Sines

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The **Law of Sines** states that in every triangle the ratio of each side to the sine of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here is a proof of the Law of Sines:



Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

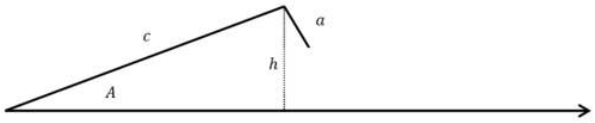
$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.



In this image side a is deliberately too short, but in most problems you will not know this. You will need to compare a to the height.

$$\sin A = \frac{h}{c}$$
$$h = c \sin A$$

This is commonly referred to testing the ambiguous case. There are four different tests to determine the number of triangles that exist given the measurements.

Case 1: $a < h$

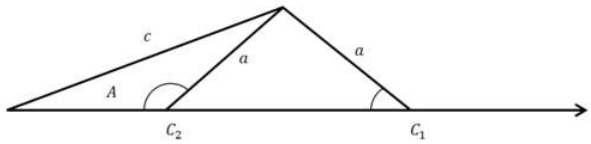
Simply put, side a is not long enough to reach the opposite side and constructing the triangle is impossible. Zero triangles exist.

Case 2: $a = h$

Side a just barely reaches the opposite side forming a $90°$ angle.

Case 3: $h < a < c$

In this case side a can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of sines to solve for angle C_1 first and then use the supplement to determine C_2.

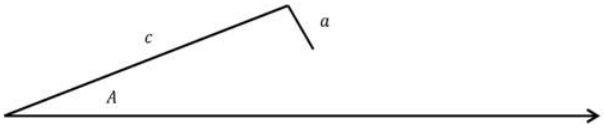


Case 4: $c \leq a$

In this case, side a can only swing towards the exterior of the triangle, only producing C_1.

For the case of SSA, you should always check how many triangles there are before starting to find measures. Take the following triangle:

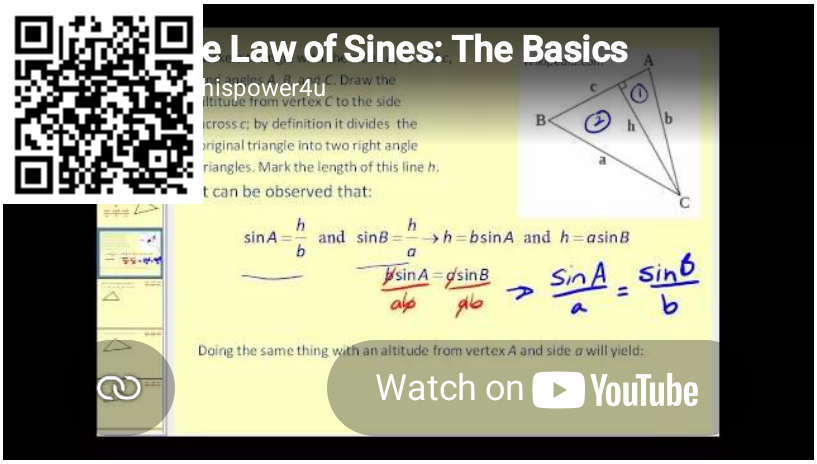
$\angle A = 40°, c = 13,$ and $a = 2$



Before trying find $\angle C$, you need to check that a triangle is possible and if there is more than one solution. Use the equation from above,

$$\sin 40° = \frac{h}{13}$$
$$h = 13 \sin 40° \approx 8.356$$

Because $a < h (2 < 8.356)$, this information does not form a proper triangle.



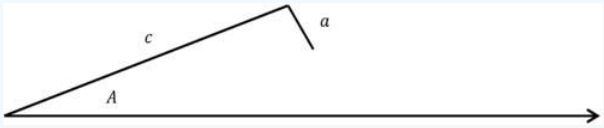
Examples

Example 1

Earlier, you were asked why SSA is not a method to show that two triangles are congruent. SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle. Sometimes there is no triangle, one triangle, or two triangles.

Example 2

$\angle A = 17°, c = 14$, and $a = 4.0932...$ If possible, find $\angle C$



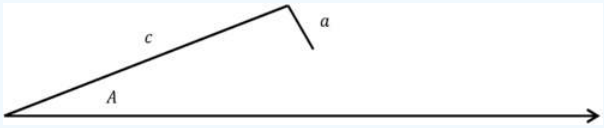
Check that a triangle is possible:

$$\sin 17° = \frac{h}{14}$$
$$h = 14 \sin 17° \approx 4.0932$$

since $a = h$, this information forms exactly one triangle and angle C must be $90°$.

Example 3

$\angle A = 22°, c = 11$ and $a = 9$. If possible, find $\angle C$



Check that a triangle is possible:

$$\sin 22° = \frac{h}{11}$$
$$h = 11 \sin 22° \approx 4.12$$

since $h < a < c$, there must be two possible angles for angle C.

Apply the Law of sines:

$$\frac{9}{\sin 22°} = \frac{11}{\sin C_1}$$
$$9 \sin C_1 = 11 \sin 22°$$
$$\sin C_1 = \frac{11 \sin 22°}{9}$$
$$C_1 = \sin^{-1}\left(\frac{11 \sin 22°}{9}\right) \approx 27.24°$$
$$C_2 = 180 - C_1 \approx 152.75°$$

Example 4

Given ΔABC where $A = 12°, B = 50°, a = 14$ find b.

$$\frac{14}{\sin 12°} = \frac{b}{\sin 50°}$$
$$b = \frac{14 \sin 50°}{\sin 12°} \approx 51.58$$

Example 5

Given ΔABC where $A = 70°, b = 8, a = 3$, find $\angle B$ if possible.

$\sin 70° = \frac{h}{8}$

$h = 8 \sin 70° \approx 7.51 \ldots$

Because $a < h$, this triangle is impossible.

Exercise 4.6.1

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30°, a = 13, b = 15$
2. $A = 22°, a = 21, b = 12$
3. $A = 42°, a = 36, b = 37$

For $4 - 7$, find all possible measures of $\angle B$ (if any exist) for each of the following triangle values.

4. $A = 86°, a = 15, b = 11$
5. $A = 30°, a = 24, b = 43$
6. $A = 48°, a = 34, b = 39$
7. $A = 80°, a = 22, b = 20$

For 8 -12, find the length of b for each of the following triangle values.

8. $A = 94°, a = 31, B = 34°$
9. $A = 112°, a = 12, B = 15°$
10. $A = 78°, a = 20, B = 16°$
11. $A = 54°, a = 15, B = 112°$
12. $A = 39°, a = 9, B = 98°$
13. In $\Delta ABC, b = 10$ and $\angle A = 39°$. What's a possible value for a that would produce two triangles?
14. In $\Delta ABC, b = 10$ and $\angle A = 39°$. What's a possible value for a that would produce no triangles?
15. In $\Delta ABC, b = 10$ and $\angle A = 39°$. What's a possible value for a that would produce one triangle?

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4.7: Area of a Triangle

From geometry you already know that the area of a triangle is $\frac{1}{2} \cdot b \cdot h$

What if you are given the sides of a triangle are 5 and 6 and the angle between the sides is $\frac{\pi}{3}$? You are not directly given the height, but can you still figure out the area of the triangle?

Finding the Area of Triangles

The sine function allows you to find the height of any triangle and substitute that value into the familiar triangle area formula.



Using the sine function, you can isolate h for height:

$$\sin C = \frac{h}{a}$$
$$a \sin C = h$$

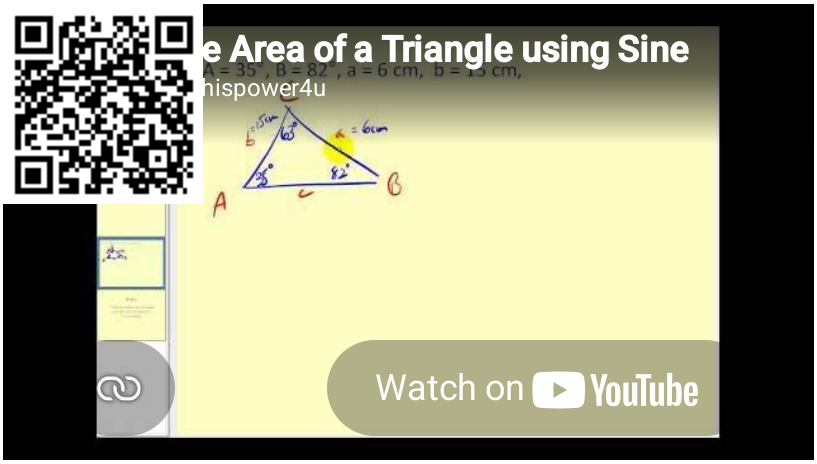
Substituting into the area formula:

$$\text{Area} = \frac{1}{2} b \cdot h$$
$$\text{Area} = \frac{1}{2} b \cdot a \cdot \sin C$$
$$\text{Area} = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

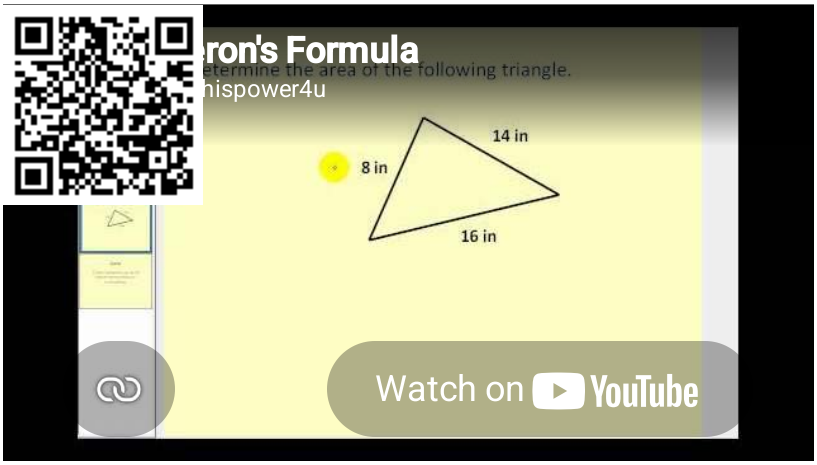
Recall that the variables used to note the sides and corresponding angles are arbitrary, so long as a side and its opposing angle share the same variable (side c has an opposing angle C). If you were given ΔABC with $A = 22°, b = 6, c = 7$ and asked to find the area, you would use the formula:

$$\text{Area} = \frac{1}{2} bc \sin A$$
$$\text{Area} = \frac{1}{2} \cdot 6 \cdot 7 \cdot \sin 22° \approx 7.86 \text{ units}^2$$

The important part is that neither given side corresponds to the given angle.



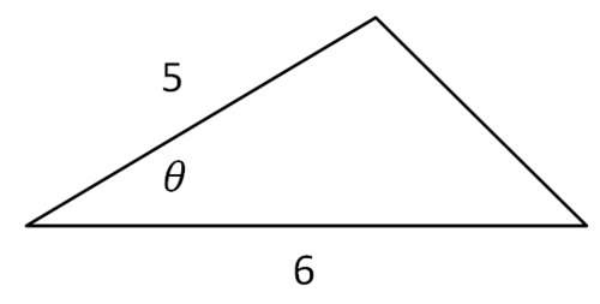
Bonus Video: There is another way of finding the area of a triangle, Heron's Formula. This will be discussed in Example 5. Heron's formula is used when three side lengths are given.



Examples

Example 1

Earlier, you were given the sides of a triangle are 5 and 6 and the angle between the sides is $\theta = \frac{\pi}{3}$ and asked to find the area.



Area $= \frac{1}{2} \cdot 5 \cdot 6 \cdot \sin \frac{\pi}{3} \approx 12.99$ units ^{2}

Example 2

Given ΔXYZ has area 28 square inches, what is the angle included between side length 8 and 9?

$$Area = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$
$$28 = \frac{1}{2} \cdot 8 \cdot 9 \cdot \sin C$$
$$\sin C = \frac{28 \cdot 2}{8 \cdot 9}$$
$$C = \sin^{-1}\left(\frac{28 \cdot 2}{8 \cdot 9}\right) \approx 51.06^\circ$$

Example 3

Given triangle ABC with $A = 12^\circ, b = 4$ and $Area = 1.7$ units^2, what is the length of side c?

$$\text{Area} = \frac{1}{2} \cdot c \cdot b \cdot \sin A$$
$$1.7 = \frac{1}{2} \cdot c \cdot 4 \cdot \sin 12^\circ$$
$$c = \frac{1.7 \cdot 2}{4 \cdot \sin 12^\circ} \approx 4.09$$

Example 4

The area of a triangle is 3 square units. Two sides of the triangle are 4 units and 5 units. What is the measure of their included angle?

$3 = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin\theta$
$\theta = \sin^{-1}\left(\frac{3 \cdot 2}{4 \cdot 5}\right) \approx 17.46°$

Example 5

What is the area of ΔXYZ with $x = 11, y = 12, z = 13$?

Since none of the angles are given, there are two possible solution paths. You could use the Law of Cosines to find one angle.

Area $= \frac{1}{2} \cdot 12 \cdot 13 \cdot \sin 52.02 \approx 61.5$ units ^{2}

The angle opposite the side of length 11 is approximately $52.02°$ therefore the area is:

Area $= \frac{1}{2} \cdot 12 \cdot 13 \cdot \sin 52.02 \approx 61.5$ units ^{2}

Another way to find the area is through the use of Heron's Formula which is:

Area$= \sqrt{s(s-a)(s-b)(s-c)}$

Where s is the semiperimeter:

$s = \frac{a+b+c}{2}$

Using Heron's formula to find the area of ΔXYZ returns the same value:

$s = \frac{a+b+c}{2}$
$s = \frac{11+12+13}{2} = \frac{36}{2} = 18$
$A = \sqrt{18(18-11)(18-12)(18-13)}$
$A = \sqrt{18 \cdot 7 \cdot 6 \cdot 5}$
$A = \sqrt{3780} \approx 61.5$ units ^{2}

Example 6

The area of a triangle is 3 square units. Two sides of the triangle are 4 units and 5 units. What is the measure of their included angle?

$3 = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin\theta$
$\theta = \sin^{-1}\left(\frac{3 \cdot 2}{4 \cdot 5}\right) \approx 17.45\ldots$

Review

For $1 - 11$, find the area of each triangle.

1. ΔABC if $a = 13, b = 15$, and $\angle C = 70°$.
2. ΔABC if $b = 8, c = 4$, and $\angle A = 58°$.
3. ΔABC if $b = 34, c = 29$, and $\angle A = 125°$.
4. ΔABC if $a = 3, b = 7$, and $\angle C = 81°$.
5. ΔABC if $a = 4.8, c = 3.7$, and $\angle B = 54°$.
6. ΔABC if $a = 12, b = 5$, and $\angle C = 22°$.
7. ΔABC if $a = 3, b = 10$, and $\angle C = 65°$.
8. ΔABC if $a = 5, b = 9$, and $\angle C = 11°$.
9. ΔABC if $a = 5, b = 7$, and $c = 8$.

10. ΔABC if $a = 7, b = 8$, and $c = 14$.

11. ΔABC if $a = 12, b = 14$, and $c = 13$.

12. The area of a triangle is 12 square units. Two sides of the triangle are 8 units and 4 units. What is the measure of their included angle?

13. The area of a triangle is 23 square units. Two sides of the triangle are 14 units and 5 units. What is the measure of their included angle?

14. Given ΔDEF has area 32 square inches, what is the angle included between side length 9 and 10?

15. Given ΔGHI has area 15 square inches, what is the angle included between side length 7 and 11?

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4.8: Applications of Basic Triangle Trigonometry

Deciding when to use SOH, CAH, TOA, Law of Cosines or the Law of Sines is not always obvious. Sometimes more than one approach will work and sometimes correct computations can still lead to incorrect results. This is because correct interpretation is still essential.

If you use both the Law of Cosines and the Law of Sines on a triangle with sides 4, 7, 10 you end up with conflicting answers. Why?

Trigonometry Applications

When applying trigonometry, it is important to have a clear toolbox of mathematical techniques to use. Some of the techniques may be review like the fact that all three angles in a triangle sum to be $180°$, other techniques may be newer like the Law of cosines. Take a look at all the tools you have in your toolbox to solve applications with trigonometry.

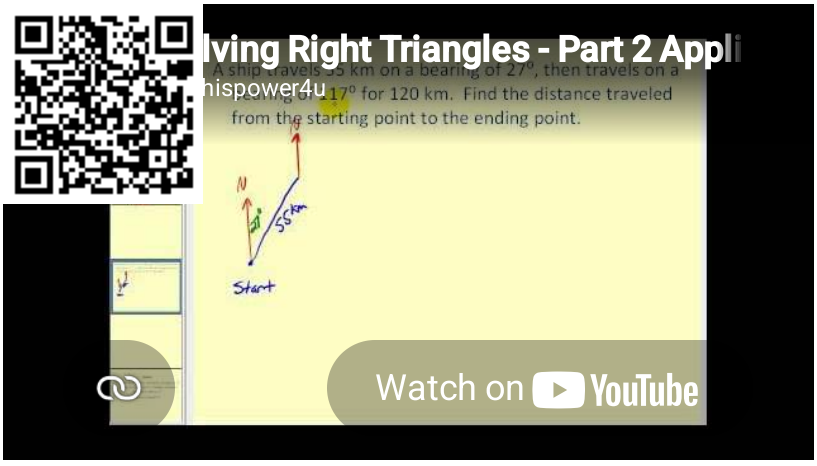
Toolbox

- The three angles in a triangle sum to be $180°$.
- There are $360°$ in a circle and this can help us interpret negative angles as positive angles.
- The Pythagorean Theorem states that for legs a, b and hypotenuse c in a right triangle, $a^2 + b^2 = c^2$
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the third side.
- The Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- The Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$ (Be careful for the ambiguous case)
- SOH CAH TOA is a mnemonic device to help you remember the three original trig functions:

$\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$

- $30-60-90$ right triangles have side ratios $x, x\sqrt{3}, 2x$
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$
- Pythagorean number triples are exceedingly common and should always be recognized in right triangle problems. Examples of triples are 3,4,5 and $5, 12, 13$.

A few definitions will also be necessary to solve applications. Angle of elevation is the angle at which you view and object above the horizon. Angle of depression is the angle at which you view and object below the horizon. Bearing is how direction is measured at sea. North is $0°$, East is $90°$, South is $180°$ and West is $270°$.

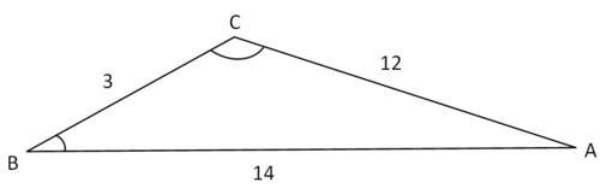


Examples

Example 1

Earlier, you were asked why you may end up with conflicting answers if you use both the Law of Sines and the Law of Cosines. Sometimes when using the Law of Sines you can get answers that do not match the Law of Cosines. Both answers

can be correct computationally, but the Law of Sines may involve interpretation when the triangle is obtuse. The Law of Cosines does not require this interpretation.



First, use Law of cosines to find $\angle B$:

$$12^2 = 3^2 + 14^2 - 2 \cdot 3 \cdot 14 \cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2-3^2-14^2}{-2 \cdot 3 \cdot 14}\right) \approx 43.43^\circ$$

Then, use Law of sines to find $\angle C$. Use the unrounded value for B even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14 \sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14 \sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of cosines to double check $\angle C$.

$$14^2 = 3^2 + 12^2 - 2 \cdot 3 \cdot 12 \cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2 \cdot 3 \cdot 12}\right) \approx 126.7^\circ$$

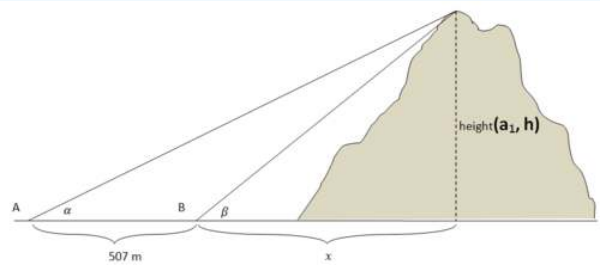
Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles.

Example 2

A surveying crew is given the job of verifying the height of a cliff. From point A, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

Note that α is just the Greek letter alpha and in this case it stands for the number 21.567°. β is the Greek letter beta and it stands for the number 25.683°.

First, sketch the image and label what you know.



Next, because the height is measured at a right angle with the ground, set up two equations. Remember that α and β are just numbers, not variables.

$$\tan \alpha = \frac{h}{507+x}$$
$$\tan \beta = \frac{h}{x}$$

Both of these equations can be solved for h and then set equal to each other to find x.

$$h = \tan\alpha(507 + x) = x\tan\beta$$
$$507\tan\alpha + x\tan\alpha = x\tan\beta$$
$$507\tan\alpha = x\tan\beta - x\tan\alpha$$
$$507\tan\alpha = x(\tan\beta - \tan\alpha)$$
$$x = \frac{507\tan\alpha}{\tan\beta - \tan\alpha} = \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 2340 \text{ meters}$$

since the problem asked for the height, you need to substitute x back and solve for h.

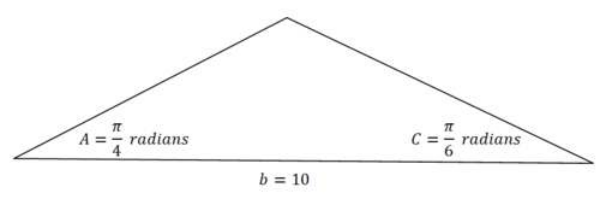
$h = x\tan\beta = 2340\tan 25.683^\circ \approx 1125.31$ meters

Example 3

Given a triangle with SSS or SAS you know to use the Law of cosines. In triangles where there are corresponding angles and sides like AAS or SSA it makes sense to use the Law of sines. What about ASA?

Given ΔABC with $A = \frac{\pi}{4}$ radians, $C = \frac{\pi}{6}$ radians and $b = 10$ in what is a ?

First, draw a picture.



The sum of the angles in a triangle is 180°. since this problem is in radians you either need to convert this rule to radians, or convert the picture to degrees.

$$A = \tfrac{\pi}{4} \cdot \tfrac{180^\circ}{\pi} = 45^\circ$$
$$C = \tfrac{\pi}{6} \cdot \tfrac{180^\circ}{\pi} = 30^\circ \tag{4.8.1}$$

The missing angle must be $\angle B = 105^\circ$. Now you can use the Law of sines to solve for a.

$$\frac{\sin 105^\circ}{10} = \frac{\sin 45^\circ}{a}$$
$$a = \frac{10\sin 45^\circ}{\sin 105^\circ} \approx 7.32in$$

Example 4

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{10}$. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

When you draw a picture, you see that the given angle $\frac{\pi}{10}$ is not directly inside the triangle between the lighthouse, the boat and the base of the lighthouse. It is complementary to the angle you need.

$$\frac{\pi}{10} + \theta = \frac{\pi}{2}$$
$$\theta = \frac{2\pi}{5}$$

Now that you have the angle, use tangent to solve for x.

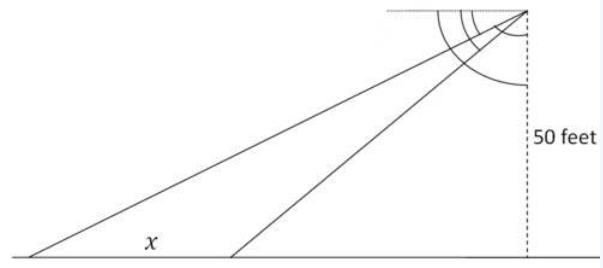
$$\tan\frac{2\pi}{5} = \frac{x}{200}$$
$$x = 200\tan\frac{2\pi}{5} \approx 615.5\text{ft}$$

Alternatively, you could have noticed that $\frac{\pi}{10}$ is alternate interior angles with the angle of elevation of the lighthouse from the boat's perspective. This would yield the same distance for x.

Example 5

From the third story of a building (50 feet) David observes a car moving towards the building driving on the streets below. If the angle of depression of the car changes from $21°$ to $45°$ while he watches, how far did the car travel?

Draw a very careful picture:



In the upper right corner of the picture there are four important angles that are marked with angles. The measures of these angles from the outside in are $90°, 45°, 21°, 69°$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore you can set up and solve an equation for x

$$\tan 69° = \frac{x+50}{50}$$
$$x = 50\tan 69° - 50 \approx 80.25\text{ft}$$

Review

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. You want to find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.

2. What methods or techniques will you use?

3. Solve the problem.

From the third story of a building (60 feet) Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from $34°$ to $62°$ while he watches. You want to know how far the car traveled.

4. Draw a picture of this situation.

5. What methods or techniques will you use?

6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.

8. What methods or techniques will you use?

9. Solve the problem.

You want to figure out the height of a building. From point A, you measure an angle of elevation to the top of the building to be $\alpha = 10°$. You move 50 feet closer to the building to point B and find that the angle to the top of the building is now $\beta = 60°$.

10. Draw a picture of this situation.

11. What methods or techniques will you use?

12. Solve the problem.

13. Given ΔABC with $A = 40°, C = 65°$ and $b = 8$ in , what is a ?

14. Given ΔABC with $A = \frac{\pi}{3}$ radians, $C = \frac{\pi}{8}$ radians and $b = 12$ in what is a ?

15. Given ΔABC with $A = \frac{\pi}{6}$ radians, $C = \frac{\pi}{4}$ radians and $b = 20$ in what is a ?

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CHAPTER OVERVIEW

5: Trigonometric Functions

5.1: The Unit Circle
5.2: The Sinusoidal Function Family
5.3: Amplitude of Sinusoidal Functions
5.4: Vertical Shift of Sinusoidal Functions
5.5: Frequency and Period of Sinusoidal Functions
5.6: Phase Shift of Sinusoidal Functions
5.7: Graphs of Other Trigonometric Functions
5.8: Graphs of Inverse Trigonometric Functions

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5.1: The Unit Circle

The **unit circle** is a circle of radius one, centered at the origin, that summarizes all the $30-60-90$ and $45-45-90$ triangle relationships that exist. When memorized, it is extremely useful for evaluating expressions like $\cos(135°)$ or $\sin\left(-\frac{5\pi}{3}\right)$. It also helps to produce the parent graphs of sine and cosine.

How can you use the unit circle to evaluate $\cos(135°)$ and $\sin\left(-\frac{5\pi}{3}\right)$?

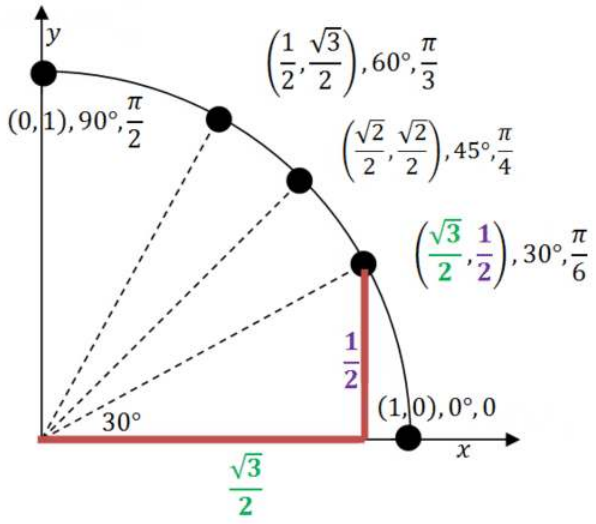
The Unit Circle

You already know how to translate between degrees and radians and the triangle ratios for 30-60-90 and 45-45-90 right triangles. In order to be ready to completely fill in and memorize a unit circle, two triangles need to be worked out. Start by finding the side lengths of a 30-60-90 triangle and a 45-45-90 triangle each with hypotenuse equal to 1.

$30°$	$60°$	$90°$
x	$x\sqrt{3}$	$2x$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

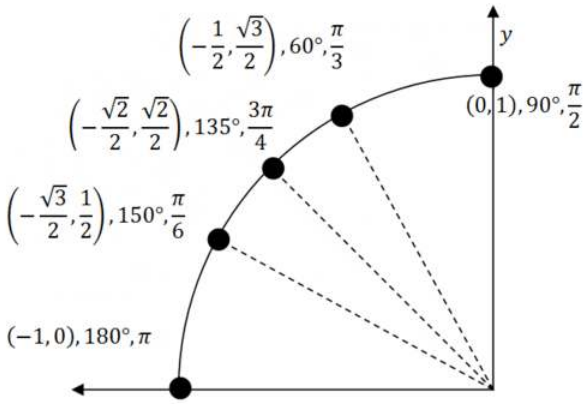
$45°$	$45°$	$90°$
x	x	$x\sqrt{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1

This is enough information to fill out the important points in the first quadrant of the unit circle. The values of the x and y coordinates for each of the key points are shown below. Remember that the x and y coordinates come from the lengths of the legs of the special right triangles, as shown specifically for the $30°$ angle. Always remember to measure the angle from the positive portion of the x -axis.



Knowing the first quadrant well is the key to knowing the entire unit circle. Every other point on the unit circle can be found using logic and this quadrant, so there is no need to memorize the whole circle.

To use your knowledge of the first quadrant of the unit circle to identify the angles and important points of the second quadrant, notice that the heights are mirrored and equal which correspond to the y values. The x values are all negative.

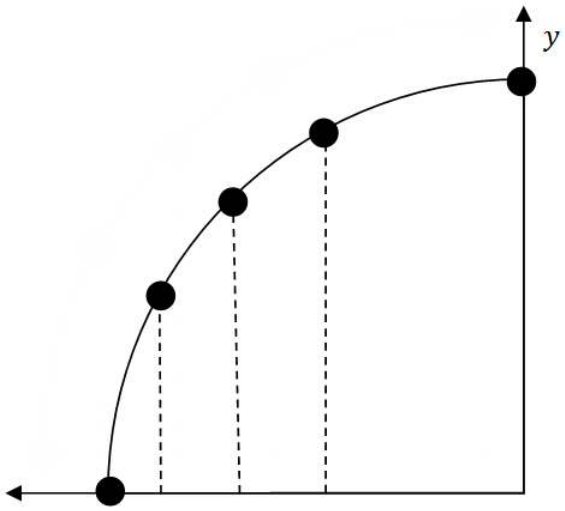


There is a pattern in the heights of the points in the first quadrant that can help you remember the points.

Notice that the heights of the points in the first quadrant are the y -coordinates: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$

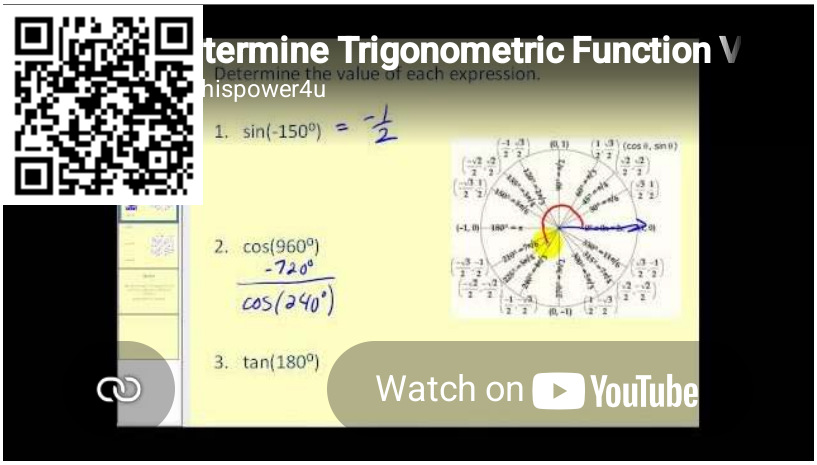
When rewritten, the pattern becomes clear: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$

The three points in the middle are the most often mixed up. This pattern illustrates how they increase in size from small $\frac{1}{2}$, to medium $\frac{\sqrt{2}}{2}$, to large $\frac{\sqrt{3}}{2}$. When you fill in the unit circle, look for the heights that are small, medium and large and this will tell you were each value should go. Notice that the heights for these five points in the second quadrant are also $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$



This technique also works for the widths. This can make memorizing the 16 points of the unit circle a matter of logic and the pattern: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$

One last item to note is that coterminal angles are sets of angles such as $90°, 450°,$ and $-270°$ that start at the positive x-axis and end at the same terminal side. since coterminal angles end at identical points along the unit circle, trigonometric expressions involving coterminal angles are equivalent: $\sin(90) = \sin(450) = \sin(-270)$



Examples

Example 1

Earlier, you were asked how you can use the unit circle to evaluate $\cos(135)$ and $\sin\left(-\frac{5\pi}{3}\right)$. The x value of a point along the unit circle corresponds to the cosine of the angle. The y value of a point corresponds to the sine of the angle. When the angles and points are memorized, simply recall the x or y coordinate. If the construction of the unit circle is understood, it becomes easier to determine the coordinates.

When evaluating $\cos(135°)$ your thought process should be something like this:

You know $135°$ goes with the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and cosine is the x portion. So, $\cos(135°) = -\frac{\sqrt{2}}{2}$

$\cos(135°) = -\frac{\sqrt{2}}{2}$

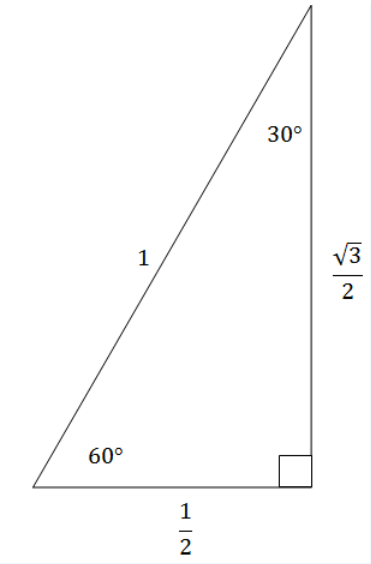
When evaluating $\sin\left(-\frac{5\pi}{3}\right)$ your thought process should be something like this:

You know $-\frac{5\pi}{3}$ goes with the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and sine is the y portion. So, $\sin\left(-\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example 2

Evaluate $\cos 60°$ using the unit circle and right triangle trigonometry. What is the connection between the x coordinate of the point and the cosine of the angle?

The point on the unit circle for $60°$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the point is one unit from the origin. This can be represented as a $30 - 60 - 90$ triangle.

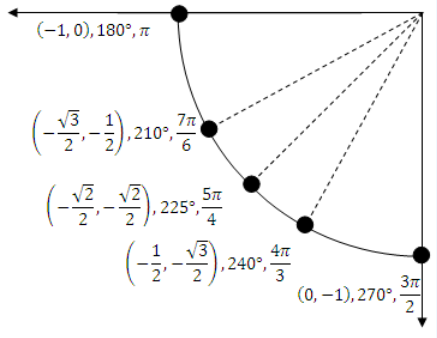


Since cosine is adjacent over hypotenuse, cosine turns out to be exactly the x coordinate $\frac{1}{2}$.

Example 3

Using knowledge of the first quadrant of the unit circle, identify the angles and important points of the third quadrant.

Both the x values and y values are negative and their respective coordinates correspond to those of the other quadrants.



Example 4

For each of the six trigonometric functions, identify the quadrants where they are positive and the quadrants where they are negative.

In quadrant I, the hypotenuse, adjacent and opposite side are all positive. Thus all 6 trigonometric functions are positive.

In quadrant II the hypotenuse and opposite sides are positive and the adjacent side is negative. This means that every trigonometric expression involving an adjacent side is negative. Sine and its reciprocal cosecant are the only two trigonometric functions that do not refer to the adjacent side which makes them the only positive ones.

In quadrant III only the hypotenuse is positive. Thus the only trigonometric functions that are positive are tangent and its reciprocal cotangent because these functions refer to both adjacent and opposite sides which will both be negative.

In quadrant IV the hypotenuse and the adjacent sides are positive while the opposite side is negative. This means that only cosine and its reciprocal secant are positive.

A mnemonic device to remember which trigonometric functions are positive and which trigonometric functions are negative is "All Students Take Calculus." All refers to all the trigonometric functions are positive in quadrant I. The letter S refers to sine

and its reciprocal cosecant that are positive in quadrant II. The letter T refers to tangent and its reciprocal cotangent that are positive in quadrant III. The letter C refers to cosine and its reciprocal secant that are positive in quadrant IV.

Example 5

Evaluate the following trigonometric expressions using the unit circle.

1. $\sin \frac{\pi}{2}$

$\sin \frac{\pi}{2} = 1$

2. $\cos 210^\circ$

$\cos 210^\circ = -\frac{\sqrt{3}}{2}$

3. $\tan 315^\circ$

$\tan 315^\circ = -1$

4. $\cot 270^\circ$

$\cot 270^\circ = 0$

5. $\sec \frac{11\pi}{6}$

$\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

6. $\csc -\frac{5\pi}{4}$

$\csc -\frac{5\pi}{4} = \frac{1}{\sin -\frac{5\pi}{4}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

Review

Name the angle between 0° and 360° that is coterminal with...

1. -20°
2. 475°
3. -220°
4. 690°
5. -45°

Use your knowledge of the unit circle to help determine whether each of the following trigonometric expressions is positive or negative.

6. $\tan 143^\circ$
7. $\cos \frac{\pi}{3}$
8. $\sin 362^\circ$
9. $\csc \frac{3\pi}{4}$

Use your knowledge of the unit circle to evaluate each of the following trigonometric expressions.

10. $\cos 120^\circ$
11. $\sec \frac{\pi}{3}$
12. $\tan 225^\circ$
13. $\cot 120^\circ$
14. $\sin \frac{11\pi}{6}$
15. $\csc 240^\circ$

16. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta > 0$.

17. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = -\frac{1}{2}$ and $\sec\theta < 0$.

18. Draw the complete unit circle (all four quadrants) and label the important points.

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5.2: The Sinusoidal Function Family

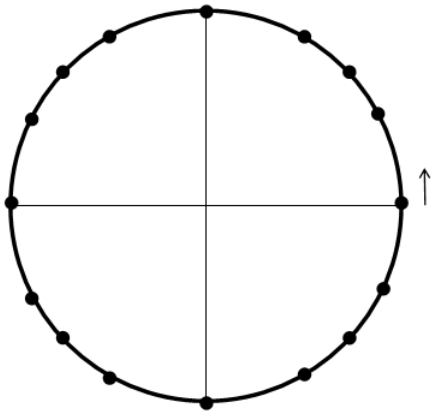
The **cosine function** is the x **coordinates** of the unit circle and the **sine function** is **the** y **coordinates**. Since the unit circle has radius one and is centered at the origin, both sine and cosine oscillate between positive and negative one.

What happens when the circle is not centered at the origin and does not have a radius of 1?

Graphs of Sinusoidal Functions

The **sinusoidal function** family refers to either sine or cosine waves since they are the same except for a horizontal shift. This function family is also called the periodic function family because the function repeats after a given period of time.

Consider a Ferris wheel that spins evenly with a radius of 1 unit. It starts at (1,0) or an angle of 0 radians and spins counterclockwise at a rate of one cycle per 2π minutes (so you can use time is equal to radians).

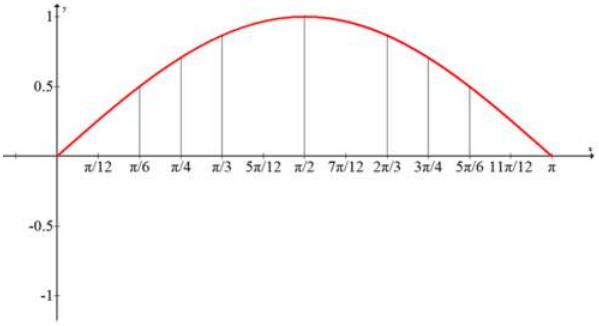


The 16 points around the circle are chosen because they correspond to the key points of the unit circle. Their heights (y -values) and widths (x -values) are already known and can be filled in.

First consider the height at each of the points as you travel around half of the circle from the starting location. Keep track of your work in a table.

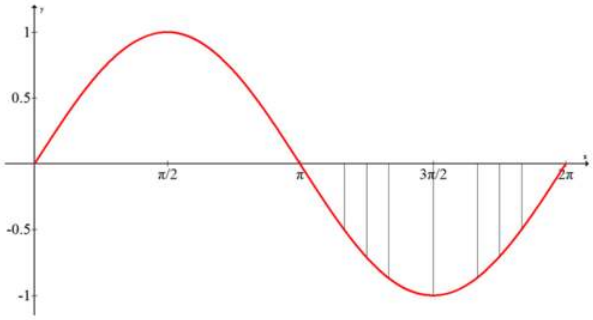
Angle (radians)	Height (units)
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.866$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2} \approx 0.707$
$\frac{5\pi}{6}$	$\frac{1}{2}$
π	0

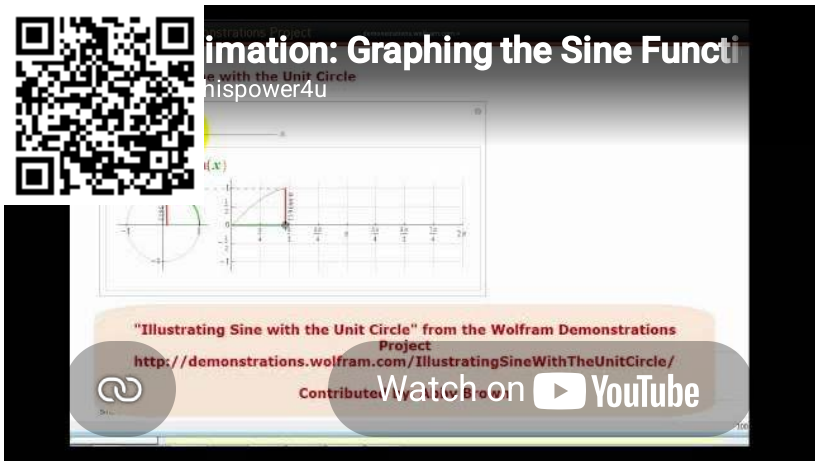
Notice the symmetry of the height around $\frac{\pi}{2}$ and see the rest of the table in the examples. Once the table is finished, you can plot these points on a regular coordinate plane where the x axis is the angle and the y axis is the height. This is the first part of the graph of the sine function.



To finish the graph of the sine function, finish the table for heights of the points in quadrants III and IV and draw an entire cycle (known as a period) of the sine function.

Angle (rodions)	Height (unite)
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.866$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2} \approx -0.707$
$\frac{11\pi}{6}$	$-\frac{1}{2}$
2π	0

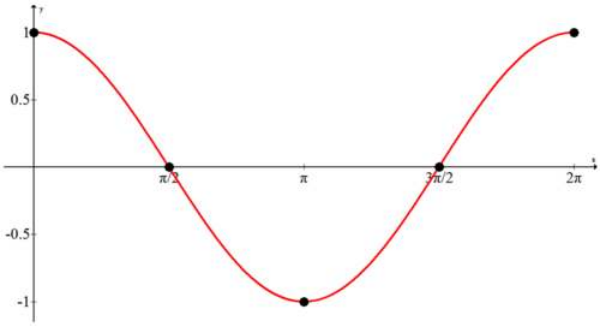




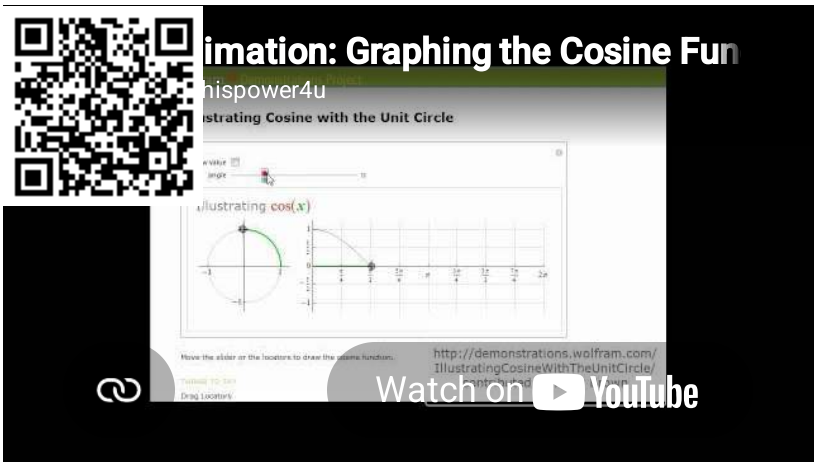
Similar to sine, you can use your knowledge of the angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ on the unit circle to get a complete cycle of the cosine graph. While the sine function uses the x -coordinates, the cosine function is the x -coordinates of the unit circle and measures width. By referring to a unit circle or your memory, you can fill out a much shorter table than before.

Angle (radians)	Width (units)
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1

First plot these five points and then connect them with a smooth curve. This will produce the cosine graph.



Determining these five main points is the key to graphing sine or cosine graphs even when the graph is shifted or stretched.



Examples

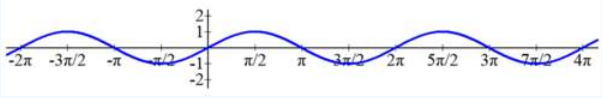
Example 1

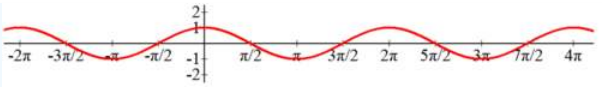
Earlier, you were asked what happens when the circle is not centered at the origin and does not have a radius of 1. The unit circle produces the parent function sine and cosine graphs. When the unit circle is shifted up or down, made wider or narrower, or spun faster or slower in either direction, then the graphs of the sine and cosine functions will be transformed using basic function transformation rules.

Example 2

What happens on either side of the sine and cosine graphs? Can you explain why?

The graphs of the sine (blue) and cosine (red) functions repeat forever in both directions.





If you think about the example with the Ferris wheel, the ride will keep on spinning and has been spinning forever. This is why the same cycle of the graph repeats over and over.

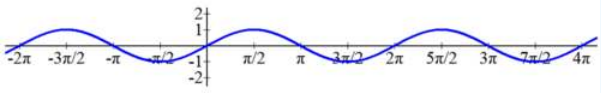
Example 3

How are the sine and cosine graphs the same and how are they different?

The sine graph is the same as the cosine graph offset by $\frac{\pi}{2}$. Besides the shift, both curves are identical due to the perfect symmetry of circles.

Example 4

Where are two maximums and two minimums of the sine graph?

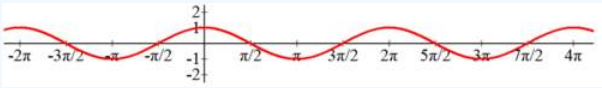


One maximum of the sine graph occurs at $(\frac{\pi}{2}, 1)$. One minimum occurs at $(\frac{3\pi}{2}, -1)$. This is one cycle of the sine graph. since it completes a cycle every 2π, when you add 2π to an x - coordinate you will be on the same point of the cycle giving you another maximum or minimum. $(\frac{5\pi}{2}, 1)$ is another maximum. $(\frac{7\pi}{2}, -1)$ is another minimum.

Example 5

In the interval $[-2\pi, 4\pi)$ where does cosine have zeroes?

Observe where the cosine curve has x -coordinates equal to zero. Note that 4π is excluded from the interval. The values are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$



Review

1. Sketch $p(x) = \sin x$ from memory.
2. Sketch $j(x) = \cos x$ from memory.
3. Where do the maximums of the cosine graph occur?
4. Where do the minimums of the cosine graph occur?
5. Find all the zeroes of the sine function on the interval $[-\pi, \frac{5\pi}{2}]$.
6. Find all the zeroes of the cosine function on the interval $(-\frac{\pi}{2}, \frac{7\pi}{2}]$.
7. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $f(x) = 2\cos x$ will look like.
8. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph $g(x) = \cos x + 2$ of will look like.
9. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $h(x) = \cos(x - \pi)$ will look like.
10. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $k(x) = -\cos x$ will look like.

...

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5.3: Amplitude of Sinusoidal Functions

The **amplitude** of the sine and cosine functions is the vertical distance between the sinusoidal axis and the maximum or minimum value of the function. In relation to sound waves, amplitude is a measure of how loud something is.

What is the most common mistake made when graphing the amplitude of a sine wave?

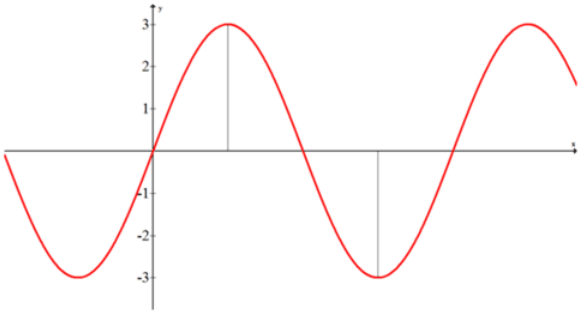
Amplitude of Sinusoidal Functions

The general form a sinusoidal function is:

f(x)=\pm a \cdot \sin (b(x+c))+d

The cosine function can just as easily be substituted and for many problems it will be easier to use a cosine equation. Since both the sine and cosine waves are identical except for a horizontal shift, it all depends on where you see the wave starting.

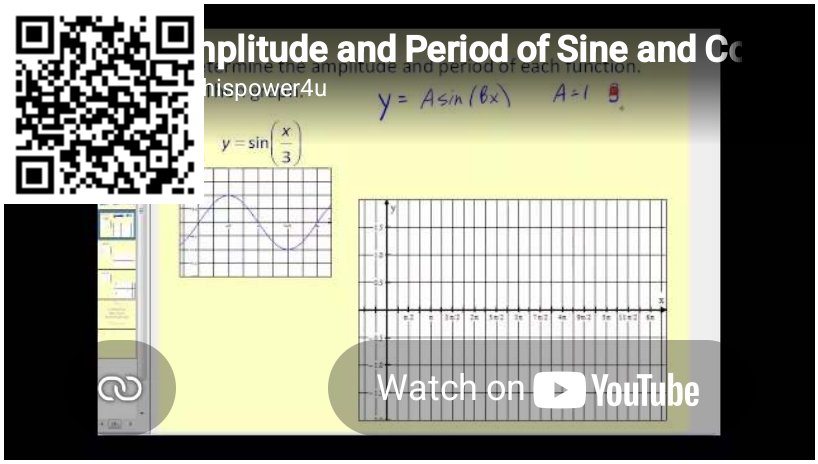
The coefficient a is the amplitude. When there is no number present, then the amplitude is 1. The best way to define amplitude is through a picture. Below is the graph of the function $f(x) = 3 \cdot \sin x$, which has an amplitude of 3.



Notice that the amplitude is 3 , not 6 . This corresponds to the absolute value of the maximum and minimum values of the function. If the function had been $f(x) = -3 \cdot \sin x$, then the whole graph would be reflected across the x axis.

Also notice that the x axis on the graph above is unlabeled. This is to show that amplitude is a vertical distance. The **sinusoidal axis** is the neutral horizontal line that lies between the **crests** and the **troughs** (or peaks and valleys if you prefer). For this function, the sinusoidal axis was just the x axis, but if the whole graph were shifted up, the sinusoidal axis would no longer be the x axis. Instead, it would still be the horizontal line directly between the crests and troughs which is also the average of the maximum and minimum values.

Watch the portion of this video discussing amplitude:



Examples

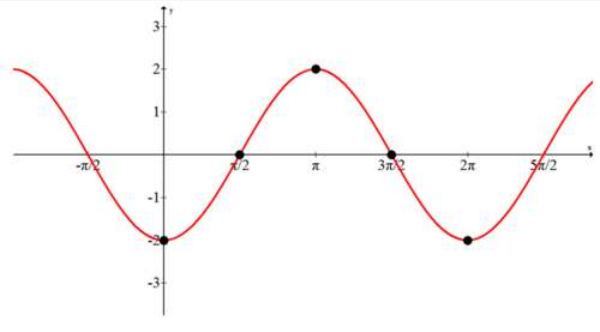
Example 1

Earlier, you were asked about the most common mistake made when graphing the amplitude of a singe wave. The most common mistake is doubling or halving the amplitude unnecessarily. Many people forget that the number a in the equation, like the 3 in $f(x) = 3 \sin x$, is the distance from the x axis to both the peak and the valley. It is not the total distance from the peak to the valley.

Example 2

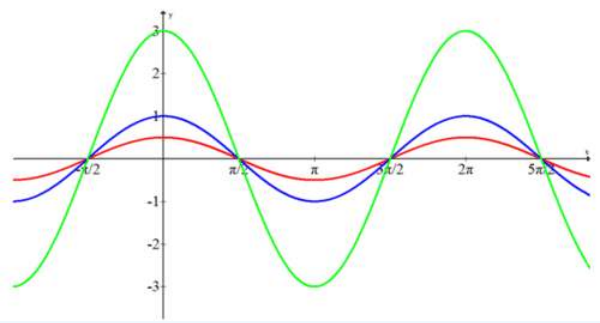
Graph the following function by plotting main points: $f(x) = -2 \cdot \cos x$.

The amplitude is 2 , which means the maximum values will be at 2 and the minimum values will be at -2. Normally with a basic cosine curve the points corresponding to $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ fall above, on or below the line in the following sequence: above, on, below, on, above. The negative sign in the equation switches above with below and below with above. The whole graph is reflected across the x -axis.



Example 3

Write a cosine equation for each of the following functions.



The amplitudes of the three functions are 3,1 and $\frac{1}{2}$ and none of them are reflected across the x axis.

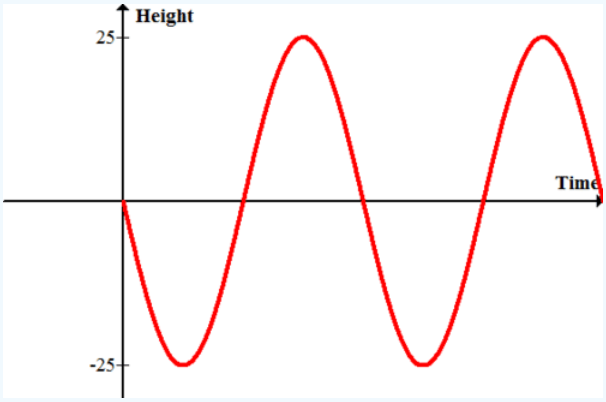
$f(x) = 3 \cdot \cos x$
$h(x) = \cos x$
$g(x) = \frac{1}{2} \cdot \cos x$

Note that amplitude itself is always positive.

Example 4

A Ferris wheel with radius 25 feet sits next to a platform. The ride starts at the platform and travels down to start. Model the height versus time of the ride.

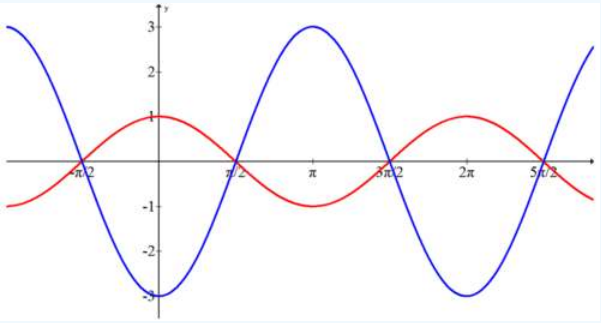
Since no information is given about the time, simply label the x axis as time. At time zero the height is zero. Initially the height will decrease as the ride goes below the platform. Eventually, the wheel will find the minimum and start to increase again all the way until it reaches a maximum.



Example 5

Find the amplitude of the function $f(x) = -3\cos x$ and use the language of transformations to describe how the graph is related to the parent function $y = \cos x$.

The new function is reflected across the x axis and vertically stretched by a factor of 3 .



Review

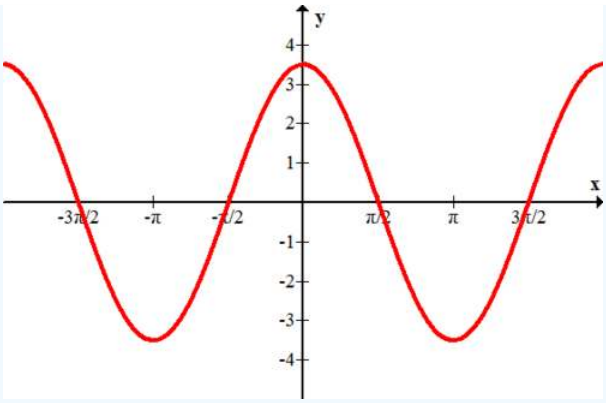
1. Explain how to find the amplitude of a sinusoidal function from its equation.

2. Explain how to find the amplitude of a sinusoidal function from its graph.

Find the amplitude of each of the following functions.

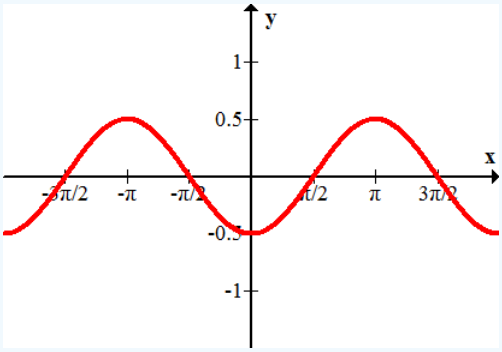
3. $g(x) = -5\cos x$

4.



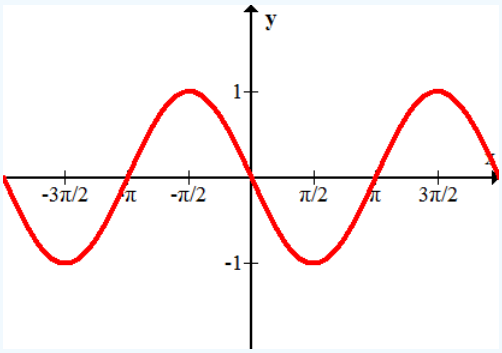
5. $f(x) = \frac{1}{2}\sin x$

6.



7. $j(x) = 3.12\cos x$

8.



Sketch each of the following functions.

9. $f(x) = 3\sin x$

10. $g(x) = -4\cos x$

11. $h(x) = \pi\sin x$

12. $k(x) = -1.2\cos x$

13. $p(x) = \frac{2}{3}\cos x$

14. $m(x) = -\frac{1}{2}\sin x$

15. Preview: $r(x) = 3\sin x + 2$

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5.4: Vertical Shift of Sinusoidal Functions

Your knowledge of transformations, specifically vertical shift, apply directly to sinusoidal functions. In practice, sketching shifted sine and cosine functions requires greater attention to detail and more careful labeling than other functions. Can you describe the following transformation in words?

$f(x) = \sin x \rightarrow g(x) = -3 \sin x - 4$

In what order do the reflection, stretch and shift occur? Is there a difference?

Vertical Shift of Sinusoidal Functions

The general form of a sinusoidal function is:

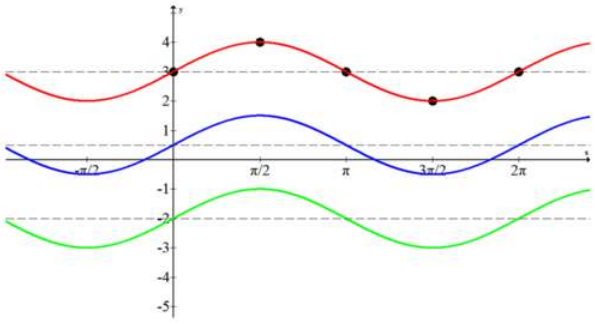
$f(x) = \pm a \cdot \sin(b(x + c)) + d$

Recall that a controls amplitude and the $\pm$ controls reflection. Here you will see how d controls the vertical shift.

The most straightforward way to think about vertical shift of sinusoidal functions is to focus on the **sinusoidal axis**, the horizontal line running through the middle of the sine or cosine wave. At the start of the problem identify the vertical shift and immediately draw the new sinusoidal axis. Then proceed to graph amplitude and reflection about that axis as opposed to the x axis.

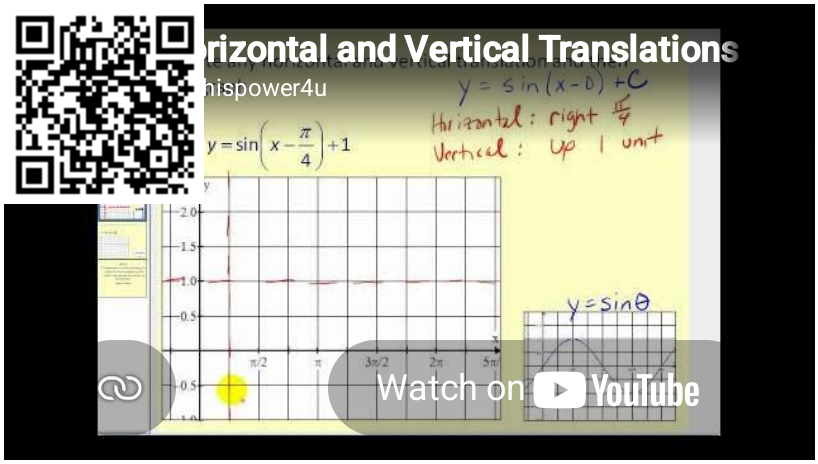
The graphs of the following three functions are shown below:

$f(x) = \sin x + 3$
$g(x) = \sin x - 2$
$h(x) = \sin x + \frac{1}{2}$



To draw these graphs, the new sinusoidal axis for each graph is drawn first. Then, a complete sine wave for each one is drawn. Note the five important points that separate each quadrant to help to get a clear sense of the graph. There are no reflections in these graphs and they all have an amplitude of 1. Right now every cycle starts at 0 and ends at 2π but this will not always be the case.

Watch the portions of the following video focused on vertical translations:



Examples

Example 1

Earlier, you were asked which order vertical shift and reflection should be performed in and if it matters. The following transformation can be described as follows.

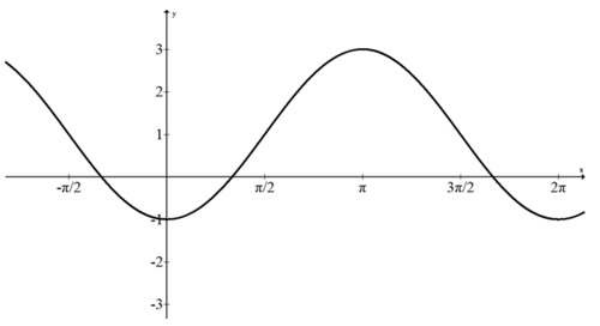
$f(x) = \sin x \rightarrow g(x) = -3 \sin x - 4$

Describe the stretching and reflecting first and then the vertical shift. This is the most logical way to discuss the transformation verbally because then the numbers like 3 and -4 can be explicitly identified in the graph.

The order in describing the transformation matters. When describing vertical transformations it is most intuitive to simply describe the transformations in the same order as the order of operations.

Example 2

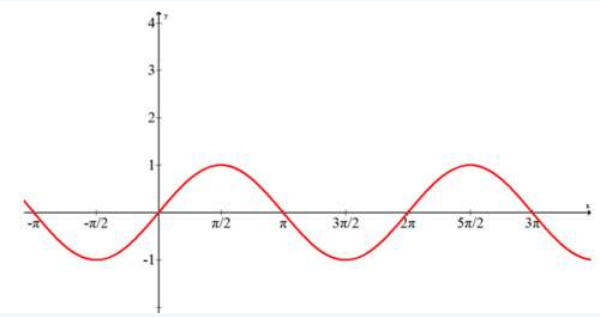
Identify the equation of the following transformed cosine graph.



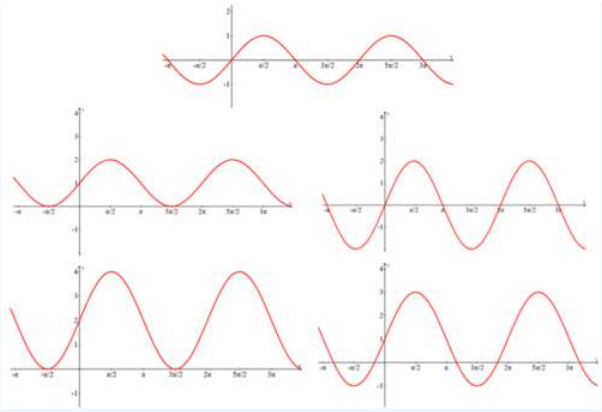
since there is no sinusoidal axis given, you must determine the vertical shift, stretch and reflection. The peak occurs at $(\pi, 3)$ and the trough occurs at (0,-1) so the horizontal line directly between +3 and -1 is $y = 1$. since the sinusoidal axis has been shifted up by one unit $d = 1$. From this height, the graph goes two above and two below which means that the amplitude is 2 . since this cosine graph starts its cycle at (0,-1) which is a lowest point, it is a negative cosine. The function is $f(x) = -2 \cos x + 1$

Example 3

Transform the following sine graph in two ways. First, transform the sine graph by shifting it vertically up 1 unit and then stretching it vertically by a factor of 2 units. Second, transform the sine graph by stretching it vertically by a factor of 2 units and then shifting it vertically up 1 unit.

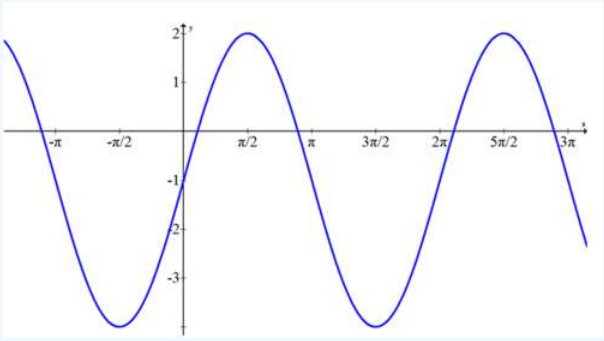


When doing ordered transformations it is good to show where you start and where you end up so that you can effectively compare and contrast the outcomes. See how both transformations start with a regular sine wave. The two columns represent the sequence of transformations that produce different outcomes.



Example 4

What equation models the following graph?

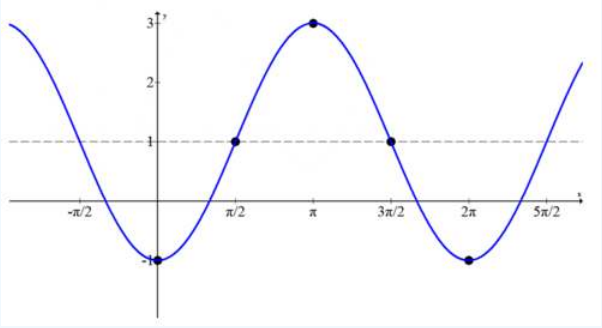


$f(x) = 3 \cdot \sin x - 1$

Example 5

Graph the following function: $f(x) = -2 \cdot \cos x + 1$

First draw the horizontal sinusoidal axis and identify the five main points for the cosine wave. Be careful to note that the amplitude is 2 and the cosine wave starts and ends at a low point because of the negative sign.



Review

Graph each of the following functions that have undergone a vertical stretch, reflection, and/or a vertical shift.

1. $f(x) = -2 \sin x + 4$
2. $g(x) = \frac{1}{2} \cos x - 1$
3. $h(x) = 3 \sin x + 2$

4. $j(x) = -1.5\cos x + \frac{1}{2}$

5. $k(x) = \frac{2}{3}\sin x - 3$

Find the minimum and maximum values of each of the following functions.

6. $f(x) = -3\sin x + 1$

7. $g(x) = 2\cos x - 4$

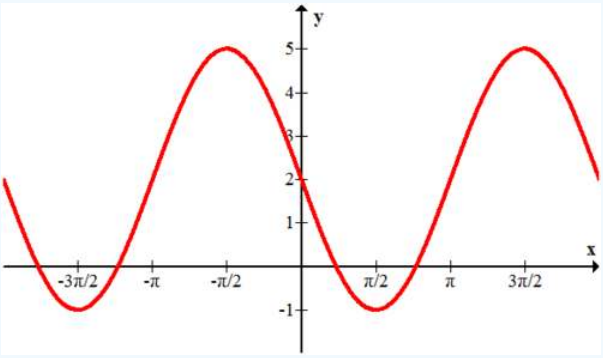
8. $h(x) = \frac{1}{2}\sin x + 1$

9. $j(x) = -\cos x + 5$

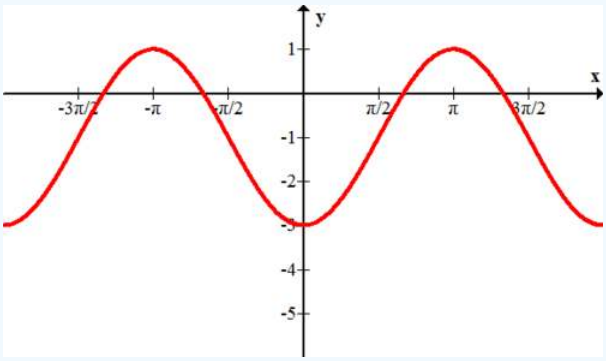
10. $k(x) = \sin(x) - 1$

Give the equation of each function graphed below.

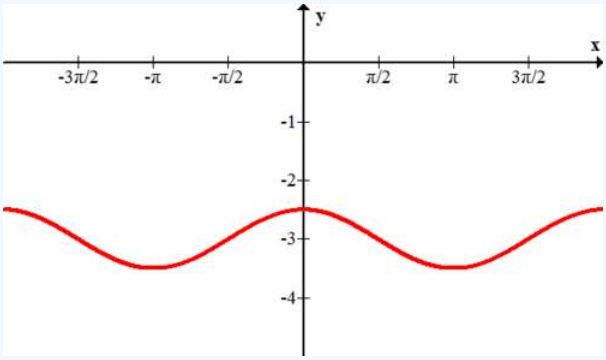
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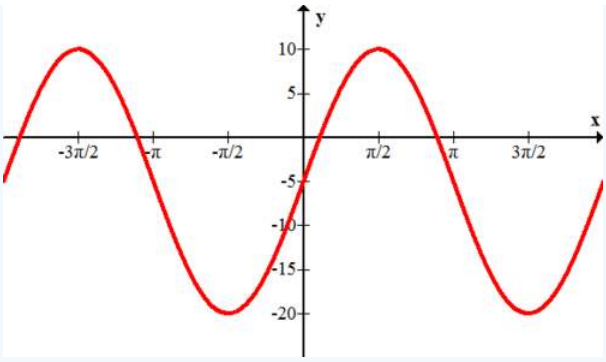
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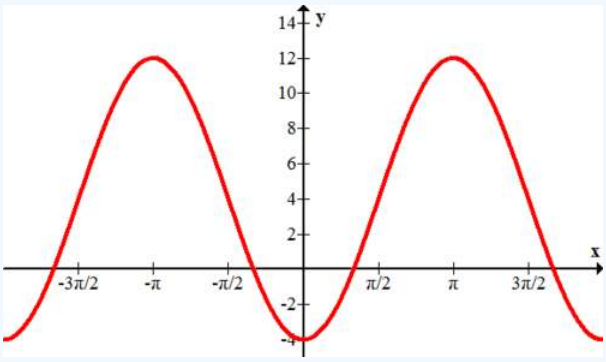
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14.



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5.5: Frequency and Period of Sinusoidal Functions

The transformation rules about horizontal stretching and shrinking directly apply to sine and cosine graphs. If a sine graph is horizontally stretched by a factor of $\frac{1}{2}$ that is the same as a horizontal compression by a factor of 2.

How does the equation change when a sine or cosine graph is stretched by a factor of 3?

Period and Frequency of Sinusoidal Functions

The general equation for a sinusoidal function is:

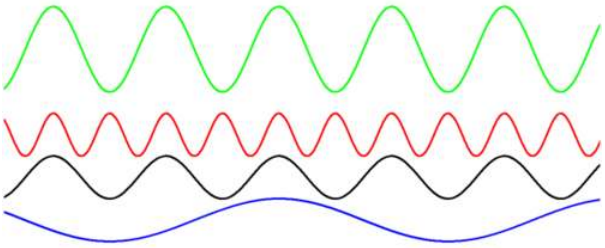
$f(x)=\pm a \cdot \sin(b(x+c))+d$

The $\pm$ controls the reflection across the x -axis. The coefficient a controls the amplitude. The constant d controls the vertical shift. Here you will see that the coefficient b controls the horizontal stretch.

Period

Horizontal stretch is measured for sinusoidal functions as their periods. This is why this function family is also called the periodic function family. The period of a sinusoid is the length of a complete cycle. For basic sine and cosine functions, the period is 2π. This length can be measured in multiple ways. In word problems and in other tricky circumstances, it may be most useful to measure from peak to peak.

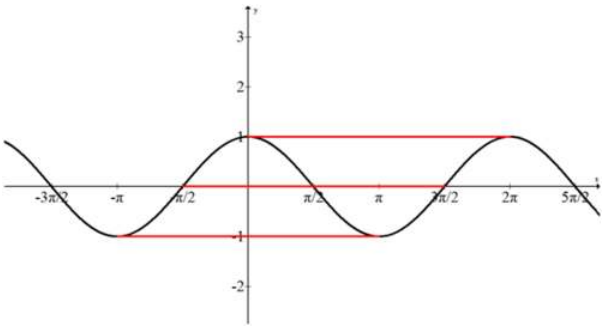
The following three waves have different periods. To rank each wave by period from shortest to longest, look at he distance between each peak.



The red wave has the shortest period.

The green and black waves have equal periods. Even though the green wave has greater amplitude than the black wave, they both have the same period.

The blue wave has the longest period.



The ability to measure the period of a function in multiple ways allows different equations to model an identical graph. In the image above, the top red line would represent a regular cosine wave. The center red line would represent a regular sine wave with a horizontal shift. The bottom red line would represent a negative cosine wave with a horizontal shift. This flexibility in perspective

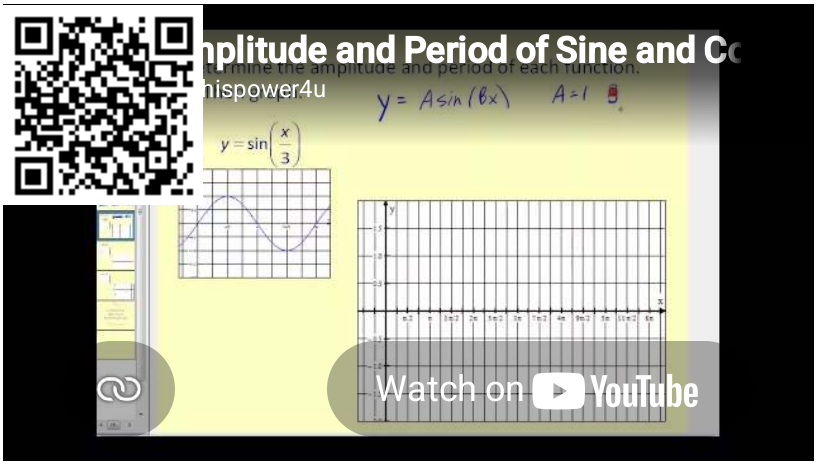
means that many of the examples, guided practice and practice problems may have multiple solutions. For now, try to always choose the function that has a period starting at $x = 0$.

Frequency

Frequency is a different way of measuring horizontal stretch. For sound, frequency is known as pitch. With sinusoidal functions, frequency is the number of cycles that occur in 2π. A shorter period means more cycles can fit in 2π and thus a higher frequency. Period and frequency are inversely related by the equation:

$$\text{period} = \frac{2\pi}{\text{frequency}}$$

The equation of a basic sine function is $f(x) = \sin x$. In this case b, the frequency, is equal to 1 which means one cycle occurs in 2π. If $b = \frac{1}{2}$, the period is $\frac{2\pi}{\frac{1}{2}}$ which means the period is 4π and the graph is stretched.



Examples

Example 1

Earlier, you were asked how an equation changes when a sine or cosine graph is stretched by a factor of 3 . If a sine graph is horizontally stretched by a factor of 3 then the general equation has $b = \frac{1}{3}$. This is because b is the frequency and counts the number (or fraction) of a period that fits in a normal period of 2π. Graphically, the sine wave will make a complete cycle in 6π. Similarly, a cosine graph will have $b = \frac{1}{3}$ and will have a period of 6π.

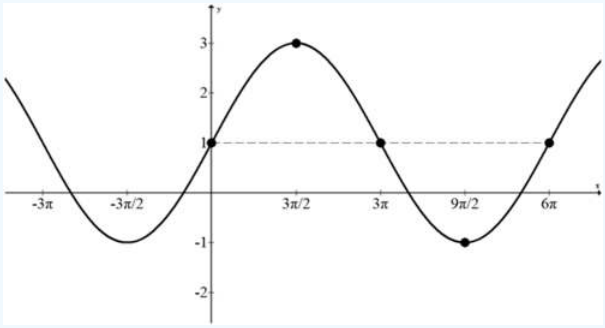
Example 2

Identify the amplitude, vertical shift, period and frequency of the following function. Then graph the function.

$f(x) = 2\sin\left(\frac{x}{3}\right) + 1$

$a = 2, b = \frac{1}{3}, d = 1$

The amplitude is 2 , the vertical shift is 1, and the frequency is $\frac{1}{3}$. The period would be $\frac{2\pi}{\frac{1}{3}}$, or 6π.



Often the most challenging part of graphing periodic functions is labeling the axes. since the period is 6π, start by drawing the sinusoidal axis shifted appropriately. Then divide the 6π into four parts so that the 5 guiding points of the sine graph can be plotted with the amplitude and reflection in mind. The very last thing to do is to draw and extend the curve.

Example 3

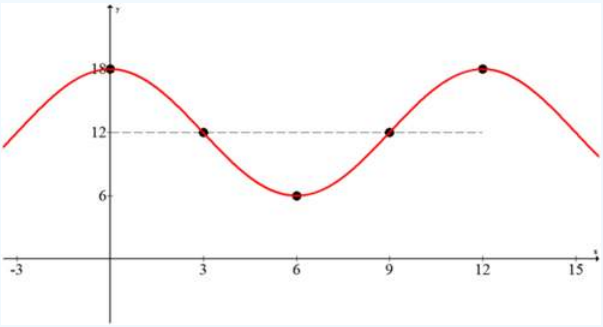
A measuring stick on a dock measures high tide to be 18 feet and low tide to be 6 feet. It takes about 6 hours for the tide to switch between low and high tides. Determine a graphical and algebraic model for the tides knowing that at $t = 0$ there is a high tide.

Usually the best course of action for word problems is to identify information, plot points, sketch and then finally come up with an equation.

From the given information you can deduce the following points. Notice how the sinusoidal axis can be assumed to be the average of the high and low tides.

Time (hours)	Water level (feet)
0	18
6	6
12	18
3	$\frac{18+6}{2} = 12$
9	12

By plotting those points and filling in the sinusoidal axis you can observe a cosine graph.



The amplitude is 6 so $a = 6$. There is no vertical reflection. since the period is 12 you can determine the frequency b :

$12 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{6}$

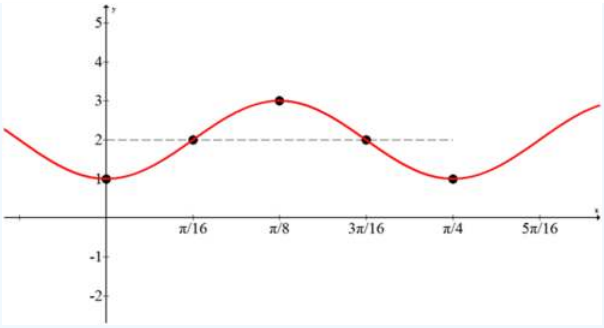
The vertical shift is 12 so $d = 12$. Thus you have all the pieces to make an algebraic model:

$f(x) = 6 \cdot \cos\left(\frac{\pi}{6}x\right) + 12$

Example 4

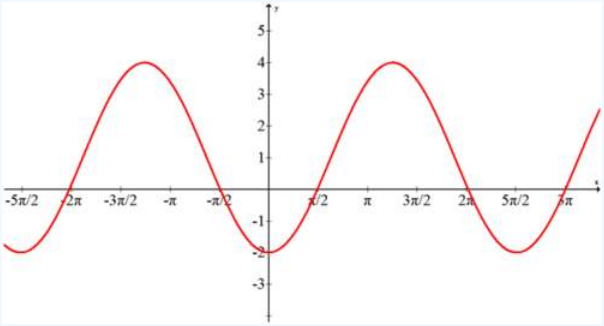
Graph the following function: $g(x) = -\cos(8x) + 2$

The labeling is the most important and challenging part of this problem. The amplitude is 1. The shape is a negative cosine. The vertical shift is 2. The period is $\frac{2\pi}{8} = \frac{\pi}{4}$



Example 5

Given the following graph, identify the amplitude, period, and frequency and create an algebraic model.



The amplitude is 3. The shape is a negative cosine. The period is $\frac{5\pi}{2}$ which implies that $b = \frac{4}{5}$. The vertical shift is 1. $f(x) = -3 \cdot \cos\left(\frac{4}{5}x\right) + 1$

Review

Find the frequency and period of each function below.

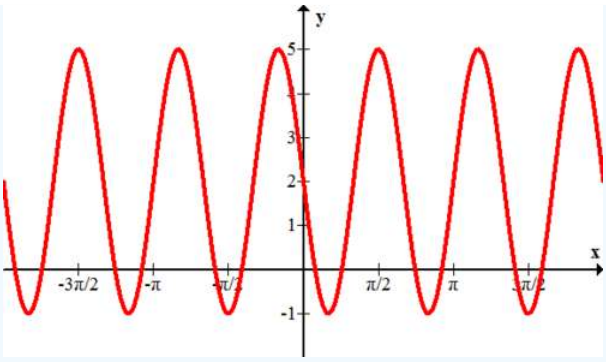
1. $f(x) = \sin(4x) + 1$
2. $g(x) = -3\cos(2x)$
3. $h(x) = \cos\left(\frac{1}{2}x\right) + 2$
4. $k(x) = -2\sin\left(\frac{3}{4}x\right) + 1$
5. $j(x) = 4\cos(3x) - 1$

Graph each of the following functions.

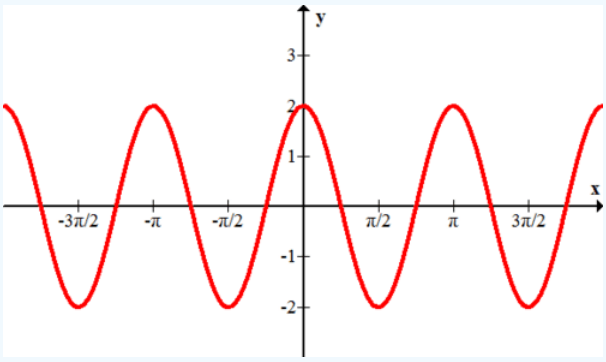
6. $f(x) = 3\sin(2x) + 1$
7. $g(x) = 2.5\cos(\pi x) - 4$
8. $h(x) = -\sin(4x) - 3$
9. $k(x) = \frac{1}{2}\cos(2x)$
10. $j(x) = -2\sin\left(\frac{3}{4}x\right) - 1$

Create an algebraic model for each of the following graphs.

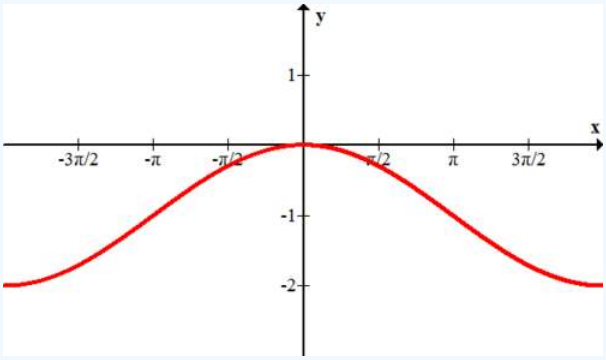
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14. At time 0 it is high tide and the water at a certain location is 10 feet high. At low tide 6 hours later, the water is 2 feet high. Given that tides can be modeled by sinusoidal functions, find a graph that models this scenario.

15. Find the equation that models the scenario in the previous problem.

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5.6: Phase Shift of Sinusoidal Functions

A periodic function that does not start at the sinusoidal axis or at a maximum or a minimum has been shifted horizontally. This horizontal movement allows for different starting points since a sine wave does not have a beginning or an end.

What are five other ways of writing the function $f(x) = 2 \cdot \sin x$?

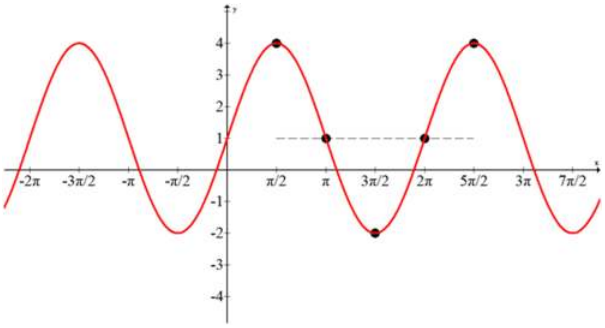
Phase Shift of Sinusoidal Functions

The general sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x + c)) + d$

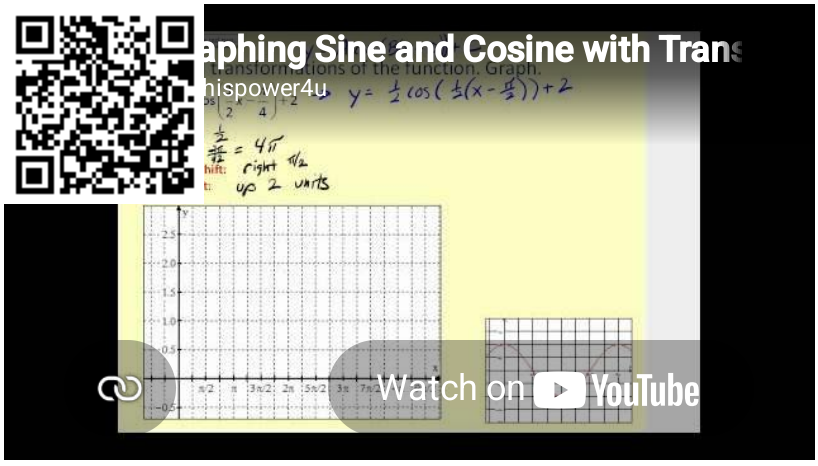
The constant c controls the phase shift. **Phase shift** is the horizontal shift left or right for periodic functions. If $c = \frac{\pi}{2}$ then the sine wave is shifted left by $\frac{\pi}{2}$. If $c = -3$ then the sine wave is shifted right by 3. This is the opposite direction than you might expect, but it is consistent with the rules of transformations for all functions.

To graph a function such as $f(x) = 3 \cdot \cos\left(x - \frac{\pi}{2}\right) + 1$, first find the start and end of one period. Then sketch only that portion of the sinusoidal axis. Finally, plot the 5 important points for a cosine graph while keeping the amplitude in mind. The graph is shown below.



Generally b is always written to be positive. If you run into a situation where b is negative, use your knowledge of even and odd functions to rewrite the function.

$\cos(-x) = \cos(x)$
$\sin(-x) = -\sin(x)$



Examples

Example 1

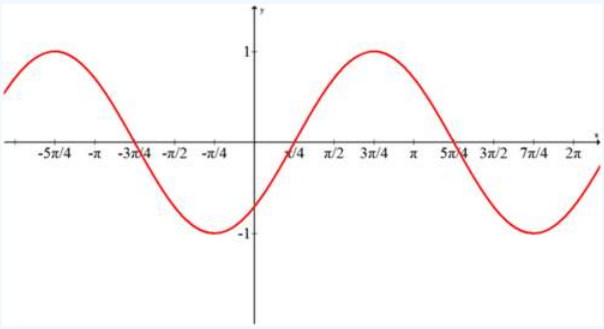
Earlier, you were asked to write $f(x) = 2 \cdot \sin x$ in five different ways. The function $f(x) = 2 \cdot \sin x$ can be rewritten an infinite number of ways.

$2 \cdot \sin x = -2 \cdot \cos\left(x + \frac{\pi}{2}\right) = 2 \cdot \cos\left(x - \frac{\pi}{2}\right) = -2 \cdot \sin(x - \pi) = 2 \cdot \sin(x - 8\pi)$

It all depends on where you choose start and whether you see a positive or negative sine or cosine graph.

Example 2

Given the following graph, identify equivalent sine and cosine algebraic models.



Either this is a sine function shifted right by $\frac{\pi}{4}$ or a cosine graph shifted left $\frac{5\pi}{4}$.

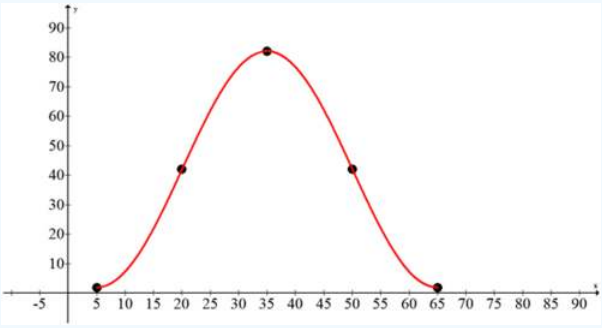
$f(x) = \sin\left(x - \frac{\pi}{4}\right) = \cos\left(x + \frac{5\pi}{4}\right)$

Example 3

At $t = 5$ minutes William steps up 2 feet to sit at the lowest point of the Ferris wheel that has a diameter of 80 feet. A full hour later he finally is let off the wheel after making only a single revolution. During that hour he wondered how to model his height over time in a graph and equation.

Since the period is 60 which works extremely well with the $360°$ in a circle, this problem will be shown in degrees.

Time (minutes)	Height (feet)
5	2
20	42
35	82
50	42
65	2



William chooses to see a negative cosine in the graph. He identifies the amplitude to be 40 feet. The vertical shift of the sinusoidal axis is 42 feet. The horizontal shift is 5 minutes to the right

The period is 60 (not 65) minutes which implies $b = 6$ when graphed in degrees.

$60 = \frac{360}{b}$

Thus one equation would be:

$f(x) = -40 \cdot \cos(6(x - 5)) + 42$

Example 4

Tide tables report the times and depths of low and high tides. Here is part of tide report from Salem, Massachusetts dated September 19, 2006.

10 : 15AM	9ft	High Tide
4 : 15PM	1ft.	Low Tide
10 : 15PM	9ft	High Tide

Find an equation that predicts the height based on the time. Choose when $t = 0$ carefully.

There are two logical places to set $t = 0$. The first is at midnight the night before and the second is at 10: 15 AM. The first option illustrates a phase shift that is the focus of this concept, but the second option produces a simpler equation. Set $t = 0$ to be at midnight and choose units to be in minutes.

Time (hours : minutes)	Time (minutes)	Tide (feet)
10 : 15	615	9
16 : 15	975	1
22 : 15	1335	9
	$\frac{615+975}{2} = 795$	5
	$\frac{1335+975}{2} = 1155$	5

These numbers seem to indicate a positive cosine curve. The amplitude is 4 and the vertical shift is 5. The horizontal shift is 615 and the period is 720.

$720 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{360}$

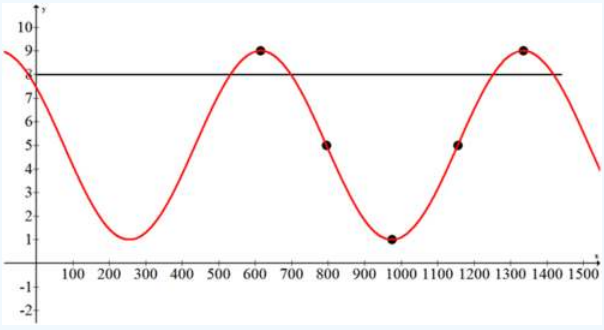
Thus one equation is:

$f(x) = 4 \cdot \cos\left(\frac{\pi}{360}(x - 615)\right) + 5$

Example 5

Use the equation from Example 4 to find out when the tide will be at exactly 8ft on September 19^{th}.

This problem gives you the y and asks you to find the x. Later you will learn how to solve this algebraically, but for now use the power of the intersect button on your calculator to intersect the function with the line $y = 8$. Remember to find all the x values between 0 and 1440 to account for the entire 24 hours.



There are four times within the 24 hours when the height is exactly 8 feet. You can convert these times to hours and minutes if you prefer.

$t \approx 532.18$ (8:52), 697.82 (11:34), 1252.18 (20:52), 1417.82 (23:38)

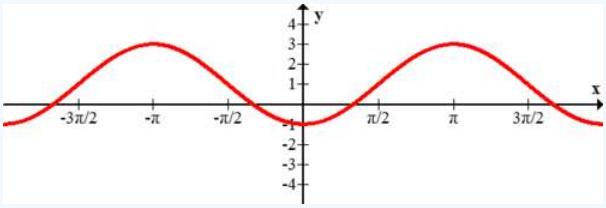
Review

Graph each of the following functions.

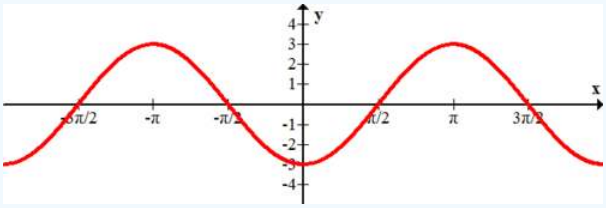
1. $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) - 1$
2. $g(x) = -\sin(x - \pi) + 3$
3. $h(x) = 3\cos(2(x - \pi))$
4. $k(x) = -2\sin(2x - \pi) + 1$
5. $j(x) = -\cos\left(x + \frac{\pi}{2}\right)$

Give one possible sine equation for each of the graphs below.

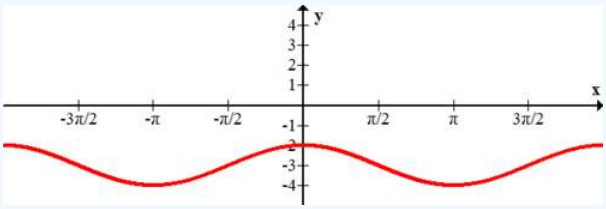
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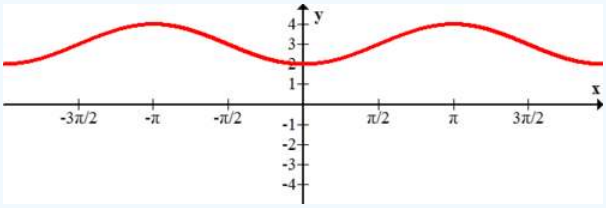


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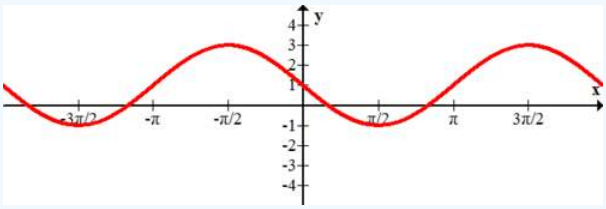


Give one possible cosine function for each of the graphs below.

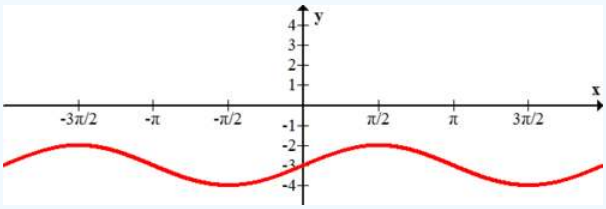
9.



10.



11.



The temperature over a certain 24 hour period can be modeled with a sinusoidal function. At 3: 00 , the temperature for the period reaches a low of $22°\mathrm{F}$. At $15:\mathrm{OO}$, the temperature for the period reaches a high of $40°F$

12. Find an equation that predicts the temperature based on the time in minutes. Choose $t = 0$ to be midnight.

13. Use the equation from #12 to predict the temperature at $4:00\mathrm{PM}$.

14. Use the equation from #12 to predict the temperature at 8: 00 AM.

15. Use the equation from #12 to predict the time(s) it will be $32°\mathrm{F}$.

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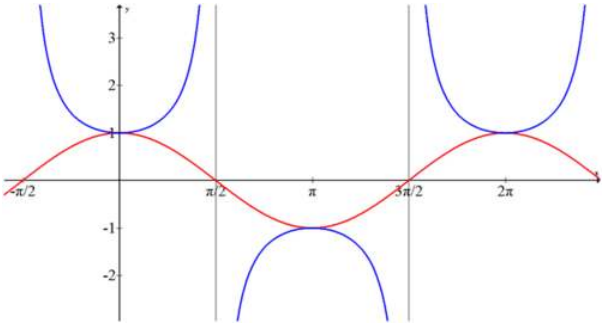
5.7: Graphs of Other Trigonometric Functions

If you already know the relationship between the equation and graph of sine and cosine functions then the other four functions can be found by identifying zeroes, asymptotes and key points. Are the four new functions transformations of the sine and cosine functions?

Graphing Other Trigonometric Functions

Secant and Cosecant

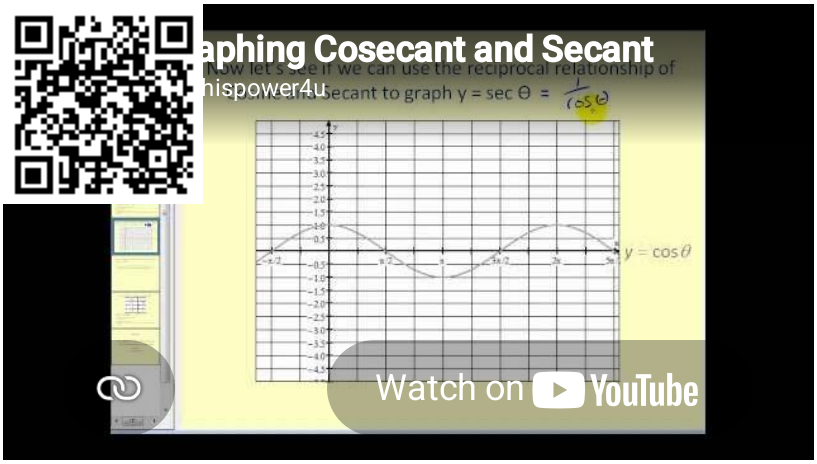
Since secant is the inverse of cosine the graphs are very closely related.



Notice wherever cosine is zero, secant has a vertical asymptote and where $\cos x = 1$ then sec $x = 1$ as well. These two logical pieces allow you to graph any secant function of the form:

$f(x) = \pm a \cdot \sec(b(x + c)) + d$

The method is to graph it as you would a cosine and then insert asymptotes and the secant curves so they touch the cosine curve at its maximum and minimum values. This technique is identical to graphing cosecant graphs. Simply use the sine graph to find the location and asymptotes.



Tangent and Cotangent

The tangent and cotangent graphs are more difficult because they are a ratio of the sine and cosine functions.

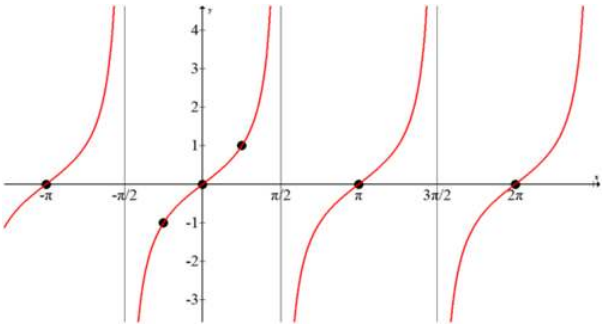
- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$

The way to think through the graph of $f(x) = \tan x$ is to first determine its asymptotes. The asymptotes occur when the denominator, cosine, is zero. This happens at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ ... The next thing to plot is the zeros which occur when the numerator, sine, is zero. This happens at $0, \pm\pi, \pm, 2\pi$ ... From the unit circle and basic right triangle trigonometry, you already know some values of $\tan x$

- $\tan \frac{\pi}{4} = 1$

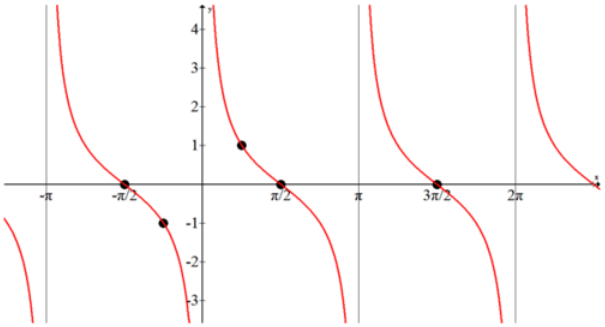
- $\tan\left(-\frac{\pi}{4}\right) = -1$

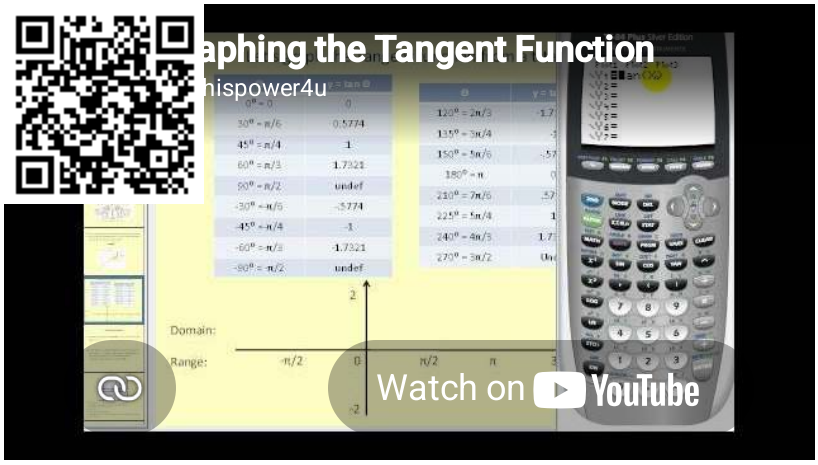
By plotting all this information, you get a very good sense as to what the graph of tangent looks like and you can fill in the rest.

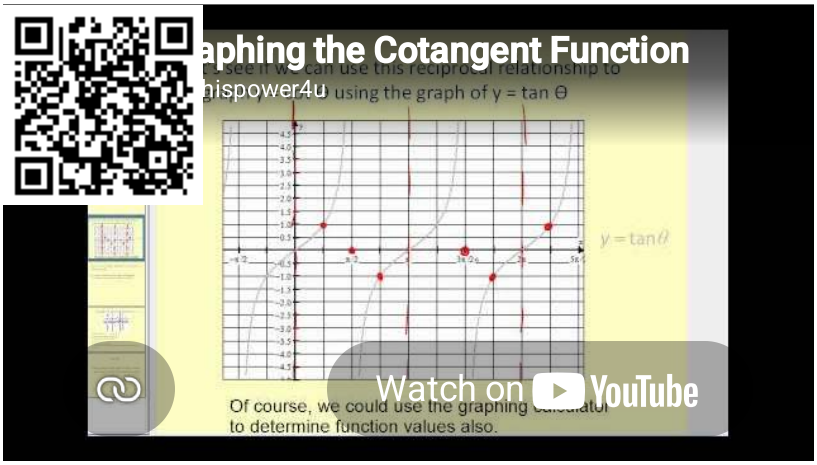


Notice that the period of tangent is π not 2π, because it has a shorter cycle.

The graph of cotangent can be found using identical logic as tangent. You know $\cot x = \frac{1}{\tan x}$ This means that the graph of cotangent will have zeros wherever tangent has asymptotes and asymptotes wherever tangent has zeroes. You also know that where tangent is 1, cotangent is also 1. Thus the graph of cotangent is:







Examples

Example 1

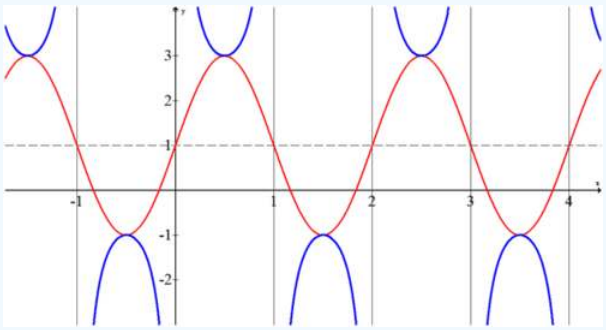
Earlier, you were asked if the four new functions are transformations of sine and cosine. The four new functions are not purely transformations of the sine and cosine functions. However, secant and cosecant are transformations of each other as are tangent and cotangent.

Example 2

Graph the function $f(x) = -2 \cdot \csc(\pi(x-1)) + 1$

Graph the function as if it were a sine function. Then insert asymptotes wherever the sine function crosses the sinusoidal axis. Lastly add in the cosecant curves.

The amplitude is 2 . The shape is negative sine. The function is shifted up one unit and to the right one unit.



Note that only the blue portion of the graph represents the given function.

Example 3

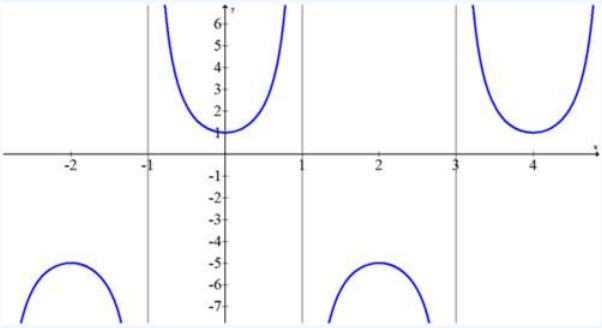
How do you write a tangent function as a cotangent function?

There are two main ways to go between a tangent function and a cotangent function. The first method was discussed in Example A: $f(x) = \tan x = \frac{1}{\cot x}$

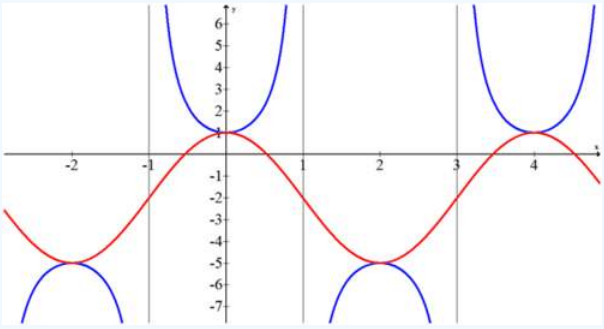
The second approach involves two transformations. Start by reflecting across the x or the y axis. Notice that this produces an identical result. Next shift the function to the right or left by $\frac{\pi}{2}$. Again this produces an identical result. $f(x) = \tan x = -\cot\left(x - \frac{\pi}{2}\right)$

Example 4

Find the equation of the function in the following graph.



If you connect the relative maximums and minimums of the function, it produces a shifted cosine curve that is easier to work with.



The amplitude is 3. The vertical shift is 2 down. The period is 4 which implies that $b = \frac{\pi}{2}$. The shape is positive cosine and if you choose to start at $x = 0$ there is no phase shift.

$f(x) = 3 \cdot \csc\left(\frac{\pi}{2}x\right) - 2$

Example 5

Where are the asymptotes for tangent and why do they occur?

Since $\tan x = \frac{\sin x}{\cos x}$ the asymptotes occur whenever $\cos x = 0$ which is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Review

1. What function can you use to help you make a sketch of $f(x) = \sec x$? Why?

2. What function can you use to help you make a sketch of $g(x) = \csc x$? Why?

Make a sketch of each of the following from memory.

3. $f(x) = \sec x$

4. $g(x) = \csc x$

5. $h(x) = \tan x$

6. $k(x) = \cot x$

Graph each of the following.

7. $f(x) = 2\csc(x) + 1$

8. $g(x) = 2\csc\left(\frac{\pi}{2}x\right) + 1$

9. $h(x) = 2\csc\left(\frac{\pi}{2}(x - 3)\right) + 1$

10. $j(x) = \cot\left(\frac{\pi}{2}x\right) + 3$
11. $k(x) = -\sec\left(\frac{\pi}{3}(x+1)\right) - 4$
12. $m(x) = -\tan(x) + 1$
13. $p(x) = -2\tan\left(x - \frac{\pi}{2}\right) + 1$
14. Find two ways to write $\sec x$ in terms of other trigonometric functions.
15. Find two ways to write $\csc x$ in terms of other trigonometric functions.

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5.8: Graphs of Inverse Trigonometric Functions

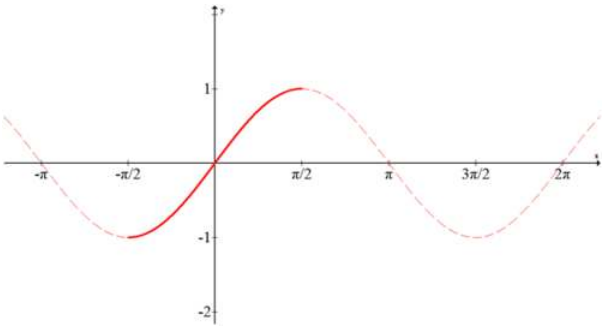
In order for inverses of functions to be functions, the original function must pass the **horizontal line test**. Though none of the trigonometric functions pass the horizontal line test, you can restrict their domains so that they can pass. Then the inverses are produced just like with normal functions. Once you have the basic inverse functions, the normal transformation rules apply.

Why is $\sin^{-1}(\sin 370°) \neq 370°$? Don't the arcsin and sin just cancel out?

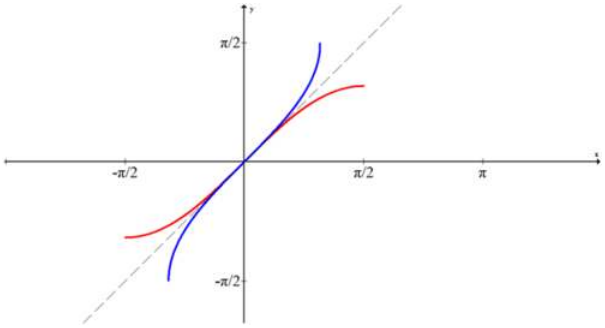
Graphs of Inverse Trigonometric Functions

Since none of the six trigonometric functions pass the horizontal line test, you must **restrict their domains** before finding inverses of these functions. This is just like the way $y = \sqrt{x}$ is the inverse of $y = x^2$ when you restrict the domain to $x \geq 0$

Consider the sine graph:

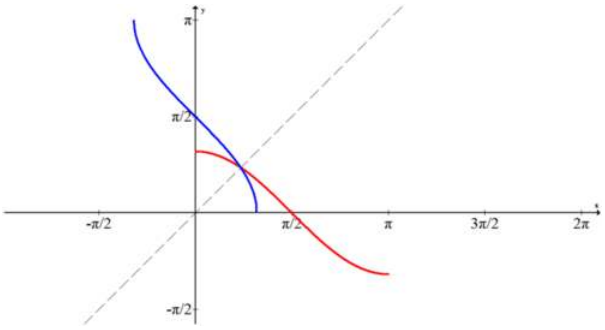


As a general rule, the restrictions to the domain are either the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ or $[0, \pi]$ to keep things simple. In this case sine is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, as shown above. To find the inverse, reflect the bold portion across the line $y = x$. The blue curve below shows $f(x) = \sin^{-1} x$.

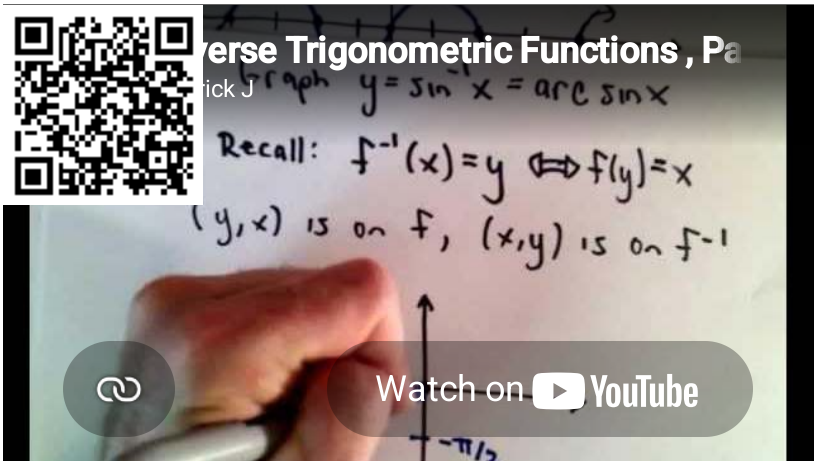


The result of this inversion is that arcsine will only ever produce angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The blue curve below shows $f(x) = \cos^{-1} x$?



The portion of cosine that fits the horizontal line test is the interval $[0, \pi]$. To find the inverse, that portion is reflected across the line $y = x$.



Examples

Example 1

Earlier, you were asked why sine and arcsine don't always just cancel out. since arcsine only produces angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ or $-90°$ to $+90°$ the result of $\sin^{-1}(\sin 370°)$ is $10°$ which is coterminal to $370°$.

Example 2

Graph the function $f(x) = 2\cos^{-1}(x-1)$

Since $f(x) = \cos^{-1} x$ was graphed earlier, now you just need to shift it right one unit and stretch it vertically by a factor of 2 . It intersected the x axis at 1 before and now it will intersect at 2 . It reached a height of π before and now it will reach a height of 2π.

Example 3

Evaluate the following expression with and without a calculator using right triangles and your knowledge of inverse trigonometric functions.

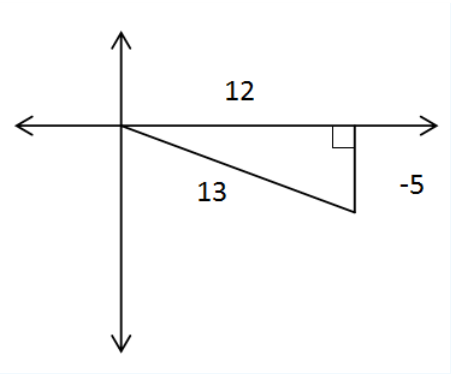
$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)$

In order to be able to effectively calculate this out it is best to write the expression explicitly only in terms of functions that your calculator does have. Be aware that some calculators have both $\sin^{-1} x$ and $(\sin x)^{-1}$

- $\csc^{-1}\left(-\frac{13}{5}\right) = \sin^{-1}\left(-\frac{5}{13}\right)$
- $\cot(\theta) = \frac{1}{\tan\theta}$

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = \frac{1}{\tan\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)} = -\frac{12}{5}$

Start with your knowledge that $\csc^{-1}\left(-\frac{13}{5}\right)$ describes an angle in the third or the fourth quadrant because those are the two quadrants where cosecant is negative. since $\csc^{-1}\theta$ has range $-\frac{\pi}{2}, \frac{\pi}{2}$, it only produces angles in quadrant I or quadrant IV. This triangle must then be in the fourth quadrant. All you need to do is draw the triangle and identify the cotangent ratio.



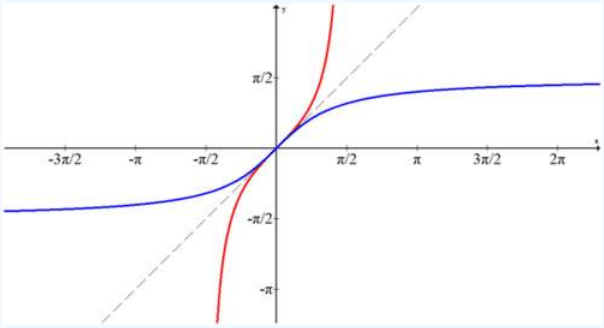
Cotangent is adjacent over opposite.

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = -\frac{12}{5}$

Example 4

What is the graph of $y = \tan^{-1} x$?

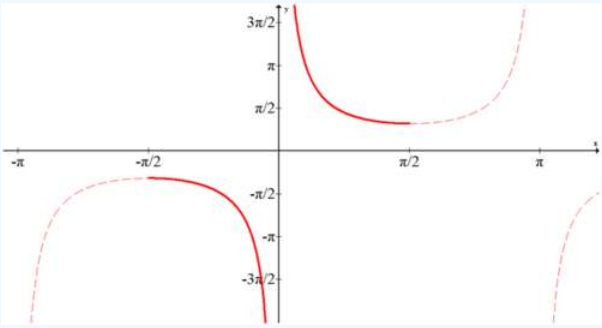
Graph the portion of tangent that fits the horizontal line test and reflect across the line $y = x$. Note that the graph of arctan is in blue.

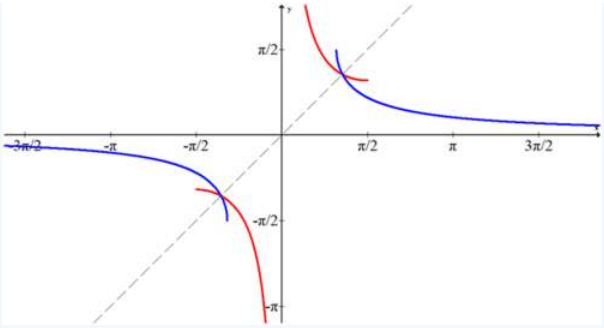


Example 5

What is the graph of $y = \csc^{-1} x$

Graph the portion of cosecant that fits the horizontal line test and reflect across the line $y = x$.





Note that $f(x) = \csc^{-1} x$ is in blue.

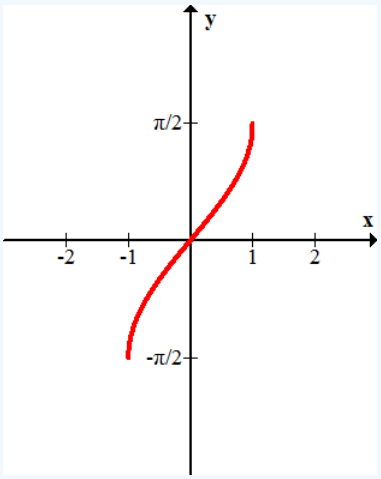
Review

1. Graph $f(x) = \cot^{-1} x$.

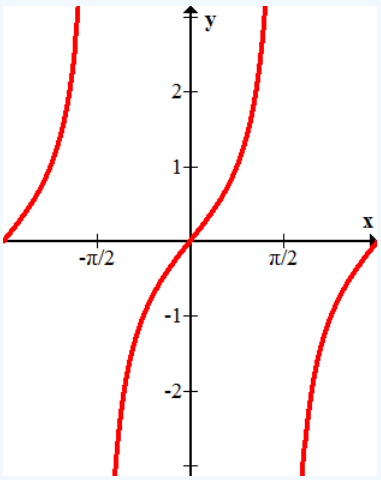
2. Graph $g(x) = \sec^{-1} x$.

Name each of the following graphs.

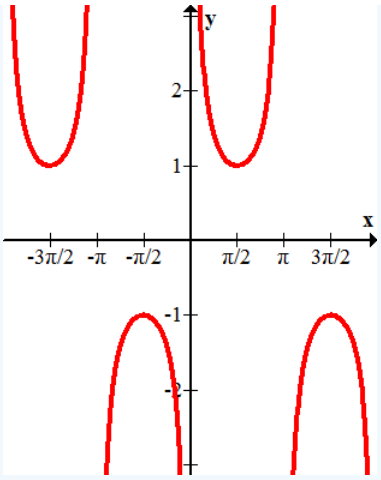
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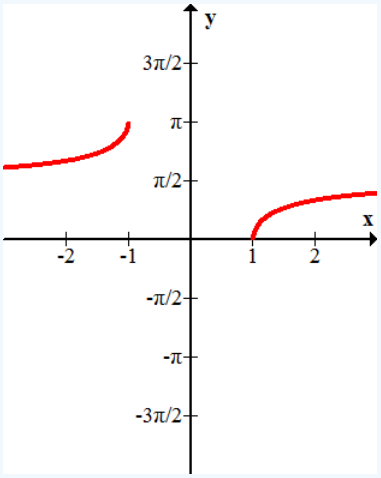
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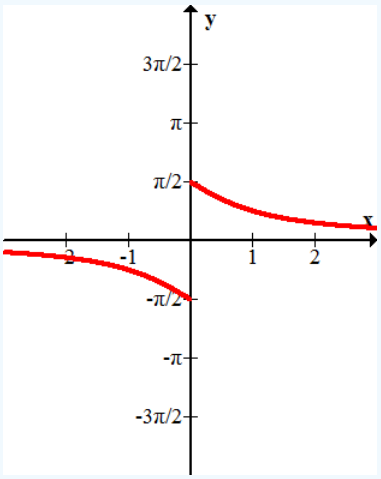
5.



6.



7.



Graph each of the following functions using your knowledge of function transformations.

8. $h(x) = 3\sin^{-1}(x+1)$

9. $k(x) = 2\sin^{-1}(x) + \frac{\pi}{2}$

10. $m(x) = -\cos^{-1}(x-2)$

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11. $j(x) = \cot^{-1}(x) + \pi$
12. $p(x) = -2\tan^{-1}(x - 1)$
13. $q(x) = \csc^{-1}(x - 2)$
14. $r(x) = -\sec^{-1}(x) + 4$
15. $t(x) = \csc^{-1}(x + 1) - \frac{3\pi}{2}$
16. $v(x) = 2\sec^{-1}(x + 2) + \frac{\pi}{2}$
17. $w(x) = -\cot^{-1}(x) - \frac{\pi}{2}$

Evaluate each expression.

18. $\sec\left(\tan^{-1}\left[\frac{3}{4}\right]\right)$
19. $\cot\left(\csc^{-1}\left[\frac{13}{12}\right]\right)$
20. $\csc\left(\tan^{-1}\left[\frac{4}{3}\right]\right)$

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CHAPTER OVERVIEW

6: Analytic Trigonometry

6.1: Basic Trigonometric Identities

6.2: Pythagorean Identities

6.3: Sum and Difference Identities

6.4: Double, Half, and Power Reducing Identities

6.5: Trigonometric Equations

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6.1: Basic Trigonometric Identities

The basic trigonometric identities are ones that can be logically deduced from the definitions and graphs of the six trigonometric functions. Previously, some of these identities have been used in a casual way, but now they will be formalized and added to the toolbox of trigonometric identities.

How can you use the trigonometric identities to simplify the following expression?

$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$

Trigonometric Identities

An identity is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side.

Reciprocal Identities

The **reciprocal identities** refer to the connections between the trigonometric functions like sine and cosecant. Sine is opposite over hypotenuse and cosecant is hypotenuse over opposite. This logic produces the following six identities.

- $\sin\theta = \frac{1}{\csc\theta}$
- $\cos\theta = \frac{1}{\sec\theta}$
- $\tan\theta = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

Quotient Identities

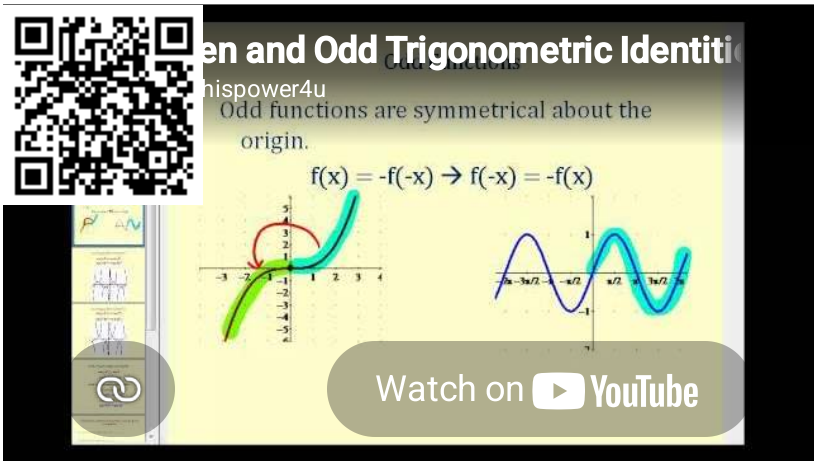
The **quotient identities** follow from the definition of sine, cosine and tangent.

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Odd/Even Identities

The **odd-even identities** follow from the fact that only cosine and its reciprocal secant are even and the rest of the trigonometric functions are odd.

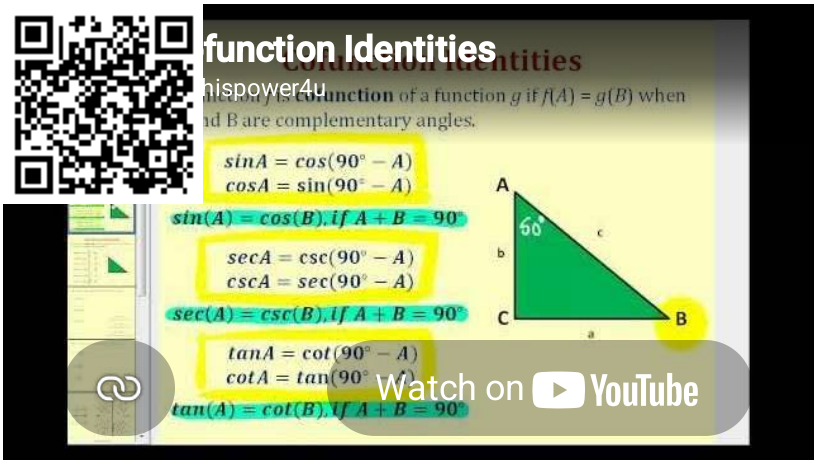
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\csc(-\theta) = -\csc\theta$



Cofunction Identities

The **cofunction identities** make the connection between trigonometric functions and their "co" counterparts like sine and cosine. Graphically, all of the cofunctions are reflections and horizontal shifts of each other.

- $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$
- $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$
- $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$
- $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$
- $\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$
- $\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$



Examples

Example 1

Earlier, you were asked how you could simplify the trigonometric expression:
$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$

It can be simplified to be equivalent to negative tangent as shown below:

$$\begin{aligned}\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1} &= \frac{\sin(-\theta)}{\sin\left(\frac{\pi}{2}-\theta\right)}\\ &= \frac{-\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

Example 2

If $\sin\theta = 0.87$, find $\cos\left(\theta - \frac{\pi}{2}\right)$

While it is possible to use a calculator to find θ, using identities works very well too.

First you should factor out the negative from the argument. Next you should note that cosine is even and apply the odd-even identity to discard the negative in the argument. Lastly recognize the cofunction identity.

$$\cos\left(\theta - \tfrac{\pi}{2}\right) = \cos\left(-\left(\tfrac{\pi}{2} - \theta\right)\right) = \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta = 0.87$$

Example 3

If $\cos\left(\theta - \frac{\pi}{2}\right) = 0.68$ then determine $\csc(-\theta)$

You need to show that $\cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - \theta\right)$

$$\begin{aligned} 0.68 &= \cos\left(\theta - \frac{\pi}{2}\right) \\ &= \cos\left(\frac{\pi}{2} - \theta\right) \\ &= \sin(\theta) \end{aligned}$$

Then, $\csc(-\theta) = -\csc\theta$

$$\begin{aligned} &= -\tfrac{1}{\sin\theta} \\ &= -(0.68)^{-1} \\ &\approx -1.47 \end{aligned}$$

Example 4

Use identities to prove the following: $\cot(-\beta)\cot\left(\frac{\pi}{2} - \beta\right)\sin(-\beta) = \cos\left(\beta - \frac{\pi}{2}\right)$.

When doing trigonometric proofs, it is vital that you start on one side and only work with that side until you derive what is on the other side. Sometimes it may be helpful to work from both sides and find where the two sides meet, but this work is not considered a proof. You will have to rewrite your steps so they follow from only one side. In this case, work with the left side and keep rewriting it until you have $\cos\left(\beta - \frac{\pi}{2}\right)$

$$\begin{aligned} \cot(-\beta)\cot\left(\frac{\pi}{2} - \beta\right)\sin(-\beta) &= -\cot\beta\tan\beta \cdot -\sin\beta \\ &= -1 \cdot -\sin\beta \\ &= \sin\beta \\ &= \cos\left(\frac{\pi}{2} - \beta\right) \\ &= \cos\left(-\left(\beta - \frac{\pi}{2}\right)\right) \\ &= \cos\left(\beta - \frac{\pi}{2}\right) \end{aligned}$$

Example 5

Prove the following trigonometric identity by working with only one side.

$$\cos x \sin x \tan x \cot x \sec x \csc x = 1$$

$$\begin{aligned} \cos x \sin x \tan x \cot x \sec x \csc x &= \cos x \sin x \tan x \cdot \frac{1}{\tan x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\sin x} \\ &= 1 \end{aligned}$$

Review

1. Prove the quotient identity for cotangent using sine and cosine.
2. Explain why $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ using graphs and transformations.
3. Explain why $\sec\theta = \frac{1}{\cos\theta}$

4. Prove that $\tan\theta \cdot \cot\theta = 1$.

5. Prove that $\sin\theta \cdot \csc\theta = 1$.

6. Prove that $\sin\theta \cdot \sec\theta = \tan\theta$

7. Prove that $\cos\theta \cdot \csc\theta = \cot\theta$

8. If $\sin\theta = 0.81$, what is $\sin(-\theta)$?

9. If $\cos\theta = 0.5$, what is $\cos(-\theta)$?

10. If $\cos\theta = 0.25$, what is $\sec(-\theta)$?

11. If $\csc\theta = 0.7$, what is $\sin(-\theta)$?

12. How can you tell from a graph if a function is even or odd?

13. Prove $\frac{\tan x \cdot \sec x}{\csc x} \cdot \cot x = \tan x$

14. Prove $\frac{\sin^2 x \cdot \sec x}{\tan x} \cdot \csc x = 1$.

15. Prove $\cos x \cdot \tan x = \sin x$

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6.2: Pythagorean Identities

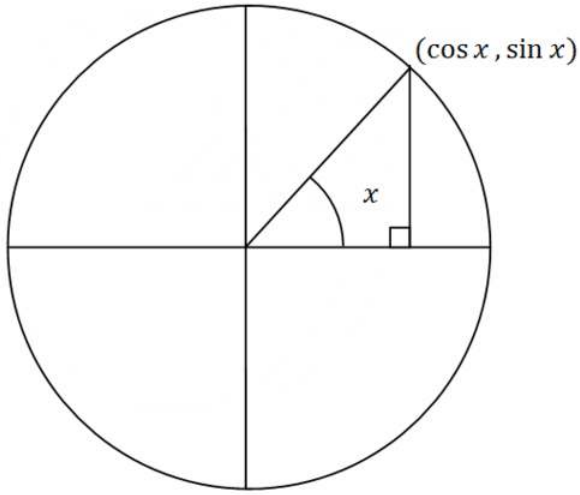
The Pythagorean Theorem works on right triangles. If you consider the x coordinate of a point along the unit circle to be the cosine and the y coordinate of the point to be the sine and the distance to the origin to be 1 then the Pythagorean Theorem immediately yields the identity:

$$y^2 + x^2 = 1$$
$$\sin^2 x + \cos^2 x = 1$$

An observant student may guess that other Pythagorean identities exist with the rest of the trigonometric functions. Is $\tan^2 x + \cot^2 x = 1$ a legitimate identity?

Pythagorean Identities

The proof of the Pythagorean identity for sine and cosine is essentially just drawing a right triangle in a unit circle, identifying the cosine as the x coordinate, the sine as the y coordinate and 1 as the hypotenuse.



$$\cos^2 x + \sin^2 x = 1$$

or

$$\sin^2 x + \cos^2 x = 1$$

The two other Pythagorean identities are:

- $1 + \cot^2 x = \csc^2 x$
- $\tan^2 x + 1 = \sec^2 x$

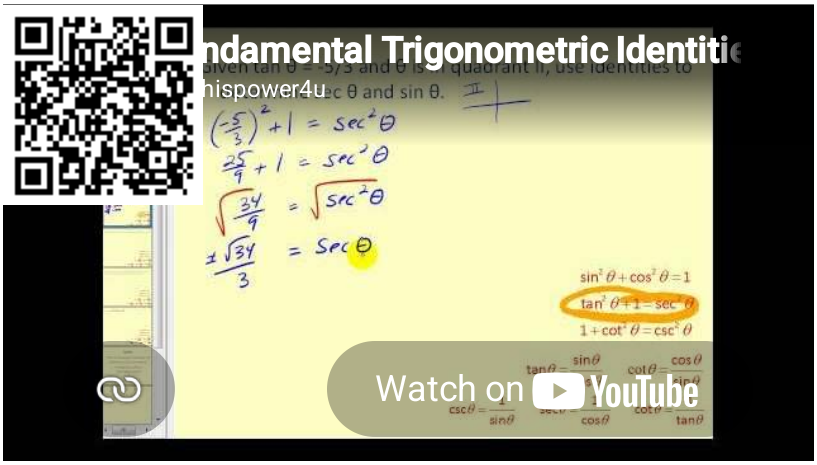
To derive these two Pythagorean identities, divide the original Pythagorean identity by $\sin^2 x$ and $\cos^2 x$ respectively.

To derive the Pythagorean identity $1 + \cot^2 x = \csc^2 x$ divide through by $\sin^2 x$ and simplify.

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$
$$1 + \cot^2 x = \csc^2 x$$

Similarly, to derive the Pythagorean identity $\tan^2 x + 1 = \sec^2 x$, divide through by $\cos^2 x$ and simplify.

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \tag{6.2.1}$$
$$\tan^2 x + 1 = \sec^2 x$$



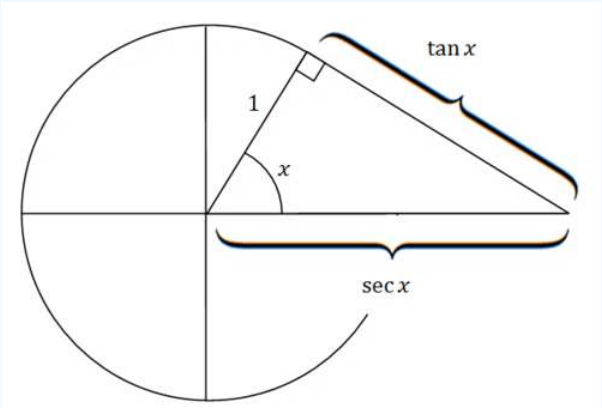
Examples

Example 1

Earlier, you were asked if $\tan^2 x + \cot^2 x = 1$ is a legitimate identity. Cofunctions are not always connected directly through a Pythagorean identity.

$\tan^2 x + \cot^2 x \neq 1$

Visually, the right triangle connecting tangent and secant can also be observed in the unit circle. Most people do not know that tangent is named "tangent" because it refers to the distance of the line tangent from the point on the unit circle to the x axis. Look at the picture below and think about why it makes sense that $\tan x$ and $\sec x$ are as marked. $\tan x = \frac{opp}{adj}$. since the adjacent side is equal to 1 (the radius of the circle), tan x simply equals the opposite side. Similar logic can explain the placement of $\sec x$.



Example 2

Simplify the following expression: $\frac{\sin x(\csc x - \sin x)}{1 - \sin x}$

$$\begin{aligned}\frac{\sin x(\csc x - \sin x)}{1 - \sin x} &= \frac{\sin x \cdot \csc x - \sin^2 x}{1 - \sin x} \\ &= \frac{1 - \sin^2 x}{1 - \sin x} \\ &= \frac{(1 - \sin x)(1 + \sin x)}{1 - \sin x} \\ &= 1 + \sin x\end{aligned}$$

Note that factoring the Pythagorean identity is one of the most powerful and common applications.

Example 3

Prove the following trigonometric identity. $(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) = 2$

Group the terms and apply a different form of the second two Pythagorean identities which are $1 + \cot^2 x = \csc^2 x$ and $\tan^2 x + 1 = \sec^2 x$

$$\begin{aligned}(\sec^2 x + \csc^2 x) - (\tan^2 x + \cot^2 x) &= \sec^2 x - \tan^2 x + \csc^2 x - \cot^2 x \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Example 4

Simplify the following expression. Note: $\sec^2 x = \frac{1}{\cos^2 x}$

$$(\sec^2 x)(1 - \sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right)$$

$$\begin{aligned}&(\sec^2 x)(1 - \sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right) \\ &= \sec^2 x \cdot \cos^2 x - (\sin^2 x + \cos^2 x) \\ &= 1 - 1 \\ &= 0\end{aligned}$$

Example 5

Simplify the following expression.

$(\cos t - \sin t)^2 + (\cos t + \sin t)^2$

Note that initially, the expression is not the same as the Pythagorean identity.

$$\begin{aligned}&(\cos t - \sin t)^2 + (\cos t + \sin t)^2 \\ &= \cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t \\ &= 1 - 2\cos t \sin t + 1 + 2\cos t \sin t \\ &= 2\end{aligned}$$

Review

Prove each of the following:

1. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$
2. $\cos x (1 - \sin^2 x) = \cos^3 x$
3. $\sin^2 x = (1 - \cos x)(1 + \cos x)$
4. $\sin x = \frac{\sin^2 x + \cos^2 x}{\csc x}$
5. $\sin^4 x - \cos^4 x = \sin^2 x - \cos^2 x$
6. $\sin^2 x \cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$

Simplify each expression as much as possible.

7. $\tan^3 x \csc^3 x$
8. $\frac{\csc^2 x - 1}{\sec^2 x}$
9. $\frac{1 - \sin^2 x}{1 + \sin x}$
10. $\sqrt{1 - \cos^2 x}$
11. $\frac{\sin^2 x - \sin^4 x}{\cos^2 x}$
12. $(1 + \tan^2 x)(\sec^2 x)$
13. $\frac{\sin^2 x + \tan^2 x + \cos^2 x}{\sec x}$

14. $\frac{1+\tan^2 x}{\csc^2 x}$
15. $\frac{1-\sin^2 x}{\cos x}$

...

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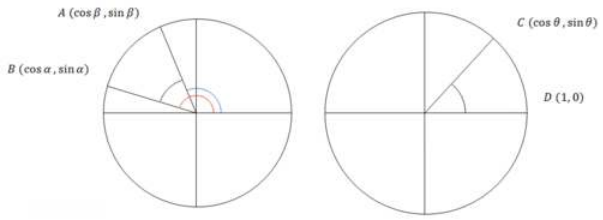
6.3: Sum and Difference Identities

With your knowledge of special angles like the sine and cosine of $30°$ and $45°$, you can find the sine and cosine of $15°$, the difference of $45°$ and $30°$, and $75°$, the sum of $45°$ and $30°$. Using what you know about the unit circle and the sum and difference identities, how do you determine $\sin 15°$ and $\sin 75°$?

Sum and Difference Identities

First look at the derivation of the **cosine difference identity**:

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$



Start by drawing two arbitrary angles α and β. In the image above α is the angle in red and β is the angle in blue. The difference $\alpha - \beta$ is noted in black as θ. The reason why there are two pictures is because the image on the right has the same angle θ in a rotated position. This will be useful to work with because the length of $\overline{AB}$ will be the same as the length of $\overline{CD}$.

$$\begin{aligned}
\overline{AB} &= \overline{CD} \\
\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} &= \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2} \\
(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2
\end{aligned}$$

$$(\cos\alpha)^2 - 2\cos\alpha\cos\beta + (\cos\beta)^2 + (\sin\alpha)^2 - 2\sin\alpha\sin\beta + (\sin\beta)^2 = (\cos\theta - 1)^2 + (\sin\theta)^2$$

$$\begin{aligned}
2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= (\cos\theta)^2 - 2\cos\theta + 1 + (\sin\theta)^2 \\
2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 1 - 2\cos\theta + 1 \\
-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= -2\cos\theta \\
\cos\alpha\cos\beta + \sin\alpha\sin\beta &= \cos\theta \\
&= \cos(\alpha - \beta)
\end{aligned}$$

You can use this identity to prove the **cosine of a sum identity**. First, start with the cosine of a difference and make a substitution. Then use the odd-even identity.

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$$

Let $\gamma = -\beta$

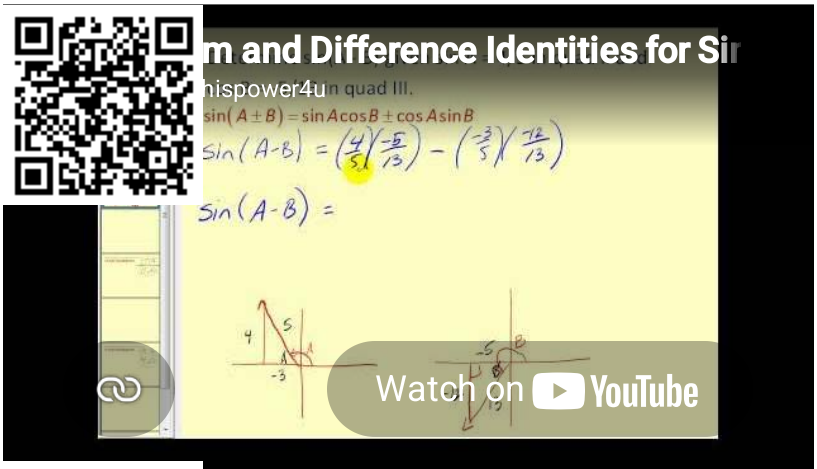
$$\cos\alpha\cos(-\gamma) + \sin\alpha\sin(-\gamma) = \cos(\alpha + \gamma)$$
$$\cos\alpha\cos\gamma - \sin\alpha\sin\gamma = \cos(\alpha + \gamma)$$

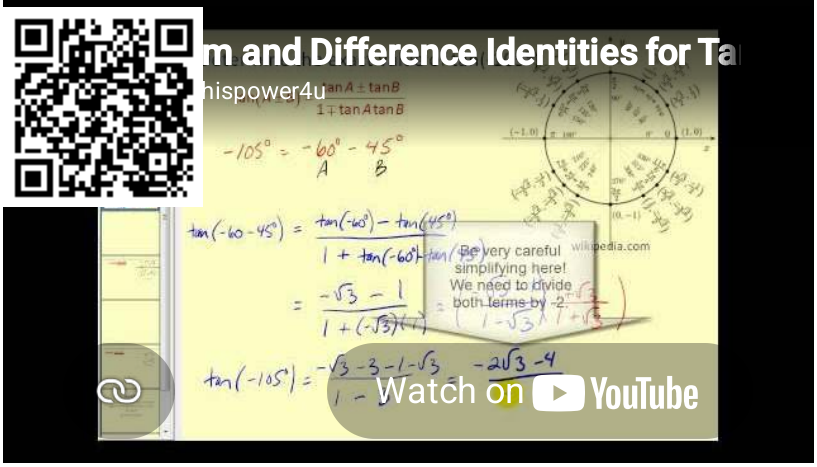
The proofs for sine and tangent are left to the videos and examples. They are listed here for your reference. Cotangent, secant and cosecant are excluded because you can use reciprocal identities to get those once you have sine, cosine and tangent.

Summary

- $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

The order of the plus or minus signs is important because for cosine of a sum, the negative sign is used on the other side of the identity. This is the opposite of sine of a sum, where a positive sign is used on the other side of the identity.





Examples

Example 1

Earlier, you were asked to evaluate $\sin 15^\circ$ and $\sin 75^\circ$ exactly without a calculator. To do this you need to use the sine of a difference and sine of a sum.

$$\sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 2

Find the exact value of $\tan 15^\circ$ without using a calculator.

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \frac{3-\sqrt{3}}{3+\sqrt{3}}$$

A final solution will not have a radical in the denominator. In this case multiplying through by the conjugate of the denominator will eliminate the radical.

$$= \frac{(3-\sqrt{3}) \cdot (3-\sqrt{3})}{(3+\sqrt{3}) \cdot (3-\sqrt{3})}$$
$$= \frac{(3-\sqrt{3})^2}{9-3}$$
$$= \frac{(3-\sqrt{3})^2}{6}$$

Example 3

Evaluate the expression exactly without using a calculator.

$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ$

Once you know the general form of the sum and difference identities then you will recognize this as cosine of a difference.

$\cos 50^\circ \cos 5^\circ + \sin 50^\circ \sin 5^\circ = \cos(50^\circ - 5^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$

Example 4

Use a sum or difference identity to find an exact value of $\cot\left(\frac{5\pi}{12}\right)$.

Start with the definition of cotangent as the inverse of tangent.

$$\begin{aligned}
\cot\left(\frac{5\pi}{12}\right) &= \frac{1}{\tan\left(\frac{5\pi}{12}\right)} \\
&= \frac{1}{\tan\left(\frac{9\pi}{12} - \frac{4\pi}{12}\right)} \\
&= \frac{1}{\tan(135^\circ - 60^\circ)} \\
&= \frac{1 + \tan 135^\circ \tan 60^\circ}{\tan 135^\circ - \tan 60^\circ} \\
&= \frac{1 + (-1)\cdot\sqrt{3}}{(-1) - \sqrt{3}} \\
&= \frac{(1-\sqrt{3})}{(-1-\sqrt{3})} \\
&= \frac{(1-\sqrt{3})^2}{(-1+\sqrt{3})\cdot(1-\sqrt{3})} \\
&= \frac{(1-\sqrt{3})^2}{-(1-3)} \\
&= \frac{(1-\sqrt{3})^2}{2}
\end{aligned}$$

Example 5

Prove the following identity:

$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

Here are the steps:

$$\begin{aligned}
\frac{\sin(x-y)}{\sin(x+y)} &= \frac{\tan x - \tan y}{\tan x + \tan y} \\
\frac{\sin x\cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} &= \\
\frac{\sin x\cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} \cdot \frac{\left(\frac{1}{\cos x \cdot \cos y}\right)}{\left(\frac{1}{\cos x \cdot \cos y}\right)} &= \\
\frac{\left(\frac{\sin x}{\cos x \cdot \cos y}\right) - \left(\frac{\cos x \sin y}{\cos x \cdot \cos y}\right)}{\left(\frac{\sin x \cos x}{\cos x \cdot \cos y}\right) + \left(\frac{\cos x}{\cos x \cdot \cos y}\right)} &= \\
\frac{\tan x - \tan y}{\tan x + \tan y} &=
\end{aligned}$$

Review

Find the exact value for each expression by using a sum or difference identity.

1. $\cos 75^\circ$

2. $\cos 105^\circ$

3. $\cos 165^\circ$

4. $\sin 105^\circ$

5. $\sec 105^\circ$

6. $\tan 75^\circ$

7. Prove the sine of a sum identity.

8. Prove the tangent of a sum identity.

9. Prove the tangent of a difference identity.

10. Evaluate without a calculator: $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$.

11. Evaluate without a calculator: $\sin 35^\circ \cos 5^\circ - \cos 35^\circ \sin 5^\circ$.

12. Evaluate without a calculator: $\sin 55^\circ \cos 5^\circ + \cos 55^\circ \sin 5^\circ$.

13. If $\cos \alpha \cos \beta = \sin \alpha \sin \beta$, then what does $\cos(\alpha + \beta)$ equal?

14. Prove that $\tan\left(x + \frac{\pi}{4}\right) = \frac{1+\tan x}{1-\tan x}$

15. Prove that $\sin(x + \pi) = -\sin x$.

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6.4: Double, Half, and Power Reducing Identities

These identities are significantly more involved and less intuitive than previous identities. By practicing and working with these advanced identities, your toolbox and fluency substituting and proving on your own will increase. Each identity in this concept is named aptly. Double angles work on finding $\sin 80^\circ$ if you already know $\sin 40^\circ$. Half angles allow you to find $\sin 15^\circ$ if you already know $\sin 30^\circ$. Power reducing identities allow you to find $\sin^2 15^\circ$ if you know the sine and cosine of 30°

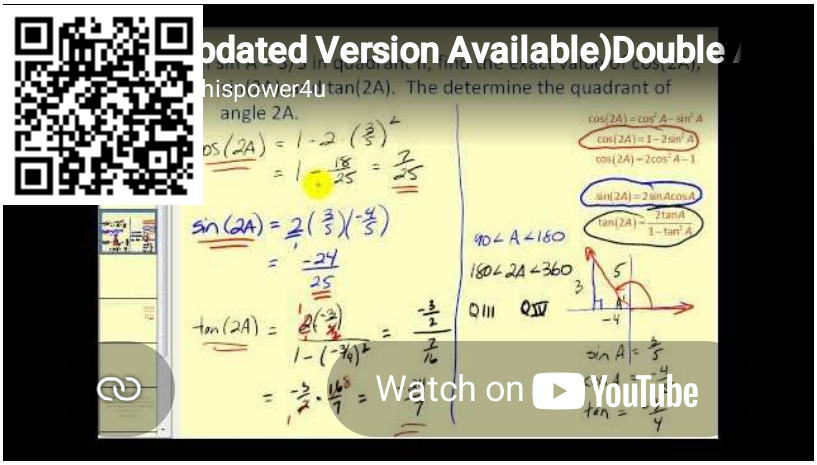
What is $\sin^2 15^\circ$?

Double Angle, Half Angle, and Power Reducing Identities

Double Angle Identities

The **double angle identities** are proved by applying the sum and difference identities. They are left as review problems. These are the double angle identities.

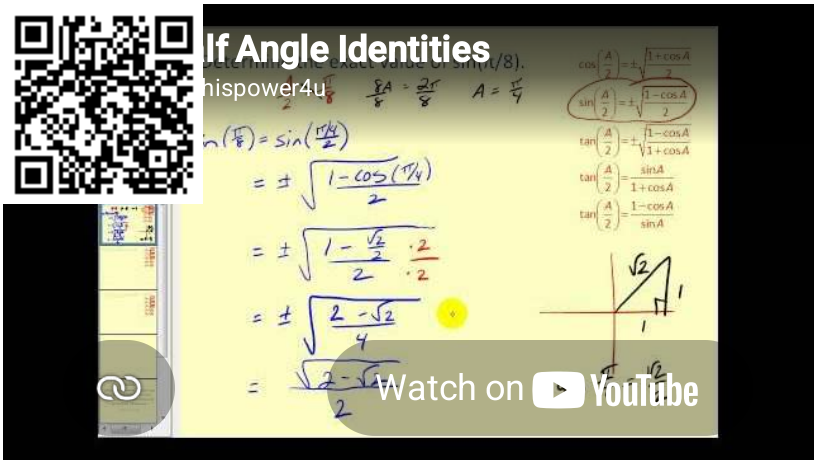
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$



Half Angle Identities

The **half angle identities** are a rewritten version of the power reducing identities. The proofs are left as review problems.

- $\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}$
- $\cos \frac{x}{2} = \pm \sqrt{\frac{1 + \cos x}{2}}$
- $\tan \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}}$



Power Reducing Identities

The **power reducing identities** allow you to write a trigonometric function that is squared in terms of smaller powers. The proofs are left as examples and review problems.

- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

Power reducing identities are most useful when you are asked to rewrite expressions such as $\sin^4 x$ as an expression without powers greater than one. While $\sin x \cdot \sin x \cdot \sin x \cdot \sin x$ does technically simplify this expression as necessary, you should try to get the terms to sum together not multiply together.

$$\begin{aligned}\sin^4 x &= \left(\sin^2 x\right)^2 \\ &= \left(\frac{1-\cos 2x}{2}\right)^2 \\ &= \frac{1-2\cos 2x+\cos^2 2x}{4} \\ &= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)\end{aligned}$$

Examples

Example 1

Earlier, you were asked to find $\sin^2 15°$. In order to fully identify $\sin^2 15°$ you need to use the power reducing formula.

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\begin{aligned}\sin^2 15° &= \frac{1-\cos 30°}{2} = \frac{1}{2}-\frac{\sqrt{3}}{4} \\ &= \frac{2-\sqrt{3}}{4}\end{aligned}$$

Example 2

Write the following expression with only $\sin x$ and $\cos x$: $\sin 2x + \cos 3x$.

$$\begin{aligned}\sin 2x+\cos 3x &= 2\sin x\cos x+\cos(2x+x) \\ &= 2\sin x\cos x+\cos 2x\cos x-\sin 2x\sin x \\ &= 2\sin x\cos x+\left(\cos^2 x-\sin^2 x\right)\cos x-(2\sin x\cos x)\sin x \\ &= 2\sin x\cos x+\cos^3 x-\sin^2 x\cos x-2\sin^2 x\cos x \\ &= 2\sin x\cos x+\cos^3 x-3\sin^2 x\cos x\end{aligned}$$

Example 3

Use half angles to find an exact value of tan $22.5°$ without using a calculator.

$$\tan\tfrac{x}{2} = \pm\sqrt{\tfrac{1-\cos x}{1+\cos x}}$$

$$\begin{aligned}\tan 22.5° &= \tan\frac{45°}{2} = \pm\sqrt{\frac{1-\cos 45°}{1+\cos 45°}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} \\ &= \pm\sqrt{\frac{(2-\sqrt{2})^2}{2}}\end{aligned}$$

Example 4

Prove the power reducing identity for sine.

$\sin^2 x = \frac{1-\cos 2x}{2}$

Using the double angle identity for cosine:

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = (1 - \sin^2 x) - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

This expression is an equivalent expression to the double angle identity and is often considered an alternate form.

Example 5

Simplify the following identity. $\sin^4 x - \cos^4 x$.

Here are the steps:

$$\begin{aligned}\sin^4 x - \cos^4 x &= \left(\sin^2 x - \cos^2 x\right)\left(\sin^2 x + \cos^2 x\right)\\ &= -\left(\cos^2 x - \sin^2 x\right)\\ &= -\cos 2x\end{aligned}$$

Review

Prove the following identities.

1. $\sin 2x = 2\sin x \cos x$
2. $\cos 2x = \cos^2 x - \sin^2 x$
3. $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
4. $\cos^2 x = \frac{1+\cos 2x}{2}$
5. $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$
6. $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
7. $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
8. $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$
9. $\csc 2x = \frac{1}{2}\csc x \sec x$
10. $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$

Find the value of each expression using half angle identities.

11. $\tan 15°$
12. $\tan 22.5°$
13. $\sec 22.5°$
14. Show that $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x}$
15. Using your knowledge from the answer to question **14**, show that $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$.

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6.5: Trigonometric Equations

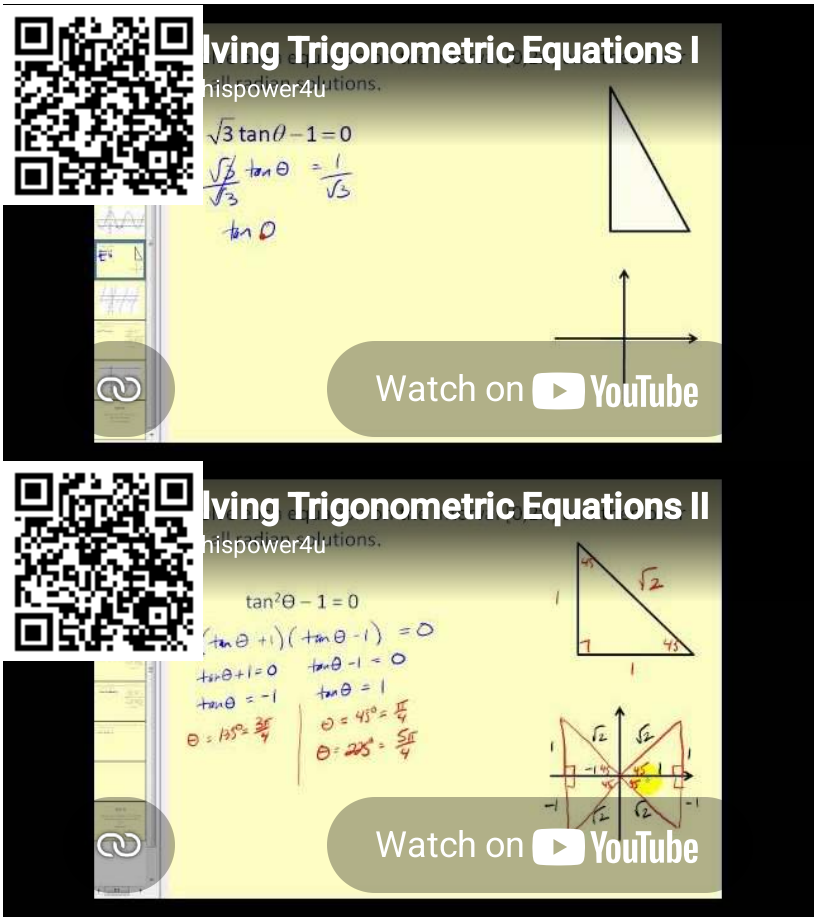
Solving a trigonometric equation is just like solving a regular equation. You will use factoring and other algebraic techniques to get the variable on one side. The biggest difference with trigonometric equations is the opportunity for there to be an infinite number of solutions that must be described with a pattern. The equation $\cos x = 1$ has many solutions including 0 and 2π. How would you describe all of them?

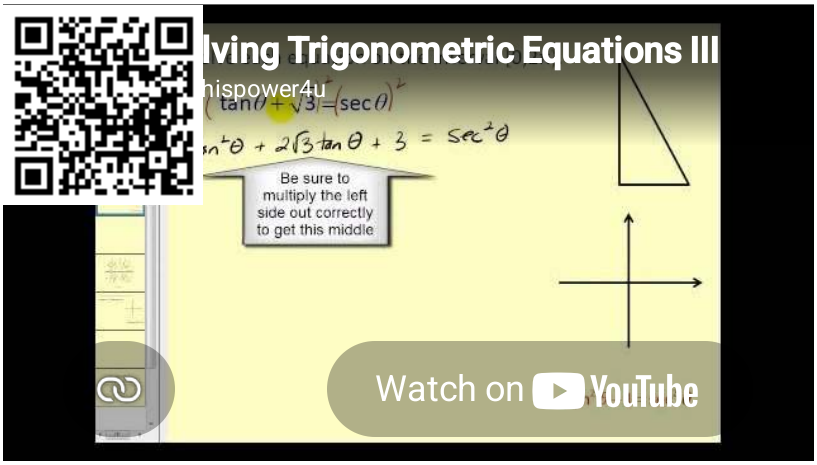
Solving Trigonometric Equations

The identities you have learned are helpful in solving **trigonometric equations**. The goal of solving an equation hasn't changed. Do whatever it takes to get the variable alone on one side of the equation. Factoring, especially with the Pythagorean identity, is critical.

Try to give exact (non-rounded) answers when solving trigonometric equations. If you are working with a calculator, keep in mind that while some newer calculators can provide exact answers like $\frac{\sqrt{3}}{2}$ most calculators will produce a decimal of $0.866\ldots$ If you see a decimal like $0.866\ldots$, try squaring it. The result might be a nice fraction like $\frac{3}{4}$. Then you can logically conclude that the original decimal must be the square root of $\frac{3}{4}$ or $\frac{\sqrt{3}}{2}$

When solving, if the two sides of the equation are always equal, then the equation is an identity. If the two sides of an equation are never equal, as with $\sin x = 3$, then the equation has no solution.





Examples

Example 1

Earlier, you were asked how you could describe the many solutions of $\cos x = 1$. When you type $\cos^{-1} 1$ on your calculator, it will yield only one solution which is 0. In order to describe all the solutions you must use logic and the graph to figure out that cosine also has a height of 1 at $-2\pi, 2\pi, -4\pi, 4\pi \ldots$ Luckily all these values are sequences in a clear pattern so you can describe them all in general with the following notation:

$x = 0 \pm n \cdot 2\pi$ where n is an integer, or $x = \pm n \cdot 2\pi$ where n is an integer.

Example 2

Solve the following equation algebraically and confirm graphically on the interval $[-2\pi, 2\pi]$.

$\cos 2x = \sin x$

$$\begin{aligned} \cos 2x &= \sin x \\ 1 - 2\sin^2 x &= \sin x \\ 0 &= 2\sin^2 x + \sin x - 1 \\ 0 &= (2\sin x - 1)(\sin x + 1) \end{aligned}$$

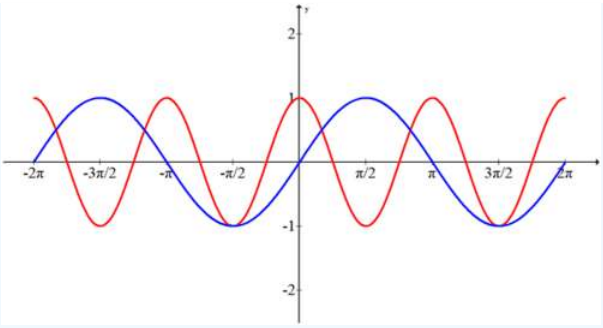
Solving the first part set equal to zero within the interval yields:

$$\begin{aligned} 0 &= 2\sin x - 1 \\ \frac{1}{2} &= \sin x \\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6} \end{aligned}$$

Solving the second part set equal to zero yields:

$$\begin{aligned} 0 &= \sin x + 1 \\ -1 &= \sin x \\ x &= -\frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

These are the six solutions that will appear as intersections of the two graphs $f(x) = \cos 2x$ and $g(x) = \sin x$



Example 3

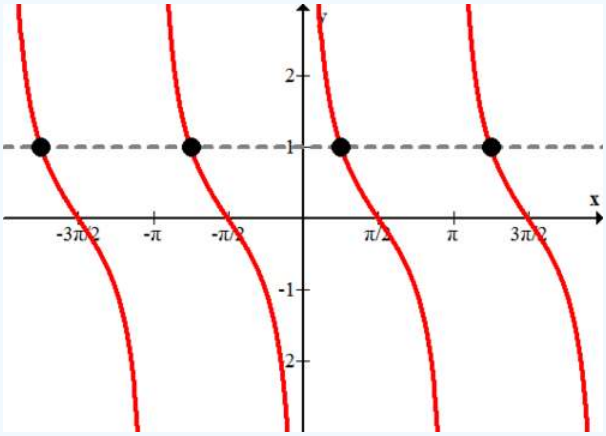
Determine the general solution to the following equation.

$\cot x - 1 = 0$

$\cot x - 1 = 0$
$\cot x = 1$

One solution is $x = \frac{\pi}{4}$. However, since this question asks for the general solution, you need to find every possible solution. You have to know that cotangent has a period of π which means if you add or subtract π from $\frac{\pi}{4}$ then it will also yield a height of 1. To capture all these other possible x values you should use this notation.

$x = \frac{\pi}{4} \pm n \cdot \pi$ where n is a integer



Notice that trigonometric equations may have an infinite number of solutions that repeat in a certain pattern because they are periodic functions. When you see these directions remember to find all the solutions by using notation like in this example.

Example 4

Solve the following equation.

$4\cos^2 x - 1 = 3 - 4\sin^2 x$

$$\begin{aligned} 4\cos^2 x - 1 &= 3 - 4\sin^2 x \\ 4\cos^2 x + 4\sin^2 x &= 3 + 1 \\ 4\left(\cos^2 x + \sin^2 x\right) &= 4 \\ 4 &= 4 \end{aligned}$$

This equation is always true which means the right side is always equal to the left side. This is an identity.

Example 5

Solve the following equation exactly.

$2\cos^2 x + 3\cos x - 2 = 0$

Note that $\cos x \neq -2$ which means only one equation needs to be solved for solutions.

$$2\cos x - 1 = 0$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, -\frac{\pi}{3}$$

These are the solutions within the interval $-\pi$ to π. since this represents one full period of cosine, the rest of the solutions are just multiples of 2π added and subtracted to these two values.

$x = \pm\frac{\pi}{3} \pm n \cdot 2\pi$ where n is an integer

Review

Solve each equation on the interval $[0, 2\pi)$.

1. $3\cos^2 \frac{x}{2} = 3$
2. $4\sin^2 x = 8\sin^2 \frac{x}{2}$

Find approximate solutions to each equation on the interval $[0, 2\pi)$

3. $3\cos^2 x + 10\cos x + 2 = 0$
4. $\sin^2 x + 3\sin x = 5$
5. $\cdot \tan^2 x + \tan x = 3$
6. $\cot^2 x + 5\tan x + 14 = 0$
7. $\sin^2 x + \cos^2 x = 1$

Solve each equation on the interval $[0, 360°)$.

8. $2\sin\left(x - \frac{\pi}{2}\right) = 1$
9. $4\cos(x - \pi) = 4$

Solve each equation on the interval $[2\pi, 4\pi)$

10. $\cos^2 x + 2\cos x + 1 = 0$
11. $3\sin x = 2\cos^2 x$
12. $\tan x \sin^2 x = \tan x$
13. $\sin^2 x + 1 = 2\sin x$
14. $\sec^2 x = 4$
15. $\sin^2 x - 4 = \cos^2 x - \cos 2x - 4$

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CHAPTER OVERVIEW

7: Vectors

7.1: Basic Properties of Vectors

7.2: Operations with Vectors

7.3: Resolution of Vectors into Components

7.4: Dot Product and Angle Between Two Vectors

7.5: Vector Projection

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7.1: Basic Properties of Vectors

An airplane being pushed off course by wind and a swimmer's movement across a moving river are both examples of vectors in action. Points in the coordinate plane describe location. **Vectors**, on the other hand, have no location and indicate only direction and magnitude. Vectors can describe the strength of forces like gravity or speed and direction of a ship at sea. Vectors are extremely useful in modeling complex situations in the real world.

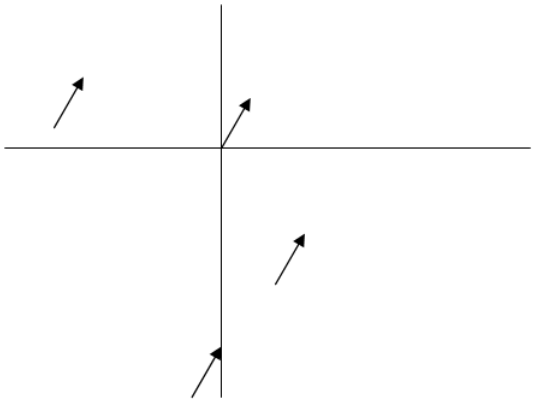
What are other differences between vectors and points?

Properties of Vectors

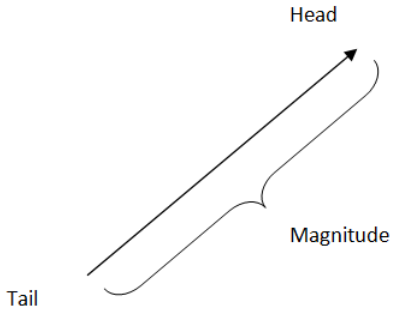
A two dimensional vector is represented graphically as an arrow with a tail and a head. The **head** is the arrow and is also called the **terminal point.** When finding the vector between two points start with the terminal point and subtract the **initial** point (the tail).



The two defining characteristics of a vector are its magnitude and its direction. The magnitude is shown graphically by the length of the arrow and the direction is indicated by the angle that the arrow is pointing. Notice how the following vector is shown multiple times on the same coordinate plane. This emphasizes that the location on the coordinate plane does not matter and is not unique. Each representation of the vector has identical direction and magnitude.



One way to define a vector is as a line segment with a direction. Vectors are said to be equal if they have the same magnitude and the same direction. The absolute value of a vector is the same as the length of the line segment or the **magnitude** of the vector. Magnitude can be found by using the Pythagorean Theorem or the distance formula.

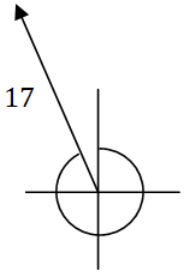


There are a few different ways to write a vector v.

$v, \vec{v}, \vec{v}$, or v with a $\sim$ underneath

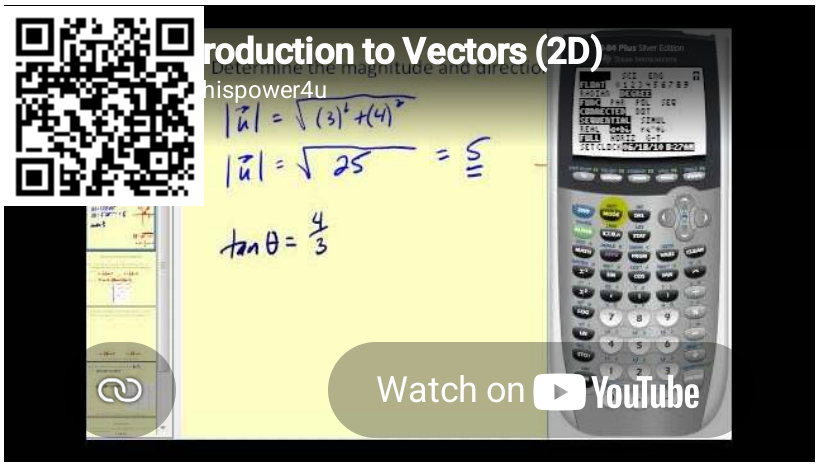
When you write about vectors algebraically there are a few ways to describe a specific vector. First, you could describe its magnitude and angle as r, θ. Second, you could describe it as an ordered pair: $< x, y >$. Notice that when discussing vectors you should use the brackets $<>$ instead of parentheses because it helps avoid confusion between a vector and a point. Vectors can be multidimensional.

Vectors are often used to describe the movement of objects. To describe the movement of a ship is traveling NNW at 17 knots (nautical **mph**) as a vector, notice that NNW is halfway between **NW** and **N**. When describing ships at sea, it is best to use bearing which has $0°$ as due North and $270°$ as due West. This makes NW equal to $315°$ and NNW equal to $337.5°$.



When you see this picture, it turns into a basic trig question to find the x and y components of the vector. Note that the reference angle that the vector makes with the negative portion of the x axis is $67.5°$

$\sin 67.5° = \frac{y}{17}, \cos 67.5° = \frac{x}{17}$
$< x, y > \approx < -6.5, 15.7 >$



Examples

Example 1

Earlier, you were asked what the differences between points and vectors are. There are many differences between points and vectors. Points are locations and vectors are made up of distance and angles. Parentheses are used for points and $\langle >$ are used for vectors. One relationship between vectors and points is that a point plus a vector will yield a new point. It is as if there is a starting place and then a vector tells you where to go from that point. Without the starting point, the vector could start from anywhere.

Example 2

Consider the points: $A(1, 3), B(-4, -6), C(5, -13)$. Find the vectors in component form of $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{CB}$

Remember that when finding the vector between two points, start with the terminal point and subtract the initial point.

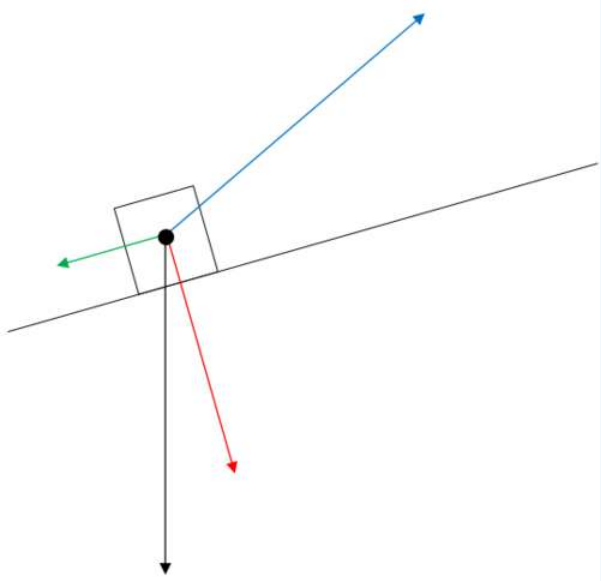
$\overrightarrow{AB} =< -5, -9 >$
$\overrightarrow{BA} =< 5, 9 >$
$\overrightarrow{AC} =< 4, -16 >$
$\overrightarrow{CB} =< -9, 7 >$

Example 3

A father is pulling his daughter up a hill. The hill has a $20°$ incline. The daughter is on a sled which sits on the ground and has a rope that the father pulls as he walks. The rope makes a $39°$ angle with the slope. A force diagram is a collection of vectors that each represent a force like gravity or wind acting on an object. Draw a force diagram showing how these forces act on the daughter's center of gravity:

a. The force of gravity.
b. The force holding the daughter in the sled to the ground.
c. The force pulling the daughter backwards down the slope.
d. The force of the father pulling the daughter up the slope.

The girl's center of gravity is represented by the black dot. The force of gravity is the black arrow straight down. The green arrow is gravity's effect pulling the girl down the slope. The red arrow is gravity's effect pulling the girl straight into the slope. The blue arrow represents the force that the father is exerting as he pulls the girl up the hill.



Notice that the father's force vector (blue) is longer than the force pulling the girl down the hill. This means that over time they will make progress and ascend the hill. Also note that the father is wasting some of his energy lifting rather than just pulling. If he could pull at an angle directly opposing the force pulling the girl down the hill, then he would be using all of his energy efficiently.

Example 4

Center the force diagram from the previous question into the origin and identify the angle between each consecutive force vector.

The x and y axis are included as reference and note that the gravity vector overlaps with the negative y axis. In order to find each angle, you must use your knowledge of supplementary, complementary and vertical angles and all the clues from the question. To check, see if all the angles sum to be $360°$.

Example 5

Given the following vectors and point, compute the sum.

$A = (1,3), \vec{v} =< 4,8 >, \vec{u} =< -1,-5 >$
$A + \vec{v} + \vec{u} =?$
$A = (1,3), \vec{v} =< 4,8 >, \vec{u} =< -1,-5 > . A + \vec{v} + \vec{u} = (4,6)$

Review

1. Describe what a vector is and give a real-life example of something that a vector could model.

Consider the points: $A(3,5), B(-2,-4), C(1,-12), D(-5,7)$. Find the vectors in component form of:

2. $\overrightarrow{AB}$

3. $\overrightarrow{BA}$

4. $\overrightarrow{AC}$

5. $\cdot\overrightarrow{CB}$

6. $\cdot\overrightarrow{AD}$

7. $\overrightarrow{DA}$

8. What is $C + \overrightarrow{CB}$? Compute this algebraically and describe why the answer makes sense.

9. Use your answer to the previous problem to help you determine $D + \overrightarrow{DA}$ without doing any algebra.

10. A ship is traveling SSW at 13 knots. Describe this ship's movement in a vector.

11. A vector that describes a ships movement is $< 5\sqrt{2}, 5\sqrt{2} >$. What direction is the ship traveling in and what is its speed in knots?

For each of the following vectors, draw the vector on a coordinate plane starting at the origin and find its magnitude.

12. $< 3,7 >$

13. $< -3,4 >$

14. $< -5,10 >$

15. $< 6,-8 >$

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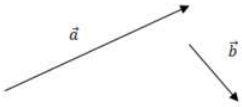
7.2: Operations with Vectors

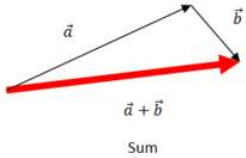
When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 miles an hour in a headwind of 2 miles an hour only makes headway at a rate of 8 miles per hour. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing NE at 2 miles per hour. How far does the bird get in one hour?

Basic Vector Operations

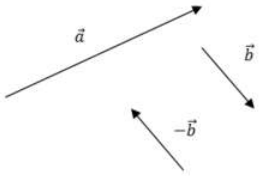
Scalar multiplication means to multiply a vector by a number. This changes the magnitude of the vector, but not its direction. If $\vec{v} =< 3, 4 >$, then $2\vec{v} =< 6, 8 >$. Scalar multiplication is fairly simple.

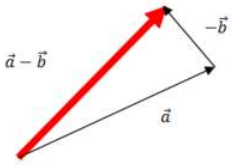
Adding and subtracting vectors is slightly more difficult. When adding vectors, place the tail of one vector at the head of the other. This is called the **tail-to-head** rule. The vector that is formed by joining the tail of the first vector with the head of the second is called the resultant **vector**.





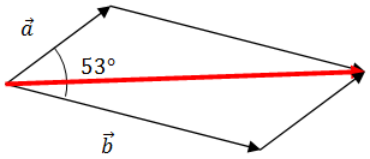
Vector subtraction reverses the direction of the second vector. $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$





Adding vectors can be done in either order (just like with regular numbers). Subtracting vectors must be done in a specific order or else the vector will be negative (just like with regular numbers).

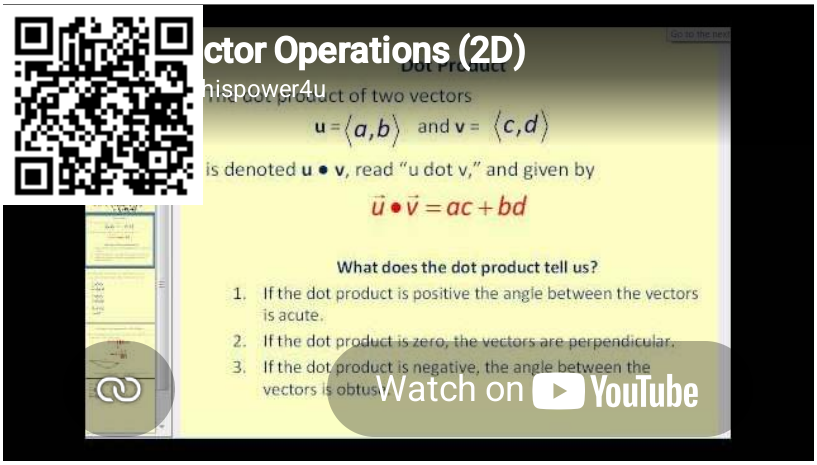
To find the length or magnitude of a resultant vector, you can use the law of cosines. To do this, you also need to know the angle between the two vectors. Say you were given two vectors $\vec{a}$ and $\vec{b}$, have magnitudes of 5 and 9 respectively and that the angle between the vectors is $53°$. To find the magnitude of $\vec{a} + \vec{b}$, which is written as $|\vec{a} + \vec{b}|$, notice that you have a parallelogram.



In order to fine the magnitude of the resulting vector in red, note that the triangle on the bottom has sides 9 and 5 with included angle $127°$ due to the properties of parallelograms. And, so applying the Law of cosines, you get:

$x^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127°$
$x \approx 12.66$

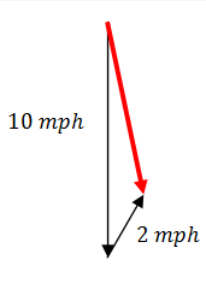
For this video, focus on scalar multiplication and adding and subtracting vectors:



Examples

Example 1

Earlier, you were asked about how fast a bird was flying. A bird flying due south at 10 miles an hour with a cross headwind of 2 mph heading NE would have a force diagram that looks like this:



The angle between the bird's vector and the wind vector is 45° which means this is a perfect situation for the Law of cosines. Let $x =$ the red vector.

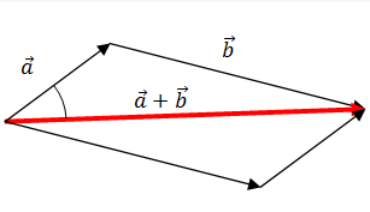
$x^2 = 10^2 + 2^2 - 2 \cdot 10 \cdot 2 \cdot \cos 45^\circ$
$x \approx 8.7$

The bird is blown slightly off track and travels only about 8.7 miles in one hour.

Example 2

Using the vectors with magnitude 5 and 9 and angle of 53° from the main section, what is the angle that the sum $\vec{a} + \vec{b}$ makes with $\vec{a}$?

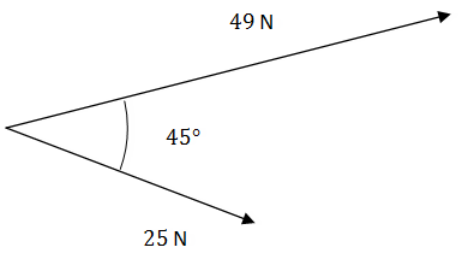
Start by drawing a good picture and labeling what you know. $|\vec{a}| = 5, |\vec{b}| = 9, |\vec{a} + \vec{b}| \approx 12.66$. since you know three sides of the triangle and you need to find one angle, this is the SSS application of the Law of cosines.



$9^2 = 12.66^2 + 5^2 - 2 \cdot 12.66 \cdot 5 \cdot \cos\theta$
$\theta = 34.6^\circ$

Example 3

Elaine started a dog walking business. She walks two dogs at a time named Elvis and Ruby. They each pull her in different directions at a 45° angle with different forces. Elvis pulls at a force of 25 N and Ruby pulls at a force of $49N$. How hard does Elaine need to pull so that she can stay balanced? Note that N stands for Newtons which is the standard unit of force.



Even though the two vectors are centered at Elaine, the forces are added which means that you need to use the tail-to-head rule to add the vectors together. Finding the angle between each component vector requires logical use of supplement angles.



$$x^2 = 49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ$$
$$x \approx 68.98 \text{ N}$$

In order for Elaine to stay balanced, she will need to counteract this force with an equivalent force of her own in the exact opposite direction.

Example 4

Consider vector $\vec{v} =< 2, 5 >$ and vector $\vec{u} =< -1, 9 >$. Determine the component form of the following: $3\vec{v} - 2\vec{u}$

Do multiplication first for each term, followed by vector subtraction.

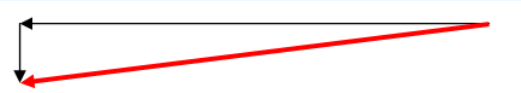
$$\begin{aligned} 3 \cdot \vec{v} - 2 \cdot \vec{u} &= 3 \cdot < 2, 5 > -2 \cdot < -1, 9 > \\ &=< 6, 15 > - < -2, 18 > \\ &=< 8, -3 > \end{aligned}$$

Example 5

An airplane is flying at a bearing of 270° at 400mph. A wind is blowing due south at 30mph. Does this cross wind affect the plane's speed?

Since the cross wind is perpendicular to the plane, it pushes the plane south as the plane tries to go directly east. As a result the plane still has an airspeed of 400 mph but the groundspeed (true speed) needs to be calculated.

$$400^2 + 30^2 = x^2$$
$$x \approx 401$$



Review

Consider vector $\vec{v} =< 1, 3 >$ and vector $\vec{u} =< -2, 4 >$

1. Determine the component form of $5\vec{v} - 2\vec{u}$.

2. Determine the component form of $-2\vec{v} + 4\vec{u}$.

3. Determine the component form of $6\vec{v} + \vec{u}$.

4. Determine the component form of $3\vec{v} - 6\vec{u}$.

5. Find the magnitude of the resultant vector from #1.

6. Find the magnitude of the resultant vector from # 2 .

7. Find the magnitude of the resultant vector from $\#3$.

8. Find the magnitude of the resultant vector from #4.

9. The vector $< 3, 4 >$ starts at the origin. What is the direction of the vector?

10. The vector $< -1, 2 >$ starts at the origin. What is the direction of the vector?

11. The vector $< 3, -4 >$ starts at the origin. What is the direction of the vector?

12. A bird flies due south at 8 miles an hour with a cross headwind blowing due east at 15 miles per hour. How far does the bird get in one hour?

13. What direction is the bird in the previous problem actually moving?

14. A football is thrown at 50 miles per hour due north. There is a wind blowing due east at 8 miles per hour. What is the actual speed of the football?

15. What direction is the football in the previous problem actually moving?

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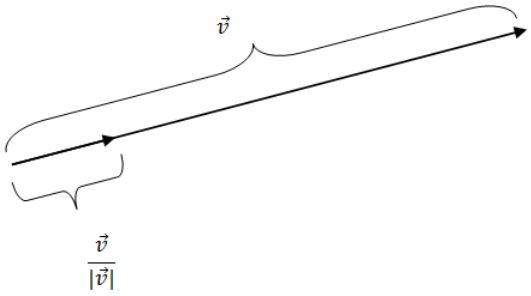
7.3: Resolution of Vectors into Components

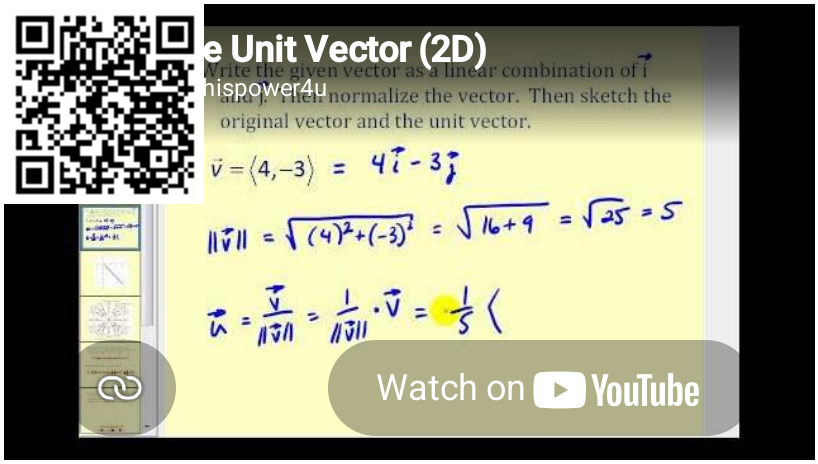
Sometimes working with horizontal and vertical components of a vector can be significantly easier than working with just an angle and a magnitude. This is especially true when combining several forces together.

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 Ib of force at an angle of $41°$. Connie pulls with 10 Ib of force at an angle of $100°$. Cynthia pulls with 12 Ib of force at an angle of $200°$. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

The Unit Vector and Component Form

A **unit vector** is a vector of length one. Sometimes you might wish to scale a vector you already have so that it has a length of one. If the length was five, you would scale the vector by a factor of $\frac{1}{5}$ so that the resulting vector has magnitude of 1. Another way of saying this is that a **unit vector in the direction of vector** $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$



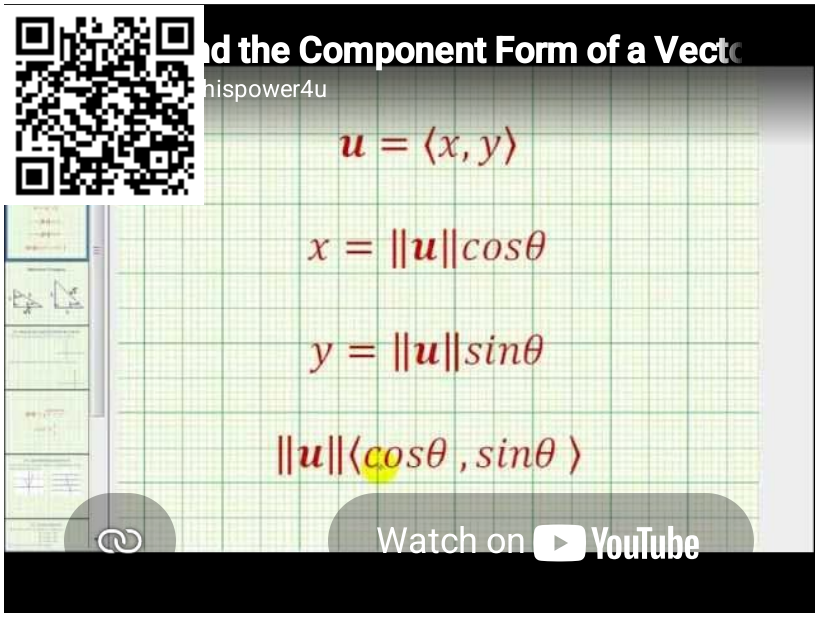


There are two standard unit vectors that make up all other vectors in the coordinate plane. They are $\vec{i}$ which is the vector $< 1, 0 >$ and $\vec{j}$ which is the vector $< 0, 1 >$. These two unit vectors are perpendicular to each other. A linear combination of $\vec{i}$ and $\vec{j}$ will allow you to uniquely describe any other vector in the coordinate plane in component form. For instance the vector $< 5, 3 >$ is the same as $5\vec{i} + 3\vec{j}$.

Often vectors are initially described as an angle and a magnitude rather than in component form. Working with vectors written as an angle and magnitude requires extremely precise geometric reasoning and excellent pictures. One advantage of rewriting the vectors in component form is that much of this work is simplified. Remember that component form is the form $< x, y >$ and to translate from magnitude r and direction θ to component form, use the relationship $< r \cdot \cos\theta, r \cdot \sin\theta > = < x, y >$. In this situation r is the magnitude and θ is the direction.

Take a plane that has a bearing of $60°$ and is going 350mph. To find the component form of the velocity of the airplane, note that a bearing of $60°$ is the same as a $30°$ on the unit circle. This corresponds to the point $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ which is the same as $(\cos 30, \sin 30)$. When written as a vector $< \frac{\sqrt{3}}{2}, \frac{1}{2} >$ is a unit vector
because it has magnitude 1. Now you just need to scale by a factor of 350 and you get your answer of $< 175\sqrt{3}, 175 >$. Thus, the

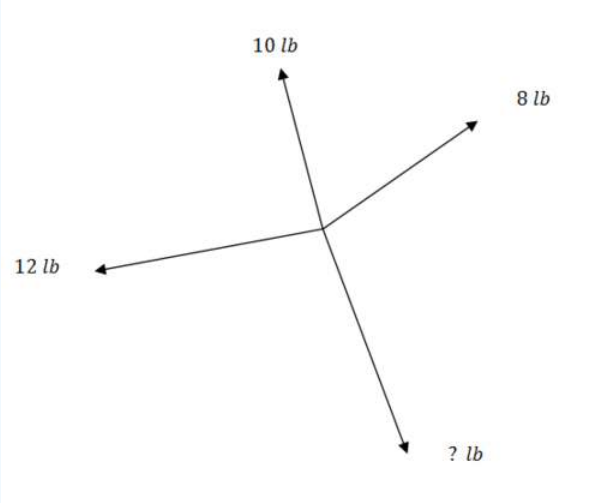
velocity of the airplane in component form is $< 175\sqrt{3}, 175 >$. This is the same as using the relationship $< r \cdot \cos\theta, r \cdot \sin\theta >=< x, y >$.



Examples

Example 1

Earlier, you were asked to consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 Ib of force at an angle of $41°$. Connie pulls with 10 Ib of force at an angle of $100°$. Cynthia pulls with 12 Ib of force at an angle of $200°$. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?



To add the three vectors together would require several iterations of the Law of Cosines. Instead, write each vector in component form and set equal to a zero vector indicating that the candy does not move.

$\vec{L} + \overrightarrow{CON} + \overrightarrow{CYN} + \vec{T} =< 0, 0 >$

$< 8 \cdot \cos 41°, 8 \cdot \sin 41° > + < 10 \cdot \cos 100°, 10 \cdot \sin 100° >$

$+ < 12 \cdot \cos 200°, 12 \cdot \sin 200° > + \vec{T} =< 0, 0 >$

Use a calculator to add all the x components and bring them to the far side and the y components and then subtract from the far side to get:

$\vec{T} \approx < 6.98, -10.99 >$

Turning this component vector into an angle and magnitude yields how hard and in what direction he would have to pull. Terry will have to pull with about 13 Ib of force at an angle of $302.4°$.

Example 2

Consider the plane that has a bearing of $60°$ and is going 350mph. If there is wind blowing with the bearing of $300°$ at 45mph, what is the component form of the total velocity of the airplane?

A bearing of $300°$ is the same as $150°$ on the unit circle which corresponds to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. You can now write and then scale the wind vector.

$45 \cdot < -\frac{\sqrt{3}}{2}, \frac{1}{2} > = < -\frac{45\sqrt{3}}{2}, \frac{45}{2} >$

Since both the wind vector and the velocity vector of the airplane are written in component form, you can simply sum them to find the component vector of the total velocity of the airplane.

$< 175\sqrt{3}, 175 > + < -\frac{45\sqrt{3}}{2}, \frac{45}{2} > = < \frac{305\sqrt{3}}{2}, \frac{395}{2} >$

Example 3

Consider the plane from Example 2 with the same wind and velocity. Find the actual ground speed and direction of the plane (as a bearing).

Since you already know the component vector of the total velocity of the airplane, you should remember that these components represent an x distance and a y distance and the question asks for the hypotenuse.

$$\left(\frac{305\sqrt{3}}{2}\right)^2 + \left(\frac{395}{2}\right)^2 = c^2$$

$$329.8 \approx c$$

The airplane is traveling at about 329.8 mph.

Since you know the x and y components, you can use tangent to find the angle. Then convert this angle into bearing.

$$\tan\theta = \frac{\left(\frac{395}{2}\right)}{\left(\frac{305\sqrt{3}}{2}\right)}$$

$$\theta \approx 36.8°$$

An angle of $36.8°$ on the unit circle is equivalent to a bearing of $53.2°$.

Note that you can do the entire problem in bearing by just switching sine and cosine, but it is best to truly understand what you are doing every step of the way and this will probably involve always going back to the unit circle.

For Examples 4 and 5, use the following information:

$\vec{v} = < 2, -5 >, \vec{u} = < -3, 2 >, \vec{t} = < -4, -3 >, \vec{r} = < 5, y >$
$B = (4, -5), P = (-3, 8)$

Example 4

Find the unit vectors in the same direction as $\vec{u}$ and $\vec{t}$.

To find a unit vector, divide each vector by its magnitude.

$\frac{\vec{u}}{|\vec{u}|} = < \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} >, \frac{\vec{t}}{|\vec{t}|} = < \frac{-4}{5}, \frac{-3}{5} >$

Example 5

Find the point 10 units away from B in the direction of P.

The vector $\overrightarrow{BP}$ is $< -7, 13 >$. First take the unit vector and then scale it so it has a magnitude of 10.

$$\frac{BP}{|BP|} =< \frac{-7}{\sqrt{218}}, \frac{13}{\sqrt{218}} >$$

$$10 \cdot \frac{BP}{|BP|} =< \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} >$$

You end up with a vector that is ten units long in the right direction. The question asked for a point from B which means you need to add this vector to point B.

$(4, -5) + < \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} > \approx (-0.74, 3.8)$

Review

Use the following defined vectors and points to answer $1 - 8$.

$\vec{v} =< 1, -3 >, \vec{u} =< 2, 5 >, \vec{t} =< 9, -1 >, \vec{r} =< 2, y >$

$A = (-3, 2), B = (5, -2)$

1. Solve for y in vector $\vec{r}$ to make $\vec{r}$ perpendicular to $\vec{t}$.

2. Find the unit vector in the some direction os $\vec{u}$.

3. Find the unit vector in the some direction as $\vec{t}$.

4. Find the unit vector in the some direction os $\vec{v}$

5. Find the unit vector in the some direction os $\vec{r}$.

6. Find the point exactly 3 units owey from A in the direction of B.

7. Find the point exoctly 6 units owry from B in the direction of A.

8. Find the point exoctly 5 units owoy from A in the direction of B.

9. Jock ond Jill went up a hill to fetch o poil of woter. When they got to the top of the hill, they were very thirsty so they each pulled on the bucket. Jill pulled ot $30°$ with 20 lbs of force. Jock pulled ot $45°$ with 28 lbs of force. What is the resulting vector for the bucket?

10. A plane is flying on a bearing of $60°$ at 400mph. Find the component form of the velocity of the plane. What does the component form tell you?

11. A baseball is thrown at a $70°$ angle with the horizontal with an initial speed of 30 mph. Find the component form of the initial velocity.

12. A plane is flying on a bearing of $200°$ at 450mph. Find the component form of the velocity of the plane.

13. A plane is flying on a bearing of $260°$ at 430 mph. At the same time, there is a wind blowing at a bearing of $30°$ at 60mph. What is the component form of the velocity of the plane?

14. Use the information from the previous problem to find the actual ground speed and direction of the plane.

15. Wind is blowing at a magnitude of 40 mph with an angle of $25°$ with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?

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7.4: Dot Product and Angle Between Two Vectors

While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. You can study more about the cross product between two vectors when you take Linear Algebra. The second type of product is the **dot product** between two vectors which results in a regular number. Other names for the dot product include **inner product** and **scalar product**. This number represents how much of one vector goes in the direction of the other. In one sense, it indicates how much the two vectors agree with each other. This concept will focus on the dot product between two vectors.

What is the dot product between $< -1, 1 >$ and $< 4, 4 >$? What does the result mean?

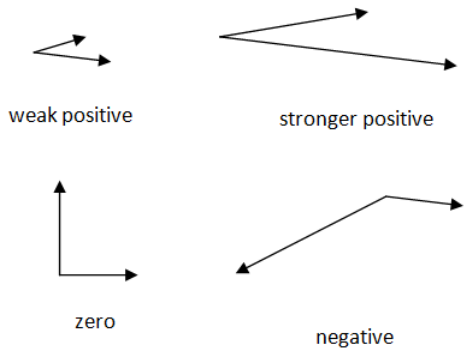
Properties of the Dot Product

The dot product is defined as:

$$u \cdot v =< u_1, u_2 > \cdot < v_1, v_2 >= u_1 v_1 + u_2 v_2$$

This procedure states that you multiply the corresponding values and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product.



Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than $90°$.

There are a many important properties related to the dot product. The two most important are 1) what happens when a vector has a dot product with itself and 2) what is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = |v|^2$
- v and u are perpendicular if and only if $v \cdot u = 0$

The commutative property, $u \cdot v = v \cdot u$, holds for the dot product between two vectors. The following proof is for two dimensional vectors although it holds for any dimensional vectors.

Start with the vectors in component form.

$u =< u_1, u_2 >$
$v =< v_1, v_2 >$

Then apply the definition of dot product and rearrange the terms. The commutative property is already known for regular numbers so we can use that.

$$u \cdot v = < u_1, u_2 > \cdot \langle v_1, v_2 \rangle$$
$$= u_1 v_1 + u_2 v_2$$
$$= v_1 u_1 + v_2 u_2$$
$$= < v_1, v_2 > \cdot < u_1, u_2 >$$
$$= v \cdot u$$

The distributive property, $u \cdot (v + w) = uv + uw$, also holds under the dot product. The following proof will work with two dimensional vectors although the property does hold in general.

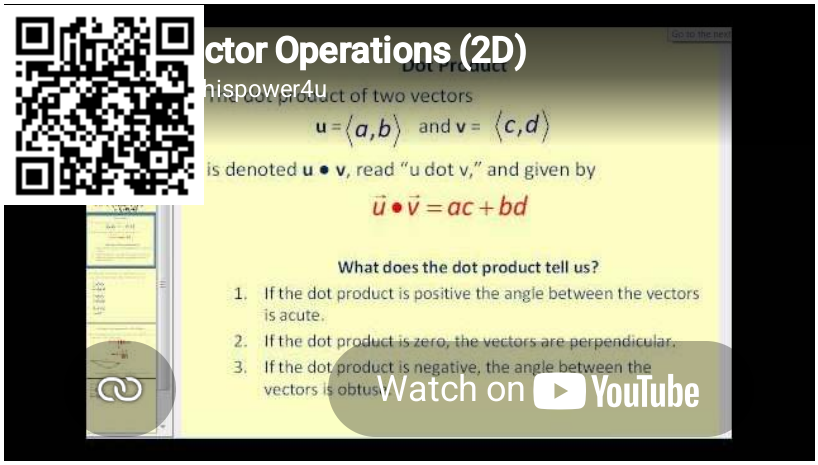
$$u = < u_1, u_2 >, v = < v_1, v_2 >, w = < w_1, w_2 >$$

$$u \cdot (v + w) = u \cdot (< v_1, v_2 > + < w_1, w_2 >)$$
$$= u \cdot < v_1 + w_1, v_2 + w_2 \rangle$$
$$= < u_1, u_2 > \cdot < v_1 + w_1, v_2 + w_2 >$$
$$= u_1 (v_1 + w_1) + u_2 (v_2 + w_2)$$
$$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$$
$$= u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2$$
$$= u \cdot v + v \cdot w$$

The dot product can help you determine the angle between two vectors using the following formula. Notice that in the numerator the dot product is required because each term is a vector. In the denominator only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

Watch the portion of this video focusing on the dot product:



Examples

Example 1

Earlier, you were asked to find the dot product between the two vectors $< -1, 1 >$ and $< 4, 4 >$. It can be computed as:

$$(-1)(4) + 1(4) = -4 + 4 = 0$$

The result of zero makes sense because these two vectors are perpendicular to each other.

Example 2

Find the dot product between the following vectors: $< 3, 1 > \cdot < 5, -4 >$

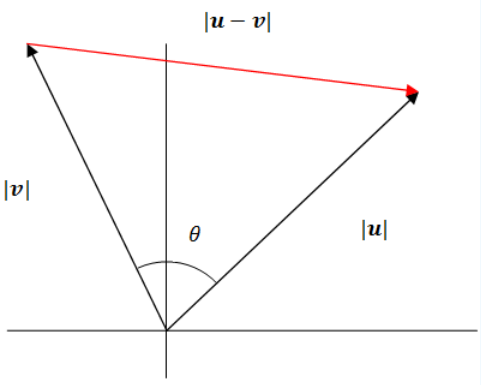
$$u = < 3, 5 >$$

Example 3

Prove the angle between two vectors formula:

$\cos\theta = \frac{u \cdot v}{|u||v|}$

Start with the law of cosines.



$$|u-v|^2 = |v|^2 + |u|^2 - 2|v||u|\cos\theta$$
$$(u-v)\cdot(u-v) =$$
$$u\cdot u - 2u\cdot v + v\cdot v =$$
$$|u|^2 - 2u\cdot v + |v|^2 =$$
$$-2u\cdot v = -2|v||u|\cos\theta$$
$$\frac{u\cdot v}{|u||v|} = \cos\theta$$

Example 4

Find the dot product between the following vectors.

$(4i - 2j)\cdot(3i - 8j)$

The standard unit vectors can be written as component vectors.

$< 4, -2 > \cdot < 3, -8 >= 12 + (-2)(-8) = 12 + 16 = 28$

Example 5

What is the angle between $u =< 3, 5 >$ and $v =< 2, 8 >$?

Use the angle between two vectors formula.

$u =< 3, 5 >$ and $v =< 2, 8 >$

$$\frac{u\cdot v}{|u||v|} = \cos\theta$$
$$\frac{< 3, 5 > \cdot < 2, 8 >}{\sqrt{34}\cdot\sqrt{68}} = \cos\theta$$
$$\frac{6 + 40}{\sqrt{34}\cdot\sqrt{68}} = \cos\theta$$
$$\cos^{-1}\left(\frac{46}{\sqrt{34}\cdot\sqrt{68}}\right) = \theta$$
$$16.93^\circ \approx \theta$$

Review

Find the dot product for each of the following pairs of vectors.

1. $< 2, 6 > \cdot < -3, 5 >$

2. $< 5, -1 > \cdot < 4, 4 >$

3. $< -3, -4 > \cdot < 2, 2 >$

4. $< 3, 1 > \cdot < 6, 3 >$

5. $< -1, 4 > \cdot < 2, 9 >$

Find the angle between each pair of vectors below.

6. $< 2, 6 > \cdot < -3, 5 >$

7. $< 5, -1 > \cdot < 4, 4 >$

8. $< -3, -4 > \cdot < 2, 2 >$

9. $< 3, 1 > \cdot < 6, 3 >$

10. $< -1, 4 > \cdot < 2, 9 >$

11. What is $v \cdot v$?

12. How can you use the dot product to find the magnitude of a vector?

13. What is \(0 \cdot v ?\

14. Show that $(cu) \cdot v = u \cdot (cv)$ where c is a constant.

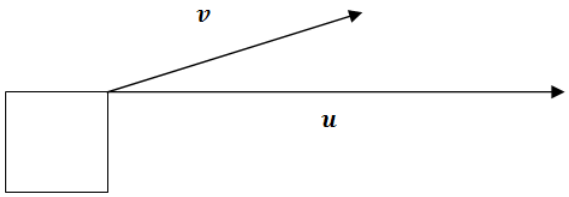
15. Show that $< 2, 3 >$ is perpendicular to $< 1.5, -1 >$.

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7.5: Vector Projection

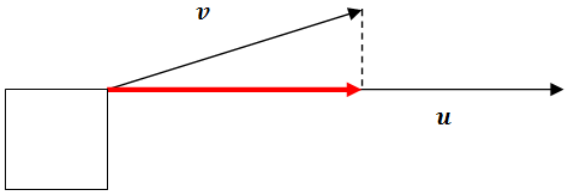
Projecting one vector onto another explicitly answers the question: “how much of one vector goes in the direction of the other vector?” The dot product is useful because it produces a scalar quantity that helps to answer this question. In this concept, you will produce an actual vector not just a scalar.

Why is vector projection useful when considering pulling a box in the direction of v instead of horizontally in the direction of u?



Projections

Consider the question from above.



The definition of **vector projection** for the indicated red vector is the called $\text{proj}_u v$. When you read proju v you should say "the vector projection of v onto u." This implies that the new vector is going in the direction of u. The vector projection is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector. The parallel vector is the vector projection. Conceptually, this means that if someone was pulling the box at an angle and strength of vector v then some of their energy would be wastedpulling the box up and some of the energy would actually contribute to pulling the box horizontally.

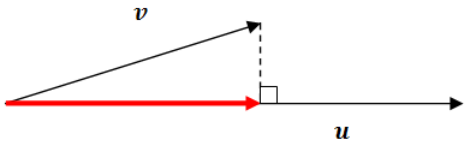
The definition of scalar projection is simply the length of the vector projection. When the scalar projection is positive it means that the angle between the two vectors is less than $90°$. When the scalar projection is negative it means that the two vectors are heading in opposite directions.

The vector projection formula can be written two ways. The version on the left is most simplified, but the version on the right makes the most sense conceptually. A.

$$proj_u v = \left(\frac{v \cdot u}{|u|^2}\right) u = \left(\frac{v \cdot u}{|u|}\right) \frac{u}{|u|}$$

The proof of the **vector projection formula** is as follows:

Given two vectors u, v, what is $proj_u v$?



First note that the projected vector in red will go in the direction of u. This means that it will be a product of the unit vector $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). In order to find the scalar projection, note the right triangle, the unknown angle θ between the two vectors and the cosine ratio.

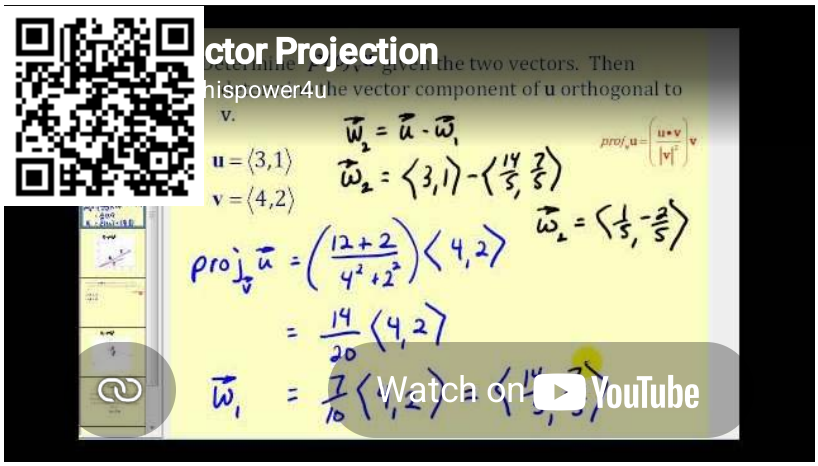
$$\cos\theta = \frac{\text{scalar projection}}{|v|}$$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now just substitute and simplify to find the length of the scalar projection.

$$\cos\theta = \frac{\text{scalar projection}}{|v|}$$
$$\frac{u \cdot v}{|u||v|} = \frac{\text{scalar projection}}{|v|}$$
$$\frac{u \cdot v}{|u|} = \text{scalar projection}$$

Now you have the length of the vector projection and the direction you want it to go:

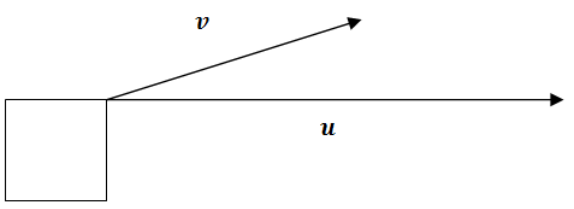
$proj_u v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$



Examples

Example 1

Earlier, you were asked why vector projection useful when considering pulling a box in the direction of instead of horizontally in the direction of u. Vector projection is useful in physics applications involving force and work.



When the box is pulled by vector v some of the force is wasted pulling up against gravity. In real life this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

Example 2

Find the scalar projection of vector $v =< 3, 4 >$ onto vector $u =< 5, -12 >$.

As noted earlier, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u \cdot v}{|u|}\right)$ where u is the vector being projected onto.

$\frac{u \cdot v}{|u|} = \frac{<5,-12> \cdot <3,4>}{13} = \frac{15-48}{13} = -\frac{33}{13}$

Example 3

Find the vector projection of vector $v =< 3, 4 >$ onto vector $u =< 5, -12 >$

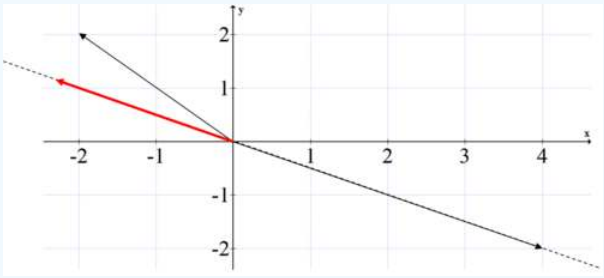
Since the scalar projection has already been found in Example 2 , you should multiply the scalar by the "onto" unit vector.

$-\frac{33}{13} < \frac{5}{13}, -\frac{12}{13} > = < -\frac{165}{169}, \frac{396}{169} >$

Example 4

Sketch the vector $< -2, -2 >$ and $< 4, -2 >$. Explain using a sketch why a negative scalar projection of $< -2, -2 >$ onto $< 4, -2 >$ makes sense.

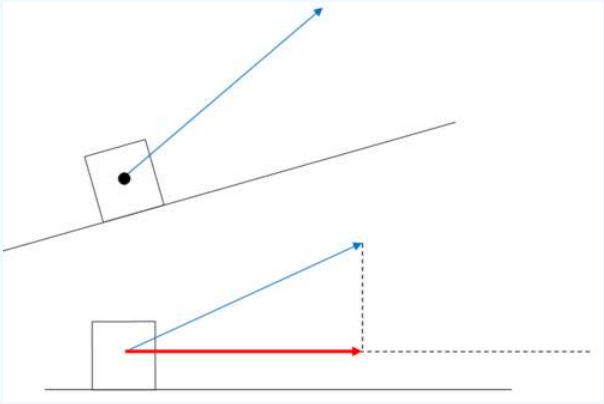
First plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $< -2, 2 >$ goes in the opposite direction of the vector $< 4, -2 >$. This will create a vector projection going in the opposite direction of $< 4, -2 >$.



Example 5

A father is pulling his daughter up a hill. The hill has a $20°$ incline. The daughter is on a sled that sits on the ground and has a rope that the father pulls with a force of 100 Ib as he walks. The rope makes a $39°$ angle with the slope. What is the effective force that the father exerts moving his daughter and the sled up the hill?

The box represents the girl and the sled. The blue arrow indicates the father's 100 Ib force. Notice that the question asks for simply the amount of force which means scalar projection. since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.



The components of the father's force vector is $100 < \cos 39°, \sin 39° >$ and the "onto" vector is any vector horizontally to the right. since we are only looking for the length of the horizontal component and you already have the angle between the two vectors, the scalar projection is:

$100 \cdot \cos 39° \approx 77.1 lb$

Review

1. Sketch vectors $< 2, 4 >$ and $< 2, 1 >$.
2. What is the vector projection of $< 2, 4 >$ onto $< 2, 1 >$? Sketch the projection.
3. Sketch vectors $< -2, 1 >$ and $< -1, 3 >$.
4. What is the vector projection of $< -1, 3 >$ onto $< -2, 1 >$? Sketch the projection.

5. Sketch vectors $< 6, 2 >$ and $< 8, 1 >$.

6. What is the vector projection of $< 6, 2 >$ onto $< 8, 1 >$? Sketch the projection.

7. Sketch vectors $< 1, 7 >$ and $< 6, 3 >$.

8. What is the vector projection of $< 1, 7 >$ onto $< 6, 3 >$? Sketch the projection.

9. A box is on the side of a hill inclined at $30°$. The weight of the box is 40 pounds. What is the magnitude of the force required to keep the box from sliding down the hill?

10. Sarah is on a sled on the side of a hill inclined at $60°$. The weight of Sarah and the sled is 125 pounds. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill.

11. $A1780$ pound car is parked on a street that makes an angle of $15°$ with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

12. $A1900$ pound car is parked on a street that makes an angle of $10°$ with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

13. $A30$ pound force that makes an angle of $32°$ with an inclined plane is pulling a box up the plane. The inclined plane makes a $20°$ angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

14. $A22$ pound force that makes an angle of $12°$ with an inclined plane is pulling a box up the plane. The inclined plane makes a $25°$ angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

15. Anne pulls a wagon on a horizontal surface with a force of 50 pounds. The handle of the wagon makes an angle of $30°$ with the ground. What is the magnitude of the effective force pulling the wagon?

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CHAPTER OVERVIEW

8: Systems and Matrices

8.1 Systems of Two Equations and Two Unknowns
8.2 Systems of Three Equations and Three Unknowns
8.3 Matrices to Represent Data
8.4 Matrix Operations
8.5 Row Operations and Row Echelon Forms
8.6 Augmented Matrices
8.10 Partial Fractions
8.7 Determinant of Matrices
8.8 Cramer's Rule
8.9 Inverse Matrices

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8.1 Systems of Two Equations and Two Unknowns

The cost of two cell phone plans can be written as a **system of equations** based on the number of minutes used and the base monthly rate. As a consumer, it would be useful to know when the two plans cost the same and when is one plan cheaper.

Plan A costs $40 per month plus $0.10 for each minute of talk time.

Plan B costs $25 per month plus $0.50 for each minute of talk time.

Plan B has a lower starting cost, but since it costs more per minute, it may not be the right plan for someone who likes to spend a lot of time on the phone. When do the two plans cost the same amount?

Solving Systems of Equations with Two Unknowns

There are many ways to solve a system that you have learned in the past including substitution and graphical intersection. Here you will focus on solving using elimination because the knowledge and skills used will transfer directly into using matrices.

When solving a system, the first thing to do is to count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. Two equations and two variables can be solved, but one equation with two variables cannot.

Here's the procedure for solving a system using the elimination method:

- **Step 1**: Write both equations with two variables in **standard form**, $Ax + By = C$. This form helps to align the variables.
- **Step 2**: Determine which variable you want to eliminate.
- **Step 3**: Scale each equation as necessary by multiplying through by constants.
- **Step 4**: Add the equations together. This should reduce both the number of equations and the number of variables leaving one equation and one variable.
- **Step 5**: Solve and substitute to determine the value of the second variable.

Here is a system of two equations and two variables in standard form: $5x + 12y = 72$ and $3x - 2y = 18$. Notice that there is an x column and a y column on the left hand side and a constant column on the right hand side when you rewrite the equations as shown. Also notice that if you add the system as written no variable will be eliminated.

Equation 1: $5x + 12y = 72$

Equation 2: $3x - 2y = 18$

Strategically choose to eliminate y by scaling the second equation by 6 so that the coefficient of y will match at 12 and -12.

$$5x + 12y = 72$$
$$18x - 12y = 108$$

Add the two equations:

$$23x = 180$$

$$x = \frac{180}{23}$$

The value for x could be substituted into either of the original equations and the result could be solved for y; however, since the value is a fraction it will be easier to repeat the elimination process in order to solve for x. This time you will take the first two equations and eliminate x by making the coefficients of x to be 15 and -15. Scale the first equation by a factor of 3 and scale the second equation by a factor of -5.

Equation 1: $15x + 36y = 216$

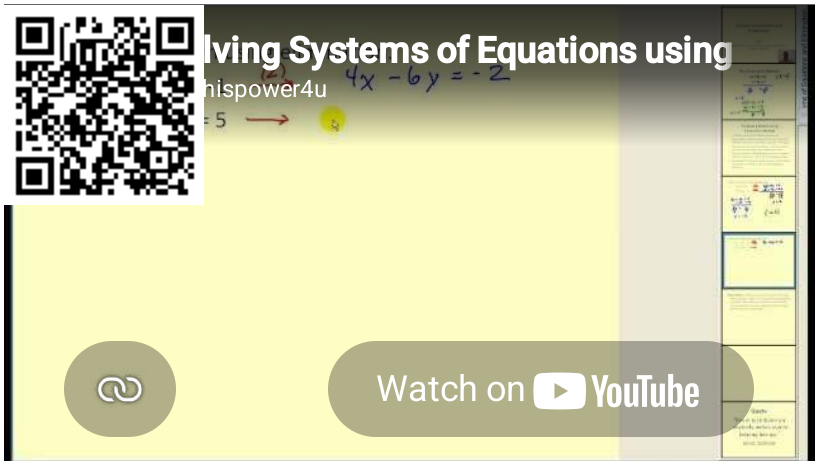
Equation $2: -15x + 10y = -90$

Adding the two equations:

$$0x + 46y = 126$$

$$y = \frac{126}{46} = \frac{63}{23}$$

The point $\left(\frac{180}{23}, \frac{63}{23}\right)$ is where these two lines intersect.



Examples

Example 1

Earlier, you were asked about two phone plans.

Plan A costs $40 per month plus $0.10 for each minute of talk time.

Plan B costs $25 per month plus $0.50 for each minute of talk time.

If you want to find out when the two plans cost the same, you can represent each plan with an equation and solve the system of equations. Let y represent cost and x represent number of minutes.

$y = 0.10x + 40$

$y = 0.50x + 25$

First you put these equations in standard form.

$x - 10y = -400$

$x - 2y = -50$

Then you scale the second equation by -1 and add the equations together and solve for y.

$$\begin{aligned} -8y &= -350 \\ y &= 43.75 \end{aligned}$$

To solve for x, you can scale the second equation by -5, add the equations together and solve for x.

$$\begin{aligned} -4x &= -150 \\ x &= 37.5 \end{aligned}$$

The equivalent costs of plan **A** and plan **B** will occur at 37.5 minutes of talk time with a cost of **$43.75**.

Example 2

Solve the following system of equations:

$$\begin{aligned} 6x - 7y &= 8 \\ 15x - 14y &= 21 \end{aligned}$$

Scaling the first equation by -2 will allow the y term to be eliminated when the equations are summed.

$$\begin{aligned} -12x + 14y &= -16 \\ 15x - 14y &= 21 \end{aligned}$$

The sum is:

$3x = 5$

$x = \frac{5}{3}$

You can substitute x into the first equation to solve for y.

$$\begin{aligned} 6 \cdot \frac{5}{3} - 7y &= 8 \\ 10 - 7y &= 8 \\ -7y &= -2 \\ y &= \frac{2}{7} \end{aligned}$$

The point $\left(\frac{5}{3}, \frac{2}{7}\right)$ is where these two lines intersect.

Example 3

Solve the following system using elimination:

$5x - y = 22$

$-2x + 7y = 19$

Start by scaling the first equation by 7 and notice that the y coefficient will immediately be eliminated when the equations are summed.

$35x - 7y = 154$

$-2x + 7y = 19$

Start by scaling the first equation by 7 and notice that the y coefficient will immediately be eliminated when the equations are summed.

$35x - 7y = 154$

$-2x + 7y = 19$

Add, solve for $x = \frac{173}{33}$. Instead of substituting, practice eliminating x by scaling the first equation by 2 and the second equation by 5.

$10x - 2y = 44$

$-10x + 35y = 95$

Add, solve for y.

Final Answer: $\left(\frac{173}{33}, \frac{139}{33}\right)$

Example 4

Solve the following system of equations:

$$\begin{aligned} 5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11 \\ \frac{1}{x} + \frac{1}{y} &= 4 \end{aligned}$$

The strategy of elimination still applies. You can eliminate the $\frac{1}{y}$ term if the second equation is scaled by a factor of -2

$5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} = 11$

$-2 \cdot \frac{1}{x} - 2 \cdot \frac{1}{y} = -8$

Add the equations together and solve for x.

$$\begin{aligned} -3 \cdot \frac{1}{x} + 0 \cdot \frac{1}{y} &= 3 \\ -3 \cdot \frac{1}{x} &= 3 \\ \frac{1}{x} &= -1 \\ x &= -1 \end{aligned}$$

Substitute into the second equation and solve for y.

$$\frac{1}{-1}+\frac{1}{y}=4$$
$$-1+\frac{1}{y}=4$$
$$\frac{1}{y}=5$$
$$y=\frac{1}{5}$$

The point $\left(-1, \frac{1}{5}\right)$ is the point of intersection between these two curves.

Example 5

Solve the following system using elimination:

$$11 \cdot \frac{1}{x}-5 \cdot \frac{1}{y}=-38$$
$$9 \cdot \frac{1}{x}+2 \cdot \frac{1}{y}=-25$$

To eliminate $\frac{1}{y}$, scale the first equation by 2 and the second equation by 5.

To eliminate $\frac{1}{x}$, scale the first equation by -9 and the second equation by 11

Final Answer: $\left(-\frac{1}{3}, 1\right)$

Review

Solve each system of equations using the elimination method.

1. $x+y=-4;-x+2y=13$
2. $\frac{3}{2}x-\frac{1}{2}y=\frac{1}{2};-4x+2y=4$
3. $6x+15y=1;2x-y=19$
4. $x-\frac{2y}{3}=\frac{-2}{3};5x-2y=10$
5. $-9x-24y=-243;\frac{1}{2}x+y=\frac{21}{2}$
6. $5x+\frac{28}{3}y=\frac{176}{3};y+x=10$
7. $2x-3y=50;7x+8y=-10$
8. $2x+3y=1;2y=-3x+14$
9. $2x+\frac{3}{5}y=3;\frac{3}{2}x-y=-5$
10. $5x=9-2y;3y=2x-3$
11. How do you know if a system of equations has no solution?
12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?
13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.
14. Solve

$$12 \cdot \frac{1}{x}-18 \cdot \frac{1}{y}=4$$
$$8 \cdot \frac{1}{x}+9 \cdot \frac{1}{y}=5$$

15. Solve

$$14 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} = -3$$
$$7 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} = 3$$

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8.2 Systems of Three Equations and Three Unknowns

Later, you will learn about matrices and how to row reduce which will allow you to solve systems of equations in a new way. In order to set you up so that using matrices is logical and helpful, it is important to first solve a few systems of three equations using a very specific type of variable elimination.

When solving systems, what are you allowed to do to each equation?

Solving Systems of Equations with Three Unknowns

A system of three equations with three unknowns represents three planes in three dimensional space. When solving the system, you are figuring out how the planes intersect. One way that three planes could intersect is in a point:

A system of equations that has at least one solution is called a **consistent system**.

It is also possible for two or more planes to be parallel or each pair of planes to intersect in a line. In either of these cases the three planes do not intersect at a single point and the system is said to have no solution. A system of equations with no solutions is called an **inconsistent system**. If the three planes intersect at a line or a plane, there are an **infinite number of solutions.**

The following system of equations has the solution (1,3,7) . You can verify this by substituting 1 for x, 3 for y, and 7 for z into each equation.

$$\begin{aligned} x+2y-z&=0\\ 7x-0y+z&=14\\ 0x+y+z&=10 \end{aligned}$$

One thing to be mindful of when given a system of equations is whether or not the equations are linearly independent. Three equations are linearly independent if each equation cannot be produced by a linear combination of the other two. Remember that a linear combination means that one equation can be written as the sum of multiples of the others.

When solving a system of three equations and three variables, there are a few general guidelines that can be helpful:

- Start by trying to eliminate the first variable in the second row.
- Next eliminate the first and second variables in the third row. This will create zero coefficients in the lower right hand corner.
- Repeat this process for the upper right hand corner and you should end up with a very nice diagonal indicating what x, y and z equal.

Take the system of equations mentioned above:

$x+2y-z=0$

$7x-0y+z=14$

$0x+y+z=10$

There are a number of ways to solve this system. Common techniques involve swapping rows,

dividing and multiplying a row by a constant and adding or subtracting a multiple of one row to

another.

Step 1: Swap rows 2 and 3 . Change -0 to $+0$.

$$\begin{aligned} x+2y-z&=0\\ 0x+y+z&=10\\ 7x+0y+z&=14 \end{aligned}$$

Step 2: Subtract 7 times row 1 from row 3 , then replace row 3 .

$$\begin{aligned} x+2y-z&=0\\ 0x+y+z&=10\\ 0x-14y+8z&=14 \end{aligned}$$

Step 3: Add 14 times row 2 to row 3 and then replace row 3.

$$x + 2y - z = 0$$
$$0x + y + z = 10$$
$$0x + 0y + 22z = 154$$

Step 4: Divide row 3 by 22 .

$$x + 2y - z = 0$$
$$0x + y + z = 10$$
$$0x + 0y + z = 7$$

Step 5: Subtract row 3 from row 2 and then replace row 2.

$$x + 2y - z = 0$$

$$0x + y + 0z = 3$$

$$0x + 0y + z = 7$$

Step 6: Add row 3 to row 1 and then replace row 1.

$$x + 2y + 0z = 7$$

$$0x + y + 0z = 3$$

$$0x + 0y + z = 7$$

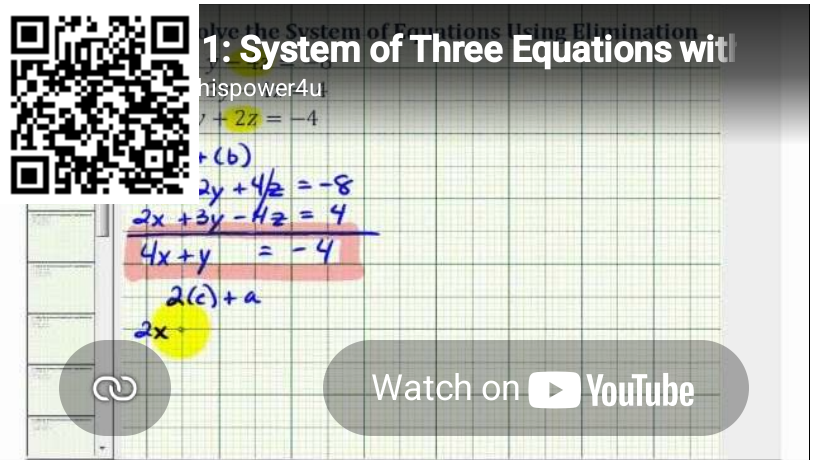
Step 7: Subtract 2 times row 2 from row 1 and then replace row 1

$$x + 0y + 0z = 1$$

$$0x + y + 0z = 3$$

$$0x + 0y + z = 7$$

The solution to the system is (1,3,7) exactly as stated earlier.



Examples

Example 1

Earlier, you were asked what you are allowed to do when solving a system of three equations. When solving a system of three equations with three unknowns, you are allowed to add and subtract rows, swap rows and scale rows. These three operations should allow you to eliminate the coefficients of the variables in a systematic way.

Example 2

Is the following system linearly independent or dependent? How do you know?

$$3x + 2y + z = 8$$

$x + y + z = 3$

$5x + 4y + 3z = 14$

$6x + 6y + 6z = 18$

With four equations and three unknowns there must be at least one dependent equation. The simplest method of seeing linearly dependence is to notice that one equation is just a multiple of the other. In this case the fourth equation is just six times the second equation and so it is dependent.

Most people will not notice that the third equation is also dependent. It is common to start doing a problem and notice somewhere along the way that all the variables in a row disappear. This means that the original equations were dependent. In this case, the third equation is the first equation plus two times the second equation. This means they are dependent.

Example 3

Reduce the following system to a system of two equations and two unknowns.

$3x + 2y + z = 7$

$4x + 0y + z = 6$

$6x - y + 0z = 5$

Strategically swapping rows so that the zero coefficients do not live on the diagonal is a clever starting move.

Step 1: Swap rows 2 and 3.

$3x + 2y + z = 7$

$6x - y + 0z = 5$

$4x + 0y + z = 6$

Step 2: Scale row 3 by a factor of 3. Subtract 2 times row 1 from row 2 and replace row 2.

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 5y - 2z &= -9 \\ 12x + 0y + 3z &= 18 \end{aligned}$$

Step 3: Subtract 4 times row 1 from row 3 and replace row 3.

$3x + 2y + z = 7$

$0x - 5y - 2z = -9$

$0x - 8y - z = -10$

Step 4: Scale the second row by 8 and the third row by 5.

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 40y - 16z &= -72 \\ 0x - 40y - 5z &= -50 \end{aligned}$$

Step 5: Subtract row 2 from row 3 and replace row 3.

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 40y - 16z &= -72 \\ 0x + 0y + 11z &= +22 \end{aligned}$$

Step 6: Divide row 3 by 11.

$$\begin{aligned} 3x + 2y + z &= 7 \\ 0x - 40y - 16z &= -72 \\ 0x + 0y + z &= 2 \end{aligned}$$

Now that $z = 2$, rewrite the system so it becomes a system of two equations and two unknowns.

$3x + 2y + 2 = 7$

$0x - 40y - 32 = -72$

Solving the second row shows that $y = 1$ which can be used to determine that $x = 1$.

Example 4

When Kaitlyn went to the store with ten dollars she saw that she had some choices about what to buy. She could get one apple, one onion and one basket of blueberries for 9 dollars. She could get two apples and two onions for 10 dollars. She could also get two onions and one basket of blueberries for 10 dollars. Write and solve a system of equations with variables a, o and b representing each of the three things she can buy.

Here is the system of equations:

$$\begin{aligned} a + o + b &= 9 \\ 2a + 2o &= 10 \\ 2o + b &= 10 \end{aligned}$$

Rewrite the system using x, y and z so that o and 0 do not get mixed up. Include coefficients of 0 so that each column represents one variable.

Step 1: Rewrite

$$1x + 1y + 1z = 9$$

$$2x + 2y + 0z = 10$$

$$0x + 2y + 1z = 10$$

Step 2: Subtract 2 times row 1 from row 2 and replace row 2.

$$1x + 1y + 1z = 9$$

$$0x + 0y - 2z = -8$$

$$0x + 2y + 1z = 10$$

Step 3: Divide row 2 by -2.

$$\begin{aligned} 1x + 1y + 1z &= 9 \\ 0x + 0y + 1z &= 4 \\ 0x + 2y + z &= 10 \end{aligned}$$

At this point you can see from the second equation that $z = 4$. From the third equation, $2y + 4 = 10$, so $y = 3$. Finally you can see from the first equation that $x + 3 + 4 = 9$ so $x = 2$. Apples cost 2 dollars each, onions cost 3 dollars each and blueberries cost 4 dollars each.

Example 5

Show that the following system is dependent.

$$x + y + z = 9$$

$$x + 2y + 3z = 22$$

$$2x + 3y + 4z = 31$$

You could notice that the third equation is simply the sum of the other two. What happens when you do not notice and try to solve the system as if it were independent?

Step 1: Rewrite the system.

$$x + y + z = 9$$

$$x + 2y + 3z = 22$$

$$2x + 3y + 4z = 31$$

Step 2: Subtract 2 times row 1 from row 3 and replace row 3.

$$x + y + z = 9$$
$$x + 2y + 3z = 22$$
$$0x + 1y + 2z = 13$$

Step 3: Subtract row 1 from row 2 and replace row 2 .

$$x + y + z = 9$$
$$0x + 1y + 2z = 13$$
$$0x + 1y + 2z = 13$$

At this point when you subtract row 2 from row 3, all the coefficients in row 3 disappear. This means that you will end up with the following system of only two equations and three unknowns. Since the unknowns outnumber the equations, the system does not have a solution of one point.

$$x + y + z = 9$$
$$0x + 1y + 2z = 13$$

Review

1. An equation with three variables represents a plane in space. Describe all the ways that three planes could interact in space.

2. What does it mean for equations to be linearly dependent?

3. How can you tell that a system is linearly dependent?

4. If you have linearly independent equations with four unknowns, how many of these equations would you need in order to get one solution?

5. Solve the following system of equations:

$$3x - 4y + z = -17$$

$$6x + y - 3z = 4$$

$$-x - y + 5z = 16$$

6. Show that the following system is dependent:

$$2x - 2y + z = 5$$
$$6x + y - 3z = 2$$
$$4x + 3y - 4z = -3$$

7. Solve the following system of equations:

$$4x + y + z = 15$$
$$-2x + 3y + 4z = 38$$
$$-x - y + 3z = 16$$

8. Solve the following system of equations:

$$3x - 2y + 3z = 6$$
$$x + 3y - 3z = -14$$
$$-x + y + 5z = 22$$

9. Solve the following system of equations:

$$3x - y + z = -10$$

$$6x - 2y + 2z = -20$$
$$-x - y + 4z = 12$$

10. Solve the following system of equations:

$$x - 3y + 6z = -30$$
$$4x + 2y - 3z = 18$$
$$-8x - 3y + 2z = -22$$

11. Solve the following system of equations.

$$x + 2y + 2z + w = 5$$
$$2x + y + 2z - 0w = 5$$
$$3x + 3y + 3z + 2w = 12$$
$$x + 0y + z + w = 1$$

A parabola goes through $(3, -9.5), (6, -32)$, and (-4,8)

12. Write a system of equations that you could use to solve to find the equation of the parabola. Hint: Use the general equation $Ax^2 + Bx + C = y$.

13. Solve the system of equations from #12.

A parabola goes through $(-2, 3), (2, 19)$, and $(1, 6)$.

14. Write a system of equations that you could use to solve to find the equation of the parabola. Hint: Use the general equation $Ax^2 + Bx + C = y$.

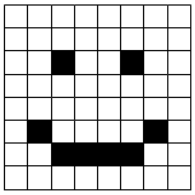
15. Solve the system of equations from #14.

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8.3 Matrices to Represent Data

A **matrix** is a rectangular array of numbers representing data in a variety of forms. Computers work very heavily with matrices because operations with matrices are efficient with memory. Matrices can represent statistical data with numbers, but also graphical data with pictures.

How might you use a matrix to write the following image as something a computer could recognize and work with?



Introduction to Matrices

A matrix is a means of storing information effectively and efficiently. The rows and columns each mean something very specific and the location of a number is just as important as its value. The following are all examples of matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The entries in a matrix can be written out using brackets like [ ], but they can also be described individually using a set of 2 subscript indices i and j that stand for the row number and the column number. Alternatively, the matrix can be named with just a capital letter like A.

$$A=[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Square matrices have the same number of rows as columns. The **order of a matrix** describes the number of rows and the number of columns in the matrix. The following matrix is said to have order 2×3 because it has two rows and three columns. A 1×1 matrix is just a regular number.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

A **symmetric matrix** is a special type of square matrix that has reflection symmetry across the main diagonal. The identity matrix is an example of a symmetric matrix.

The **identity matrix** of order $n \times n$ has zeros everywhere except along the main diagonal where it has ones. Just like the number 1 has an important property with numbers, the identity matrix of any order has special properties as well.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When you turn the rows of a matrix into the columns of a new matrix, the two matrices are

transpositions of one another. The superscript T stands for transpose. Sometimes using the

transpose of a matrix is more useful than using the matrix itself.

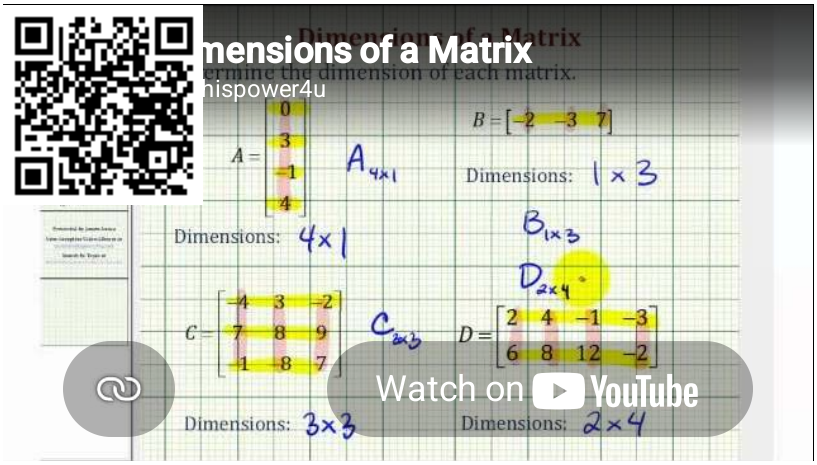
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A **triangular matrix** is not a matrix in the shape of a triangle. Rather, a **lower triangular matrix** is a square matrix where every entry below the diagonal is zero. An **upper triangular matrix** is a square matrix where every entry above the diagonal is zero. The following is a lower triangular matrix. When you work with solving matrices, look for triangular matrices because they are much easier to solve.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

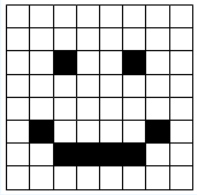
A **diagonal matrix** is both upper and lower triangular which means all the entries except those along the diagonal are zero. The identity matrix is a special case of a diagonal matrix.



Examples

Example 1

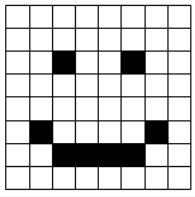
Earlier, you were asked how you might use a matrix to write the following image as something a computer could recognize and work with.



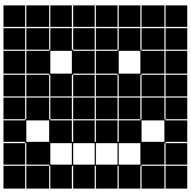
By writing every hollow square as a 0 and a blank square as a 1 a computer could read the picture:

When you use computers to manipulate images, the computer just manipulates the numbers. In this case, if you swap zeros and ones, you get the negative image.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Real photos and computer images have matrices that are much larger and include more numbers than just zero and one to account for more colors.

Example 2

Kate runs three bakeries and each bakery sells bagels and muffins. The rows represent the bakeries and the columns represent bagels (left) and muffins (right) sold. Answer the following questions about Kate's sales.

$$K=\begin{bmatrix} 144 & 192 \\ 115 & 127 \\ 27 & 34 \end{bmatrix}$$

1. What does 127 represent?

a. 127 represents the number of muffins that Kate sold in her second location. You know this because it is in the muffin column and the second row.

2. How many muffins did Kate sell in total?

a. The total muffins sold is equal to the sum of the right hand column. $192+127+34=353$

3. How many bagels did Kate sell in her first location?

a. Kate sold 144 bagels at her first location.

4. Which location is doing poorly?

a. The third location is doing much worse than the other two locations.

Example 3

Identify the order of the following matrices

$$A=\begin{bmatrix} 1 & 3 & 4 & 7 \end{bmatrix}, \quad B=\begin{bmatrix} 21 & 45 & 1 \\ 34 & 1 & 5 \end{bmatrix}, \quad C=\begin{bmatrix} 25 & 235 \\ 562 & 562 \\ 4 & 413 \\ 454 & 33 \\ 1 & 141 \end{bmatrix}$$

A is $1\times 4, B$ is $2\times 3, C$ is 5×2. Note that $4\times 1, 3\times 2, 2\times 5$ are not the same orders and would be incorrect.

Example 4

Write out the 5×4 matrix whose entries are $a_{ij}=\frac{i+j}{j}$

$$\begin{bmatrix} 2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \frac{6}{5} \\ 3 & 2 & \frac{5}{3} & \frac{3}{2} & \frac{7}{5} \\ 4 & \frac{5}{2} & 2 & \frac{7}{4} & \frac{8}{5} \\ 5 & 3 & \frac{7}{3} & 2 & frac95 \end{bmatrix}$$

Example 5

Create a 3×3 matrix for each of the following:

a. Diagonal Matrix

Possible answer:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

b. Lower Triangular

Possible answer:

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 14 \\ 0 & 0 & 5 \end{bmatrix}$$

c. Symmetric

Possible answer

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 14 \\ 1 & 14 & 5 \end{bmatrix}$$

d. Identity:

Answer:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that while the identity matrix does technically work for all the parts of this problem, it does not highlight the differences between each definition.

Review

State the order of each of the following matrices:

1. $A = \begin{bmatrix} 4 & 2 & 4 & 7 \\ 5 & 2 & 1 & 0 \end{bmatrix}$
2. $B = \begin{bmatrix} 0 & 1 \\ 34 & 1 \end{bmatrix}$
3. $C = \begin{bmatrix} 2 & 62 \\ 14 & 3 \\ 4 & 3 \\ 1 & 11 \end{bmatrix}$
4. $D = \begin{bmatrix} 12 & 0 & 2 \\ 0 & 3 & 3 \\ 4 & 0 & 1 \\ 1 & 4 & 0 \end{bmatrix}$
5. $E = \begin{bmatrix} 1 & 11 \end{bmatrix}$
6. Give an example of a 1×1 matrix.
7. Give an example of a 3×2 matrix.
8. If a symmetric matrix is also lower triangular, what type of matrix is it?
9. Write out the 2×3 matrix whose entries are $a_{ij} = i - j$.

Morgan worked for three weeks during the summer earning money on Mondays, Tuesdays, Wednesdays, Thursdays, and Fridays. The following matrix represents his earnings.

$$\begin{bmatrix} 24 & 22 & 32 \\ 25 & 28 & 30 \\ 30 & 28 & 32 \\ 10 & 15 & 19 \\ 35 & 32 & 30 \end{bmatrix}$$

10. What do the rows and columns represent?

11. How much money did Morgan make in the first week?

12. How much money did Morgan make on Tuesdays?

13. What day of the week was most profitable?

14. What day of the week was least profitable?

15. Is the following a matrix? Explain.

$$\begin{bmatrix} \text{dogs} & 0 \\ \text{cats} & 3 \\ \text{sheep} & 0 \\ \text{ducks} & 4 \end{bmatrix}$$

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8.4 Matrix Operations

Algebra refers to your ability to manipulate variables and unknowns based on rules and properties. Matrix algebra is extremely similar to the algebra you already know for numbers with a few important differences. What are these differences?

Algebra with Matrices

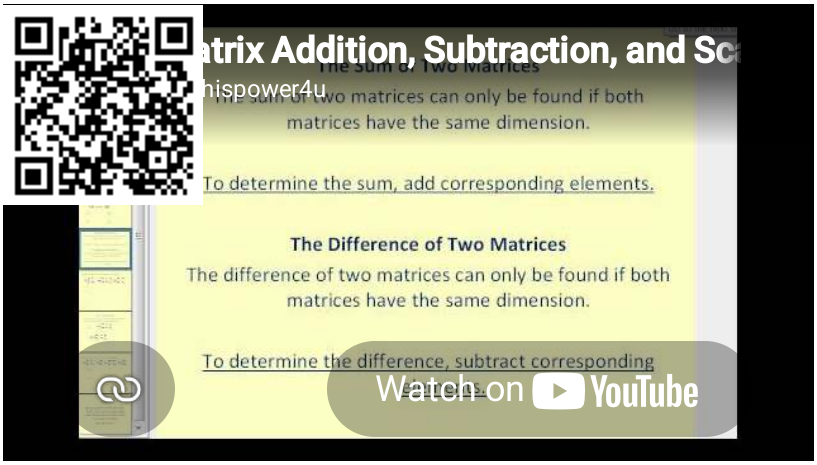
Addition and Subtraction

Two matrices of the **same order** can be added by summing the entries in the corresponding positions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

Two matrices of the **same order** can be subtracted by subtracting the entries in the corresponding positions.

$$\begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$



Multiplication

You can find the product of matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$ if the number of columns in matrix $\boldsymbol{A}$ matches the number of rows in matrix $\boldsymbol{B}$. Another way to remember this is when you write the orders of matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$ next to each other they must be connected by the same number. The resulting matrix has the number of rows from the first matrix and the number of columns from the second matrix.

$(2 \times 3) \cdot (3 \times 5) = (2 \times 5)$

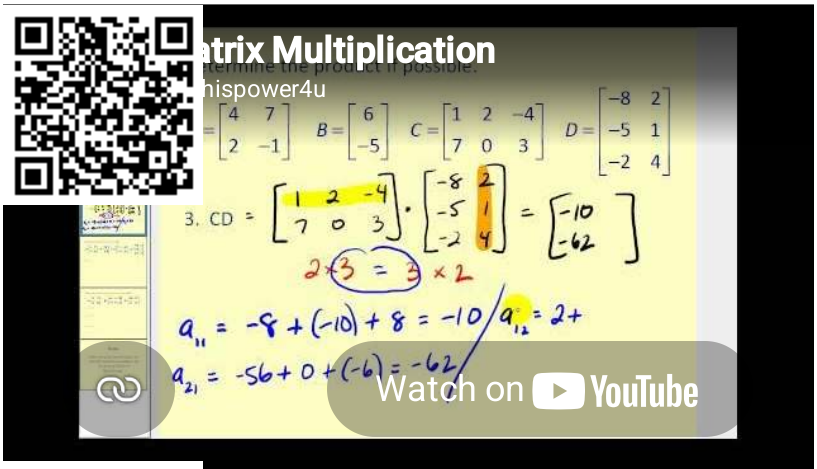
To compute the first entry of the resulting 2×5 matrix you should match the first row from the first matrix and the first column of the second matrix. The arithmetic operation to combine these numbers is identical to taking the dot product between two vectors.

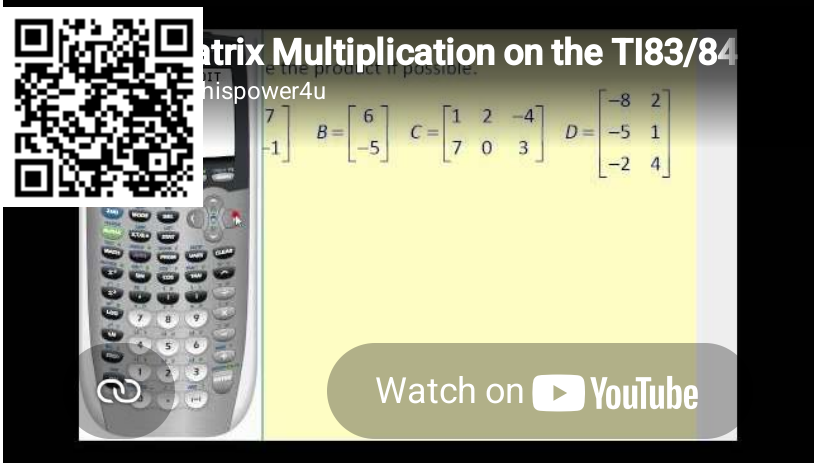
$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & ? & ? & ? & ? \\ 21 & ? & ? & ? & ? \end{bmatrix}$$

- The entry in the first row first column of the new matrix is computed as $1 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$.
- - The entry in the second row first column of the new matrix is computed as $5 \cdot 0 + 6 \cdot 2 + 9 \cdot 1 = 21$
- - The entry in the first row second column of the new matrix is computed as $1 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 = 4$
- - The entry in the second row second column of the new matrix is computed as $5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 = 14$

Continue this pattern and you will find that the solution to this multiplication is:

$$C = \begin{bmatrix} 11 & 4 & 12 & 9 & 7 \\ 21 & 14 & 42 & 17 & 15 \end{bmatrix}$$





Other Properties of Matrix Algebra

- Commutativity holds for matrix addition. This means that when matrices A and B can be added (when they have matching orders), then: $A + B = B + A$
- Commutativity does not hold in general for matrix multiplication.
- Associativity holds for both multiplication and addition. $(AB)C = A(BC), (A + B) + C = A + (B + C)$
- Distribution over addition and subtraction holds. $A(B \pm C) = AB \pm AC$

Examples

Example 1

Earlier, you were asked what the differences between matrix and regular algebra are. The main difference between matrix algebra and regular algebra with numbers is that matrices do not have the commutative property for multiplication. There are other complexities that matrices have, but many of them stem from the fact that for most matrices $AB \neq BA$.

Example 2

Show the commutative property does not hold by demonstrating $AB \neq BA$

$$A = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 2 & 0 \\ 4 & 3 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 1 \\ 4 & 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 22 & -1 \\ 5 & 9 & 3 \\ 58 & 62 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 9 & 12 & 20 \\ 6 & 5 & 28 \\ 3 & 2 & 32 \end{bmatrix}$$

Example 3

Compute the following matrix arithmetic: $10 \cdot (2A - 3C) \cdot B$.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 12 & 0 \\ 1 & 3 \end{bmatrix}$$

When a matrix is multiplied by a scalar (such as with $2A$), multiply each entry in the matrix by the scalar.

$$2A = \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix}$$

$$-3C = \begin{bmatrix} -36 & 0 \\ -3 & -9 \end{bmatrix}$$

$$2A - 3C = \begin{bmatrix} -34 & 4 \\ 5 & 1 \end{bmatrix}$$

Since the associative property holds, you can either distribute the ten or multiply by matrix B next.

$$(2A - 3C) \cdot B = \begin{bmatrix} 16 & -22 & -60 \\ 4 & 8 & 12 \end{bmatrix}$$

$$10 \cdot (2A - 3C) \cdot B = \begin{bmatrix} 160 & -220 & -600 \\ 40 & 80 & 120 \end{bmatrix}$$

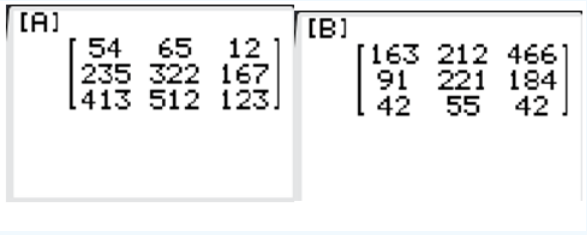
Example 4

Use your calculator to input and compute the following matrix operations.

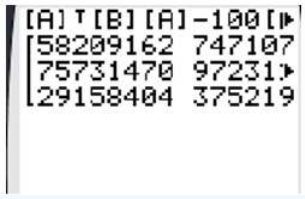
$$A = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}, \quad B = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$$

$A^T \cdot B \cdot A - 100A$

Most graphing calculators like the TI-84 can do operations on matrices. Find where you can enter matrices and enter the two matrices.



Then type in the appropriate operation and see the result. The TI-84 has a built in Transpose button.



The actual numbers on this guided practice are less important than the knowledge that your calculator can perform all of the matrix algebra demonstrated in this concept. It is useful to fully know the capabilities of the tools at your disposal, but it should not replace knowing why the calculator does what it does.

Example 5

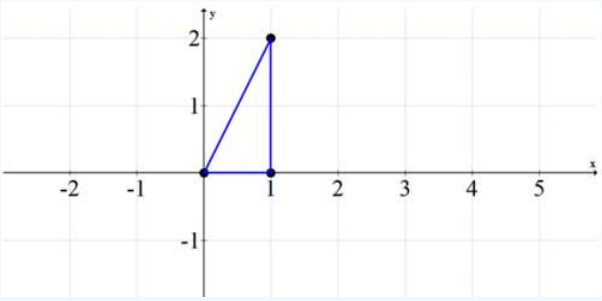
Matrix multiplication can be used as a transformation in the coordinate system. Consider the triangle with coordinates (0, 0) (1, 2) and (1, 0) the following matrix:

\left[\begin{array}{cc}

\cos 90^{\circ} & \sin 90^{\circ} \

-\sin 90 & \cos 90

\end{array}\right]



What does the new picture look like?

The matrix simplifies to become:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

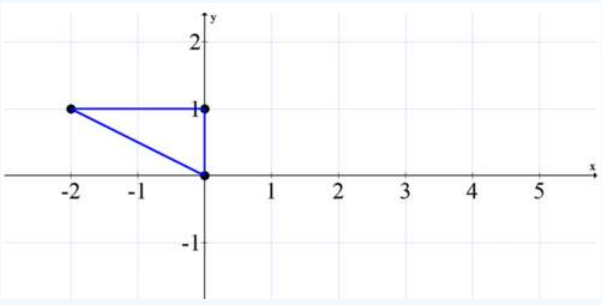
When applied to each point as a transformation, a new point is produced. Note that $\begin{bmatrix} x & y \end{bmatrix}$ is a matrix representing each original point and $[x'y']$ is the new point. The x' is read as "x prime" and is a common way to refer to a result after a transformation.

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$



Notice how the matrix transformation rotates graphs in a counterclockwise direction 90°.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} -y & x \end{bmatrix}$$

The matrix transformation applied in the following order will rotate a graph clockwise 90°.

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

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Review

Do #1-#11 without your calculator.

$A = \begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}, C = \begin{bmatrix} 14 & 6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$

1. Find AC. If not possible, explain.
2. Find BA. If not possible, explain.
3. Find CA. If not possible, explain.
4. Find $4B^T$. If not possible, explain.
5. Find $A + C$. If not possible, explain.
6. Find $D - A$. If not possible, explain.
7. Find $2(A + C - D)$. If not possible, explain.
8. Find $(A + C)B$. If not possible, explain.
9. Find $B(A + C)$. If not possible, explain.
10. Show that $A(C + D) = AC + AD$
11. Show that $A(C - D) = AC - AD$.

Practice using your calculator for #12-#15.

$E = \begin{bmatrix} 312 & 59 & 34 \\ 342 & 156 & 189 \\ 783 & 23 & 133 \end{bmatrix}, F = \begin{bmatrix} 33 & 72 & 21 \\ 93 & 41 & 94 \\ 62 & 75 & 72 \end{bmatrix}, G = \begin{bmatrix} 11 & 735 & 67 \\ 93 & 456 & 2 \\ 94 & 34 & 0 \end{bmatrix}$

12. Find $E + F + G$
13. Find $2E$
14. Find $4F$
15. Find $(E + F)G$

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8.5 Row Operations and Row Echelon Forms

Applying row operations to reduce a matrix is a procedural skill that takes lots of writing, rewriting and careful arithmetic. The payoff for being able to transform a matrix into a simplified form will become clear later. For now, what does the simplified form mean for a matrix?

Heading Row Operations and Row Echelon Forms

There are only three operations that are permitted to act on matrices. They are the exact same operations that are permitted when solving a system of equations.

Add a multiple of one row to another row.

Scale a row by multiplying through by a non-zero constant.

Swap two rows.

Using these three operations, your job is to simplify matrices into row echelon form. **Row echelon form** must meet three requirements.

1. The leading coefficient of each row must be a one.

2. All entries in a column below a leading one must be zero.

3. All rows that just contain zeros are at the bottom of the matrix.

Here are some examples of matrices in row echelon form:

$$\begin{bmatrix} 1 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form also has one extra stipulation compared with row echelon form.

4. Every leading coefficient of 1 must be the only non-zero element in that column.

Here are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Putting a matrix into reduced row echelon form is a result of performing Gauss-Jordan elimination. The process illustrated in this concept is named after those two mathematicians.

To put the a matrix into reduced row echelon form, use the row operations to change the matrix. Take the following matrix:

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

In each step of reducing the matrix, only one of the three row operations will be used. Specific shorthand will be introduced.

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

$$3R_2 \rightarrow \begin{bmatrix} 3 & 7 \\ 6 & 15 \end{bmatrix}$$

$$-2R_1 + R_2 \rightarrow \begin{bmatrix} 3 & 7 \\ 0 & 1 \end{bmatrix}$$

$$-7R_2 + R_1 \rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{3}R \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that the $3R_2$ indicates that the second row of the matrix is scaled by a factor of 3. The $-2R_1 + R_2$ before the third matrix indicates that the second row has two times the first row subtracted from it.

Row reducing a 2×2 matrix to become the identity matrix illustrates the fact that the rows of the original matrix are linearly independent.

Examples

Example 1

Earlier, you were asked what it means for a matrix to be simplified. There are two forms of a matrix that are most simplified. The most important is reduced row echelon form that follows the four stipulations from the guidance section. An example of a matrix in reduced row echelon form is:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 43 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 98 & 5 \end{bmatrix}$$

Example 2

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$R_1 \cdot -\tfrac{1}{2} + R_3 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \div 4 \rightarrow \begin{bmatrix} 2 & 4 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \div 2, R_2 \div 3 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \cdot -\tfrac{1}{3} + R_2 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \cdot -2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that two operations were used in the fourth row to produce the fourth matrix. This is acceptable when the operations do not interfere or interact with each other.

Again, row reducing a 3×3 matrix to become the identity matrix is just an exercise that illustrates the fact that the rows were linearly independent.

Example 3

Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix}$$

Switch Rows $\rightarrow \begin{bmatrix} 2 & 6 & 8 \\ 0 & 4 & 5 \end{bmatrix}$

$R_1 \div 2 \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 5 \end{bmatrix}$

$R_2 \cdot \frac{1}{4} \rightarrow \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & \frac{5}{4} \end{bmatrix}$

$R_2 \cdot -3 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{5}{4} \end{bmatrix}$

Example 4

Reduce the following matrix to row echelon form.

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 17 \end{bmatrix}$$

$R_1 \div 3, R_2 \div 2 \rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 5 & 17 \end{bmatrix}$

$R_1 \cdot -1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 5 & 17 \end{bmatrix}$

$R_1 \cdot -5 + R_3 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 7 \end{bmatrix}$

Switch R_2 and $R_3 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$

$R_2 \div 7 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$R_2 \cdot -2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Example 5

Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$$

$R_1 \cdot 5 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix}$

$R_2 \cdot 3 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 15 & -3 & 0 & 3 \end{bmatrix}$

$R_2 - R_1 \rightarrow \begin{bmatrix} 15 & 20 & 5 & 0 \\ 0 & -23 & -5 & -3 \end{bmatrix}$

$R_1 \div 15 \rightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & -23 & -5 & 3 \end{bmatrix}$

$R_2 \div -23 \rightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{23} & -\frac{3}{23} \end{bmatrix}$

$R_2 \cdot -\frac{4}{3} + R_1 \rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{23} & \frac{4}{23} \\ 0 & 1 & \frac{5}{23} & -\frac{3}{23} \end{bmatrix}$

Notice how fractions were avoided until the final step. Adding and subtracting large numbers in a matrix is easier to handle than adding and subtracting small numbers because then you don't need to find a common denominator.

Review

1. Give an example of a matrix in row echelon form.

2. Give an example of a matrix in reduced row echelon form.

3. What are the three row operations you are allowed to perform when reducing a matrix?

4. If a square matrix reduces to the identity matrix, what does that mean about the rows of the original matrix?

Use the following matrix for $5 - 6$

$$A = \begin{bmatrix} -3 & -4 & -12 \\ 4 & 4 & 12 \\ -11 & -12 & -35 \end{bmatrix}$$

5. Reduce matrix A to row echelon form.

6. Reduce matrix A to reduced row echelon form. Are the rows of matrix A linearly independent?

Use the following matrix for $7 - 8$.

$$B = \begin{bmatrix} 3 & -4 & 8 \\ 9 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

7. Reduce matrix B to row echelon form.

8. Reduce matrix B to reduced row echelon form. Are the rows of matrix B linearly independent?

Use the following matrix for $9 - 10$.

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 3 & 6 & -3 & 1 \\ 6 & 12 & -7 & 0 \end{bmatrix}$$

9. Reduce matrix C to row echelon form.

10. Reduce matrix C to reduced row echelon form. Are the rows of matrix C linearly independent?

Use the following matrix for $11 - 12$.

$$D = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{bmatrix}$$

11. Reduce matrix D to row echelon form.

12. Reduce matrix D to reduced row echelon form. Are the rows of matrix D linearly independent?

Use the following matrix for $13 - 14$

$$E = \begin{bmatrix} -5 & -6 & -12 \\ -1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$$

13. Reduce matrix E to row echelon form.

14. Reduce matrix E to reduced row echelon form. Are the rows of matrix E linearly independent?

Use the following matrix for $15 - 16$.

$$F = \begin{bmatrix} -23 & 6 & 3 \\ 2 & -\frac{1}{2} & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

15. Reduce matrix F to row echelon form.

16. Reduce matrix F to reduced row echelon form. Are the rows of matrix F linearly independent?

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8.6 Augmented Matrices

The reason why the rules for row reducing matrices are the same as the rules for eliminating coefficients when solving a system of equations is because you are essentially doing the same thing in each case. When you write and rewrite the equation every time you end up writing down lots of extra information. Matrices take care of this information by embedding it in the location of each entry. How would you use matrices to write the following system of equations?

$5x + y = 6$

$x + y = 10$

HEADING Solving Systems of Equations with Augmented Matrices

In order to represent a system as a matrix equation, first write all the equations in standard form so that the coefficients of the variables line up in columns. Then copy down just the coefficients in a **coefficient matrix array**. Next copy the variables in a **variable matrix** and the constants into a **constant matrix**.

$x + y + z = 9$

$x + 2y + 3z = 22$

$2x + 3y + 4z = 31$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

The reason why this works is because of the way matrix multiplication is defined.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1x + 1y + 1z \\ 1x + 2y + 3z \\ 2z + 3y + 4z \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \end{bmatrix}$$

Notice how putting brackets around the two matrices on the right does very little to hide the fact that this is just a regular system of 3 equations and 3 variables.

Once you have your system represented as a matrix you can solve it using an **augmented matrix**. An augmented matrix is two matrices that are joined together and operated on as if they were a single matrix. In the case of solving a system, you need to augment the coefficient matrix and the constant matrix. The vertical line indicates the separation between the coefficient matrix and the constant matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

To solve, reduce the matrix to reduced row echelon form.

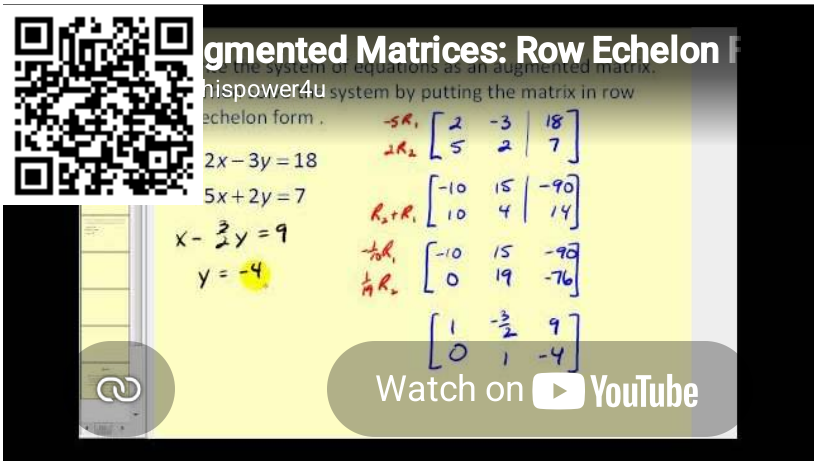
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

$$R_1 \cdot -1 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

$$R_1 \cdot -2 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$$R_2 \cdot -1 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Because the last row is all 0's, this system is dependent. Therefore, there are an infinite number of solutions.



Examples

Example 1

Earlier, you were asked how to write a system of equations as a matrix equation. If you were to write the system as a matrix equation, you could write:

$$\begin{aligned} 5x+y&=6\\ x+y&=10 \end{aligned}$$

$$\begin{bmatrix} 5 & 1\\ 1 & 1 \end{bmatrix}\cdot\begin{bmatrix} x\\ y \end{bmatrix}=\begin{bmatrix} 6\\ 10 \end{bmatrix}$$

Example 2

Solve the following system using an augmented matrix.

$x+y+z=6$

$x-y-z=-4$

$x+2y+3z=14$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6\\ 1 & -1 & -1 & -4\\ 1 & 2 & 3 & 14 \end{array}\right]$$

$$R_1\cdot -1+R_2\rightarrow\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6\\ 0 & -2 & -2 & -10\\ 1 & 2 & 3 & 8 \end{array}\right]$$

$$R_1\cdot -1+R_3\rightarrow\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6\\ 0 & -2 & -2 & -10\\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$R_3\cdot -1+R_1\rightarrow\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & -2 & -2 & -10\\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$R_3\cdot 3+R_2\rightarrow\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & 1 & 4 & 14\\ 0 & 1 & 2 & 8 \end{array}\right]$$

$$R_2\cdot -1+R_3\rightarrow\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2\\ 0 & 1 & 4 & 14\\ 0 & 0 & -2 & -6 \end{array}\right]$$

$R_3 \div -2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_3 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$R_3 \cdot -4 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

Every matrix can be interpreted as its own linear system. The final augmented matrix can be interpreted as:

$1x + 0y + 0z = 1$

$0x + 1y + 0z = 2$

$0x + 0y + 1z = 3$

Which means $x = 1, y = 2, z = 3$.

Example 3

Solve the following system using augmented Matrices.

$w + x + z = 11$

$w + x = 9$

$x + y = 7$

$y + z = 5$

While substitution would work in this problem, the idea is to demonstrate how augmented matrices will work even with larger matrices.

$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right]$

Switch R_2, R_3, and $R_4 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 1 & 1 & 0 & 0 & 9 \end{array}\right]$

$R_1 \cdot -1 + R_4 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$

$R_4 \cdot -1 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$R_4 \cdot -1 + R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$R_4 \cdot -1 + R_3 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$R_3 \cdot -1 + R_2 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$R_2 \cdot -1 + R_1 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

Thus, $w = 5, x = 4, y = 3, z = 2$

Example 4

Use an augmented matrix to solve the following system.

$3x + y = -15$

$x + 2y = 15$

The row reduction steps are not shown, only the initial and final augmented matrices.

$\left[\begin{array}{cc|c} 3 & 1 & -15 \\ 1 & 2 & 15 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 12 \end{array}\right]$

Example 5

Use an augmented matrix to solve the following system.

$-a + b - c = 0$
$2a - 2b - 3c = 25$
$3a - 4b + 3c = 2$

The row reduction steps are not shown, only the initial and final augmented matrices.

$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & -2 & -3 & 25 \\ 3 & -4 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

Review

Solve the following systems of equations using augmented matrices. If one solution does not exist, explain why not.

$$\begin{aligned} 4x - 2y &= -20 \\ x - 3y &= -15 \end{aligned} \quad (1)$$

$$\begin{aligned} 3x + 5y &= 33 \\ -x - 2y &= -13 \end{aligned} \quad (2)$$

$$\begin{aligned} x + 4y &= 11 \\ 3x + 12y &= 33 \end{aligned} \quad (3)$$

$$\begin{aligned} -3x + y &= -7 \\ -x + 4y &= 5 \end{aligned} \quad (4)$$

$$\begin{aligned} 3x + y &= 6 \\ -6x - 2y &= 10 \end{aligned} \quad (5)$$

$$\begin{aligned} 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned} \quad (6)$$

$$\begin{aligned} 4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17 \end{aligned} \tag{7}$$

$$\begin{aligned} 3x + 2y - 3z &= 7 \\ -x + 5y + 2z &= 29 \\ x + 2y + z &= 15 \end{aligned} \tag{8}$$

$$\begin{aligned} 2x + y - 2z &= 4 \\ -4x - 2y + 4z &= -8 \\ 3x + y - z &= 5 \end{aligned} \tag{9}$$

$$\begin{aligned} -x + 3y + z &= 11 \\ 3x + y + 2z &= 27 \\ 5x - y - z &= 5 \end{aligned} \tag{10}$$

$$\begin{aligned} 3x + 2y + 4z &= 21 \\ -2x + 3y + z &= -11 \\ x + 2y - 3z &= -3 \end{aligned} \tag{11}$$

$$\begin{aligned} -x + 2y - 6z &= 4 \\ 8x + 5y + 3z &= -8 \\ 2x - 4y + 12z &= 5 \end{aligned} \tag{12}$$

$$\begin{aligned} 3x + 5y + 8z &= 37 \\ -6x + 3y + z &= 42 \\ x + 3y - 2z &= 5 \end{aligned} \tag{13}$$

$$\begin{aligned} 4x + y - 6z &= -38 \\ 2x + 7y + 8z &= 108 \\ -3x + 2y - 3z &= -15 \end{aligned} \tag{14}$$

$$\begin{aligned} 6x + 3y - 2z &= -22 \\ -4x - 2y + 4z &= 28 \\ 3x + 3y + 2z &= 7 \end{aligned} \tag{15}$$

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8.10 Partial Fractions

When given a rational expression like $\frac{4x-9}{x^2-3x}$ it is very helpful in calculus to be able to write it as the sum of two simpler fractions like $\frac{3}{x}+\frac{1}{x-3}$. The challenging part is trying to get from the initial rational expression to the simpler fractions.

You may know how to add fractions and go from two or more separate fractions to a single fraction, but how do you go the other way around?

HEADING Partial Fraction Decomposition

Partial fraction decomposition is a procedure that reverses adding fractions with unlike denominators. The most challenging part is coming up with the denominators of each individual partial fraction. See if you can spot the pattern.

$$\frac{6x-1}{x^2(x-1)(x^2+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{Dx+E}{x^2+2}$$

In this example each individual factor of the denominator must be represented. Linear factors that are raised to a power greater than one must have each successive power included as a separate denominator. Quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of x. Take a look at the examples to see partial fraction decomposition put into practice.

Examples

Example 1

Earlier, you were asked how to go from one fraction to multiple simpler fractions. To decompose the rational expression into the sum of two simpler fractions you need to use partial fraction decomposition.

$$\begin{aligned}\frac{4x-9}{x^2-3x}&=\frac{A}{x}+\frac{B}{x-3}\\ 4x-9&=A(x-3)+Bx\\ 4x-9&=Ax-3A+Bx\end{aligned}$$

Notice that the constant term -9 must be equal to the constant term $-3A$ and that the terms with x must be equal as well.

$$\begin{aligned}-9&=-3A\\ 4&=A+B\end{aligned}$$

Solving this system yields:

$$A=3,\quad B=1$$

Therefore,

$$\frac{4x-9}{x^2-3x}=\frac{3}{x}+\frac{1}{x-3}$$

Example 2

Use partial fractions to decompose the following rational expression.

$$\frac{7x^2+x+6}{x^3+3x}$$

First factor the denominator and identify the denominators of the partial fractions.

$$\frac{7x^2+x+6}{x(x^2+3)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$$

When the fractions are eliminated by multiplying through by the LCD the equation becomes:

$$7x^2+x+6=A\left(x^2+3\right)+x(Bx+C)$$

$$7x^2+x+6=Ax^2+3A+Bx^2+Cx$$

Notice the squared term, linear term and constant term form a system of three equations with three variables

$$A+B=7$$

$$C=1$$

$3A = 6$

In this case it is easy to see that $A = 2, B = 5, C = 1$. Often, the resulting system of equations is more complex and would benefit from your knowledge of solving systems using matrices.

$\frac{7x^2+x+6}{x(x^2+3)} = \frac{2}{x} + \frac{5x+1}{x^2+3}$

Example 3

Decompose the following rational expression.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}$

First identify the denominators of the partial fractions.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x} + \frac{E}{x^2}$

When the entire fraction is multiplied through by $(x-1)^3x^2$ the equation results to.

$5x^4 - 3x^3 - x^2 + 4x - 1$

$= A(x-1)^2x^2 + B(x-1)x^2 + Cx^2 + D(x-1)^3x + E(x-1)^3$

Multiplication of each term can be done separately to be extra careful.

$Ax^4 - 2Ax^3 + Ax^2$

$Bx^3 - Bx^2$

Cx^2

$Dx^4 - 3Dx^3 + 3Dx^2 - Dx$

$Ex^3 - 3Ex^2 + 3Ex - E$

Group terms with the same power of x and set equal to the corresponding term.

$$\begin{aligned} 5x^4 &= Ax^4 + Dx^4 \\ -3x^3 &= -2Ax^3 + Bx^3 - 3D^3 + Ex^3 \\ -x^2 &= Ax^2 - Bx^2 + Cx^2 + 3Dx^2 - 3Ex^2 \\ 4x &= -Dx + 3Ex \\ -1 &= -E \end{aligned}$$

From these 5 equations, every x can be divided out. Assume that $x \neq 0$ because if it were, then the original expression would be undefined.

$$\begin{aligned} 5 &= A + D \\ -3 &= -2A + B - 3D + E \\ -1 &= A - B + C + 3D - 3E \\ 4 &= -D + 3E \\ -1 &= E \end{aligned}$$

This is a system of equations of five variables and 5 equations. Some of the equations can be solved using logic and substitution like $E = -1, D = -7, A = 12$. You can use any method involving determinants or matrices. In this case it is easiest to substitute known values into equations with one unknown value to get more known values and repeat.

$$\begin{aligned} B &= 1 \\ C &= 6 \end{aligned}$$

$$\frac{5x^4 - 3x^3 - x^2 + 4x - 1}{(x-1)^3x^2} = \frac{12}{x-1} + \frac{1}{(x-1)^2} + \frac{6}{(x-1)^3} + \frac{-7}{x} + \frac{-1}{x^2}$$

Example 4

Decompose the following rational expression.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}$

First identify the denominators of the partial fractions.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x} + \frac{E}{x^2}$

When the entire fraction is multiplied through by $(x-1)^3x^2$ the equation results to.

$5x^4 - 3x^3 - x^2 + 4x - 1$

$= A(x-1)^2x^2 + B(x-1)x^2 + Cx^2 + D(x-1)^3x + E(x-1)^3$

Multiplication of each term can be done separately to be extra careful.

$Ax^4 - 2Ax^3 + Ax^2$

$Bx^3 - Bx^2$

Cx^2

$Dx^4 - 3Dx^3 + 3Dx^2 - Dx$

$Ex^3 - 3Ex^2 + 3Ex - E$

Group terms with the same power of x and set equal to the corresponding term.

$$\begin{aligned} 5x^4 &= Ax^4 + Dx^4 \\ -3x^3 &= -2Ax^3 + Bx^3 - 3D^3 + Ex^3 \\ -x^2 &= Ax^2 - Bx^2 + Cx^2 + 3Dx^2 - 3Ex^2 \\ 4x &= -Dx + 3Ex \\ -1 &= -E \end{aligned}$$

From these 5 equations, every x can be divided out. Assume that $x \neq 0$ because if it were, then the original expression would be undefined.

$$\begin{aligned} 5 &= A + D \\ -3 &= -2A + B - 3D + E \\ -1 &= A - B + C + 3D - 3E \\ 4 &= -D + 3E \\ -1 &= E \end{aligned}$$

This is a system of equations of five variables and 5 equations. Some of the equations can be solved using logic and substitution like $E = -1, D = -7, A = 12$. You can use any method involving determinants or matrices. In this case it is easiest to substitute known values into equations with one unknown value to get more known values and repeat.

$$\begin{aligned} B &= 1 \\ C &= 6 \end{aligned}$$

$$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} = \frac{12}{x-1} + \frac{1}{(x-1)^2} + \frac{6}{(x-1)^3} + \frac{-7}{x} + \frac{-1}{x^2}$$

Example 5

Use matrices to help you decompose the following rational expression. Confirm the solution by adding the partial fractions.

$\frac{5x-2}{(2x-1)(3x+4)}$

\(\begin{aligned} \frac{5 x-2}{(2 x-1)(3 x+4)} &=\frac{A}{2 x-1}+\frac{B}{3 x+4} \\ 5 x-2 &=A(3 x+4)+B(2 x-1) \\ 5 x-2 &=3 A x+4 A+2 B x-B \\ 5 &=3 A+2 B \\ -2 &=4 A-B \end{aligned}\)

$$\left[\begin{array}{cc|c} 3 & 2 & 5 \\ 4 & -1 & -2 \end{array}\right] \rightarrow \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \\ \cdot 4 \\ \cdot 3 \end{array} \begin{array}{l} \\ \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 12 & 8 & 20 \\ 12 & -3 & -6 \end{array}\right] \rightarrow \begin{array}{c} \\ -I \end{array} \begin{array}{l} \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 12 & 8 & 20 \\ 0 & -11 & -26 \end{array}\right] \rightarrow \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} 11 \\ .8 \end{array} \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 132 & 88 & 220 \\ 0 & -88 & -208 \end{array}\right]$$

$\rightarrow II \quad \rightarrow$

$$\rightarrow \left[\begin{array}{cc|c} 132 & 0 & 12 \\ 0 & -88 & -208 \end{array}\right] \rightarrow \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \begin{array}{c} \div 132 \\ \div -88 \end{array} \begin{array}{l} \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{11} \\ 0 & 1 & \frac{26}{11} \end{array}\right]$$

$A = \frac{1}{11}, B = -\frac{26}{11}$

$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$

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To confirm, add the fractions.

$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$$

$$5x-2 = \frac{1}{11}(3x+4) + \frac{26}{11}(2x-1)$$

$$55x-22 = 3x+4+26(2x-1)$$

$$55x-22 = 3x+4+52x-26$$

$$55x-22 = 55x-22$$

? Review

Decompose the following rational expressions. Practice using matrices with at least one of the problems.

1. $\frac{3x-4}{(x-1)(x+4)}$
2. $\frac{2x+1}{x^2(x-3)}$
3. $\frac{x+1}{x(x-5)}$
4. $\frac{x^2+3x+1}{x(x-3)(x+6)}$
5. $\frac{3x^2+2x-1}{x^2(x+2)}$
6. $\frac{x^2+1}{x(x-1)(x+1)}$
7. $\frac{4x^2-9}{x^2(x-4)}$
8. $\frac{2x-4}{(x+7)(x-3)}$
9. $\frac{3x-4}{x^2(x^2+1)}$
10. $\frac{2x+5}{(x-3)(x^2+4)}$
11. $\frac{3x^2+2x-5}{x^2(x-3)(x^2+1)}$
12. Confirm your answer to **#1** by adding the partial fractions.
13. Confirm your answer to #3 by adding the partial fractions.
14. Confirm your answer to **#6** by adding the partial fractions.
15. Confirm your answer to #9 by adding the partial fractions.

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8.7 Determinant of Matrices

A **determinant** is a number computed from the entries in a square matrix. It has many properties and interpretations that you will explore in linear algebra. This concept is focused on the procedure of calculating determinants. Once you know how to calculate the determinant of a 2×2 matrix, then you will be able to calculate the determinant of a 3×3 matrix. Once you know how to calculate the determinant of a 3×3 matrix you can calculate the determinant of a 4×4 and so on.

A logical question about determinants is where does the procedure come from? Why are determinants defined in the way that they are?

The Determinant

The determinant of a matrix A is written as $|A|$. For a 2×2 matrix A, the value is calculated as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If you substitute numbers for the letters and try to calculate $\det A$ for $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$, you get:

$$\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3 \cdot 5 - 2 \cdot 1 = 15 - 2 = 13$$

Notice how the diagonals are multiplied and then subtracted.

The determinant of a 3×3 matrix is more involved.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Usually you will start by looking at the top row, although any row or column will work. Then use the checkerboard pattern for signs (shown below) and create smaller 2×2 matrices.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The smaller 2×2 matrices are the entries that remain when the row and column of the coefficient you are working with are ignored.

$$\det B = |B| = +a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Next take the determinant of the smaller 2×2 matrices and you get a long string of computations.

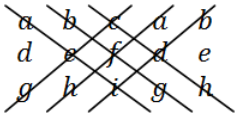
$$= +a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

$$= aei + bfg + cdh - ceg - afh - bdi$$

Most people do not remember this sequence. A French mathematician named Sarrus demonstrated a great device to memorize the computation of the determinant for 3×3 matrices. The first step is simply to copy the first two columns to the right of the matrix. Then draw three diagonal lines going down and to the right.

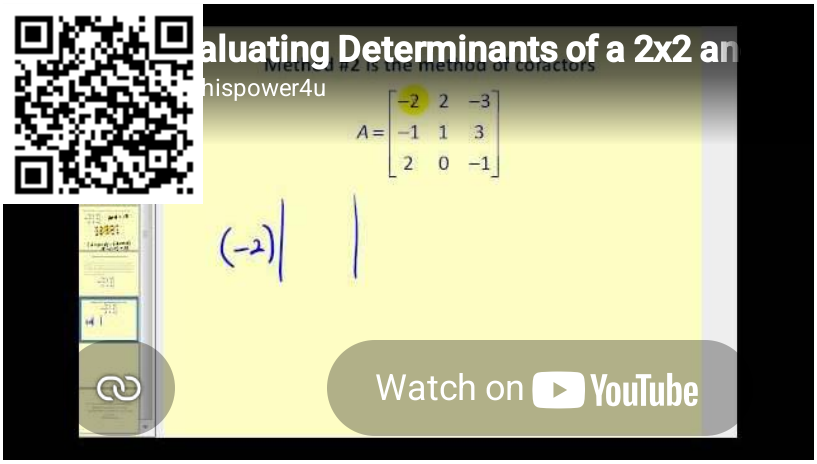
$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



Notice that they correspond exactly to the three positive terms of the determinant demonstrated above. Next draw three diagonals going up and to the right. These diagonals correspond exactly to the three negative terms.

$\det B = aei + bfg + cdh - ceg - afh - bdi$

Sarrus's Rule does not work for the determinants of matrices that are not of order 3×3.



Examples

Example 1

Earlier, you were asked where the procedure for finding the determinants came from. Determinants for 2×2 matrices are defined the way they are because of the general solution to a system of 2 variables and 2 equations.

$ax + by = e$

$cx + dy = f$

To eliminate the x, scale the first equation by c and the second equation by a.

$acx + bcy = ec$
$acx + ady = af$

Subtract the second equation from the first and solve for y.

$ady - bcy = af - ec$
$y(ad - bc) = af - ec$
$y = \frac{af - ec}{ad - bc}$

When you solve for x you also get $ad - bc$ in the denominator of the general solution. This pattern led people to start using this strategy in solving systems of equations. The determinant is defined in this way so it will always be the denominator of the general solution of either variable.

Example 2

Find the determinant of the following matrix.

$C = \begin{bmatrix} -4 & 12 \\ 1 & -3 \end{bmatrix}$

$$\det C = \begin{vmatrix} -4 & 12 \\ 1 & -3 \end{vmatrix} = 12 - 12 = 0$$

Example 3

Find $\det B$ for $B = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix}$

$$\begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= 3(0 \cdot 5 - 2 \cdot 1) - 2(5 \cdot 5 - 2 \cdot 2) + 1(5 \cdot 1 - 2 \cdot 0)$$

$$= -6 - 42 + 5 = -43$$

Example 4

Find the determinant of B from example B using Sarrus's Rule.

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ 5 & 0 & 2 & 5 & 0 \\ 2 & 1 & 5 & 2 & 1 \end{matrix}$$

$$\det B = 0 + 8 + 5 - 0 - 6 - 50 = -43$$

As you can see, Sarrus's Rule is efficient and much of the calculations can be done mentally. Additionally, zero values make much of the multiplication easier.

Example 5

Find the determinant of the following 4×4 matrix by carefully choosing the row or column to work with.

$$E = \begin{bmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{bmatrix}$$

Notice that the third column is made up with zeros and a one. Choose this column to make up the coefficients because then instead of having to evaluate the determinant of four individual

3×3 matrices, you only need to do one.

$$\begin{vmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{vmatrix} = 0 \cdot \begin{vmatrix} -1 & -3 & 3 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ 4 & 8 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix}$$

$$= 4 \cdot (-3) \cdot 9 + 5 \cdot 3 \cdot (-3) + 2 \cdot (-1) \cdot 2 - 18 - 24 - (-45)$$

$$= -154$$

Review

Find the determinants of each of the following matrices.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

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2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 & 4 & 6 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ -6 & 2 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 0 & 0 & 1 \\ 2 & 1 & 8 & 3 \\ 9 & 3 & 2 & 6 \\ -4 & 2 & 5 & 1 \end{bmatrix}$

15. Can you find the determinant for any matrix? Explain.

16. The following matrix has a determinant of zero: $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. If the determinant of a matrix is zero, what does that say about the rows of the matrix?

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8.8 Cramer's Rule

A system of equations can be represented and solved in general using matrices and determinants. This method can be significantly more efficient than eliminating variables in equations. What does it mean for a solution method to be more efficient? Is Cramer's Rule the most efficient means of solving a system of equations?

Using Cramer's Rule

The determinant is defined in a seemingly arbitrary way, however, when you look at the general solution for a 2×2 matrix, the reasoning why it is defined this way is apparent

$ax + by = e$

$cx + dy = f$

When you solve the system above for y and x, you get the following:

$y = \frac{af - ce}{ad - bc}$

$x = \frac{bf - de}{ad - bc}$

Note that the system can be represented by the matrix and the solutions can be written as ratios of two determinants. The determinant in the denominator is of the coefficient matrix. **Cramer's Rule** states that for two equations, the numerator of the x solution is the determinant of the new matrix whose columns are made up of the y coefficients and the solution coefficients. The numerator of the y solution is the determinant of the new matrix made up of the x coefficients and the solution coefficients.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

This is a fantastic improvement over solving systems using substitution or elimination. Cramer's Rule also works with larger order matrices. For a system of 3 variables and 3 equations the reasoning is identical.

$ax + by + cz = j$

$dx + ey + fz = k$

$gx + hy + iz = l$

The system can be represented as a matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

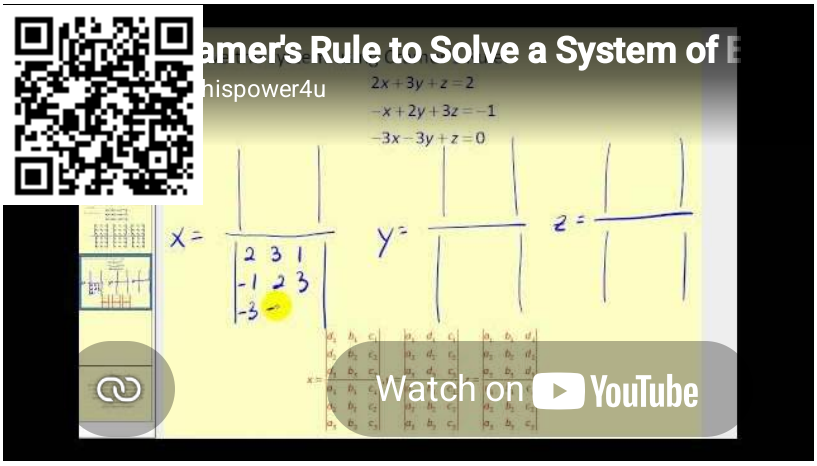
The three solutions can be represented as a ratio of determinants.

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

Remember that evaluating the determinants of 3×3 matrices using Sarrus's rule is very efficient.



Examples

Example 1

Earlier, you were asked about efficient solutions. You have seen that using traditional row reduction to solve a system of equations can take a while and use up a lot of paper. Efficiency partly means requiring less time and space. If this was all that efficiency meant then it would not make sense to solve systems of two equations with two unknowns using matrices because the solution could be found more quickly using substitution. However, the other part of efficiency is minimizing the number of decisions that have to be made. A computer is very good at adding, subtracting and multiplying numbers, but not very good at deciding whether eliminating x or eliminating y would be better. This is why a definite algorithm using matrices and Cramer's Rule is more efficient.

Example 2

Represent the following system of equations as a matrix equation and solve using Cramer's Rule.

$$y - 13 = -3x$$
$$x = 19 - 4y$$

First write each equation in standard form.

$$3x + y = 13$$

$$x + 4y = 19$$

Then write as a coefficient matrix times a variable matrix equal to a solution matrix.

$$\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} 13 & 1 \\ 19 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{13 \cdot 4 - 19 \cdot 1}{3 \cdot 4 - 1 \cdot 1} = \frac{33}{11} = 3$$

$$y = \frac{\begin{vmatrix} 3 & 13 \\ 1 & 19 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{3 \cdot 19 - 13}{11} = \frac{44}{11} = 4$$

Example 3

What is y equal to in the following system?

$x + 2y - z = 0$

$7x - 0y + z = 14$

$0x + y + z = 10$

If you attempted to solve this using elimination, it would take over a page of writing and rewriting to solve. Cramer's Rule speeds up the solving process.

$$\begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 10 \end{bmatrix}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 7 & 14 & 1 \\ 0 & 10 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{14 + 0 + (-70) - 0 - 10 - 0}{0 + 0 + (-7) - 0 - 1 - 14} = \frac{-66}{-22} = 3$$

Example 4

Solve the following system using Cramer's Rule.

$5x + 12y = 72$
$18x - 12y = 108$

$$x = \frac{\begin{vmatrix} 72 & 12 \\ 108 & -12 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{72 \cdot (-12) - 12 \cdot 108}{5 \cdot (-12) - 12 \cdot 18} = \frac{-2160}{-276} = \frac{180}{23}$$

$$y = \frac{\begin{vmatrix} 5 & 72 \\ 18 & 108 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{5 \cdot 108 - 72 \cdot 18}{-276} = \frac{-756}{-276} = \frac{63}{23}$$

Example 5

What is the value of z in the following system?

$3x + 2y + z = 7$

$4x + 0y + z = 6$

$6x - y + 0z = 5$

$$z = \frac{\begin{vmatrix} 3 & 2 & 7 \\ 4 & 0 & 6 \\ 6 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 6 & -1 & 0 \end{vmatrix}} = \frac{0+2\cdot 6\cdot 6+7\cdot 4\cdot(-1)-0-(-1)\cdot 6\cdot 3-5\cdot 4\cdot 2}{0+2\cdot 1\cdot 6+1\cdot 4\cdot(-1)-0-(-1)\cdot 1\cdot 3-0}$$

$$= \frac{22}{11}$$

$$= 2$$

? Review

Solve the following systems of equations using Cramer's Rule. If one solution does not exist, explain.

$$\begin{aligned} 4x - 2y &= -20 \\ x - 3y &= -15 \end{aligned} \tag{1}$$

$$\begin{aligned} 3x + 5y &= 33 \\ -x - 2y &= -13 \end{aligned} \tag{2}$$

$$\begin{aligned} x + 4y &= 11 \\ 3x + 12y &= 33 \end{aligned} \tag{3}$$

$$\begin{aligned} -3x + y &= -7 \\ -x + 4y &= 5 \end{aligned} \tag{4}$$

$$\begin{aligned} 3x + y &= 6 \\ -6x - 2y &= 10 \end{aligned} \tag{5}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(x\backslash)\ in\ the\ following\ system: \\ 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned} \tag{6}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(y\backslash)\ in\ the\ following\ system: \\ 4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17 \end{aligned} \tag{7}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(z\backslash)\ in\ the\ following\ system: \\ 3x + 2y - 3z &= 7 \\ -x + 5y + 2z &= 29 \\ x + 2y + z &= 15 \end{aligned} \tag{8}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(x\backslash)\ in\ the\ following\ system: \\ 2x + y - 2z &= -5 \\ -4x - 2y + 3z &= 2 \\ 3x + y - z &= 3 \end{aligned} \tag{9}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(y\backslash)\ in\ the\ following\ system: \\ -x + 3y + z &= 11 \\ 3x + y + 2z &= 27 \\ 5x - y - z &= 5 \end{aligned} \tag{10}$$

$$\begin{aligned} &Use\ Cramer's\ Rule\ to\ solve\ for\ \backslash(z\backslash)\ in\ the\ following\ system: \\ 3x + 2y + 4z &= 21 \\ -2x + 3y + z &= -11 \\ x + 2y - 3z &= -3 \end{aligned} \tag{11}$$

Solve the following systems of equations using Cramer's Rule. Practice using your calculator to help with at least one problem. If one solution does not exist, explain.

$$\begin{aligned} -x+2y-6z &= 4 \\ 8x+5y+3z &= -8 \\ 2x-4y+12z &= 5 \end{aligned} \tag{12}$$

$$\begin{aligned} 3x+5y+8z &= 37 \\ -6x+3y+z &= 42 \\ x+3y-2z &= 5 \end{aligned} \tag{13}$$

$$\begin{aligned} 4x+y-6z &= -38 \\ 2x+7y+8z &= 108 \\ -3x+2y-3z &= -15 \end{aligned} \tag{14}$$

$$\begin{aligned} 6x+3y-2z &= -22 \\ -4x-2y+4z &= 28 \\ 3x+3y+2z &= 7 \end{aligned} \tag{15}$$

When using Cramer's Rule to solve a system of equations you will occasionally find that the determinant of the coefficient matrix is zero. When this happens, how can you tell whether your system has no solution or infinite solutions (16)

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8.9 Inverse Matrices

Two numbers are multiplicative inverses if their product is 1. Every number besides the number 0 has a multiplicative inverse. For matrices, two matrices are inverses of each other if they multiply to be the identity matrix.

What kinds of matrices do not have inverses?

Inverses of Matrices

Multiplicative inverses are two numbers or matrices whose product is one or the identity matrix. Consider a matrix A that has inverse A^{-1}. How do you find matrix A^{-1} if you just have matrix A ?

A=\left[\begin{array}{ccc}

1 & 2 & 3 \\

1 & 0 & 1 \\

0 & 2 & -1

\end{array}\right], A^{-1}=?

The answer is that you augment matrix A with the identity matrix and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \cdot -1 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 + R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right]$$

$$R_2 \div -2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right]$$

$$R_3 \div -3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_3 \cdot -3 + R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_3 \cdot -1 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$R_2 \cdot -2 + R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

The matrix on the right is the inverse matrix A^{-1}.

$$A^{-1} = \left[\begin{array}{ccc} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

Fractions are usually unavoidable when computing inverses.

One reason why inverses are so powerful is because they allow you to solve systems of equations with the same logic as you would solve a single linear equation. Consider the following system based on the coefficients of matrix A from above.

$$x + 2y + 3z = 96$$
$$x + 0y + z = 36$$
$$0x + 2y - z = -12$$

By writing this system as a matrix equation you get:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

If this were a normal linear equation where you had a constant times the variable equals a constant, you would multiply both sides by the multiplicative inverse of the coefficient. Do the same in this case.

$$A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

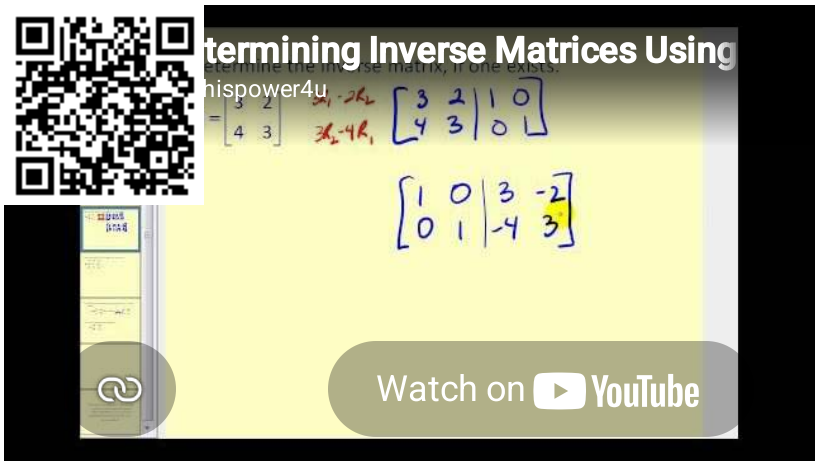
All that is left is for you to substitute in and to perform the matrix multiplication to get the solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \cdot 96 + \frac{4}{3} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{6} \cdot 96 - \frac{1}{6} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{3} \cdot 96 - \frac{1}{3} \cdot 36 - \frac{1}{3} \cdot (-12) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix}$$



Examples

Example 1

Earlier, you were asked what types of matrices do not have inverses. Non-square matrices do not generally have inverses. Square matrices that have determinants equal to zero do not have inverses.

Example 2

Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 6 \\ 4 & 24 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 4 & 24 & 0 & 1 \end{array}\right]$$

$$R_1 \cdot -4 + R_2 \rightarrow \left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{array}\right]$$

This matrix is not invertible because its rows are not linearly independent. To test to see if a square matrix is invertible, check whether or not the determinant is zero. If the determinant is zero then the matrix is not invertible because the rows are not linearly independent.

Example 3

Confirm matrix A and A^{-1} are inverses by computing $A^{-1} \cdot A$ and $A \cdot A^{-1}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$A^{-1} \cdot A = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$a_{11} = -\frac{1}{3} \cdot 1 + \frac{4}{3} \cdot 1 + \frac{1}{3} \cdot 0 = 1$

$a_{22} = \frac{1}{6} \cdot 2 - \frac{1}{6} \cdot 0 + \frac{1}{3} \cdot 2 = 1$

$a_{33} = \frac{1}{3} \cdot 3 - \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = 1$

Note that the rest of the entries turn out to be zero. This is left for you to confirm.

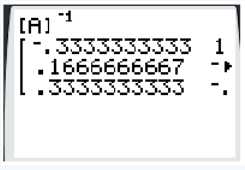
Example 4

Use a calculator to compute A^{-1}, compute $A^{-1} \cdot A$, compute $A \cdot A^{-1}$ and compute

$$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

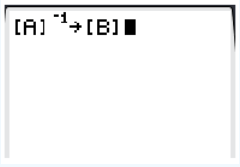
Start by entering just matrix A into the calculator.

To compute matrix A^{-1} use the inverse button programmed into the calculator. Do not try to raise the matrix to the negative one exponent. This will not work.

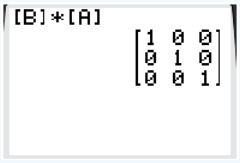


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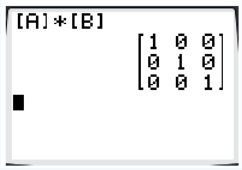
Note that the calculator may return decimal versions of the fractions and will not show the entire matrix on its limited display. You will have to scroll to the right to confirm that A^{-1} matches what you have already found. Once you have found A^{-1} go ahead and store it as matrix B so you do not need to type in the entries.



$A^{-1} \cdot A = B \cdot A$

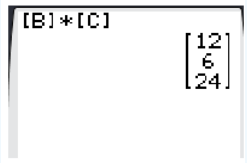


$A \cdot A^{-1} = A \cdot B$



$$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot C$$

You need to create matrix $C = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$



Being able to effectively use a calculator should improve your understanding of matrices and allow you to check all the work you do by hand.

Example 5

The identity matrix happens to be its own inverse. Find another matrix that is its own inverse.

Helmert came up with a very clever matrix that happens to be its own inverse. Here are the 2×2 and the 3×3 versions.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

Review

Find the inverse of each of the following matrices, if possible. Make sure to do some by hand and some with your calculator.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. Show that Helmert's 2×2 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

14. Show that Helmert's 3×3 matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$

15. Non-square matrices sometimes have left inverses, where $\boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$, or right inverses, where $\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{I}$. Why can't non-square matrices have "regular" inverses?

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CHAPTER OVERVIEW

9: Conics

9.1 General Form of a Conic
9.2 Parabolas
9.3 Circles
9.4 Ellipses
9.5 Hyperbolas
9.6 Degenerate Conics

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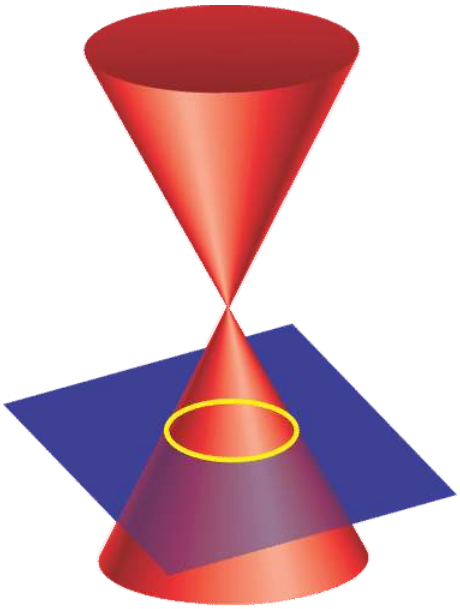
9.1 General Form of a Conic

Conics are a family of graphs that include parabolas, circles, ellipses and hyperbolas. All of these graphs come from the same general equation and by looking and manipulating a specific equation you can learn to tell which conic it is and how it can be graphed.

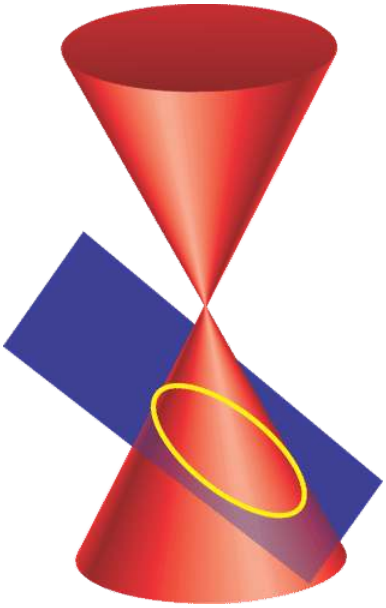
What is the one essential skill that enables you to manipulate the equation of a conic in order to sketch its graph?

Introduction to Conics

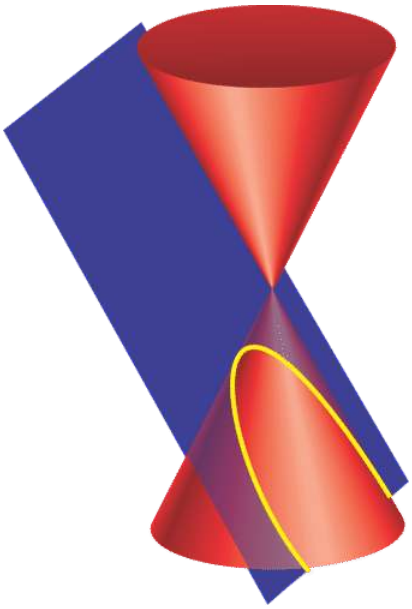
The word conic comes from the word cone which is where the shapes of parabolas, circles, ellipses and hyperbolas originate. Consider two cones that open up in opposite directions and a plane that intersects it horizontally. A flat intersection would produce a perfect circle.



To produce an ellipse, tilt the plane so that the circle becomes elongated and oval shaped. Notice that the angle that the plane is tilted is still less steep than the slope of the side of the cone.



As you tilt the plane even further and the slope of the plane equals the slope of the cone edge you produce a parabola. Since the slopes are equal, a parabola only intersects one of the cones.



Lastly, if you make the plane steeper still, the plane ends up intersecting both the lower cone and the upper cone creating the two parts of a hyperbola.



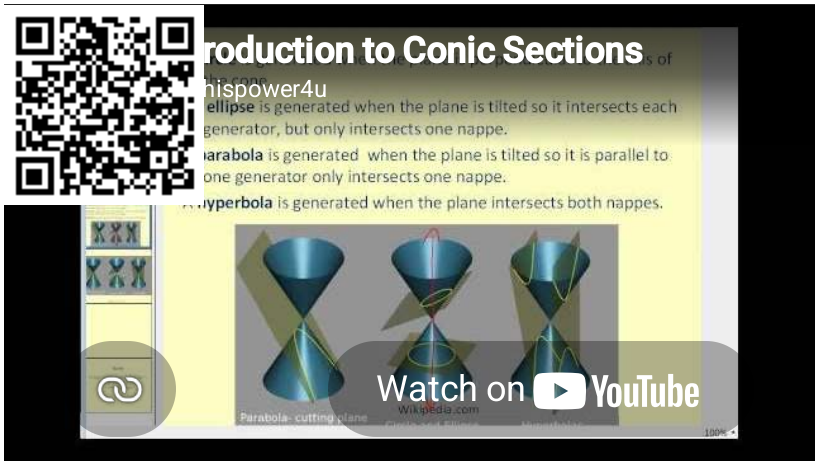
The intersection of three dimensional objects in three dimensional space to produce two dimensional graphs is quite challenging. In practice, the knowledge of where conics come from is not widely used. It will be more important for you to be able to manipulate an equation into standard form and graph it in a regular coordinate plane. The regular form of a conic is:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Before you start manipulating the general form of a conic equation you should be able to recognize whether it is a circle, ellipse, parabola or hyperbola. In standard form, the two coefficients to examine are A and C.

- For circles, the coefficients of x^2 and y^2 are the same sign and the same value: $A = C$
- For ellipses, the coefficients of x^2 and y^2 are the same sign and different values:
- $A, C > 0, A \neq C$

- For hyperbolas, the coefficients of x^2 and y^2 are opposite signs: $C < 0 < A$ or $A < 0 < C$
- For parabolas, either the coefficient of x^2 or y^2 must be zero: $A = 0$ or $C = 0$



Each specific type of conic has its own graphing form, but in all cases the technique of completing the square is essential.

For review, let's complete the square in the expression $x^2 + 6x$. and demonstrate graphically what completing the square represents.

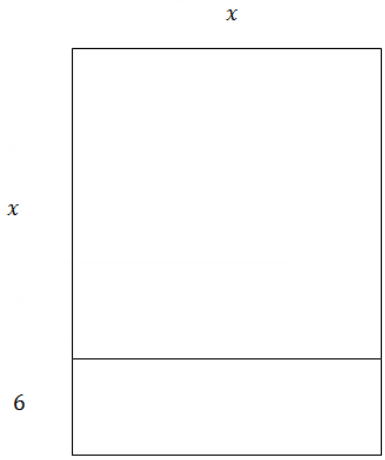
Algebraically, completing the square just requires you to divide the coefficient of x by 2 and square the result. In this case $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Since you cannot add nine to an expression without changing its value, you must simultaneously add nine and subtract nine so the net change will be zero.

$x^2 + 6x + 9 - 9$

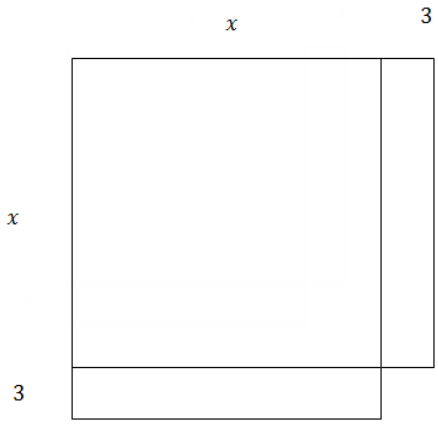
Now you can factor by recognizing a perfect square.

$(x + 3)^2 - 9$

Graphically the original expression $x^2 + 6x$ can be represented by the area of a rectangle with sides x and $(x + 6)$



The term "complete the square" has visual meaning as well algebraic meaning. The rectangle can be rearranged to be more square-like so that instead of small rectangle of area $6x$ at the bottom, there is a rectangle of area $3x$ on two sides of the x^2 square.



Notice what is missing to make this shape a perfect square? A little corner square of 9 is missing which is why the 9 should be added to make the perfect square of $(x+3)(x+3)$.

Examples

Example 1

Earlier, you were asked what skills you need for conics. The one essential skill that you need for conics is completing the square. If you can complete the square with two variables then you will be able to graph every type of conic.

Example 2

What type of conic is each of the following relations?

1. $5y^2 - 2x^2 = -25$

a. Parabola (sideways) because the x^2 term is missing.

3. $4x^2 + 6y^2 = 36$

a. Ellipse because the x^2 and y^2 coefficients are different values but the same sign.

4. $x^2 - \frac{1}{4}y = 1$

a. Parabola (upright) because the y^2 term is missing.

5. $-\frac{x^2}{8} + \frac{y^2}{4} = 1$

a. Hyperbola because the x^2 and y^2 coefficients are different signs.

6. $-x^2 + 99y^2 = 12$

a. Hyperbola because the x^2 and y^2 coefficients are different signs.

Example 3

Complete the square for both the x and y terms in the following equation.

$x^2 + 6x + 2y^2 + 16y = 0$

First write out the equation with space so that there is room for the terms to be added to both sides. Since this is an equation, it is appropriate to add the values to both sides instead of adding and subtracting the same value simultaneously. As you rewrite with spaces, factor out any coefficient of the x^2 or y^2 terms since your algorithm for completing the square only works when this coefficient is one.

$x^2 + 6x + \ldots + 2\left(y^2 + 8y + \ldots\right) = 0$

Next complete the square by adding a nine and what looks like a 16 on the left (it is actually a 32 since it is inside the parentheses).

$x^2+6x+9+2\left(y^2+8y+16\right)=9+32$

Factor

$(x+3)^2+2(y+4)^2=41$

Example 4

Identify the type of conic in each of the following relations.

1. $3x^2=3y^2+18$

a. The relation is a hyperbola because when you move the $3y^2$ to the left hand side of the equation, it becomes negative and then the coefficients of x^2 and y^2 have opposite signs.

2. $y=4(x-3)^2+2$

a. Parabola

3. $x^2+y^2=4$

a. Circle

4. $y^2+2y+x^2-6x=12$

a. Circle

5. $\frac{x^2}{6}+\frac{y^2}{12}=1$

a. Ellipse

6. $x^2-y^2+4=0$

a. Hyperbola

Example 5

Complete the square for both x and y in the following equation.

$-3x^2-24x+4y^2-32y=8$

$-3x^2-24x+4y^2-32y=8$

$-3\left(x^2+8x+\right)+4\left(y^2-8y+\right)=8$

$-3\left(x^2+8x+16\right)+4\left(y^2-8y+16\right)=8-48+64$

$-3(x+4)^2+4(y-4)^2=24$

Review

Identify the type of conic in each of the following relations.

1. $3x^2+4y^2=12$
2. $x^2+y^2=9$
3. $\frac{x^2}{4}+\frac{y^2}{9}=1$
4. $y^2+x=11$
5. $x^2+2x-y^2+6y=15$
6. $x^2=y-1$

Complete the square for x and/or y in each of the following expressions

7. $x^2 + 4x$

8. $y^2 - 8y$

9. $3x^2 + 6x + 4$

10. $3y^2 + 9y + 15$

11. $2x^2 - 12x + 1$

Complete the square for x and/or y in each of the following equations.

12. $4x^2 - 16x + y^2 + 2y = -1$

13. $9x^2 - 54x + y^2 - 2y = -81$

14. $3x^2 - 6x - 4y^2 = 9$

15. $y = x^2 + 4x + 1$

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9.2 Parabolas

When working with parabolas in the past you probably used vertex form and analyzed the graph by finding its roots and intercepts. There is another way of defining a parabola that turns out to be more useful in the real world. One of the many uses of parabolic shapes in the real world is satellite dishes. In these shapes it is vital to know where the receptor point should be placed so that it can absorb all the signals being reflected from the dish.

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

Graphing Parabolas

The definition of a **parabola** is the collection of points equidistant from a point called the **focus** and a line called the **directrix**.

Notice how the three points P_1, P_2, P_3 are each connected by a blue line to the focus point F and the directrix line L

$\overline{FP_1} = \overline{P_1Q_1}$

$\overline{FP_2} = \overline{P_2Q_2}$

$\overline{FP_3} = \overline{P_3Q_3}$

There are two graphing equations for parabolas that will be used in this concept. The only difference is one equation graphs parabolas opening vertically and one equation graphs parabolas opening horizontally. You can recognize the parabolas opening vertically because they have an x^2 term. Likewise, parabolas opening horizontally have a y^2 term.

The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$. The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.

Note that the vertex is still (h, k). The parabola opens upwards or to the right if the $4p$ is positive. The parabola opens down or to the left if the $4p$ is negative. The focus is just a point that is distance p away from the vertex. The directrix is just a line that is distance p away from the vertex in the opposite direction. You can sketch how wide the parabola is by noting the focal width is $|4p|$.

Once you put the parabola into this graphing form you can sketch the parabola by plotting the vertex, identifying p and plotting the focus and directrix and lastly determining the focal width and sketching the curve.

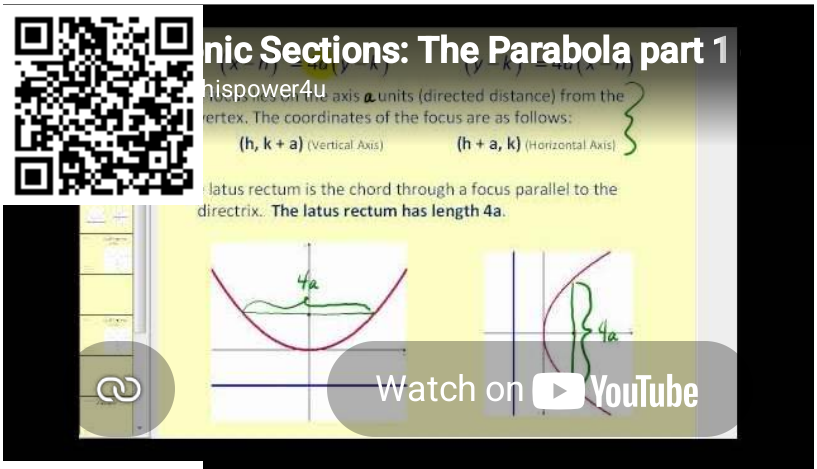
Take the conic:

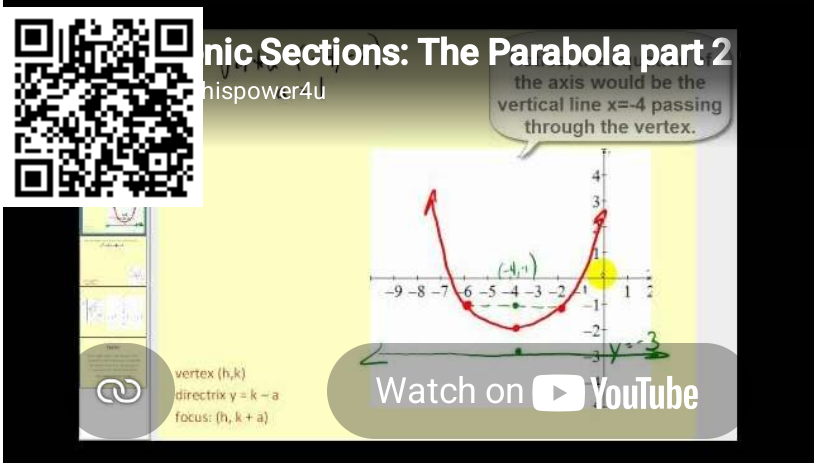
$2x^2 + 16x + y = 0$

This is a parabola because the y^2 coefficient is zero.

$$x^2 + 8x = -\frac{1}{2}y$$
$$x^2 + 8x + 16 = -\frac{1}{2}y + 16$$
$$(x+4)^2 = -\frac{1}{2}(y-32)$$
$$(x+4)^2 = -4 \cdot \frac{1}{8}(y-32)$$

The vertex is $(-4, 32)$. The focal length is $p = \frac{1}{8}$. This parabola opens down which means that the focus is at $\left(-4, 32 - \frac{1}{8}\right)$ and the directrix is horizontal at $y = 32 + \frac{1}{8}$. The focal width is $\frac{1}{2}$.





Examples

Example 1

Earlier, you were asked where the receptor should be located on a satellite dish that is four feet wide and nine inches deep.

Since real world problems do not come with a predetermined coordinate system, you can choose to make the vertex of the parabola at (0, 0). Then, if everything is done in inches, another point on the parabola will be (24, 9). (Many people might mistakenly believe the point (48, 9) is on the parabola but remember that half this width stretches to (-24, 9) as well.) Using these two points, the focal width can be found.

$$(x-0)^2 = 4p(y-0)$$
$$(24-0)^2 = 4p(9-0)$$
$$\frac{24^2}{4 \cdot 9} = p$$
$$16 = p$$

The receptor should be sixteen inches away from the vertex of the parabolic dish.

Example 2

Sketch the following parabola and identify the important pieces of information.

$(y+1)^2 = 4 \cdot \frac{1}{2} \cdot (x+3)$

The vertex is at (-3,-1) . The parabola is sideways because there is a y^2 term. The parabola opens to the right because the $4p$ is positive. The focal length is $p = \frac{1}{2}$ which means the focus is $\frac{1}{2}$ to the right of the vertex at (-2.5,-1) and the directrix is $\frac{1}{2}$ to the

left of the vertex at $x = -3.5$. The focal width is 2 which is why the width of the parabola stretches from (-2.5,0) to (-2.5,-2) .

Example 3

What is the equation of a parabola that has a focus at (4,3) and a directrix of $y = -1$?

It would probably be useful to graph the information that you have in order to reason about where the vertex is.

The vertex must be halfway between the focus and the directrix. This places it at (4, 1). The focal length is 2. The parabola opens upwards. This is all the information you need to create the equation.

$(x-4)^2 = 4 \cdot 2 \cdot (y-1)$

OR $(x-4)^2 = 8(y-1)$

Example 4

What is the equation of a parabola that opens to the right with focal width from (6,-7) to (6,12)? The focus is in the middle of the focal width. The focus is $(6, \frac{5}{2})$. The focal width is 19 which is four times the focal length so the focal length must be $\frac{19}{4}$. The vertex must be a focal length to the left of the focus, so the vertex is at $(6 - \frac{19}{4}, \frac{5}{2})$. This is enough information to write the equation of the parabola.

$(y - \frac{5}{2})^2 = 4 \cdot \frac{19}{4} \cdot (x - 6 + \frac{19}{4})$

Example 5

Sketch the following conic by putting it into graphing form and identifying important information.

$y^2 - 4y + 12x - 32 = 0$

$y^2 - 4y = -12x + 32$
$y^2 - 4y + 4 = -12x + 32 + 4$
$(y-2)^2 = -12(x-3)$
$(y-2)^2 = -4 \cdot 3 \cdot (x-3)$

The vertex is at (3,2) . The focus is at (0,2) . The directrix is at $x = 6$.

Review

1. What is the equation of a parabola with focus at (1,4) and directrix at $y = -2$?
2. What is the equation of a parabola that opens to the left with focal width from (-2,5) to (-2,-7)?
3. What is the equation of a parabola that opens to the right with vertex at (5,4) and focal width of 12?
4. What is the equation of a parabola with vertex at (1,8) and directrix at $y = 12$?
5. What is the equation of a parabola with focus at (-2,4) and directrix at $x = 4$?
6. What is the equation of a parabola that opens downward with a focal width from (-4,9) to (16,9)?
7. What is the equation of a parabola that opens upward with vertex at (1,11) and focal width of 4?

Sketch the following parabolas by putting them into graphing form and identifying important information.

8. $y^2 + 2y - 8x + 33 = 0$
9. $x^2 - 8x + 20y + 36 = 0$
10. $x^2 + 6x - 12y - 15 = 0$

11. $y^2 - 12y + 8x + 4 = 0$
12. $x^2 + 6x - 4y + 21 = 0$
13. $y^2 + 14y - 2x + 59 = 0$
14. $x^2 + 12x - \frac{8}{3}y + \frac{92}{3} = 0$
15. $x^2 + 2x - \frac{4}{5}y + 1 = 0$

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9.3 Circles

A **circle** is the collection of points that are the same distance from a single point. What is the connection between the Pythagorean Theorem and a circle?

Graphing Circles

The single point that all the points on a circle are equidistant from is called the **center** of the circle. A circle does not have a focus or a directrix, instead it simply has a center. Circles can be recognized immediately from the general equation of a conic when the coefficients of x^2 and y^2 are the same sign and the same value. Circles are not functions because they do not pass the vertical line test. The distance from the center of a circle to the edge of the circle is called the **radius** of the circle. The distance from one end of the circle through the center to the other end of the circle is called the **diameter**. The diameter is equal to twice the radius.

The graphing form of a circle is:

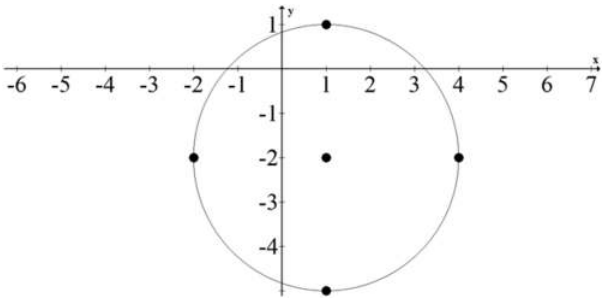
$$(x-h)^2+(y-k)^2=r^2$$

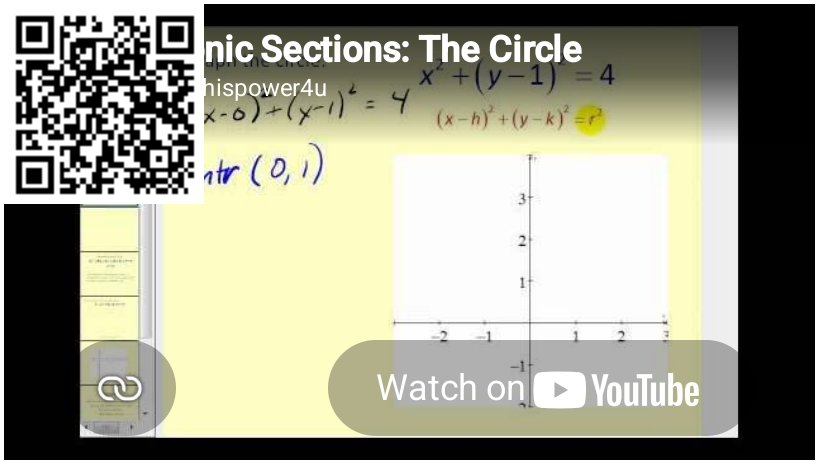
The center of the circle is at (h,k) and the radius of the circle is r. Note that this looks remarkably like the Pythagorean Theorem.

To graph a circle, first plot the center and then apply the radius. Take the following equation for a circle:

$$(x-1)^2+(y+2)^2=9$$

The center is at (1,-2) . Plot that point and the four points that are exactly 3 units from the center.





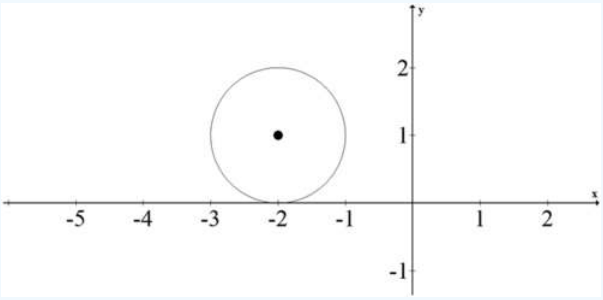
Examples

Example 1

Earlier, you were asked about the connection between circles and the Pythagorean Theorem. The reason why the graphing form of a circle looks like the Pythagorean Theorem is because each x and y coordinate along the outside of the circle forms a perfect right triangle with the radius as the hypotenuse.

Example 2

Graph the following conic: $(x+2)^2+(y-1)^2=1$



Example 3

Turn the following equation into graphing form for a circle. Identify the center and the radius.

$36x^2+36y^2-24x+36y-275=0$

Complete the square and then divide by the coefficient of x^2 and y^2

$36x^2-24x+36y^2+36y=275$

$36\left(x^2-\frac{2}{3}x+\_\right)+36\left(y^2+y+\underline{x}\right)=275$

$36\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+36\left(y^2+y+\frac{1}{4}\right)=275+4+9$

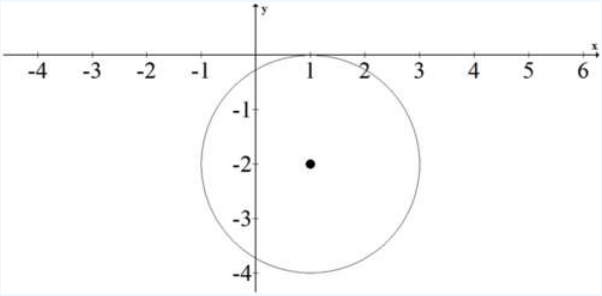
$36\left(x-\frac{1}{3}\right)^2+36\left(y+\frac{1}{2}\right)^2=288$

$\left(x-\frac{1}{3}\right)^2+\left(y+\frac{1}{2}\right)^2=8$

The center is $\left(\frac{1}{3},-\frac{1}{2}\right)$. The radius is $\sqrt{8}=2\sqrt{2}$.

Example 4

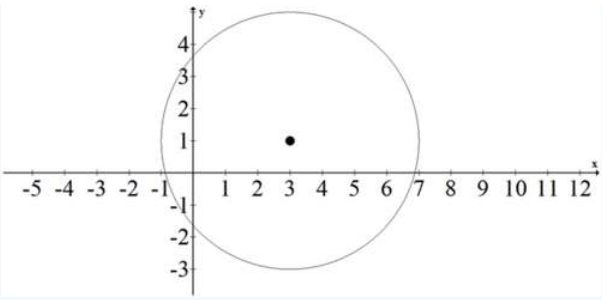
Write the equation for the following circle.



$(x-1)^2+(y+2)^2=4$

Example 5

Write the equation of the following circle.



The center of the circle is at (3,1) and the radius of the circle is $r = 4$. The equation is $(x-3)^2 + (y-1)^2 = 16$

Review

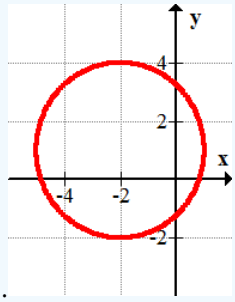
Graph the following conics:

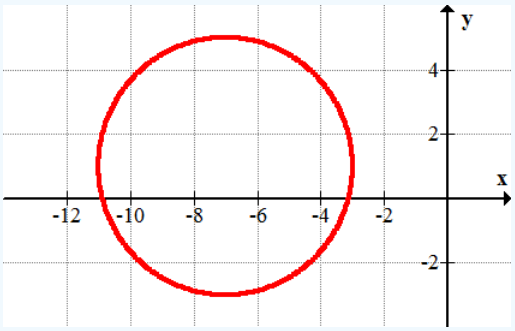
1. $(x+4)^2 + (y-3)^2 = 1$
2. $(x-7)^2 + (y+1)^2 = 4$
3. $(y+2)^2 + (x-1)^2 = 9$
4. $x^2 + (y-5)^2 = 8$
5. $(x-2)^2 + y^2 = 16$

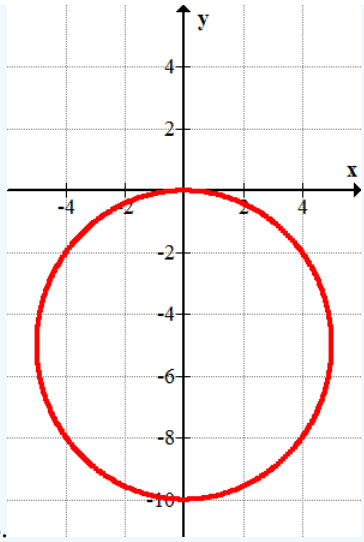
Translate the following conics from standard form to graphing form.

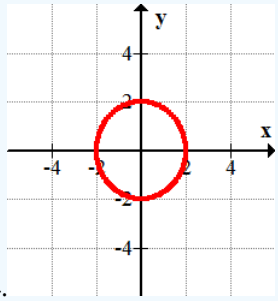
6. $x^2 - 4x + y^2 + 10y + 18 = 0$
7. $x^2 + 2x + y^2 - 8y + 1 = 0$
8. $x^2 - 6x + y^2 - 4y + 12 = 0$
9. $x^2 + 2x + y^2 + 14y + 25 = 0$
10. $x^2 - 2x + y^2 - 2y = 0$

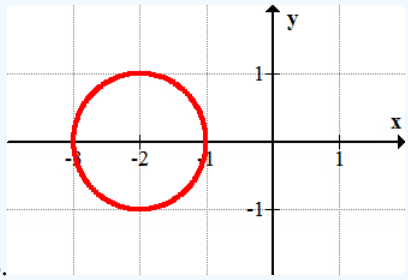
Write the equations for the following circles.

11. 

12. 

13. 

14. 

15. 

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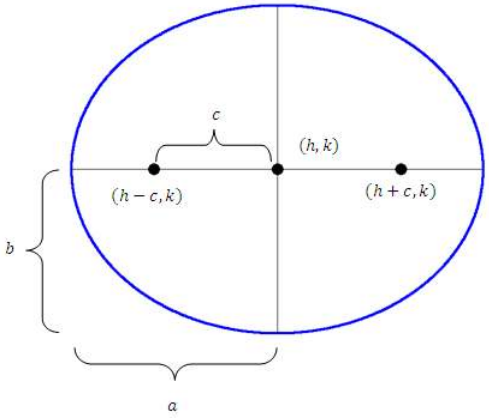
9.4 Ellipses

An **ellipse** is commonly known as an oval. Ellipses are just as common as parabolas in the real world with their own uses. Rooms that have elliptical shaped ceilings are called whisper rooms because if you stand at one focus point and whisper, someone standing at the other focus point will be able to hear you.

Ellipses look similar to circles, but there are a few key differences between these shapes. Ellipses have both an x -radius and a y -radius while circles have only one radius. Another difference between circles and ellipses is that an ellipse is defined as the collection of points that are a set distance from two focus points while circles are defined as the collection of points that are a set distance from one center point. A third difference between ellipses and circles is that not all ellipses are similar to each other while all circles are similar to each other. Some ellipses are narrow and some are almost circular. How do you measure how strangely shaped an ellipse is?

Graphing Ellipses

An ellipse has two foci. For every point on the ellipse, the sum of the distances to each foci is constant. This is what defines an ellipse. Another way of thinking about the definition of an ellipse is to allocate a set amount of string and fix the two ends of the string so that there is some slack between them. Then use a pencil to pull the string taught and trace the curve all the way around both fixed points. You will trace an ellipse and the fixed end points of the string will be the foci. Foci is the plural form of focus. In the picture below, (h, k) is the center of the ellipse and the other two marked points are the foci.



The major axis is the longest distance from end to end of an ellipse and is twice as long as the semi-major axis. The semi-major axis is the distance from the center of the ellipse to the furthest point on the ellipse and the semi-minor axis is the distance from the center to the edge of the ellipse on the axis that is perpendicular to the semi-major axis.

The general equation for an ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

In this case the major axis is horizontal because a, the x -radius, is larger. If the y -radius were larger, then a and b would reverse. In other words, the coefficient a always comes from the length of the semi major axis (the longer one) and the coefficient b always comes from the length of the semi minor axis (the shorter one).

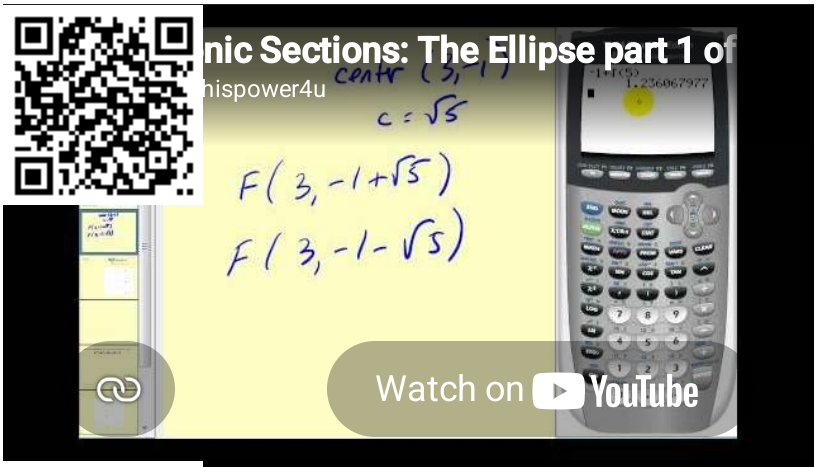
In order to find the locations of the two foci, you will need to find the focal radius represented as c using the following relationship:

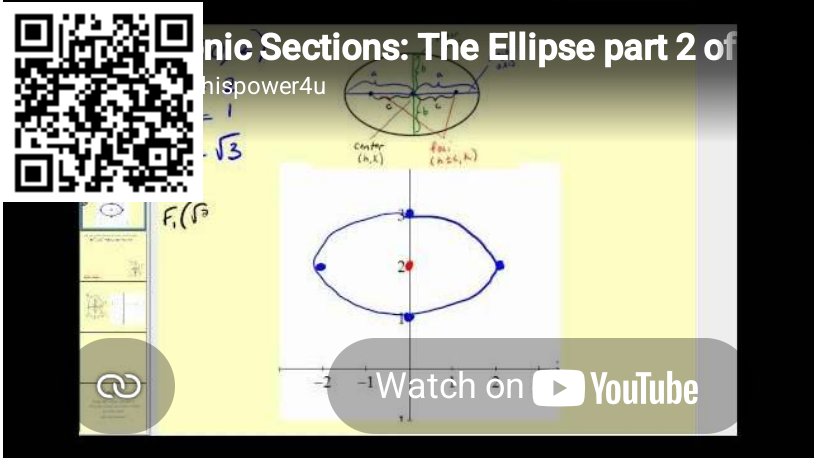
$$a^2 - b^2 = c^2$$

Once you have the focal radius, measure from the center along the major axis to locate the foci. The general shape of an ellipse is measured using eccentricity. **Eccentricity** is a measure of how oval or how circular the shape is. Ellipses can have an eccentricity between 0 and 1 where a number close to 0 is extremely circular and a number close to 1 is less circular. Eccentricity is calculated by:

$$e = \frac{c}{a}$$

Ellipses also have two directrix lines that correspond to each focus but on the outside of the ellipse. The distance from the center of the ellipse to each directrix line is $\frac{a^2}{c}$

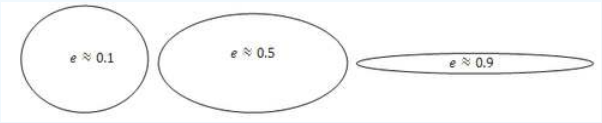




Examples

Example 1

Earlier, you were asked how you measure how strangely shaped an ellipse is. Ellipses are measured using their eccentricity. Here are three ellipses with estimated eccentricity for you to compare.



Eccentricity is the ratio of the focal radius to the semi major axis: $e = \frac{c}{a}$

Example 2

Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{16} = 1$

The center of the ellipse is at (2,-1) . The major axis is vertical which means the semi major axis is $a = 4$. The vertices are (2,3) and (2,-5)

$$16^2 - 4^2 = c^2$$
$$4\sqrt{15} = \sqrt{240} = c$$

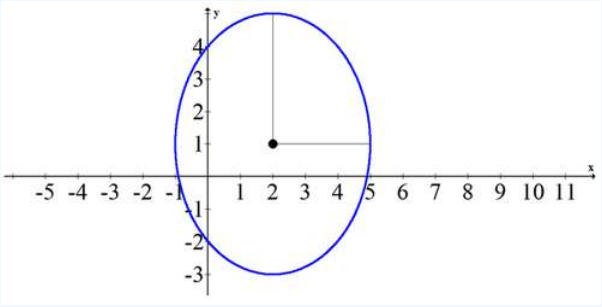
Thus the foci are $(2, -1 + 4\sqrt{15})$ and $(2, -1 - 4\sqrt{15})$

Example 3

Sketch the following ellipse.

$\frac{(y-1)^2}{16} + \frac{(x-2)^2}{9} = 1$

Plotting the foci are usually important, but in this case the question simply asks you to sketch the ellipse. All you need is the center, x -radius and y -radius.



Example 4

Put the following conic into graphing form.

$25x^2 - 150x + 36y^2 + 72y - 639 = 0$

$$25x^2 - 150x + 36y^2 + 72y - 639 = 0$$
$$25\left(x^2 - 6x\right) + 36\left(y^2 + 2y\right) = 639$$
$$25\left(x^2 - 6x + 9\right) + 36\left(y^2 + 2y + 1\right) = 639 + 225 + 36$$
$$25(x-3)^2 + 36(y+1)^2 = 900$$
$$\frac{25(x-3)^2}{900} + \frac{36(y+1)^2}{900} = \frac{900}{900}$$
$$\frac{(x-3)^2}{36} + \frac{(y+1)^2}{25} = 1$$

Example 5

Put the following conic into graphing form.

$9x^2 - 9x + 4y^2 + 12y + \frac{9}{4} = -8$

$9x^2 - 9x + 4y^2 + 12y + \frac{9}{4} = -8$

$9x^2 - 9x + \frac{9}{4} + 4y^2 + 12y = -8$

$9\left(x^2 - x - \frac{1}{4}\right) + 4\left(y^2 + 3y\right) = -8$

$9\left(x - \frac{1}{2}\right)^2 + 4\left(y^2 + 3y + \frac{9}{4}\right) = -8 + 4 \cdot \frac{9}{4}$

$9\left(x - \frac{1}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 = 1$

$\frac{\left(x-\frac{1}{2}\right)^2}{\frac{1}{9}} + \frac{\left(y+\frac{3}{2}\right)^2}{\frac{1}{4}} = 1$

Review

Find the vertices, foci, and eccentricity for each of the following ellipses.

1. $\frac{(x-1)^2}{4} + \frac{(y+5)^2}{16} = 1$
2. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$

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3. $(x-2)^2+\frac{(y-1)^2}{4}=1$

Now sketch each of the following ellipses (note that they are the same as the ellipses in #1-#3).

4. $\frac{(x-1)^2}{4}+\frac{(y+5)^2}{16}=1$

5. $\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1$

6. $(x-2)^2+\frac{(y-1)^2}{4}=1$

Put each of the following equations into graphing form.

7. $x^2+2x+4y^2+56y+197=16$

8. $x^2-8x+9y^2+18y+25=9$

9. $9x^2-36x+4y^2+16y+52=36$

Find the equation for each ellipse based on the description.

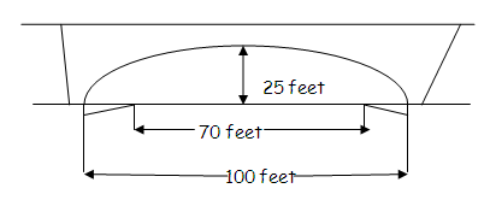
10. An ellipse with vertices (4,-2) and (4,8) and minor axis of length 6.

11. An ellipse with minor axis from (4,-1) to (4,3) and major axis of length 12 .

12. An ellipse with minor axis from (-2,1) to (-2,7) and one focus at (2,4) .

13. An ellipse with one vertex at $(6,-15)$, and foci at (6,10) and (6,-14)

A bridge over a roadway is to be built with its bottom the shape of a semi-ellipse 100 feet wide and 25 feet high at the center. The roadway is to be 70 feet wide.



14. Find one possible equation of the ellipse that models the bottom of the bridge.

15. What is the clearance between the roadway and the overpass at the edge of the roadway?

...

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9.5 Hyperbolas

Hyperbolas are relations that have asymptotes. When graphing rational functions you often produce a hyperbola. In this concept, hyperbolas will not be oriented in the same way as with rational functions, but the basic shape of a hyperbola will still be there.

Hyperbolas can be oriented so that they open side to side or up and down. One of the most common mistakes that you can make is to forget which way a given hyperbola should open. What are some strategies to help?

Graphing Hyperbolas

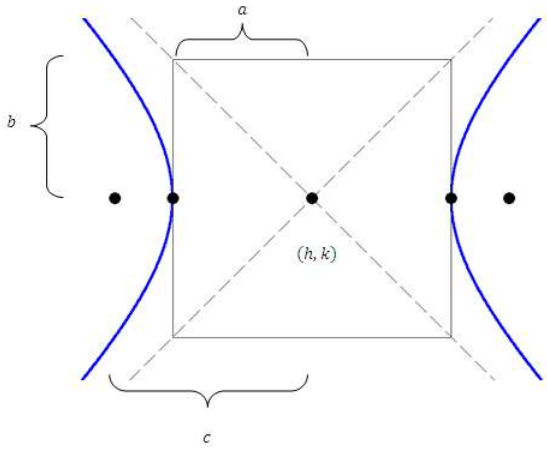
A hyperbola has two foci. For every point on the hyperbola, the difference of the distances to each foci is constant. This is what defines a hyperbola. The graphing form of a hyperbola that opens side to side is:

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

A hyperbola that opens up and down is:

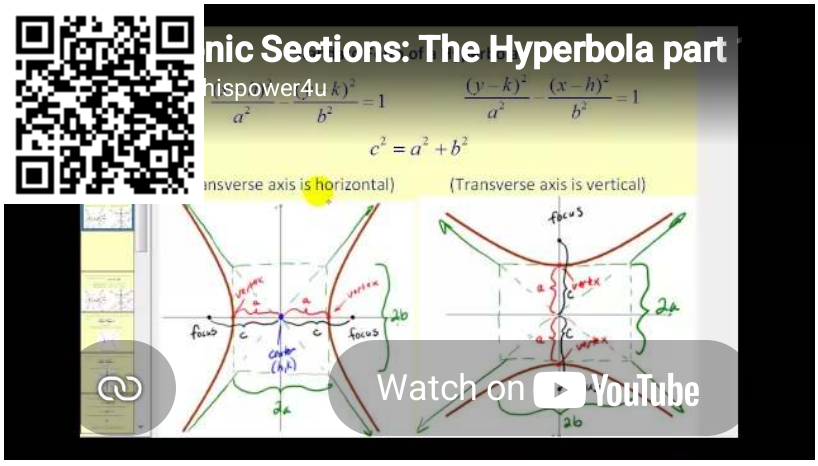
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

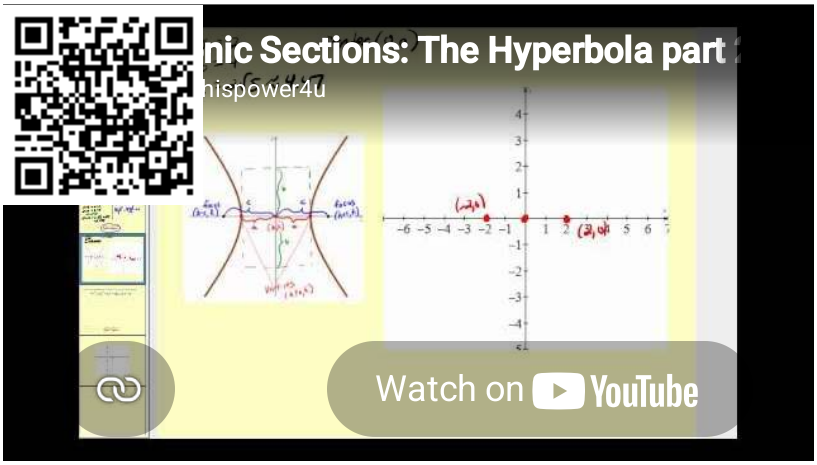
Notice that for hyperbolas, a goes with the positive term and b goes with the negative term. It does not matter which constant is larger.



When graphing, the constants a and b enable you to draw a rectangle around the center. The transverse axis travels from vertex to vertex and has length $2a$. The conjugate axis travels perpendicular to the transverse axis through the center and has length $2b$. The foci lie beyond the vertices so the eccentricity, which is measured as $e = \frac{c}{a}$, is larger than 1 for all hyperbolas. Hyperbolas also have two directrix lines that are $\frac{a^2}{c}$ away from the center (not shown on the image).

The focal radius is $a^2 + b^2 = c^2$





Examples

Example 1

Earlier, you were asked how to determine the direction that a hyperbola opens. The best strategy to remember which direction the hyperbola opens is often the simplest. Consider the hyperbola $x^2 - y^2 = 1$. This hyperbola opens side to side because x can clearly never be equal to zero. This is a basic case that shows that when the negative is with the y value then the hyperbola opens up side to side.

Example 2

Put the following hyperbola into graphing form, list the components, and sketch it.

$$9x^2 - 4y^2 + 36x - 8y - 4 = 0$$

$$9\left(x^2 + 4x\right) - 4\left(y^2 + 2y\right) = 4$$

$$9\left(x^2 + 4x + 4\right) - 4\left(y^2 + 2y + 1\right) = 4 + 36 - 4$$

$$9(x+2)^2 - 4(y+1)^2 = 36$$

$$\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} = 1$$

Shape: Hyperbola that opens horizontally.

Center: (-2,-1)

$a = 2$

$b = 3$

$c = \sqrt{13}$

$e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

$d = \frac{a^2}{c} = \frac{4}{\sqrt{13}}$

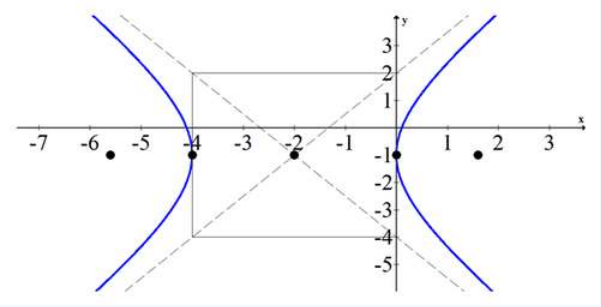
Foci: $\left(-2 + \frac{\sqrt{13}}{2}, -1\right), \left(-2 - \frac{\sqrt{13}}{2}, -1\right)$

Vertices: (-4,-1),(0,-1)

Equations of asymptotes: $\pm\frac{3}{2}(x+2) = (y+1)$

Note that it is easiest to write the equations of the asymptotes in point-slope form using the center and the slope.

Equations of directrices: $y = -2 \pm \frac{4}{\sqrt{13}}$



Example 3

Find the equation of the hyperbola with foci at (-3,5) and (9,5) and asymptotes with slopes of $\pm\frac{4}{3}$.

The center is between the foci at (3,5) . The focal radius is $c = 6$. The slope of the asymptotes is always the rise over run inside the box. In this case since the hyperbola is horizontal and a is in the x direction the slope is $\frac{b}{a}$. This makes a system of equations.

$\frac{b}{a} = \pm\frac{4}{3}$

$a^2 + b^2 = 6^2$

When you solve,you get $a = \sqrt{13}, b = \frac{4}{3}\sqrt{13}$.

$\frac{(x-3)^2}{13} - \frac{(y-5)^2}{\frac{16}{9}\cdot 13} = 1$

Example 4

Find the equation of the conic that has a focus point at (1,2) , a directrix at $x = 5$, and an eccentricity equal to $\frac{3}{2}$. Use the property that the distance from a point on the hyperbola to the focus is equal to the eccentricity times the distance from that same point to the directrix:

$\overline{PF} = e\overline{PD}$

This relationship bridges the gap between ellipses which have eccentricity less than one and hyperbolas which have eccentricity greater than one. When eccentricity is equal to one, the

shape is a parabola.

$\sqrt{(x-1)^2 + (y-2)^2} = \frac{3}{2}\sqrt{(x-5)^2}$

Square both sides and rearrange terms so that it is becomes a hyperbola in graphing form.

$$x^2 - 2x + 1 + (y-2)^2 = \frac{9}{4}\left(x^2 - 10x + 25\right)$$

$$x^2 - 2x + 1 - \frac{9}{4}x^2 + \frac{90}{4}x - \frac{225}{4} + (y-2)^2 = 0$$

$$-\frac{5}{4}x^2 + \frac{92}{4}x + (y-2)^2 = \frac{221}{4}$$

$$-5x^2 + 92x + 4(y-2)^2 = 221$$

$$-5\left(x^2 - \frac{92}{5}x\right) + 4(y-2)^2 = 221$$

$$-5\left(x^2 - \frac{92}{5}x + \frac{92^2}{10^2}\right) + 4(y-2)^2 = 221 - \frac{2116}{5}$$

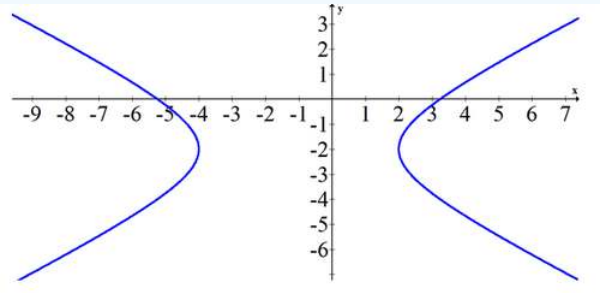
$$-5\left(x - \frac{92}{10}\right)^2 + 4(y-2)^2 = -\frac{1011}{5}$$

$$\left(x - \frac{92}{10}\right)^2 - (y-2)^2 = \frac{1011}{100}$$

$$\frac{\left(x - \frac{92}{10}\right)^2}{\left(\frac{1011}{100}\right)} - \frac{(y-2)^2}{\left(\frac{1011}{100}\right)} = 1$$

Example 5

Given the following graph, estimate the equation of the conic.



Since exact points are not marked, you will need to estimate the slope of asymptotes to get an approximation for a and b. The slope seems to be about $\pm\frac{2}{3}$. The center seems to be at $(-1, -2)$. The transverse axis is 6 which means $a = 3$.

$\frac{(x+1)^2}{9} - \frac{(y+2)^2}{4} = 1$

Review

Use the following equation for #1-#5: $x^2 + 2x - 4y^2 - 24y - 51 = 0$

1. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
2. Identify whether the hyperbola opens side to side or up and down.
3. Find the location of the vertices.
4. Find the equations of the asymptotes.
5. Sketch the hyperbola.

Use the following equation for $\#6\text{–}\#10: -9x^2 - 36x + 16y^2 - 32y - 164 = 0$

6. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
7. Identify whether the hyperbola opens side to side or up and down.
8. Find the location of the vertices.
9. Find the equations of the asymptotes.
10. Sketch the hyperbola.

Use the following equation for #11 - #15: $x^2 - 6x - 9y^2 - 54y - 81 = 0$

11. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
12. Identify whether the hyperbola opens side to side or up and down.
13. Find the location of the vertices.
14. Find the equations of the asymptotes.
15. Sketch the hyperbola.

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9.6 Degenerate Conics

The general equation of a conic is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This form is so general that it encompasses all regular lines, singular points and degenerate hyperbolas that

look like an $\mathbf{X}$. This is because there are a few special cases of how a plane can intersect a two sided cone. How are these degenerate shapeGraphing Degenerate Conicss formed?

Graphing Degenerate Conics

A **degenerate conic** is a conic that does not have the usual properties of a conic. Degenerate conic equations simply cannot be written in graphing form. There are three types of degenerate conics:

1. **A singular point**, which is of the form: $\frac{(x-h)^2}{a} + \frac{(y-k)^2}{b} = 0$. You can think of a singular point as a circle or an ellipse with an infinitely small radius.

2. **A line**, which has coefficients $A = B = C = 0$ in the general equation of a conic. The remaining portion of the equation is $Dx + Ey + F = 0$, which is a line.

3. **A degenerate hyperbola**, which is of the form: $\frac{(x-h)^2}{a} - \frac{(y-k)^2}{b} = 0$. The result is two intersecting lines that make an "X" shape. The slopes of the intersecting lines forming the $\mathbf{X}$ are $\pm\frac{b}{a}$. This is because b goes with the y portion of the equation and is the rise, while a goes with the x portion of the equation and is the run.

Examples

Example 1

Earlier, you were asked how degenerate conics are formed. When you intersect a plane with a two sided cone where the two cones touch, the intersection is a single point. When you intersect a plane with a two sided cone so that the plane touches the edge of one cone, passes through the central point and continues touching the edge of the other conic, this produces a line. When you intersect a plane with a two sided cone so that the plane passes vertically through the central point of the two cones, it produces a degenerate hyperbola.

Example 2

Transform the conic equation into standard form and sketch.

$0x^2 + 0xy + 0y^2 + 2x + 4y - 6 = 0$

This is the line $y = -\frac{1}{2}x + \frac{3}{2}$.

Example 3

Transform the conic equation into standard form and sketch.

$3x^2 - 12x + 4y^2 - 8y + 16 = 0$

$3x^2 - 12x + 4y^2 - 8y + 16 = 0$

$$3\left(x^2 - 4x\right) + 4\left(y^2 - 2y\right) = -16$$
$$3\left(x^2 - 4x + 4\right) + 4\left(y^2 - 2y + 1\right) = -16 + 12 + 4$$
$$3(x-2)^2 + 4(y-1)^2 = 0$$
$$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{3} = 0$$

The point (2,1) is the result of this degenerate conic.

Example 4

Transform the conic equation into standard form and sketch.

$16x^2 - 96x - 9y^2 + 18y + 135 = 0$

$16x^2 - 96x - 9y^2 + 18y + 135 = 0$

$$16\left(x^2 - 6x\right) - 9\left(y^2 - 2y\right) = -135$$
$$16\left(x^2 - 6x + 9\right) - 9\left(y^2 - 2y + 1\right) = -135 + 144 - 9$$
$$16(x-3)^2 - 9(y-1)^2 = 0$$
$$\frac{(x-3)^2}{9} - \frac{(y-1)^2}{16} = 0$$

This is a degenerate hyperbola.

Example 5

1. Create a conic that describes just the point (4,7)).

$(x-4)^2 + (y-7)^2 = 0$

Review

1. What are the three degenerate conics?

Change each equation into graphing form and state what type of conic or degenerate conic it is.

2. $x^2 - 6x - 9y^2 - 54y - 72 = 0$

3. $4x^2 + 16x - 9y^2 + 18y - 29 = 0$

4. $9x^2 + 36x + 4y^2 - 24y + 72 = 0$

5. $9x^2 + 36x + 4y^2 - 24y + 36 = 0$

6. $0x^2 + 5x + 0y^2 - 2y + 1 = 0$

7. $x^2 + 4x - y + 8 = 0$

8. $x^2 - 2x + y^2 - 6y + 6 = 0$

9. $x^2 - 2x - 4y^2 + 24y - 35 = 0$

10. $x^2 - 2x + 4y^2 - 24y + 33 = 0$

Sketch each conic or degenerate conic.

11. $\frac{(x+2)^2}{4} + \frac{(y-3)^2}{9} = 0$

12. $\frac{(x-3)^2}{9} + \frac{(y+3)^2}{16} = 1$

13. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{4} = 1$

14. $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{4} = 0$

15. $3x + 4y = 12$

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CHAPTER OVERVIEW

10: Polar and Parametric Equations

10.1 Polar and Rectangular Coordinates

10.2 Polar Equations of Conics

10.3 Parameters and Parameter Elimination

10.4 Parametric Inverses

10.5 Applications of Parametric Equations

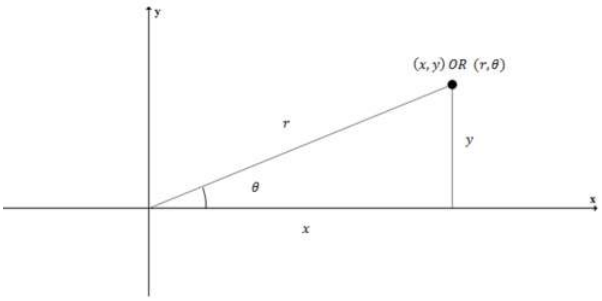
This page titled 10: Polar and Parametric Equations is shared under a CK-12 license and was authored, remixed, and/or curated by CK12 via source content that was edited to the style and standards of the LibreTexts platform.

10.1 Polar and Rectangular Coordinates

In the rectangular coordinate system, points are identified by their distances from the x and y axes. In the polar coordinate system, points are identified by their angle on the unit circle and their distance from the origin. You can use basic right triangle trigonometry to translate back and forth between the two representations of the same point. How are lines and other functions affected by this new coordinate system?

Polar and Rectangle Coordinates

Rectangular coordinates are the ordinary (x, y) coordinates that you are used to.



Polar coordinates represent the same point, but describe the point by its distance from the origin (r) and its angle on the unit circle (θ). To translate back and forth between polar and rectangular coordinates you should use the basic trig relationships:

$\sin\theta = \frac{y}{r} \rightarrow r \cdot \sin\theta = y$

$\cos\theta = \frac{x}{r} \rightarrow r \cdot \cos\theta = x$

$\tan\theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\frac{y}{x}$

You can also express the relationship between x, y and r using the Pythagorean Theorem.

$x^2 + y^2 = r^2$

Note that coordinates in polar form are not unique. This is because there are an infinite number of coterminal angles that point towards any given (x, y) coordinate.

For example, the point (3,4) can be written in polar coordinates in at least three different ways. To find θ, use the third equation from above and to find r use the pythagorean theorem.

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1°$

$r^2 = 3^2 + 4^2$

$r = 5$

Three equivalent polar coordinates for the point (3,4) are:

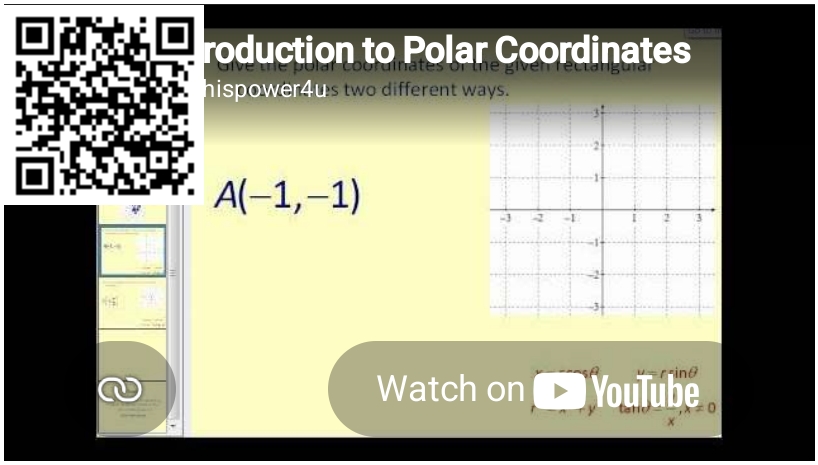
$(5, 53.1°), \quad (5, 413.1°), \quad (-5, 233.1°)$

Notice how the third coordinate points in the opposite direction and has a seemingly negative radius. This means go in the opposite direction of the angle.

Once you can translate back and forth between points, use the same substitutions to change equations too. A polar equation is written with the radius as a function of the angle. This means an equation in polar form should be written in the form $r =$ ___.

To write an equation in polar form, use the conversion equations to substitute. For example, to convert $y = -x + 1$ to polar form make substitutions for y and x. Then, solve for r.

$$r \cdot \sin\theta = -r \cdot \cos\theta + 1$$
$$r \cdot \sin\theta + r \cdot \cos\theta = 1$$
$$r(\sin\theta + \cos\theta) = 1$$
$$r = \frac{1}{\sin\theta + \cos\theta}$$



Examples

Example 1

Earlier, you were asked how lines can be represented in the polar coordinate system. The general way to express a line $y = mx + b$ in polar form is $r = \frac{b}{\sin\theta - m \cdot \cos\theta}$.

Example 2

Express the following equation using rectangular coordinates: $r = \frac{8}{1+2\cos\theta}$.

Use the fact that $r = \pm\sqrt{x^2 + y^2}$ and $r\cos\theta = x$.

$$r + 2r \cdot \cos\theta = 8$$
$$\pm\sqrt{x^2 + y^2} + 2x = 8$$
$$\pm\sqrt{x^2 + y^2} = 8 - 2x$$
$$x^2 + y^2 = 64 - 32x + 4x^2$$
$$-3x^2 + 32x + y^2 - 64 = 0$$

This is the equation of a hyperbola.

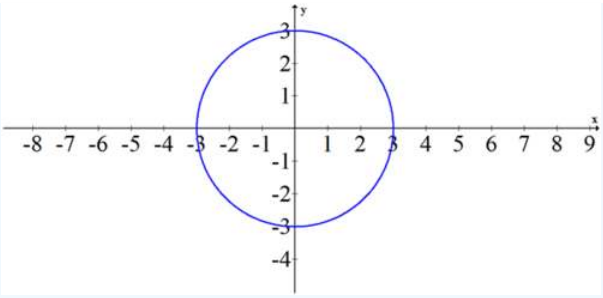
Example 3

Sketch the following polar equation: $r = 3$.

Since theta is not in the equation, it can vary freely. This simple equation produces a perfect

circle of radius 3 centered at the origin.

You can show this equation is equivalent to $x^2 + y^2 = 9$

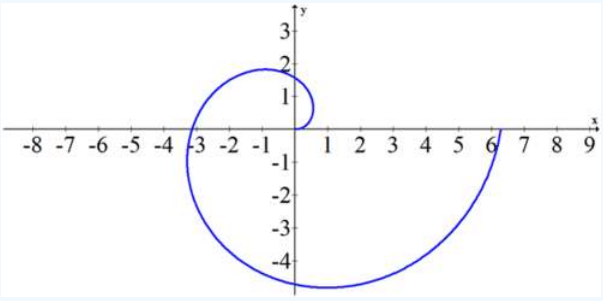


Example 4

Sketch the following polar equation: $r = \theta$ with $\theta : 0 \leq \theta \leq 2\pi$.

The equation $r = \theta$ is an example of a polar equation that cannot be easily expressed in rectangular form. In order to sketch the graph, identify a few key points:

$(0, 0), \left(\frac{\pi}{2}, \frac{\pi}{2}\right), (\pi, \pi), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right), (2\pi, 2\pi)$. You should see that the shape is very recognizable as a spiral.



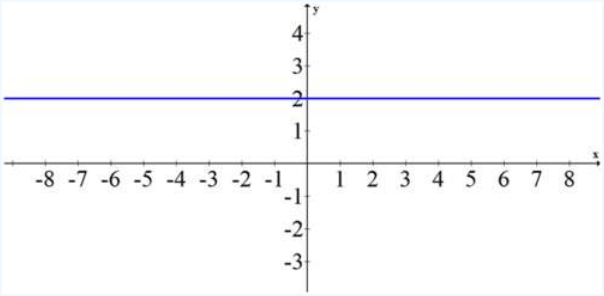
Example 5

Translate the following polar expression into rectangular coordinates and then graph.

$r = 2 \cdot \sec\left(\theta - \frac{\pi}{2}\right)$

Simplify the polar equation first before converting to rectangular coordinates.

$$r = 2 \cdot \sec\left(\theta - \frac{\pi}{2}\right)$$
$$r \cdot \cos\left(\theta - \frac{\pi}{2}\right) = 2$$
$$r \cdot \cos\left(\frac{\pi}{2} - \theta\right) = 2$$
$$r \cdot \sin\theta = 2$$
$$y = 2$$



Review

Plot the following polar coordinates.

1. $\left(3, \frac{5\pi}{6}\right)$
2. $\left(2, \frac{\pi}{2}\right)$
3. $\left(4, -\frac{7\pi}{6}\right)$
4. $\left(-2, \frac{5\pi}{3}\right)$

Give two alternate sets of coordinates for each point.

5. $(2, 60°)$
6. $(5, 330°)$
7. $(2, 210°)$

Graph each equation.

8. $r = 4$
9. $\theta = \frac{\pi}{4}$
10. $r = 2\theta$ with $\theta : 0 \leq \theta \leq 2\pi$.

Convert each point to rectangular form.

11. $\left(4, \frac{2\pi}{3}\right)$
12. $\left(3, \frac{\pi}{4}\right)$
13. $\left(5, \frac{\pi}{3}\right)$

Convert each point to polar form using radians where $0 \leq \theta < 2\pi$.

14. (1,3)
15. (1,-4)
16. (2,6)

Convert each equation to polar form.

17. $x = 3$
18. \(2 x+4 y=2\

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10.2 Polar Equations of Conics

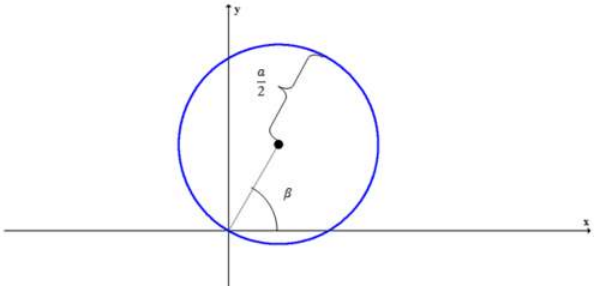
Polar coordinates allow you to extend your knowledge of conics in a new context. Calculators are an excellent tool for graphing polar conics. What settings do you need to know in order to properly use your calculator?

Polar Equations of Conics

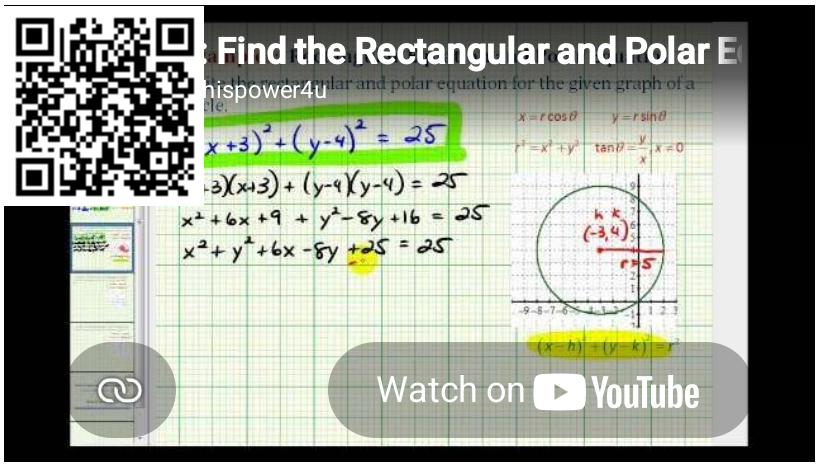
Polar equations refer to the radius r as a function of the angle θ. There are a few typical polar equations you should be able to recognize and graph directly from their polar form.

The following polar function is a circle of radius $\frac{a}{2}$ passing through the origin with a center at angle β.

$r = a \cdot \cos(\theta - \beta)$

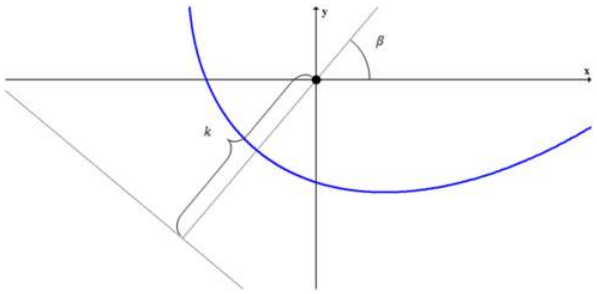


There are other ways of representing a circle like this using cofunction identities and coterminal angles.



Ellipses, parabolas and hyperbolas have a common general polar equation. Just like with the circle, there are other ways of representing these relations using cofunction and coterminal angles; however, this general form is easiest to use because each parameter can be immediately interpreted in a graph. A **parameter** is a constant in a general equation that takes on a specific value in a specific equation.

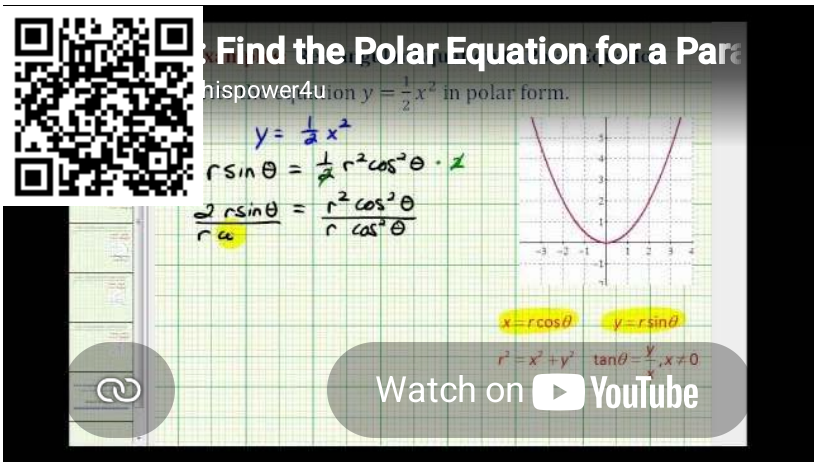
$r = \frac{k \cdot e}{1 - e \cdot \cos(\theta - \beta)}$



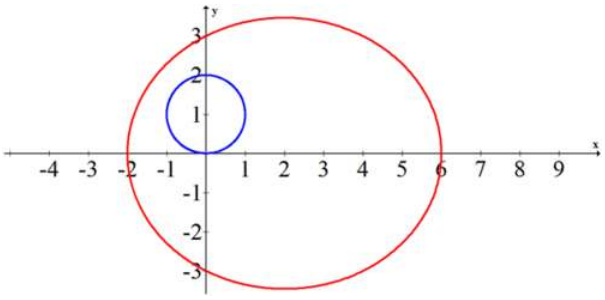
One of the focus points of a conic written in this way is always at the **pole** (the origin). The angle β indicates the angle towards the center if the conic is an ellipse, the opening direction if the conic is a parabola and the angle away from the center if the conic is a hyperbola. The eccentricity e should tell you what conic it is. The constant k is the distance from the focus at the pole to the nearest directrix. This directrix lies in the opposite direction indicated by β.

There are many opportunities for questions involving partial information with polar conics. A few relationships that are often useful for solving these questions are:

- $e = \frac{c}{a} = \frac{\overline{PF}}{\overline{PD}} \rightarrow \overline{PF} = e \cdot \overline{PD}$
- Ellipses: $k = \frac{a^2}{c} - c$
- Hyperbolas: $k = c - \frac{a^2}{c}$



A great way to discover new types of graphs in polar coordinates is to experiment on your own with your calculator. Try to come up with equations and graphs that look similar to the following two polar functions.



The circle in blue has a center at $90°$ and has a diameter of 2 . Its equation is $r = 2\cos(\theta - 90°)$.

The red ellipse appears to have center at (2,0) with $a = 4$ and $c = 2$. This means the eccentricity is $e = \frac{1}{2}$. In order to write the equation in polar form you still need to find k.

$k = \frac{a^2}{c} - c = \frac{4^2}{2} - 2 = 8 - 2 = 6$

Thus the equation for the ellipse is:

$r = \frac{6 \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \cos(\theta)}$

Examples

Example 1

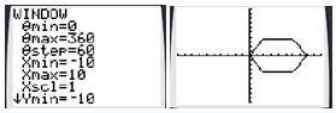
Earlier, you were asked about how to use your calculator to graph polar equations. Most calculators have a polar coordinate mode. On the $\mathbf{TI-84}$, the mode can be switched to polar in the mode menu. This changes the graphing features. You can choose to be in radians or degrees and graphs will look the same. When you graph a circle of the form $r = 8 \cdot \cos\theta$. you should see the following on your calculator.



When you go to the window setting you should notice that in addition to $X_{\min}, X_{\max}$ there are new settings called $\theta_{\min}, \theta_{\max}$ and θ_{step}.

If $\theta_{\min}$ and $\theta_{\max}$ do not span an entire period, you may end up missing part of your polar graph.

The θ_{step} controls how accurate the graph should be. If you put θ_{step} at a low number like 0.1 the graph will plot extremely slowly because the calculator is doing 3600 cosine calculations. On the other hand if $\theta_{\text{step}} = 30$ then the calculator will do fewer calculations producing a rough circle, but probably not accurate enough for your purposes.



Example 2

Identify the center, foci, vertices and equations of the directrix lines for the following conic:

$r = \frac{20}{4-5\cdot\cos\left(\theta-\frac{3\pi}{4}\right)}$

First the polar equation needs to be in graphing form. This means that the denominator needs to look like $1 - e \cdot \cos(\theta - \beta)$.

$r = \frac{20}{4-5\cdot\cos\left(\theta-\frac{3\pi}{4}\right)} \cdot \frac{\frac{1}{4}}{\frac{1}{4}} = \frac{5}{1-\frac{5}{4}\cdot\cos\left(\theta-\frac{3\pi}{4}\right)} = \frac{4\cdot\frac{5}{4}}{1-\frac{5}{4}\cdot\cos\left(\theta-\frac{3\pi}{4}\right)}$

$e = \frac{5}{4}, \quad k = 4, \quad \beta = \frac{3\pi}{4} = 135°$

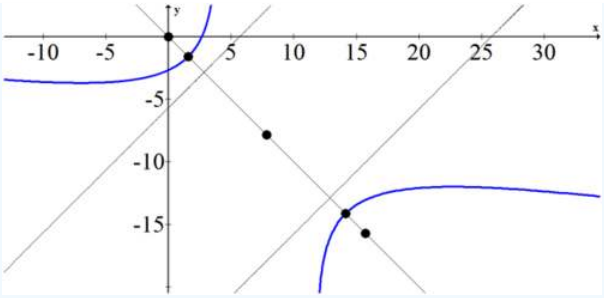
Using this information and the relationships you were reminded of in the guidance, you can set up a system and solve for a and c.

$$4 = c - \frac{a^2}{c}$$
$$\frac{5}{4} = \frac{c}{a} \rightarrow \frac{4}{5} = \frac{a}{c} \rightarrow \frac{4c}{5} = a$$
$$4 = c - \left(\frac{4c}{5}\right)^2 \cdot \frac{1}{c}$$
$$4 = c - \frac{16c^2}{25c}$$
$$4 = \frac{9c}{25}$$
$$\frac{100}{9} = c$$
$$\frac{80}{9} = a$$

The center is the point $\left(\frac{100}{9}, \frac{7\pi}{4}\right)$ which is much more convenient to write in polar coordinates. The closest directrix is the line $r = 4 \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. The other directrix is the line $r = \left(2 \cdot \frac{100}{9} - 4\right) \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. One focus is at the pole, the other focus is the point $\left(\frac{200}{9}, \frac{7\pi}{4}\right)$. The vertices are at the center plus or minus a in the same angle:

$\left(\frac{100}{9} \pm \frac{80}{9}, \frac{7\pi}{4}\right)$

Putting all this information together, the graph of the conic is:



Example 3

Convert the following conic from polar form to rectangular form.

$r = \frac{3}{2-\cos\theta}$

There are many ways to convert from polar form to rectangular form. You should become comfortable with the algebra.

$$
\begin{aligned}
r &= \frac{3}{2-\cos\theta} \\
r(2-\cos\theta) &= 3 \\
2r - r\cdot\cos\theta &= 3 \\
2r = 3 + r\cdot\cos\theta &= 3 + y \\
4r^2 &= 9 + 6y + y^2 \\
4\left(x^2+y^2\right) &= 9 + 6y + y^2 \\
4x^2 + 4y^2 &= 9 + 6y + y^2 \\
4x^2 + 3y^3 + 6y &= 9 \\
4x^2 + 3\left(y^2 + 2y + 1\right) &= 9 + 3 \\
4x^2 + 3(y+1)^2 &= 12 \\
\frac{x^2}{3} + \frac{(y+1)^2}{4} &= 1
\end{aligned}
$$

Example 4

Graph the following conic.

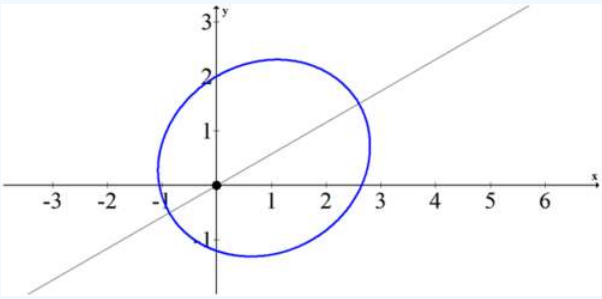
$r = \frac{3}{2-\cos(\theta-30°)}$

Convert to the standard conic form.

$$r = \frac{3}{2-\cos\left(\theta-30°\right)}$$

$$r = \frac{3}{2-\cos\left(\theta-30°\right)}\cdot\frac{\frac{1}{2}}{\frac{1}{2}} = \frac{\frac{3}{2}}{1-\frac{1}{2}\cdot\cos\left(\theta-30°\right)} = \frac{3\cdot\frac{1}{2}}{1-\frac{1}{2}\cdot\cos\left(\theta-30°\right)}$$

$$k = 3, \quad e = \frac{1}{2}, \quad \beta = 30°$$



Example 5

Translate the following conic to polar form.

$(x-3)^2+(y+4)^2=25$

Expand the original equation and then translate to polar coordinates:

$(x-3)^2+(y+4)^2=25$

$x^2-6x+9+y^2+8y+16=25$

$r^2-6x+8y=0$

$r^2-6r\cdot\cos\theta+8r\cdot\sin\theta=0$

$r-6\cos\theta+8\sin\theta=0$

$r=6\cos\theta-8\sin\theta$

Review

Convert the following conics from polar form to rectangular form. Then, identify the conic.

1. $r=\frac{5}{3-\cos\theta}$
2. $r=\frac{4}{2-\cos\theta}$
3. $r=\frac{2}{2-\cos\theta}$
4. $r=\frac{3}{2-4\cos\theta}$
5. $r=5\cos(\theta)$

Graph the following conics.

6. $r=\frac{5}{4-2\cos(\theta-90°)}$
7. $r=\frac{5}{3-7\cos(\theta-60°)}$
8. $r=\frac{3}{3-3\cos(\theta-30°)}$
9. $r=\frac{1}{2-\cos(\theta-60°)}$
10. $r=\frac{3}{6-3\cos(\theta-45°)}$

Translate the following conics to polar form.

11. $\frac{(x-1)^2}{4}+\frac{y^2}{3}=1$
12. $(x-5)^2+(y+12)^2=169$
13. $x^2+(y+1)^2=1$
14. $(x-1)^2+y^2=1$
15. $-3x^2-4x+y^2-1=0$

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10.3 Parameters and Parameter Elimination

In a **parametric equation**, the variables x and y are not dependent on one another. Instead, both variables are dependent on a third variable, t. This is the **parameter** or a number that affects the behavior of the equation. Usually t will stand for time. A real world example of the relationship between x, y and t is the height, weight and age of a baby.

Both the height and the weight of a baby depend on time, but there is also clearly a positive relationship between just the height and weight of the baby. By focusing on the relationship between the height and the weight and letting time hide in the background, you create a parametric relationship between the three variables.

What other types of real world situations are modeled with parametric equations?

Eliminating the Parameter

In your graphing calculator there is a parametric mode. Once you put your calculator into parametric mode, on the graphing screen you will no longer see $y =$ ___, instead, you will see:

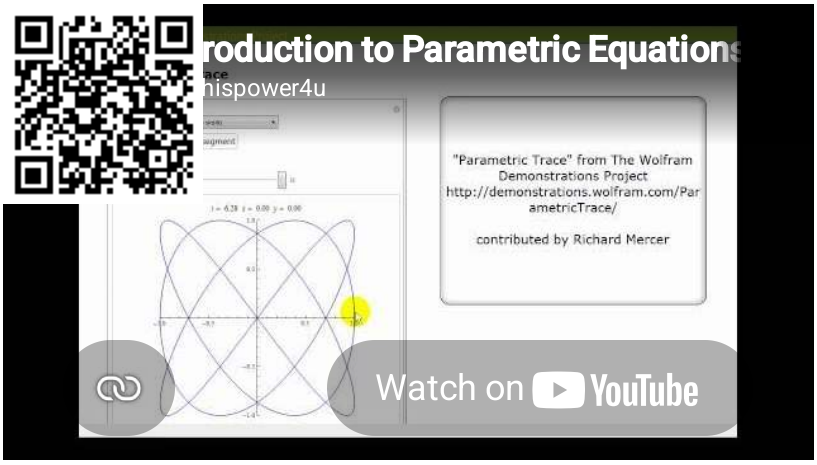


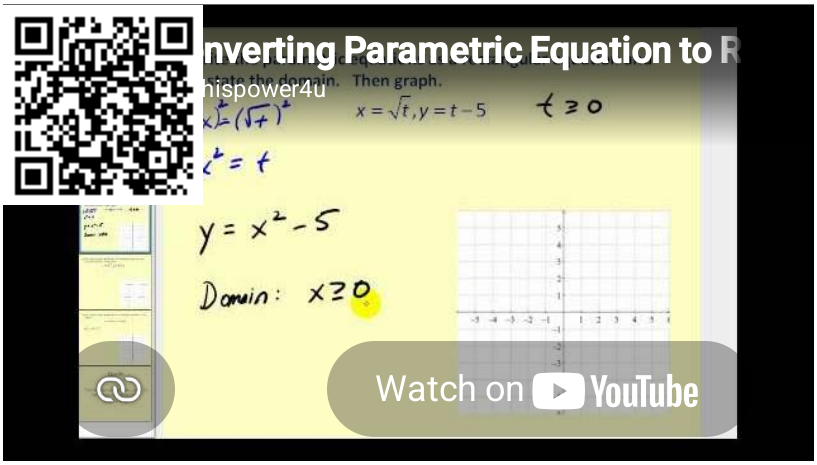
Notice how for plot one, the calculator is asking for two equations based on variable T :

$x_{1T} = f(t)$

$y_{1T} = g(t)$

This is called parametric form. **Parametric form** refers to a relationship that includes $x = f(t)$ and $y = g(t)$. In order to transform a parametric equation into a normal one, you need to do a process called "eliminating the parameter." **"Eliminating the parameter"** is a phrase that means to turn a parametric equation that has $x = f(t)$ and $y = g(t)$ into just a relationship between y and x. You are eliminating t. To do this, you must solve the $x = f(t)$ equation for $t = f^{-1}(x)$ and substitute this value of t into the y equation. This will produce a normal function of y based on x.





There are two major benefits of graphing in parametric form. First, it is straightforward to graph a portion of a regular function using the T_{min}, T_{max} and T_{step} in the window setting. Second, parametric form enables you to graph projectiles in motion and see the effects of time.

A tortoise and a hare start 202 feet apart and then race to a flag halfway between them. The hare decides to take a nap and give the tortoise a 21 second head start. The hare runs at 9.8 feet per second and the tortoise hustles along at 3.2 feet per second. This situation can be represented by parametric equations and we can use the equations to determine who wins this epic race and by how much.

First draw a picture and then represent each character with a set of parametric equations.



The tortoise's position is (-101,0) at $t = 0$ and (-97.8,0) at $t = 1$. You can deduce that the equation modeling the tortoise's position is:

$x_1 = -101 + 3.2 \cdot t$

$y_1 = 0$

The hare's position is (101,0) at $t = 21$ and (91.2,0) at $t = 22$. Note that it does not make sense to make equations modeling the hare's position before 21 seconds have elapsed because the Hare is napping and not moving. You can set up an equation to solve for the hare's theoretical starting position had he been running the whole time.

$x_2 = b - 9.8t$

$101 = b - 9.8 \cdot 21$

$305.8 = b$

The hare's position equation after $t = 21$ can be modeled by:

$x_2 = 305.8 - 9.8 \cdot t$

$y_2 = 0$

The tortoise crosses $x = 0$ when $t \approx 31.5$. The hare crosses $x = 0$ when $t \approx 31.2$. The hare wins by about 1.15 feet.

Now, use your calculator to display these parametric equations. T

here are many settings you should know for parametric equations that bring questions like this to life. The TI-84 has features that allow you to see the race happen.

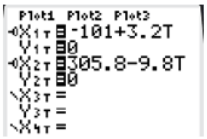
First, set the mode to simultaneous graphing. This will show both the tortoise and hare's position at the same time.



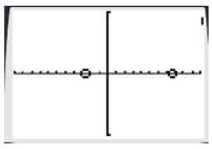
Next, change the graphing window so that t varies between 0 and 32 seconds. The T_{step} determines how often the calculator will calculate points. The larger the T_{step}, the faster and less accurately the graph will plot. Also change the x to vary between -110 and 110 so you can see the positions of both characters.

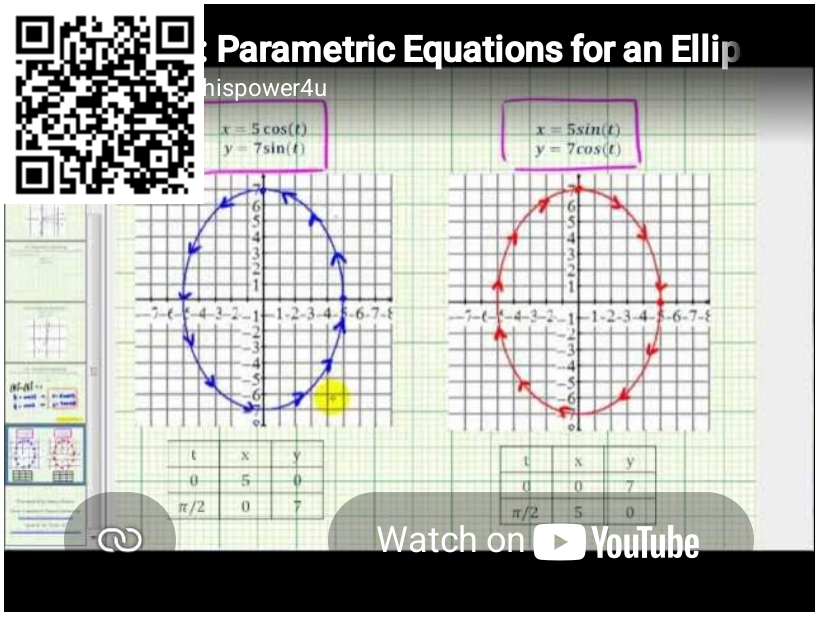


Input the parametric equations. Toggle to the left of the x and change the cursor from a line to a line with a bubble at the end. This shows their position more clearly.



Now when you graph you should watch the race unfold as the two position graphs race towards each other.





Examples

Example 1

Earlier, you were asked what types of real world situations can be modeled by parametric equations. Parametric equations are often used when only a portion of a graph is useful. By limiting the domain of t, you can graph the precise interval of the function you want. Parametric equations are also useful when two different variables jointly depend on a third variable and you

wish to look at the relationship between the two dependent variables. This is very common in statistics where an underlying variable may actually be the cause of a problem and the observer can only examine the relationship between the outcomes that they see. In the physical world, parametric equations are exceptional at graphing position over time because the horizontal and vertical vectors of objects in free motion are each dependent on time, yet independent of one another.

Example 2

Eliminate the parameter in the following equations.

$x = 6t - 2$

$y = 5t^2 - 6t$

$x = 6t - 2$ So $\frac{x+2}{6} = t$. Now, substitute this value for t into the second equation:

$y = 5\left(\frac{x+2}{6}\right)^2 - 6\left(\frac{x+2}{6}\right)$

Example 3

For the given parametric equation, graph over each interval of t.

$x = t^2 - 4$

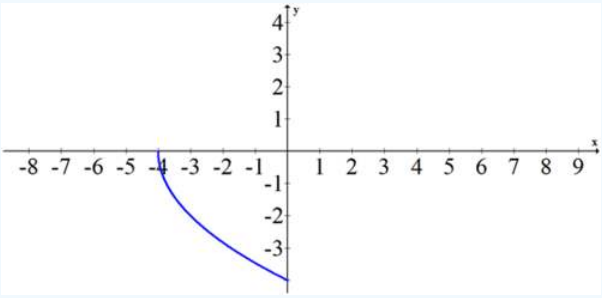
$y = 2t$

1. $-2 \leq t \leq 0$

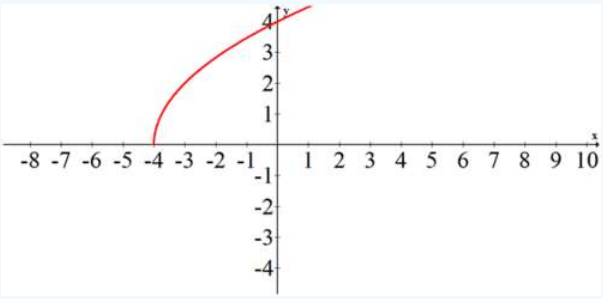
2. $0 \leq t \leq 5$

3. $-3 \leq t \leq 2$

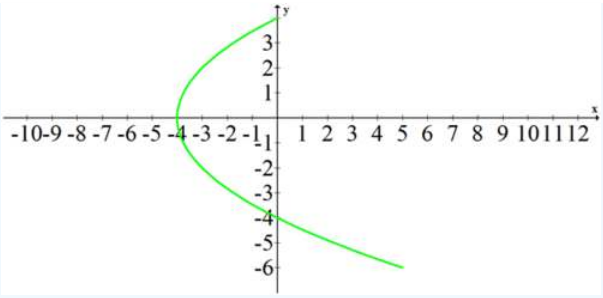
a. A good place to start is to find the coordinates where t indicates the graph will start and end. For $-2 \leq t \leq 0, t = -2$ and $t = 0$ indicate that the points (0,-4) and (-4,0) are the endpoints of the graph.



b. $0 \leq t \leq 5$



c. $-3 \leq t \leq 2$



Example 4

Eliminate the parameter and graph the following parametric curve.

$x = 3 \cdot \sin t$

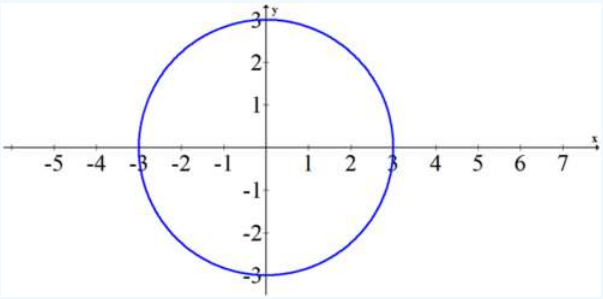
$y = 3 \cdot \cos t$

When parametric equations involve trigonometric functions you can use the Pythagorean Identity, $\sin^2 t + \cos^2 t = 1$. In this problem, $\sin t = \frac{x}{3}$ (from the first equation) and $\cos t = \frac{y}{3}$ (from the second equation). Substitute these values into the Pythagorean Identity and you have:

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$x^2 + y^2 = 9$

This is a circle centered at the origin with radius 3.



Example 5

Find the parameterization for the line segment connecting the points (1,3) and $(4, 8)$.

Use the fact that a point plus a vector yields another point. A vector between these points is $< 4-1, 8-3 > = < 3, 5 >$

Thus the point (1,3) plus t times the vector $< 3, 5 >$ will produce the point (4,8) when $t = 1$ and the point (1,3) when $t = 0$.

$(x, y) = (1, 3) + t \cdot < 3, 5 >$, for $0 \leq t \leq 1$

You than then break up this vector equation into parametric form.

$x = 1 + 3t$

$y = 3 + 5t$

$0 \leq t \leq 1$

Review

Eliminate the parameter in the following sets of parametric equations.

1. $x = 3t - 1; y = 4t^2 - 2t$

2. $x = 3t^2 + 6t; y = 2t - 1$

3. $x = t + 2; y = t^2 + 4t + 4$

4. $x = t - 5; y = t^3 + 1$

5. $x = t + 4; y = t^2 - 5$

For the parametric equation $x = t, y = t^2 + 1$, graph over each interval of t.

6. $-2 \leq t \leq -1$

7. $-1 \leq t \leq 0$

8. $-1 \leq t \leq 1$

9. $-2 \leq t \leq 2$

10. $-5 \leq t \leq 5$

11. Eliminate the parameter and graph the following parametric curve:

$x = \sin t, y = -4 + 3\cos t$

12. Eliminate the parameter and graph the following parametric curve:

$x = 1 + 2\cos t, y = 1 + 2\sin t$

13. Using the previous problem as a model, find a parameterization for the circle with center (2, 4) and radius 3.

14. Find the parameterization for the line segment connecting the points (2,7) and $(1, 4)$.

15. Find a parameterization for the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Use the fact that $\cos^2 t + \sin^2 t = 1$. Check your answer with your calculator.

16. Find a parameterization for the ellipse $\frac{(x-4)^2}{9} + \frac{(y+1)^2}{36} = 1$. Check your answer with your calculator.

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10.4 Parametric Inverses

You have learned that a graph and its inverse are reflections of each other across the line $y = x$. You have also learned that in order to find an inverse algebraically, you can switch the x and y variables and solve for y. Parametric equations actually make finding inverses easier because both the x and y variables are based on a third variable t. All you need to do to find the inverse of a set of parametric equations and switch the functions for x and y.

Is the inverse of a function always a function?

Inverses of Parametric Equations

To find the inverse of a parametric equation you must switch the function of x with the function of y. This will switch all the points from (x, y) to (y, x) and also has the effect of visually reflecting the graph over the line $y = x$.

Similar to the inverses regular functions, the inverses of parametric equations are often restricted so that they are also functions. Take the following parametric equations:

$x = 2t$

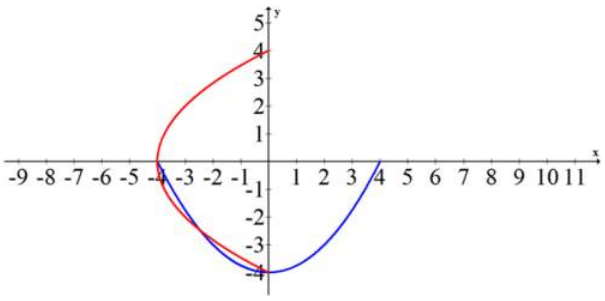
$y = t^2 - 4$

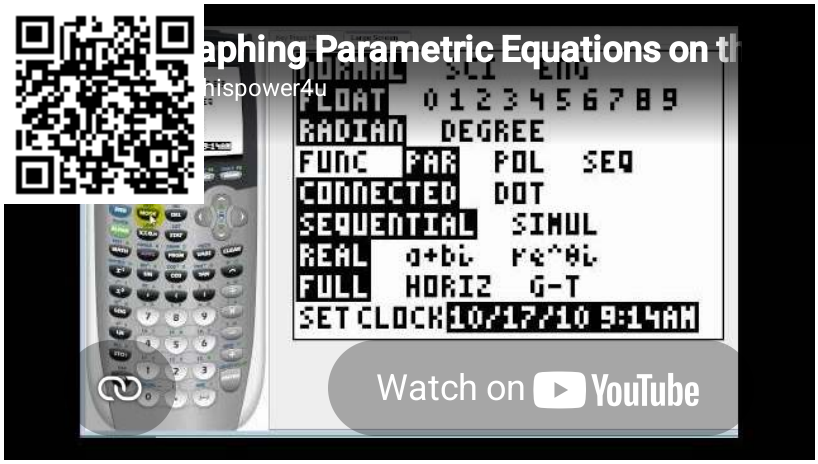
To find and graph the inverse of the parametric function on the domain $-2 < t < 2$, first switch the x and y functions and graph.

$x = t^2 - 4$

$y = 2t$

The original function is shown in blue and the inverse is shown in red.





Examples

Example 1

Earlier, you were asked if the inverse of a function is always a function. The inverse of a function is not always a function. In order to see whether the inverse of a function will be a function, you must perform the horizontal line test on the original function. If the function passes the horizontal line test then the inverse will be a function. If the function does not pass the horizontal line test then the inverse produces a relation rather than a function.

Example 2

Is the point (4,8) in the following function or its inverse?

$x = 2t^2 - 2$

$y = t^2 - 1$

Try to solve for a matching t in the original function.

The point does not satisfy the original function. Check to see if it satisfies the inverse.

The point does satisfy the inverse of the function.

Example 3

Parameterize the following function and then graph the function and its inverse.

$f(x) = x^2 + x - 4$

For the original function, the parameterization is:

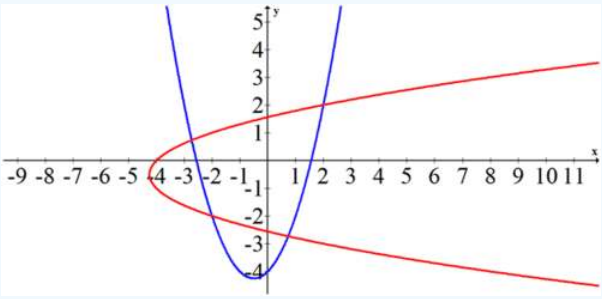
$x = t$

$y = t^2 + t - 4$

The inverse is:

$x = t^2 + t - 4$

$y = t$



Example 4

An **intersection** for two sets of parametric equations happens when the points exist at the same x, y and t. Find the points of intersection of the function and its inverse from Example 2.

The parameterized function is:

$x_1 = t$

$y_1 = t^2 + t - 4$

The inverse is:

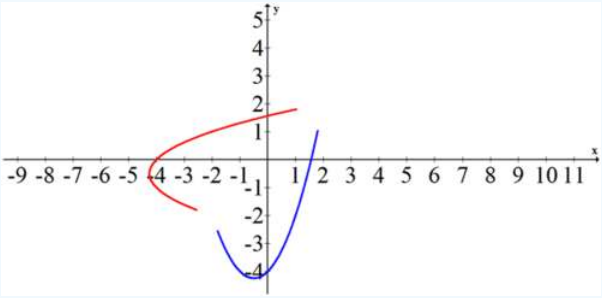
$x_2 = t^2 + t - 4$

$y_2 = t$

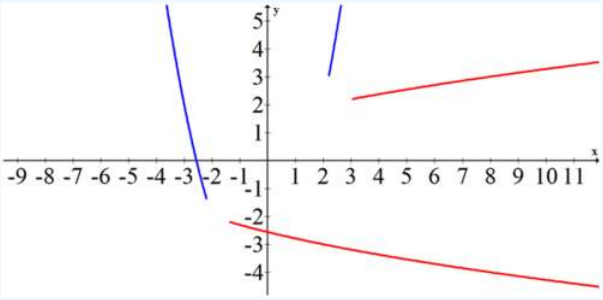
To find where these intersect, set $x_1 = x_2$ and $y_1 = y_2$ and solve.

$t = t^2 + t - 4$
$t^2 = 4$
$t = \pm 2$

You still need to actually calculate the points of intersection on the graph. You can tell from the graph in Example C that there seem to be four points of intersection. Since t can mean time, the question of intersection is more complicated than simply overlapping. It means that the points are at the same x and y coordinate at the same time. Note what the graphs look like when $-1.8 < t < 1.8$



Note what the graphs look like $t > 2.2$ or $t < -2.2$



Notice how when these partial graphs are examined there is no intersection at anything besides $t = \pm 2$ and the points (2,2) and (-2,-2) While the paths of the graphs intersect in four places, they intersect at the same time only twice.

Example 5

Identify where the following parametric function intersects with its inverse.

$x = 4t$

$y = t^2 - 16$

$x_1 = 4t; y_1 = t^2 - 16$ The inverse is:

$x_2 = t^2 - 16$

$y_2 = 4t$

Solve for t when $x_1 = x_2$ and $y_1 = y_2$.

$4t = t^2 - 16$
$0 = t^2 - 4t - 16$
$t = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot (-16)}}{2} = \frac{4 \pm 4\sqrt{5}}{2} = 2 \pm 2\sqrt{5}$

The points that correspond to these two times are:

$x = 4(2 + 2\sqrt{5}), y = (2 + 2\sqrt{5})^2 - 16$

$x = 4(2 - 2\sqrt{5}), y = (2 - 2\sqrt{5})^2 - 16$

? Review

Use the function $x = t - 4; y = t^2 + 2$ for #1- #3.

1. Find the inverse of the function.

2. Does the point (-2,6) live on the function or its inverse?

3. Does the point (0,1) live on the function or its inverse?

Use the relation $x = t^2; y = 4 - t$ for $\#4 - \#6$.

4. Find the inverse of the relation.

5. Does the point (4,0) live on the relation or its inverse?

6 . Does the point (0,4) live on the relation or its inverse?

Use the function $x = 2t + 1; y = t^2 - 3$ for $\#7 - \#9$.

7. Find the inverse of the function.

8. Does the point (1,5) live on the function or its inverse?

9. Does the point (9,13) live on the function or its inverse?

Use the function $x = 3t + 14; y = t^2 - 2t$ for $\#10 - \#11$

10. Find the inverse of the function.

11. Identify where the parametric function intersects with its inverse.

Use the relation $x = t^2; y = 4t - 4$ for $\#12 - \#13$.

12. Find the inverse of the relation.

13. Identify where the relation intersects with its inverse.

14. Parameterize $f(x) = x^2 + x - 6$ and then graph the function and its inverse.

15. Parameterize $f(x) = x^2 + 3x + 2$ and then graph the function and its inverse.

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10.5 Applications of Parametric Equations

A regular function has the ability to graph the height of an object over time. Parametric equations allow you to actually graph the complete position of an object over time. For example, parametric equations allow you to make a graph that represents the position of a point on a Ferris wheel. All the details like height off the ground, direction, and speed of spin can be modeled using the parametric equations.

What is the position equation and graph of a point on a Ferris wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds?

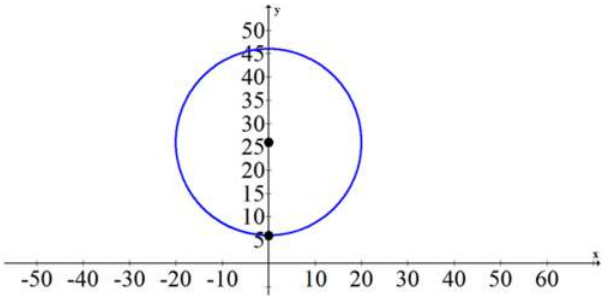
Applying Parametric Equations

There are two types of parametric equations that are typical in real life situations. The first is

circular motion as was described in the concept problem. The second is projectile motion.

Circular Motion

Parametric equations that describe circular motion will have x and y as periodic functions of sine and cosine. Either x will be a sine function and y will be a cosine function or the other way around. The best way to come up with parametric equations is to first draw a picture of the circle you are trying to represent.



Next, it is important to note the starting point, center point and direction. You should already

have the graphs of sine and cosine memorized so that when you see a pattern in words or as a graph, you can identify what you see as $+\sin, -\sin, +\cos, -\cos$.

Take the example given above with the Ferris Wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds. The the vertical component starts at a low point of 6 , travels to a middle point of 26 and then a height of 46 and back down. This is a - cos pattern. The amplitude of the $-\cos$ is 20 and the vertical shift is 26 . Lastly, the period is 60 . You can use the period to help you find b.

$$60 = \frac{2\pi}{b}$$
$$b = \frac{\pi}{30}$$

Thus the vertical parameterization is:

$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$$

The horizontal parameterization is found by noticing that the x values start at 0, go up to 20, go back to 0 , then down to -20 , and finally back to 0 . This is a $+\sin$ pattern with amplitude 20. The period is the same as with the vertical component.

Thus parametric equations for the point on the wheel are:

$$x = 20\sin\left(\frac{\pi}{30}t\right)$$

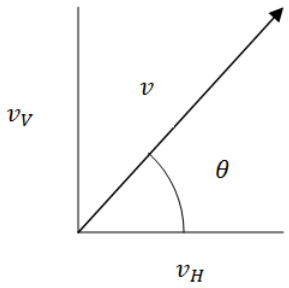
$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$$

Note that **horizontal and vertical components** of parametric equations are the $x =$ and

$y =$ functions respectively or the horizontal and vertical parameterization.

Projectile Motion

Projectile motion has a vertical component that is quadratic and a horizontal component that is linear. This is because there are 3 parameters that influence the position of an object in flight: starting height, initial velocity, and the force of gravity. The horizontal component is independent of the vertical component. This means that the starting horizontal velocity will remain the horizontal velocity for the entire flight of the object.



Note that gravity, g, has a force of about -32ft/s^2 or -9.81 m/s^2. The examples and practice questions in this concept will use feet.

If an object is launched from the origin at a velocity of v then it has horizontal and vertical components that can be found using basic trigonometry.

$\sin\theta = \frac{v_V}{v} \rightarrow v \cdot \sin\theta = v_V$

$\cos\theta = \frac{v_H}{v} \rightarrow v \cdot \cos\theta = v_H$

The horizontal component is basically finished. The only adjustments that would have to be made are if the starting location is not at the origin, wind is added or if the projectile travels to the left instead of the right. See Example A.

$x = t \cdot v \cdot \cos\theta$

The vertical component also needs to include gravity and the starting height. The general

equation for the vertical component is:

$y = \frac{1}{2} \cdot g \cdot t^2 + t \cdot v \cdot \sin\theta + k$

The constant g represents gravity, t represents time, v represents initial velocity and k represents starting height. You will explore this equation further in calculus and physics. Note that in this concept, most answers will be found and confirmed using technology such as your graphing calculator.

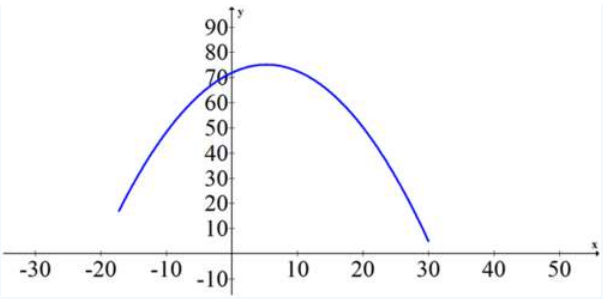
Examples

Example 1

A ball is thrown from the point (30,5) at an angle of $\frac{4\pi}{9}$ to the left at an initial velocity of 68ft/s. Model the position of the ball over time using parametric equations. Use your graphing calculator to graph your equations for the first four seconds while the ball is in the air.

The horizontal component is $x = -t \cdot 68 \cdot \cos\left(\frac{4\pi}{9}\right) + 30$. Note the negative sign because the object is traveling to the left and the +30 because the object starts at (30,5).

The vertical component is $y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 68 \cdot \sin\left(\frac{4\pi}{9}\right) + 5$. Note that $g = -32$ because gravity has a force of -32ft/s^2 and the +5 because the object starts at (30,5).

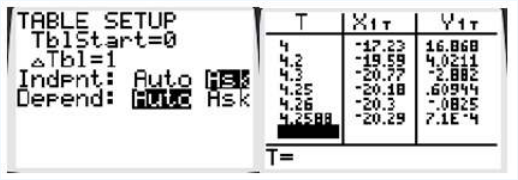


Example 2

When does the ball from Example 1 reach its maximum and when does the ball hit the ground? How far did the person throw the ball? T

o find when the function reaches its maximum, you can find the vertex of the parabola. Analytically this is messy because of the decimal coefficients in the quadratic. Use your calculator to approximate the maximum after you have graphed it. Depending on how small you make your T_{step} should find the maximum height to be about 75 feet.

To find out when the ball hits the ground, you can set the vertical component equal to zero and solve the quadratic equation. You can also use the table feature on your calculator to determine when the graph goes from having a positive vertical value to a negative vertical value. The benefit for using the table is that it simultaneously tells you the x value of the zero.



After about 4.2588 seconds the ball hits the ground at (-20.29, 0). This means the person threw the ball from (30, 5) to (-20.29, 0), a horizontal distance of just over 50 feet.

Example 3

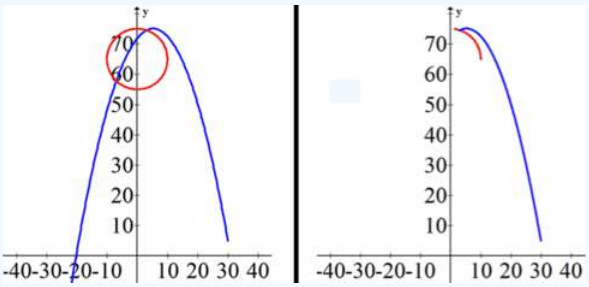
Kieran is on a Ferris wheel and his position is modeled by the parametric equations:

$x_K = 10 \cdot \cos\left(\frac{\pi}{5}t\right)$

$y_K = 10 \cdot \sin\left(\frac{\pi}{5}t\right) + 65$

Jason throws the ball modeled by the equation in Example 1 towards Kieran who can catch the ball if it gets within three feet. Does Kieran catch the ball?

This question is designed to demonstrate the power of your calculator. If you simply model the two equations simultaneously and ignore time you will see several points of intersection. This graph is shown below on the left. These intersection points are not interesting because they represent where Kieran and the ball are at the same place but at different moments in time.



When the T_{max} is adjusted to 2.3 so that each graph represents the time from 0 to 2.3 , you get a better sense that at about 2.3 seconds the two points are close. This graph is shown above on the right.

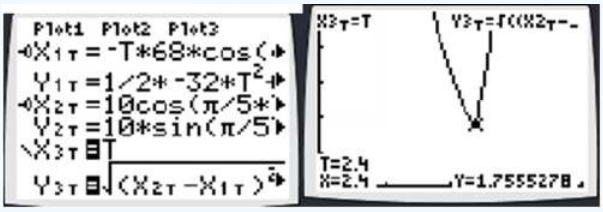
You can now use your calculator to help you determine if the distance between Kieran and the ball actually does go below 3 feet. Start by plotting the ball's position in your calculator as

x_1 and y_1 and Kieran's position as x_2 and y_2. Then, plot a new parametric equation that compares the distance between these two points over time. You can put this under x_3 and y_3. A calculator can reference internal variables like x_1, y_1 that have already been set in the calculator's memory to form new variables like x_3, y_3. Note that you can find the x_1, x_2, y_1, y_2 entries in the vars and parametric menu.

$x_3 = t$

$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Now when you graph, you should change your window settings and let t vary between 0 and 4, the x window show between 0 and 4 and the y window show between 0 and 5. This way it should be clear if the distance truly does get below 3 feet.



Depending on how accurate your T_{step} is, you should find that the distance is below 3 feet. Kieran does indeed catch the ball.

Example 4

At what velocity does a football need to be thrown at a $45°$ angle in order to make it all the way across a football field?

A football field is 100 yards or 300 feet. The parametric equations for a football thrown from (300,0) back to the origin at speed v are:

$x = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$

$y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot \sin\left(\frac{\pi}{4}\right)$

Substituting the point (0,0) in for (x, y) produces a system of two equations with two variables v, t

$0 = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$

$0 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot \sin\left(\frac{\pi}{4}\right)$

You can solve this system many different ways.

$t = \frac{5\sqrt{3}}{2} \approx 4.3$ seconds, $v = 40\sqrt{6} \approx 97.98 \text{ft/s}$

In order for someone to throw a football at a $45°$ angle all the way across a football field, they would need to throw at about 98ft/s which is about 66.8mph.

$\frac{98 \text{ feet}}{1 \text{ sec}} \cdot \frac{3600 \text{ sec}}{1 \text{ hour}} \cdot \frac{1 \text{ mile}}{5280 \text{ feet}} \approx \frac{66.8 \text{ miles}}{1 \text{ hour}}$

Example 5

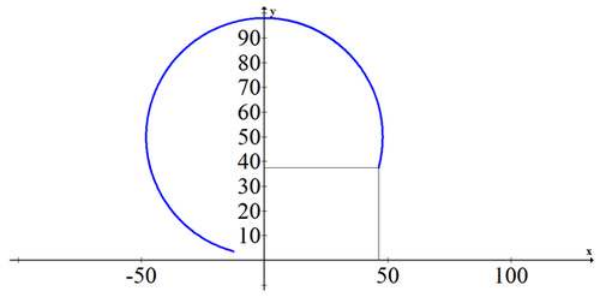
Nikki got on a Ferris wheel ten seconds ago. She started 2 feet off the ground at the lowest point of the wheel and will make a complete cycle in four minutes. The ride reaches a maximum height of 98 feet and spins clockwise. Write parametric equations that model Nikki's position over time. Where will Nikki be three minutes from now?

Don't let the 10 second difference confuse you. In order to deal with the time difference, use $\left(t + \frac{1}{6}\right)$ instead of t in each equation. When $t = 0$, ten seconds $\left(\frac{1}{6}\right.$ of a minute $\left.\right)$ have already elapsed.

$x = -48 \cdot \sin\left(\frac{\pi}{2}\left(t + \frac{1}{6}\right)\right)$

$y = -48 \cdot \cos\left(\frac{\pi}{2}\left(t + \frac{1}{6}\right)\right) + 50$

At $t = 3, x \approx 46.36$ and $y \approx 37.58$



Review

Candice gets on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 103 feet, making a complete cycle in 5 minutes.

1. Write a set of parametric equations to model Candice's position.

2. Where will Candice be in two minutes?

3. Where will Candice be in four minutes?

One minute ago Guillermo got on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 83 feet, making a complete cycle in 6 minutes.

4. Write a set of parametric equations to model Guillermo's position.

5. Where will Guillermo be in two minutes?

6. Where will Guillermo be in four minutes?

Kim throws a ball from (0,5) to the right at 50 mph at a $45°$ angle.

7. Write a set of parametric equations to model the position of the ball.

8. Where will the ball be in 2 seconds?

9. How far does the ball get before it lands?

David throws a ball from (0,7) to the right at 70 mph at a $60°$ angle. There is a 6 mph wind in David's favor.

10. Write a set of parametric equations to model the position of the ball.

11. Where will the ball be in 2 seconds?

12. How far does the ball get before it lands?

Suppose Riley stands at the point (250,0) and launches a football at 72 mph at an angle of $60°$ towards Kristy who is at the origin. Suppose Kristy also throws a football towards Riley at 65 mph at an angle of $45°$ at the exact same moment. There is a 6 mph breeze in Kristy's favor.

13. Write a set of parametric equations to model the position of Riley's ball.

14. Write a set of parametric equations to model the position of Kristy's ball.

15. Graph both functions and explain how you know that the footballs don't collide even though the two graphs intersect.

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CHAPTER OVERVIEW

11: Complex Numbers

11.1 Fundamental Theorem of Algebra

11.2 Arithmetic with Complex Numbers

11.3 Trigonometric Polar Form of Complex Numbers

11.4 De Moivre's Theorem and nth Roots

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11.1 Fundamental Theorem of Algebra

You have learned that a quadratic has at most two real zeroes and a cubic has at most three real zeros. You may have noticed that the number of real zeros is always less than or equal to the degree of the polynomial. By looking at a graph you can see when a parabola crosses the x axis 0,1 or 2 times, but what does this have to do with complex numbers?

The Fundamental Theorem of Algebra

A real number is any rational or irrational number. When a real number is squared, it will always produce a non-negative value. Complex numbers include real numbers and another type of number called imaginary numbers. Unlike real numbers, imaginary numbers may produce a negative value when squared. The square root of negative one is defined to be the imaginary number i.

$i = \sqrt{-1}$ and $i^2 = -1$

Complex numbers are written with a real component and an imaginary component. All complex numbers can be written in the form $a + bi$. When the imaginary component is zero, the number is simply a real number. This means that real numbers are a subset of complex numbers.

The Fundamental Theorem of Algebra states that an n^{th} degree polynomial with real or complex coefficients has, with multiplicity, exactly n complex roots. This means a cubic will have exactly 3 roots, some of which may be complex.

Take the following polynomial.

$f(x) = x^2 + 9$

At first you may think that this does not have any roots but the Fundamental Theorem of Algebra states that it must have 2 roots. Both roots for this polynomial are complex.

To find the roots for this complex polynomial, set $y = 0$ and solve for x. This will give you the two zeros.

$$\begin{aligned} 0 &= x^2 + 9 \\ -9 &= x^2 \\ \pm 3i &= x \end{aligned}$$

Thus the linear factorization of the function is:

$f(x) = (x - 3i)(x + 3i)$

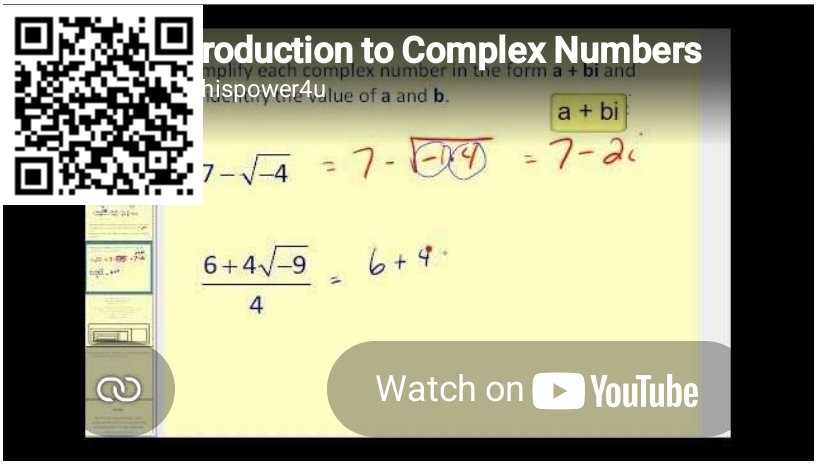
Multiplicity refers to when a root counts more than once. For example, in the following function the only root occurs at $x = 3$.

$f(x) = (x - 3)^2$

The Fundamental Theorem of Algebra states that this 2^{nd} degree polynomial must have exactly 2 roots with multiplicity. This means that the root $x = 3$ has multiplicity 2 . One way to determine the multiplicity is to simply look at the degree of each of the factors in the factorized polynomial.

$g(x) = (x - 1)(x - 3)^4(x + 2)$

This function has 6 roots. The first two roots $x = 1$ and $x = -2$ have multiplicities of 1 because the power of each of their binomial factors is 1 . The third root $x = 3$ has a multiplicity of 4 because the power of its binomial factor is 4 . Keep in mind that all polynomials can be written in factorized form like the above polynomial, due to a theorem called the Linear Factorization Theorem.



Examples

Example 1

Earlier, you were asked about the graphs of parabolas and their relationships to the zeroes of the parabolas. When a parabola fails to cross the x axis it still has 2 roots. These two roots happen to be imaginary numbers. The function $f(x) = x^2 + 4$ does not cross the x axis, but its roots are $x = \pm 2i$.

Example 2

Identify the polynomial that has the following five roots. $x = 0, 2, 3, \pm\sqrt{5}i$

Write the function in factorized form.

$f(x) = (x-0)(x-2)(x-3)(x-\sqrt{5}i)(x+\sqrt{5}i)$

When you multiply through, it will be helpful to do the complex conjugates first. Remember that **Complex conjugates** are pairs of complex numbers with real parts that are identical and imaginary parts that are of equal magnitude but opposite signs. The complex conjugates in this equation are $(x-\sqrt{5}i)(x+\sqrt{5}i)$.

$f(x) = x\left(x^2 - 5x + 6\right)\left(x^2 - 5 \cdot (-1)\right)$

$f(x) = \left(x^3 - 5x^2 + 6x\right)\left(x^2 + 5\right)$

$f(x) = x^5 - 5x^4 + 6x^3 + 5x^3 - 25x^2 + 30x$

$f(x) = x^5 - 5x^4 + 11x^3 - 25x^2 + 30x$

Example 3

Write the polynomial that has the following roots: 4 (with multiplicity 3), 2 (with multiplicity 2) and 0.

$f(x) = (x-4)^3 \cdot (x-2)^2 \cdot x$

Example 4

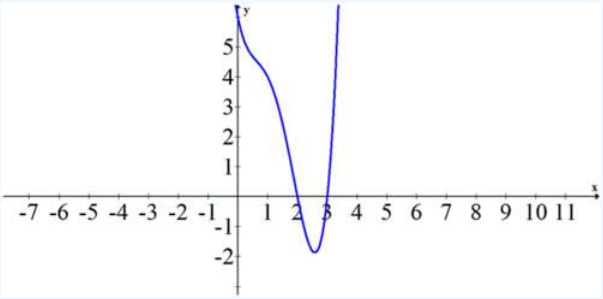
Factor the following polynomial into its linear factorization and state all of its roots.

$f(x) = x^4 - 5x^3 + 7x^2 - 5x + 6$

You can use polynomial long division to obtain the following factorization.

$f(x) = (x-3)(x-2)(x-i)(x+i)$

If you need a place to start, it is helpful to look at the graph of the polynomial and notice that the graph shows you exactly where the real roots appear.



Example 5

Can you create a polynomial with real coefficients that has one imaginary root? Why or why not?

No, if a polynomial has real coefficients then either it has no imaginary roots, or the imaginary roots come in pairs of complex conjugates (so that the imaginary portions cancel out when the factors are multiplied).

Review

For $\mathbf{1-4}$, find the polynomial with the given roots.

1.2 (with multiplicity 2), 4 (with multiplicity 3), $\mathbf{1, \sqrt{2}i, -\sqrt{2}i}$.

2. 1,-3 (with multiplicity 3), $\mathbf{-1, \sqrt{3}i, -\sqrt{3}i}$

3. 5 (with multiplicity 2), -1 (with multiplicity 2), $\mathbf{2i, -2i}$

4. $\mathbf{i, -i, \sqrt{2}i, -\sqrt{2}i}$

For each polynomial, factor into its linear factorization and state all of its roots.

5. $\mathbf{f(x) = x^5 + 4x^4 - 2x^3 - 14x^2 - 3x - 18}$

6. $\mathbf{g(x) = x^4 - 1}$

7. $\mathbf{h(x) = x^6 - 12x^5 + 61x^4 - 204x^3 + 532x^2 - 864x + 576}$

8. $\mathbf{j(x) = x^7 - 11x^6 + 49x^5 - 123x^4 + 219x^3 - 297x^2 + 243x - 81}$

9. $\mathbf{k(x) = x^5 + 3x^4 - 11x^3 - 15x^2 + 46x - 24}$

10. $\mathbf{m(x) = x^6 - 12x^4 + 23x^2 + 36}$

11. $\mathbf{n(x) = x^6 - 3x^5 - 10x^4 - 32x^3 - 81x^2 - 85x - 30}$

12. $\mathbf{p(x) = x^6 + 4x^5 + 7x^4 + 12x^3 - 16x^2 - 112x - 112}$

13. How can you tell the number of roots that a polynomial has from its equation?

14. Explain the meaning of the term "multiplicity".

15. A polynomial with real coefficients has one root that is $\sqrt{3}i$. What other root(s) must the polynomial have?

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11.2 Arithmetic with Complex Numbers

The idea of a complex number can be hard to comprehend, especially when you start thinking about absolute value. In the past you may have thought of the absolute value of a number as just the number itself or its positive version. How should you think about the absolute value of a complex number?

Arithmetic Operations with Complex Numbers

Complex numbers follow all the same rules as real numbers for the operations of adding, subtracting, multiplying and dividing. There are a few important ideas to remember when working with complex numbers:

1. When simplifying, you must remember to combine imaginary parts with imaginary parts and real parts with real parts. For example, $4+5i+2-3i=6+2i$

2. If you end up with a complex number in the denominator of a fraction, eliminate it by multiplying both the numerator and denominator by the complex conjugate of the denominator.

3. The powers of i are:

- $i=\sqrt{-1}$
- $i^2=-1$
- $i^3=-\sqrt{-1}=-i$
- $i^4=1$
- $i^5=i$
- ... and the pattern repeats

Consider this complex expression:

$(2+3i)(1-5i)-3i+8$

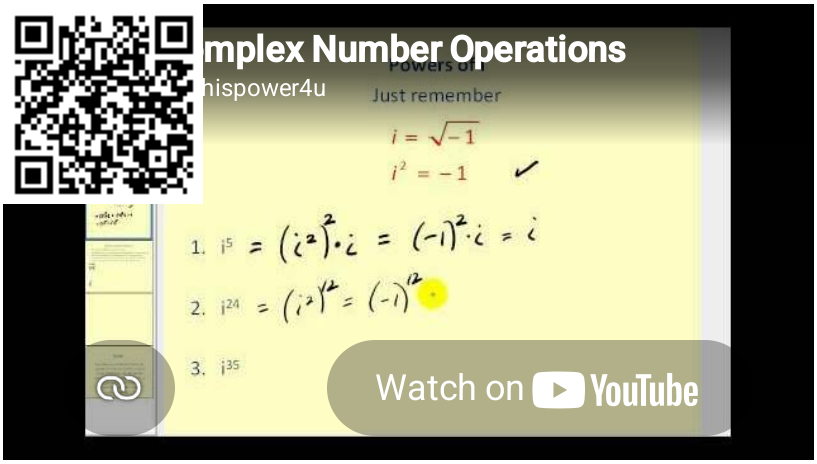
First, multiply the two binomials and then combine the imaginary parts with imaginary parts and real parts with real parts.

$=2-10i+3i-15i^2-3i+8$

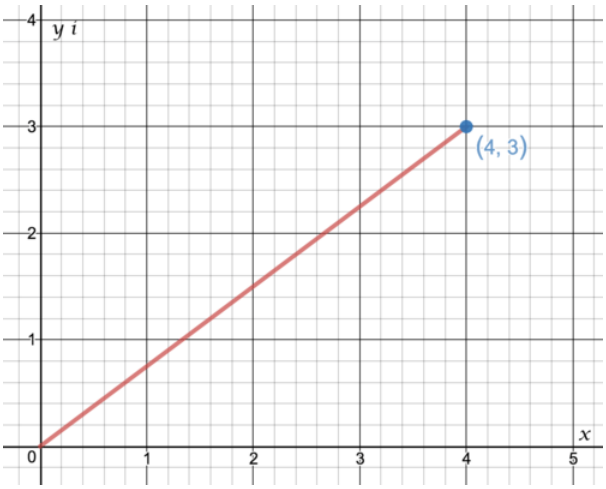
$=10-10i+15$

$=25-10i$

Note that a power higher than 1 of i can be simplified using the pattern above.



The **complex plane** is set up in the same way as the regular x, y plane, except that real numbers are counted horizontally and complex numbers are counted vertically. The following is the number $4+3i$ plotted in the complex number plane. Notice how the point is four units over and three units up.



The absolute value of a complex number like $|4+3i|$ is defined as the distance from the complex number to the origin. You can use the Pythagorean Theorem to get the absolute value. In this case, $|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5$.

Examples

Example 1

Earlier, you were asked how to think about the absolute value of a complex number. A good way to think about the absolute value for all numbers is to define it as the distance from a number to zero. In the case of complex numbers where an individual number is actually a coordinate on a plane, zero is the origin.

Example 2

Compute the following power by hand and use your calculator to support your work.

$(\sqrt{3}+2i)^3$

$(\sqrt{3}+2i)\cdot(\sqrt{3}+2i)\cdot(\sqrt{3}+2i)$

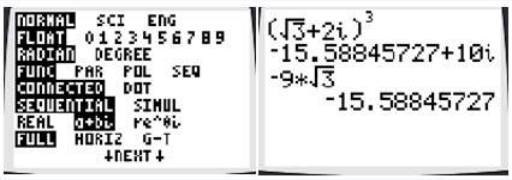
$= (3+4i\sqrt{3}-4)(\sqrt{3}+2i)$

$= (-1+4i\sqrt{3})(\sqrt{3}+2i)$

$= -\sqrt{3}-2i+12i-8\sqrt{3}$

$= -9\sqrt{3}+10i$

A TI-84 can be switched to imaginary mode and then compute exactly what you just did. Note that the calculator will give a decimal approximation for $-9\sqrt{3}$.



Example 3

Simplify the following complex expression.

$\frac{7-9i}{4-3i} + \frac{3-5i}{2i}$

To add fractions you need to find a common denominator.

$\frac{(7-9i)\cdot 2i}{(4-3i)\cdot 2i} + \frac{(3-5i)\cdot(4-3i)}{2i\cdot(4-3i)}$

$= \frac{14i+18}{8i+6} + \frac{12-20i-9i-15}{8i+6}$

$= \frac{15-15i}{8i+6}$

Lastly, eliminate the imaginary component from the denominator by using the conjugate.

$= \frac{(15-15i)\cdot(8i-6)}{(8i+6)\cdot(8i-6)}$

$= \frac{120i-90+120+90i}{100}$

$= \frac{30i+30}{100}$

$= \frac{3i+3}{10}$

Example 4

Simplify the following complex number.

i^{2013}

When simplifying complex numbers, i should not have a power greater than 1 . The powers of i repeat in a four part cycle:

$i^5 = i = \sqrt{-1}$

$i^6 = i^2 = -1$

$i^7 = i^3 = -\sqrt{-1} = -i$

$i^8 = i^4 = 1$

Therefore, you just need to determine where 2013 is in the cycle. To do this, determine the remainder when you divide 2013 by 4 . The remainder is 1 so $i^{2013} = i$.

Example 5

Plot the following complex number on the complex coordinate plane and determine its absolute value.

$-12 + 5i$

The sides of the right triangle are 5 and 12, which you should recognize as a Pythagorean

triple with a hypotenuse of 13. $|-12 + 5i| = 13$.

Review

Simplify the following complex numbers.

1. i^{252}
2. i^{312}
3. i^{411}
4. i^{2345}

For each of the following, plot the complex number on the complex coordinate plane and

determine its absolute value.

5. $6 - 8i$
6. $2 + i$
7. $4 - 2i$

8. $-5i+1$

Let $c=2+7i$ and $d=3-5i$

9. What is $c+d$?

10. What is $c-d$?

11. What is $c \cdot d$?

12. What is $2c-4d$?

13. What is $2c \cdot 4d$?

14. What is $\frac{c}{d}$?

15. What is c^2-d^2?

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11.3 Trigonometric Polar Form of Complex Numbers

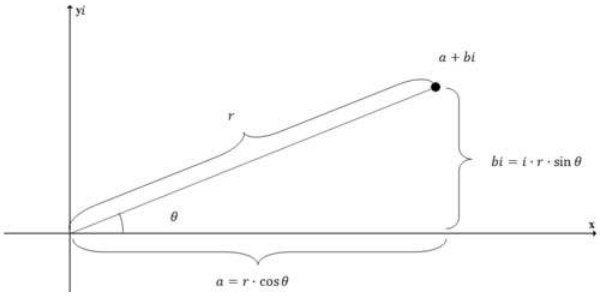
You already know how to represent complex numbers in the complex plane using rectangular coordinates and you already know how to multiply and divide complex numbers. Representing these points and performing these operations using trigonometric polar form will make your computations more efficient.

What are the two ways to multiply the following complex numbers?

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i)$

Trigonometric Polar Form of Complex Numbers

Any point represented in the complex plane as $a+bi$ can be represented in polar form just like any point in the rectangular coordinate system. Trigonometric polar form of a complex number describes the location of a point on the complex plane using the angle and the radius of the point. You will use the distance from the point to the origin as r and the angle that the point makes as θ.



As you can see, the point $a+bi$ can also be represented as $r\cdot\cos\theta+i\cdot r\cdot\sin\theta$. The trigonometric polar form can be abbreviated by factoring out the r and noting the first letters:

$r(\cos\theta+i\cdot\sin\theta)\rightarrow r\cdot\operatorname{cis}\theta$

The abbreviation $r\cdot\operatorname{cis}\theta$ is read as " r kiss theta." It allows you to represent a point as a radius and an angle.

Take the following complex number in rectangular form.

$1-\sqrt{3}i$

To convert the following complex number from rectangular form to trigonometric polar form, find the radius using the absolute value of the number.

$r^2=1^2+(-\sqrt{3})^2\rightarrow r=2$

The angle can be found with basic trig and the knowledge that the opposite side is always the

imaginary component and the adjacent side is always the real component.

$\tan\theta=-\frac{\sqrt{3}}{1}\rightarrow\theta=60^\circ$

Thus the trigonometric form is 2 cis 60°.

One great benefit of the cis form is that it makes multiplying and dividing complex numbers

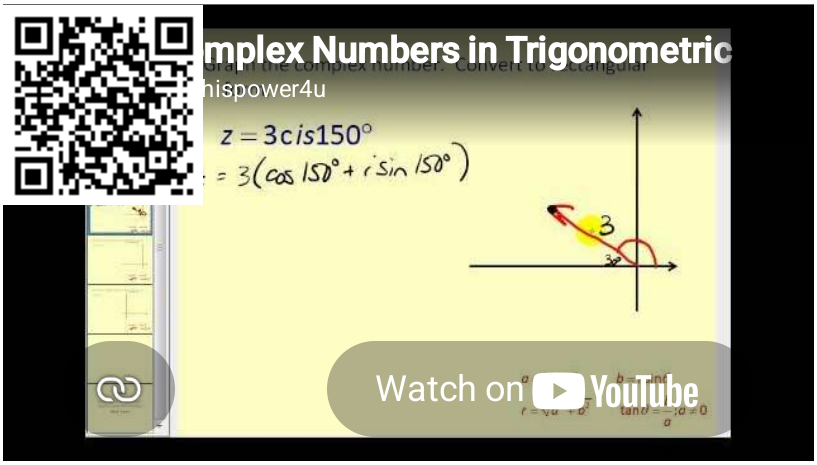
extremely easy. For example:

Let: $z_1=r_1\cdot\operatorname{cis}\theta_1, z_2=r_2\cdot\operatorname{cis}\theta_2$ with $r_2\neq 0$.

Then:

$$z_1\cdot z_2=r_1\cdot r_2\cdot\operatorname{cis}(\theta_1+\theta_2)$$
$$z_1\div z_2=\frac{r_1}{r_2}\cdot\operatorname{cis}(\theta_1-\theta_2)$$

For basic problems, the amount of work required to compute products and quotients for complex numbers given in either form is roughly equivalent. For more challenging questions, trigonometric polar form becomes significantly advantageous.



Examples

Example 1

Earlier, you were asked how to multiply the complex numbers $(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i)$.

In rectangular coordinates:

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i) = \sqrt{2}-\sqrt{2}i+\sqrt{6}i+\sqrt{6}$

In trigonometric polar coordinates, $1+\sqrt{3}i = 2\,\text{cis}\,60°$ and $\sqrt{2}-\sqrt{2}i = 2$ cis $-45°$. Therefore:

$(1+\sqrt{3}i)(\sqrt{2}-\sqrt{2}i) = 2\,\text{cis}\,60° \cdot 2\text{cis} - 45° = 4$ cis $105°$

Example 2

Convert the following complex number from trigonometric polar form to rectangular form.

$4\,\text{cis}\left(\frac{3\pi}{4}\right)$

$4\,\text{cis}\left(\frac{3\pi}{4}\right) = 4\left(\cos\left(\frac{3\pi}{4}\right)+i\cdot\sin\left(\frac{3\pi}{4}\right)\right) = 4\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = -2\sqrt{2}+2\sqrt{2}i$

Example 3

Divide the following complex numbers.

$\frac{4\text{ cis }32°}{2\,\text{cis}\,2°}$

$\frac{4\,\text{cis}\,32°}{2\,\text{cis}\,2°} = \frac{4}{2}\text{cis}\,(32°-2°) = 2\,\text{cis}\,(30°)$

Example 4

Translate the following complex number from rectangular form into trigonometric polar form:

8

$8 = 8\,\text{cis}\,0°$

Note that this has no complex part and therefore has no angle.

Example 5

Multiply the following complex numbers in trigonometric polar form.

4 cis $34° \cdot 5$ cis $16° \cdot \frac{1}{2}$ cis $100°$

$4 \text{ cis } 34^\circ \cdot 5 \text{ cis } 16^\circ \cdot \frac{1}{2} \text{ cis } 100^\circ$

$= 4 \cdot 5 \cdot \frac{1}{2} \cdot \text{ cis } (34^\circ + 16^\circ + 100^\circ) = 10 \text{ cis } 150$

Note how much easier it is to do products and quotients in trigonometric polar form.

? Review

Translate the following complex numbers from trigonometric polar form to rectangular form.

1. 5 cis 270°
2. 2 cis 30°
3. -4 cis $\frac{\pi}{4}$
4. $6 \text{ cis } \frac{\pi}{3}$
5. 2 cis $\frac{5\pi}{2}$

Translate the following complex numbers from rectangular form into trigonometric polar form.

6. $2 - i$
7. $5 + 12i$
8. $6i + 8$
9. i

Complete the following calculations and simplify.

10. 2 cis $22^\circ \cdot \frac{1}{5} \text{cis } 15^\circ \cdot 3 \text{ cis } 95^\circ$
11. 9 cis $98^\circ \div 3 \text{ cis } 12^\circ$
12. $15 \text{ cis } \frac{\pi}{4} \cdot 2$ cis $\frac{\pi}{6}$
13. $-2 \text{ cis } \frac{2\pi}{3} \div 15 \text{ cis } \frac{7\pi}{6}$

Let $z_1 = r_1 \cdot \text{cis } \theta_1$ and $z_2 = r_2 \cdot \text{cis } \theta_2$ with $r_2 \neq 0$.

14. Use the trigonometric sum and difference identities to prove that $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \text{cis } (\theta_1 + \theta_2)$
15. Use the trigonometric sum and difference identities to prove that $z_1 \div z_2 = \frac{r_1}{r_2} \cdot \text{cis } (\theta_1 - \theta_2)$

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11.4 De Moivre's Theorem and nth Roots

You know how to multiply two complex numbers together and you've seen the advantages of using trigonometric polar form, especially when multiplying more than two complex numbers at the same time. Because raising a number to a whole number power is repeated multiplication, you also know how to raise a complex number to a whole number power. What is a geometric interpretation of squaring a complex number?

De Moivre's Theorem and nth Roots

Recall that if $z_1 = r_1 \cdot \text{cis}\,\theta_1$ and $z_2 = r_2 \cdot \text{cis}\,\theta_2$ with $r_2 \neq 0$, then $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \text{cis}\,(\theta_1 + \theta_2)$

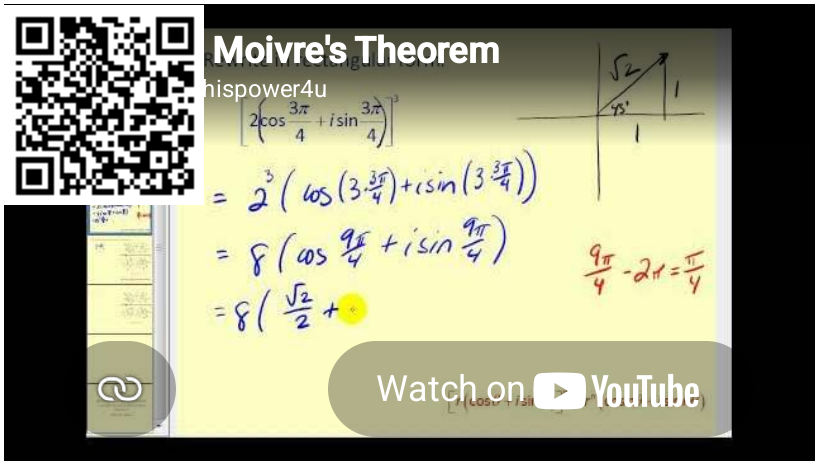
If $z_1 = z_2 = z = r\,\text{cis}\,\theta$ then you can determine z^2 and z^3 :

$z^2 = r \cdot r \cdot \text{cis}(\theta + \theta) = r^2\,\text{cis}(2 \cdot \theta)$

$z^3 = r^3\,\text{cis}(3 \cdot \theta)$

De Moivre's Theorem simply generalizes this pattern to the power of any positive integer.

$z^n = r^n \cdot \text{cis}(n \cdot \theta)$



In addition to raising a complex number to a power, you can also take square roots, cube roots and n^{th} roots of complex numbers. Suppose you have complex number $z = r\,\text{cis}\,\theta$ and you want to take the n^{th} root of z. In other words, you want to find a number $v = s \cdot \text{cis}\,\beta$ such that $v^n = z$. Do some substitution and manipulation:

$$v^n = z$$
$$(s \cdot \text{cis}\,\beta)^n = r \cdot \text{cis}\,\theta$$
$$s^n \cdot \text{cis}(n \cdot \beta) = r \cdot \text{cis}\,\theta$$

You can see at this point that to find s you need to take the n^{th} root of r. The trickier part is to find the angles, because $n \cdot \beta$ could be any angle coterminal with θ. This means that there are n different n^{th} roots of z.

$n \cdot \beta = \theta + 2\pi k$

$\beta = \frac{\theta + 2\pi k}{n}$

The number k can be all of the counting numbers including zeros up to $n - 1$. So if you are taking the 4^{th} root, then $k = 0, 1, 2, 3$.

Thus the n^{th} root of a complex number requires n different calculations, one for each root:

$v = \sqrt[n]{r} \cdot \text{cis}\left(\frac{\theta + 2\pi k}{n}\right)$ for $\{k \in I \mid 0 \leq k \leq n - 1\}$

To apply this formula, find the cube root of the number 8 . Most students know that $2^3 = 8$ and so know that 2 is the cube root of 8 . However, they don't realize that there are two other cube roots that they are missing. Remember to write out $k = 0, 1, 2$ and use the unit circle whenever possible to help you to find all three cube roots.

$$8 = 8 \text{ cis } 0 = (s \cdot \text{cis}\, \beta)^3$$

$$z_1 = 2 \cdot \text{cis}\left(\frac{0 + 2\pi \cdot 0}{3}\right) = 2 \text{ cis } 0 = 2(\cos 0 + i \cdot \sin 0) = 2(1 + 0) = 2$$

$$z_2 = 2 \cdot \text{cis}\left(\frac{0 + 2\pi \cdot 1}{3}\right) = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$$

$$= 2\left(\cos\left(\frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + i\sqrt{3}$$

$$z_3 = 2 \cdot \text{cis}\left(\frac{0 + 2\pi \cdot 2}{3}\right) = 2\,\text{cis}\left(\frac{4\pi}{3}\right)$$

$$= 2\left(\cos\left(\frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{4\pi}{3}\right)\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - i\sqrt{3}$$

The cube roots of 8 are $2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$.

To check, that they are the cube roots, cube them all simplify.

$$z_1^3 = 2^3 = 8$$

$$z_2^3 = (-1 + i\sqrt{3})^3$$

$$= (-1 + i\sqrt{3}) \cdot (-1 + i\sqrt{3}) \cdot (-1 + i\sqrt{3})$$

$$= (1 - 2i\sqrt{3} - 3) \cdot (-1 + i\sqrt{3})$$

$$= (-2 - 2i\sqrt{3}) \cdot (-1 + i\sqrt{3})$$

$$= 2 - 2i\sqrt{3} + 2i\sqrt{3} + 6$$

$$= 8$$

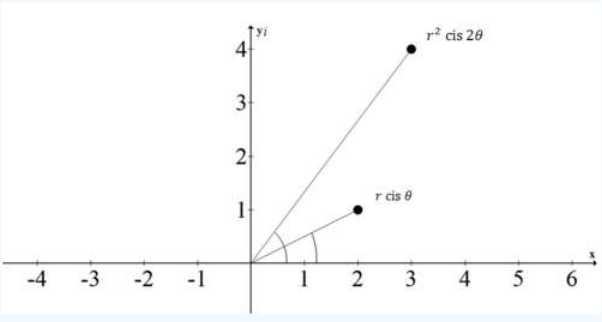
Note how many steps and opportunities there are for making a mistake when multiplying multiple terms in rectangular form. When you check z_3, use trigonometric polar form.

$$z_3^3 = 2^3 \text{ cis}\left(3 \cdot \frac{4\pi}{3}\right)$$
$$= 8(\cos 4\pi + i \cdot \sin 4\pi)$$
$$= 8(1 + 0)$$
$$= 8$$

Examples

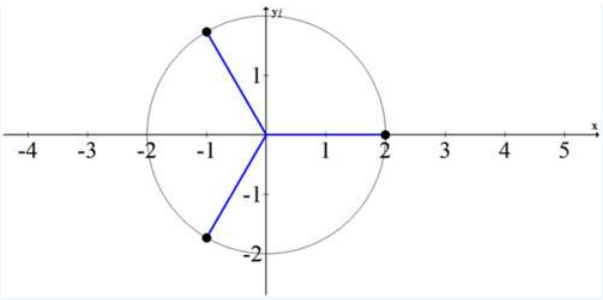
Example 1

Earlier, you were asked what a geometric interpretation of squaring a complex number is. Squaring a complex number produces a new complex number. The angle gets doubled and the magnitude gets squared, so geometrically you see a rotation.



Example 2

Plot the roots of 8 graphically and discuss any patterns you notice.



The three points are equally spaced around a circle of radius 2. Only one of the points, $2+0i$, is made up of only real numbers. The other two points have both a real and an imaginary component which is why they are off of the x axis.

As you become more comfortable with roots, you can just determine the number of points that need to be evenly spaced around a certain radius circle and find the first point. The rest is just logic.

Example 3

What are the fourth roots of 16 cis $48°$?

There will be 4 points, each $90°$ apart with the first point at 2 cis $(12°)$.

$2\operatorname{cis}(12°), 2\operatorname{cis}(102°), 2\operatorname{cis}(192°), 2\operatorname{cis}(282°)$

Example 4

Solve for z by finding the nth root of the complex number.

$z^3 = 64 - 64\sqrt{3}i$

First write the complex number in cis form. Remember to identify $k = 0, 1, 2$. This means the roots will appear every $\frac{360°}{3} = 120°$.

$z^3 = 64 - 64\sqrt{3}i = 128 \cdot \operatorname{cis} 300°$

$z_1 = 128^{\frac{1}{3}} \cdot \operatorname{cis}\left(\frac{300}{3}°\right) = 128^{\frac{1}{3}} \cdot \operatorname{cis}(100°)$

$z_2 = 128^{\frac{1}{3}} \cdot \operatorname{cis}(220°)$

$z_3 = 128^{\frac{1}{3}} \cdot \operatorname{cis}(340°)$

Example 5

Use De Moivre's Theorem to evaluate the following power.

$(\sqrt{2} - \sqrt{2}i)^6$

First write the number in trigonometric polar form, then apply De Moivre's Theorem and simplify.

$$\begin{aligned}(\sqrt{2} - \sqrt{2}i)^6 &= (2\operatorname{cis} 315°)^6 \\ &= 2^6 \cdot \operatorname{cis}(6 \cdot 315°) \\ &= 64 \cdot \operatorname{cis}(1890°) \\ &= 64 \cdot \operatorname{cis}(1890°) \\ &= 64 \cdot \operatorname{cis}(90°) \\ &= 64(\cos 90° + i \cdot \sin 90°) \\ &= 64(0 + i) \\ &= 64i\end{aligned}$$

Review

Use De Moivre's Theorem to evaluate each expression. Write your answers in rectangular form.

1. $(1+i)^5$
2. $(1-\sqrt{3}i)^3$
3. $(1+2i)^6$
4. $(\sqrt{3}-i)^5$
5. $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^4$
6. Find the cube roots of $3+4i$.
7. Find the 5^{th} roots of $32i$.
8. Find the 5^{th} roots of $1+\sqrt{5}i$.
9. Find the 6^{th} roots of -64 and plot them on the complex plane.
10. Use your answers to #9 to help you solve $x^6+64=0$.

For each equation: a) state the number of roots, b) calculate the roots, and c) represent the roots graphically.

11. $x^3=1$
12. $x^8=1$
13. $x^{12}=1$
14. $x^4=16$
15. $x^3=27$

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12: Discrete Math

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14: Concepts of Calculus

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15: Concepts of Statistics

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16: Logic and Set Theory

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Index

Glossary

Sample Word 1 | Sample Definition 1

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By Page

- Precalculus - *CK-12 License*
  - Front Matter - *CK-12 License*
    - TitlePage - *CK-12 License*
    - InfoPage - *CK-12 License*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Functions and Graphs - *CK-12 License*
    - 1.1: 1.1 Function Families - *CK-12 License*
    - 1.2: 1.2 Graphical Transformations - *CK-12 License*
    - 1.3: 1.3 Point Notation and Function Notation - *CK-12 License*
    - 1.4: 1.4 Domain and Range - *CK-12 License*
    - 1.5: 1.5 Maximums and Minimums - *CK-12 License*
    - 1.6: 1.6 Symmetry - *CK-12 License*
    - 1.7: 1.7 Increasing and Decreasing - *CC BY-NC 4.0*
    - 1.8: 1.8 Zeroes and Intercepts of Functions - *CC BY-NC 4.0*
    - 1.9: 1.9 Asymptotes and End Behavior - *CC BY-NC 4.0*
    - 1.10: 1.10 Continuity and Discontinuity - *CC BY-NC 4.0*
    - 1.11: 1.11 Function Composition - *CC BY-NC 4.0*
    - 1.12: 1.12 Inverses of Functions - *CC BY-NC 4.0*
  - 2: Polynomials and Rational Functions - *CK-12 License*
    - 2.1: 2.1 Factoring Review - *CK-12 License*
    - 2.2: Advanced Factoring - *CK-12 License*
    - 2.3: Polynomial Expansion and Pascal's Triangle - *CK-12 License*
    - 2.4: Rational Expressions - *CK-12 License*
    - 2.5: Polynomial Long Division and Synthetic Division - *CK-12 License*
    - 2.6: Solving Rational Equations - *CK-12 License*
    - 2.7 Holes in Rational Functions - *CC BY-NC 4.0*
    - 2.8 Zeroes of Rational Functions - *CC BY-NC 4.0*
    - 2.9 Vertical Asymptotes - *CC BY-NC 4.0*
    - 2.10 Horizontal Asymptotes - *CC BY-NC 4.0*
    - 2.11 Oblique Asymptotes - *CC BY-NC 4.0*
    - 2.12 Sign Test for Rational Function Graphs - *CC BY-NC 4.0*
    - 2.13 Graphs of Rational Functions by Hand - *CC BY-NC 4.0*
  - 3: Logs and Exponents - *CK-12 License*
    - 3.1: Exponential Functions - *CK-12 License*
    - 3.2: Properties of Exponents - *CK-12 License*
    - 3.3: Scientific Notation - *CK-12 License*
    - 3.4: Properties of Logs - *CK-12 License*
    - 3.5: Change of Base - *CK-12 License*
    - 3.6: Exponential Equations - *CK-12 License*
    - 3.7: Logistic Functions - *CC BY-NC 4.0*
  - 4: Basic Triangle Trigonometry - *CK-12 License*
    - 4.1: Angles in Radians and Degrees - *CK-12 License*
    - 4.2: Circular Motion and Dimensional Analysis - *CK-12 License*
    - 4.3: Special Right Triangles - *CK-12 License*
    - 4.4: Right Triangle Trigonometry - *CK-12 License*
    - 4.5: Law of Cosines - *CK-12 License*
    - 4.6: Law of Sines - *CK-12 License*
    - 4.7: Area of a Triangle - *CC BY-NC 4.0*
    - 4.8: Applications of Basic Triangle Trigonometry - *CC BY-NC 4.0*
  - 5: Trigonometric Functions - *CK-12 License*
    - 5.1: The Unit Circle - *CK-12 License*
    - 5.2: The Sinusoidal Function Family - *CK-12 License*
    - 5.3: Amplitude of Sinusoidal Functions - *CK-12 License*
    - 5.4: Vertical Shift of Sinusoidal Functions - *CK-12 License*

- 5.5: Frequency and Period of Sinusoidal Functions - *CK-12 License*
- 5.6: Phase Shift of Sinusoidal Functions - *CK-12 License*
- 5.7: Graphs of Other Trigonometric Functions - *CC BY-NC 4.0*
- 5.8: Graphs of Inverse Trigonometric Functions - *CC BY-NC 4.0*

6: Analytic Trigonometry - *CK-12 License*
- 6.1: Basic Trigonometric Identities - *CK-12 License*
- 6.2: Pythagorean Identities - *CK-12 License*
- 6.3: Sum and Difference Identities - *CK-12 License*
- 6.4: Double, Half, and Power Reducing Identities - *CK-12 License*
- 6.5: Trigonometric Equations - *CK-12 License*

7: Vectors - *CK-12 License*
- 7.1: Basic Properties of Vectors - *CK-12 License*
- 7.2: Operations with Vectors - *CK-12 License*
- 7.3: Resolution of Vectors into Components - *CK-12 License*
- 7.4: Dot Product and Angle Between Two Vectors - *CK-12 License*
- 7.5: Vector Projection - *CK-12 License*

8: Systems and Matrices - *CK-12 License*
- 8.1 Systems of Two Equations and Two Unknowns - *CK-12 License*
- 8.2 Systems of Three Equations and Three Unknowns - *CK-12 License*
- 8.3 Matrices to Represent Data - *CK-12 License*
- 8.4 Matrix Operations - *CK-12 License*
- 8.5 Row Operations and Row Echelon Forms - *CK-12 License*
- 8.6 Augmented Matrices - *CK-12 License*
- 8.7 Determinant of Matrices - *CC BY-NC 4.0*
- 8.8 Cramer's Rule - *CC BY-NC 4.0*
- 8.9 Inverse Matrices - *CC BY-NC 4.0*
- 8.10 Partial Fractions - *CC BY-NC 4.0*

9: Conics - *CK-12 License*
- 9.1 General Form of a Conic - *CK-12 License*
- 9.2 Parabolas - *CK-12 License*
- 9.3 Circles - *CK-12 License*
- 9.4 Ellipses - *CK-12 License*
- 9.5 Hyperbolas - *CK-12 License*
- 9.6 Degenerate Conics - *CK-12 License*

10: Polar and Parametric Equations - *CK-12 License*
- 10.1 Polar and Rectangular Coordinates - *CK-12 License*
- 10.2 Polar Equations of Conics - *CK-12 License*
- 10.3 Parameters and Parameter Elimination - *CK-12 License*
- 10.4 Parametric Inverses - *CK-12 License*
- 10.5 Applications of Parametric Equations - *CK-12 License*

11: Complex Numbers - *CK-12 License*
- 11.1 Fundamental Theorem of Algebra - *CK-12 License*
- 11.2 Arithmetic with Complex Numbers - *CK-12 License*
- 11.3 Trigonometric Polar Form of Complex Numbers - *CK-12 License*
- 11.4 De Moivre's Theorem and nth Roots - *CK-12 License*

12: Discrete Math - *Undeclared*
- 12.1: Section 1- - *CK-12 License*
- 12.2: Section 2- - *CK-12 License*
- 12.3: Section 3- - *CK-12 License*
- 12.4: Section 4- - *CK-12 License*
- 12.5: Section 5- - *CK-12 License*
- 12.6: Section 6- - *CK-12 License*

13: Finance - *Undeclared*
- 13.1: Section 1- - *CK-12 License*
- 13.2: Section 2- - *CK-12 License*
- 13.3: Section 3- - *CK-12 License*
- 13.4: Section 4- - *CK-12 License*
- 13.5: Section 5- - *CK-12 License*
- 13.6: Section 6- - *CK-12 License*

14: Concepts of Calculus - *Undeclared*
- 14.1: Section 1- - *CK-12 License*
- 14.2: Section 2- - *CK-12 License*
- 14.3: Section 3- - *CK-12 License*
- 14.4: Section 4- - *CK-12 License*
- 14.5: Section 5- - *CK-12 License*
- 14.6: Section 6- - *CK-12 License*

15: Concepts of Statistics - *Undeclared*
- 15.1: Section 1- - *CK-12 License*
- 15.2: Section 2- - *CK-12 License*
- 15.3: Section 3- - *CK-12 License*
- 15.4: Section 4- - *CK-12 License*
- 15.5: Section 5- - *CK-12 License*
- 15.6: Section 6- - *CK-12 License*

16: Logic and Set Theory - *Undeclared*
- 16.1: Section 1- - *CK-12 License*
- 16.2: Section 2- - *CK-12 License*
- 16.3: Section 3- - *CK-12 License*
- 16.4: Section 4- - *CK-12 License*
- 16.5: Section 5- - *CK-12 License*

- 16.6: Section 6- - *CK-12 License*
- Back Matter - *CK-12 License*
- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*