ANALYSIS

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SECTION OVERVIEW

1.1: Relations and Functions

Topic hierarchy

1.1.1: Relations and Functions

1.1.2: Intervals and Interval Notation

1.1.3: Symmetry

1.1.4: Zeroes and Intercepts

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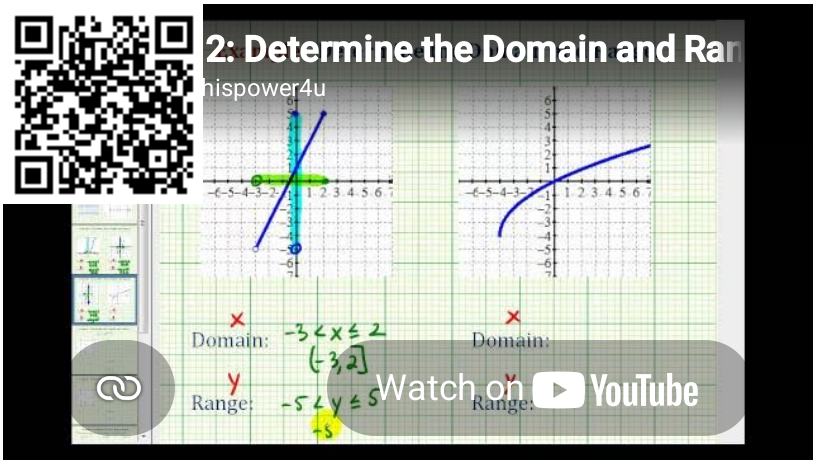
1.1.1: Relations and Functions

Finding the Domain and Range of Functions

You just got a new part-time job at the mall that pays a base rate of $150/week plus $5/sale. Your boss encourages you to make as many sales as possible but she will cap your weekly earnings at $250. What are the domain and range of the **function** represented by this situation?

Domain and Range of a Function

The input and output of a function is also called the domain and range. The **domain** of a function is the set of all input values. The **range** of a function is the set of all output values. Sometimes, a function is a set of points. In this case, the domain is all the x− values and the range is all the y− values. Functions can also be linear and polynomial equations. In these instances, you need to graph the function to see where it is defined. You may notice that some functions are defined for "all real numbers." The symbol, $\mathbb{R}$, is used to denote the set of all real numbers.



Let's determine if {(9, 2), (7, -3), (4, -6), (-10, 4), (-2, -7)} is a function. If so, we will find the domain and range.

First, this is a function because the x− values do not repeat. To find the domain, we need to list all the x− values. The range is all the y− values. Typically, you would list the values in the order they appear. Notice the notation.

$$x \in \{9,7,4,-10,-2\}$$

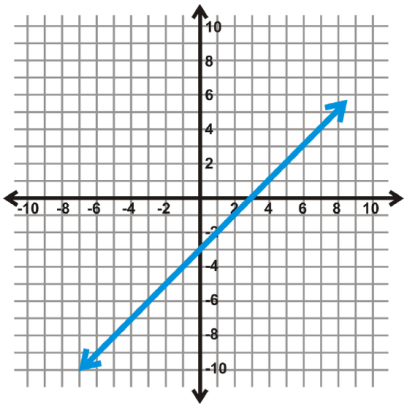
$$y \in \{2,-3,-6,4,-7\}$$

The $\in$ symbol means "an element of/in." The braces, { }, around the x and y−values, indicate that each is a set. In words, you would say, "x is an element in the set 9, 7, 4, -10, and 2." Depending on the text, you may see ":" (colon) interchanged with the "$\in$" symbol and sets without { } around them.

Let's find the domain and range for the following problems.

1. $y=x-3$

Because this is a linear equation we also know that it is a linear function. All lines continue forever in both directions, as indicated by the arrows.



[Figure1]

Notice the line is solid, there are no dashes or breaks. This means that it is continuous. A **continuous** function has a value for every x, or the domain is all real numbers. Can you plug in ANY value for x and get a y−value? Yes. There are a few ways to write this.

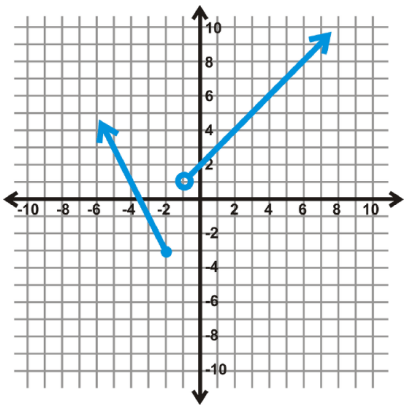
Domain: $x \in \mathbb{R}$, $x \in (-\infty, \infty)$, x is all reals

In words, x is an element in the set of real numbers.

The second option, $(-\infty, \infty)$, is an interval, not a point. The parenthesis indicate that infinity, ∞, and negative infinity, $-\infty$, are not included in the interval, but every number between them is. To include an endpoint in the interval, use [or] brackets. This is called **interval notation.**

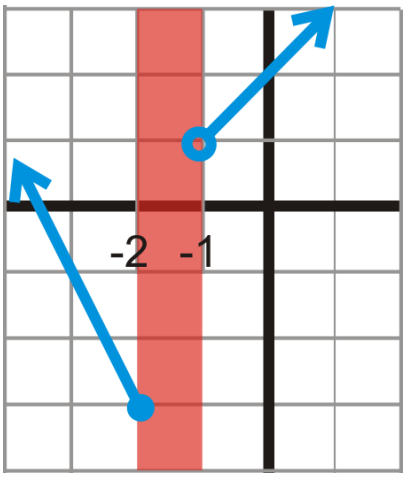
The range of this function is also continuous. Therefore, the range is also the set of all real numbers. We can write the range in the same ways we wrote the domain, but with y instead of x.

Range: $y \in \mathbb{R}$ or $y \in (-\infty, \infty)$



This is a function, even though it might not look like it. This type of function is called a **piecewise function** because it pieces together two or more parts of other functions.

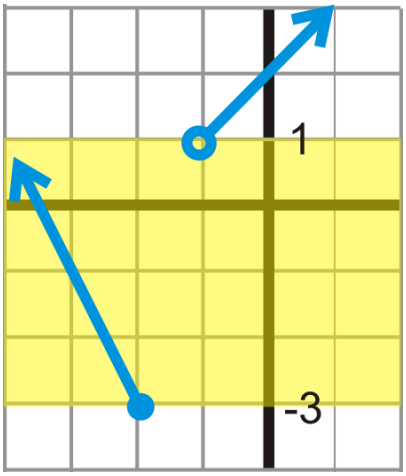
To find the domain, look at the possible x−values. Notice that when x is between -2 and -1 it is not defined, or there are no x−values.



[Figure3]

Mathematically, this would be written: $x \in (-\infty,-2] \cup (-1,\infty)$. The $\cup$ symbol means "**union**." In words, the domain is "all real numbers except those between -2 and -1." Notice that -2 is included in the domain because the dot at -2 is closed. To find the range, we need to look at the possible y−values. Changing our viewpoint to look at the y−axis, at first glance, it looks like the function is not defined from 1 to -3.

However, upon further investigation, the branch on the left does pass through the yellow region, where we though the function was not defined. This means that the function is defined between 1 and -3 and thus for all real numbers. However, below -3, there are no y−values. The range is $y \in [-3,\infty)$.



[Figure4]

Examples

Example 1

Earlier, you were asked to find the domain and range of the function of your sales, where you make a base rate of $150/week plus $5/sale (your weekly earnings are capped at $250/week).

Solution

The function represented by this situation can be written as $y=150+5x$, where x is the number of sales you make. You can't make a negative number of sales, so the least amount of sales you can make is zero. To find the maximum number of sales before you reach the cap, we must plug in $250 for y.

$250=150+5x$

100=5x

x=20

Therefore, the domain of the function is $0 \leq x \leq 20$.

To find the range, plug the two extremes of the domain into the equation. When x equals 0, y equals 150, and when x equals 20, y equals 250.

Therefore the range of the function is $150 \leq y \leq 250$.

Example 2

Find the domain and range of the following function: {(8, 3), (-4, 2), (-6, 1), (5, 7)}.

Solution

Domain: $x \in \{8, -4, -6, 5\}$ Range: $y \in \{3, 2, 1, 7\}$

Example 3

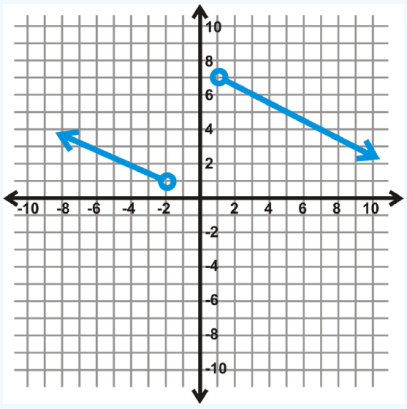
Find the domain and range of the following function: $y = -\frac{1}{2}x + 4$.

Solution

Domain: $x \in \mathbb{R}$ Range: $y \in \mathbb{R}$

Example 4

Find the domain and range of the following function.



[Figure5]

Solution

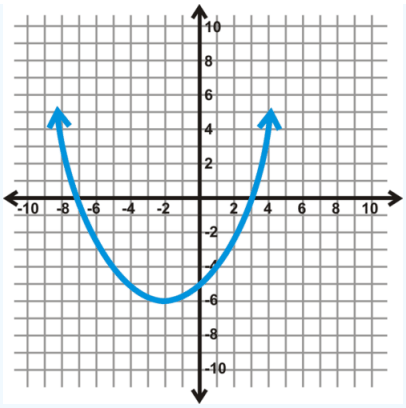
This is a piecewise function. The x−values are not defined from -2 to 1. The range looks like it is not defined from 1 to 7, but the lines continue on, filling in that space as x gets larger, both negatively and positively.

Domain: $x \in (-\infty, -2) \cup (1, \infty)$ Range: $y \in \mathbb{R}$

Example 5

Find the domain and range of the following function.

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[Figure6]

Solution

This is a parabola, the graph of a quadratic equation. Even though it might not look like it, the ends of the graph continue up, infinitely, and x keeps growing. In other words, x is not limited to be between -9 and 5. It is all real numbers. The range, however, seems to start at -6 and is all real numbers above that value.

Domain: $x \in \mathbb{R}$ Range: $y \in [-6, \infty)$

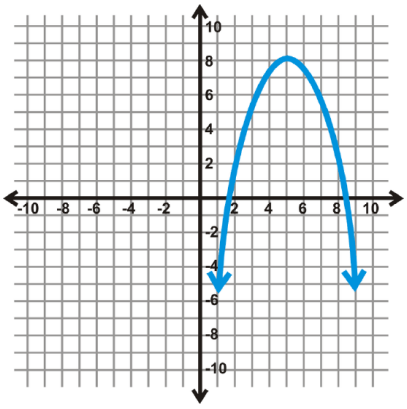
Review

Determine if the following sets of points are functions. If so, state the domain and range.

1. {(5, 6), (-1, 5), (7, -3), (0, 9)}
2. {(9, 8), (-7, 8), (-7, 9), (8, 8)}
3. {(6, 2), (-5, 6), (-5, 2)}
4. {(-1, 2), (-6, 3), (10, 7), (8, 11)}
5. {(5, 7), (3, 7), (5, 8), (8, 1)}
6. {(-3, -4), (-5, -6), (1, 2), (2, -6)}

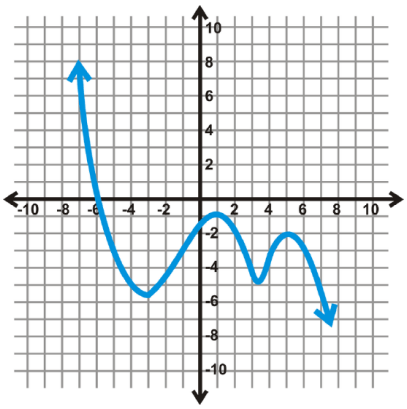
Find the domain and range of the following functions.

7. $y=3x-7$
8. $6x-2y=10$



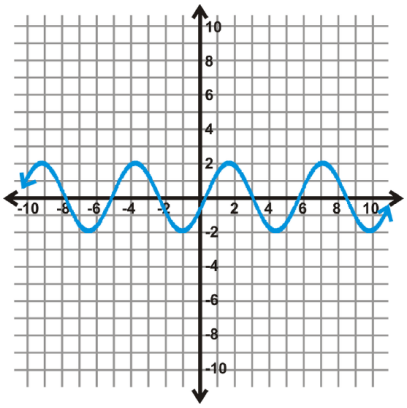
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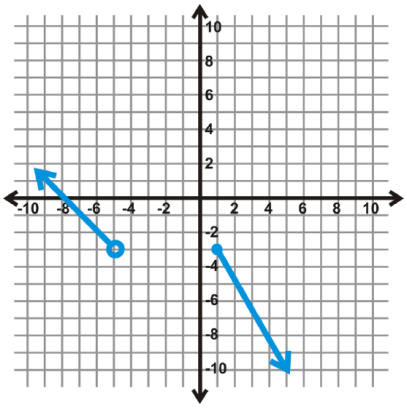
[Figure7]

10.



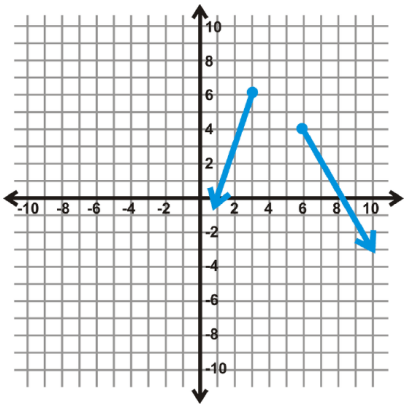
[Figure8]

11.



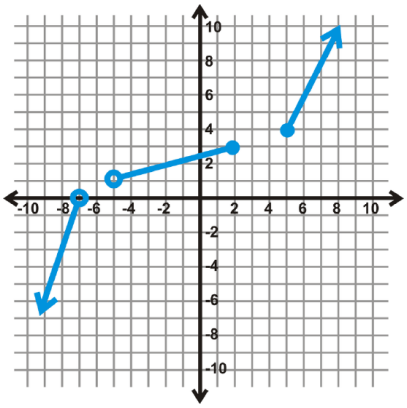
[Figure9]

12.



[Figure10]

13.



14. **Challenge**

15. **Writing** Make a general statement about the domain and range of all linear functions. Use the proper notation.

Vocabulary

Term	Definition
closed interval	A closed interval includes the minimum and maximum values (endpoints) of the interval.
Continuous Function	A continuous function is a function without breaks or gaps. It contains an infinite, uncountable number of values.
Function	A function is a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.
open interval	An open interval does not include the endpoints of the interval.
Piecewise Function	A piecewise function is a function that pieces together two or more parts of other functions to create a new function.
Relation	A relation is any set of ordered pairs (x,y). A relation can have more than one output for a given input.
union	$\cup$ is a symbol that stands for union and is used to connect two groups together. It is associated with the logical term OR.

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[Figure 7]
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Source: CK-12 Foundation

[Figure 8]
Credit: Laura Guerin;CK-12 Foundation
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[Figure 9]
Credit: Laura Guerin;CK-12 Foundation
Source: CK-12 Foundation

[Figure 10]
Credit: CK-12 Foundation, Larame Spence
Source: Desmos Graphing Calculator

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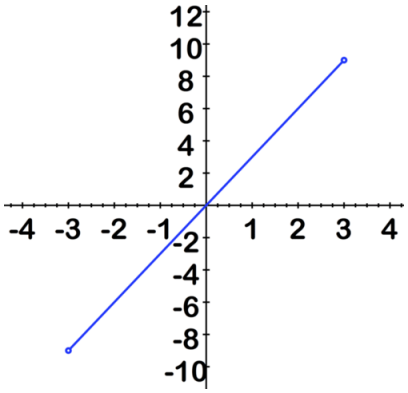
1.1.2: Intervals and Interval Notation

Intervals and Interval Notation

Suppose you and 2 of your friends were out for lunch and decide to buy tacos. Together you have $15 to spend on lunch, and tacos are $1.25 each. It is clear that the total cost could be graphed as a function of the number of tacos purchased, but how would you specify that the graph should not include values greater than $15 or less than $3.75 (one taco each)?

Intervals and Interval Notation

A function is defined as a **real function** if both the **domain** and the **range** are sets of real numbers. Many of the functions you have likely encountered before are real functions, and many of these functions have Domain=$\mathbb{R}$. Consider, for example, the function y=3x. A section of the graph of this function is shown below.



[Figure1]

You may already be familiar with the graphs of lines. In particular, you may already be in the habit of placing arrows at the ends. We do this in order to indicate that the line will continue forever in both the positive and negative directions, both in terms of the domain and the range. The line above, however, only shows the function y=3x on the **interval** [-3, 3]. The square brackets indicate that the graph includes the endpoints of the interval, where x=−3 and x=3. We call this a closed interval. A **closed interval** contains its endpoints. In contrast, an **open interval** does not contain its endpoints. We indicate an open interval with parentheses. For example, (-3, 3) indicates the set of numbers between -3 and 3, not including -3 and 3. You may have noticed that the open interval notation looks like the notation for a point (x,y) in the plane. It is important to read an example or a homework problem carefully to avoid confusing a point with an interval! The difference is generally quite clear from the context.

The table below summarizes the kinds of intervals you may need to consider while studying functions and their domains (and ranges):

Interval notation	Inequality notation	Description
[a,b]	$a \le x \le b$	The value of x is between a and b, including a and b, where a, b are real numbers.
(a,b)	$a < x < b$	The value of x is between a and b, not including a and b.
[a,b)	$a \le x < b$	The value of x is between a and b, including a, but not including b.
(a,b]	$a < x \le b$	The value of x is between a and b, including b, but not including a.
(a,∞)	$x > a$	The value of x is strictly greater than a.
[a,∞)	$x \ge a$	The value of x is greater than or equal to a.
(−∞,a)	x	The value of x is strictly less than a.
(−∞,a]	$x \le a$	The value of x is less than or equal to a.

Using the table above, you can identify the sets following three sets:

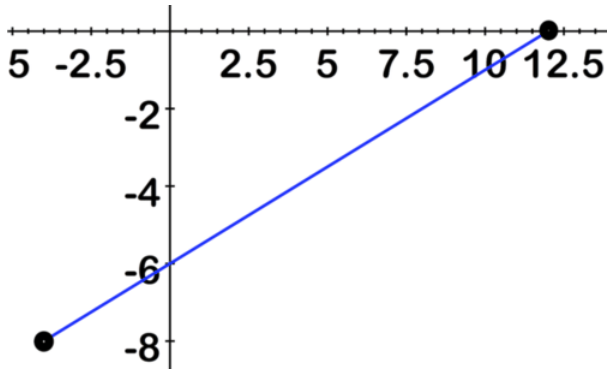
1. (−3,9]
2. [−23,12]
3. (−∞,0)

The set (−3,9] describes the set of numbers between -3 and 9, “not including” the actual value of -3, but “including” 9.

The set [−23,12] describes the set of numbers between -23 and 12, “including” the values -23 and 12.

The set (−∞,0) describes all numbers less than 0, not including 0 itself.

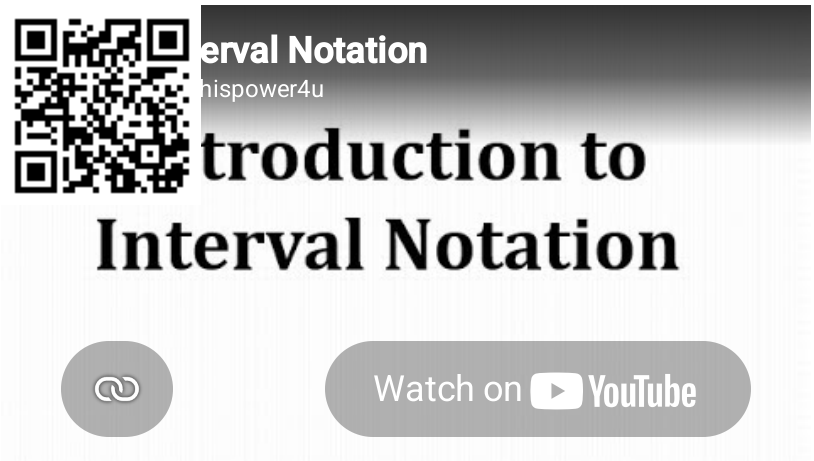
When asked to graph a function on a given interval, be careful to restrict the graph to the interval. Take the following function $f(x)=\frac{1}{2}x-6$ on the interval [-4, 12]. The graph is:



[Figure2]

The part of the function that is graphed is the line between -4 and 12 with -4 and 12 included.

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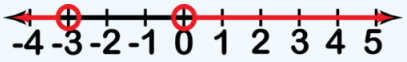
Examples

Example 1

Earlier, you were asked about how to specify the cost of lunch for you and your 2 friends if you have $15 together to spend and tacos are $1.25 each. The least you could spend on lunch is $3.75 which includes one taco each. To describe the only the values values between $3.75 and $15.00, specify the interval of the domain as: [3.75, 15]. Note that these are closed brackets because the values in the brackets are included.

Example 2

Describe the set shown in the image using interval notation:



[Figure3]

Solution

The set is opened with "(", since neg infinity cannot be reached, then closed with ")", since 3 is not included. The set is re-opened with "(" since 0 is not included, and finally closed with ")" since pos infinity cannot be reached either.

Example 3

Describe the specified intervals, use interval notation:

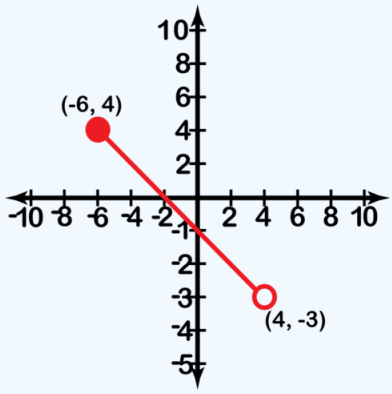
Solution

1. All negative numbers.
   a. Zero is neither positive nor negative, so the ")" is used to specify that zero is "not" included. Since there is no maximum negative number, we specify that infinity is the lower value, and use "(" since it cannot be reached.
2. The numbers between five and twelve, including five, but not twelve.
   a. The "[" is used to open the set, since 5 is included, but ")" is used to close, since 12 is not.
3. Negative numbers down to negative six, zero, and all positive numbers.
   a. The "(" denotes that negative infinity cannot be reached, and "]" on the other end specifies that 9 is included in the set.

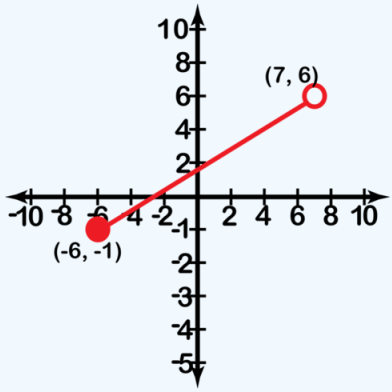
Example 4

Describe the domain in the sets in the graphs using interval notation:

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[Figure4]



[Figure5]

Solution

For graph 1, the domain is the set of x values starting with the included -6 and ending at 4, which is not included: [-6, 4).

For graph 2, for the same reasoning as above, the domain is [-6, 7).

Example 5

Describe the range in the sets in the images in Example 4 using interval notation.

Solution

The range for graph 1 is the set of y values from -3 (not included) to 4 (included): (-3, 4]

The range for graph 2 is: [-1, 6)

Review

For #1-4, write the inequality in interval notation.

1. $-3 \leq x < 1$
2. $0 < x < 2$
3. $x > -3$
4. $x \leq 2$

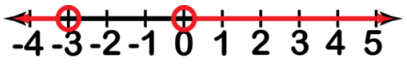
For #5-6, solve and put your answer in interval notation.

5. $-2x+3<1$
6. $7x+4 \leq 2x-6$

For #7-10, write the given set of numbers in interval notation.

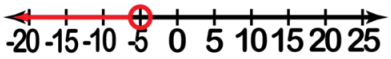
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7.



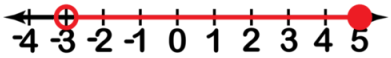
[Figure6]

8.



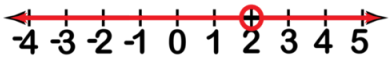
[Figure7]

9.



[Figure8]

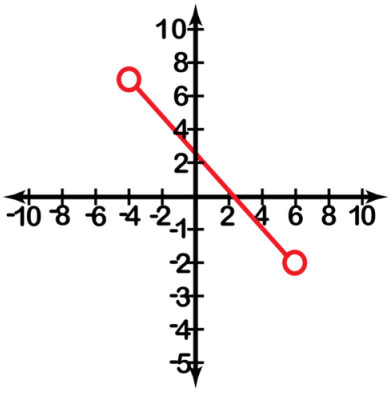
10.



[Figure9]

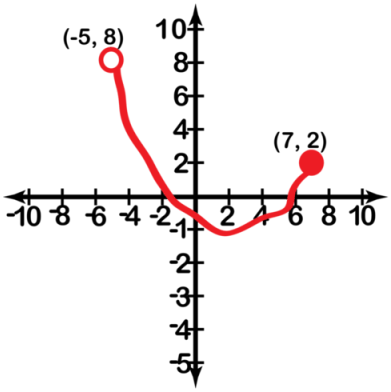
For #11-12, name the domain and range for each relation using interval notation.

11.



[Figure10]

12.



For #13-16, express in interval notation and then sketch them on a number line.

13. {x:−1≤x≤3}
14. {x:−2≤x<1}
15. A is the set of all numbers bigger than 2 but less than or equal to 5.

16. {x:−3<x<∞}

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.3.

Vocabulary

Term	Definition
domain	The domain of a function is the set of x-values for which the function is defined.
Interval Notation	Interval notation is the notation [a,b), where a function is defined between a and b. Use (or) to indicate that the end value is not included and [or] to indicate that the end value is included. Never use [or] with infinity or negative infinity.
Range	The range of a function is the set of y values for which the function is defined.
Real Number	A real number is a number that can be plotted on a number line. Real numbers include all rational and irrational numbers.

Image Attributions

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[Figure 2]
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[Figure 3]
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[Figure 4]
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[Figure 5]
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[Figure 11]
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1.1.3: Symmetry

Symmetry

Some functions, like the sine function, the absolute value function and the squaring function, have reflection symmetry across the line x=0. Other functions like the cubing function and the reciprocal function have rotational symmetry about the origin.

Why is the first group categorized as even functions while the second group is categorized as odd functions?

Even and Odd Functions

Even Functions

Functions symmetrical across the line x=0 (the y axis) are called even. **Even functions** have the property that when a negative value is substituted for x, it produces the same value as when the positive value is substituted for the x. In other words, the equation f(−x)=f(x) holds true for even functions.

To show that the function $f(x)=3x^4-5x^2+1$ is even, show that f(−x)=f(x).

$f(-x)=3(-x)^4-5(-x)^2+1$

$=3x^4-5x^2+1$

$=f(x)$

The property that both positive and negative numbers raised to an even power are always positive is the reason why the term even is used. It does not matter that the coefficients are even or odd, just the exponents.

Odd Functions

Functions that have rotational symmetry about the origin are called odd functions. **Odd functions** have the property that when a negative x value is substituted into the function, it produces a negative version of the function evaluated at a positive value. In other words, the equationf(−x)=−f(x) holds true for odd functions.

This property becomes increasingly important in problems and proofs of Calculus and beyond, but for now it is sufficient to identify functions that are even, odd or neither and show why.

To show that $f(x)=4x^3-x$ is odd, show that f(−x)=−f(x).

$f(-x)=4(-x)^3-x$

$=-4x^3+x$

$=-(4x^3+x)$

$=-f(x)$

Just like even functions are named, odd functions are named because negative signs don't disappear and can always be factored out of odd functions.

Even and odd functions describe different types of symmetry, but both derive their name from the properties of exponents. A negative number raised to an even number will always be positive. A negative number raised to an odd number will always be negative.

Examples

Example 1

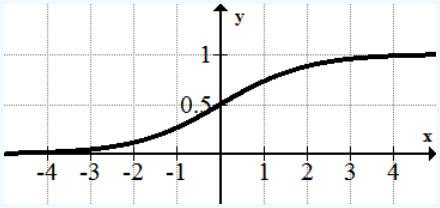
Which of the basic functions are even, which are odd and which are neither?

Solution

Even Functions: The squaring function and the absolute value function.

Odd Functions: The identity function, the cubing function, the reciprocal function, the sine function.

Neither: The square root function, the exponential function and the log function. The logistic function is also neither because it is rotationally symmetric about the point $(0, \frac{1}{2})$ as opposed to the origin.



[Figure1]

Example 2

Suppose h(x) is an **even function** and g(x) is an odd function. f(x)=h(x)+g(x). Is f(x) even or odd?

Solution

If h(x) is even then h(−x)=h(x). If g(x) is odd then g(−x)=−g(x).

Therefore: f(−x)=h(−x)+g(−x)=h(x)−g(x)

This does not match f(x)=h(x)+g(x) nor does it match −f(x)=−h(x)−g(x).

This is a proof that shows the sum of an even function and an odd function will never itself be even or odd.

Example 3

Determine whether the following function is even, odd, or neither.

$f(x)=x(x^2-1)(x^4+1)$

Identify whether the function is even, odd or neither and explain why.

Solution

$f(x)=x(x^2-1)(x^4+1)$

$f(-x)=(-x)((-x)^2-1)((-x)^4+1)$

$=-x(x^2-1)(x^4+1)$

$=-f(x)$

The function is odd because f(−x)=−f(x) holds true.

$f(x)=4x^3-|x|$

$f(-x)=4(-x)^3-x$

$=-4x^3-x$

This does not seem to match either $f(x)=4x^3-|x|$ or $-f(x)=-4x^3+|x|$. Therefore, this function is neither even nor odd.

Note

This function is a difference of an odd function and an even function. This should be a clue that the resulting function is neither even nor odd.

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Review

Determine whether the following functions are even, odd, or neither.

1. $f(x)=-4x^2+1$
2. $g(x)=5x^3-3x$
3. $h(x)=2x^2-x$
4. $j(x)=(x-4)(x-3)^3$
5. $k(x)=x(x2-1)^2$
6. $f(x)=2x^3-5x^2-2x+1$
7. $g(x)=2x^2-4x+2$
8. $h(x)=-5x^4+x^2+2$
9. Suppose h(x) is even and g(x) is odd. Show that $f(x)=h(x)-g(x)$ is neither even nor odd.
10. Suppose h(x) is even and g(x) is odd. Show that $f(x) = \frac{h(x)}{g(x)}$ is odd.
11. Suppose h(x) is even and g(x) is odd. Show that $f(x)=h(x)\cdot g(x)$ is odd.
12. Is the sum of two even functions always an even function? Explain.
13. Is the sum of two odd functions always an odd function? Explain.
14. Why are some functions neither even nor odd?
15. If you know that a function is even or odd, what does that tell you about the symmetry of the function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.6.

Vocabulary

Term	Definition
Even Function	An even function is a function with a graph that is symmetric with respect to the y-axis and has the property that $f(-x)=f(x)$.
reflection symmetry	A figure has reflection symmetry if it can be reflected across a line and look exactly the same as it did before the reflection.
Rotation Symmetry	A figure has rotational symmetry if it can be rotated less than 360° around its center point and look exactly the same as it did before the rotation.

Image Attributions

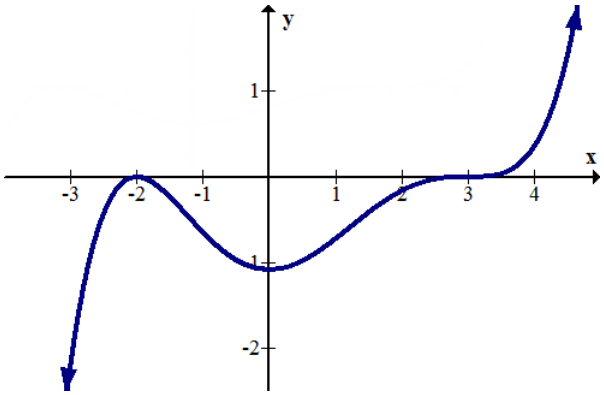
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1.1.4: Zeroes and Intercepts

Zeroes and Intercepts of Functions

An **intercept** in mathematics is where a function crosses the x or y axis. What are the **intercepts** of this function?

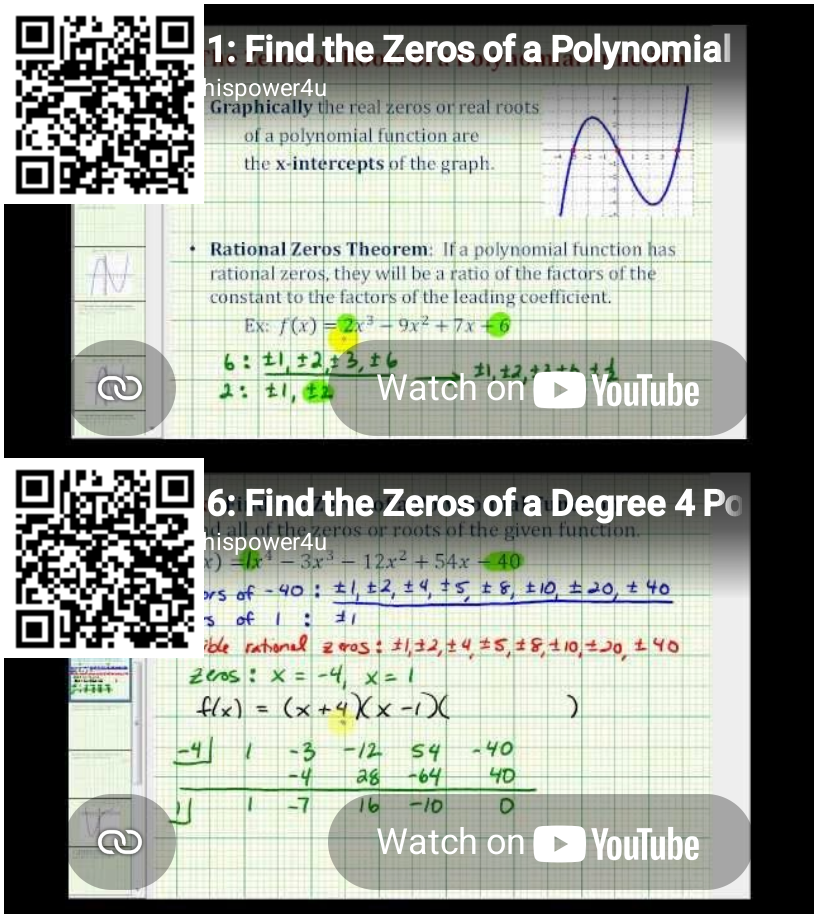


[Figure1]

X and Y Intercepts

The first type of intercept you may have learned is the y-intercept when you learned the slope intercept form of a line: y=mx+b. A **y-intercept** is the unique point where a function crosses the y axis. It can be found algebraically by setting x=0 and solving for y.

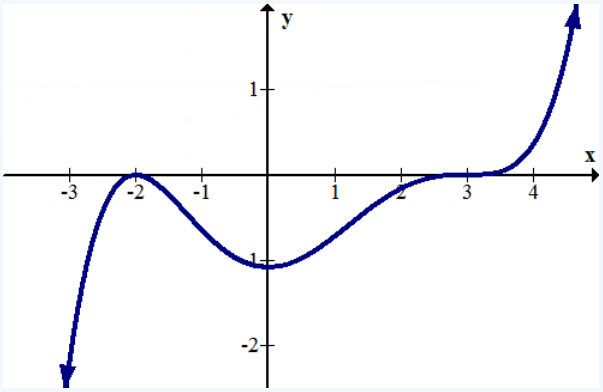
x-intercepts are where functions cross the x axis and where the height of the function is zero. They are also called **roots**, solutions and **zeroes** of a function. They are found algebraically by setting y=0 and solving for x. Watch the videos below for practice:



Examples

Example 1

Earlier, you were asked what the intercepts of the graph below are.



[Figure2]

Solution

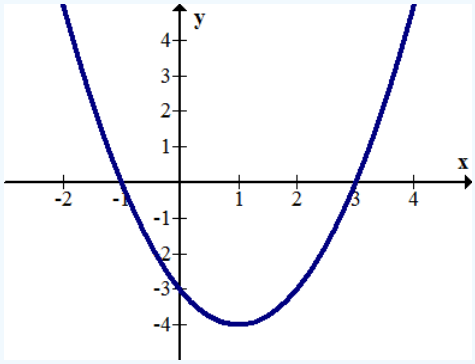
Graphically the function has zeroes at -2 and 3 with a y intercept at about -1.1.

Note: In order for a function to pass the **vertical line test**, it must only have one y-intercept, but it may have multiple x-intercepts.

Example 2

What are the zeroes and y-intercepts of the parabola $y=x^2-2x-3$?

Using a graph:



[Figure3]

Solution

The zeroes are at (-1, 0) and (3, 0). The y-intercept is at (0, -3).

Using Algebra:

Substitute 0 for y to find zeroes.

$0=x^2-2x-3=(x-3)(x+1)$

$y=0, x=3, -1$

Substitute 0 for x to find the y-intercept.

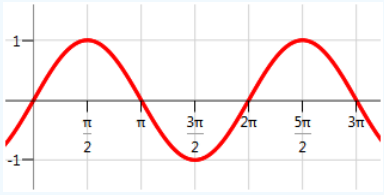
$y=(0)^2-2(0)-3=-3$

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x=0,y=−3

Example 3

Identify the zeroes and y-intercepts for the sine function.



[Figure4]

Solution

The y-intercept is (0, 0). There are four zeroes visible on this portion of the graph. One thing you know about the sine graph is that it is periodic and repeats forever in both directions. In order to capture every x-intercept, you must identify a pattern instead of trying to write out every single one.

The visible x-intercepts are 0,π,2π,3π. The pattern is that there is an x-intercept every multiple of π including negative multiples. In order to describe all of these values you should write:

The x-intercepts are ±nπ where n is an integer {0,±1,±2,…}.

Example 4

Identify the intercepts and zeroes of the function: $f(x) = \frac{1}{100}(x-3)^3(x+2)^2$

Solution

To find the y-intercept, substitute 0 for x:

$y = \frac{1}{100}(0-3)^3(0+2)^2 = \frac{1}{100}(-27)(4) = -\frac{108}{100} = -1.08$

To find the x-intercepts, substitute 0 for y:

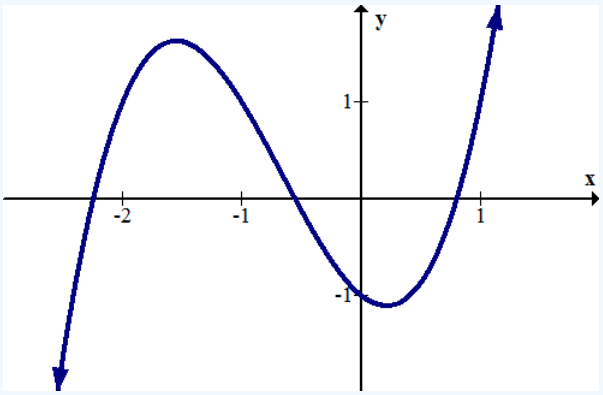
$0 = \frac{1}{100}(x-3)^3(x+2)^2$

x=3,−2

Thus the y-intercept is (0, -1.08) and the x-intercepts are (3, 0) and (-2, 0).

Example 5

Determine the intercepts of the following function graphically.



[Figure5]

Solution

The y-intercept is approximately (0, -1). The x-intercepts are approximately (-2.3, 0), (-0.4, 0) and (0.7, 0). When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

Review

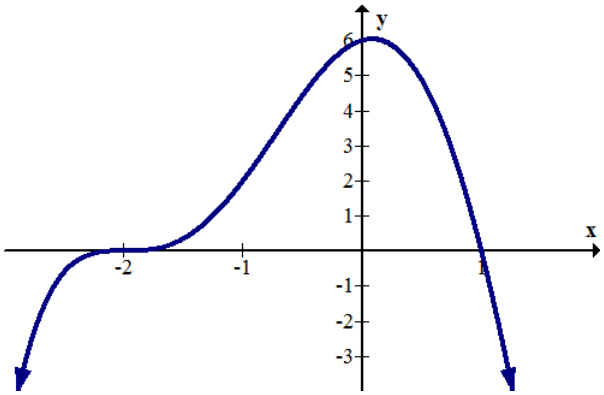
1. Determine the zeroes and y-intercept of the following function using algebra:

$f(x)=(x+1)^3(x-4)$

2. Determine the roots and y-intercept of the following function using algebra or a graph:

$g(x)=x^4-2x^3-7x^2+20x-12$

3. Determine the intercepts of the following function graphically:



[Figure6]

Find the intercepts for each of the following functions.

4. $y=x^2$

5. $y=x^3$

6. $y=\ln(x)$

7. $y=\frac{1}{x}$

8. $y=e^x$

9. $y=\sqrt{x}$

10. Are there any functions without a y-intercept? Explain.

11. Are there any functions without an x-intercept? Explain.

12. Explain why it makes sense that an x-intercept of a function is also called a “zero” of the function.

Determine the intercepts of the following functions using algebra or a graph.

13. $h(x)=x^3-6x^2+3x+10$

14. $j(x)=x^2-6x-7$

15. $k(x)=4x^4-20x^3-3x^2+14x+5$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.8.

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Vocabulary

Term	Definition
Intercept	The intercepts of a curve are the locations where the curve intersects the x and y axes. An x intercept is a point at which the curve intersects the x-axis. A y intercept is a point at which the curve intersects the y-axis.
Intercepts	The intercepts of a curve are the locations where the curve intersects the x and y axes. An x intercept is a point at which the curve intersects the x-axis. A y intercept is a point at which the curve intersects the y-axis.
Roots	The roots of a function are the values of *x* that make *y* equal to zero.
Vertical Line Test	The vertical line test says that if a vertical line drawn anywhere through the graph of a relation intersects the relation in more than one location, then the relation is *not* a function.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.

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4. [Figure 4]
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5. [Figure 5]
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SECTION OVERVIEW

1.2: Average Rate of Change

Topic hierarchy

1.2.1: Minimums and Maximums

1.2.2: Discrete and Continuous Functions

1.2.3: Increasing and Decreasing Functions

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1.2.1: Minimums and Maximums

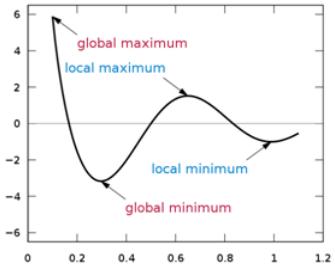
Maximums and Minimums

When riding a roller coaster there is always one point that is the absolute highest off the ground. There are usually many other places that reach fairly high, just not as high as the first. There are also places where the roller coaster dips with one being the absolute lowest the roller coaster can go. How do you identify and distinguish between these different peaks and valleys in a precise way?

Finding Maximums and Minimums

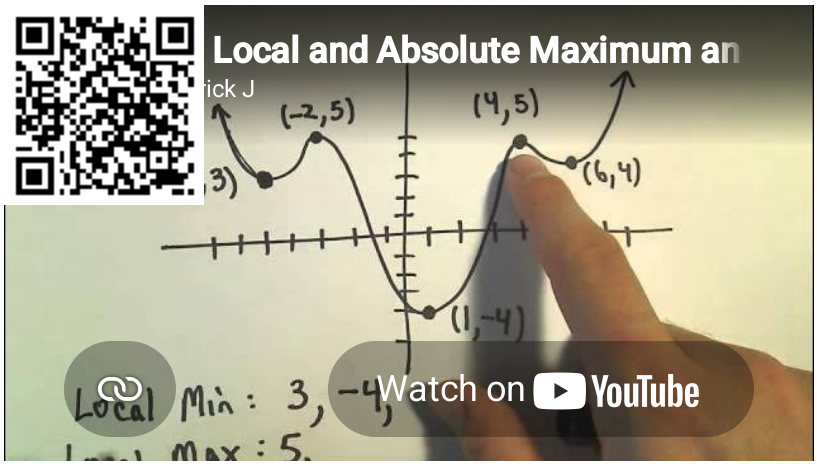
A **global maximum** refers to the point with the largest y value possible on a **function**. A **global minimum** refers to the point with the smallest y value possible. Together these two values are referred to as **global extrema**. There can only be one global maximum and only one global minimum. Global refers to entire space where the function is defined. Global extrema are also called **absolute extrema**.

In addition to global maximums and global minimums, there are also **local extrema** or **relative maximums** and **relative minimums**. The word relative is used because in relation to some neighborhood, these values stand out as being the highest or the lowest.



[Figure1]

Calculus uses advanced analytic tools to compute extreme values, but for the purposes of PreCalculus it is sufficient to be able to identify and categorize extreme values graphically or through the use of technology. For example, the TI-84 has a maximum finder when you select <2nd> then <trace>.



Examples

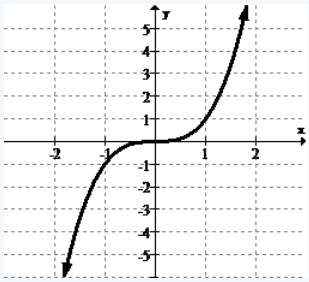
Example 1

Solution

Earlier, you were asked to identify and distinguish between different peaks and valleys on a graph. Maximums and minimums should be intuitive because they simply identify the highest points and the lowest points, or the peaks and the valleys, in a graph. There is a formal distinction about whether a maximum is the highest on some local open interval (does not matter how small), or whether it is simply the highest overall.

Example 2

Identify and categorize all extrema.



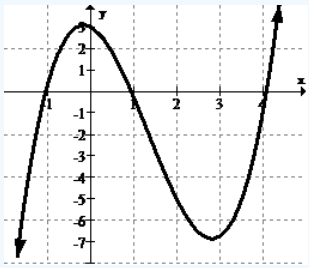
[Figure2]

Solution

There are no global or local maximums or minimums. The function flattens, but does not actually reach a peak or a valley.

Example 3

Identify and categorize all the extrema in the graph below:



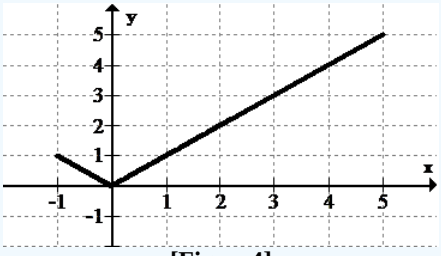
[Figure3]

Solution

Since the function appears from the arrows to increase and decrease beyond the display, there are no global extrema. There is a **local maximum** at approximately (0, 3) and a **local minimum** at approximately (2.8, -7).

Example 4

Identify and categorize all the extrema in the graph below:



[Figure4]

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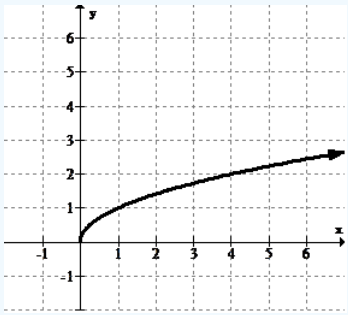
Solution

Since the function seems to abruptly end at the end points and does not go beyond the display, the endpoints are important.

There is a global minimum at (0, 0). There is a local maximum at (-1, 1) and a global maximum at (5, 5).

Example 5

Identify and categorize all the extrema in the graph below:



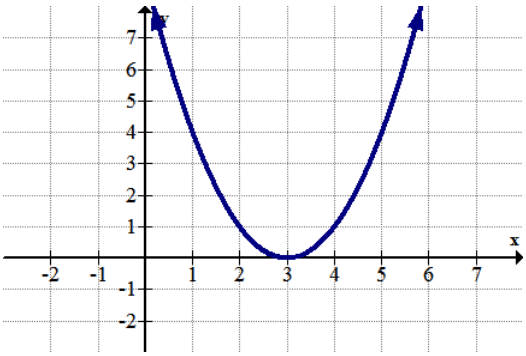
[Figure5]

Solution

Since this function appears to increase to the right as indicated by the arrow there is no global maximum. There are not any other high points either, so there are no local maximums. There is only the end point at (0, 0) which is a global minimum.

Review

Use the graph below for 1-2.

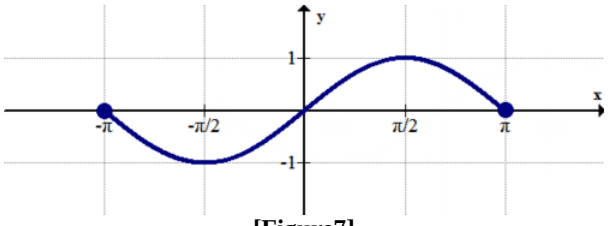


[Figure6]

1. Identify any global extrema.

2. Identify any local extrema.

Use the graph below for 3-4.

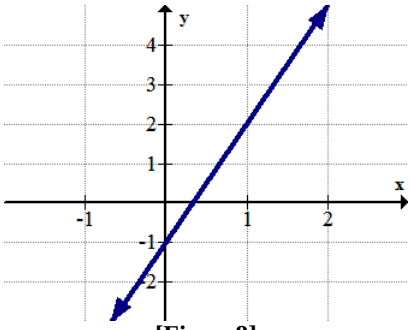


[Figure7]

3. Identify any global extrema.

4. Identify any local extrema.

Use the graph below for 5-6.

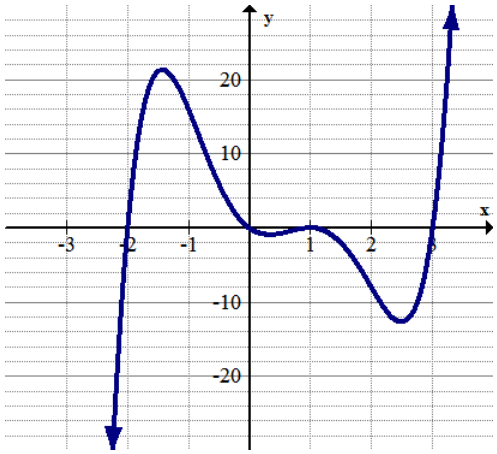


[Figure8]

5. Identify any global extrema.

6. Identify any local extrema.

Use the graph below for 7-8.

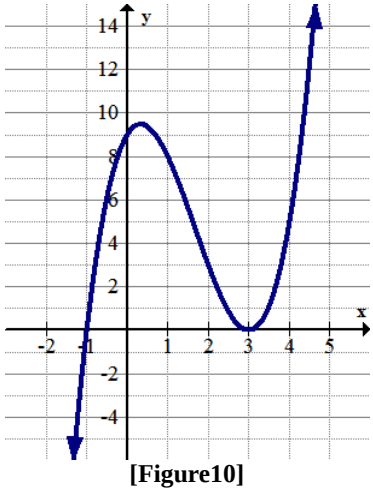


[Figure9]

7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10.



[Figure10]

9. Identify any global extrema.

10. Identify any local extrema.

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11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use your graphing calculator to identify and categorize the extrema of:

$f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24$

15. Use your graphing calculator to identify and categorize the extrema of:

$g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.5.

Vocabulary

Term	Definition
absolute extrema	The absolute extrema of a function are the points with the y values that are the highest or the lowest of the entire function.
End behavior	End behavior is a description of the trend of a function as input values become very large or very small, represented as the 'ends' of a graphed function.
Function	A function is a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.
global extrema	The global extrema of a function are the points with the y values that are the highest or the lowest of the entire function.
local extrema	The local extrema of a function are the points of the function with y values that are the highest or lowest of a local neighborhood of the function.
Local Maximum	A local maximum is the highest point relative to the points around it. A function can have more than one local maximum.
Local Minimum	A local minimum is the lowest point relative to the points around it. A function can have more than one local minimum.
Maximum	The maximum is the highest point of a graph. The maximum will yield the largest value of the range.
Minimum	The minimum is the lowest point of a graph. The minimum will yield the smallest value of the range.
relative extrema	The relative extrema of a function are the points of the function with y values that are the highest or lowest of a local neighborhood of the function.

Image Attributions

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1.2.2: Discrete and Continuous Functions

Continuity and Discontinuity

Continuity is a property of functions that can be drawn without lifting your pencil. Some functions, like the reciprocal functions, have two distinct parts that are unconnected. Functions that are unconnected are discontinuous. What are the three ways functions can be discontinuous and how do they come about?

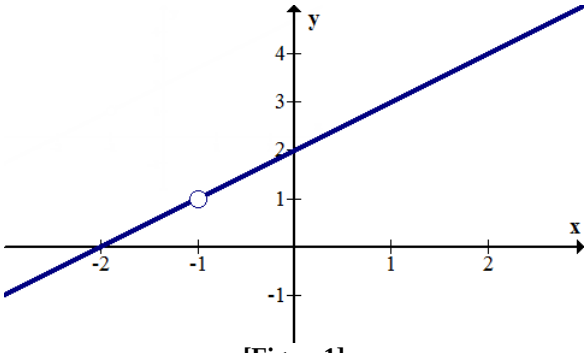
Continuity and Discontinuity of Functions

Functions that can be drawn without lifting up your pencil are called **continuous functions**. You will define continuous in a more mathematically rigorous way after you study limits.

There are three types of **discontinuities**: Removable, Jump and Infinite.

Removable Discontinuities

Removable discontinuities occur when a rational **function** has a factor with an x that exists in both the numerator and the denominator. Removable discontinuities are shown in a graph by a hollow circle that is also known as a hole. Below is the graph for $f(x) = \frac{(x+2)(x+1)}{x+1}$. Notice that it looks just like y=x+2 except for the hole at x=−1. When graphing function, you should cancel the removable factor, graph like usual and then insert a hole in the appropriate spot at the end. There is a hole at x=−1 because when $x = -1, f(x) = \frac{0}{0}$



[Figure1]

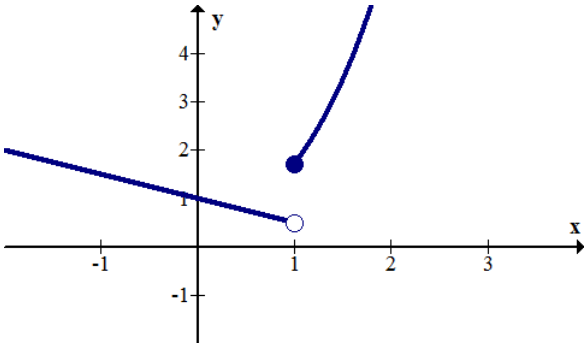
Removable discontinuities can be “filled in” if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make f(x) continuous you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

Jump Discontinuities

Jump discontinuities occur when a function has two ends that don’t meet even if the hole is filled in. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity.

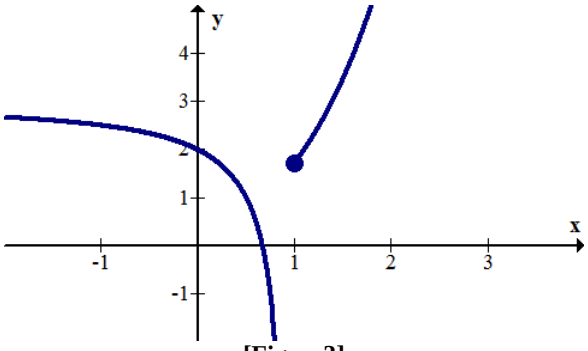
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[Figure2]

Infinite Discontinuities

I**nfinite discontinuities** occur when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below at x=1.



[Figure3]

Examples

Example 1

Solution

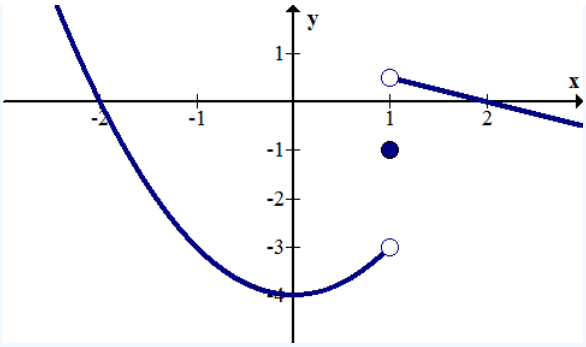
Earlier you were asked how functions can be discontinuous. There are three ways that functions can be discontinuous. When a rational function has a vertical asymptote as a result of the denominator being equal to zero at some point, it will have an **infinite discontinuity** at that point. When the numerator and denominator of a rational function have one or more of the same factors, there will be removable discontinuities corresponding to each of these factors. Finally, when the different parts of a piecewise function don't "match", there will be a jump discontinuity.

Example 2

Identify the discontinuity of the piecewise function graphically.

$$f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$$

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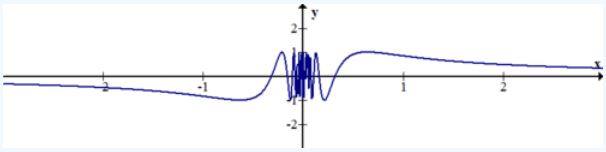
[Figure4]

Solution

There is a jump discontinuity at x=1. The piecewise function describes a function in three parts; a parabola on the left, a single point in the middle and a line on the right.

Example 3

Describe the continuity or discontinuity of the function $f(x) = \sin\left(\frac{1}{x}\right)$



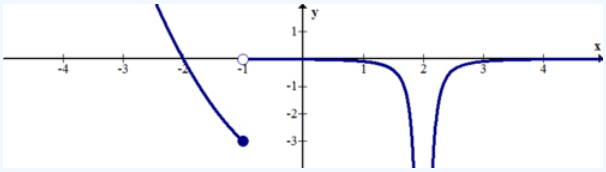
[Figure5]

Solution

The function seems to oscillate infinitely as x approaches zero. One thing that the graph fails to show is that 0 is clearly not in the domain. The graph does not shoot to infinity, nor does it have a simple hole or jump discontinuity. Calculus and Real Analysis are required to state more precisely what is going on.

Example 4

Describe the discontinuities of the function below.



[Figure6]

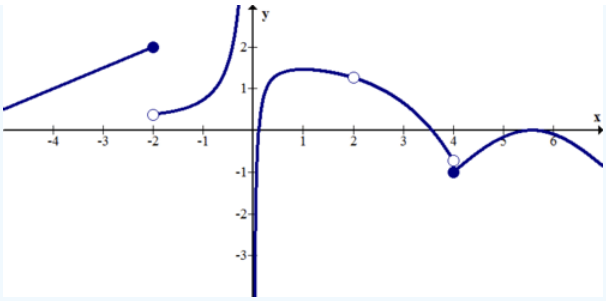
Solution

There is a jump discontinuity at x=−1 and an infinite discontinuity at x=2.

Example 5

Describe the discontinuities of the function below.

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[Figure7]

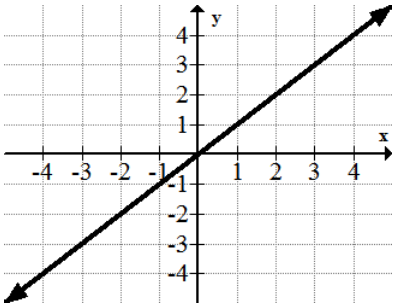
Solution

There are jump discontinuities at $x=-2$ and $x=4$. There is a **removable discontinuity** at $x=2$. There is an infinite discontinuity at $x=0$.

Review

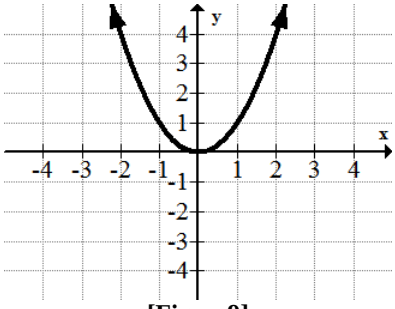
Describe any discontinuities in the functions below:

1. $y=x$



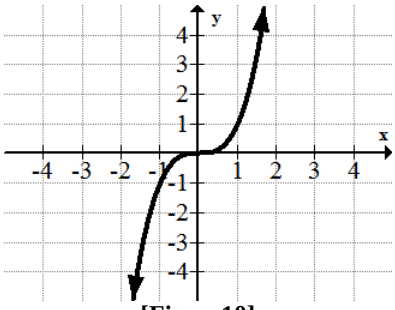
[Figure8]

2. $y=x^2$



[Figure9]

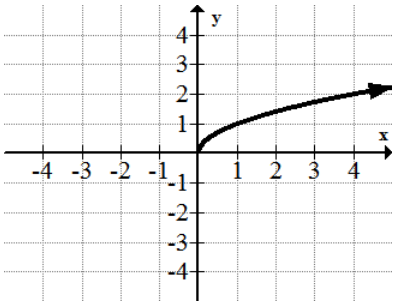
3. $y=x^3$



[Figure10]

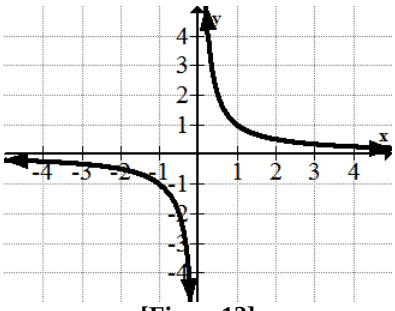
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4. $y = \sqrt{x}$



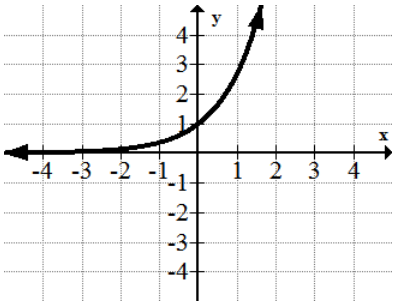
[Figure11]

5. $y = \frac{1}{x}$



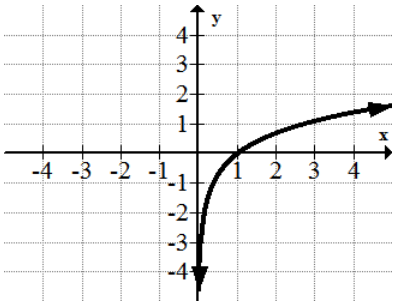
[Figure12]

6. $y=e^x$



[Figure13]

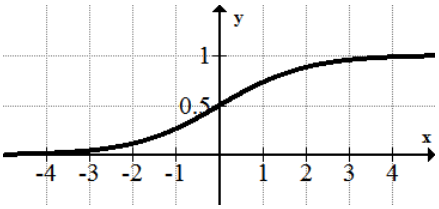
7. y=ln(x)



[Figure14]

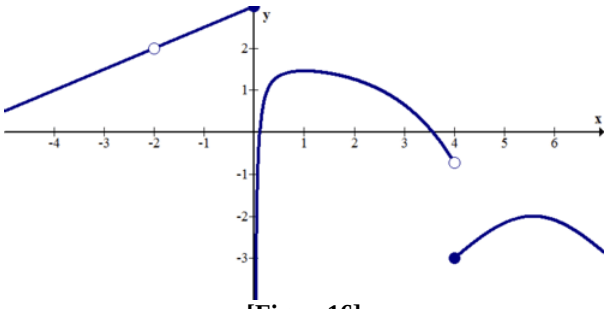
8. $y = \frac{1}{1+e^{-x}}$

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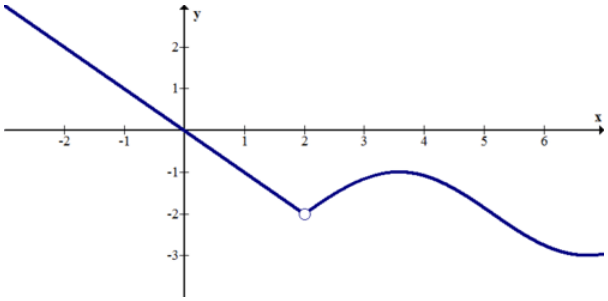
[Figure15]

9.



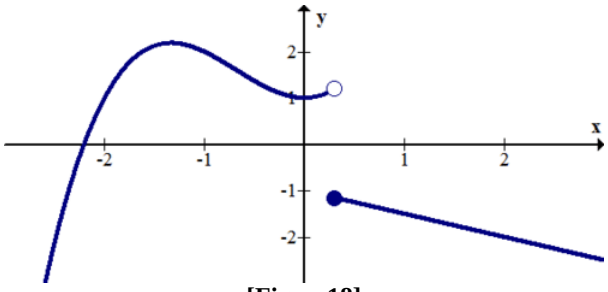
[Figure16]

10.



[Figure17]

11.



[Figure18]

12. f(x) has a jump discontinuity at x=3, a removable discontinuity at x=5, and another jump discontinuity at x=6. Draw a picture of a graph that could be f(x).

13. g(x) has a jump discontinuity at x=−2, an infinite discontinuity at x=1, and another jump discontinuity at x=3. Draw a picture of a graph that could be g(x).

14. h(x) has a removable discontinuity at x=−4, a jump discontinuity at x=1, and another jump discontinuity at x=7. Draw a picture of a graph that could be h(x).

15. j(x) has an infinite discontinuity at x=0, a removable discontinuity at x=1, and a jump discontinuity at x=4. Draw a picture of a graph that could be j(x).

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Review (Answers)

To see the Review answers, open this PDF file and look for section 1.10.

Vocabulary

Term	Definition
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
discontinuities	The points of discontinuity for a function are the input values of the function where the function is discontinuous.
Discrete	A relation is said to be discrete if there are a finite number of data points on its graph. Graphs of discrete relations appear as dots.
Function	A function is a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.
Infinite discontinuities	Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This will happen when a factor in the denominator of the function is zero.
Infinite discontinuity	Infinite discontinuities occur when a function has a vertical asymptote on one or both sides. This will happen when a factor in the denominator of the function is zero.
interval	An interval is a specific and limited part of a function.
Jump discontinuities	Inverse functions are functions that 'undo' each other. Formally f(x) and g(x) are inverse functions if f(g(x))=g(f(x))=x.
Removable discontinuities	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.

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1.2.3: Increasing and Decreasing Functions

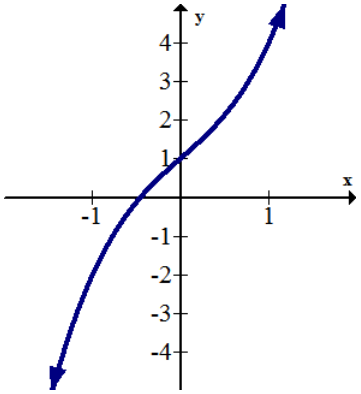
Increasing and Decreasing

It is important to be able to distinguish between when functions are **increasing** and when they are decreasing. In business this could mean the difference between making money and losing money. In physics it could mean the difference between speeding up and slowing down.

How do you decide when a **function** is increasing or decreasing?

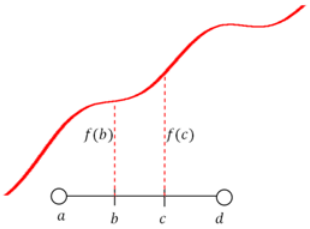
Increasing and Decreasing Functions

Increasing means places on the graph where the slope is positive.



[Figure1]

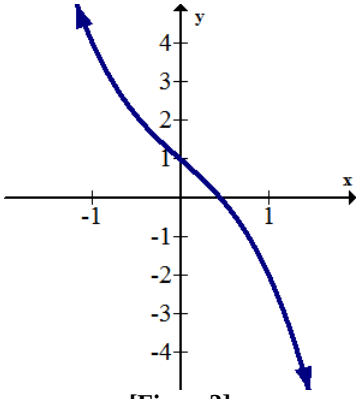
The formal definition of an increasing interval is: an open interval on the x axis of (a,d) where every b,c∈(a,d) with b<c has f(b)≤f(c).



[Figure2]

A interval is said to be **strictly** increasing if $f(b)<f(c)$ is substituted into the definition.

Decreasing means places on the graph where the slope is negative. The formal definition of decreasing and strictly decreasing are identical to the definition of increasing with the inequality sign reversed.



[Figure3]

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A function is called **monotonic** if the function only goes in one direction and never switches between increasing and decreasing.

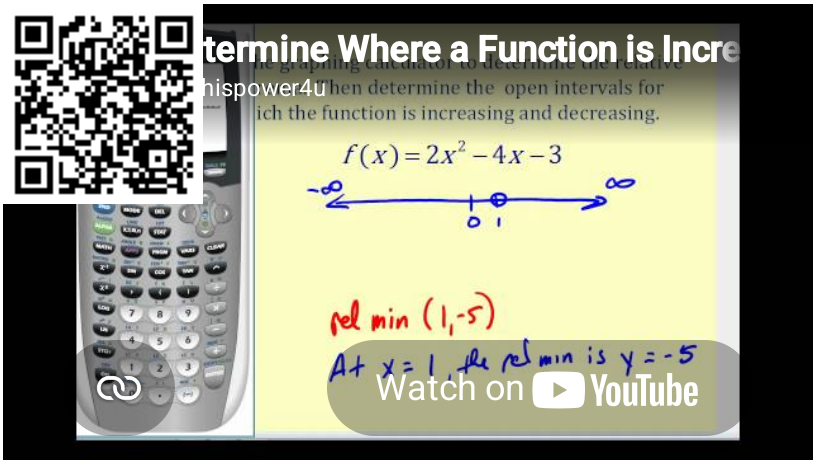
Out of the basic functions, the monotonically increasing functions are:

$f(x) = x, f(x) = x^3, f(x) = \sqrt{x}, f(x) = e^x, f(x) = \ln x, f(x) = \frac{1}{1+e^{-x}}$

The only basic functions that are not monotonically increasing are:

$f(x) = x^2, f(x) = |x|, f(x) = \frac{1}{x}, f(x) = \sin x$

Identifying analytically where functions are increasing and decreasing often requires Calculus. For PreCalculus, it will be sufficient to be able to identify intervals graphically and through your knowledge of what the parent functions look like.



Examples

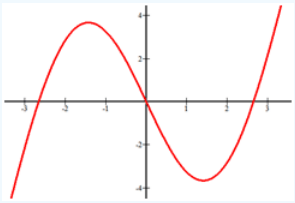
Example 1

Solution

Earlier, you were asked how to determine if a function is increasing or decreasing. Increasing is where the function has a positive slope and decreasing is where the function has a negative slope. A common misconception is to look at the squaring function and see two curves that symmetrically increase away from zero. Instead, you should always read functions from left to right and draw slope lines and decide if they are positive or negative.

Example 2

Estimate where the following function is increasing and decreasing.



[Figure4]

Solution

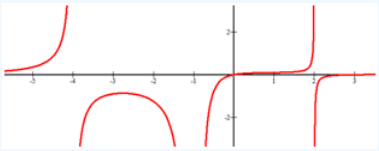
Increasing: $x \in (-\infty, -1.5) \cup (1.5, \infty)$.

Decreasing: $x \in (-1.5, 1.5)$

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Example 2

Estimate where the following function is increasing and decreasing.



[Figure5]

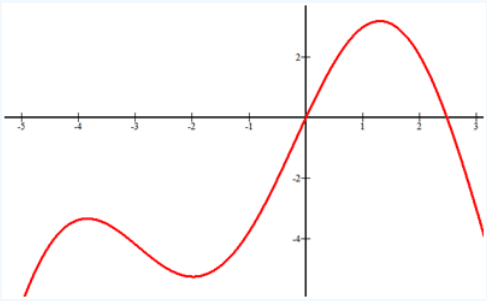
Solution

Increasing $x \in (-\infty,-4) \cup (-4,-2.7) \cup (-1,2) \cup (2,\infty)$.

Decreasing $x \in (-2.7,-1)$

Example 3

Estimate the intervals where the function is increasing and decreasing.



[Figure6]

Solution

Increasing: $x \in (-\infty,-4) \cup (-2,1.5)$

Decreasing: $x \in (-4,-2) \cup (1.5,\infty)$

Notice that open intervals are used because at $x=-4,-2,1.5$ the slope of the function is zero. This is where the slope transitions from being positive to negative. The reason why open parentheses are used is because the function is not actually increasing or decreasing at those specific points.

Example 4

A **continuous** function has a **global maximum** at the point (3, 2), a **global minimum** at (5, -12) and has no **relative extrema** or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

Solution

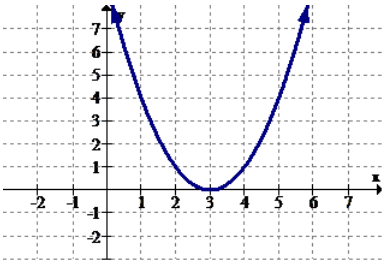
Increasing $x \in (-\infty,3) \cup (5,\infty)$.

Decreasing $x \in (3,5)$

Note: The y coordinates are not used in the intervals. A common mistake is to want to use the y coordinates.

Review

Use the graph below for 1-2.

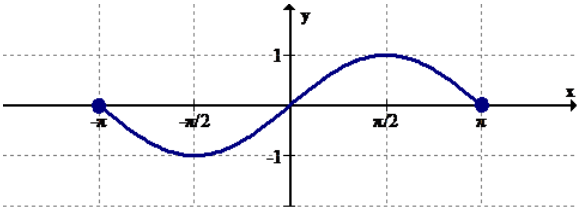


[Figure7]

1. Identify the intervals (if any) where the function is increasing.

2. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 3-4.

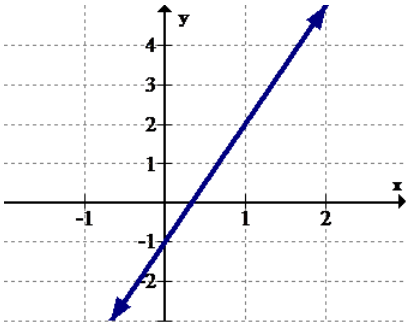


[Figure8]

3. Identify the intervals (if any) where the function is increasing.

4. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 5-6.

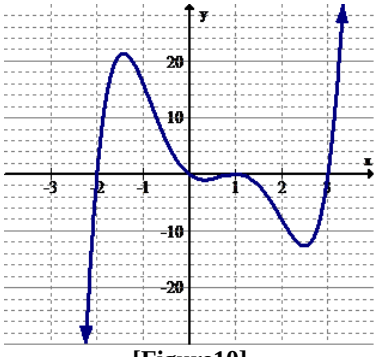


[Figure9]

5. Identify the intervals (if any) where the function is increasing.

6. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 7-8.



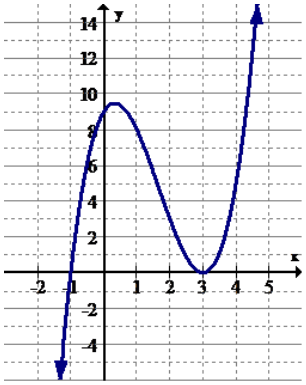
[Figure10]

7. Identify the intervals (if any) where the function is increasing.

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8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10.



[Figure11]

9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A continuous function has a global maximum at the point (1, 4), a global minimum at (3, -6) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

14. A continuous function has a global maximum at the point (1, 1) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

15. A continuous function has a global minimum at the point (5, -15) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.7.

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Vocabulary

Term	Definition
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Function	A function is a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.
Global Maximum	The global maximum of a function is the largest value of the entire function. Symbolically, it is the highest point on the entire graph.
Global Minimum	The global minimum of a function is the smallest value of the entire function. Symbolically, it is the lowest point on the entire graph.
increasing	A function is increasing over an interval if its y values are getting larger over the interval. The graph will go up from left to right over the interval.
Interval Notation	Interval notation is the notation [a,b), where a function is defined between a and b. Use (or) to indicate that the end value is not included and [or] to indicate that the end value is included. Never use [or] with infinity or negative infinity.
monotonic	A function is monotonic if it does not switch between increasing and decreasing at any point.
relative extrema	The relative extrema of a function are the points of the function with y values that are the highest or lowest of a local neighborhood of the function.
strictly	Strictly is an adjective that alters increasing and decreasing to exclude any flatness or periods where y values are staying constant.

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SECTION OVERVIEW

1.3: Limits

Topic hierarchy

1.3.1: Limits and Asymptotes

1.3.2: Infinite and Non-Existent Limits

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1.3.1: Limits and Asymptotes

Asymptotes and End Behavior

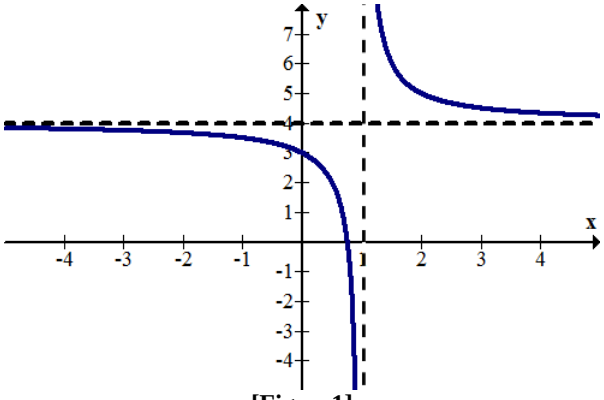
Most functions continue beyond the viewing window in our calculator or computer. People often draw an arrow next to a dotted line to indicate the pattern specifically. How can you recognize these asymptotes?

Asymptotes and End Behavior of Functions

A **vertical asymptote** is a vertical line such as x=1 that indicates where a function is not defined and yet gets infinitely close to.

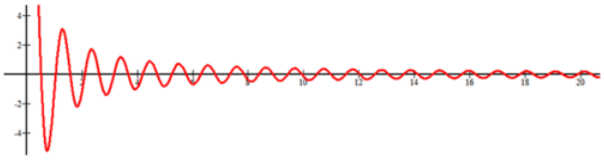
A **horizontal asymptote** is a horizontal line such as y=4 that indicates where a function flattens out as x gets very large or very small. A function may touch or pass through a horizontal asymptote.

The reciprocal function has two asymptotes, one vertical and one horizontal. Most computers and calculators do not draw the asymptotes and so they must be inserted by hand as dotted lines.



[Figure1]

Many students have the misconception that an asymptote is a line that a function gets infinitely close to but does not touch. This is not true. Take the following function:

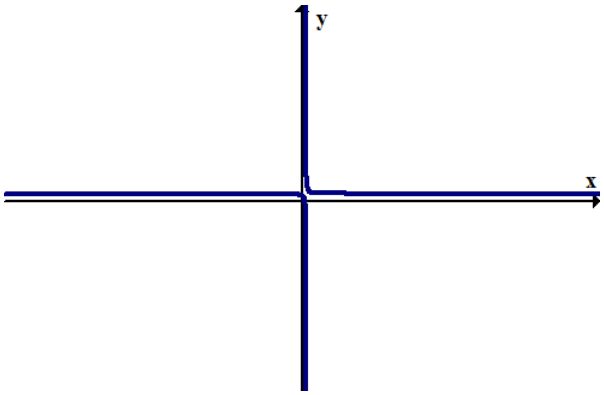


[Figure2]

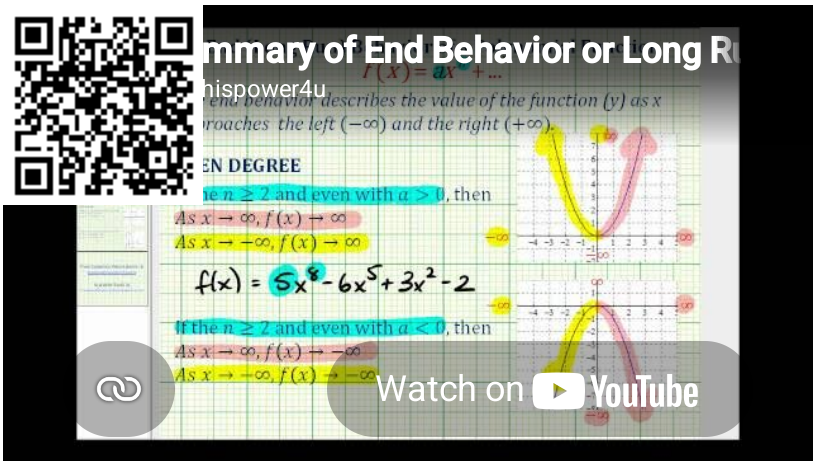
The graph appears to flatten as x grows larger. Thus, the horizontal asymptote is y=0 even though the function clearly passes through this line an infinite number of times.

The reason why asymptotes are important is because when your perspective is zoomed way out, the asymptotes essentially become the graph.

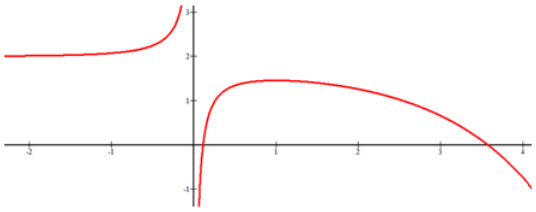
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[Figure3]



To find the asymptotes and end behavior of the function below, examine what happens to x and y as they each increase or decrease.



[Figure4]

The function has a horizontal asymptote y=2 as x approaches negative infinity. There is a vertical asymptote at x=0. The right hand side seems to decrease forever and has no asymptote.

Note that slant asymptotes do exist and are called **oblique asymptotes**.

Examples

Example 1

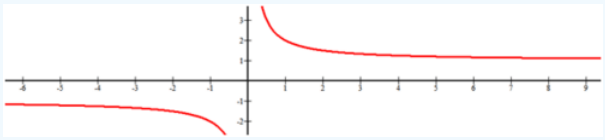
Solution

Earlier, you were asked how to identify asymptotes on a graph. Asymptotes written by hand are usually identified with dotted lines next to the function that indicate how the function will behave outside the viewing window. The equations of these vertical and horizontal dotted lines are of the form x=———————————— and y=————————————. When problems ask you to find the asymptotes of a function, they are asking for the equations of these horizontal and vertical lines.

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Example 2

Identify the horizontal and vertical asymptotes of the following function.



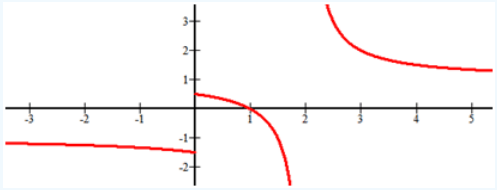
[Figure5]

Solution

There is a vertical asymptote at x=0. As x gets infinitely small, there is a horizontal asymptote at y=−1. As x gets infinitely large, there is another horizontal asymptote at y=1.

Example 3

Identify the horizontal and vertical asymptotes of the following function.



[Figure6]

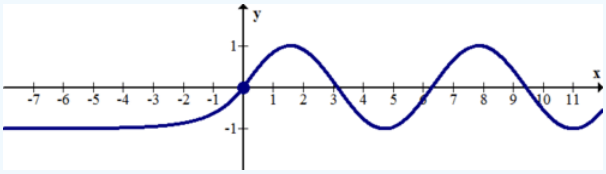
Solution

There is a vertical asymptote at x=2. As x gets infinitely small there is a horizontal asymptote at y=−1. As x gets infinitely large, there is a horizontal asymptote at y=1.

Example 4

Identify the horizontal and vertical asymptotes of the following **piecewise function**:

$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$



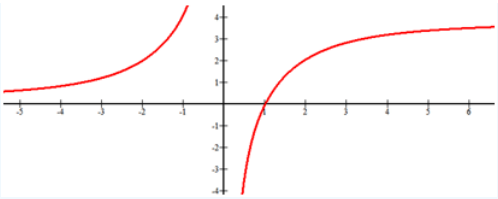
[Figure7]

Solution

There is a horizontal asymptote at y=−1 as x gets infinitely small. This is because e raised to the power of a very small number becomes 0.000000... and basically becomes zero.

Example 5

Identify the asymptotes and end behavior of the following function.



[Figure8]

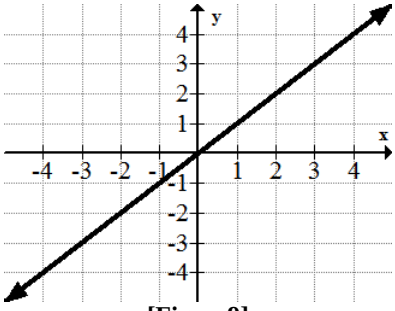
Solution

There is a vertical asymptote at x=0. The end behavior of the right and left side of this function does not match. The horizontal asymptote as x approaches negative infinity is y=0 and the horizontal asymptote as x approaches positive infinity is y=4. At this point you can only estimate these heights because you were not given the function or the tools to find these values analytically.

Review

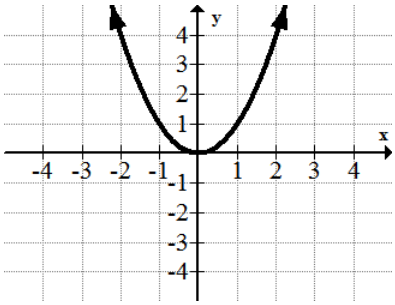
Identify the asymptotes and end behavior of the following functions.

1. y=x



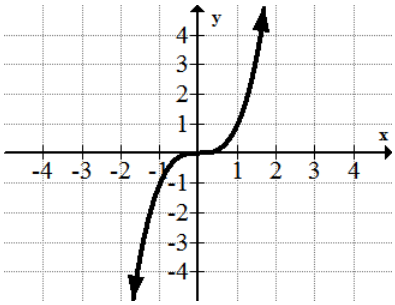
[Figure9]

2. $y=x^2$



[Figure10]

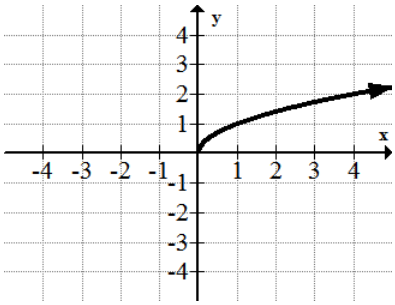
3. $y=x^3$



[Figure11]

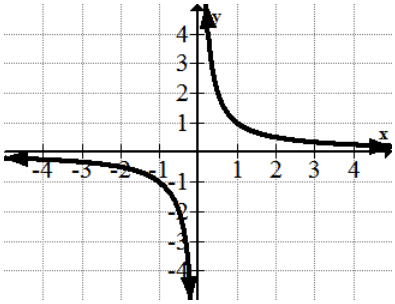
4. y= $\sqrt{x}$

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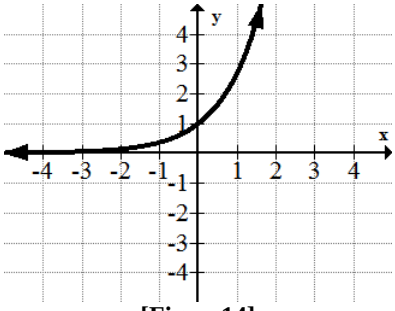
[Figure12]

5. y= $\frac{1}{2}$



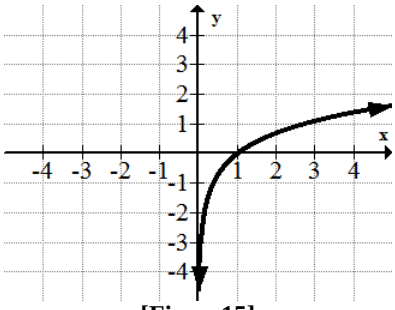
[Figure13]

6. $y=e^x$



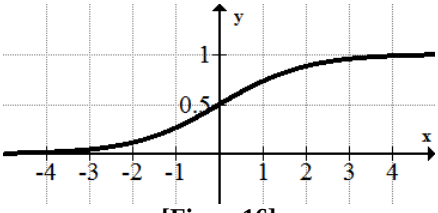
[Figure14]

7. y=ln(x)



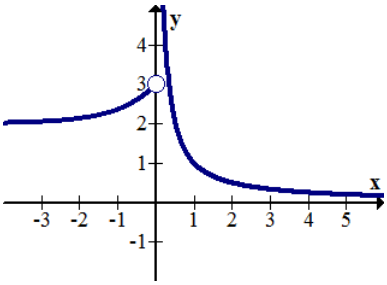
[Figure15]

8. $y = \frac{1}{1+e^{-x}}$



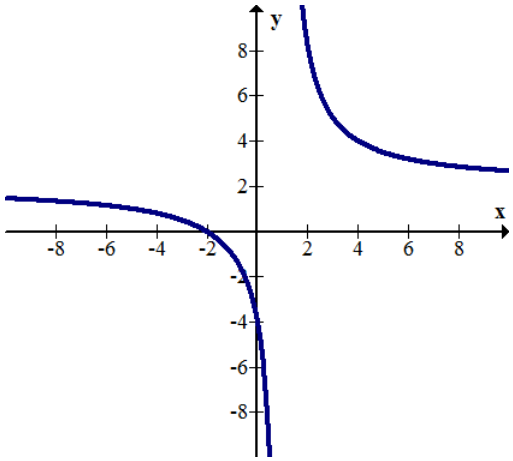
[Figure16]

9.



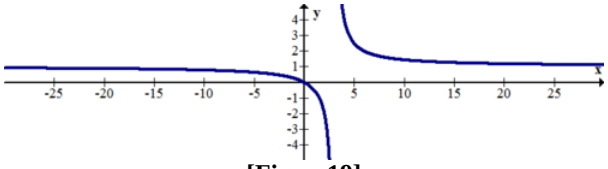
[Figure17]

10.



[Figure18]

11.



[Figure19]

12. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x}$ has a vertical asymptote at x=0.

13. Vertical asymptotes occur at x values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x+3}$ has a vertical asymptote at x=−3.

14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = \frac{1}{x-2}$.

15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = \frac{2}{x+4}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.

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Vocabulary

Term	Definition
Asymptotic	A function is asymptotic to a given line if the given line is an asymptote of the function.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
Oblique Asymptote	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Oblique Asymptotes	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Piecewise Function	A piecewise function is a function that pieces together two or more parts of other functions to create a new function.
Slant Asymptote	A slant asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but will never reach. A slant asymptote exists when the numerator of the function is exactly one degree greater than the denominator. A slant asymptote may be found through long division.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.

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1.3.2: Infinite and Non-Existent Limits

Infinite and Non-Existent Limits

You have heard time and again that it is "against the rules" to divide by the number *0*. Even the most basic calculator will return some form of "ERROR" if you try to divide even the smallest of numbers by *0*.

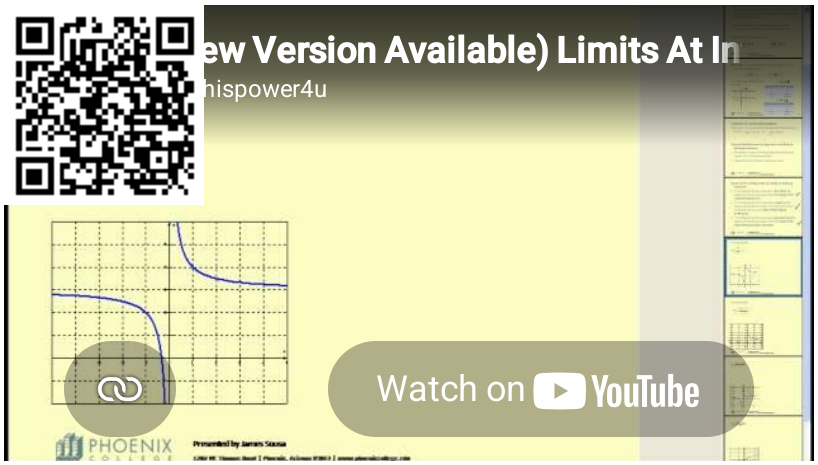
Do you really understand *why* it is "against the rules"? What is really wrong with dividing by nothing?

Infinite and Non-Existent Limits

Functions can exhibit a number of different behaviors as the input value gets very large or very small.

As x approaches ∞, some functions output values get closer and closer to a single number, some approach zero, and some continue to get larger and larger or smaller and smaller without **limit**.

In this lesson, we will explore functions of the last type, functions with **infinite limits**, and the different types of **asymptotes** they may have.



Examples

Example 1

Solution

Earlier, you were given a question about dividing by zero.

Dividing by zero is "against the rules" because there is no definition for the answer you would get.

Consider what happens as you take a given value and divide it by smaller and smaller numbers:

2 / 10 = 1/5

2 / 1 = 2

2 / .1 = 20

2 / .001 = 2,000

2 / .000000001 = 2,000,000,000

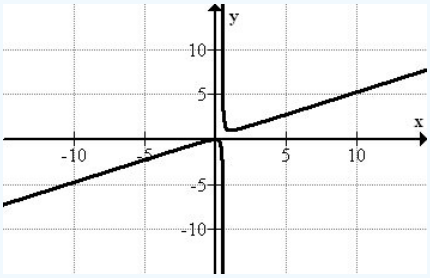
As we divide by progressively smaller numbers, the quotient gets larger and larger. Also, we can see that in each case, the problem could be reversed by multiplying the product by the dividend to get the divisor, for instance: 2 / .1 = 20 ==> 20 * .1 = 2.

Unfortunately, this doesn't work if you actually divide by 0, even if you assume that dividing by zero resulted in **infinity**! No matter how big the number you multiply by zero, even infinity, you will never be able to get back to 2.

$\therefore \frac{x}{0} =$ **undefined**

Example 2

Evaluate the function $h(x) = \frac{x^2}{2x-1}$



Solution

To evaluate this function, consider the behavior of the function as larger and larger values are inserted for x. As x approaches ∞, the function values also approach ∞. Therefore the limit of the function as x approaches ∞ is: $\lim_{x\to\infty} \frac{x^2}{2x-1} = \infty$. Similarly, as x approaches −∞, f(x) approaches −∞. Therefore we have $\lim_{x\to-\infty} \frac{x^2}{2x-1} = -\infty$.

We can also understand this limit if we analyze the equation for h(x). As x gets larger and larger, the value of the expression 2x - 1 gets closer and closer to the value of the expression 2x. That is, for sufficiently large values of x, 2x - 1 ≈ 2x. Therefore the values of h(x) approach $\frac{x^2}{2x} = \frac{x}{2}$.

As x gets larger and larger, so does $\frac{x}{2}$. For large values of x, the function h(x) gets closer and closer to $\frac{x}{2}$. Therefore the limit is infinity.

Example 3

Approximate the function $f(x) = x^2 + 2x - 3$.

Solution

This function has an **infinite limit** as x approaches infinity. However, as x gets larger and larger, $f(x) \approx x^2$, since the x^2 value grows much more quickly than the 2x value, particularly apparent at very large +/- values of x. If this is not immediately apparent, evaluate the function for x = 1,000,000, and you will quickly get the idea!

Therefore we can use the function $y = x^2$ to describe the end behavior of f(x).

Example 4

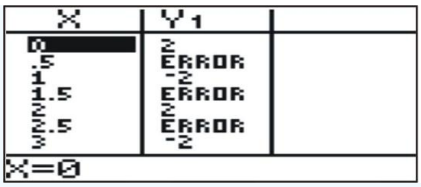
Describe the end behavior of each graph. That is, determine if the function has a limit L, if the limit is infinite, or if the limit does not exist.

a. $y=x^2$
b. $y=2(-1)^x$
c. $y = 1 - \frac{1}{|x|}$

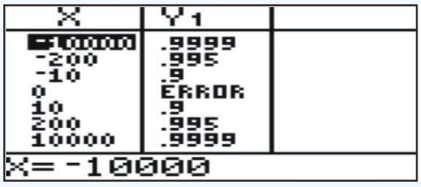
Solution

a. As x approaches +∞, x^2 also approaches +∞. As x approaches −∞, x^2 approaches +∞. Therefore $\lim_{x\to\infty} x^2 = \lim_{x\to-\infty} x^2 = \infty$.
b. This function is difficult to understand without producing a graph. The table shows that the function only takes on two values: 2, and -2, and is undefined at non-integer values of x. As x approaches +∞ or −∞, the function values alternate between 2 and -2. Therefore the limit does not exist.

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c. If you look at the table of this function, which has negative and positive values of x, you can see that as x approaches $+\infty$ or $-\infty$, the function values approach 1.



Therefore $\lim_{x\to\infty}\left(1-\frac{1}{|x|}\right)=\lim_{x\to-\infty}\left(1-\frac{1}{|x|}\right)=1$.

We can also determine this limit analytically. For large values of x, |x| is also large, and so $\frac{1}{|x|}$ is small (since dividing 1 by a large number results in a very small number). Therefore, for large values of $x, 1-\frac{1}{|x|}\approx 1-0=1$. We can make the same argument for x approaching $-\infty$.

Example 5

Evaluate $\lim_{x\to\infty}\frac{x^2+2x}{x-3}$.

Solution

The numerator is of greater degree than the denominator, so the function does not approach a limit. The denominator is x - 3, so the graph cannot include 3.

Example 6

Evaluate $-2x^3-5x^2+8x$.

Solution

$-2x^3-5x^2+8x$ is a 3rd degree equation, so it will turn twice, since it is not a rational function, there are no concerns about numerator or denominator. The function will have no limits, and will grow without bound in both the positive and negative directions. If you use a graphing calculator to graph the function, you will see that $y=x^3$ can be used to approximate it.

Review

For questions 1-3: The limit of f(x) as $x\to t$ cannot exist if which conditions are true? List three conditions.

1. One condition is...
2. A second condition is...
3. A third condition is...
4. Give an example of a limit that does not exist

Questions 5-7: Assuming that f(x) is a rational function:

5. What is $\lim_{x\to\infty} f(x)$ when the degree of the numerator is less than the degree of the denominator?
6. What is $\lim_{x\to\infty} f(x)$ when the degrees of the numerator and the denominator are equal?
7. What is $\lim_{x\to\infty} f(x)$ when the degree of the numerator is greater than the degree of the denominator?
8. In general, if *r* is a positive real number, what is $\lim_{x\to\infty}\frac{1}{x^r}$?
9. In general, if *r* is a positive real number, what is $\lim_{x\to\infty} x^r$?

Questions 10-13: Let a and b be real numbers and let t be a positive integer. Complete each of the following properties of limits.

10. $\lim_{x \to a} x^t =$

11. If f is a polynomial, $\lim_{x \to a} f(x) =$

12. $\lim_{x \to a} k \cdot f(x) =$

13. $\lim_{x \to \infty} t^x =$

Questions 14-16: Evaluate.

14. $\lim_{x \to -5} \frac{(5+x)^2 - 25}{x}$

15. $\lim_{x \to 3} \frac{x^3 - 6x + 2}{x^2 + 2x - 3}$

16. $\lim_{x \to 0} \frac{(3+3y)^{-1} - 3y^{-1}}{x}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.9.

Vocabulary

Term	Definition
∞	The symbol "∞" means "infinity", and is an abstract concept describing a value greater than any countable number.
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
infinite limit	A function has an infinite limit if it's output approaches infinity or negative infinity as very large or very small values are calculated for the input variable (usually "x").
infinity	Infinity is an unbounded quantity that is greater than any countable number. The symbol for infinity is ∞.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.

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SECTION OVERVIEW

1.4: Function Families

Topic hierarchy

1.4.1: Linear and Absolute Value Function Families

1.4.2: Square and Cube Function Families

1.4.3: Graphing Cube Root Functions

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1.4.1: Linear and Absolute Value Function Families

Linear and Absolute Value Function Families

On Tuesday, Mr. Varner's math class filed into the room, and gawked at the message on the whiteboard: "The first student to add together all of the numbers between 1 and 100 wins four free movie tickets to the theater next Friday!" Everyone grabbed a pencil and started adding: 1 + 2 + 3 + 4 + 5... No one was further than about 20 when Brian walked in, late as usual, looked at the white board for about 15 secs, and wrote: "5050" on the bottom. The surprised Mr. Varner handed Brian the tickets and told him to take his seat.

How did he come up with the answer so fast?

Linear and Absolute Value Function Families

In this Concept we will examine several families of functions. A **family** of functions is a set of functions whose equations have a similar form. The **parent** of the family is the equation in the family with the simplest form. For example, $y = x^2$ is a parent to other functions, such as $y = 2x^2 - 5x + 3$.

Linear Function Family

An equation is a member of the **linear function** family if it contains no powers of x greater than

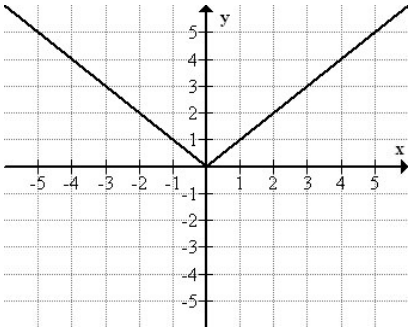
1. For example, $y = 2x$ and $y = 2$ are linear equations, while $y = x^2$ and $y = \frac{1}{x}$ are non-linear.

Linear equations are called linear because their graphs form straight lines. As you may recall from your earlier studies of algebra, we can describe any line by its average rate of change, or slope, and its *y*-intercept. (In fact, it is the constant slope of a line that makes it a line!) These aspects of a line are easiest to identify if the equation of the line is written in slope-intercept form, or $y = mx + b$. The slope of the line is equal to the coefficient *m*, and the *y*-intercept of the line is the point (0, b).

Note that a line can be a member of a family such as the family of "linear functions", and also a member of a sub-family of linear functions with the same slope. The graphs of this subfamily will be a set of parallel lines. One particular subfamily of linear functions is the **constant function** subfamily. The line $x = 5$ is a constant function, as the function values are constant, or unchanging. The constant functions “sub-family” of linear functions is composed of functions whose graphs are horizontal lines.

Absolute Value Function Family

Let’s first consider the parent of the family: $y = |x|$. Because the **absolute value** of a number is that number’s distance from zero, all of the function values of an absolute value function will be non-negative. If $x = 0$, then $y = |0| = 0$. If x is positive, then the function value is equal to x. For example, the graph contains the points (1, 1), (2, 2), (3, 3), etc. However, when *x* is negative, the function value will be the opposite of the number. For example, the graph contains the points (-1, 1), (-2, 2), (-3, 3), etc. As you can see in the graph below, the absolute value function forms a “V” shape.



There are two important things to note about the graph of this kind of function. First, the absolute value graph has a vertex (a highest or lowest point) and a line of symmetry (a line that splits the function into equal and opposite 'halves'). For example, the graph of $y = |x|$ has its vertex at (0, 0) and it is symmetric across the *y*-axis. Second, note that the graph is not curved, but composed

of two straight portions. Every absolute value graph will take this shape, as long as the expression inside the absolute value is linear.

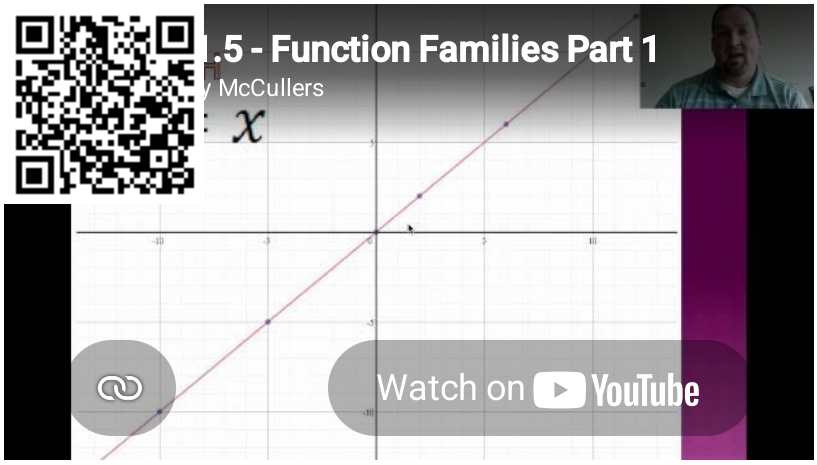
Piece-Wise Defined Functions

Consider again the function $y = |x|$. For positive x values, the graph resembles the identity function $y = x$. For negative x values, the graph resembles the function $y = -x$. We can express this relationship by defining the absolute value function in two pieces:

$$f(x) = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

We can read this notation as: the function values are equal to $-x$ if x is negative. The function values are equal to x if x is 0 or positive.

A piece-wise defined function does not have to represent a function that can already be written as a single equation, such as the absolute value function. For example, one “piece” may be from one function family, while another piece is from a different function family.



Examples

Example 1

Solution

Earlier, you were given a problem about Mr. Varner's math class giveaway.

He wrote the following message on the whiteboard: "The first student to add together all of the numbers between 1 and 100 wins four free movie tickets to the theater next Friday!" Everyone grabbed a pencil and started adding: 1 + 2 + 3 + 4 + 5... No one was further than about 20 when Brian walked in, looked at the white board for about 15 secs, and wrote: "5050" on the bottom. Mr. Varner handed Brian the tickets and told him to take his seat. How did he come up with the answer so fast?

Brian recognized that he didn't need to add each of the numbers individually, only the pairs: 100 + 0 = 100, so does 99 + 1, and 98 + 2, and so on. Since there are 50 pairs of 100, that adds up to 5000. The only number without a pair is 50, so it gets added to the total: 5050.

Function families represent this same sort of time-saver. By recognizing common bits of information and combining them in different ways, we can 'automate' some very complex-seeming processes.

Example 2

Identify the slope and the y-intercept of each line.

a. $y = \frac{2}{3}x - 1$
b. $y = 5$

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c. $x = -2$

d. $y = \frac{2}{3}x + 3$

Solution

a. This line has slope (2/3) and the *y*-intercept is the point (0, -1).

b. This is a horizontal line. The slope is 0, and the *y*-intercept is (0,5).

c. This is a vertical line. The slope is undefined, and the line does not cross the *y*-axis. (Note that this line is not a function!)

d. The slope of this line is 2/3, and the *y*-intercept is the point (0, 3).

Example 3

Graph the following: $y = |2x - 1|$ and $y = |2x^2 - 1|$.

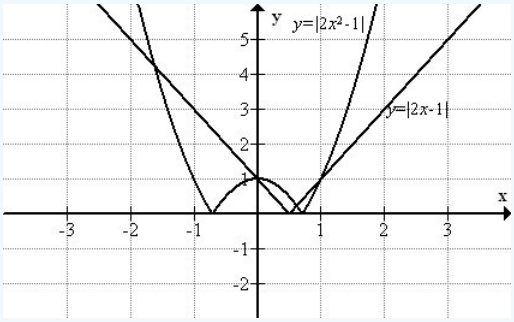
Solution

The graph of $y = |2x - 1|$ makes a "V" shape, much like $y = |x|$.

The function inside the absolute value, $2x + 1$, is linear, so the graph is composed of straight lines.

The graph of $y = |2x^2 - 1|$ is curved, and it does not have a single vertex, but two "cusps."

The function inside the absolute value is NOT linear, therefore the graph contains curves.

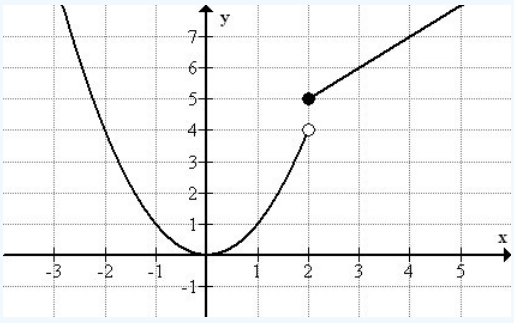


Example 4

Sketch a graph of the function

$$f(x) = \begin{cases} x^2, x < 2 \\ x + 3, x \geq 2 \end{cases}$$

Solution



It is important to note that the pieces of a piece-wise defined function may or may not meet up. For example, in the graph of *f*(*x*) above, the function value is 4 at *x* = -2, but the piece of the graph that is defined by *x* + 3 is headed to the y value of 1. Therefore the two pieces do not meet.

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Example 5

Find the x-intercepts of the function $f(x) = 8|x - 7| - 64$.

Solution

To find the *x*-intercepts, set *f(x)* equal to *0*, and solve for *x*:

$0=8|x-7|-64$

$64=8|x-7|$

$8=|x-7|$

$8=(x-7)$ *or* $8=-(x-7)$

$15=x$ or $-1=x$

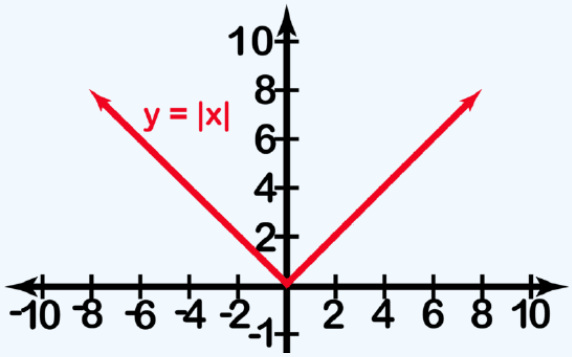
∴ the *x*-intercepts are *15 and -1*

Example 6

What is the graph of y=|x|? How is that graph related to the graph of y=a|x−h|+k? What happens to the graph of y=|x| when the equation changes to y=|x|−5?

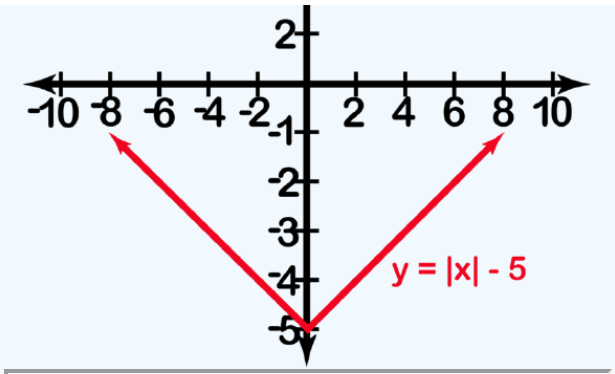
Solution

The graph of y=|x| is shown below. You can either use a graphing tool, or plot points, noting that every positive *x* has a matching *y*, and every negative *x* matches with its positive equivalent as *y*. y=|x| is the simplest example of a graph in the **absolute value function family**, of which y=a|x−h|+k is the parent. Changes to a, h, and k shift the graph of y=|x| in different ways.



The graph of y=|x|−5 is below. It is clear that the *-5* after the absolute value causes the graph to shift down 5 places.

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Review

For questions 1-5, identify the family that each function belongs to.

1. y=|x−7|
2. y=3x−4
3. f(x)=|x^2|
4. |x|−2=y
5. $f(x) = x + \frac{3x}{2}$
6. Graph the following piecewise function by hand: $f(x) = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases}$
7. On your graphing calculator, graph the function f(x)=|x|−2, and answer the following questions: a. What is the shape of the graph? b. Compare the graph to the graph in the problem above. What is the difference between the two graphs? c. What is the slope of the two lines that create the graph?

For each equation that follows, identify the coordinates of the vertex of the graph, without actually graphing.

8. f(x)=|6x|
9. f(x)=|x−6|+8
10. f(x)=|x+7|−8
11. f(x)=|x+5|
12. The graph of p(x)=|x| is shown below. If t(x)=−|x|−3, how will the graph of t(x) be different from the graph of p(x)?
13. Graph the absolute value equation, create your own table to justify values: f(x)=|x−3|
14. Graph the absolute value equation, create your own table to justify values: g(x)=|x+3|

Identify the **parent function** for each set of linear functions. Graph each set of functions using a graphing calculator. Identify similarities and differences of each set.

15. a. f(x)=x−7, b. f(x)=x−2, c. f(x)=x+1, d. f(x)=x+5, e. f(x)=x+10

    Parent Function:

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Similarities:

Differences:

16. a. $f(x) = \frac{2}{11}x$, b. $f(x) = \frac{1}{2}x$, c. $f(x) = \frac{2}{3}x$

    Parent Function:

    Similarities:

    Differences:

17. a. f(x)=x−7, b. f(x)=2x, c. f(x)=4x, d. f(x)=2x+5, e. f(x)=6x−10

    Parent Function:

    Similarities:

    Differences:

    Use the standard form of a **linear equation**: f(x)=ax+b and your investigations above to help you answer the following questions.

18. How does the a value affect the graph?
19. How does the b value affect the graph?
20. How are the domain values similar/different?
21. How are the range values similar/different?
22. Does the a and/or b value affect the domain?
23. Does the a and/or b value affect the range?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.10.

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Vocabulary

Term	Definition
Absolute Value	The absolute value of a number is the distance the number is from zero. Absolute values are never negative.
Absolute value function	An absolute value function graphs a V shape, and is in the form y=|x|.
Function families	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
Function Family	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
linear equation	A linear equation is an equation between two variables that produces a straight line when graphed.
Linear Function	A linear function is a relation between two variables that produces a straight line when graphed.
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
Piecewise defined function	A piecewise defined function is a function that pieces together two or more parts of other functions to create a new function.

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1.4.2: Square and Cube Function Families

Square and Cube Function Families

Often, the most challenging part of completing a math problem is just getting started. Once you have a 'feel' for how a particular problem should be solved, crunching the numbers is generally not very difficult.

An understanding of **function families** can be a big help with this, as it gives you an idea of what a more complex function should look like once it has been graphed, just by identifying the **parent** or most simplified version of the function.

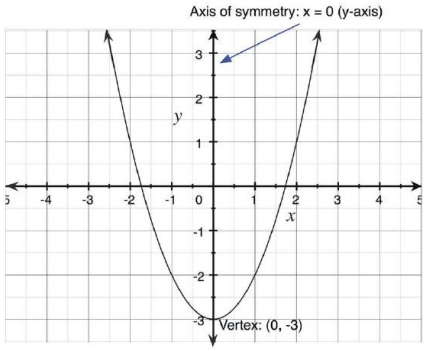
Do you know the **parent functions** for the **Square**, **Cube**, Square Root, and Reciprocal function families?

Square and Cube Function Families

Function Family: Square Functions

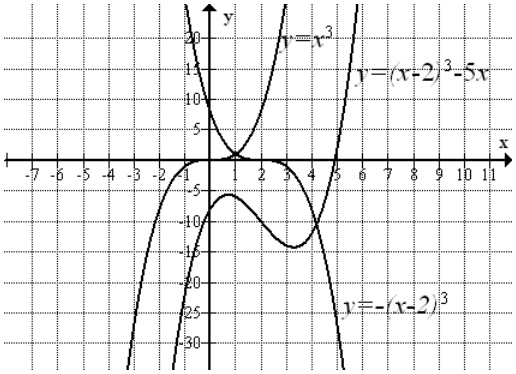
A **square function** is a 2nd degree equation, meaning it has an x^2.

The graph of every square function is a **parabola.** A parabola has a **vertex,** and an **axis of symmetry.** The graph below shows these aspects of the graph of $y = x^2 - 3$.



Function Family: Cubic Functions

A cube function is a third-degree equation: x^3 and which does not contain negative or fractional exponents. In general, the graphs of cube functions have a particular shape, illustrated by the graph shown here:



Cubic functions have a similar shape. However, only some cubic functions will have a relative maximum and minimum. For example, the graph of $y = (x - 2)^3 - 5x$ shown above, has a relative maximum around $x = 0.7$, and a relative minimum around $x = 3.3$. The shape of the cubic graph means that we can predict end behavior: one end will approach ∞, and the other will approach $-\infty$.

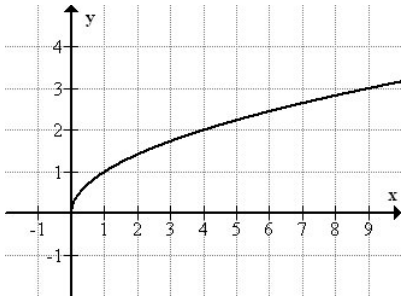
It is important to note here that the **cubic function** grows faster than an associated quadratic function. For example, $y = x^3$ grows faster than $y = x^2$.

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Function Family: Square Root Functions

Consider the parent of the family, $y = \sqrt{x}$. The domain of the function is limited to real numbers ≥ 0, as the square root of a negative number is not a real number. Similarly, the range of the function is limited to real numbers ≥ 0. This may seem confusing if you think of squares having two roots. For example, 9 has two roots: 3, and -3. However, for $y = \sqrt{x}$, we have to define the function value as the **principal root**, which means the positive root.

The function $y = \sqrt{x}$ is shown below:



The same kind of limitations of domain and range will exist for any **square root function**.

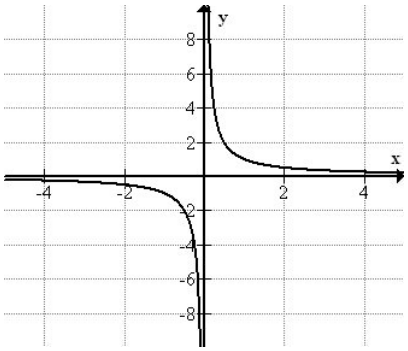
Function Family: Reciprocal Functions

The function $y = \frac{1}{x}$ has a rather surprising graph. First, the domain cannot include 0, as the fraction $\frac{1}{0}$ is undefined. The range also does not include 0, as a fraction can only be zero if the numerator is zero, and the numerator of $y = \frac{1}{x}$ is always 1.

In order to understand what these limitations mean for the graph, we will consider function values near $x = 0$ and $y = 0$. First, consider very small values of x. For example, consider x = 0.001. This yields $y = \frac{1}{x} = \frac{1}{0.001} = 1000$.

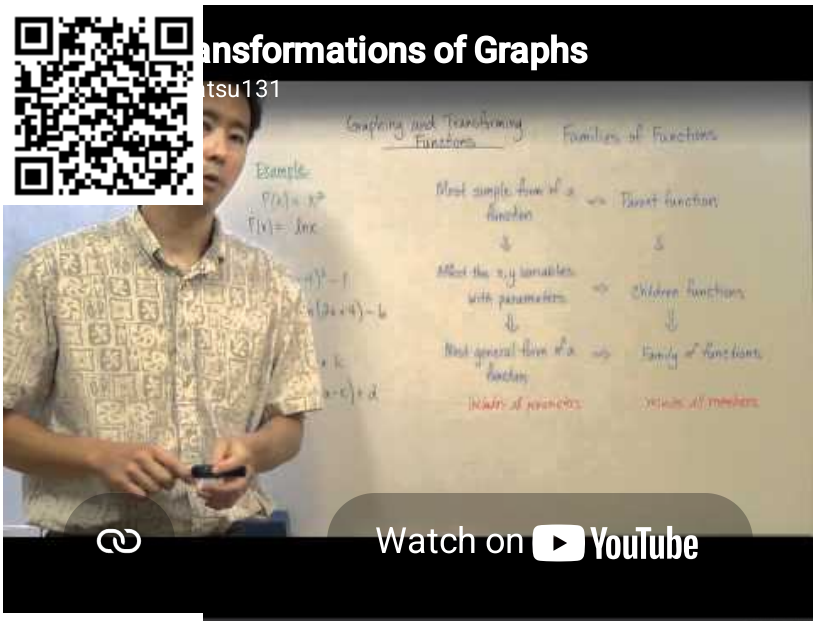
As we get closer and closer to $x = 0$, the function values approach ∞. On the other side of the x-axis the function values will approach −∞. We can see this behavior in the graph as a vertical asymptote: the graph is asymptotic to the y-axis.

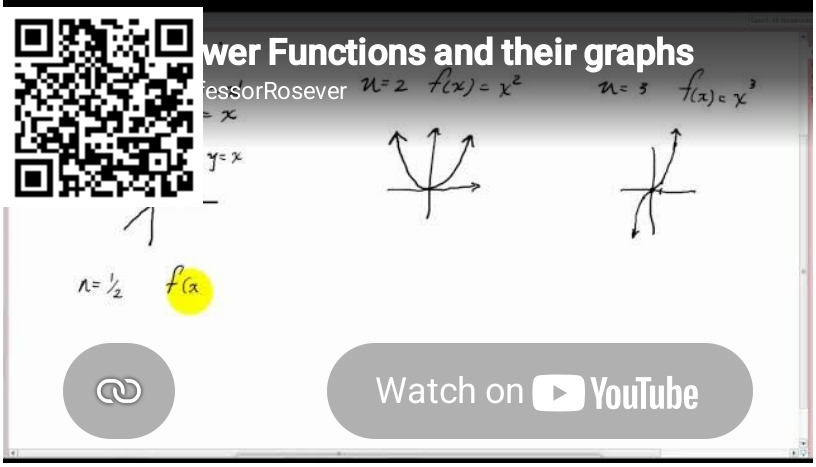
We can also see in the graph that as x approaches +∞ or −∞, the function values approach 0. The exclusion of $y = 0$ from the range means that the function is asymptotic to the x-axis.



[Figure1]

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Examples

Example 1

Solution

Earlier, you were given a question about identifying the parent functions of various function families.

Can you identify the parent functions of the square, cube, square root, and reciprocal functions now?

Square: $y = x^2$

Cube: $y = x^3$

Square Root: $y = \sqrt{x}$

Reciprocal: $y = 1/x$

Learning the function families is one of the fastest way to graph complex equations. Using **parent functions** and **transformations** (which are detailed in another set of lessons), you can graph very complex equations rather easily.

Example 2

Describe the end behavior of each function, and identify the **parent function** for each.

a. $y = x^2 - 1$
b. $y = -x^2 + 1$

Solution

a. The parent of this function is $y = x^2$.

The graph of this function is a parabola that opens up. Therefore $\lim_{x \to \pm\infty} (x^2 - 1) = \infty$.

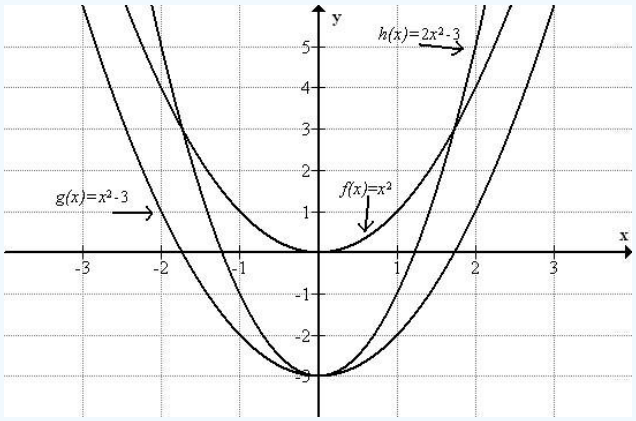
b. The parent of this function is $y = x^2$

The graph of this function is a parabola that opens down. Therefore $\lim_{-x \to \pm\infty} (x^2 - 1) = -\infty$.

All square functions have either a global maximum or minimum. The location of the maximum or minimum is always the vertex of the parabola. Square functions also share behavior in terms of their average rate of change. Consider for example the functions $f(x) = x^2, g(x) = x^2 - 3$ and $g(x) = 2x^2 - 3$. The table below shows the average rate of change (ARC) of each function on several intervals.

Note: The ARC of a function on the interval (a, b) is $\frac{f(b) - f(a)}{b - a}$.

Interval	ARC of $f(x)$	ARC of $g(x)$	ARC of $h(x)$
(-1, 0)			
(0, 1)	1	1	2
(1, 2)	3	3	6
(2, 3)	5	5	10



Notice that the average rate of change of $f(x)$ and $g(x)$ is the same on each interval, and the average rate of change of $h(x)$ is twice that of the other two functions. You may also notice that the average rate of change follows a linear pattern: on each interval the rate increases at a constant rate of 2. While linear functions have a constant rate of change, quadratic functions have an average rate of change that follows a linear pattern.

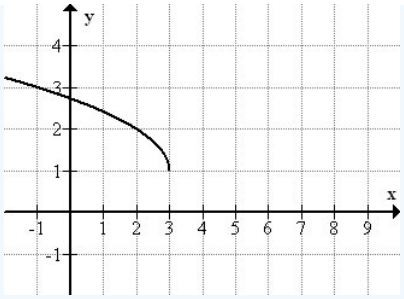
Example 3

Graph the function $y = \sqrt{3 - x} + 1$, identify the parent function, and state the domain and range of the function.

Solution

The parent function is $y = \sqrt{x}$

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From the graph you can see that the function does not take on any x values above 3. (Why not?) Therefore the domain is limited to real numbers ≤ 3. The function's lowest value is 1, so the range is limited to all real numbers ≥ 1 .

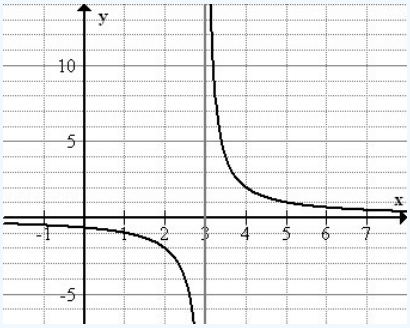
It is important to note that while the graph of a square root function might look as if it has horizontal asymptote, it does not. The function values will grow without bound (though relatively slowly!).

Example 4

Graph the function $f(x) = \frac{2}{x-3}$, identify the parent function, and identify horizontal and vertical asymptotes.

Solution

The parent function is $y = 1/x$



The graph is asymptotic to the *x*-axis ($y = 0$) and to the line $x = 3$.

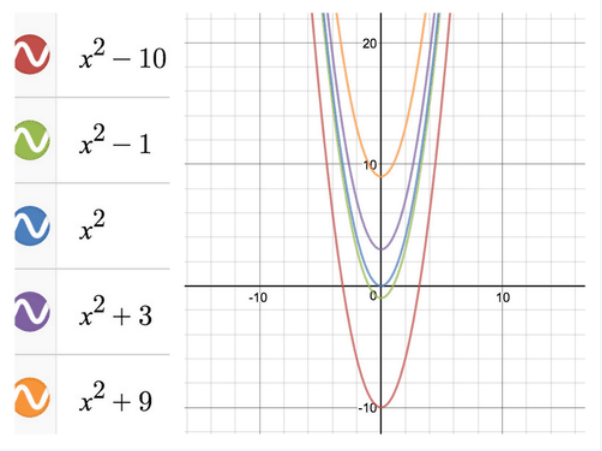
Example 5

Identify the parent function within the following set of functions. Graph the set of functions using a graphing calculator. Identify similarities and differences of the set.

$f(x)=x^2-10$	$f(x)=x^2-1$	$f(x)=x^2$
$f(x)=x^2+3$	$f(x)=x^2+9$	

Solution

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[Figure2]

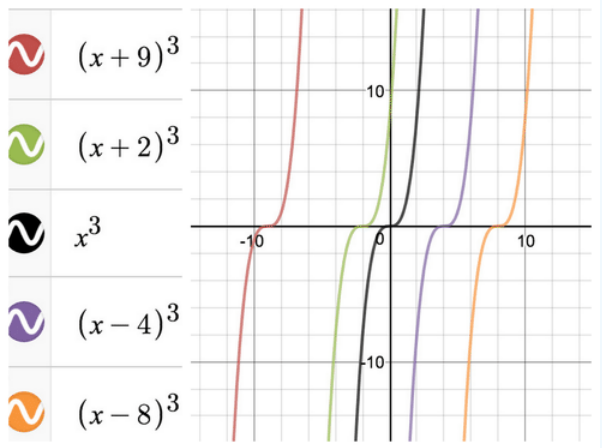
a. Parent Function: The parent function of this group of quadratic functions is the most basic function in the set: $f(x) = x^2$
b. Similarities: width, shape, end behavior, and degree
c. Differences: *x* and *y* intercepts

Example 6

Identify the parent function within the following set of functions. Graph the set of functions using a graphing calculator. Identify similarities and differences of the set.

$f(x)=(x+9)^3$	$f(x)=(x+2)^3$	$f(x)=x^3$
$f(x)=(x-4)^3$	$f(x)=(x-8)^3$	

Solution



[Figure3]

a. Parent Function: The parent function of this group of quartic functions is the most basic function in the set: $y = x^3$
b. Similarities: end behavior, domain and range, direction, and width
c. Differences: *x* and *y* intercepts, increasing and decreasing intervals

Review

1. Explain what a square function is:
2. What is a cube function?
3. Describe the rate of growth of a cubic function related to the growth of a squared function

4. For square root functions we have to define the function value as the positive root, also known as what?
5. Why are reciprocal functions asymptotic to the x-axis?

Identify the parent function within each set of squared functions. Graph each set of functions using a graphing calculator. Identify similarities and differences of each set.

Set 1:

$f(x)=x^2-10$	$f(x)=x^2-1$	$f(x)=x^2$
$f(x)=x^2+3$	$f(x)=x^2+9$	

6. Set 1:
   a. Parent Function:
   b. Similarities:
   c. Differences:

   Set 2:

$f(x)=(x+10)^2$	$f(x)=(x+4)^2$	$f(x)=x^2$
$f(x)=(x-2)^2$	$f(x)=(x-5)^2$	

7. Set 2:
   a. Parent Function:
   b. Similarities:
   c. Differences:

   Set 3:

$f(x)=-x^2$	$f(x)=x^2$

8. Set 3:
   a. Parent Function:
   b. Similarities:
   c. Differences:

Use the above information and the vertex form of a quadratic equation: $f(x)=a(x-h)^2+k$ to help you answer the following questions:

9. How does the a value affect the graph?
10. How does the h value affect the graph?
11. How does the k value affect the graph?
12. How are domain values similar/different?
13. How are range values similar/different?
14. Does the a, h, and/or k value affect the domain?
15. Does the a, h, and/or k value affect the range?

Cubic Functions:

Circle the parent function within each set of cubic functions. Graph each set of functions using a graphing calculator. Identify similarities and differences of each set.

Set 4:

$f(x)=x^3-5$	$f(x)=x^3-3$	$f(x)=x^3$
$f(x)=x^3+1$	$f(x)=x^3+6$	

16. Set 4:
    a. Parent Function:
    b. Similarities:
    c. Differences:

    Set 5:

$f(x)=-x^3$	$f(x)=x^3$

17. Set 5:
    a. Parent Function:
    b. Similarities:
    c. Differences:

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.11.

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Vocabulary

Term	Definition
∞	The symbol "∞" means "infinity", and is an abstract concept describing a value greater than any countable number.
axis of symmetry	The axis of symmetry of a parabola is a vertical line that passes through the vertex of the parabola. The parabola is symmetrical about this line.
Cube	The cube of a number is the number multiplied by itself three times. For example, "two-cubed" = 2^3=2×2×2=8.
Cubic Function	A cubic function is a function containing an x^3 term as the highest power of x.
Function families	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
Function Family	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
Parabola	A parabola is the characteristic shape of a quadratic function graph, resembling a "U".
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
principal root	The principal root is the positive root of a number.
Reciprocal Function	A reciprocal function is a function with the parent function $y = \frac{1}{x}$.
square	The square of a number is the number multiplied by itself.
square function	A square function is a quadratic function. Its parent function is $y=x^2$ and its graph is a parabola.
square root function	A square root function is a function with the parent function $y = \sqrt{x}$.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.
Vertex	The vertex of a parabola is the highest or lowest point on the graph of a parabola. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.

Image Attributions

1. [Figure 1]
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   Source: Desmos Graphing Calculator
2. [Figure 2]
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3. [Figure 3]
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1.4.3: Graphing Cube Root Functions

Graphing Cubed Root Functions

Mrs. Garcia assigns her student the cube root function $y = -\sqrt[3]{(x+1)}$ to graph for homework. The following day, she asks her students which quadrant(s) their graph is in.

Alendro says that because of the negative sign, all *y* values are negative. Therefore his graph is only in the third and fourth quadrants quadrant.

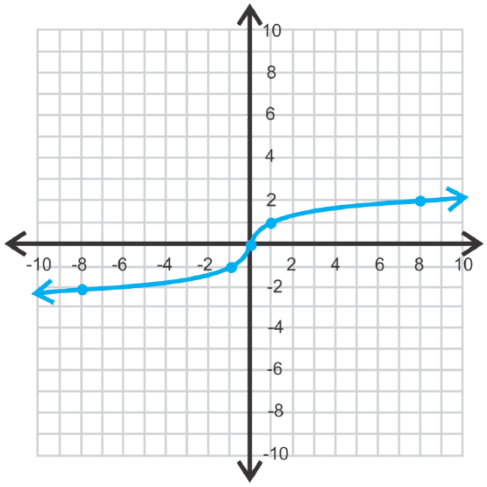
Dako says that his graph is in the third and fourth quadrants as well but it is also in the second quadrant.

Marisha says they are both wrong and that her graph of the function is in all four quadrants.

Which one of them is correct?

Graphing Cubed Root Functions

A cubed root function is different from that of a square root. Their general forms look very similar, $y = a\sqrt[3]{x-h}+k$ and the parent graph is $y = \sqrt[3]{x}$. However, we can take the cubed root of a negative number, therefore, it will be defined for all values of x. Graphing the parent graph, we have:

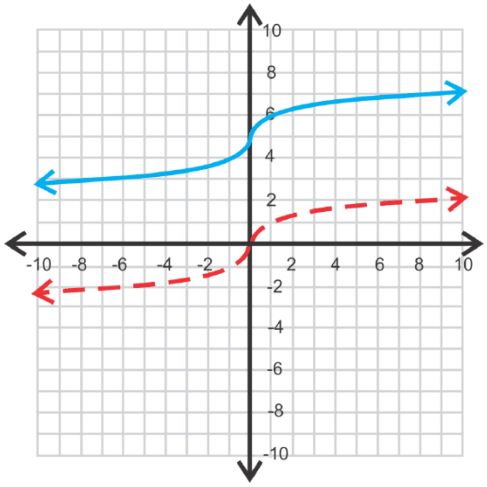


[Figure1]

x	*y*
-27	-3
-8	-2
-1	-1
0	0
1	1
8	2
27	3

For $y = \sqrt[3]{x}$, the output is the same as the input of $y = x^3$. The domain and range of $y = \sqrt[3]{x}$ are all real numbers. Notice there is no "starting point" like the square root functions, the (h,k) now refers to the point where the function bends, called a **point of inflection**.

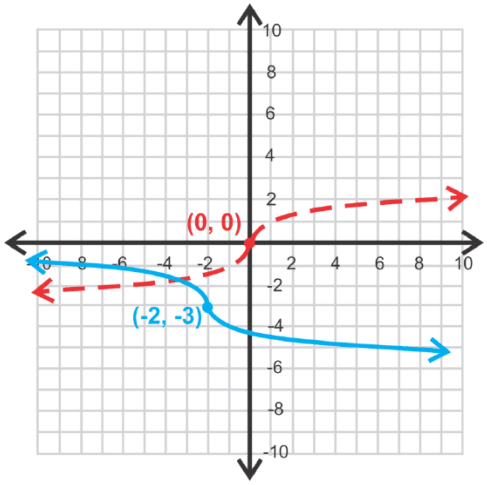
Let's describe how to obtain the graph of $y = \sqrt[3]{x}+5$ from $y = \sqrt[3]{x}$.



[Figure2]

We know that the +5 indicates a vertical shift of 5 units up. Therefore, this graph will look exactly the same as the parent graph, shifted up five units.

Now, let's graph $f(x) = -\sqrt[3]{x+2} - 3$ and find the domain and range.



[Figure3]

From the previous problem, we know that from the parent graph, this function is going to shift to the left two units and down three units. The negative sign will result in a reflection.

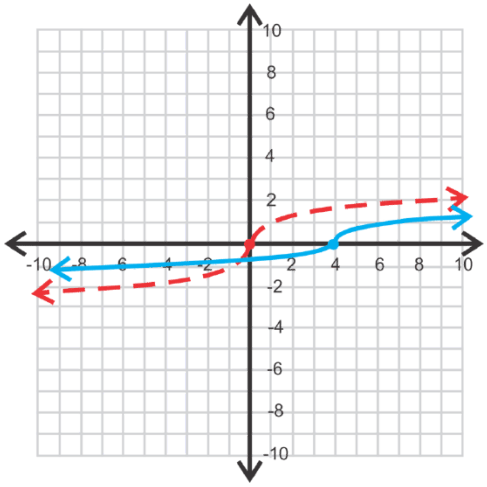
Alternate Method: If you want to use a table, that will also work. Here is a table, then plot the points. (h,k) should always be the middle point in your table.

x	*y*
6	-5
-1	-4
-2	-3
-3	-2
-10	-1

Finally, let's graph $f(x) = \frac{1}{2}\sqrt[3]{x-4}$.

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The -4 tells us that, from the parent graph, the function will shift to the right four units. The $\frac{1}{2}$ effects how quickly the function will "grow". Because it is less than one, it will grow slower than the parent graph.



Using the graphing calculator: If you wanted to graph this function using the TI-83 or 84, press Y= and clear out any functions. Then, press (1÷2), **MATH** and scroll down to **4**: \sqrt[3]{ } and press **ENTER**. Then, type in the rest of the function, so that $Y = \left(\frac{1}{2}\right)\sqrt[3]{(X-4)}$. Press **GRAPH** and adjust the window.

Important Note: The domain and range of all cubed root functions are both all real numbers.

Example 1

Solution

Earlier, you were asked to determine which student was correct.

If you graph the function $y = -\sqrt[3]{(x+1)}$, you see that the domain is all real numbers, which makes all quadrants possible. However, for all positive values of *x*, *y* is negative because of the negative sign in front of the cube root. That rules out the first quadrant. Therefore, Dako is correct.

Example 2

Evaluate $y = \sqrt[3]{x+4} - 11$ when $x = -12$.

Solution

Plug in $x = -12$ and solve for y.

$y = \sqrt[3]{-12+4} - 11 = \sqrt[3]{-8} + 4 = -2 + 4 = 2$

Example 3

Describe how to obtain the graph of $y = \sqrt[3]{x+4} - 11$ from $y = \sqrt[3]{x}$.

Solution

Starting with $y = \sqrt[3]{x}$, you would obtain $y = \sqrt[3]{x+4} - 11$ by shifting the function to the left four units and down 11 units.

Graph the following cubed root functions. Check your graphs on the graphing calculator.

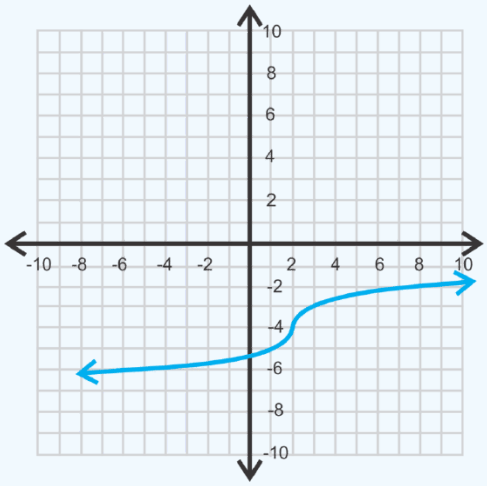
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Example 4

$y = \sqrt[3]{x-2} - 4$

Solution

This function is a horizontal shift to the right two units and down four units.



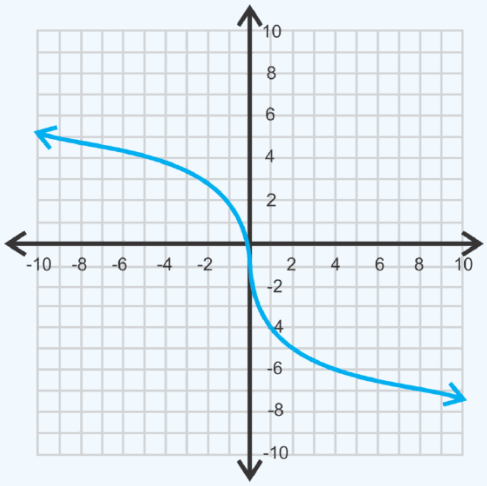
[Figure4]

Example 5

$f(x) = -3\sqrt{x} - 1$

Solution

This function is a reflection of $y = \sqrt[3]{x}$ and stretched to be three times as large. Lastly, it is shifted down one unit.



[Figure5]

Review

Evaluate $f(x) = \sqrt[3]{2x-1}$ for the following values of x.

1. f(14)
2. f(−62)
3. f(20)

Graph the following cubed root functions. Use your calculator to check your answers.

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4. $y = \sqrt[3]{x} + 4$
5. $y = \sqrt[3]{x - 3}$
6. $f(x) = \sqrt[3]{x + 2} - 1$
7. $g(x) = -\sqrt[3]{x} - 6$
8. $f(x) = 2\sqrt[3]{x + 1}$
9. $h(x) = -3\sqrt[3]{x} + 5$
10. $y = \frac{1}{2}\sqrt[3]{1 - x}$
11. $y = 2\sqrt[3]{x + 4} - 3$
12. $y = -\frac{1}{3}\sqrt[3]{x - 5} + 2$
13. $g(x) = \sqrt[3]{6 - x} + 7$
14. $f(x) = -5\sqrt[3]{x - 1} + 3$
15. $y = 4\sqrt[3]{7 - x} - 8$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 7.5.

Vocabulary

Term	Definition
General Equation for a Cubed Root Function	The general equation for a cubed root function is $f(x) = a\sqrt[3]{x - h} + k$, where h is the horizontal shift and k is the vertical shift.

Image Attributions

1. [Figure 1]
   Credit: Sean MacEntee
   Source: https://www.flickr.com/photos/smemon/17323618152/in/photolist-soQ58A-5rouLK-67pQ1V-a54ocN-51ysYT-pP1VGr-6gshc-d8RVdu-nBXVi8-Nv9px-qtm3yn-dUUJmL-6ojCxV-DejJUS-51yqZg-51yqk4-5NrFcF-51CE4o-51CD2w-51CEao-51yqf8-fdo2Jf-51ypZi-51CDos-F9SKny-51CEmJ-jnS68-51ysyc-4t1z5J-51yqat-51CCQU-d9VTUC-51ypzB-2hVbrU-29Gjnh-4Q2TH4-ayVtXQ-6ooPqQ-51CEA9-agFDN7-51ypHi-51CGdY-51CFMG-51ysUK-anJz1-51yrta-51yuA8-51CFPY-51ysrz-51CFG1
2. [Figure 2]
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3. [Figure 3]
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4. [Figure 4]
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51CEao-51yqf8-fdo2Jf-51ypZi-51CDos-F9SKny-51CEmJ-jnS68-51ysyc-4t1z5J-51yqat-51CCQU-d9VTUC-51ypzB-2hVbrU-29Gjnh-4Q2TH4-ayVtXQ-6ooPqQ-51CEA9-agFDN7-51ypHi-51CGdY-51CFMG-51ysUK-anJz1-51yrta-51yuA8-51CFPY-51ysrz-51CFG1

5. [Figure 5]

   Credit: Sean MacEntee

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SECTION OVERVIEW

1.5: Transformations

Topic hierarchy

1.5.1: Vertical and Horizontal Transformations

1.5.2: Stretching and Reflecting Transformations

1.5.3: Combining Transformations

1.5.4: Notation for Transformations

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1.5.1: Vertical and Horizontal Transformations

Vertical and Horizontal Transformations

Horizontal and vertical **transformations** are two of the many ways to convert the basic **parent functions** in a **function family** into their more complex counterparts.

What vertical and/or horizontal **shifts** must be applied to the **parent function** of $y = x^2$ in order to graph $g(x) = (x-3)^2 + 4$?

Vertical and Horizontal Transformations

Have you ever tried to draw a picture of a rabbit, or cat, or dog? Unless you are talented, even the most common animals can be a bit of a challenge to draw accurately (or even recognizably!). One trick that can help even the most "artistically challenged" to create a clearly recognizable basic sketch is demonstrated in nearly all "learn to draw" courses: **start with basic shapes**. By starting your sketch with simple circles, ellipses, rectangles, etc., the basic outline of the more complex figure is easily arrived at, then details can be added as necessary, but the figure is already recognizable for what it is.

The same trick works when graphing equations. By learning the basic shapes of different types of function graphs, and then adjusting the graphs with different types of **transformations**, even complex graphs can be sketched rather easily. This lesson will focus on two particular types of transformations: **vertical shifts** and **horizontal shifts**.

We can express the application of **vertical shifts** this way:

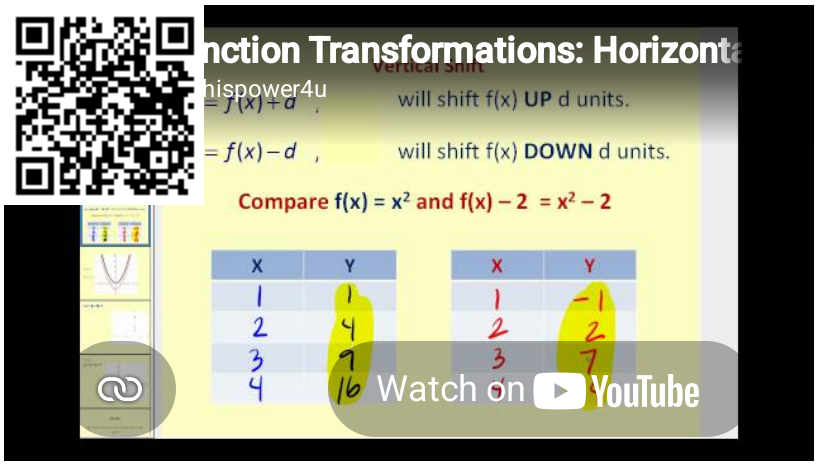
Formally: For any function f(x), the function g(x) = f(x) + c has a graph that is the same as f(x), shifted c units vertically. If c is positive, the graph is shifted up. If c is negative, the graph is shifted down.

Informally: Adding a positive number after the x **outside the parentheses** shifts the graph up, adding a negative (or subtracting) shifts the graph down.

We can express the application of **horizontal shifts** this way:

Formally: given a function f(x), and a constant a > 0, the function g(x) = f(x - a) represents a **horizontal shift** a units to the right from f(x). The function h(x) = f(x + a) represents a horizontal **shift** a units to the left.

Informally: Adding a positive number after the x **inside the parentheses** shifts the graph left, adding a negative (or subtracting) shifts the graph right.



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Examples

Example 1

Solution

Earlier, you were given a question about applying vertical and/or horizontal shifts to a parent function in order to graph a different function in the same function family.

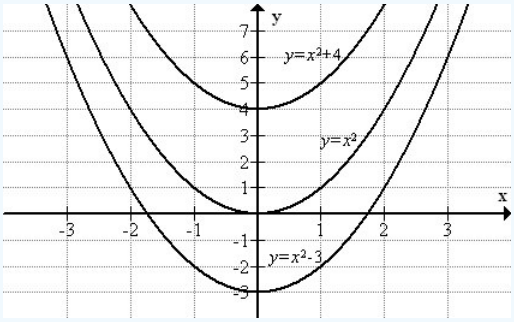
What transformations must be applied to $y = x^2$, in order to graph $g(x) = (x - 3)^2 + 4$?

The graph of $g(x) = (x - 3)^2 + 4$ is the graph of $y = x^2$ shifted 3 units to the right, and 4 units up.

Example 2

What must be done to the graph of $y = x^2$ to convert it into the graphs of $y = x^2 - 3$ and $y = x^2 + 4$?

Solution



At first glance, it may seem that the graphs have different widths. For example, it might look like $y = x^2 + 4$, the uppermost of the three parabolas, is thinner than the other two parabolas. However, this is not the case. The parabolas are **congruent**.

If we shifted the graph of $y = x^2$ up four units, we would have the exact same graph as $y = x^2 + 4$. If we shifted $y = x^2$ down three units, we would have the graph of $y = x^2 - 3$.

Example 3

Identify the transformation(s) involved in converting the graph of $f(x) = |x|$ into $g(x) = |x - 3|$.

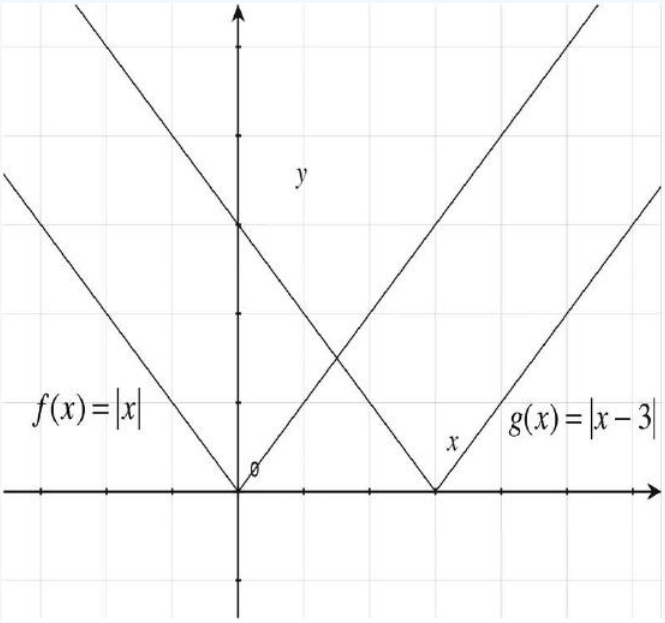
Solution

From the examples of vertical shifts above, you might think that the graph of $g(x)$ is the graph of f(x), shifted 3 units to the left. However, this is not the case. The graph of $g(x)$ is the graph of $f(x)$, shifted 3 units to the right.

The direction of the shift makes sense if we look at specific function values.

x	$g(x) = \text{abs}(x - 3)$
0	3
1	2
2	1
3	0
4	1
5	2
6	3

From the table we can see that the vertex of the graph is the point (3, 0). The function values on either side of $x = 3$ are symmetric, and greater than 0.



Example 4

What transformations must be applied to $y=x^2$, in order to graph $g(x)=(x+2)^2-2$?

Solution

The graph of $g(x)=(x+2)^2-2$ is the graph of $y=x^2$ shifted 2 units to the left, and 2 units down.

Example 5

Use the parent function $f(x) = x^2$ to graph $f(x) = x^2 + 3$.

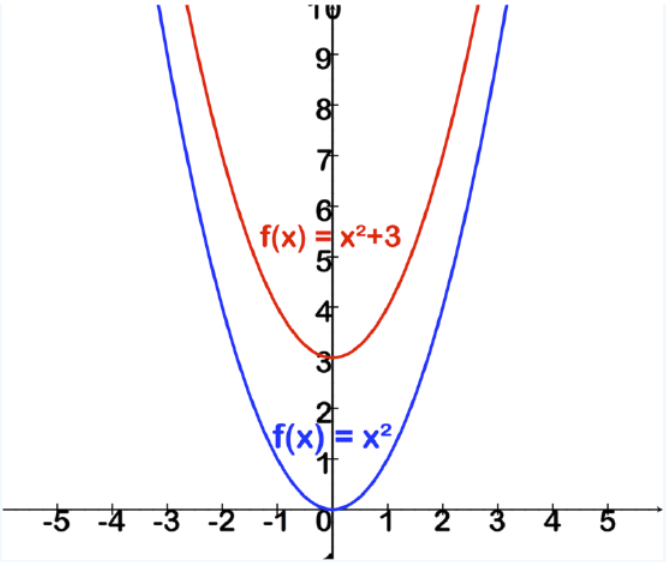
Solution

The function $f(x) = x^2$ is a parabola with the vertex at (0, 0).

Adding outside the parenthesis shifts the graph vertically.

Therefore, $f(x) = x^2 + 3$ will be a parabola with the vertex 3 units up.g

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Example 6

Use the parent function f(x) = |x| to graph f(x) = |x - 4|.

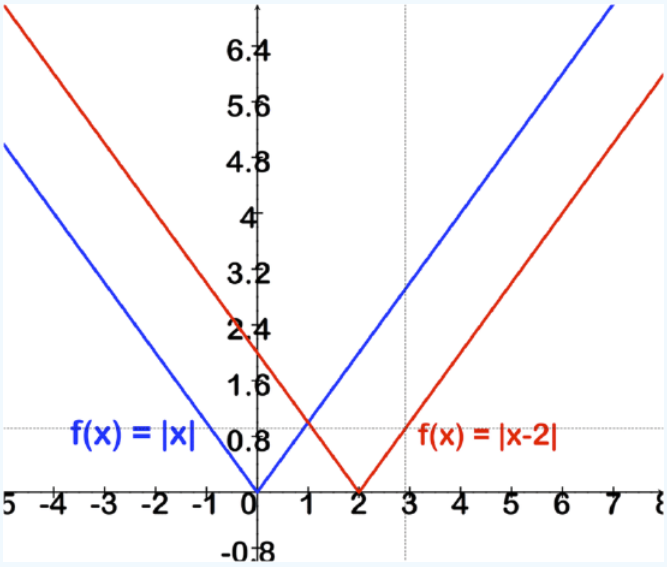
Solution

The graph of the **absolute value function** family parent function f(x) = |x| is a large "V" with the vertex at the origin.

Adding or subtracting inside the parenthesis results in horizontal movement.

Recall that the horizontal shift is right for negative numbers, and left for positive numbers.

Therefore f(x) = |x - 4| is a large "V" with the vertex 4 units to the right of the origin.



Review

1. Graph the function f(x)=2|x−1|−3 without a calculator.
2. What is the vertex of the graph and how do you know?
3. Does it open up or down and how do you know?
4. For the function: f(x)=|x|+c if c is positive, the graph shifts in what direction?
5. For the function: f(x)=|x|+c if c is negative, the graph shifts in what direction?

6. The function g(x)=|x−a| represents a shift to the right or the left?
7. The function h(x)=|x+a| represents a shift to the right or the left?
8. If a graph is in the form a·f(x). What is the effect of changing the *a*?

Describe the transformation that has taken place for the parent function f(x)=|x|.

9. f(x)=|x|−5
10. f(x)=5|x+7|

Write an equation that reflects the transformation that has taken place for the parent function $g(x) = \frac{1}{x}$, for it to move in the following ways:

11. Move two spaces up
12. Move four spaces to the right
13. Stretch it by 2 in the y-direction

Write an equation for each described transformation.

14. a V-shape shifted down 4 units.
15. a V-shape shifted left 6 units
16. a V-shape shifted right 2 units and up 1 unit.

The following graphs are transformations of the parent function f(x)=|x| in the form of f(x)=a|x−h|=k. Graph or sketch each to observe the type of transformation.

17. f(x)=|x|+2. What happens to the graph when you add a number to the function? (i.e. f(x) + k).
18. f(x)=|x|−4. What happens to the graph when you subtract a number from the function? (i.e. f(x) - k).
19. f(x)=|x−4|. What happens to the graph when you subtract a number in the function? (i.e. f(x - h)).
20. f(x)=|x+2|. What happens to the graph when you add a number in the function? (i.e. f(x + h)).

Practice: Graph the following.

21. f(x)=2|x|
22. $f(x) = \frac{5}{2}|x|$
23. $f(x) = \frac{1}{2}|x|$
24. $f(x) = \frac{2}{5}|x|$
25. Let $f(x)=x^2$. Let g(x) be the function obtained by shifting the graph of f(x) two units to the right and then up three units. Find a formula for g(x) and then draw its graph

Suppose H(t) gives the height of high tide in Hawaii(H) on a Tuesday, (t) of the year. Use shifts of the function H(t) to find formulas of each of the following functions:

26. F(t), the height of high tide on Fiji on Tuesday (t), given that high tide in Fiji is always one foot higher than high tide in Hawaii.
27. S(d), the height of high tide in Saint Thomas on Tuesday (t), given that high tide in Saint Thomas is the same height as the previous day's height in Hawaii.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.12.

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Vocabulary

Term	Definition
Absolute value function	An absolute value function graphs a V shape, and is in the form y=|x|.
Function families	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
Function Family	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
Horizontal shift	A horizontal shift is the result of adding a constant term to the function inside the parentheses. A positive term results in a shift to the left and a negative term in a shift to the right.
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
shift	A shift, also known as a translation or a slide, is a transformation applied to the graph of a function that does not change the shape or orientation of the graph, only the location of the graph.
shifts	A shift, also known as a translation or a slide, is a transformation applied to the graph of a function that does not change the shape or orientation of the graph, only the location of the graph.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.
Translation	A translation is a transformation that slides a figure on the coordinate plane without changing its shape, size, or orientation.
Vertical shift	A vertical shift is the result of adding a constant term to the value of a function. A positive term results in an upward shift, and a negative term in a downward shift.

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1.5.2: Stretching and Reflecting Transformations

Stretching and Reflecting Transformations

Understanding how changes in the equation of a function result in **stretching** and/or **reflecting** the graph of the function is a great way to take some of the mystery out of graphing more complicated equations. By recognizing the family to which a more complex equation belongs, and then identifying what changes have been made to the parent of that family, the graph of even quite detailed functions can be made much more understandable.

See if you can identify what parts of the equation: $y = -\frac{1}{5}x^2$ represent either a **stretch** or a **reflection** of the **parent function** $y=x^2$ before the examples in this section.

Stretching and Reflecting Transformations

Stretching and Compressing Graphs

If we multiply a function by a coefficient, the graph of the function will be stretched or compressed.

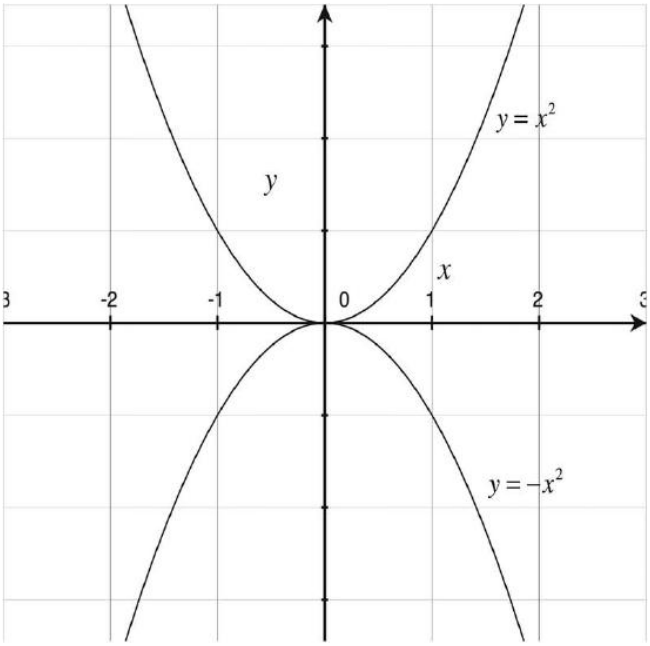
Given a function f(x), we can formalize compressing and stretching the graph of f(x) as follows:

- A function g(x) represents a vertical stretch of f(x) if g(x) = cf(x) and c > 1.
- A function g(x) represents a vertical **compression** of f(x) if g(x) = cf(x) and 0 < c < 1.
- A function h(x) represents a horizontal compression of f(x) if h(x) = f(cx) and c > 1.
- A function h(x) represents a horizontal stretch of f(x) if h(x) = f(cx) and 0 < c < 1.

Notice that a vertical compression or a horizontal stretch occurs when the coefficient is a number between 0 and 1.

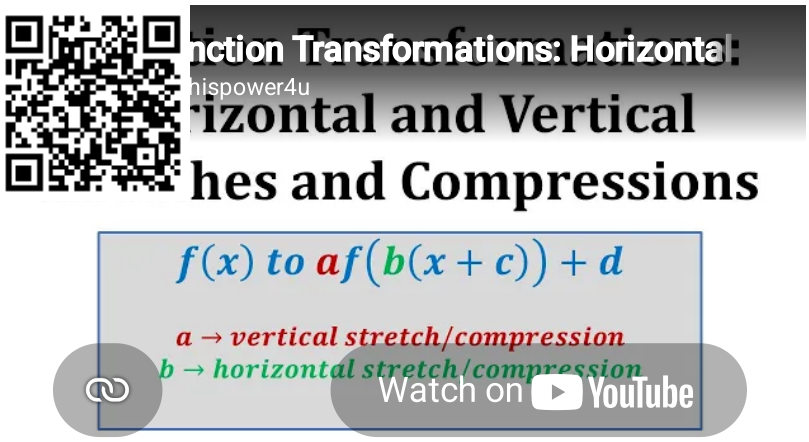
Reflecting Graphs Over the y-axis and x-axis

Consider the graphs of the functions $y = x^2$ and $y = -x^2$, shown below.



The graph of $y = -x^2$ represents a **reflection** of $y = x^2$, over the x-axis. That is, every function value of $y = -x^2$ is the negative of a function value of $y = x^2$. In general, g(x) = -f(x) has a graph that is the graph of f(x), reflected over the x-axis.

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Example 1

Solution

Earlier, you were asked a question about identifying transformations.

The function $y = -\frac{1}{5}x^2$ is the result of transforming $y=x^2$ by reflecting it over the *x* axis, because of the negative co-efficient on the *x*, and vertically compressing it (making it *wider*), because the co-efficient on the *x* is a fraction between 0 and 1.

Example 2

Identify the graph of the function $y = (3x)^2$.

Solution

We have multiplied x by 3. This should affect the graph horizontally. However, if we simplify the equation, we get $y = 9x^2$. Therefore the graph if this parabola will be taller/thinner than $y = x^2$. Multiplying x by a number greater than 1 creates a horizontal compression, which looks like a vertical stretch.

Example 3

Identify the transformation described by $y = ((1/2)x)^2$.

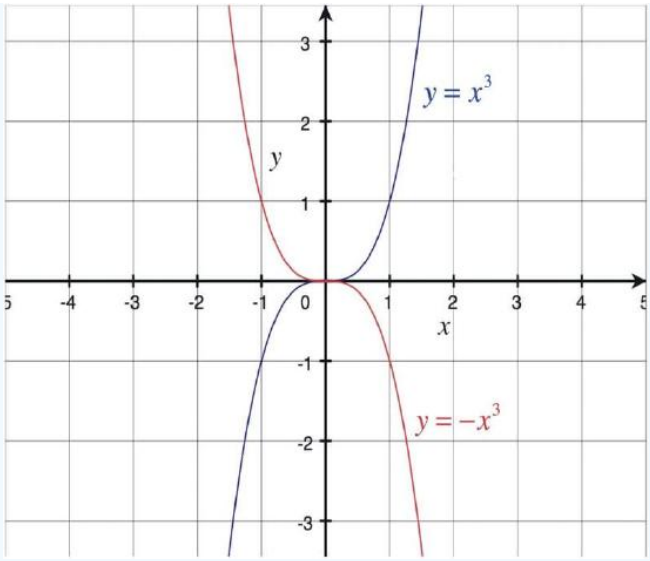
Solution

If we simplify this equation, we get $y = (1/4)\ x^2$. Therefore multiplying x by a number between 0 and 1 creates a horizontal stretch, which looks like a vertical compression. That is, the parabola will be shorter/wider.

Example 4

Sketch a graph of $y = x^3$ and $y = -x^3$ on the same axes.

Solution



At first the two functions might look like two parabolas. If you graph by hand, or if you set your calculator to sequential mode (and not simultaneous), you can see that the graph of $y = -x^3$ is in fact a reflection of $y = x^3$ over the x-axis.

However, if you look at the graph, you can see that it is a reflection over the y-axis as well. This is the case because in order to obtain a reflection over the y-axis, we negate x. In other words, h(x) = f(-x) is a reflection of f(x) over the y-axis. For the function $y = x^3$, $h(x) = (-x)^3 = (-x)(-x)(-x) = -x^3$. This is the same function as the one we have already graphed.

It is important to note that this is a special case. The graph of $y = x^2$ is also a special case. If we want to reflect $y = x^2$ over the y-axis, we will just get the same graph! This can be explained algebraically: $y = (-x)^2 = (-x)(-x) = x^2$.

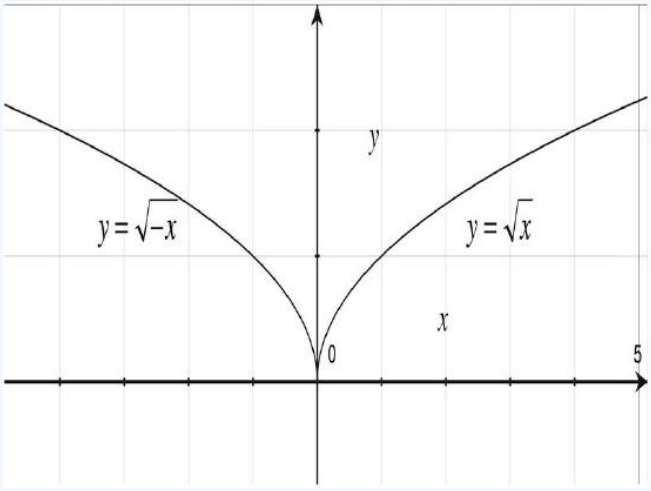
Example 5

Graph the functions $y = \sqrt{x}$ and $y = \sqrt{-x}$.

Solution

The equation $y = \sqrt{-x}$ might look confusing because of the *-x* under the square root. It is important to keep in mind that *- x* means the *opposite* of *x*. Therefore the domain of this function is restricted to values ≤ 0. For example, if $x = -4, y = \sqrt{-(-4)} = \sqrt{4} = 2$. It is this domain, which includes all real numbers not in the domain of $y = \sqrt{x}$ plus zero, that gives us a graph that is a reflection over the *y*-axis.

In sum, a graph represents a reflection over the *x*-axis if the function has been negated (i.e. the *y* has been negated if we think of *y* = *f*(*x*)). The graph represents a reflection over the *y*-axis if the variable *x* has been negated.



Example 6

Identify the function and sketch the graph of $y = \sqrt{x}$ reflected over both axes.

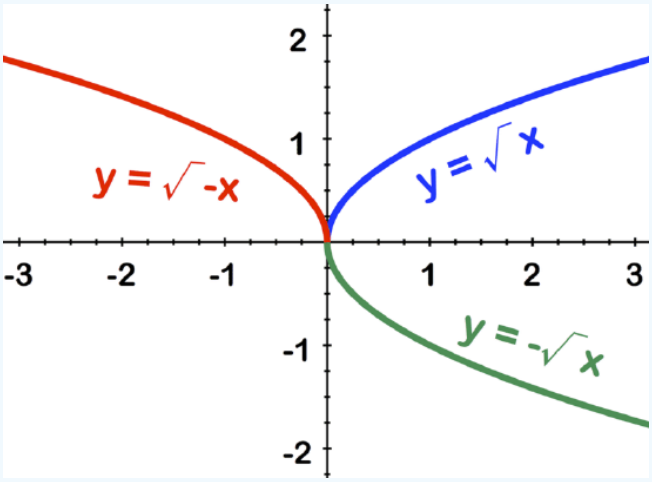
Solution

To reflect the graph of $y = \sqrt{x}$ over both axes, the function must be *negated* both outside and inside the root: $y = -\sqrt{-x}$. The negation (negative) *outside* of the root has the effect of reflecting the graph vertically, and the negation *inside* of the root reflects the graph horizontally. The image below shows three versions:

BLUE: $y = \sqrt{x}$

GREEN: $y = -\sqrt{x}$

RED: $y = \sqrt{-x}$



403 ERROR

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Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

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Generated by cloudfront (CloudFront)
Request ID: 9mnnpfKg9njAP68oiD82hO04dwgFEhjTba4L6OOHl1Pu6lGqhPESew==
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Review

1. If a function is multiplied by a coefficient, what will happen to the graph of the function?
2. What does multiplying x by a number greater than one create?
3. What happens when we multiply x by a number between 0 and 1?
4. In order to obtain a reflection over the y axis what do we have to do to x?
5. How do we obtain a reflection over the x-axis?
6. Write a function that will create a horizontal compression of the following:f(x)=x2+3
7. Write a function that will horizontally stretch the following: f(x)=x^2-6
8. Rewrite the function $f(x) = -\sqrt{x}$ to get a reflection over the x-axis.
9. Rewrite the function $f(x) = \sqrt{x}$ to get a reflection over the y-axis.

Graph each of the following using transformations. Identify the translations and reflections.

10. f(x)=|x|−2
11. $h(x) = \sqrt{x+3}$
12. $g(x) = \frac{1}{x+1}$
13. f(x)=$-4x^3$
14. h(x)=$(x+3)^3+1$
15. $f(x) = \frac{1}{3}(x-3)^2 + 1$
16. $f(x) = -4\sqrt{x+1} - 2$
17. $f(x) = \frac{2}{3(x-2)} + \frac{1}{4}$

Let y=f(x) be the function defined by the line segment connecting the points (-1, 4) and (2, 5). Graph each of the following transformations of y=f(x).

18. y=f(x)+1
19. y=f(x+2)
20. y=f(−x)
21. y=f(x+3)−2

The graph of y = x is shown below. Sketch the graph of each of the following transformations of y = x

22. y=x+3
23. y=x−2
24. y=−x

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.13.

Vocabulary

Term	Definition
compression	A stretch or compression is a function transformation that makes a graph narrower or wider, without translating it horizontally or vertically.
Function Family	Function families are groups of functions with similarities that make them easier to graph when you are familiar with the parent function, the most basic example of the form.
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
Reflecting	Reflecting a graph means to transform the graph in order to produce a "mirror image" of the original graph by flipping it across a line.
Reflection	Reflections are transformations that result in a "mirror image" of a parent function. They are caused by differing signs between parent and child functions.
stretch	A stretch or compression is a function transformation that makes a graph narrower or wider.
stretching	Stretching a graph means to make the graph narrower or wider.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.

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1.5.3: Combining Transformations

Function Graphs: Combined Transformations

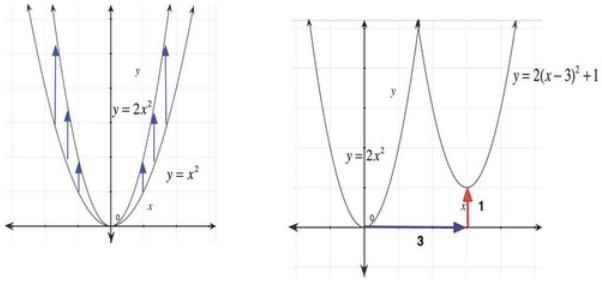
How do the different forms of **transformations** result in the differences between the basic parent functions we have explored and some of the more complex graphs you may have seen? It has likely occurred to you that these individual transformations are not enough to result in such significant differences. How do we then apply the individual transformations so that the more complex graphs may be understood?

Function Transformations: Combined Transformations

By combining **shifts**, **reflections**, and vertical and horizontal **stretches** and compression, a simple **parent function** graph can represent a much more advanced function.

Consider the equation $y=2(x-3)^2+1$. We can compare the graph of this function to the graph of the parent $y=x^2$: the graph represents a vertical stretch by a factor of 2, a horizontal shift 3 units to the right, and a vertical shift of 1 unit.

We can use this relationship to graph the function $y=2(x-3)^2+1$. You can start by sketching $y=x^2$ or $y=2x^2$. Then you can shift the graph 3 units to the right, and up 1 unit.



[Figure1]

Using your knowledge of the **parent function** $y=|x|$ and your knowledge of transformations, graph the following function:

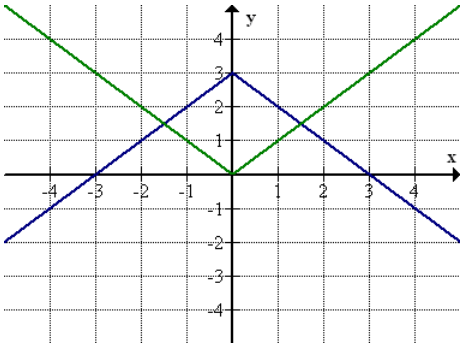
$f(x)=-|x|+3$

The parent graph of this function is the graph of $y=|x|$, reflected over the x-axis, and shifted up 3 units. The question is: which transformation do you perform first?

We can answer this question if we consider a few key function values. The table below shows several function values for $f(x)=-|x|+3$:

x	f(x)=−abs(x)+3
-3	−abs(−3)+3=−(+3)+3=−3+3=0
-2	−abs(−2)+3=−(+2)+3=−2+3=1
-1	2
0	3
1	2
3	0

From the function values in the table we can see that the function increases until a vertex at (0, 3), and then it decreases again. This tells us that we can obtain the graph if we first reflect $y=|x|$ over the x-axis (turn the “v” upside down), and then shift the graph up 3 units.

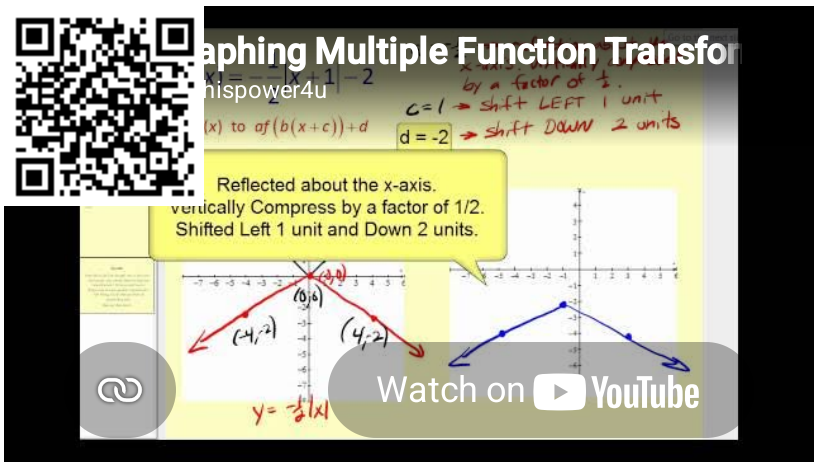


[Figure2]

We can also justify this ordering of the transformations of we think about the order of operations. To find any function value we take an x value, find its absolute value, find the negative of that number, and then add 3. This is the same as the order of the transformation: reflection comes before shifting up.

When describing the relationship between a transformed graph and its parent graph, it is important to be careful with the order of operations. Take the graph of $f(x)=3(x+7)^3+5$ and its parent function $g(x)=x^3$.

The graph of $f(x)=4(x+8)^3-3$ is the graph of $g(x)=x^3$, stretched vertically (made narrower) by a factor of 3, shifted 7 units to the left, and then shifted 5 units up.



Examples

Example 1

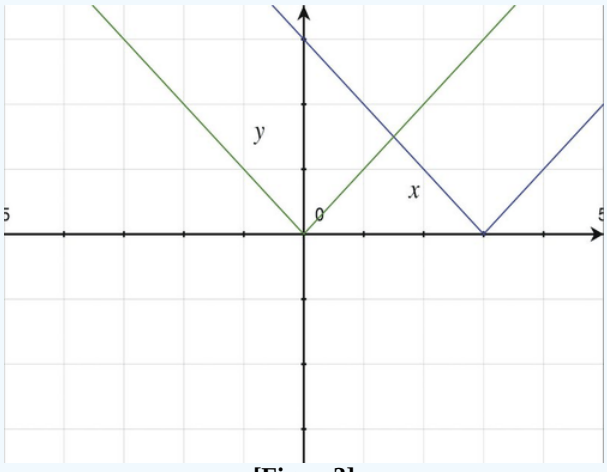
Graph the function below using your knowledge of the parent function $y=|x|$ and your knowledge of transformations.
$g(x)=|-x+3|$

Solution

This function represents a horizontal shift of y=|x|, and a reflection over the x-axis. Before graphing, consider a few function values:

x	g(x)=abs(−x+3)
-3	abs(−(−3)+3)=abs(3+3)=abs(6)=6
-2	abs(−(−2)+3)=(2+3)=abs(5)=5
0	3
1	2
3	0
4	1

From the values in the table, we can see that the vertex of the graph is at (3, 0). The graph is shown below.



[Figure3]

The graph looks the same as the graph of y=|x−3|. This is the case because y=|−x+3|=|−(x−3)|, and because |−a|=|a| for all values of a, then |−(x−3)|=|x−3|. So the original function is equal to |x−3|.

We can still think of this graph as a reflection: if we reflect y=|x| over the x-axis, the graph remains the same, as it is symmetric over the x-axis. Then we shift the graph 3 units to the right. What is important to note here is that in order to "read" the equation as a horizontal shift, the entire expression inside the function (in this case, inside the absolute value) must be negated.

Remember that the key to multiple transformations is to do them in order. Another way to keep track of which operations to do in which order is to just do them in the order they appear in the equation, left to right.

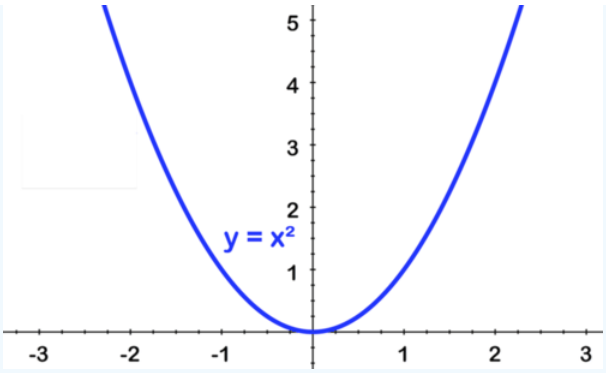
Example 2

Sketch the graph of $y=-3(x+2)^2+4$

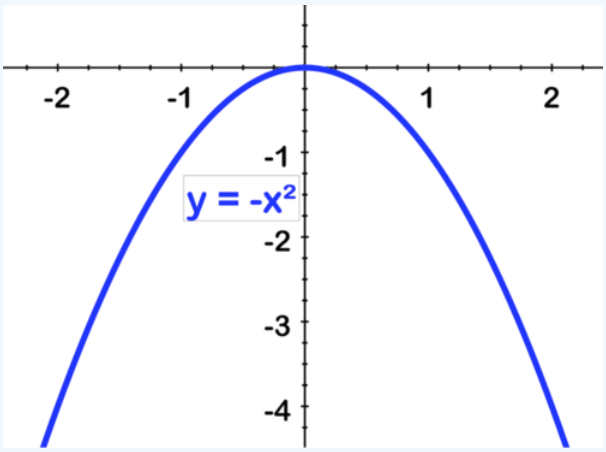
Solution

To sketch $y=-3(x+2)^2+4$, we start with the parent function, $y=x^2$.

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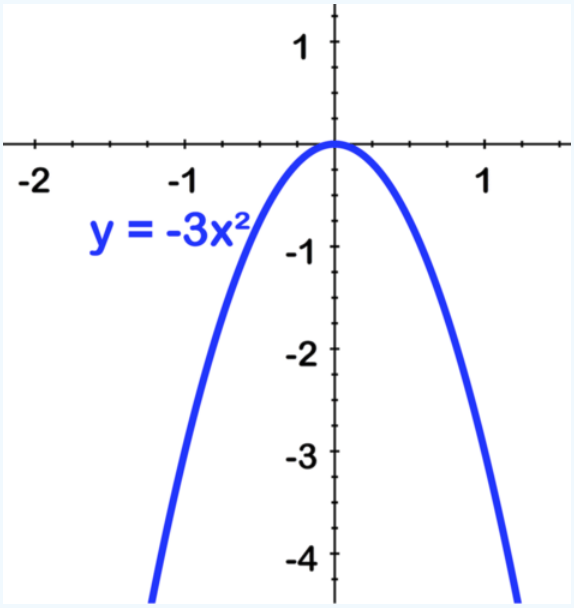


[Figure4]



[Figure5]

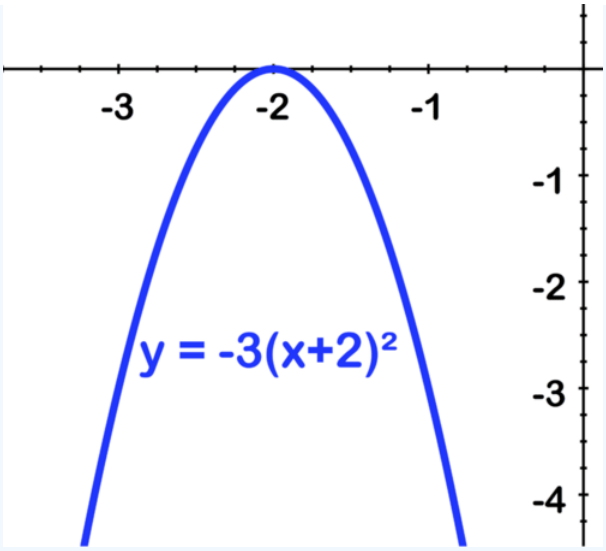
Next we stretch by 3:



[Figure6]

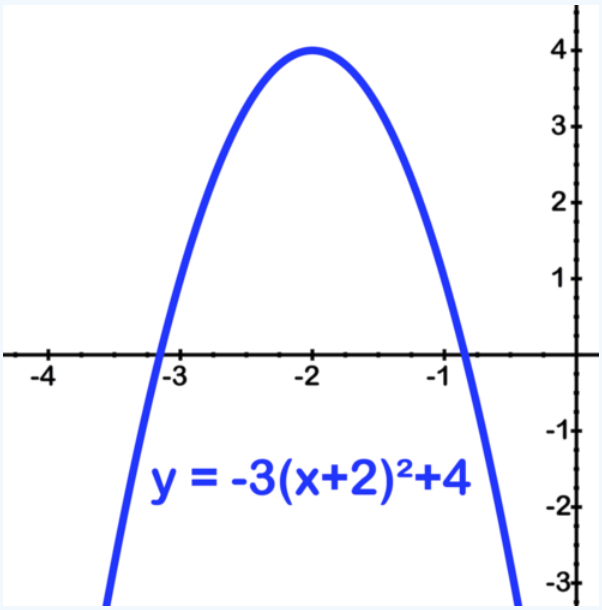
Shift left by 2:

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[Figure7]

Finally shift up by 4:



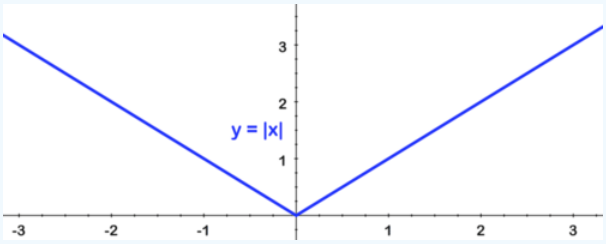
[Figure8]

Example 3

Sketch the graph of $y=-1|x+2|-3$

Solution

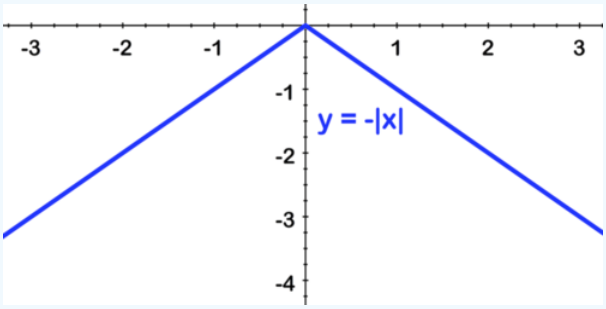
To sketch $y=-1|x+2|-3$ start with the parent equation: $f(x)=|x|$ and complete the transformations left to right:



[Figure9]

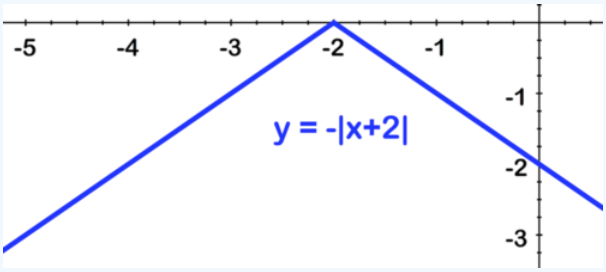
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First reflect over the x axis:



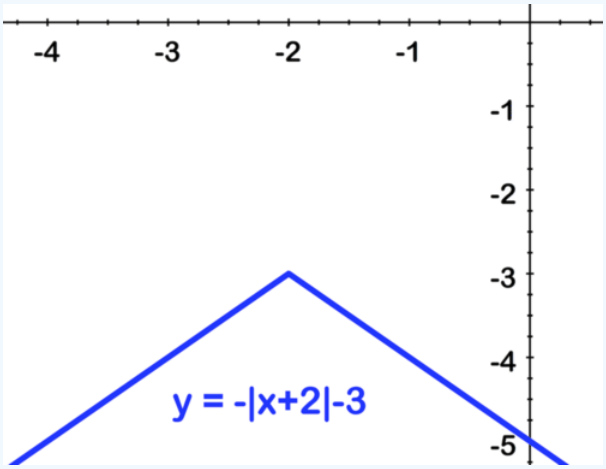
[Figure10]

Second, shift left by 2:



[Figure11]

Finally, shift down by 3:



[Figure12]

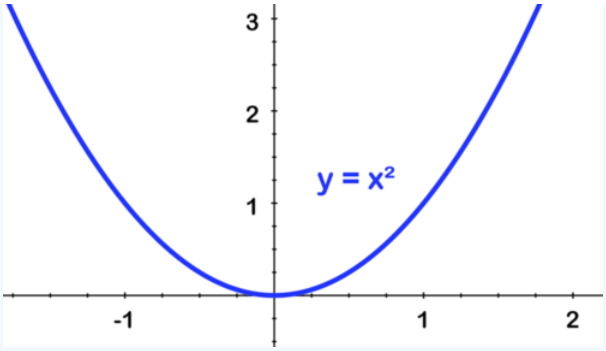
Example 4

Sketch the graph of $f(x)=2(x-1)^2$

Solution

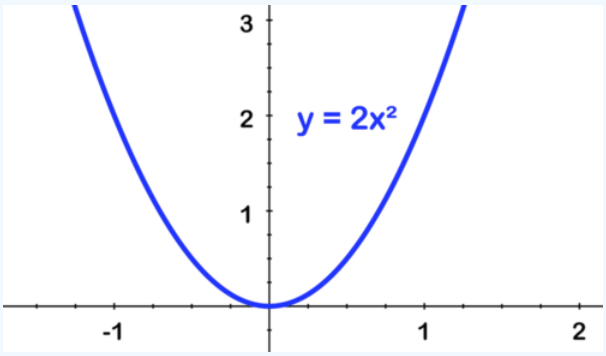
To sketch $f(x)=2(x-1)^2$ start with the parent $f(x)=x^2$

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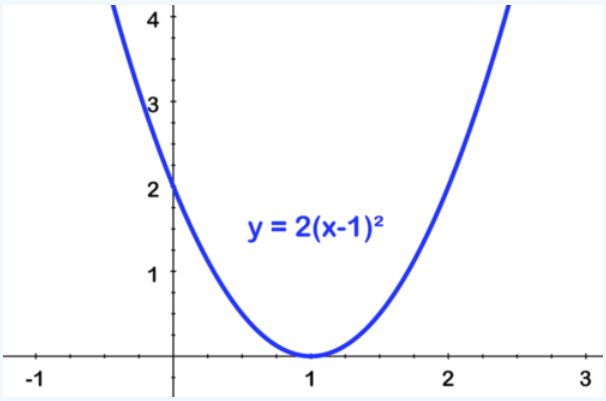
[Figure13]

First stretch by 2:



[Figure14]

Second, shift right by 1:



[Figure15]

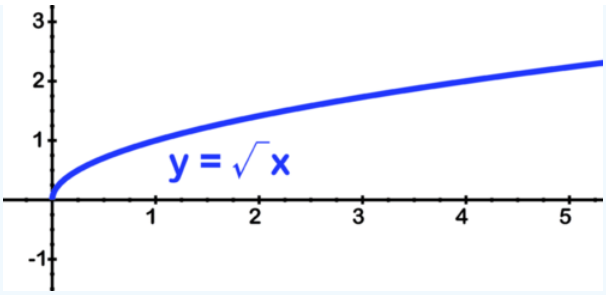
Example 5

Sketch the graph of $f(x) = -2\sqrt{x-1}$

Solution

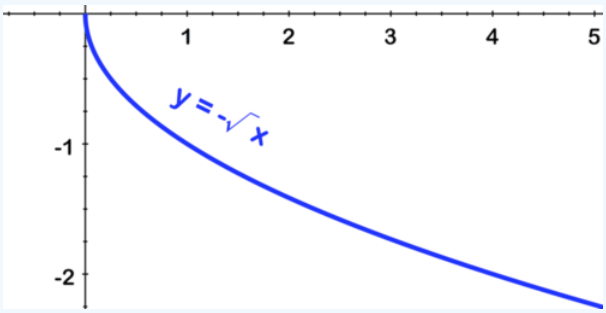
To sketch the graph of $f(x) = -2\sqrt{x-1}$ start with the parent $y = \sqrt{x}$

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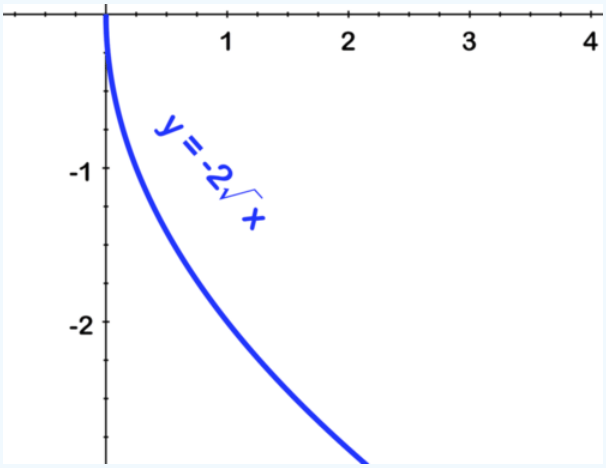
[Figure16]

First reflect over the x axis:



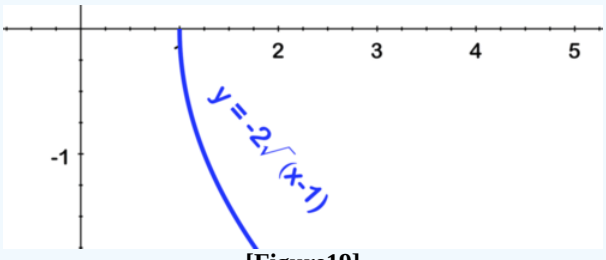
[Figure17]

Second, stretch by 2:



[Figure18]

Third, shift right by 1:



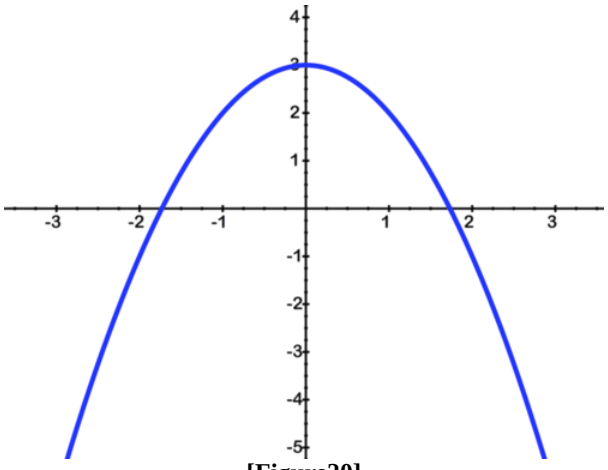
[Figure19]

Review

1. What part of the function $g(x)=-(f(x)+1)=-(x^3+1)$ shifts the graph of f(x) vertically?
2. What part of the function $g(x)=-(f(x)+1)$ reflects the graph of f(x) across the x-axis?

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3. What is different between the functions $g(x)=-(x^3+1.0)$ and $h(x)=-x^3+1.0$ that changes the appearance of the graph?
4. Given the function $g(x)=3.0(f(x)+2.0)=3.0(x^2+2.0)$, what is it that *shifts* the graph of f(x) vertically?
    a. 3.0
    b. x^2
    c. 2.0
5. Given the function $g(x)=3.0(f(x)+2.0)=3.0(x^2+2.0)$, what is it that *stretches* the graph of f(x) vertically?
    a. 3.0
    b. x^2
    c. 2.0
6. What part of the equation $k(x)=-(x+1)^3$ shifts the graph of $j(x)=x^3$ horizontally?
7. What part of the equation $k(x)=-(x+1)^3$ reflects the graph of $j(x)=x^3$ across the x-axis?
8. Given the function $g(x)=3.0(f(x)+2.0)=3.0(x+2.0)^3$, what is it that *shifts* the graph of f(x) horizontally?
9. Given the function $g(x)=3.0(f(x)+2.0)=3.0(x^2+2.0)^3$, what is it that *stretches* the graph of f(x) vertically?
10. The graph of g(x) is f(x) reflected across the x-axis. The graph of h(x) is f(x) reflected across the y-axis. The graph of j(x) is f(x) reflected across the x-axis and the y-axis. When graphing j(x) does the order in which the reflections occur matter? (Does it matter which axis we reflect the graph across first?)
11. Given the function $f(x)=x^3$, write a function g(x) that is: f(x) reflected across the y-axis and then stretched vertically by 8.
12. How do you transform the graph of: $f(x)=x^3$ so that it looks like the graph of: $f(x)=4x^3+6$?
    a. Stretch it by a factor of $\frac{1}{4}$ and shift it up 6 units.
    b. Stretch it by a factor of 6 and shift it left 4 units.
    c. Stretch it by a factor of 4 and shift it down 6 units.
    d. Stretch it by a factor of 4 and shift it up 6 units.
13. How do you transform the graph of: $f(x)=\sqrt{x}$ so that it looks like the graph of: $f(x)=-\sqrt{x}-4$?
    a. Reflect it across the x-axis and shift it 4 units down.
    b. Reflect it across the y-axis and shift it 4 units up.
    c. Reflect it across the x-axis and shift it 4 units up.
    d. Reflect it across the y-axis and shift it 4 units down.
14. The graph below is a transformation of a common function. What is the common function that has been transformed?
    a. y=|x|
    b. y= $\sqrt{x}$
    c. $y=x^2$
    d. $y=x^3$



[Figure20]

15. How has the function in Q# 14 been transformed?
    a. Reflected across the y-axis and shifted right 3 units.

b. Stretched vertically by a factor of 3 and shifted right 1 unit.
c. Reflected across the x-axis and shifted up 3 units.
d. Reflected across the x-axis and shifted left 4 units.

16. Write a function g(x) whose graph looks like the graph of f(x)=|x| reflected across the x-axis and shifted up 1 unit.
17. Choose the function whose graph looks like the graph of $f(x)=x^3$ shifted to the right 2 units and reflected across the y-axis.
    a. $f(x)=(-x-2)^3$
    b. $f(x)=(-x+2)^3$
    c. $f(x)=-(x-2)^3$
    d. $f(x)=(x-2)^3$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.7.

Vocabulary

Term	Definition
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.

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   Source: https://www.flickr.com/photos/devinmoore/187215281/in/photolist-hxwzF-9Y2dT2-8R5iar-9Y2cLM-ePfP1W-eP4pca-epUxUk-ePfP4W-ceh8k3-3dMceZ-aehC8w-gmZd-e3dfr7-ac7c3c-dMr5kM-6bfCWV-6vpzyo-eKFqm1-iEv4o-aYNoV6-aWuFqv-9Uomhm-aWuFb4-eLWJx6-9Y2d4v-aWuFg4-633FZc-7L1mSC-aWuFn6-eKFqkC-7nAuor-9yD6vQ-7jAj2m-aWuFja-bXcHpR-7q9MQ4-eqQMBS-4V9irC-4V9isQ-ePfPis-9mMCH4-7qeaJq-ePfNzy-b3SGm-eP4ok2-eP4oZD-7q9Y2k-ePfP9o-eKFxDm-7qdvyU
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10. [Figure 10]
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12. [Figure 12]
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1.5.4: Notation for Transformations

Point Notation and Function Notation

When performing multiple transformations, it is very easy to make a small error. This is especially true when you try to do every step mentally. **Point notation** is a useful tool for concentrating your efforts on a single point and helps you to avoid making small mistakes.

What would f(3x)+7 look like in point notation and why is it useful?

Using Function Notation and Point Notation

A transformation can be written in function notation and in point notation. Function notation is very common and practical because it allows you to graph any function using the same basic thought process it takes to graph a parabola in vertex form.

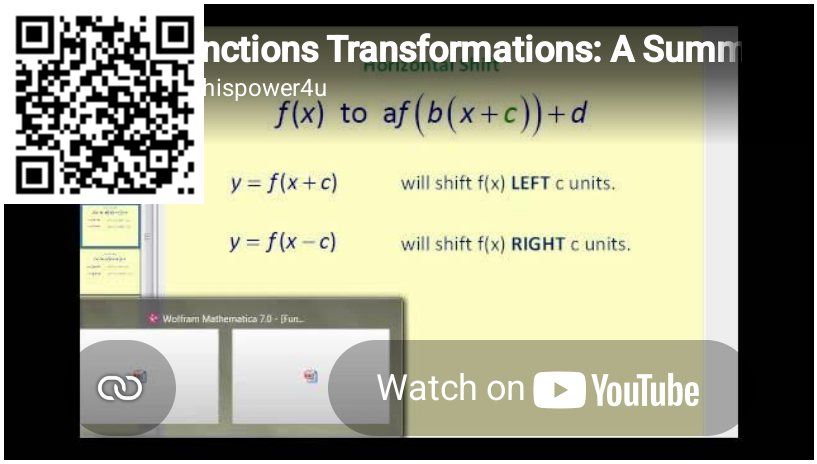
Another way to graph a function is to transform each point one at a time. This method works well when a table of x,y values is available or easily identified from the graph.

Essentially, it takes each coordinate (x,y) and assigns a new coordinate based on the transformation.

$(x,y) \longrightarrow$(new x,new y)

This notation is called **point notation**. The new y coordinate is straightforward and is directly from what takes place outside f(x) because f(x) is just another way to write y. For example, $f(x) \rightarrow 2f(x)-1$ would have a new y coordinate of $2y-1$.

The new x coordinate is trickier. It comes from undoing the operations that affect x. For example, $f(x) \rightarrow f(2x-1)$ would have a new x coordinate of $\frac{x+1}{2}$.



The function notation and point notation representations of the transformation "Horizontal shift right three units, vertical shift up 4 units" are

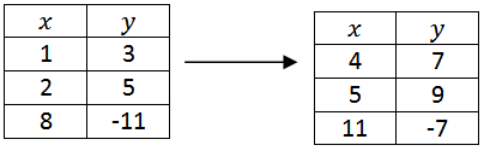
$f(x) \rightarrow f(x-3)+4$

$(x,y) \rightarrow (x+3,y+4)$

Notice that the operations with the x are different.

Apply the transformation above to the following table of points.

x	y
1	3
2	5
8	−11



x	y
1	3
2	5
8	-11

x	y
4	7
5	9
11	-7

[Figure1]

Notice that point notation greatly reduces the mental visualization required to keep all the transformations straight at once.

Examples

Example 1

Solution

Earlier, you were asked what the function f(3x)+7 would be when written in point notation. When written in point notation, it would be written as $(x, y) \rightarrow \left(\frac{x}{3}, y+7\right)$. This is useful because it becomes obvious that the x values are all divided by three and the y values all increase by 7.

Example 2

Convert the following function in point notation to words and then function notation.

(x,y)→(3x+1,−y+7)

Solution

Horizontal stretch by a factor of 3 and then a horizontal shift right one unit. Vertical reflection over the x axis and then a vertical shift 7 units up.

$f(x) \rightarrow -f\left(\frac{1}{3}x - \frac{1}{3}\right) + 7$

Example 3

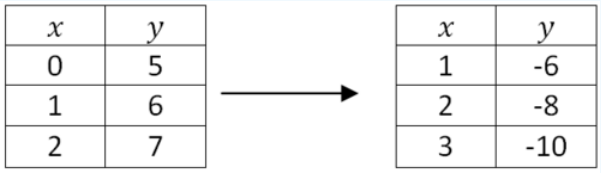
Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

f(x)→−2f(x−1)+4

Solution

Vertical reflection across the x axis. Vertical stretch by a factor of 2. Vertical shift 4 units. Horizontal shift right one unit.

(x,y)→(x+1,−2y+4)



x	y
0	5
1	6
2	7

x	y
1	-6
2	-8
3	-10

[Figure2]

Example 4

Convert the following function notation into point notation and apply it to the included table of points

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$f(x) \rightarrow \frac{1}{4}f(-x-3)-1$

x	y
1	4
2	8

Solution

The y component can be directly observed. For the x component you need to undo the argument. $(x,y) \rightarrow (-x-3, \frac{1}{4}y-1)$

x	y
0	0
1	4
2	8

→

x	y
-3	-1
-4	0
-5	1

[Figure3]

Example 5

Convert the following point notation to words and to function notation and then apply the transformation to the included table of points.

$(x+3,y-1) \rightarrow (2x+6,-y)$

x	y
10	8
12	7
14	6

Solution

This problem is different because it seems like there is a transformation happening to the original left point. This is an added layer of challenge because the transformation of interest is just the difference between the two points. Notice that the x coordinate has simply doubled and the y coordinate has gotten bigger by one and turned negative. This problem can be rewritten as:

$(x,y) \rightarrow (2x,-(y+1))=(2x,-y-1)$

$f(x) \rightarrow -f\left(\frac{x}{2}\right)-1$

x	y
10	8
12	7
14	6

→

x	y
5	-9
6	-8
7	-7

[Figure4]

Review

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

x	y
	5
1	6
2	7

1. $f(x) \rightarrow -\frac{1}{2}f(x+1)$
2. g(x)→2g(3x)+2
3. h(x)→−h(x−4)−3
4. j(x)→3j(2x−4)+1
5. k(x)→−k(x−3)

Convert the following functions in point notation to function notation.

6. $(x, y) \rightarrow \left(\frac{1}{2}x + 3, y - 4\right)$
7. (x,y)→(2x+4,−y+1)
8. (x,y)→(4x,3y−5)
9. (2x,y)→(4x,−y+1)
10. (x+1,y−2)→(3x+3,−y+3)

Convert the following functions in function notation to point notation.

11. f(x)→3f(x−2)+1
12. g(x)→−4g(x−1)+3
13. $h(x) \rightarrow \frac{1}{2}h(2x+2) - 5$
14. $j(x) \rightarrow 5j\left(\frac{1}{2}x - 2\right) - 1$
15. $k(x) \rightarrow \frac{1}{4}k(2x - 4)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.3.

Vocabulary

Term	Definition
function notation	In the context of transformations, function notation describes how a function f(x) changes as a result of the transformation.
point notation	In the context of transformations, point notation describes how each coordinate (x,y) changes as a result of the transformation.

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SECTION OVERVIEW

1.6: Operations on Functions

Topic hierarchy

1.6.1: Composition of Functions

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1.6.1: Composition of Functions

Composition of Functions

If f(x) = x + 2, and g(x) = 2x + 4, what is f(g(x))?

A **function** can be conceptualized as a 'black box'. The **input**, or *x* value is placed into the box, and the box performs a specific set of operations on it. Once the operations are complete, the **output** (the "*f(x)*" or "*y*" value) is retrieved. Once the output is retrieved, the box is ready to work on the next input.

Using this idea, **function composition** can be seen as a box inside of a box. The input *x* value goes into the inner box, and then the output of the *inner* box is used as the input of the *outer* box.

Composition of Functions

Functions are often described in terms of “input” and “output.” For example, consider the function f(x) = 2x + 3. When we input an x value, we output a y value, or a function value. We find the output by taking the input x, multiplying by 2, and adding 3. We can do this for any value of x. Now consider a second function g(x) = 5x. For this function too, we can take an x value, input the x into g(x), and obtain an output. What happens if we take the output of g and use it as the input of f?

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Examples

Example 1

Solution

Earlier, you were given a problem about finding a **composite function**.

If f(x) = x + 2, and g(x) = 2x + 4, what is f(g(x))?

f(g(x)) = f(2x + 4) = (2x + 4) + 2 = 2x + 6

Example 2

Given the function definition above, g(x) = 5x. Therefore if x = 4, then we have g(4) = 5(4) = 20. What happens if we then take the output of 20 and use it as the input of f?

Solution

Substituting 20 in for x in f(x) = 2x + 3 gives: f(20) = 2(20) + 3 = 43.

The table below shows several examples of this same process:

x	Output from *g*	Output from *f*
2	10	23
3	15	33
4	20	43
5	25	53

Examining the values in the table, we can see a pattern: all of the final output values from f are 3 more than 10 times the initial input. We have created a new function called h(x) out of f(x) = 2x + 3 in which g(x) = 5x is the input:

h(x) = f(5x) = 2(5x) + 3 = 10x + 3

When we input one function into another, we call this the **composition** of the two functions. Formally, we write the composed function as f(g(x)) = 10x + 3 or write it as (f o g)x = 10x + 3

Example 3

Find f(g(x)) and g(f(x)):

a. f(x) = 3x + 1 and g(x) = x^2
b. f(x) = 2x + 4 and g(x) = (1/2) x - 2

Solution

a. f(g(x)) = f(x^2) = 3(x^2) + 1 = 3x^2 + 1

   g(f(x)) = g(3x + 1) = (3x + 1)^2 = 9x^2 + 6x + 1

   In both cases, the resulting function is quadratic.

b. f(g(x)) = 2((1/2)x - 2) + 4 = (2/2)x - 4 + 4 = (2/2)x = x

   g(f(x)) = g(2x + 4) = (1/2)(2x + 4) - 2 = x+ 2 - 2 = x.

   In this case, the composites were equal to each other, and they both equal x, the original input into the function. This means that there is a special relationship between these two functions. We will examine this relationship in Chapter 3. It is important to note, however, that f(g(x) is not necessarily equal to g(f(x)).

Example 4

Decompose the function f(x) = (3x - 1)^2 - 5 into a quadratic function g(x) and a linear function h(x).

Solution

When we compose functions, we are combining two (or more) functions by inputting the output of one function into another. We can also *decompose* a function. Consider the function f(x) = (2x + 1)^2. We can decompose this function into an “inside” and

an “outside” function. For example, we can construct $f(x) = (2x+ 1)^2$ with a linear function and a quadratic function. If $g(x) = x^2$ and $h(x) = (2x + 1)$, then $f(x) = g(h(x))$. The linear function $h(x) = (2x + 1)$ is the inside function, and the quadratic function $g(x) = x^2$ is the outside function.

Let $h(x) = 3x - 1$ and $g(x) = x^2 - 5$. Then $f(x) = g(h(x))$ because $g(h(x)) = g(3x - 1) = (3x - 1)^2 - 5$.

The decomposition of a function is not necessarily unique. For example, there are many ways that we could express a linear function as the composition of other linear functions.

Example 5

Given:

$f(x)=5x+3$

$g(x)=3x^2$

Find: $f(g(4))$

Solution

To find $f(g(4))$, we need to know what $g(4)$ is, so we know what to substitute into $f(x)$:

Substitute 4 for x for the function $g(x)$, giving: $3\cdot4^2$

Simplify: $3\cdot16=48$

$\therefore g(4)=48$

Substitute 48 for the x in the function $f(x)$ giving: $5(48)+3$

Simplify: $240+3=243$

$\therefore f(g(4))=243$

Example 6

Given:

$h(n)=7n+1+4(g(n))$

$g(t)=-t$

$f(x)=-2x+g(x)$

Find: $f(h(-5))$

Solution

First, let's solve for the value of the inner function, $h(-5)$. Then we'll know what to plug into the outer function.

$h(-5)=(7)(-5)+1+4(g(-5))$

To solve for the value of h, we need to solve $g(-5)$

$g(-5)=-(-5)$

$\therefore g(-5)=5$

Now we have: $h(-5)=(7)(-5)+1+(4)(5)$

Simplify to get: $h(-5)=-14$

Now we know that $h(-5)=-14$. That tells us that $f(h(-5))$ is $f(-14)$

Find $f(-14)=(-2)(-14)+g(-14)$

So to solve for the value of $f(-14)$, we need to solve for the value of $g(-14)$

$g(-14)=-(-14)$

$\therefore g(-14)=14$

Now we can finish up!

$f(-14)=(-2)(-14)+14$

$\therefore f(-14)=42$

Review

For problems 1-4:

$f(x)=2x-1$ $g(x)=3x$ $h(x)=x^2+1$

1. Find: $f(g(-3))$
2. Find: $f(h(7))$
3. Find: $h(g(-4))$
4. Find: $f(g(h(2)))$

Evaluate each composition below:

5. Given: $f(x) = -5x + 2$ and $g(x) = \frac{1}{2}x + 4$. Find $f(g(12))$.
6. Given: $g(x) = -3x + 6$ and $h(x) = 9x + 3$. Find $g\left(h\left(\frac{1}{3}\right)\right)$.
7. Given: $f(x) = -\frac{1}{5}x + 4$ and $g(x) = 4x^2$. Find $f(g(10))$.
8. Given: $g(x) = 3|x - 4| + 6$ and $h(x) = -x^3$. Find $h(g(4))$.
9. Given: $f(x) = \sqrt{x + 2}$ and $g(x) = |2x|$. Find $g(f(-7))$.
10. Given: $f(x) = -3x + 2$ and given $g(x) = 2x^2$ and given $h(x) = 4|7 - x| + 6$. Find $f(g(h(1)))$.
11. Given: $f(x) = (-3)$ and given $g(x) = \sqrt{2x}$ and given $h(x) = |4x| - 12$. Find $f(h(g(18)))$.
12. Are compositions commutative? In other words, does $f(g(x)) = g(f(x))$?
13. Given: $f(x) = -2^2 - 5x$ and $h(x) = 3x + 2$. Find $f(h(x))$.
14. Two functions are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$ If $f(x) = x + 3$, find its inverse: $g(x)$
15. A toy manufacturer has a new product to sell. The number of units to be sold, n, is a function of the price p such that: $n(p) = 30 - 25p$. The revenue r earned from the sales is a function of the number of units sold n such that: $r(n) = 1000 - \frac{1}{4}x^2$. Find the function for revenue in terms of price, p.

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.16.

Vocabulary

Term	Definition
composite function	A composite function is a function h(x) formed by using the output of one function g(x) as the input of another function f(x). Composite functions are written in the form h(x)=f(g(x)) or h=f∘g.
domain	The domain of a function is the set of x-values for which the function is defined.
Function	A function is a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.
Function composition	Function composition involves 'nested functions' or functions within functions. Function composition is the application of one function to the result of another function.
input	The input of a function is the value on which the function is performed (commonly the x value).
Output	The output of a function is the result of the operations performed on the independent variable (commonly x). The output values are commonly the values of y or f(x).
Range	The range of a function is the set of y values for which the function is defined.

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SECTION OVERVIEW

1.7: Models of Functions

Topic hierarchy

1.7.1: Linear and Quadratic Models

1.7.2: Cubic Models

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1.7.1: Linear and Quadratic Models

Linear and Quadratic Models

Learning to express real-life situations as mathematical functions allows seemingly complex ideas and actions to be broken down into smaller, simpler parts and analyzed.

How might you express the following mathematically?

Two brothers decide to race home from school, taking different routes. The second brother leaves 5 minutes after the first, and both arrive at home at the same time.

Linear and Quadratic Models

Linear Models

The simplest functions are generally linear models. For instance, the equation $y = 3x$ could be used to represent how much money you would bring in if you sold x boxes of cookies for \$3 per box. Many situations can be modeled with linear functions. The key idea is that some quantity in the situation has a constant rate of change. In the cookie-selling example, every box costs \$3.00. Therefore the profits increase at a constant rate. In sum, linear functions are used to **model** a situation of constant change, either increase or decrease.

Quadratic Models

Quadratic functions may also be used as models. For example, the function $A(x) = 50x - \frac{x^2}{2}$ could be used to represent the area of a rectangular plot of land enclosed on three sides by 100 feet of fence. This model could be used to identify the maximum possible area for the plot of land.

Another example is a situation in which a ball is tossed into the air. The ball will travel up, and then it will travel down until it hits the ground. How high will the ball go? When will it reach the ground? This kind of situation can be modeled by a function of the form $h(t) = -16t^2 + v_0t + h_0$. The variable t represents the time since the ball was thrown. The coefficient v_0 represents the initial velocity of the ball, and the constant h_0 represents the initial height of the ball. The constant -16 comes from the force of gravity which pulls the ball down, which is why it is negative. Example 3 shows a specific situation of this form.



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Examples

Example 1

Solution

Earlier, you were asked to come up with a mathematical model for the following statement.

Two brothers decide to race home from school, taking different routes. The second brother leaves 5 minutes after the first, and both arrive at home at the same time.

There are a number of different ways to model the information, depending on what part(s) of the information you choose to use. A couple of examples include:

If t = the time the second brother took to get home, then $t + 5$ = the time the first brother took.

If t = the time the first brother took to get home, then $(t + (t - 5))/2$ represents the average time to run home.

Your model may be similar, or may be written differently, but should compare different values given in the story problem.

Example 2

You run a lawn mowing business, and charge $15 per lawn. Write a linear function to describe the amount of money made as a function of the number of lawns mowed.

Solution

If you express the number of lawns mowed as l, then $15 multiplied by l would represent the total money made based on the number of lawns, therefore:

total income = 15*l*

Example 3

You drop a rock off the edge of a cliff, and time the fall in seconds. Determine the function that models the height of the cliff as a function of falling time in seconds.

Solution

With an initial velocity of 0, the height of the cliff may be calculated with the formula: $h(t) = 16t^2$.

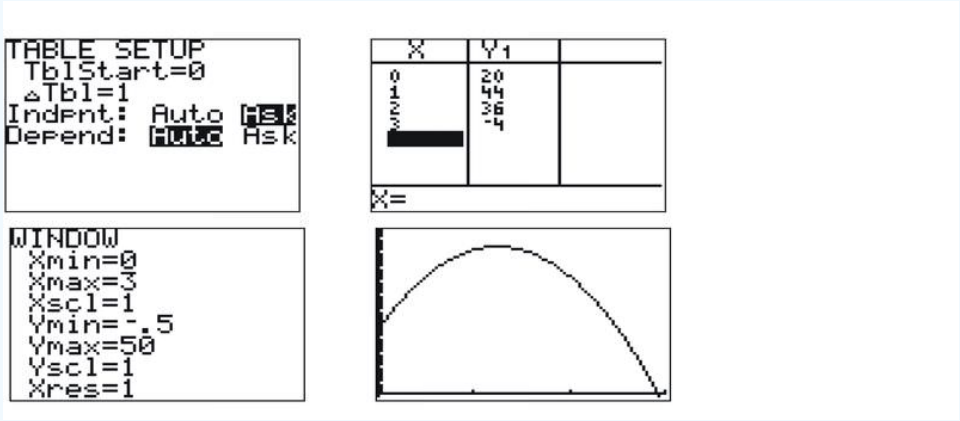
Example 4

You are standing on the roof of a building that is 20 feet above the ground. You toss a ball into the air with an initial vertical velocity of 40 ft/sec, so that it will land on the ground, not on the roof. How high will the ball go, and when will it reach its maximum height? When will the ball hit the ground?

Solution

First we need to write a function to model the situation. Using the general form of the equation given above, we can write the function $h(t) = -16t^2 + 40t + 20$, where h(t) represents the height above the ground.

To answer the first question, we need to examine the graph of the function. If you graph this equation on your calculator, you will need to determine a good viewing window. One way to start to determine a good window is to take into account the *y*-intercept of the function. In this case, the *y*-intercept is (0, 20). Think about what kind of function this is: a parabola, facing downwards. This fact should lead you to think that we need to look at *y*-values *well above 20*. It is often useful to look at a table of values. Using the "ask" capability of the calculator, if you input *x* values of 1, 2, and 3, you will see that the function goes up to 44 at $x = 1$. The maximum value is most likely somewhere near $x = 1$. Press WINDOW, and set xmin = 0, xmax = 3. Then set Ymin = 0 (or a little less, if you want to see the *y*-axis). Ymax should be no less than 44, though you may want to make it larger, such as 50 or more, just to be sure that you can see the vertex. Once you have set the window, press GRAPH.



Now you should see the parabola. To identify the coordinates of the vertex, you can use the MAX function in the CALC menu. Remember that the calculator will ask you to input a left bound, a right bound, and a guess for the maximum. If you use the MAX function, you should find that the coordinates of the vertex are (1.25, 45). This means that 1.25 seconds after the ball is thrown into the air, it reaches a maximum height of 45 feet.



Now, to answer the second part of the question, we need to determine when the height of the ball is 0. Graphically, we are looking for the *x*-intercept of the parabola. If you return to the GRAPH screen, you should see that the *x*-intercept is around 3. If we want to determine the exact value, or at least a good approximation of the *x*-intercept, we can use the ZERO function. Press 2nd TRACE to get the CALC menu, and choose option 2, ZERO. Like the MAX function, you need to input a left bound, a right bound, and a guess, although the guess is optional – just press ENTER. (Note that the calculator works this way because it is asking you to identify which *x*-intercept to calculate. The parabola has two x-intercepts, and other functions may have more.) Using the ZERO function, you should find that the *x*-intercept is approximately 2.93. This means that the ball reaches the ground in just under 3 seconds.

Example 5

The Arlington Freshmen class wants to have a fundraiser. The class wants to buy a number of $4.00 flip-flops and $5.00 baseball hats, and has a total of $100 to spend.

Solution

To express this information as a function, remember that the question specified that there was $100 to spend, and that any money not spent on hats (at $5 ea) was spent on flip-flops (at $4 ea).

a. If *f* represents the number of flip-flops and *b* represents the number of baseball hats, write a function to represent the number of flip-flops purchased as a function of unspent monies from baseball hats.

$$\therefore f = \frac{\$100 - 5h}{4}$$

b. Using your equation from (a), determine the number of baseball hats that can be bought if 10 flip flops were purchased.

To calculate how many hats could be bought if 10 pairs of flip flops were purchased, substitute 10 in for *f*, and solve for *h*:

$$10 = \frac{100 - 5h}{4}$$
$$40 = 100 - 5h$$
$$5h = 60$$
$$h = 12$$

Therefore, if 10 pairs of flip flops were purchased, there would be money left over to buy 12 baseball caps.

Example 6

Studies of the metabolism of alcohol consistently show that blood alcohol content (BAC) declines linearly, after rising rapidly after initial ingestion. In one study, BAC in a fasting person rose to about 0.018 % after a single drink. After an hour the level had dropped to 0.010 %.

Solution

In order to answer the question, you must express the relationship as an equation and then use the equation.

First, define the variables in the function and create a table.

The two variables are time and BAC.

Time	BAC
0	0.018%
1	0.010%

Next, calculate the rate of change.

Time	BAC	Rate of change
0	0.018%	0
1	0.010%	(0.008%)

This rate of change means when the time increases by 1, the BAC *decreases* (since the rate of change is negative) by .008. In other words, the BAC is decreasing .008% every hour. Since we are told that BAC declines linearly, we can assume that figure stays constant.

a. Write an equation relating BAC to time in hours after drinking t.

Now we can write an equation with b representing BAC and t, the time in hours:

$b=-.008t+.018$

b. Assuming that BAC continues to decline linearly (meaning at a constant rate of change), approximately when will BAC drop to 0.002%?

To learn when will the BAC reach .002%, substitute .002 in for b and solve for t.

$.002=-.008t+.018$

$-.016=-.008t$

$t=2$

Therefore the BAC will reach .002% after 2 hours.

Review

1. From 2002 - 2009 the number of gas stations in a certain country increased by 100 stations per year. In 2004 there were 1100 gas stations. Write a linear equation for the number of gas stations, (y), as a function of time, (t,) where t = 0 represents the year 2002.

Find the vertex of the following quadratic functions and then graph them.

2. $f(x)=2x^2-6x+11$
3. $f(x)=3(x+5)^2-2$

At the local downtown 4th of July fireworks celebration, the fireworks are shot by remote control into the air from a pit in the ground that is 12 feet below the earth's surface.

4. Find an equation that models the height of an aerial bomb t seconds after it is shot upwards with an initial velocity of 80 ft/sec. Assume that the bomb decelerates at a rate of 30 ft/sec each second
5. Find the vertex of the quadratic function.
6. What is the maximum height above ground level that the aerial bomb will reach?
7. How many seconds after it was launched will it take to reach that height?
8. A rock is thrown from the top of a 763ft tall building. The distance, in feet, between the rock and the ground t seconds after it is thrown is given by $d=-16t^2-2t+763$. How long after the rock is thrown is it 430 feet from the ground?
9. Use the vertical motion formula $h=-16t^2+v_0t+s$ to find the number of seconds it takes for a rocket launched from the ground with a starting velocity of 96 ft/s to reach an altitude of 45 ft. Round answers to the nearest tenth.
10. The function $P=0.0089t^2+1.1149t+78.4491$ models the United States population in millions since 1900. Use the function P to predict the year in which the population exceeds 1 billion.
11. For which value of x is $f(x)=-10$ if $f(x)=-4x^2+3x$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.17.

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Vocabulary

Term	Definition
linear model	A linear model is a linear function representing a situation that involves a constant rate of change.
Model	A model is a mathematical expression or function used to describe a physical item or situation.
quadratic model	A quadratic model is a function representing a situation with a rate of change that is a squared value. The graph of a quadratic function is a parabola.

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1.7.2: Cubic Models

Cubic Models

How do you write a math problem to represent a physical object?

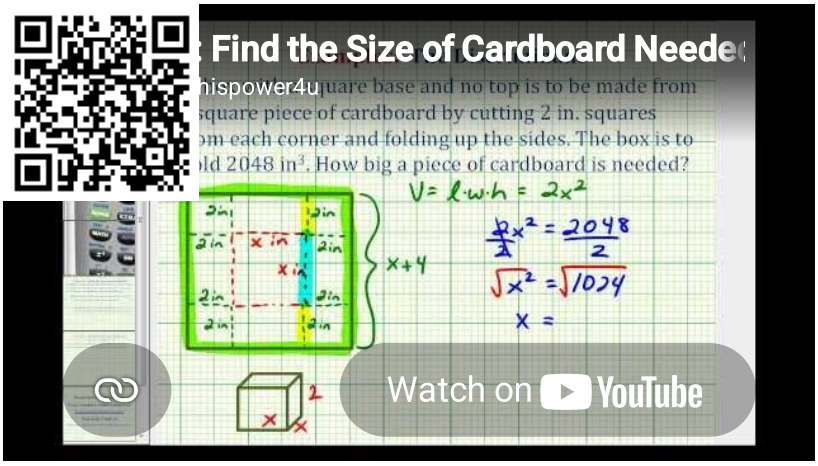
The video below, and the first portion of the lesson thereafter, both describe different challenges involving modeling a box created by cutting corners out of a flat sheet of cardboard and then folding up the sides. In the video, the specific problem is to identify the size of the cardboard needed to result in a given volume. In the lesson, the challenge is to identify the greatest volume possible from a given size sheet of cardboard.

How would you mathematically **model** a divider to split the volume of the box described in the lesson into multiple spaces?

Cubic Models

Cubic functions and piece-wise functions can be used to model real-world situations, allowing you to identify missing bits of information you may need to complete a project. Cubic functions are commonly used to model three-dimensional objects to allow you to identify a missing dimension or explore the result of changes to one or more dimensions.

Piece-wise functions may be used to model the interactions of multiple items each previously modeled by a simpler function.



Examples

Example 1

Solution

Earlier, you were asked to create a different model describing the cardboard box.

There are a number of different possibilities, for instance:

In the case of the box in the video above: $t(v) = v/h$ could be used to describe the area of the top of the box as a function of the volume.

In the case of the box in the lesson text: d(x)=(12−x)x or d(x)=(8−x)x could be used to find the area of a divider run the long way or the wide way across the box as a function of the length of one side of the square corner cut-outs.

Example 2

Consider a situation in which a rectangular piece of cardboard is folded into a box. The folding is made possible by cutting squares out of the four corners of the cardboard.

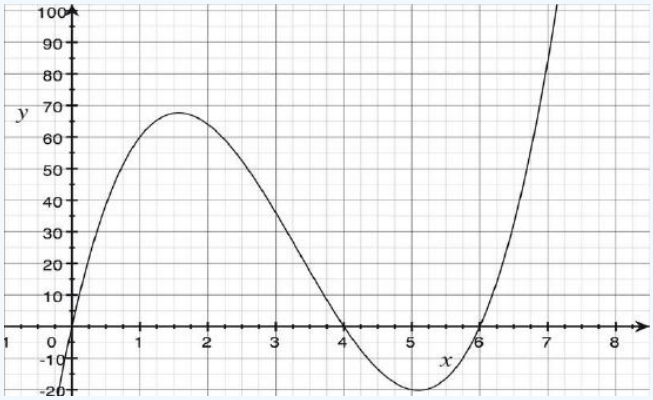
Calculate the maximum volume possible of a box made from a sheet of cardboard 12" x 8".

The function $v(x) = (12 - 2x)(8 - 2x)x$ could be used to represent the volume of the box as a function of *x*, the side-length of the squares cut out of the corners. If we multiply out the factors of this function we can verify that this is a cubic function:

$$v(x) = (12 - 2x)(8 - 2x)x => 4x^3 - 40x^2 + 96x$$

Solution

This function could be used to find the maximum possible volume of the box. We can also analyze the graph to understand how the volume changes as a function of *x*.



When analyzing the function to determine the maximum volume of the box, we only look at the portion of the graph that looks "parabolic". This is because the function ceases to model the situation if *x* is more than 4. If we cut out {4x4} squares, we would cut out the entire short side of the cardboard rectangle, and we would not be able to make a box. Focusing then on the interval (0, 4) we can see that the volume of the box increases, and then decreases.

If we are using a graphing calculator and want to know the volume of a box with particular dimensions, we can trace on the graph, input values into the table, or take advantage of the graph being in trace mode. That is, if you press GRAPH to view the graph, and then press TRACE, you can input *x* values. For example, say that you wanted to cut out squares of side-length 2.5. Press TRACE, then press 2.5, then press ENTER. At the bottom of the screen you will see $x = 2.5$ and $y = 52.5$. This tells you that the volume of the box will be $52.5 in^3$.

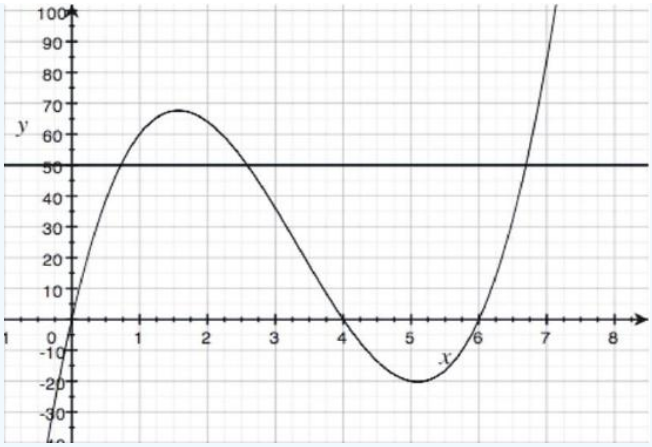
Example 3

Using the information given in Example 2 above, calculate the size of the square corners to cut out to result specifically in a volume of $50\ in^3$.

Solution

One way to determine the value of x is to graph the constant function $y = 50$, and find the point where the volume function intersects $y = 50$. Press Y= and enter 50 into *Y2*. Now press GRAPH. You should see the horizontal line $y = 50$ intersecting the volume function in several places. We are interested in the two intersection points in the interval (0, 4).

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To find a good approximation of an intersection point, trace close to one of the two points. If you trace close to the first point, you will see that it is around $x = 0.8$. Now press 2nd, CALC, and choose option 5, INTERSECT. The calculator will send you back to the graph screen, and ask you to choose the first curve. (The calculator does this in case you have more than two functions graphed at the same time.) You should see the cubic equation at the top of the screen. Press ENTER, and the calculator will ask you for the second curve. You should see $y = 50$ at the top of the screen. Press ENTER, and then enter a guess. (If you have already traced close to an intersection point, and you only have two functions graphed, you can simply press ENTER three times.) You should now see the coordinates of the intersection point at the bottom of the screen: ***x* is approximately 0.723**. (You can use the same process to estimate the other intersection point.)

A cubic function can be used to model situations that involve volume, but can also be used to model situations that follow particular growth patterns.

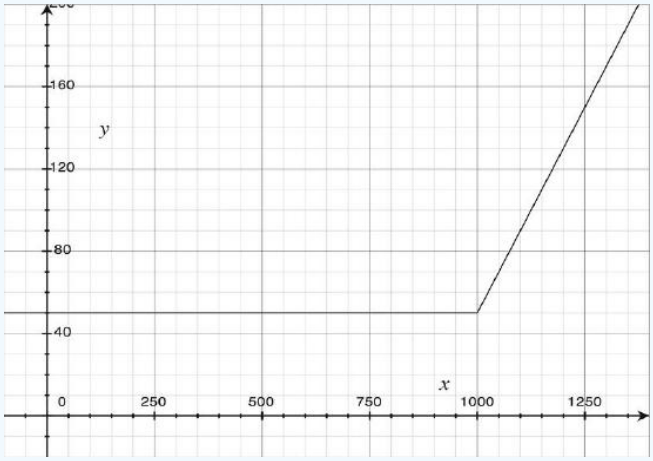
Example 4

Piece-wise functions can be used to describe situations in which quantitative relationships are different in different intervals within the domain of the function. For example, consider a situation in which a wireless provider offers customers a monthly plan that costs $50, but then charges $0.40 cents per minute for every minute over 1000 included daytime minutes.

Solution

Model the monthly cost, C, of the plan as a function of m, the number of daytime minutes you use:

$$C(m) = \begin{cases} 50, & m \leq 1000 \\ 50 + 0.40(m - 1000), & m > 1000 \end{cases}$$



This function is comprised of a constant function, and a linear function with slope 0.40. If in a given month you use 1000 minutes or fewer, your monthly cost is a constant $50. If you use more than 1000, each additional minute influences the value of C. For example, if you use 1,020 minutes, your cost is:

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		$C(1020) = 50 + 0.40(1020 - 1000)$
		$= 50 + 0.4(20)$
		$= 50 + 8$
		$= \$58.00$

It is important to note that in this kind of situation, the time used may to be rounded to the nearest minute. So, for example, if you use 20.5 minutes, you will be charged for 21 minutes. This is an example of a non-continuous, or discontinuous, function, where there are definite *steps* from value to value rather than a smooth line connecting all possible values.

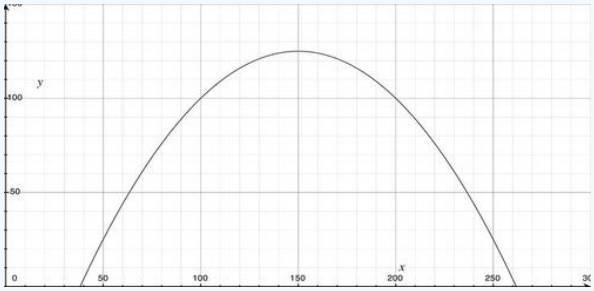
Example 5

The profits for a business can be determined by subtracting the costs from the revenue. Suppose the revenue of a business is modeled by the function $R(x)=5x-0.01x^2$, and the costs of manufacturing the product is modeled by $C(x)=100+2x$, where x is the number of units of the product.

a. Write a function $P(x)$ to model the company's profits.
b. Graph $P(x)$ and determine the maximum profit.

Solution

a. $P(x) = R(x) - C(x) = -0.01x^2 + 3x - 100$
b.



[Figure1]

The maximum profit is 125 (usually in thousands, or another larger unit!)

Example 6

Express the following situation as a composition of functions: You are running a small business making wooden jewelry boxes. It costs you \$5.00 per unit to produce wooden boxes, plus an initial investment of \$300 in other materials. It then costs you an additional \$2.00 per box to decorate the boxes.

Solution

Initial cost function: $C_1(x) = 5x + 300$

Second cost function $C_2(x) = 2x$

Composition: $C(x) = C_1(C_2(x)) = 5(2x) + 300 = 10x + 300$

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Review

1. A box is to be made by cutting squares out of the corners of a rectangular piece of cardboard. The dimensions of the cardboard are *n* inches by *m* inches. Assume that $n > m$. a. Write a model for the volume of the box. b. What is the largest square that can be cut out of the corners of the cardboard?
2. Suppose you were told that you could use a single sheet of paper for notes on your math final. The instructor says that you may use any sheet you like, but the shape must be a quadrangle and the perimeter may not exceed 45in. a) What dimensions should you use to provide the greatest area for your notes? b) How does a graphing calculator help to simplify this question?
3. Is $f(x) = -3^{4x}$ a power function?
4. Is $f(x) = \sqrt[3]{8x^5}$ a power function?
5. Is $g(x) = 7 \cdot 2^x$ a power function? If not, why not?
6. Is $h(x) = 2x^{-5}$ a power function? If not, why not?
7. The volume *v* of a sphere varies directly as the cube of the radius *r*. When the radius of a sphere is 6 cm, the volume is 904.779cm^3. What is the radius of a sphere whose volume is 268.083cm^3?
8. The force of gravity (F) acting on an object is inversely proportional to the square of the distance *d* from the object to the center of the earth. Write an equation that models this situation.

Sue and Betty gathered the data in the table below using a 100-watt light bulb and a Calculator-Based Laboratory(CBL) with a light-intensity probe.

Light Intensity Data for a 100w light bulb

Distance (m)	Intensity (W/m^2)
1.0	7.95
1.5	3.53
2.0	2.01
2.5	1.27
3.0	0.90

9. Use your calculator to find the power regression model of the data.
10. Describe the relationship between the intensity and distance modeled with the equation in Q #9
11. Use the regression model from Q #9 to predict the intensity of an object 2.75 meters away.

Number of births to females in the US under 15 years of age for years 1990 - 2005

Year	CPI Annual Change %	Year	CPI Annual Change %	Year	CPI Annual Change %
1990	5.4	1996	3.0	2002	1.6
1991	4.2	1997	2.3	2003	2.3
1992	3.0	1998	1.6	2004	2.7
1993	3.0	1999	2.2	2005	3.4
1994	2.6	2000	3.4		
1995	2.8	2001	2.8		

12. Find the cubic function that models the data, where (y) is the number of thousands of births and (x) is the number of years from 1970.
13. Use the function to estimate the number of such births that occured in 2005.
14. When do we know this model is no longer valid?

Review (Answers)

To see the Review answers, open this PDF file and look for section 1.18.

Vocabulary

Term	Definition
Cubic model	A cubic model is a mathematical function including an x^3 term, used to describe a real-world situation, such as the volume of a three-dimensional object.
Model	A model is a mathematical expression or function used to describe a physical item or situation.
piecewise functions	A piecewise function is a function that pieces together two or more parts of other functions to create a new function.

Image Attributions

1. [Figure 1]
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CHAPTER OVERVIEW

Unit 2: Polynomial and Rational Functions

2.1: Quadratic Functions

2.1.1: Methods for Solving Quadratic Functions

2.1.2: Graphs of Quadratic Functions

2.2: Graphing Polynomials

2.2.1: Graphs of Polynomials Using Transformations

2.2.2: Graphs of Polynomials Using Zeros

2.2.3: Graphing Calculator to Analyze Polynomial Functions

2.3: Analysis of Graphs of Rational Functions

2.3.1: Graphs of Basic Rational Functions

2.3.2: Graphs of Rational Functions when the Degrees are Equal

2.3.3: Graphs of Rational Functions when the Degrees are not Equal

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2.6: Finding Zeros of Polynomials

2.6.1: Long Division of Polynomials

2.6.2: Synthetic Division of Polynomials

2.6.3: Real Zeros of Polynomials

2.6.4: Intermediate Value Theorem

2.6.5: Fundamental Theorem of Algebra

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SECTION OVERVIEW

2.1: Quadratic Functions

Topic hierarchy

2.1.1: Methods for Solving Quadratic Functions

2.1.2: Graphs of Quadratic Functions

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2.1.1: Methods for Solving Quadratic Functions

Factoring Polynomials in Quadratic Form

The volume of a rectangular prism is $10x^3-25x^2-15x$. What are the lengths of the prism's sides?

Factoring Polynomials in Quadratic Form

The last type of factorable polynomial are those that are in quadratic form. **Quadratic form** is when a polynomial looks like a trinomial or binomial and can be factored like a quadratic. One example is when a polynomial is in the form ax^4+bx^2+c. Another possibility is something similar to the difference of squares, a^4-b^4. This can be factored to $(a^2-b^2)(a^2+b^2)$ or $(a-b)(a+b)(a^2+b^2)$. Always keep in mind that the greatest common factors should be factored out first.

Let's factor the following polynomials.

1. $2x^4-x^2-15$

   This particular polynomial is factorable. First, $ac=-30$. The factors of -30 that add up to -1 are -6 and 5. Expand the middle term and then use factoring by grouping.

   $2x^4-x^2-15$

   $2x^4-6x^2+5x^2-15$

   $2x^2(x^2-3)+5(x^2-3)$

   $(x^2-3)(2x^2+5)$

   Both of the factors are not factorable, so we are done.

2. $81x^4-16$

   Treat this polynomial equation like a difference of squares.

   $81x^4-16$

   $(9x^2-4)(9x^2+4)$

   Now, we can factor $9x^2-4$ using the difference of squares a second time.

   $(3x-2)(3x+2)(9x^2+4)$

   $9x^2+4$ cannot be factored because it is a sum of squares. This will have imaginary solutions.

   Now, let's find all the real-number solutions of $6x^5-51x^3-27x=0$.

   First, pull out the GCF among the three terms.

   $6x^5-51x^3-27x=0$

   $3x(2x^4-17x^2-9)=0$

   Factor what is inside the parenthesis like a quadratic equation. $ac=-18$ and the factors of -18 that add up to -17 are -18 and 1. Expand the middle term and then use factoring by grouping.

   $6x^5-51x^3-27x=0$

   $3x(2x^4-17x^2-9)=0$

   $3x(2x^4-18x^2+x^2-9)=0$

   $3x[2x^2(x^2-9)+1(x^2-9)]=0$

   $3x(x^2-9)(2x^2+1)=0$

   Factor x^2-9 further and solve for x where possible. $2x^2+1$ is not factorable.

   $3x(x^2-9)(2x^2+1)=0$

$3x(x-3)(x+3)(2x^2+1)=0$

$x=-3,0,3$

Examples

Example 1

Earlier, you were asked to find the lengths of the prism's sides.

Solution

To find the lengths of the prism's sides, we need to factor $10x^3-25x^2-15x$.

First, pull out the GCF among the three terms.

$10x^3-25x^2-15x$

$5x(2x^2-5x-3)$

Factor what is inside the parenthesis like a quadratic equation. $ac=-6$ and the factors of -6 that add up to -5 are -6 and 1.

$5x(2x^2-5x-3)=5x(2x+1)(x-3)$

Therefore, the lengths of the rectangular prism's sides are $5x$, $2x+1$, and $x-3$.

Example 2

Factor: $3x^4+14x^2+8$.

Solution

$ac=24$ and the factors of 24 that add up to 14 are 12 and 2.

$3x^4+14x^2+8$

$3x^4+12x^2+2x^2+8$

$3x^2(x^2+4)+2(x^4+4)$

$(x^2+4)(3x^2+2)$

Example 3

Factor: $36x^4-25$.

Solution

Factor this polynomial like a difference of squares.

$36x^4-25$

$(6x^2-5)(6x^2+5)$

6 and 5 are not square numbers, so this cannot be factored further.

Example 4

Find all the real-number solutions of $8x^5+26x^3-24x=0$.

Solution

Pull out a 2x from each term.

$8x^5+26x^3-24x=0$

$2x(4x^4+13x-12)=0$

$2x(4x^4+16x^2-3x^2-12)=0$

$2x[4x^2(x^2+4)-3(x^2+4)]=0$

$2x(x^2+4)(4x^2-3)=0$

Set each factor equal to zero.

$4x^2-3=0$

$2x=0$

$x^2+4=0$

and $x^2=\frac{3}{4}$

$x=0$

$x^2=-4$

$x=\pm\frac{\sqrt{3}}{2}$

Notice the second factor will give imaginary solutions.

Review

Factor the following quadratics completely.

1. x^4-6x^2+8
2. x^4-4x^2-45
3. x^4-18x^2+45
4. $4x^4-11x^2-3$
5. $6x^4+19x^2+8$
6. x^4-81
7. $16x^4-1$
8. $6x^5+26x^3-20x$
9. $4x^6-36x^2$
10. $625-81x^4$

Find all the real-number solutions to the polynomials below.

11. $2x^4-5x^2-12=0$
12. $x^4-16=0$
13. $16x^4-49=0$
14. $12x^6+69x^4+45x^2=0$
15. $3x^4+17x^2-6=0$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.8.

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Vocabulary

Term	Definition
Factor to Solve	"Factor to Solve" is a common method for solving quadratic equations accomplished by factoring a trinomial into two binomials and identifying the values of x that make each binomial equal to zero.
factored form	The factored form of a quadratic function f(x) is $f(x)=a(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of the function.
Factoring	Factoring is the process of dividing a number or expression into a product of smaller numbers or expressions.
Quadratic form	A polynomial in quadratic form looks like a trinomial or binomial and can be factored like a quadratic expression.
quadratic function	A quadratic function is a function that can be written in the form $f(x)=ax^2+bx+c$, where a, b, and c are real constants and $a\neq 0$.
Roots	The roots of a function are the values of *x* that make *y* equal to zero.
standard form	The standard form of a quadratic function is $f(x)=ax^2+bx+c$.
Vertex form	The vertex form of a quadratic function is $y=a(x-h)^2+k$, where (h,k) is the vertex of the parabola.
Zeroes of a Polynomial	The zeroes of a polynomial f(x) are the values of x that cause f(x) to be equal to zero.

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2.1.2: Graphs of Quadratic Functions

Graphs of Quadratic Functions

In prior lessons, we have discussed the **standard form** of a **quadratic function**: $f(x)=ax^2+bx+c$. You may have seen other forms, such as the **vertex form**, or the **factored form**. Why are there so many common ways to write the same equations? Why should we learn all these different forms if the standard form is the most common anyway?

Graphing Quadratic Functions

Quadratic Functions

A function f defined by $f(x)=ax^2+bx+c$, where a,b, and c are real numbers and $a\neq 0$, is called a **quadratic function**.

The defining characteristic of a quadratic function is that it is a polynomial whose highest exponent is 2.

There are several ways to write quadratic functions:

- **standard form**, the form of the quadratic function above: $f(x)=ax^2+bx+c$
- **vertex form**, commonly used for quick sketching: $f(x)=a(x-h)^2+k$
- **factored form**, excellent for finding *x*-intercepts: $f(x)=a(x-r_1)(x-r_2)$

The **y−*intercept*** of a quadratic function in standard form is (0,c) and it is found by substituting for x in $f(x)=ax^2+bx+c$.

Summary of Vertex Form

Given a quadratic function in **vertex form**: $f(x)=a(x-h)^2+k$:

- The vertex is at (h,k)
- The **parabola** opens up if $a>0$
- The parabola opens down if $a<0$
- The parabola is narrower than $y=x^2$ if $|a|>1$
- The parabola opens wider than $y=x^2$ if $|a|<1$

Vertex of a Parabola

In the **standard form** of a quadratic function, the x−coordinate of the vertex of the parabola is given by the equation:

$$x=-\frac{b}{2a}$$

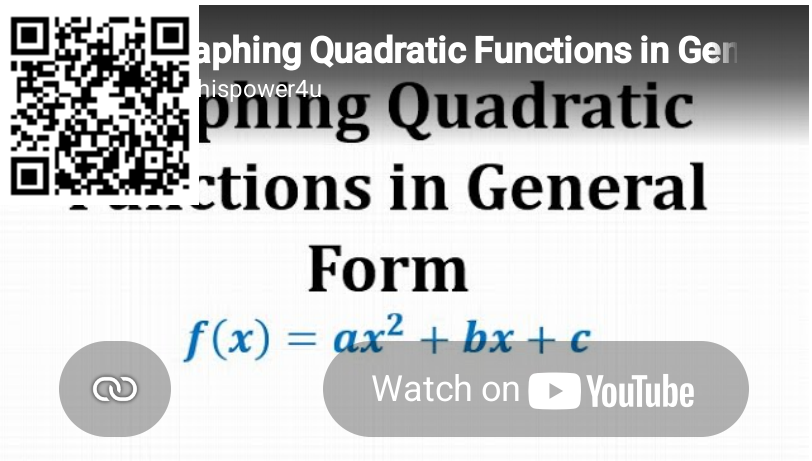
The y−coordinate of the vertex is found with:

$$y=f\left(-\frac{b}{2a}\right)$$

Axis of Symmetry of a Parabola

A parabola has reflective symmetry about a vertical line through the vertex.

The vertical line $x=-\frac{b}{2a}$ is also the parabola's **axis of symmetry**.



Examples

Example 1

Graph $g(x)=x^2+6x+7$ using **transformations**.

Solution

First, complete the square to write this function in vertex form. Add and subtract $\left(\frac{b}{2}\right)^2$ to the right hand side of the equation:

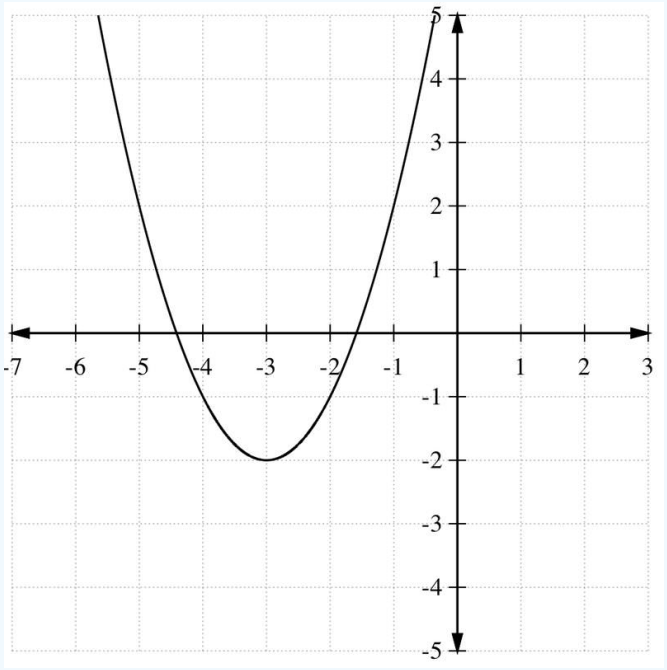
$g(x)=x^2+6x+7$

$=x^2+6x+9+7-9$

Now, factor the right hand side:

$g(x)=(x+3)^2-2$

Thus, a=1 and the vertex of this parabola is (-3, -2). We know that the parabola opens up with the same width as $y=x^2$ and it has a **minimum** value at the vertex. The graph of the parabola is below.



Example 2

Find the vertex and graph the quadratic function $g(x)=x^2-8x+12$.

Solution

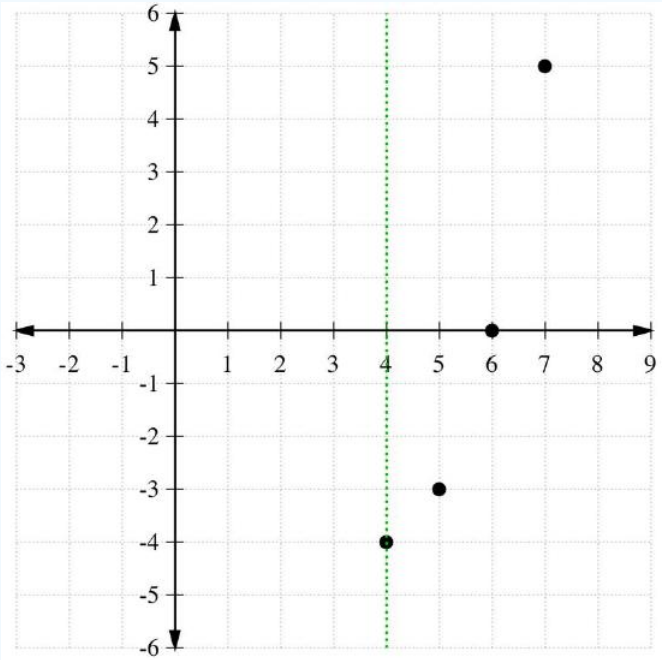
The x−coordinate of the vertex is $x=-\frac{-8}{2}=4$.

The y−coordinate of the vertex is $g(4)=(4)^2-8(4)+12=16-32+12=-4$

Thus the vertex is at (4, -4).

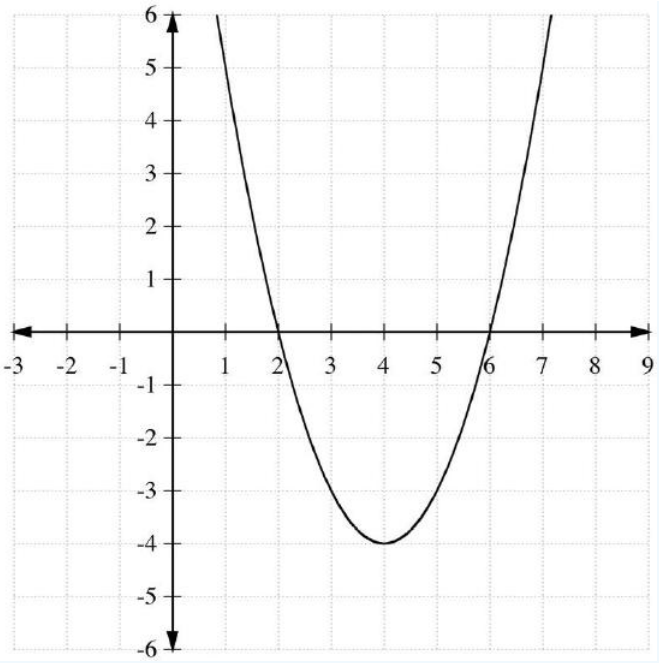
To graph the parabola, we will make a table of points staring with the x−coordinate of 4:

x	y=g(x)
4	-4
5	-3
6	0
7	5



Now we can use the symmetry of g(x) to fill in this table for g(3). Note that g(3)=g(5)=−3. Likewise, g(2)=g(6)=0. The final graph is below.

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Example 3

Sketch the graph of the function $y=f(x)=x^2+2x-3$.

Solution

Let's first find the intercepts. For the y−intercept, if x=0, then f(0)=−3, or y=−3, so the y−intercept point is (0, -3).

Now, for the x−intercepts, if y=f(x)=0, then $x^2+2x-3=0$, or $x^2+2x-3=(x+3)(x-1)=0$

so that x=−3 and x=1 are the x−intercepts, that is (-3, 0) and (1, 0).

The vertex (extreme point) is at

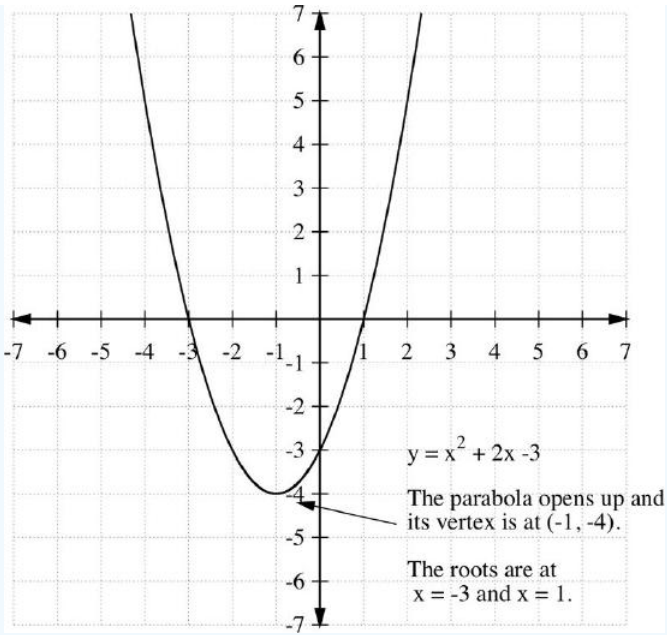
$$x=\frac{-b}{2a}=\frac{-2}{2(1)}=-1$$

Since

$$f(-1)=(-1)^2+2(-1)-3 \\ =-4$$

The vertex is (-1, -4).

Since the coefficient of x^2 is positive, a>0, the extreme point is a minimum and the parabola opens up. From this information, we can make a rough sketch of the parabola containing the points determined above. Notice that the range of the function is y≥−4.



Example 4

Sketch the graph of the quadratic function $f(x)=-x^2+4x$.

Solution

To find the y−intercept, set x=0, and $f(0)=-(0)^2+4(0)=0$

Thus the parabola intercepts the y−axis at the origin.

The x−intercept is obtained by setting y=0. Thus, $-x^2+4x=0$.

Factoring,

$-x^2+4x=-x(x-4)=0$

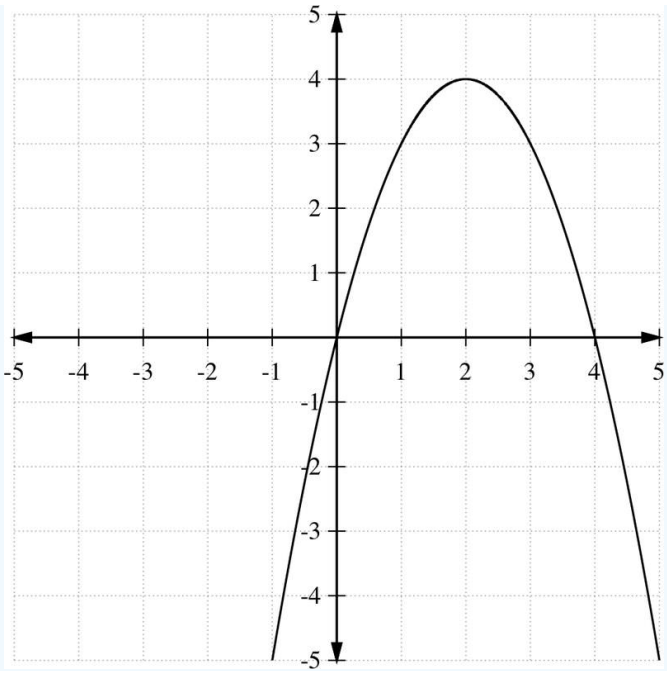
so that x=0 and x=4 are the x−intercepts.

We have a=−1 and b=4, so that the extreme point occurs when

$$x=\frac{-b}{2a}=\frac{-4}{2(-1)}=2$$

Since $f(2)=-(2)^2+4(2)=-4+8=4$, then (2, 4) is the extreme point. It is a **maximum** point since a=−1<0 and the parabola opens down. Finally, the graph can be obtained by sketching a parabola through the points determined above. From graph, the range of the function is y≤4.

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Example 5

Sketch the graph of $y=-3(x-2)^2+1$.

Solution

The equation $y=-3(x-2)^2+1$ is already in vertex form, so graphing is relatively easy:

Recall the when the equation is written in vertex form like this, the vertex is the point (h,k):

vertex form: $y=a(x-h)^2+k$

Our equation: $y=-3(x-2)^2+1$

Examining our equation, we can see the vertex of the parabola is at (2, 1).

To find another point on the parabola solve for an x value.

Since the vertex is at $x = 2$, let's try one unit to the right: $x = 3$.

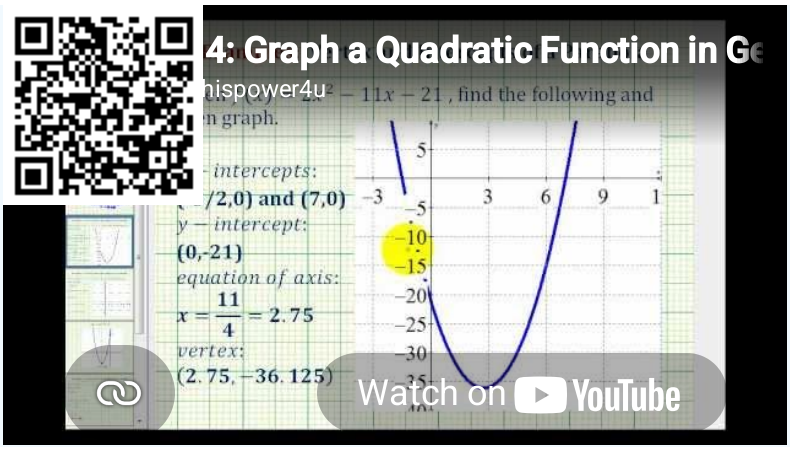
$-3(3-2)^2+1=-2$

$\therefore$ There is a point on the parabola at (3, -2)

Since a parabola has an axis of symmetry that passes through its vertex, we can reflect the point (3, -2) across the axis of symmetry to get another point, (1, -2) also on the parabola.

The graph of $y=-3(x-2)2+1$ is shown below, using vertex (2, 1) and points (3, -2) and (1, -2).

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Review

1. What is the U-Shaped graph of a quadratic function called?
2. Which direction does a parabola open if the leading coefficient (a) is positive?
3. For $y^2=x$ If the coefficient of y is positive, which way does the parabola open?
4. What is the name of the lowest point of a parabola that opens up and the highest point of a parabola that opens down.
5. What is the name of the line passing through the vertex that divides the parabola into two symmetric parts?
6. Sketch the graph of $y=x^2+3$
7. Sketch the graph of $y=-x^2+4x-4$
8. Sketch the graph of $y=2x^2+8x$
9. Consider the following quadratic function: $y=-x^2-2x+1$ a) Which direction does it open? b) What is the vertex? c) Is it stretched in any way?
10. Consider the quadratic functions: $y=2x^2$ $y=4x^2$ $y=6x^2$ Which quadratic function would you expect to have the narrowest parabola? Explain your answer.

Sketch the graph of each function:

11. $y=-x^2$
12. $y=3x^2+6x+1$
13. $y=12x^2+2x+4$
14. $y=(x-3)^2+4$
15. $y=-x^2-8x-17$

The quadratic function $y=-0.05x^2+1.5x$ can be used to represent the path of a football kicked 30 yards down the field. The variable x represents the distance, in yards, the ball has traveled down the field. The height, in yards, of the football in the air is represented by the variable (y).

Use the quadratic function to calculate the height of the ball as it travels down the field. Round your answers to the nearest hundredth of a yard.

Distance Down the Field (yds)	Height in the Air (yds)
0.0	
5.0	
10.0	
15.0	
20.0	
25.0	
30.0	

16. What is the maximum height of the football during the kick?
17. How far down the field has the football traveled when it reaches its maximum height?
18. Use the information in the table to graph the path of the football kick.
19. If you were shown only the graph of this quadratic function, how could you determine the maximum height of the football during the kick and how far down the field the football has traveled when it reaches its maximum height?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.2.

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Vocabulary

Term	Definition
axis of symmetry	The axis of symmetry of a parabola is a vertical line that passes through the vertex of the parabola. The parabola is symmetrical about this line.
factored form	The factored form of a quadratic function f(x) is $f(x)=a(x-r_1)(x-r_2)$, where r_1 and r_2 are the roots of the function.
Intercept	The intercepts of a curve are the locations where the curve intersects the x and y axes. An x intercept is a point at which the curve intersects the x-axis. A y intercept is a point at which the curve intersects the y-axis.
Maximum	The maximum is the highest point of a graph. The maximum will yield the largest value of the range.
Maximum/Minimum	The maximum is the highest point of a function and the minimum is the lowest point of a function.
Minimum	The minimum is the lowest point of a graph. The minimum will yield the smallest value of the range.
Parabola	A parabola is the characteristic shape of a quadratic function graph, resembling a "U".
quadratic function	A quadratic function is a function that can be written in the form $f(x)=ax^2+bx+c$, where a, b, and c are real constants and $a\neq 0$.
standard form	The standard form of a quadratic function is $f(x)=ax^2+bx+c$.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.
Vertex	The vertex of a parabola is the highest or lowest point on the graph of a parabola. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.
Vertex form	The vertex form of a quadratic function is $y=a(x-h)^2+k$, where (h,k) is the vertex of the parabola.

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SECTION OVERVIEW

2.2: Graphing Polynomials

Topic hierarchy

2.2.1: Graphs of Polynomials Using Transformations

2.2.2: Graphs of Polynomials Using Zeros

2.2.3: Graphing Calculator to Analyze Polynomial Functions

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2.2.1: Graphs of Polynomials Using Transformations

Penny has been commissioned to paint a large mural on the side of the humane society. The society has a mascot, an African lion that the society supports by donating money to the local zoo for his care. Penny is expected to sketch the lion, submit the sketch to the board of the humane society for approval, and then scale up the drawing to 22 square feet so it will cover the side of the building.

Shortly after Penny actually begins the job of painting the mural, she is visited by the chairman of the board of the humane society. He tells Penny that the Fire Chief has just notified the society that the additional activity the mural is expected to attract means that the building will need another entrance/exit for fire safety. Unfortunately, that means that the mural will need to be moved up and to the right about 5 feet.

What kinds of **transformations** of complex lines (like the ones in this lesson) will Penny have used by the time she completes the job?

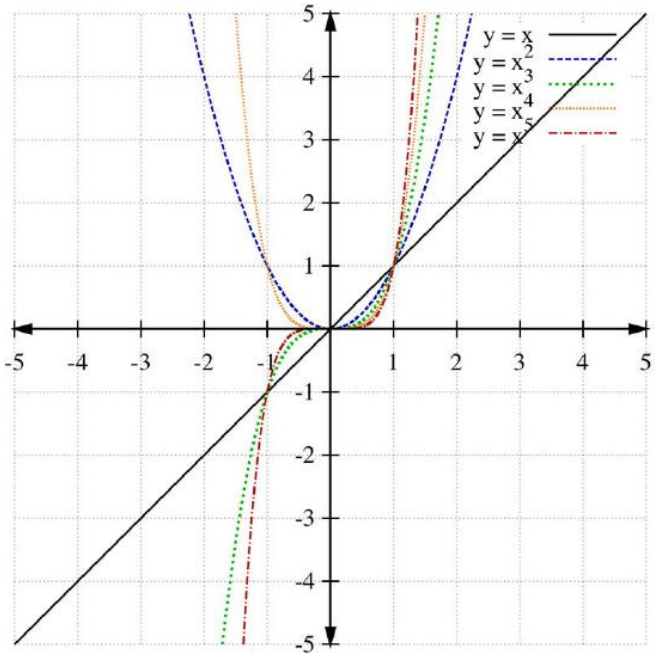
Graphing Polynomials Using Transformations

Polynomial Functions

You have already studied many different kinds of functions, for example linear functions, constant functions, and quadratic functions. All three these functions belong to a larger group of functions called the **polynomial functions**.

The most simple polynomial is called a **power function**. A **power function** is a polynomial of the form $f(x)=ax^n$ where a is a real number and n is an integer with $n \geq 1$.

If n is even, then the power function is also called “even,” and if n is odd, then the power function is “odd.” The graphs of the first five power functions are shown below.



Notice that each power function has only one x− and y−intercept, the origin (0, 0).

The **end behavior** of a function describes the y−values as x gets very large ($x \to \infty$ in symbols) or as x gets very small ($x \to -\infty$).

- For even powers n, the power function $f(x)=ax^n$ is U-shaped (like a parabola) and as $x \to \infty, f(x) \to \infty$. Likewise as $x \to -\infty, f(x) \to \infty$.
- For odd powers n, the power function goes from the third quadrant to the first quadrant (like the line y=x). As $x \to \infty, f(x) \to \infty$, and as $x \to -\infty, f(x) \to -\infty$.

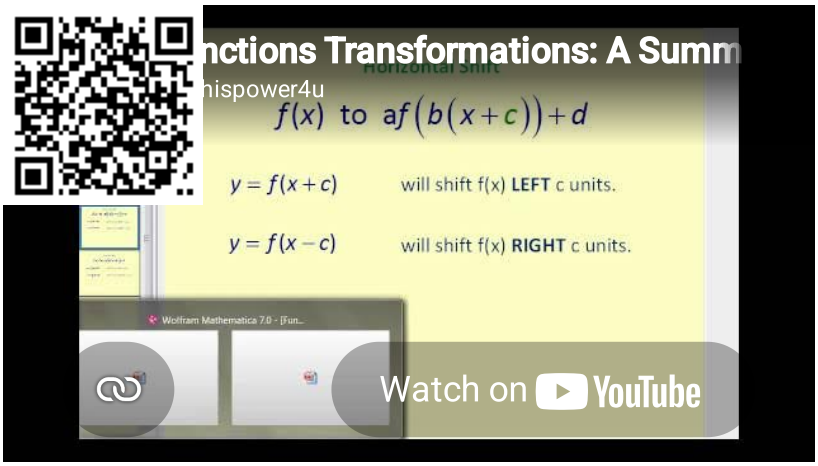
As with quadratics and polynomials, the leading coefficient a changes the vertical “**stretching**” of power functions.

Graph Polynomial Functions Using Transformations

Just like quadratics, polynomial functions can be graphed using transformations of a known graph. The basic transformations are vertical and horizontal **shifts** and reflections about the x− and y−axis.

Given a polynomial p(x) and constant real numbers c and a

- p(x)+c is a *vertical* ***shift*** of the graph of p(x) by c units *up* (so the function shifts down if c<0).
- p(x−c) is a *horizontal* shift of the graph of p(x) by c units to the *right*. (So the function shifts left if c<0).
- −p(x) is a **reflection** of the graph of p(x) about the x−axis.
- p(−x) is a reflection of the graph of p(x) about the y−axis.
- ap(x) is a vertical **stretch** by a multiple of a.
- p(ax) is a horizontal **compression** by a multiple of a.



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Examples

Example 1

Earlier, you were asked a question about the kinds of transformations of complex lines that Penny will have used in her sketch by the time she completes it.

Solution

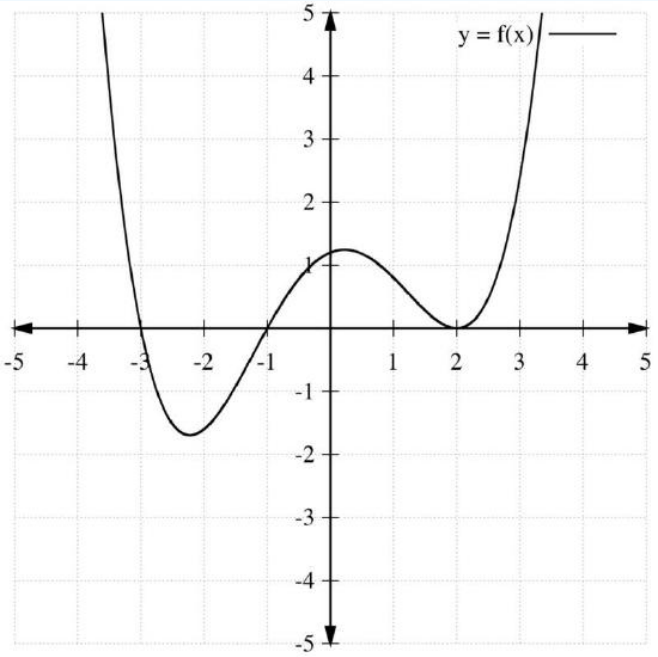
When Penny first sketches the lion, she will be (probably unconsciously) applying a vertical and horizontal compression to the sketch, unless she is using a very large piece of paper!

Once the sketch is approved, Penny will then need to greatly stretch the image vertically and horizontally to make the image big enough to cover the side of the building.

After the Fire Chief visit, Penny was required to apply a horizontal and vertical shift to move the image out of the way of the new door.

Example 2

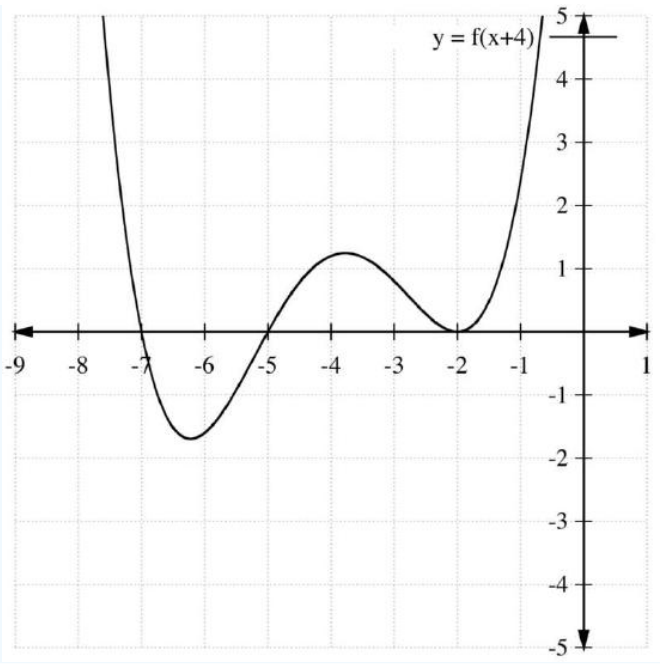
The graph of f(x) is shown below. Use the graph of f(x) to graph f(x+4).



Solution

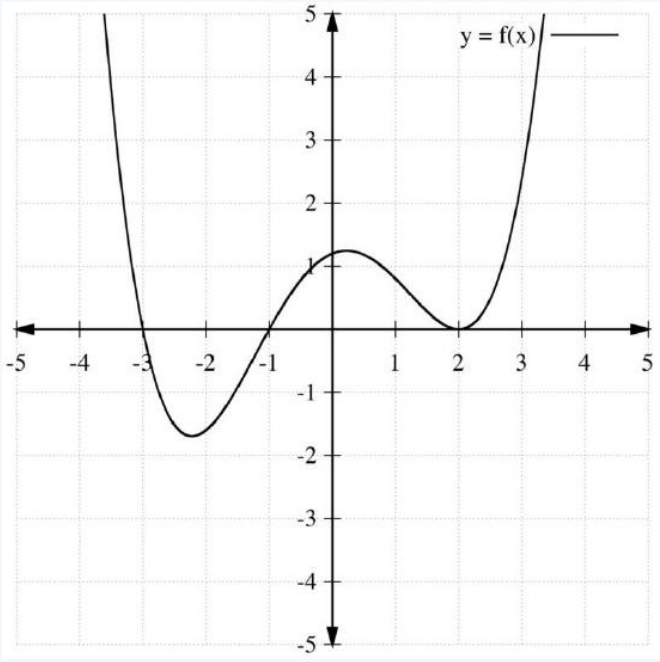
This is a horizontal shift of f(x) to the left by 4 units.

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Example 3

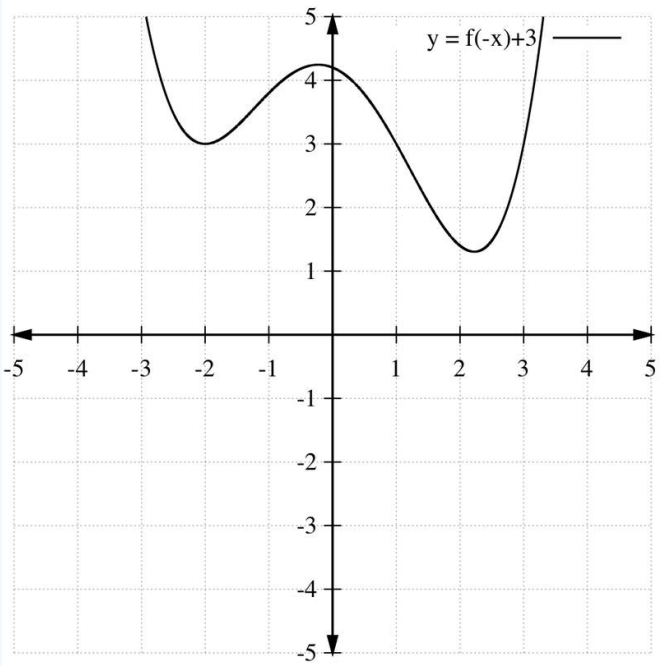
The graph of f(x) is shown below. Use the graph of f(x) to graph f(−x)+3.



Solution

This is a reflection of f(x) about the y−axis and a vertical shift up by 3 units.

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Example 4

Describe the transformations to a graph of the function $y=x^2$ necessary to make a reasonable approximation of $f(x)=-x^4$.

Solution

Since $f(x)=-x^4$ is an even function, the function $f(x)=-x^2$ is a reasonable approximation, requiring only a reflection of $y=x^2$ across the x-axis.

Example 5

Describe the transformations necessary to make a graph of the reference function $f(x)=x^3$ look like the graph of $y=-2x^3+2$.

Solution

The transformations required to replicate the function $y=-2x^3+2$ are reflect f(x)=x3 across the x-axis, stretch $f(x)=-x^3$ by 2, and shift $f(x)=-2x^3$ upward by 2.

Example 6

Describe the end behavior of $f(x)=-7x^3+6x^2-3x$ using the leading coefficient test.

Solution

According to the leading coefficient test, given $f(x)=ax^n$ where *a* is the leading coefficient and *n* is the degree, if *n* is odd and *a* is negative, the graph goes up on the left, and down on the right.

$\therefore f(x)=-7x^3+6x^2-3x$ grows without bound toward ∞ in Quadrant II and grows negatively without bound in Quadrant IV.

Review

1. Given: $P(x)=7x^4-5x^3+x^2-7x+6$

   State:

   a. The leading term:
   b. The degree of the polynomial:

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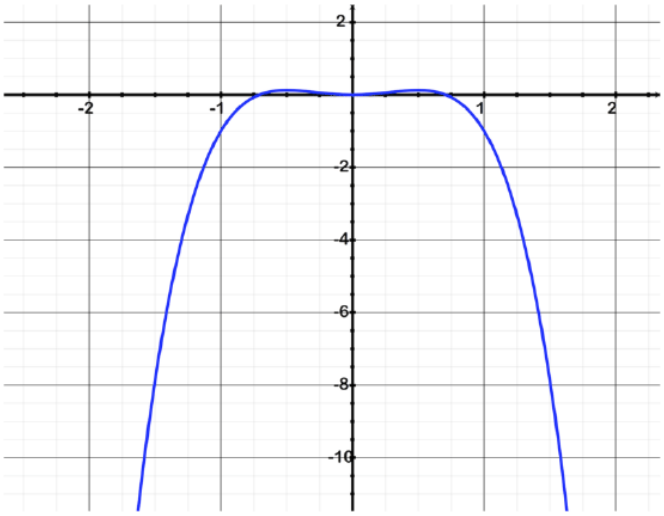
c. The leading coefficient:

Describe the transformation given in each question below:

2. Original Function: $g(x)=3x^3$ Transformed Function: $f(x)=3x^3+3$
3. Original Function: $g(x)=2x^3+3$ Transformed Function: $f(x)=2x^3+7$
4. Original Function: $g(x)=x^4+2$ Transformed Function: $f(x)=3(x^4+2)$
5. Original Function: $g(x)=5x^3$ Transformed Function: $f(x)=\frac{1}{2}(x^3)$

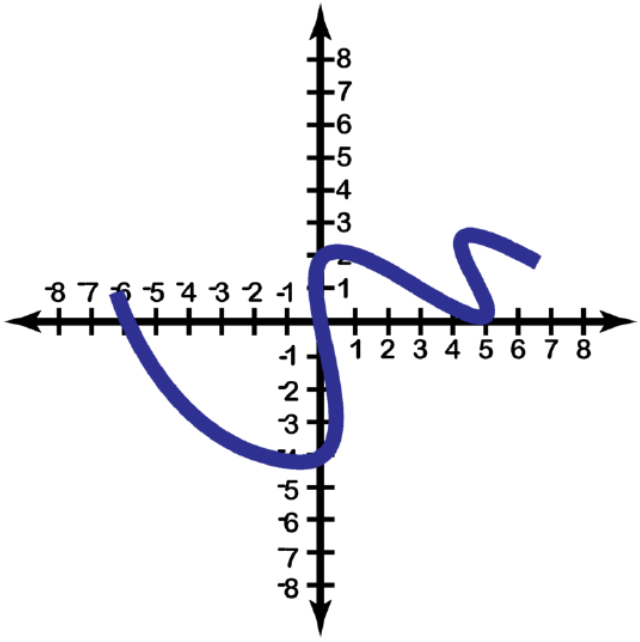
Graph the following by using transformations of parent functions:

6. $f(x)=2x^5-4$
7. $f(x)=(x-4)^3+6$
8. The graph of $f(x)=-2x^4+x^2$ is shown below.

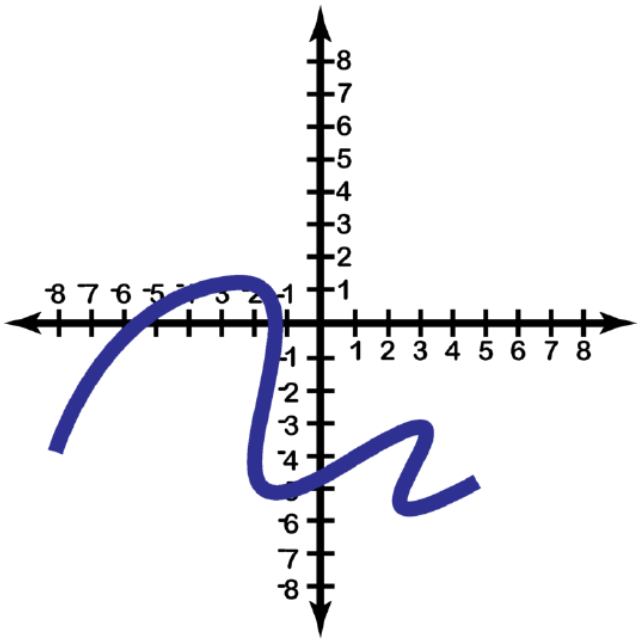


Describe each transformation based on the images below:

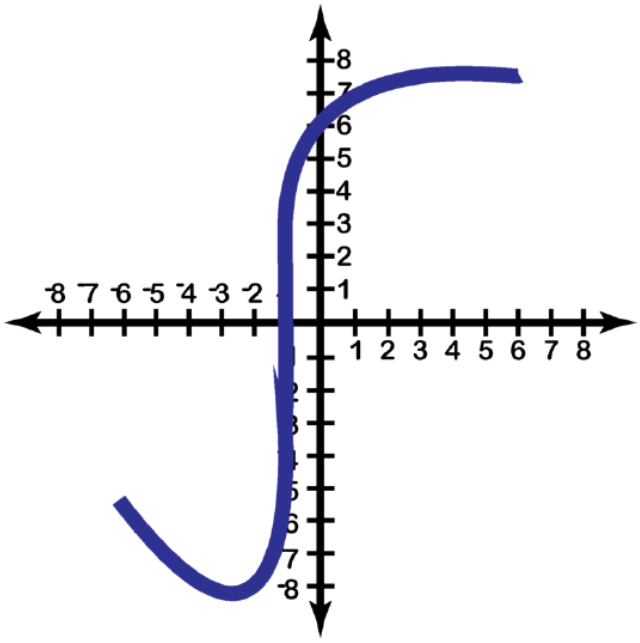
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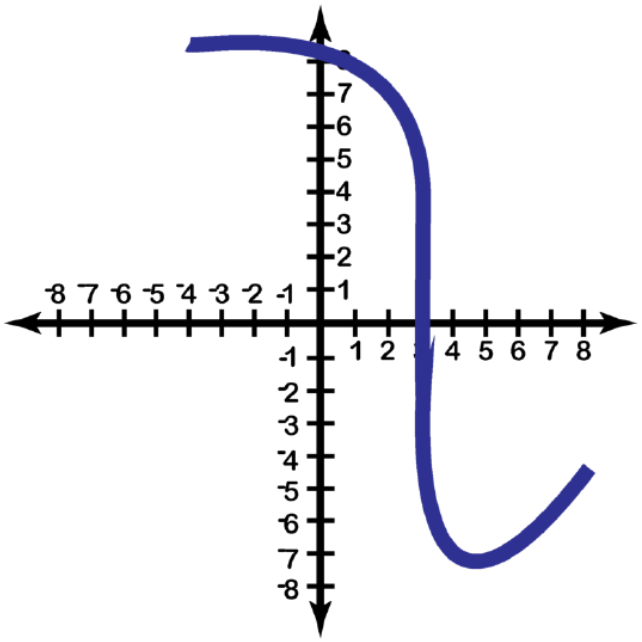
9. From: To



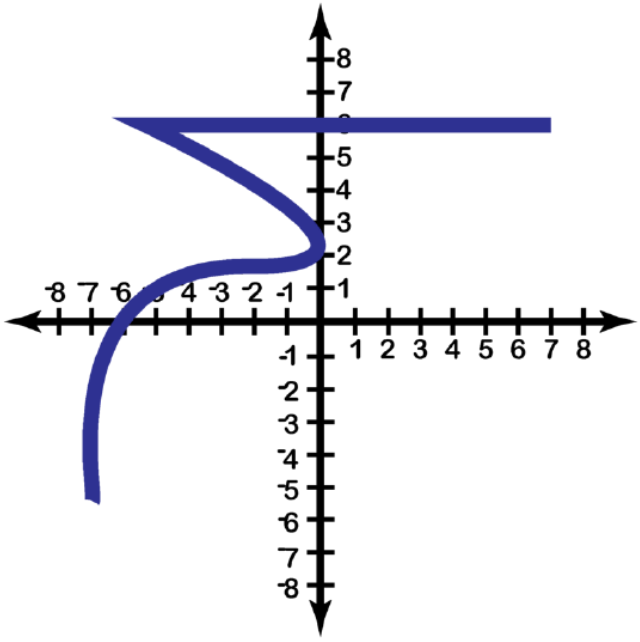
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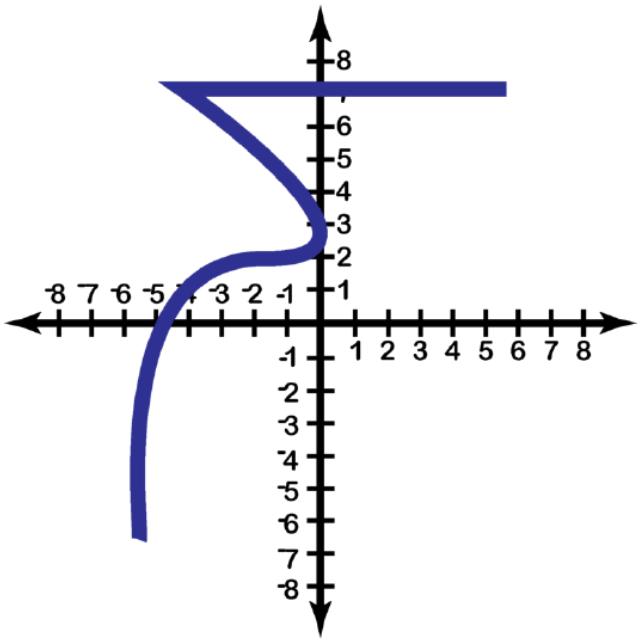
10. From: To



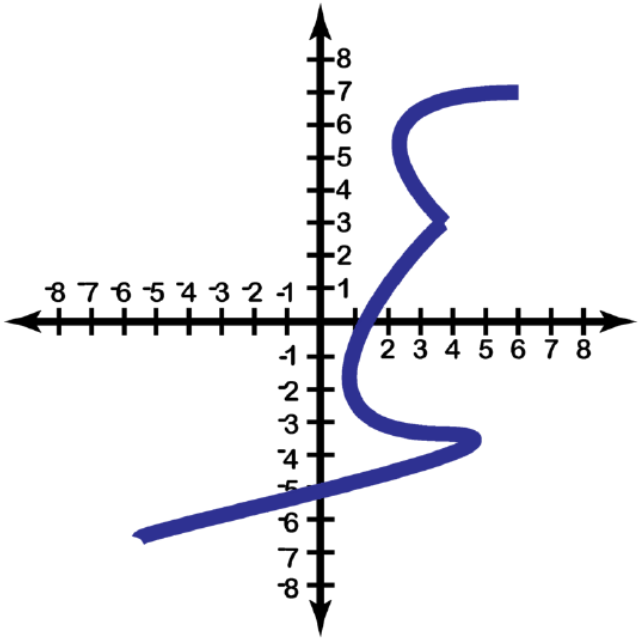
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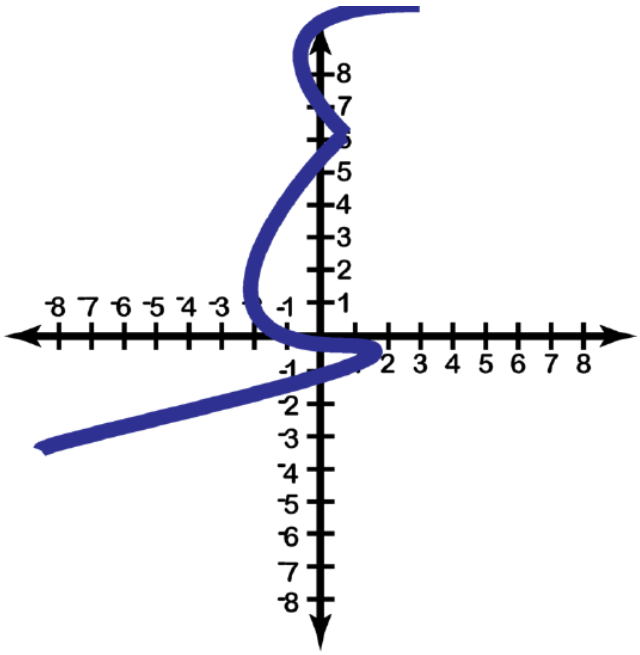
11. From: To



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12. From: To



Review (Answers)

To see the Review answers, open this PDF file and look for section 2.3.

Vocabulary

Term	Definition
compression	A stretch or compression is a function transformation that makes a graph narrower or wider, without translating it horizontally or vertically.
Odd Power Function	An odd power function is a polynomial of the form $f(x)=ax^n$ where a is a real number and n is an odd integer.
parent function	A parent function is the simplest form of a particular type of function. All other functions of this type are usually compared to the parent function.
Polynomial	A polynomial is an expression with at least one algebraic term, but which does not indicate division by a variable or contain variables with fractional exponents.
Polynomial Graph	A polynomial graph is the graph of a polynomial function. The term is most commonly used for polynomial functions with a degree of at least three.
Power function	A power function is a polynomial of the form $f(x)=ax^n$ where a is a real number and n is an integer with $n \geq 1$.
Reflection	Reflections are transformations that result in a "mirror image" of a parent function. They are caused by differing signs between parent and child functions.
shift	A shift, also known as a translation or a slide, is a transformation applied to the graph of a function that does not change the shape or orientation of the graph, only the location of the graph.
shifts	A shift, also known as a translation or a slide, is a transformation applied to the graph of a function that does not change the shape or orientation of the graph, only the location of the graph.
stretch	A stretch or compression is a function transformation that makes a graph narrower or wider.
stretching	Stretching a graph means to make the graph narrower or wider.
Transformations	Transformations are used to change the graph of a parent function into the graph of a more complex function.

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2.2.2: Graphs of Polynomials Using Zeros

Graphs of Polynomials Using Zeros

How is finding and using the **zeroes** of a higher-degree **polynomial** related to the same process you have used in the past on quadratic functions?

Graphing Polynomials Using Zeros

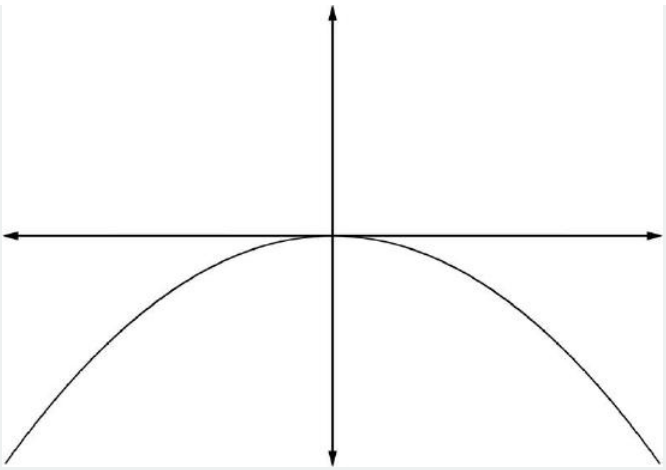
The following procedure can be followed when graphing a polynomial function.

- Use the **leading-term test** to determine the end behavior of the graph.
- Find the x−**intercept**(s) of f(x) by setting f(x)=0 and then solving for x.
- Find the y−intercept of f(x) by setting y=f(0) and finding y.
- Use the x−intercept(s) to divide the x−axis into intervals and then choose test points to determine the sign of f(x) on each **interval**.
- Plot the test points.
- If necessary, find additional points to determine the general shape of the graph.

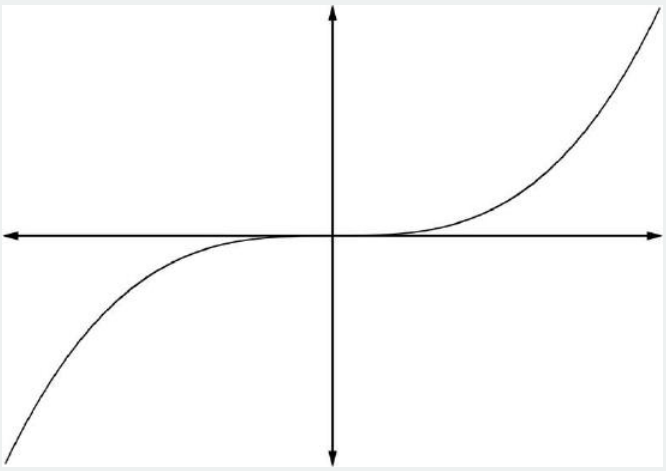
The Leading-Term Test

If $a_n x^n$ is the leading term of a polynomial. Then the behavior of the graph as $x \to \infty$ or $x \to -\infty$ can be known by one the four following behaviors:

2. If $a_n<0$ and n even:

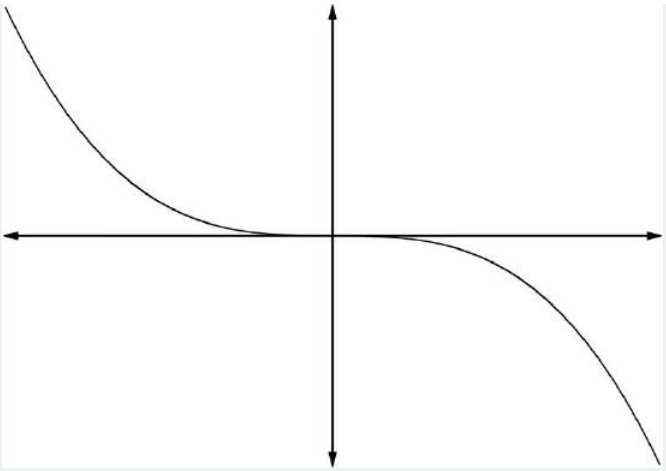


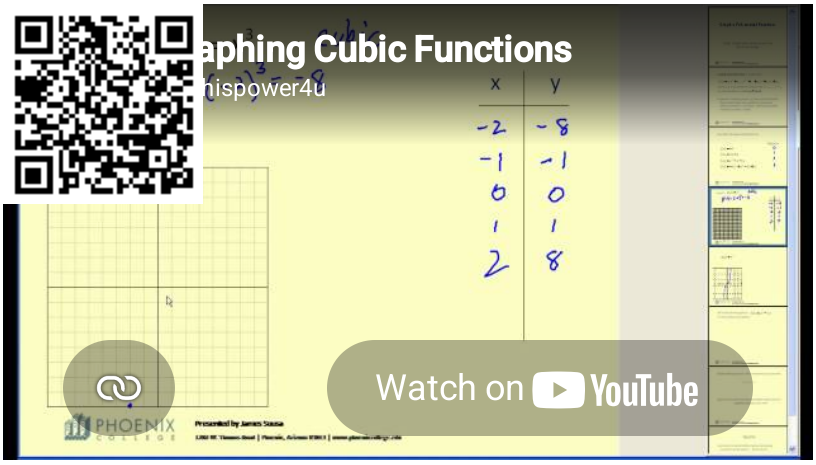
3. If $a_n>0$ and n odd:



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4. If $a_n<0$ and n odd:





Examples

Example 1

Earlier, you were asked to identify some similarities in graphing using zeroes between quadratic functions and higher-degree polynomials.

Solution

Despite the more complex nature of the graphs of higher-degree polynomials, the general process of graphing using zeroes is actually very similar. In both cases, your goal is to locate the points where the graph crosses the x or y axis. In both cases, this is done by setting the y value equal to zero and solving for x to find the x axis intercepts, and setting the x value equal to zero and solving for y to find the y axis intercepts.

Example 2

Find the **roots** (zeroes) of the polynomial:

$h(x)=x^3+2x^2-5x-6$

Solution

Start by factoring:

$h(x)=x^3+2x^2-5x-6=(x+1)(x-2)(x+3)$

To find the zeros, set $h(x)=0$ and solve for x.

$(x+1)(x-2)(x+3)=0$

This gives

$x+1=0$

$x-2=0$

$x+3=0$

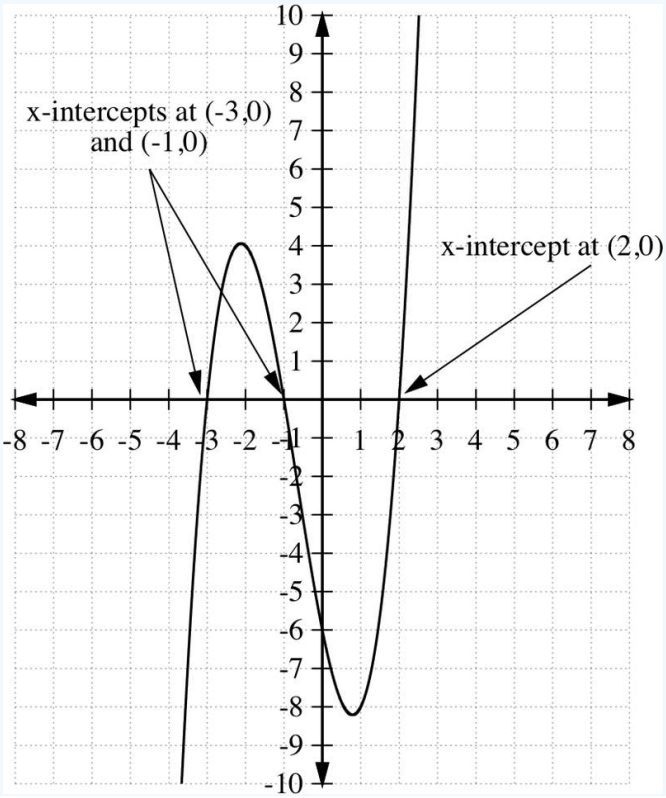
or

$x=-1$

$x=2$

$x=-3$

So we say that the solution set is $\{-3,-1,2\}$. They are the zeros of the function h(x). The zeros of h(x) are the x−intercepts of the graph y=h(x) below.



Example 3

Find the zeros of $g(x)=-(x-2)(x-2)(x+1)(x+5)(x+5)(x+5)$.

Solution

The polynomial can be written as

$g(x)=-(x-2)^2(x+1)(x+5)^3$

To solve the equation, we simply set it equal to zero

$-(x-2)^2(x+1)(x+5)^3=0$

this gives

$x-2=0$

$x+1=0$

$x+5=0$

or

$x=2$

$x=-1$

$x=-5$

Notice the occurrence of the zeros in the function. The factor $(x-2)$ occurred twice (because it was squared), the factor $(x+1)$ occurred once and the factor $(x+5)$ occurred three times. We say that the zero we obtain from the factor $(x-2)$ has a *multiplicity* $k=2$ and the factor $(x+5)$ has a *multiplicity* $k=3$.

Example 4

Graph the polynomial function $f(x)=-3x^4+2x^3$.

Solution

Since the leading term here is $-3x^4$ then $a_n=-3<0$, and $n=4$ even. Thus the end behavior of the graph as $x\to\infty$ and $x\to-\infty$ is that of Box #2, item 2.

We can find the zeros of the function by simply setting $f(x)=0$ and then solving for x.

$-3x^4+2x^3=0$

$-x^3(3x-2)=0$

This gives

$x=0$ or $x=\frac{2}{3}$

So we have two x−intercepts, at $x=0$ and at $x=\frac{2}{3}$, with multiplicity $k=3$ for $x=0$ and multiplicity $k=1$ for $x=\frac{2}{3}$

To find the y−intercept, we find $f(0)$, which gives

$f(0)=0$

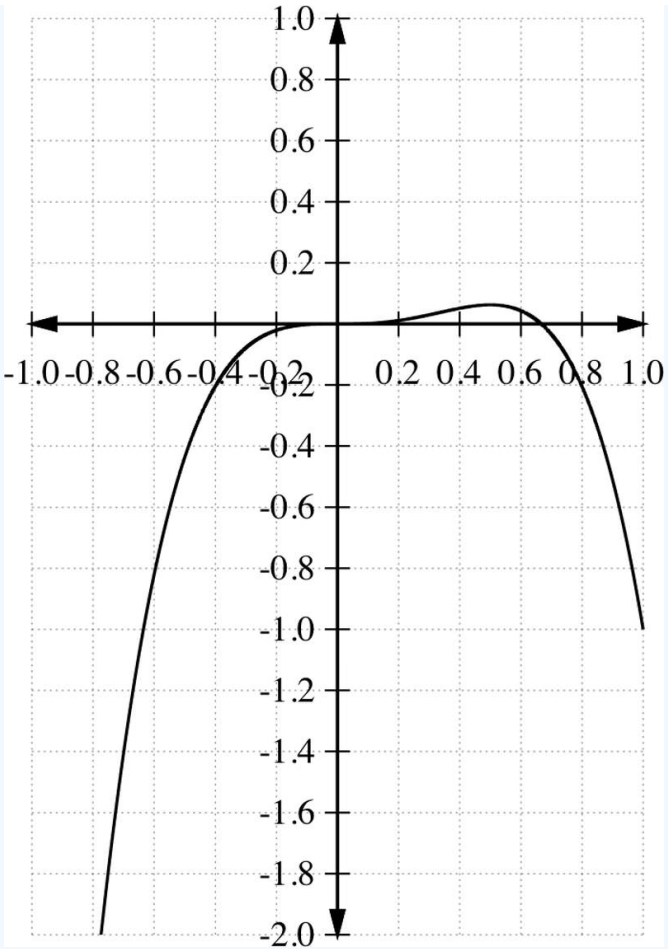
So the graph passes the y−axis at $y=0$.

Since the x−intercepts are 0 and $\frac{2}{3}$, they divide the x−axis into three intervals: $(-\infty, 0)$, $(0, \frac{2}{3})$, and $(\frac{2}{3}, \infty)$. Now we are interested in determining at which intervals the function f(x) is negative and at which intervals it is positive. To do so, we construct a table and choose a test value for x from each interval and find the corresponding f(x) at that value.

Interval	**Test Value x**	f(x)	**Sign of f(x)**	**Location of points on the graph**
$(-\infty, 0)$	-1	-5	-	below the x−axis
$(0, \frac{2}{3})$	$\frac{1}{2}$	$\frac{1}{16}$	+	above the x−axis
$(\frac{2}{3}, \infty)$	1	-1	-	below the x−axis

Those test points give us three additional points to plot: $(-1, -5)$, $(\frac{1}{2}, \frac{1}{16})$, and $(1, -1)$. Now we are ready to plot our graph. We have a total of three intercept points, in addition to the three test points. We also know how the graph is behaving as $x\to-\infty$ and $x\to+\infty$. This information is usually enough to make a rough sketch of the graph. If we need additional points, we can simply select more points to complete the graph.

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Example 5

Find the zeros and sketch a graph of the polynomial

$f(x)=x^4-x^2-56$

Solution

This is a factorable equation,

$f(x)=x^4-x^2-56$

$=(x^2-8)(x^2+7)$

Setting $f(x)=0$,

$(x^2-8)(x^2+7)=0$

the first term gives

$x^2-8=0$

$x^2=0$

$x=\pm\sqrt{8}$

$=\pm 2\sqrt{2}$

and the second term gives

$x^2+7=0$

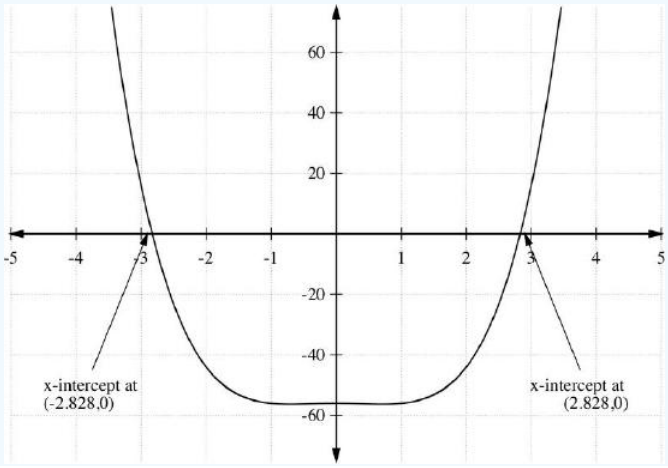
$x^2=-7$

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$x = \pm\sqrt{-7}$

$= \pm i\sqrt{7}$

So the solutions are $\pm 2\sqrt{2}$ and $\pm i\sqrt{7}$, a total of four zeros of f(x). Keep in mind that only the *real* zeros of a function correspond to the x−intercept of its graph.



Example 6

Graph $g(x) = -(x-2)^2(x+1)(x+5)^3$.

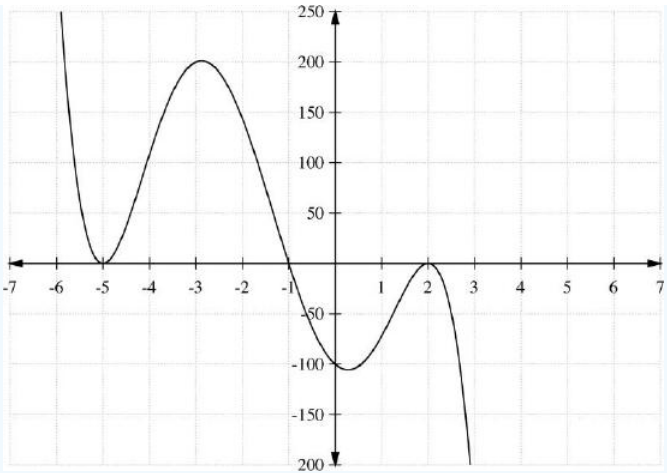
Solution

Use the zeros to create a table of intervals and see whether the function is above or below the x−axis in each interval:

Interval	Test value x	g(x)	Sign of g(x)	Location of graph relative to x−axis
$(-\infty, -5)$	-6	320	+	Above
$x = -5$	-5	0	NA	
(-5, -1)	-2	144	+	Above
$x = -1$	-1	0	NA	
(-1, 2)	0	-100	-	Below
$x = 2$	2	0	NA	
$(2, \infty)$	3	-256	-	Below

Finally, use this information and the test points to sketch a graph of g(x).

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Review

1. If *c* is a zero of *f*, then *c* is a/an _____ of the graph of *f*.
2. If *c* is a zero of *f*, then (*x* - *c*) is a factor of _____?
3. Find the zeros of the polynomial: $P(x)=x^3-5x^2+6x$

Consider the function: $f(x)=-3(x-3)^4(5x-2)(2x-1)^3(4-x)^2$.

4. How many zeros (x-intercepts) are there?
5. What is the leading term?

Find the zeros and graph the polynomial. Be sure to label the *x*-intercepts, *y*-intercept (if possible) and have correct end behavior. You may use technology for questions 9-12.

6. $P(x)=-2(x+1)^2(x-3)$
7. $P(x)=x^3+3x^2-4x-12$
8. $f(x)=-2x^3+6x^2+9x+6$
9. $f(x)=-4x^2-7x+3$
10. $f(x)=2x^5+4x^3+8x^2+6x$
11. $f(x)=x^4-3x^2$
12. $g(x)=x^2-|x|$
13. Given: $P(x)=(3x+2)(x-7)^2(9x+2)^3$

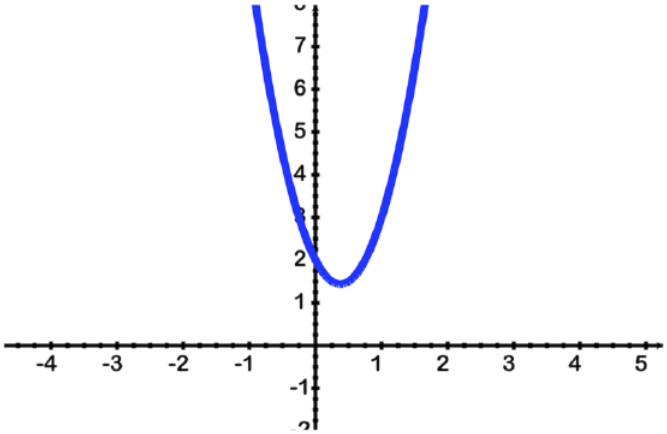
State:

a. The leading term:
b. The degree of the polynomial:
c. The leading coefficient:

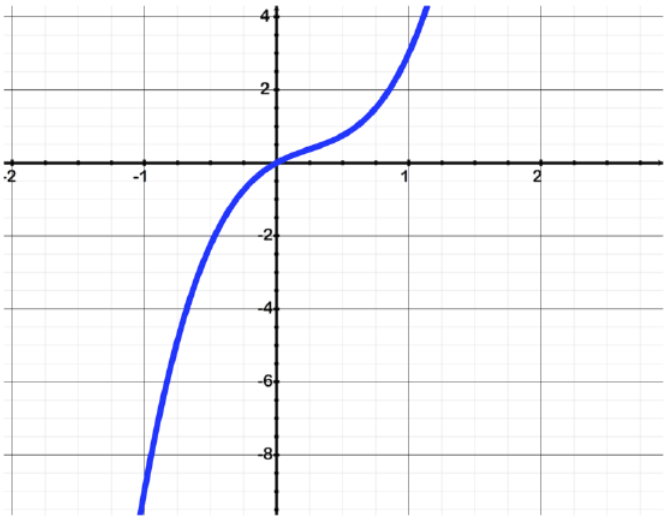
Determine the equation of the polynomial based on the graph:

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14.



15.



Review (Answers)

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Vocabulary

Term	Definition
Cubic Function	A cubic function is a function containing an x^3 term as the highest power of x.
Intercept	The intercepts of a curve are the locations where the curve intersects the x and y axes. An x intercept is a point at which the curve intersects the x-axis. A y intercept is a point at which the curve intersects the y-axis.
interval	An interval is a specific and limited part of a function.
Leading-Term Test	The leading-term test is a test to determine the end behavior of a polynomial function.
Polynomial	A polynomial is an expression with at least one algebraic term, but which does not indicate division by a variable or contain variables with fractional exponents.
Polynomial Graph	A polynomial graph is the graph of a polynomial function. The term is most commonly used for polynomial functions with a degree of at least three.
Quartic Function	A quartic function is a function f(x) containing an x^4 term as the highest power of "x".
Roots	The roots of a function are the values of *x* that make *y* equal to zero.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.

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2.2.3: Graphing Calculator to Analyze Polynomial Functions

Graphing Polynomial Functions with a Graphing Calculator

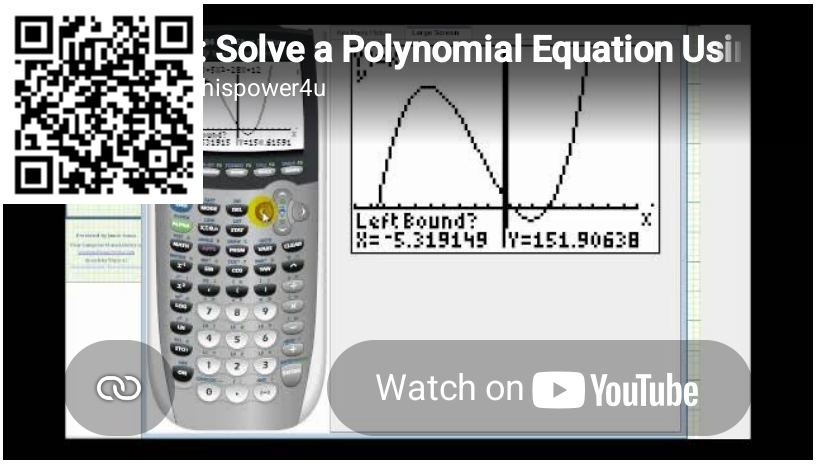
To make a fair race between a dragster and a funny car, a scientist devised the following polynomial equation:

$f(x)=71.682x-60.427x^2+84.710x^3-27.769x^4+4.296x^5-0.262x^6$. What is the maximum point of this function's graph?

Source: Rice University

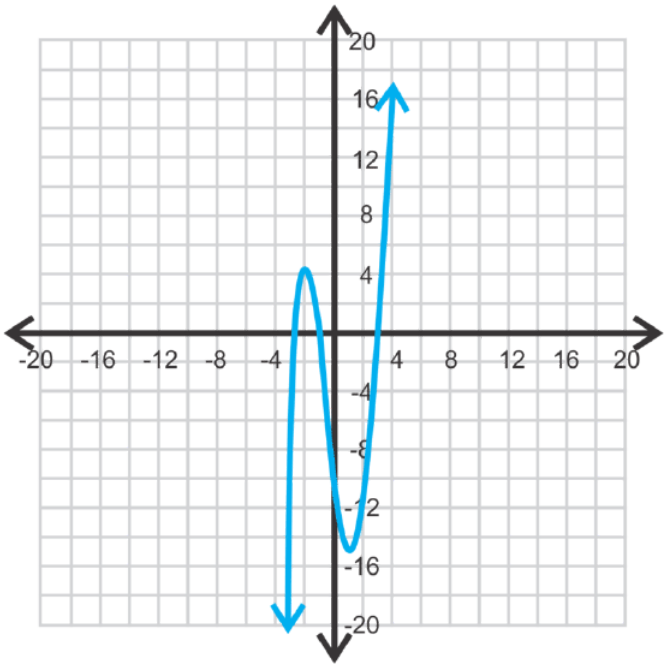
Graphing Polynomial Functions with a Calculator

You have already used the graphing calculator to graph parabolas. Now, we will expand upon that knowledge and graph higher-degree polynomials. Then, we will use the graphing calculator to find the zeros, maximums and minimums.



Let's graph $f(x)=x^3+x^2-8x-8$ using a graphing calculator.

These instructions are for a TI-83 or 84. First, press Y=. If there are any functions in this window, clear them out by highlighting the = sign and pressing **ENTER.** Now, in Y1, enter in the polynomial. It should look like: x^3+x^2−8x−8. Press **GRAPH**.



To adjust the window, press **ZOOM**. To get the typical -10 to 10 screen (for both axes), press **6:ZStandard.** To zoom out, press **ZOOM, 3:ZoomOut, ENTER, ENTER.** For this particular function, the window needs to go from -15 to 15 for both x and y. To

manually input the window, press WINDOW and change the Xmin,Xmax,Ymin, and Ymax so that you can see the zeros, minimum and maximum. Your graph should look like the one above.

Now, let's find the zeros, maximum, and minimum of the function from the previous problem above.

To find the zeros, press 2nd **TRACE** to get the **CALC** menu. Select **2:Zero** and you will be asked "Left Bound?" by the calculator. Move the cursor (by pressing the ↑ or ↓) so that it is just to the left of one zero. Press **ENTER.** Then, it will ask "Right Bound?" Move the cursor just to the right of that zero. Press **ENTER.** The calculator will then ask "Guess?" At this point, you can enter in what you think the zero is and press **ENTER** again. Then the calculator will give you the exact zero. For the graph from the previous problem above, you will need to repeat this three times. The zeros are -2.83, -1, and 2.83.

To find the minimum and maximum, the process is almost identical to finding zeros. Instead of selecting **2:Zero**, select **3:min** or **4:max**. The minimum is (1.33, -14.52) and the maximum is (-2, 4).

Finally, let's find the y−intercept of the graph from the same problem above.

If you decide not to use the calculator, plug in zero for x and solve for y.

$f(0)=0^3+0^2-8\cdot 0-8$

Using the graphing calculator, press 2nd **TRACE** to get the **CALC** menu. Select **1:value.** X= shows up at the bottom of the screen. If there is a value there, press **CLEAR** to remove it. Then press **0** and **ENTER.** The calculator should then say "Y=−8."

Examples

Example 1

Earlier, you were asked to find the maximum point of the function's graph.

Solution

If you plug the equation $f(x)=71.682x-60.427x^2+84.710x^3-27.769x^4+4.296x^5-0.262x^6$ into your calculator, you find that the maximum occurs when x=6.15105. At that value of *x*, *f(x)* equals 1754.43. Therefore the maximum point of the function's graph is (6.15105, 1754.43).

Graph and find the critical values of the following functions.

Example 2

$f(x)=-\frac{1}{3}x^4-x^3+10x^2+25x-4$

Solution

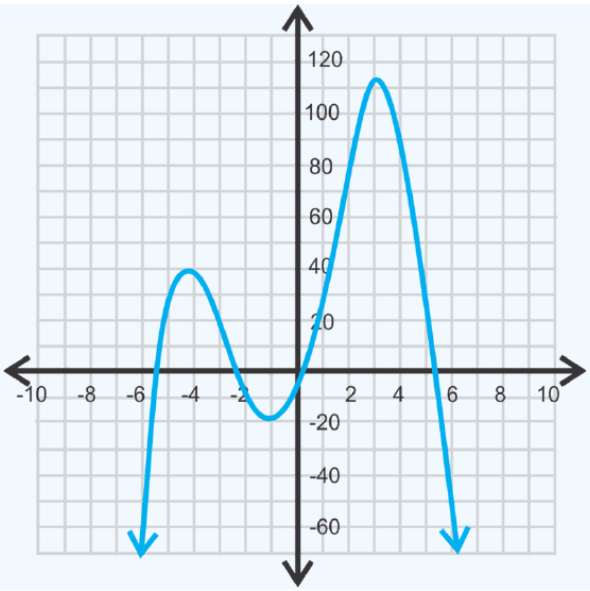
zeros: -5.874, -2.56, 0.151, 5.283

y−intercept: (0, -4)

minimum: (-1.15, -18.59)

local maximum: (-4.62, 40.69)

absolute maximum: (3.52, 113.12)



Example 3

$g(x)=2x^5-x^4+6x^3+18x^2-3x-8$

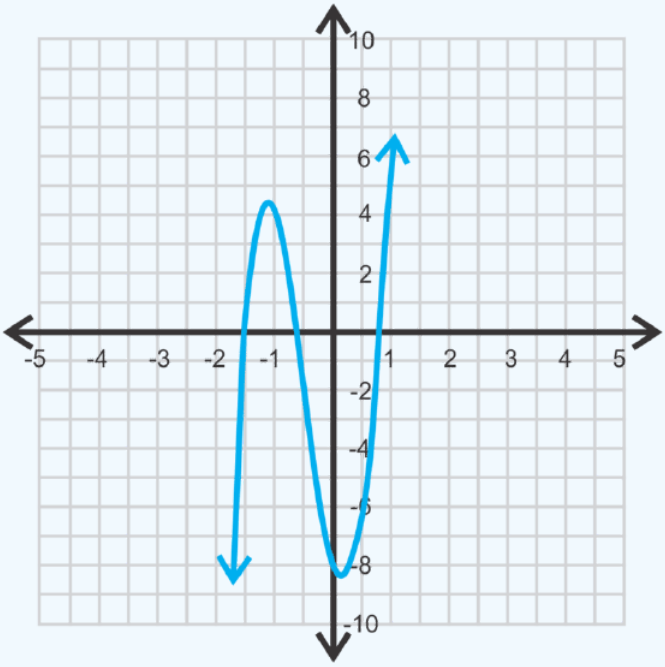
Solution

zeros: -1.413, -0.682, 0.672

y−intercept: (0, -8)

minimum: (-1.11, 4.41)

maximum: (0.08, -8.12)



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Example 4

Find the domain and range of Examples 2 and 3.

Solution

The domain of Example 2 is all real numbers and the range is all real numbers less than the maximum; $(-\infty,113.12]$. The domain and range of Example 3 are all real numbers.

Review

Graph questions 1-6 on your graphing calculator. Sketch the graph in an appropriate window. Then, find all the critical values, domain, range, and describe the end behavior.

1. $f(x)=2x^3+5x^2-4x-12$
2. $h(x)=-\frac{1}{4}x^4-2x^3-\frac{13}{4}x^2-8x-9$
3. $y=x^3-8$
4. $g(x)=-x^3-11x^2-14x+10$
5. $f(x)=2x^4+3x^3-26x^2-3x+54$
6. $y=x^4+2x^3-5x^2-12x-6$
7. What are the types of solutions in #2?
8. Find the two imaginary solutions in #3.
9. Find the exact values of the irrational **roots** in #5.

Determine if the following statements are SOMETIMES, ALWAYS, or NEVER true. Explain your reasoning.

10. The range of an even function is $(-\infty,max]$, where *max* is the maximum of the function.
11. The domain and range of all odd functions are all real numbers.
12. A function can have exactly three imaginary solutions.
13. An nth degree polynomial has n real solutions.
14. The parent graph of any polynomial function has one zero.
15. **Challenge** The exact value for one of the zeros in #2 is $-4+\sqrt{7}$. What is the exact value of the other root? Then, use this information to find the imaginary roots.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.14.

Vocabulary

Term	Definition
Roots	The roots of a function are the values of *x* that make *y* equal to zero.

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SECTION OVERVIEW

2.3: Analysis of Graphs of Rational Functions

Topic hierarchy

2.3.1: Graphs of Basic Rational Functions

2.3.2: Graphs of Rational Functions when the Degrees are Equal

2.3.3: Graphs of Rational Functions when the Degrees are not Equal

2.3.4: Sign Test for Rational Function Graphs

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2.3.1: Graphs of Basic Rational Functions

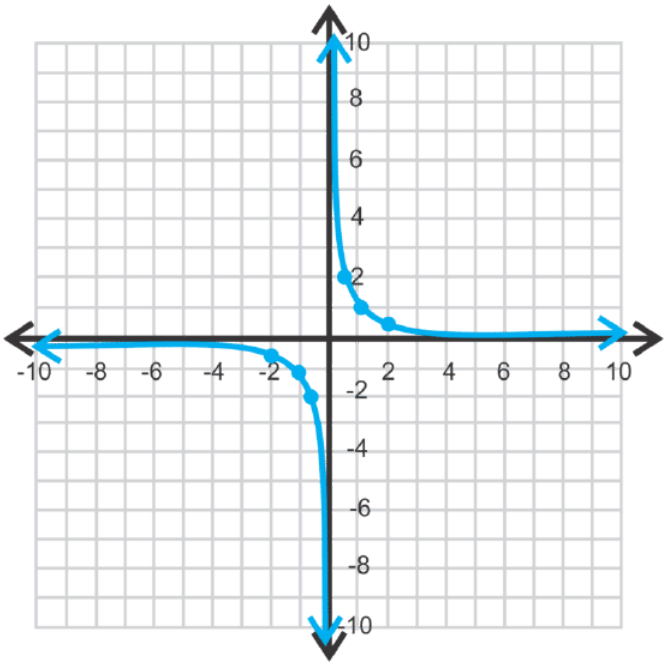
Graphing Rational Functions in Standard Form

The standard form of a **rational function** is given by the equation $f(x) = \frac{a}{x-h} + k$. What are the asymptotes of this equation?

Rational Function

A **rational function** is in the form $\frac{p(x)}{q(x)}$ where $p(x)$ are polynomials and $q(x) \neq 0$. The parent graph for rational functions is $y = \frac{1}{x}$, and the shape is called a **hyperbola**.

x	y
−4	$-\frac{1}{4}$
−2	$-\frac{1}{2}$
−1	−1
$-\frac{1}{2}$	−2
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$



[Figure1]

Notice the following properties of this hyperbola: the x-axis is a horizontal asymptote, the y-axis is a vertical asymptote, and the domain and range are all real numbers except where the asymptotes are. Recall that the vertical asymptote is the value that makes the denominator zero because we cannot divide by zero. For the horizontal asymptote, it is the value where the range is not defined.

The two parts of the graph are called **branches**. In the case with a hyperbola, the branches are always symmetrical about the point where the asymptotes intersect. In this example, they are symmetrical about the origin.

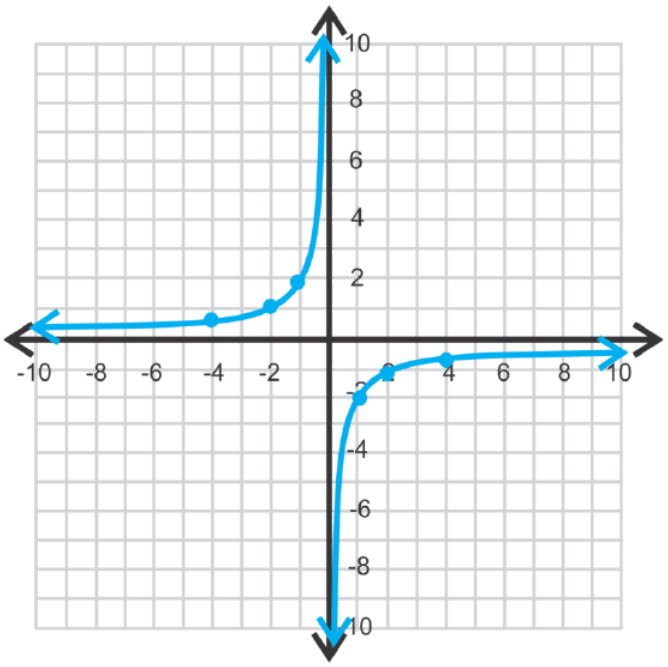
In this lesson, all the rational functions will have the form $f(x) = \frac{a}{x-h} + k$

Let's graph $f(x) = \frac{-2}{x}$ and find any asymptotes, the domain, range, and any zeros.

Let's make a table of values.

x	y
1	−2
2	−1
4	$-\frac{1}{2}$

Notice that these branches are in the second and fourth quadrants. This is because of the negative sign in front of the 2, or a. The horizontal and vertical asymptotes are still the x and y-axes. There are no zeros, or x-intercepts, because the x-axis is an asymptote. The domain and range are all non-zero real numbers (all real numbers except zero).

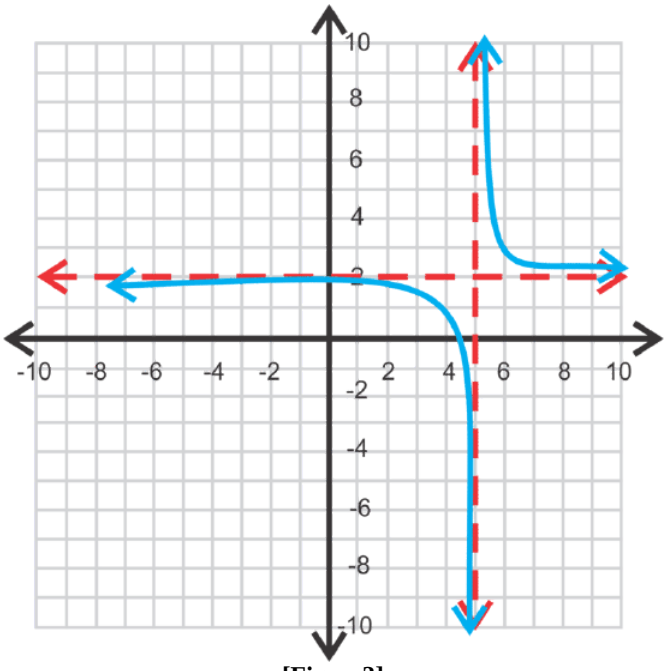


[Figure2]

Now, let's graph $y = \frac{1}{x-5} + 2$ and find all asymptotes, zeros, the domain and range.

For $y = \frac{1}{x-5} + 2$, the vertical asymptote is x=5 because that would make the denominator zero and we cannot divide by zero. When x=5, the value of the function would be $y = \frac{1}{0} + 2$, making the range undefined at $y = 2$. The shape and location of the branches are the same as the parent graph, just shifted to the right 5 units and up 2 units.

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[Figure3]

Therefore, for the general form of a rational function, $y = \frac{a}{x-h} + k, x = h$ is the vertical asymptote and $y = k$ is the horizontal asymptote.

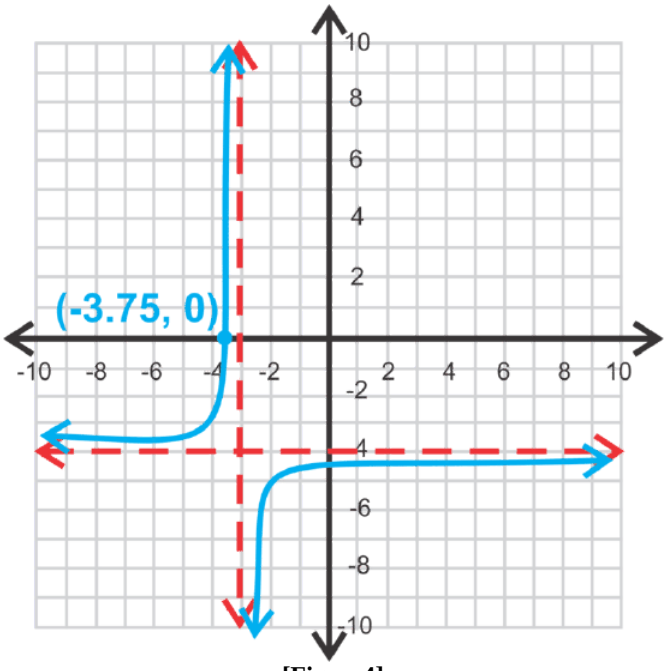
The domain is all real numbers; $x \neq 5$ and the range is all real numbers; $y \neq 2$. To find the zero, set the function equal to zero and solve for x.

$$\begin{aligned} 0 &= \frac{1}{x-5} + 2 \\ -2 &= \frac{1}{x-5} \\ -2x + 10 &= 1 \\ -2x &= -9 \\ x &= \frac{9}{2} = 4.5 \end{aligned}$$

To find the y-intercept, set x=0, and solve for y. $y = \frac{1}{0-5} + 2 = -\frac{1}{5} + 2 = 1\frac{4}{5}$.

Finally, let's find the equation of the hyperbola below.

We know that the numerator will be negative because the branches of this hyperbola are in the second and fourth quadrants. The asymptotes are x=−3 and y=−4. So far, we know $y = \frac{a}{x+3} - 4$. In order to determine a, we can use the given x-intercept.



[Figure4]

$$0 = \frac{a}{-3.75+3} - 4$$
$$4 = \frac{a}{-0.75}$$
$$-3 = a$$

The equation is $y = \frac{-3}{x+3} - 4$

Examples

Example 1

Earlier, you were asked to find the asymptotes of the equation $f(x) = \frac{a}{x-h} + k$.

Solution

From the previous problems, we've seen that the vertical asymptote occurs when the denominator of the equation equals zero and the horizontal asymptote occurs when the range is undefined.

When $x = h$, the denominator of $f(x) = \frac{a}{x-h} + k$ is zero, so $x = h$ is the vertical asymptote.

The value of the function at $x = h$ would be $y = \frac{a}{0} + k$, making the range undefined at $y = k$.

Therefore, $y = k$ is the horizontal asymptote.

Example 2

What are the asymptotes for $f(x) = \frac{-1}{x+6} + 9$ Is $(-5,-8)$ on the graph?

Solution

The asymptotes are $x = -6$ and $y = 9$. To see if the point $(-5,-8)$ is on the graph, substitute it in for x and y.

$$-8 = \frac{-1}{-5+6} + 9 \quad -8 \neq 8, \text{ therefore, the point } (-5,-8) \text{ is not on the graph.}$$
$$-8 = -1 + 9$$

For Examples 3 & 4, graph the rational functions. Find the zero, y-intercept, asymptotes, domain and range.

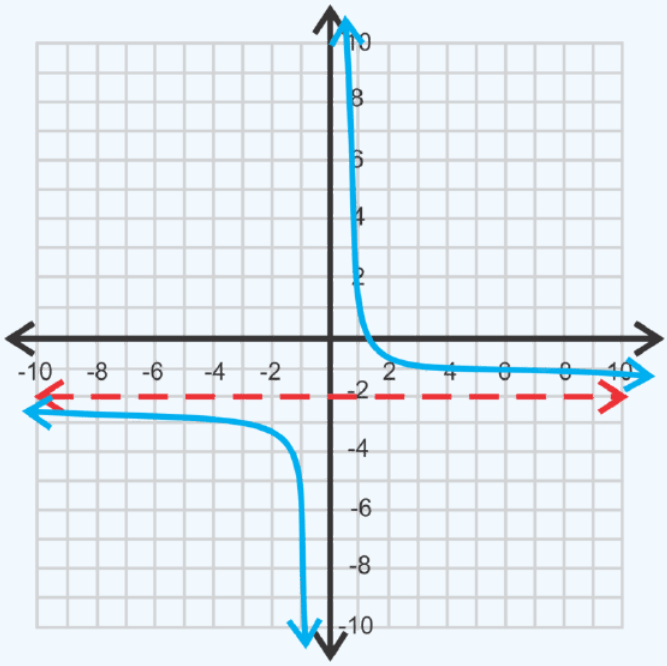
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Example 3

$y = \frac{4}{x} - 2$

Solution

There is no y-intercept because the y-axis is an asymptote. The other asymptote is y=−2. The domain is all real numbers; x≠0. The range is all real numbers; y≠−2. The zero is:



$$0 = \frac{4}{x} - 2$$
$$2 = \frac{4}{x}$$
$$2x = 4$$
$$x = 2$$

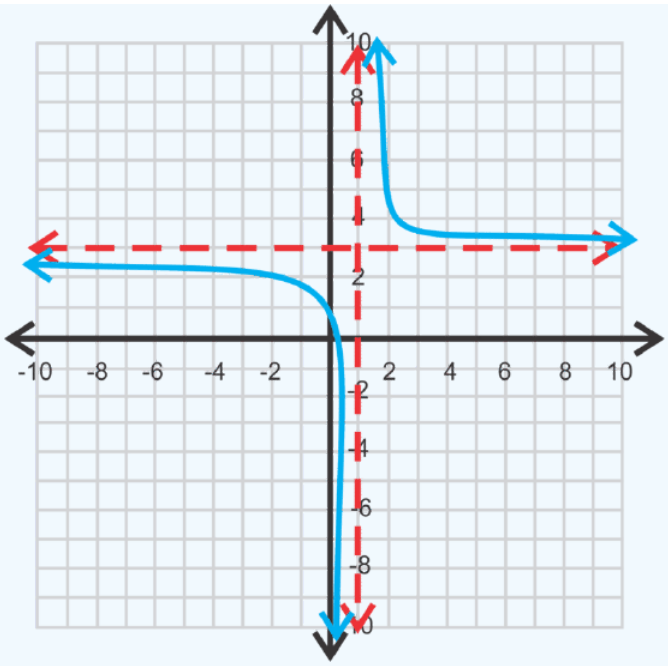
Example 4

$y = \frac{2}{x-1} + 3$

Solution

The asymptotes are x=1 and y=3. Therefore, the domain is all real numbers except 1 and the range is all real numbers except 3. The y-intercept is $y = \frac{2}{0-1} + 3 = -2 + 3 = 1$ and the zero is:

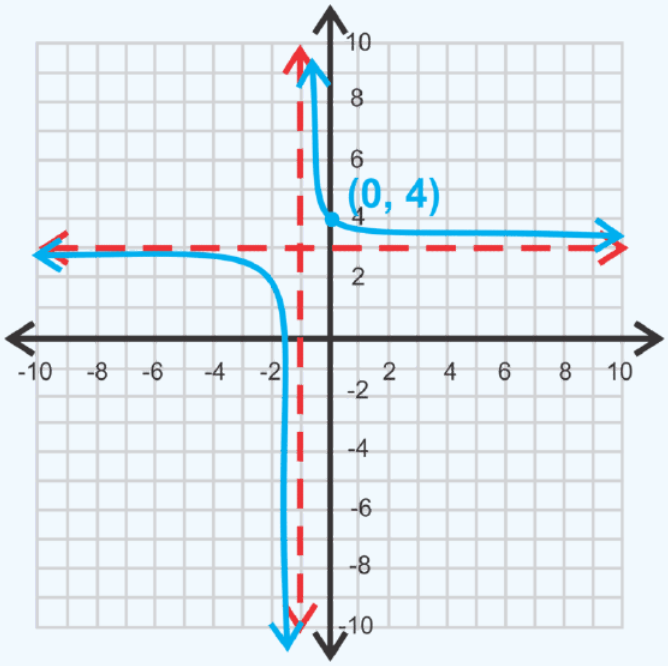
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$$0 = \frac{2}{x-1} + 3$$
$$-3 = \frac{2}{x-1}$$
$$-3x + 3 = 2$$
$$-3x = -1 \rightarrow x = \frac{1}{3}$$

Example 5

Determine the equation of the hyperbola.



Solution

The asymptotes are x=−1,y=3, making the equation $y = \frac{a}{x+1} + 3$. Taking the y-intercept, we can solve for a.

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$4 = \frac{a}{0+1} + 3$ The equation is $y = \frac{1}{x+1} + 3$

$1 = a$

Review

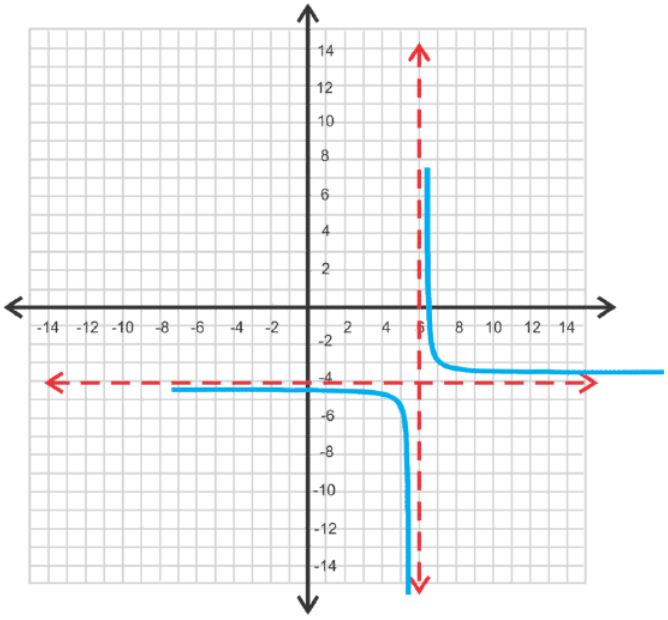
1. What are the asymptotes for $y = \frac{2}{x+8} - 3$?
2. Is $(-6,-2)$ a point on the graph from #1?
3. What are the asymptotes for $y = 6 - \frac{1}{x-4}$?
4. Is (5,4) a point on the graph from #3?

For problems 5-13, graph each rational function, state the equations of the asymptotes, the domain and range and the intercepts.

5. $y = \frac{3}{x}$
6. $y = \frac{1}{x} + 6$
7. $y = -\frac{1}{x}$
8. $y = -\frac{1}{x+3}$
9. $y = \frac{1}{x+5}$
10. $y = \frac{1}{x-3} - 4$
11. $y = \frac{2}{x+4} - 3$
12. $y = \frac{5}{x} + 2$
13. $y = 3 - \frac{1}{x+2}$

Write the equations of the hyperbolas. You may assume that a = 1.

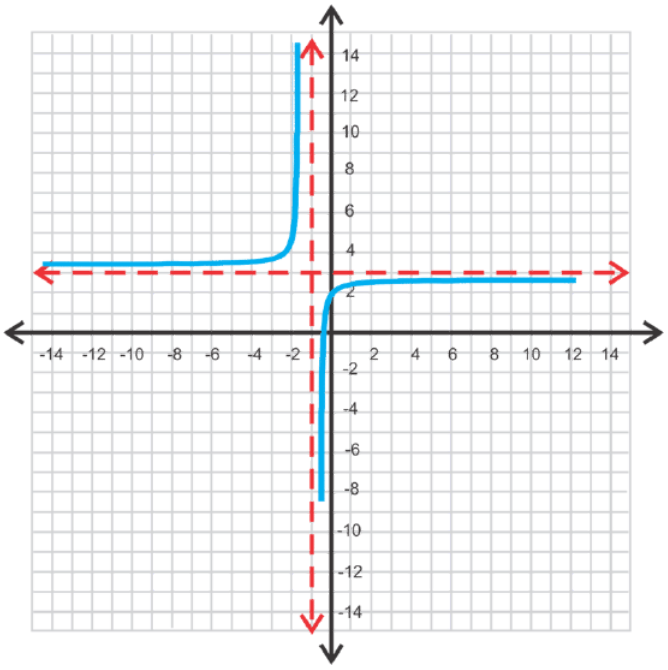
14.



[Figure5]

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15.



[Figure6]

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.4.

Vocabulary

Term	Definition
Branches	The two curves of a hyperbola are sometimes called branches.
hyperbola	A hyperbola is a conic section formed when the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shaped curves.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.

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3. [Figure 3]
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5. [Figure 5]
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6. [Figure 6]
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2.3.2: Graphs of Rational Functions when the Degrees are Equal

Graphing when the Degrees of the Numerator and Denominator are the Same

Darnell says that the function $y = \frac{2x^4+5}{x^4-16}$ has two vertical asymptotes, Barb says it has only one, and Aruna says it has four. Which one of them is correct?

Graphing Rational Functions

We have already graphed functions in the form $y = \frac{1}{x-h} + k$, where x=h and y=k are the asymptotes. In this concept, we will extend graphing rational functions when both the denominator and numerator are linear or both quadratic. So, there will be no "k" term in this concept. Let's go through some practice problems to determine any patterns in graphing this type of rational function.

Let's graph $f(x) = \frac{2x-1}{x+4}$ and find asymptotes, x and y intercepts, domain and range.

To find the vertical asymptote, it is the same as before, the value that makes the denominator zero. In this case, x=−4. Also the same is how to find the x and y intercepts.

y-intercept (when x=0): $y = \frac{2 \cdot 0-1}{0+4} = -\frac{1}{4}$

x-intercept (when y=0):

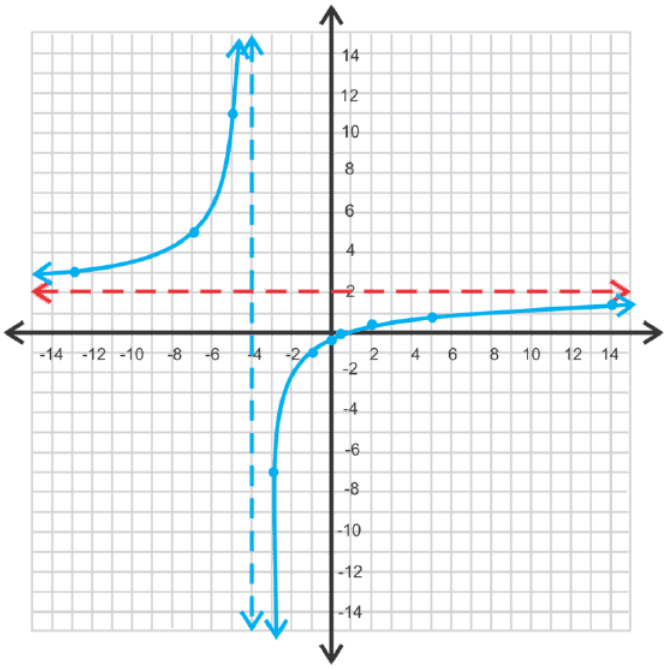
$$0 = \frac{2x-1}{x+4}$$
$$0 = 2x - 1$$
$$1 = 2x$$
$$\frac{1}{2} = x$$

When solving for the x-intercept, to get the denominator out, we multiplied both sides by x+4. But, when we multiply anything by 0, it remains 0. Therefore, to find the x-intercept, we only need to set the numerator equal to zero and solve for x.

The last thing to find is the horizontal asymptote. We know that the function is positive, so the branches will be in the first and third quadrants. Let's make a table.

x	y
−13	3
−7	5
−5	11
−3	−7
−1	−1
0	−0.25
2	0.5
5	1
14	1.5

It looks like the horizontal asymptote is y=2 because both branches seem to approach 2 as x gets larger, both positive and negative. If we plug in x=86,y=1.9 and when x=−94,y=2.1. As you can see, even when x is very large, the function is still approaching 2.



[Figure1]

Looking back at the original equation, $f(x)=\frac{2x-1}{x+4}$, extract the leading coefficients and leave them numerator over denominator, $\frac{2}{1}$. This is the horizontal asymptote. We can generalize this pattern for all rational functions. *When the **degree** of the numerator is equal to the degree of the denominator, the horizontal asymptote is the ratio of the leading coefficients.*

Finally, the domain is all real numbers; $x\neq-4$ and the range is all real numbers; $y\neq2$.

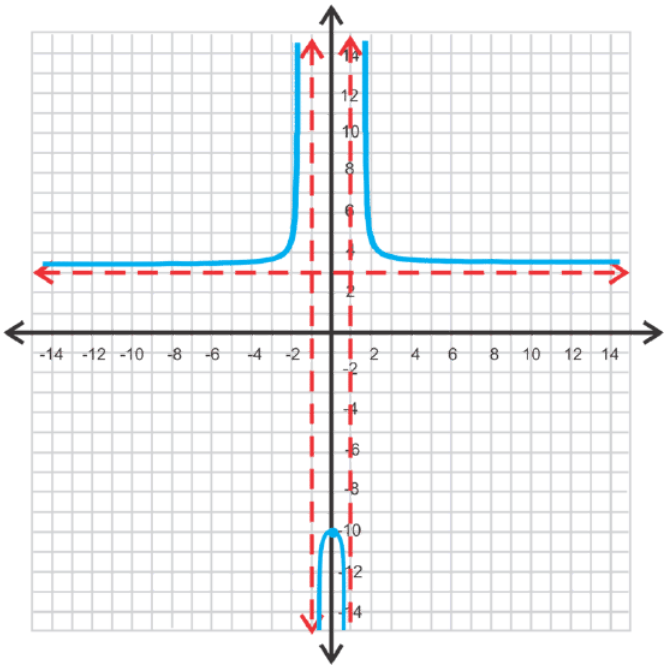
Now, let's graph $y=\frac{3x^2+10}{x^2-1}$ and find the asymptotes, intercepts, domain, and range.

From the previous problem above, we can conclude that the horizontal asymptote is at y=3. Because the denominator is squared, there will be two vertical asymptotes because x^2-1 factors to $(x-1)(x+1)$. Therefore, the vertical asymptotes are x=1 and x=−1. As for the intercepts, there are no x-intercepts because there is no real solution for $3x^2+10=0$.

Solving for the y-intercept, we have $y=\frac{10}{-1}=-10$.

At this point, put the equation in your calculator to see the general shape. To graph this function using a TI-83 or 84, enter the function into Y= like this: $\frac{(3x\text{^}2+10)}{(x\text{^}2-1)}$ and press **GRAPH**. You will need to expand the window to include the bottom portion of the graph. The final graph is below.

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The domain is still all real numbers except the vertical asymptotes. For this function, that would be all real numbers; $x \neq -1, x \neq 1$.

The range is a bit harder to find. Notice the gap in the range from the horizontal asymptote and the y-intercept. Therefore, the range is $(-\infty, -10] \cup (3, \infty)$.

The notation above is one way to write a range of numbers called **interval notation** and was already introduced. The $\cup$ symbol means "union." Notice that $-\infty$ and ∞ are not included in the range.

In general, rational functions with quadratics in the denominator are split into six regions and have branches in three of them, like the problem above. However, there are cases when there are no zeros or vertical asymptotes and those look very different. You should always graph the function in a graphing calculator after you find the critical values and make as accurate a sketch as you can.

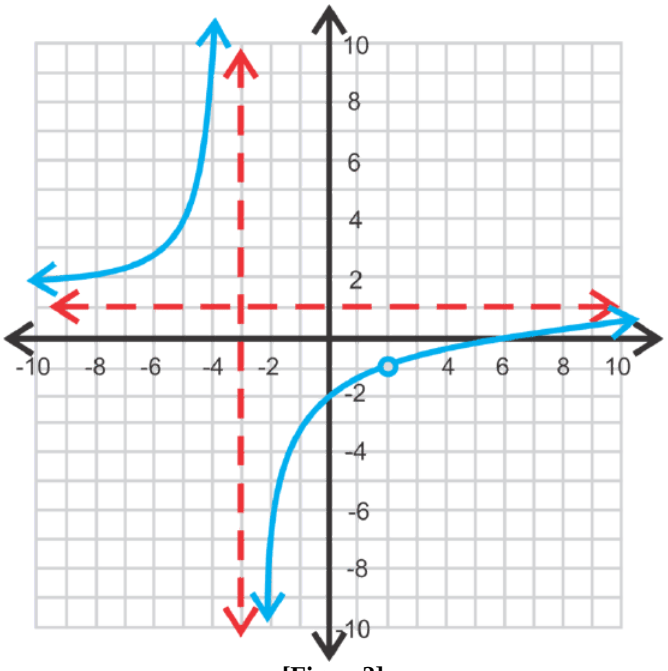
Finally, let's graph $f(x) = \frac{x^2 - 8x + 12}{x^2 - x - 6}$ and find the intercepts, asymptotes, domain and range.

Let's factor the numerator and denominator to find the intercepts and vertical asymptotes.

$$f(x) = \frac{x^2 - 8x + 12}{x^2 + x - 6} = \frac{(x-6)(x-2)}{(x+3)(x-2)}$$

Notice that the numerator and denominator both have a factor of $(x-2)$. When this happens, a **hole** is created because $x=2$ is both a zero and an asymptote. Therefore, $x=2$ is a hole and neither a zero nor an asymptote.

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[Figure2]

There is a vertical asymptote at x=−3 and a zero at x=6. The horizontal asymptote is at y=1. The graph of $f(x)=\frac{x^2-8x+12}{x^2-x-6}$ will look like the graph of $f(x)=\frac{x-6}{x+3}$, but with a hole at x=2. A hole is not part of the domain. And, the output value that corresponds with the hole is not part of the range. In this problem, $f(2)=\frac{2-6}{2+3}=\frac{-4}{6}=-\frac{2}{3}$ is not part of the range. If you were to graph this function on your graphing calculator, the calculator would not show there is a hole.

The domain is x∈ℝ; x≠2, −3 and the range is y∈ℝ; y≠1, $-\frac{2}{3}$.

Examples

Example 1

Earlier, you were asked to determine which student is correct.

Solution

The vertical asymptote(s) occur(s) when the denominator of the function equals zero. For the function $y=\frac{2x^4+5}{x^4-16}$, the denominator equals zero when $x^4-16=0$.

$$x^4-16=0$$
$$x^4=16$$
$$x=2 \text{ or } x=-2$$

Therefore, there are two vertical asymptotes and Darnell is correct.

Graph the following functions. Find all intercepts, asymptotes, the domain and range. Double-check your answers with a graphing calculator.

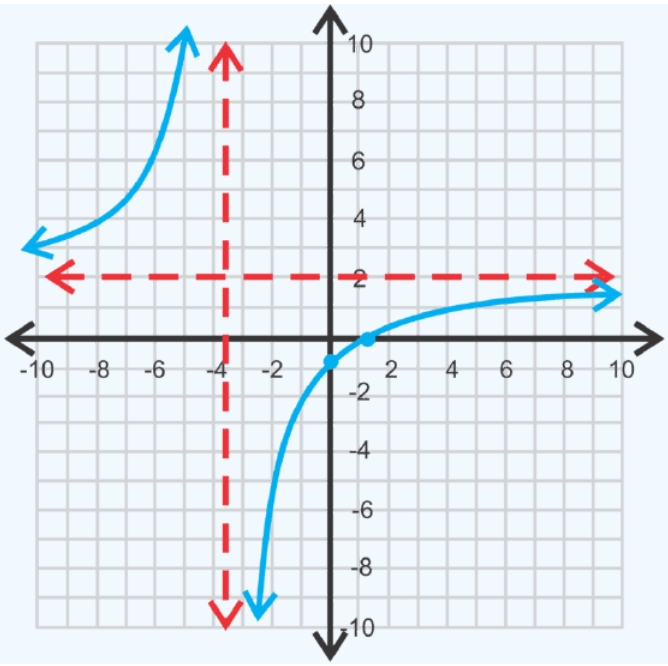
Example 2

$y=\frac{4x-5}{2x+7}$

Solution

y-intercept: $y=\frac{-5}{7}=-\frac{5}{7}$, x-intercept: $0=4x-5\rightarrow x=\frac{5}{4}$, horizontal asymptote: $y=\frac{4}{2}=2$, vertical asymptote: $2x+7=0\rightarrow x=-\frac{7}{2}$, domain: $\mathbb{R}; x\neq-\frac{7}{2}$, range: \mathbb{R}: y \neq 2

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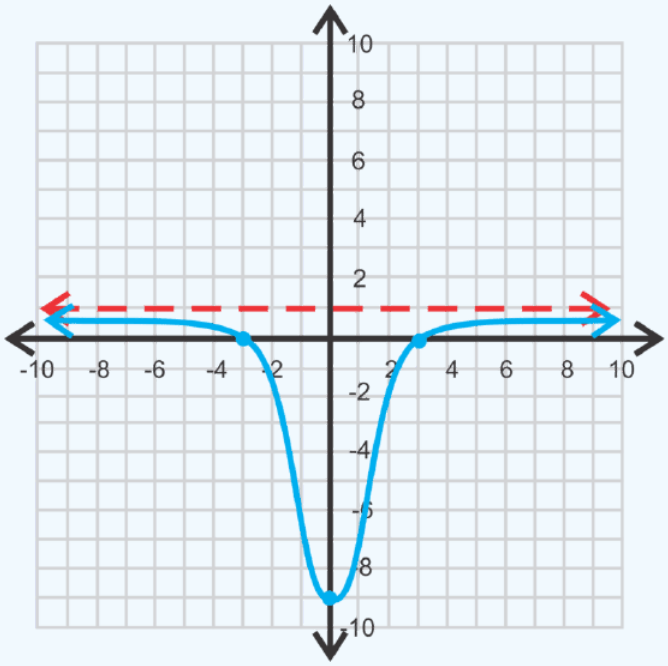
Example 3

$f(x) = \frac{x^2-9}{x^2+1}$

Solution

y-intercept: $y = \frac{-9}{1} = -9$, x-intercepts: $0=x^2-9 \rightarrow x=\pm 3$, horizontal asymptote: $y=1$, vertical asymptote: none, domain: $\mathbb{R}$, range: $\mathbb{R}$; $y \neq 1$

Special Note: When there are no vertical asymptotes and the numerator and denominator are both quadratics, this is the general shape. It could also be reflected over the horizontal asymptote.



[Figure3]

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Example 4

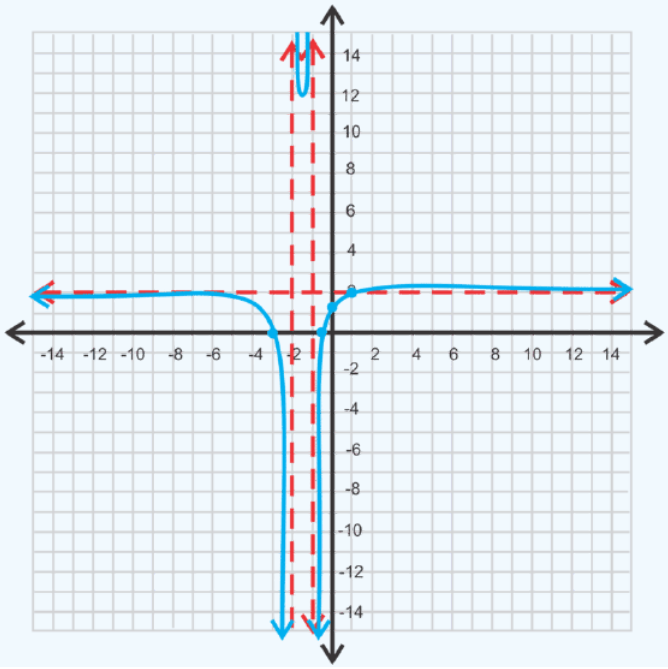
$y = \frac{2x^2+7x+3}{x^2+3x+2}$

Solution

y-intercept: $(0, \frac{3}{2})$, x-intercepts: $(-3, 0)$ and $(-\frac{1}{2}, 0)$, horizontal asymptote: y=2, vertical asymptotes: x=−2, x=−1.

domain: $\mathbb{R}$; x≠−1, −2

range: y∈(−∞, 2.1] ∪ [12, ∞)



Example 5

$y = \frac{x^2-4}{2x^2-5x+2}$

Solution

horizontal asymptote: $y = \frac{1}{2}$, y-intercept: (0,−2)

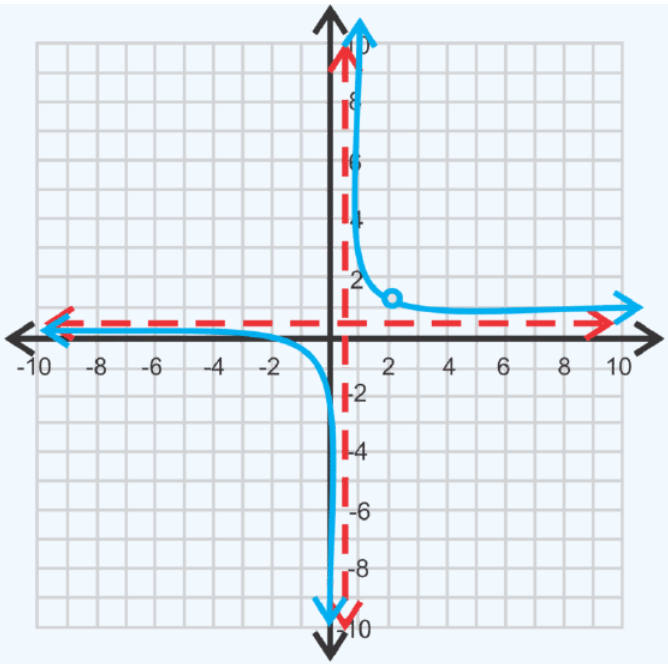
vertical asymptotes: $x = \frac{1}{2}$, x-intercept: (−2,0)

hole: $x = 2, f(2) = \frac{4}{3}$

domain: $\mathbb{R}; x \neq \frac{1}{2}, 2$

range: $\mathbb{R}; y \neq \frac{1}{2}, \frac{4}{3}$

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Review

1. What are the vertical and horizontal asymptotes for $y = \frac{x-2}{x+7}$?
2. What is the domain of this function?
3. What is the range of this function?
4. Are there any *x*-intercepts? If so, what are they?
5. Is there a *y*-intercept? If so, what is it?

Graph the following rational functions. Write down the equations of the asymptotes, the domain and range, x and y intercepts and identify any holes.

6. $y = \frac{x+3}{x-5}$
7. $y = \frac{5x+2}{x-4}$
8. $y = \frac{3-x}{2x+10}$
9. $y = \frac{x^2+5x+6}{x^2-8x+12}$
10. $y = \frac{x^2+4}{2x^2+x-3}$
11. $y = \frac{2x^2-x-10}{3x^2+10x+8}$
12. $y = \frac{x^2-4}{x^2+3x-10}$
13. $y = \frac{6x^2-7x-3}{4x^2-1}$
14. $y = \frac{x^3-8}{x^3+x^2-4x-4}$
15. Graph $y = \frac{1}{x-2} + 3$ and $y = \frac{3x-5}{x-2}$ on the same set of axes. Compare the two. What do you notice? Explain your results.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.5.

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Vocabulary

Term	Definition
Degree	The degree of a polynomial is the largest exponent of the polynomial.
Hole	A hole exists on the graph of a rational function at any input value that causes both the numerator and denominator of the function to be equal to zero.
Interval Notation	Interval notation is the notation [a,b), where a function is defined between a and b. Use (or) to indicate that the end value is not included and [or] to indicate that the end value is included. Never use [or] with infinity or negative infinity.

Image Attributions

1. [Figure 1]
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2.3.3: Graphs of Rational Functions when the Degrees are not Equal

Graphing when the Degrees of the Numerator and Denominator are Different

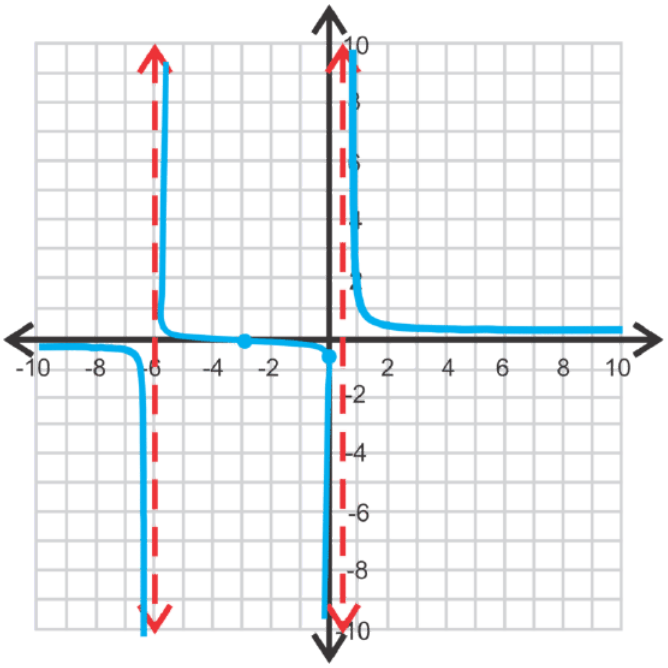
Xerxes says that the function $y = \frac{x-2}{4x^2+7}$, has a horizontal asymptote of $y = \frac{1}{4}$, Yolanda says the function has no horizontal asymptote, Zeb says that it does have a horizontal asymptote but it's at y=0. Which one of them is correct?

Graphing Rational Functions

In this concept we will touch on the different possibilities for the remaining types of rational functions. You will need to use your graphing calculator throughout this concept to ensure your sketches are correct.

Let's graph $y = \frac{x+3}{2x^2+11x-6}$ and find all asymptotes, intercepts, and the domain and range.

In this problem *the degree of the numerator is less than the degree of the denominator.* Whenever this happens the horizontal asymptote will be y=0, or the x-axis. Now, even though the x-axis is the horizontal asymptote, there will still be a zero at x=−3 (solving the numerator for x and setting it equal to zero). The vertical asymptotes will be the solutions to $2x^2+11x-6=0$. Factoring this quadratic, we have (2x−1)(x+6)=0 and the solutions are $x = \frac{1}{2}$ and −6. The y-intercept is $\left(0, -\frac{1}{2}\right)$. At this point, we can plug our function into the graphing calculator to get the general shape.

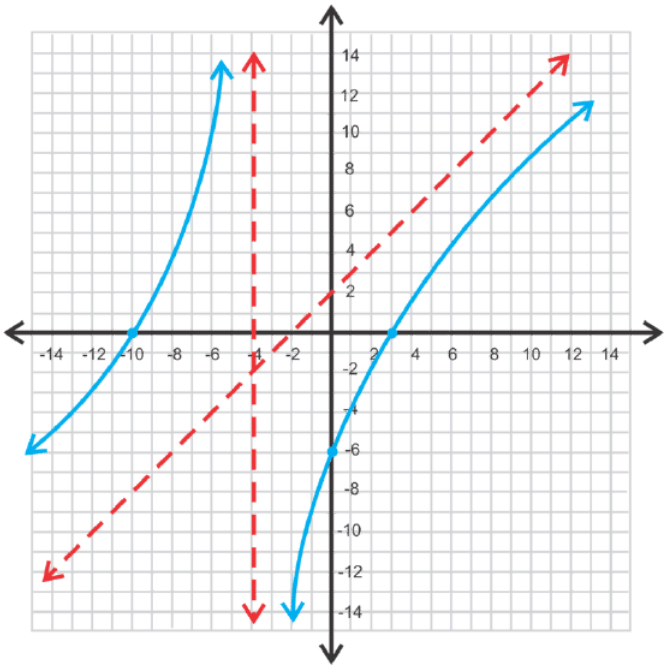


Because the middle portion crosses over the horizontal asymptote, the range will be all real numbers. The domain is $x \in \mathbb{R}; x = -6; x \neq \frac{1}{2}$.

Be careful when graphing any rational function. This function does not look like the graph to the left in a TI-83/84. This is because the calculator does not have the ability to draw the asymptotes separately and wants to make the function continuous. Make sure to double-check the table ($2^{nd} \rightarrow$ GRAPH) to find where the function is undefined.

Now, let's graph $f(x) = \frac{x^2+7x-30}{x+5}$ and find all asymptotes, intercepts, and the domain and range.

In this problem *the degree of the numerator is greater than the degree of the denominator*. When this happens, there is no horizontal asymptote. Instead there is a **slant asymptote**. Recall that this function represents division. If we were to divide $x^2+7x-30$ by x+5, the answer would be $x + 2 - \frac{20}{x+5}$. The slant asymptote would be the answer, minus the remainder. Therefore,for this problem the slant asymptote is y=x+2. Everything else is the same. The y-intercept is $\frac{-30}{5} \rightarrow (0, -6)$ and the x-intercepts are the solutions to the numerator, $x^2+7x-30=0 \rightarrow (x+10)(x-3) \rightarrow x=-10,3$. There is a vertical asymptote at x=−5. At this point, you can either test a few points to see where the branches are or use your graphing calculator.

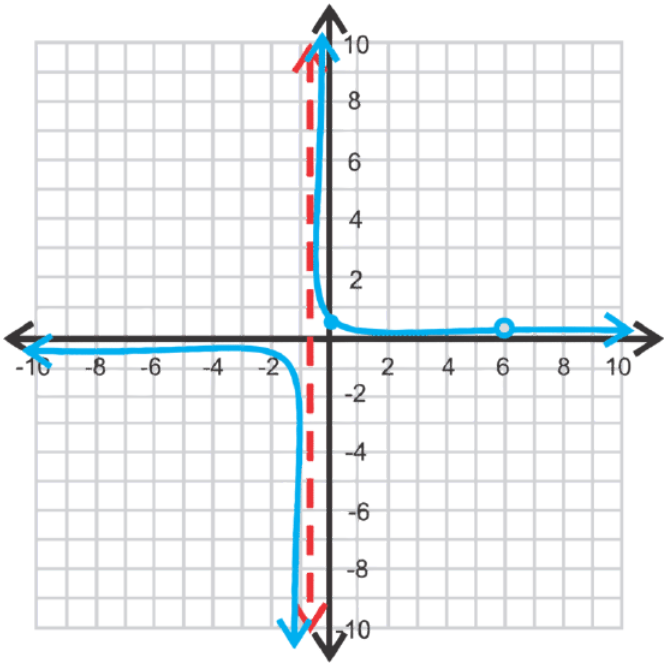


[Figure1]

The domain would be all real numbers; x≠−5. Because of the slant asymptote, there are no restrictions on the range. It is all real numbers.

Finally, let's graph $y = \frac{x-6}{3x^2-16x-12}$ and find the asymptotes and intercepts.

Because the degree of the numerator is less than the degree of the denominator, there will be a horizontal asymptote along the x-axis. Next, let's find the vertical asymptotes by factoring the denominator; (x−6)(3x+2). Notice that the denominator has a factor of (x−6), which is the entirety of the numerator. That means there will be a *hole* at x=6.



Therefore, the graph of $y = \frac{x-6}{3x^2-16x-12}$ will be the same as $y = \frac{1}{3x+2}$ except with a hole at x=6. There is no x-intercept, the vertical asymptote is at x=-\frac{2}{3} and the y-intercept is $(0, \frac{1}{2})$.

Recap

For a rational function; $f(x)=\frac{p(x)}{q(x)}=\frac{a_m x^m+...+a_0}{b_n x^n+...+b_0}$

1. If m<n, then there is a horizontal asymptote at y=0.
2. If m=n, then there is a horizontal asymptote at $y=\frac{a_m}{b_n}$ (ratio of the leading coefficients).
3. If m>n, then there is a slant asymptote at $y=(a_m x^m+...+a_0)\div(b_n x^n+...+b_0)$ without the remainder. In this concept, we will only have functions where m is one greater than n.

Examples

Example 1

Earlier, you were asked to determine which student is correct.

Solution

The degree of the numerator x−2 is less than the degree of the denominator $4x^2+7$. We know that whenever this happens the horizontal asymptote will be y=0, or the x-axis.

Therefore, Zeb is correct.

Graph the following functions. Find any intercepts and asymptotes.

Example 2

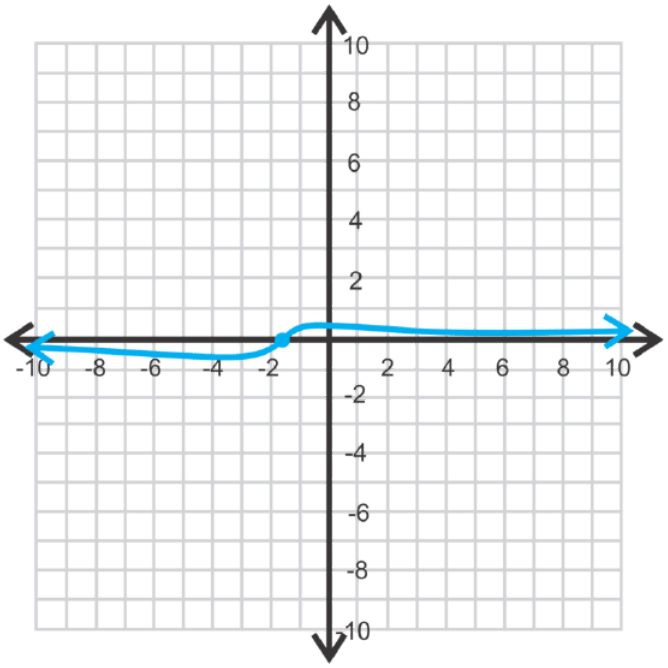
$y=\frac{3x+5}{2x^2+9x+20}$

Solution

x-intercept: $(-\frac{5}{3},0)$, y-intercept: $(0,\frac{1}{4})$

horizontal asymptote: y=0

vertical asymptotes: none



[Figure2]

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Example 3

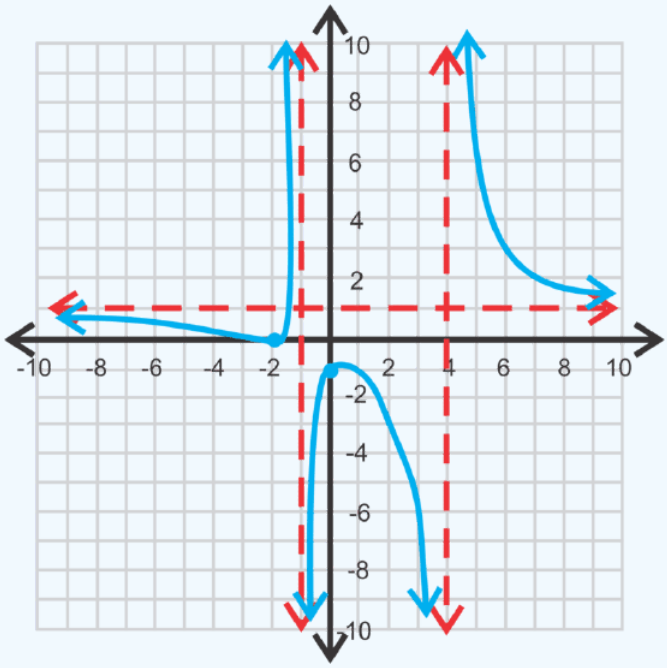
$f(x) = \frac{x^2+4x+4}{x^2-3x-4}$

Solution

x-intercept: (−2,0), y-intercept: (0,−1)

horizontal asymptote: y=1

vertical asymptotes: x=4 and x=−1



[Figure3]

Example 4

$g(x) = \frac{x^2-16}{x+3}$

Solution

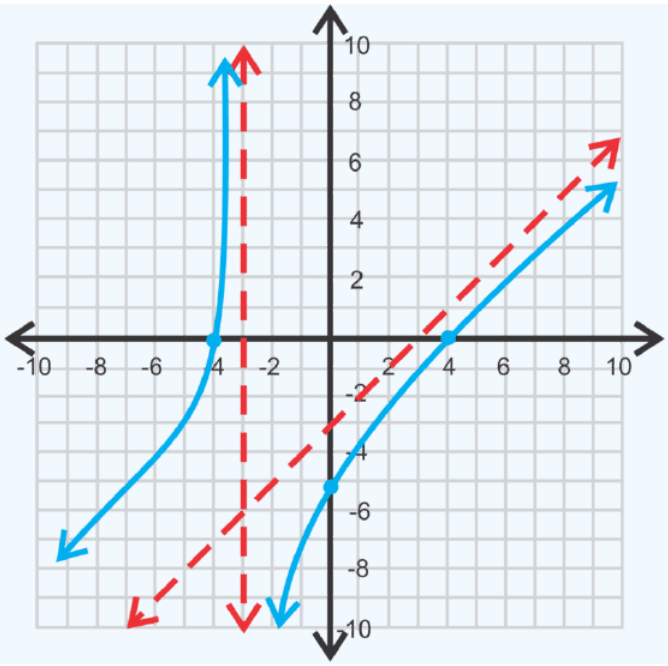
x-intercepts: (−4,0) and (4,0)

y-intercept: $\left(0, -\frac{16}{3}\right)$

slant asymptote: y=x−3

vertical asymptotes: x=−3

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Example 5

$y = \frac{2x+3}{6x^2-x-15}$

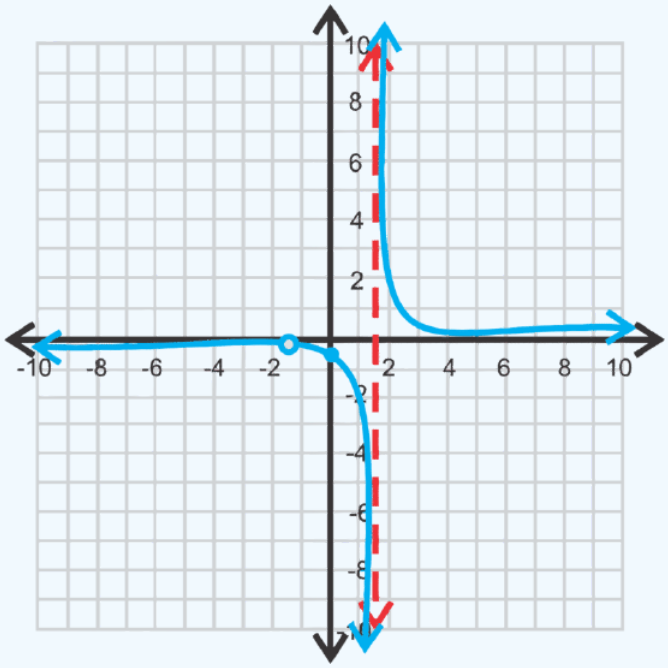
Solution

x-intercepts: none, hole at $x = -\frac{3}{2}$

y-intercept: $\left(0, -\frac{1}{5}\right)$

horizontal asymptote: y=0

vertical asymptote: $x = \frac{5}{3}$



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Review

Find all asymptotes of the following functions.

1. $y = \frac{x-2}{x^2+6x+8}$
2. $y = \frac{x^2-4}{x+5}$
3. $y = \frac{x^2}{x-3}$
4. Find the x-intercepts of the function in #2.
5. Find the x-intercepts of the function in #3.

Graph the following functions. Find any intercepts, asymptote and holes.

6. $y = \frac{x+1}{x^2-x-12}$
7. $f(x) = \frac{x^2+3x-10}{x-3}$
8. $y = \frac{x-7}{2x^2-11x-21}$
9. $g(x) = \frac{2x^2-2}{3x+5}$
10. $y = \frac{x^2+x-30}{x+6}$
11. $f(x) = \frac{x^2+x-30}{2x^3-5x^2-4x+3}$
12. $y = \frac{x^3-2x^2-3x}{x^2-5x+6}$
13. $f(x) = \frac{2x+5}{x^2+5x-6}$
14. $g(x) = \frac{-x^2+3x+4}{2x-6}$
15. Determine the slant asymptote of $y = \frac{3x^2-x-10}{3x+5}$. Now, graph this function. Is there really a slant asymptote? Can you explain your results?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 9.6.

Vocabulary

Term	Definition
Slant Asymptote	A slant asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but will never reach. A slant asymptote exists when the numerator of the function is exactly one degree greater than the denominator. A slant asymptote may be found through long division.

Image Attributions

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3. [Figure 3]
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2.3.4: Sign Test for Rational Function Graphs

Sign Test for Rational Function Graphs

The asymptotes of a rational function provide a very rigid structure in which the function must live. Once the asymptotes are known you must use the sign testing procedure to see if the function becomes increasingly positive or increasingly negative near the asymptotes. A driving question then becomes how close does near need to be in order for the **sign test** to work?

Sign Test for Rational Functions

Consider mentally substituting the number 2.99999 into the following rational expression.

$$f(x) = \frac{(x-1)(x+3)(x-5)(x+10)}{(x+2)(x-4)(x-3)}$$

Without doing any of the arithmetic, simply note the sign of each term:

$$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(-)}$$

The only term where the value is close to zero is (x−3) but careful subtraction still indicates a negative sign. The product of all of these signs is negative. This is strong evidence that this function approaches negative infinity as x approaches 3 from the left.

Next consider mentally substituting 3.00001 and going through the same process.

$$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(+)}$$

The product of all of these signs is positive which means that from the right this function approaches positive infinity instead. This technique is called the sign test. The **sign test** is a procedure for determining only whether a function is above or below the axis at a particular value.

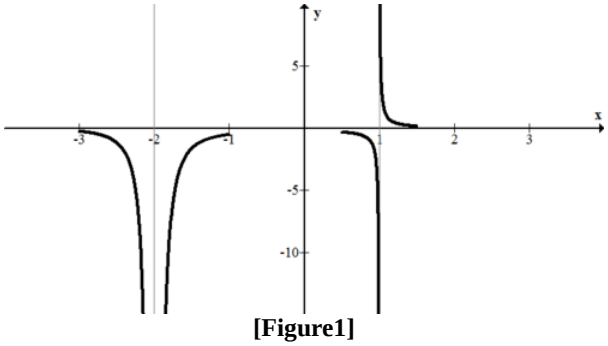
The sign test helps you sketch and graph a function. Look at the following function:

$$f(x) = \frac{1}{(x+2)^2\cdot(x-1)}$$

Your first step for sketching this is to identify the vertical asymptotes. The vertical asymptotes occur at x=−2 and x=1. Then, you use the asymptotes to perform a sign test. The points to use the sign testing procedure with are -2.001, -1.9999, 0.9999, 1.00001. The number of decimals does matter so long as the number is sufficiently close to the asymptote. Note that any real number squared is positive.

$$f(-2.001) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(0.9999) = \frac{(+)}{(+)\cdot(-)} = -$$
$$f(1.0001) = \frac{(+)}{(+)\cdot(+)} = +$$

Later when you sketch everything you will use your knowledge of zeroes and intercepts. For now, focus on just the portions of the graph near the asymptotes. Note that the graph below is NOT complete.



[Figure1]

Examples

Example 1

Earlier, you were asked how close the numbers need to be to perform the sign test.

Solution

In order to truly answer the question about how close the numbers need to be, calculus should be used. For the purposes of PreCalculus, the testing number should be closer to the **vertical asymptote** than any other number in the problem. If the vertical asymptote occurs at 3 and 3.01 is in the problem elsewhere, do not choose 3.1 as a sign test number.

Example 2

Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

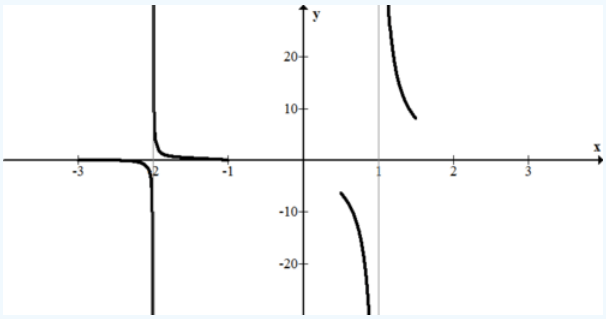
$$f(x) = \frac{(x+1)(x-4)^2(x-1)(x+3)^3}{100(x-1)^2(x+2)}$$

Solution

Note that x=−2 is clearly an asymptote. It may be initially unclear whether x=1 is an asymptote or a hole. Just like holes have priority over zeroes, asymptotes have priority over holes. The four values to use the sign testing procedure are -2.001, -1.9999, 0.9999, 1.00001.

$$f(-2.001) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = +$$
$$f(0.9999) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = -$$
$$f(1.0001) = \frac{(+)\cdot(+)\cdot(+)\cdot(+)}{(+)\cdot(+)} = +$$

A sketch of the behavior of this function near the asymptotes is:



[Figure2]

Example 3

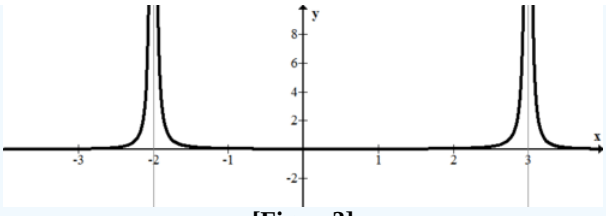
Create a function with two vertical asymptotes at 3 and -2 such that the function approaches positive infinity from both directions at both vertical asymptotes.

Solution

Earlier, there was a function that approached negative infinity from both sides of the asymptote. This occurred because the term was squared in the denominator. An even powered term will always produce a positive term.

$$f(x) = \frac{1}{(x-3)^2(x+2)^2}$$

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[Figure3]

Example 4

Create a function with three vertical asymptotes such that the function approaches negative infinity for large and small values of x and has an **oblique asymptote**.

Solution

There are an infinite number of possible solutions. The key is to create a function that may work and then use the sign testing procedure to check. Here is one possibility.

$$f(x) = \frac{-x^7}{10(x-1)^2(x-2)^2(x-4)^2}$$

Example 5

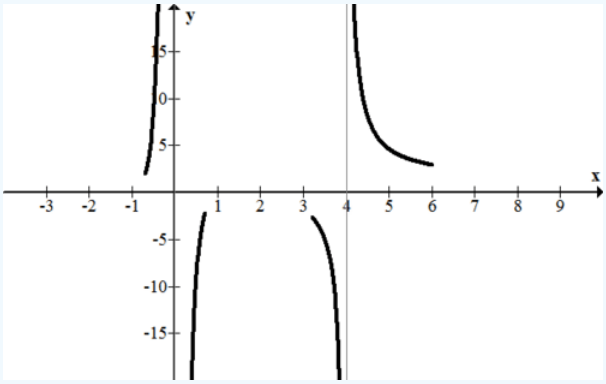
Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x-2)^3(x-1)^2(x+1)(x+3)}{x^3(x+\frac{1}{2})(x-1)(x-2)^2}$$

Solution

The vertical asymptotes occur at $x = 0, -\frac{1}{2}$. Therefore the x values to sign test are -.001, 0.001, 3.999, 4.0001.

$$f(-0.001) = +$$
$$f(0.001) = -$$
$$f(3.999) = -$$
$$f(4.0001) = +$$



[Figure4]

Review

Consider the function below for questions 1-4.

$$f(x) = \frac{(x-2)^4(x+1)(x+3)}{x^3(x+3)(x-4)}$$

1. Identify the vertical asymptotes.
2. Will this function have an oblique asymptote? A **horizontal asymptote**? If so, where?
3. What values will you need to use the sign test with in order to help you make a sketch of the graph?
4. Use the sign test and sketch the graph near the vertical asymptotes.

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Consider the function below for questions 5-8.

$g(x) = \frac{3(x-2)^2(x-1)^2(x+1)(x+3)}{15x^2(x+5)(x+1)(x-3)^2}$

5. Identify the vertical asymptotes.

6. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

7. What values will you need to use the sign test with in order to help you make a sketch of the graph?

8. Use the sign test and sketch the graph near the vertical asymptote(s).

Consider the function below for questions 9-12.

$h(x) = \frac{9x^4-102x^3+349x^2-340x+100}{x^3-9x^2+24x-16}$

9. Identify the vertical asymptotes.

10. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

11. What values will you need to use the sign test with in order to help you make a sketch of the graph?

12. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 13-16.

$k(x) = \frac{2x^3-5x^2-11x-4}{3x^3+11x^2+5x-3}$

13. Identify the vertical asymptotes.

14. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?

15. What values will you need to use the sign test with in order to help you make a sketch of the graph?

16. Use the sign test and sketch the graph near the vertical asymptotes.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.12.

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Vocabulary

Term	Definition
End behavior	End behavior is a description of the trend of a function as input values become very large or very small, represented as the 'ends' of a graphed function.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
Oblique Asymptote	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Oblique Asymptotes	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
sign test	The sign test is a procedure for determining only whether a function is above or below the x-axis at a particular x value. The specific height is not calculated.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.

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SECTION OVERVIEW

2.4: Analysis of Rational Functions

Topic hierarchy

2.4.1: Holes in Rational Functions

2.4.2: Vertical Asymptotes

2.4.3: Horizontal Asymptotes

2.4.3.1: Oblique Asymptotes of Rational Functions

2.4.4: Solving Rational Equations

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2.4.1: Holes in Rational Functions

Holes in Rational Functions

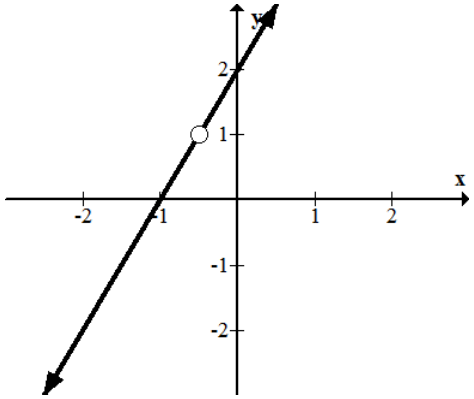
In a function like $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$, you should note that the factor (x−1) clearly cancels leaving only 3x−1. This appears to be a regular line. What happens to this line at x=1?

Holes and Rational Functions

A **hole** on a graph looks like a hollow circle. It represents the fact that the function approaches the point, but is not actually defined on that precise x value.

Take a look at the graph of the following equation:

$f(x) = (2x + 2) \cdot \frac{(x+\frac{1}{2})}{(x+\frac{1}{2})}$

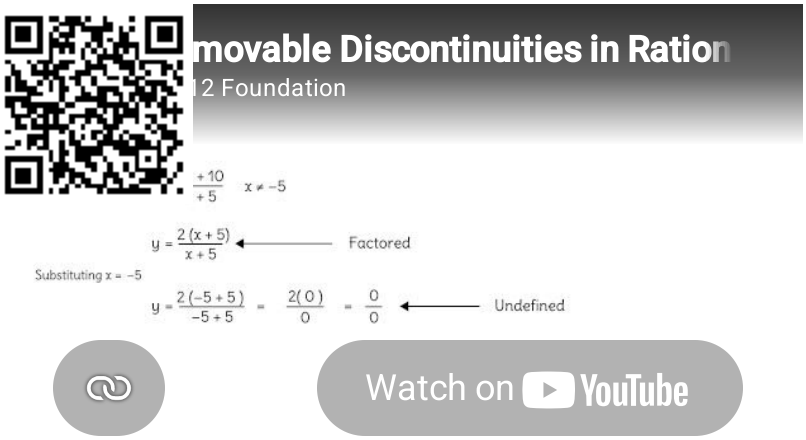


[Figure1]

The reason why this function is not defined at $-\frac{1}{2}$ is because $-\frac{1}{2}$ is not in the domain of the function. As you can see, $f\left(-\frac{1}{2}\right)$ is undefined because it makes the denominator of the rational part of the function zero which makes the whole function undefined. Also notice that once the factors are canceled/removed then you are left with a normal function which in this case is 2x+2. The hole in this situation is at $\left(-\frac{1}{2}, 1\right)$ because after removing the factors that cancel, $f\left(-\frac{1}{2}\right) = 1$.

This is the essence of dealing with holes in rational functions. You should cancel what you can and graph the function like normal making sure to note what x values make the function undefined. Once the function is graphed without holes go back and insert the hollow circles indicating what x values are removed from the domain. This is why holes are called **removable discontinuities**.

Watch the first part of this video and focus on holes in rational equations.



Examples

Example 1

Earlier, you were asked what happens to the equation $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$ at $x = 1$.

Solution

Since this function that is not defined at x=1 there is a **removable discontinuity** that is represented as a hollow circle on the graph. Otherwise the function behaves precisely as 3x+1.

Example 2

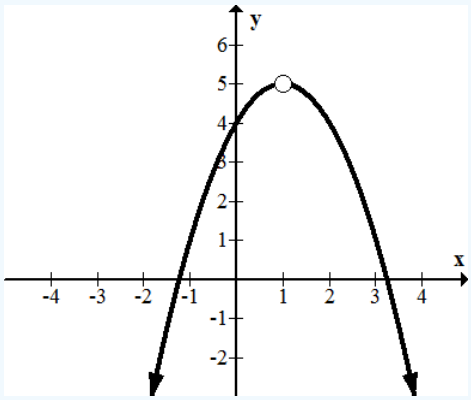
Graph the following **rational function** and identify any removable discontinuities.

$f(x) = \frac{-x^3+3x^2+2x-4}{x-1}$

Solution

This function requires some algebra to change it so that the removable factors become obvious. You should suspect that (x−1) is a factor of the numerator and try polynomial or synthetic division to factor. When you do, the function becomes:

$f(x) = \frac{(-x^2+2x+4)(x-1)}{(x-1)}$



[Figure2]

The removable discontinuity occurs at (1, 5).

Example 3

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{x^3-7x-6}$

Solution

This is probably one of the most challenging rational expressions with only holes that people ever try to graph by hand. There are multiple ways to start, but a good habit to get into is to factor everything you possibly can initially. The denominator seems less complicated with possible factors (x±1),(x±2),(x±3),(x±6). Using polynomial division, you will find the denominator becomes:

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{(x+1)(x+2)(x-3)}$

The factors of the denominator are strong hints as to the factors of the numerator so use polynomial division and try each. When you fully factor the numerator you will have:

$f(x) = \frac{(x^3-6x^2+12x-9)(x+1)(x+2)(x-3)}{(x+1)(x+2)(x-3)}$

Note the factors that cancel (x=−1,−2,3) and then work with the cubic function that remains.

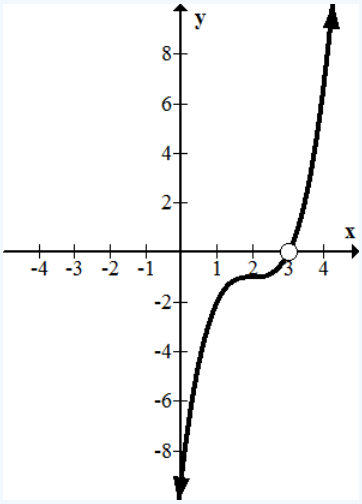
$f(x) = x^3 - 6x^2 + 12x - 9$

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At this point it is probably reasonable to make a table and plot points to get a sense of where this cubic function lives. You also could notice that the coefficients are almost of the pattern 1 3 3 1 which is the binomial expansion. By separating the -9 into -8 -1 you can factor the first four terms.

$$f(x) = x^3 - 6x^2 + 12x - 8 - 1 = (x-2)^3 - 1$$

This is a cubic function that has been shifted right by two units and down one unit.



[Figure3]

Note that there are two holes that do not fit in the graph window. When this happens you still need to note where they would appear given a properly sized window. To do this, substitute the invalid x values: x=−1,−2,3 into the factored cubic that remained after canceling.

$$f(x) = (x-2)^3 - 1$$

Holes: (3, 0); (-1, -28); (-2, -65)

Example 4

Without graphing, identify the location of the holes of the following function.

$$f(x) = \frac{x^3+4x^2+x-6}{x^2+5x+6}$$

Solution

First factor everything. Then, identify the x values that make the denominator zero and use those values to find the exact location of the holes.

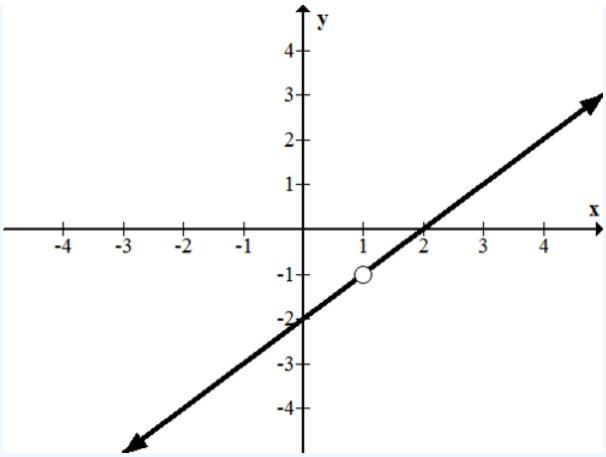
$$f(x) = \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)}$$

Holes: (-3, -4); (-2, -3)

Example 5

What is a possible equation for the following rational function?

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[Figure4]

Solution

The function seems to be a line with a removable discontinuity at (1, -1). The line is has slope 1 and y-intercept of -2 and so has the equation:

$f(x) = x - 2$

The removable discontinuity must not allow the x to be 1 which implies that it is of the form $\frac{x-1}{x-1}$. Therefore, the function is:

$f(x) = \frac{(x-2)(x-1)}{x-1}$

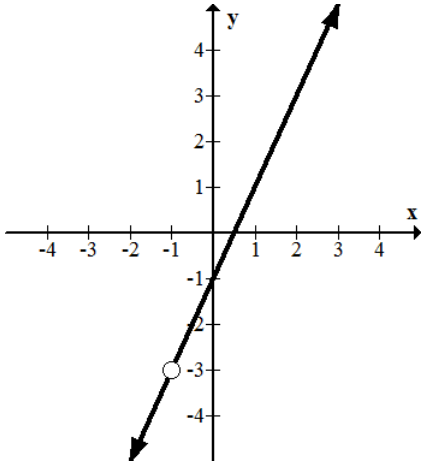
Review

1. How do you find the holes of a rational function?
2. What's the difference between a hole and a removable discontinuity?
3. If you see a hollow circle on a graph, what does that mean?

Without graphing, identify the location of the holes of the following functions.

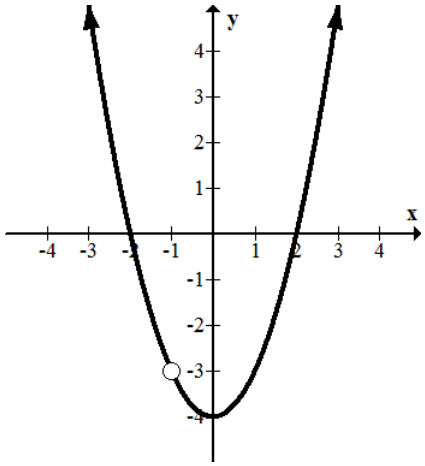
4. $f(x) = \frac{x^2+3x-4}{x-1}$
5. $g(x) = \frac{x^2+8x+15}{x+3}$
6. $h(x) = \frac{x^3+6x^2+2x-8}{x^2+x-2}$
7. $k(x) = \frac{x^3+6x^2+2x-8}{x^2-3x+2}$
8. $j(x) = \frac{x^3+4x^2-17x-60}{x^2-9}$
9. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-5x+4}$
10. $g(x) = \frac{x^3-4x^2-19x-14}{x^2-8x+7}$
11. What is a possible equation for the following rational function?

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[Figure5]

12. What is a possible equation for the following rational function?



[Figure6]

Sketch the following rational functions.

13. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-x-12}$
14. $g(x) = \frac{x^3+4x^2-17x-60}{x^2+8x+15}$
15. $h(x) = \frac{x^3-4x^2-19x-14}{x^2-6x-7}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.7.

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Vocabulary

Term	Definition
Hole	A hole exists on the graph of a rational function at any input value that causes both the numerator and denominator of the function to be equal to zero.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.
Removable discontinuities	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.

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2.4.2: Vertical Asymptotes

Vertical Asymptotes

The basic **rational function** $f(x) = \frac{1}{x}$ is a hyperbola with a **vertical asymptote** at x=0. More complicated rational functions may have multiple vertical asymptotes. These asymptotes are very important characteristics of the function just like holes. Both holes and vertical asymptotes occur at x values that make the denominator of the function zero. A driving question is: what makes vertical asymptotes different from holes?

Finding Vertical Asymptotes

Vertical asymptotes occur when a factor of the denominator of a rational expression does not cancel with a factor from the numerator. When you have a factor that does not cancel, instead of making a **hole** at that x value, there exists a vertical asymptote. The vertical asymptote is represented by a dotted vertical line. Most calculators will not identify vertical asymptotes and some will incorrectly draw a steep line as part of a function where the asymptote actually exists.

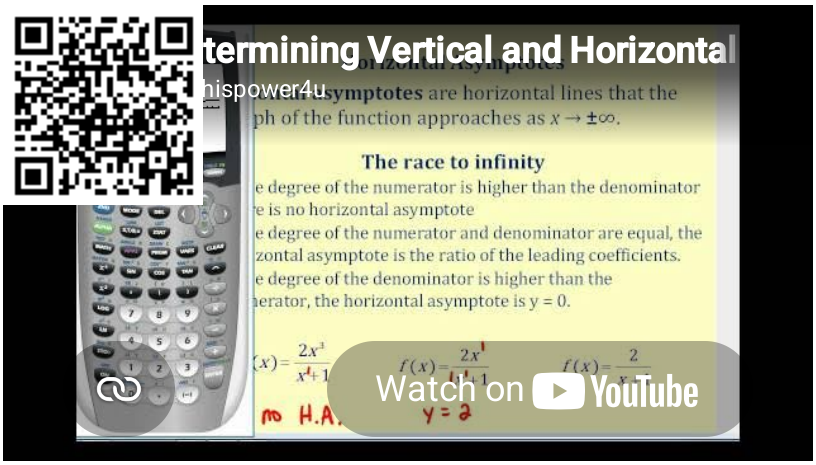
Your job is to be able to identify vertical asymptotes from a function and describe each asymptote using the equation of a vertical line.

Take the following rational function:

$f(x) = \frac{(2x-3)(x+1)(x-2)}{(x+2)(x+1)}$

To identify the holes and the equations of the vertical asymptotes, first decide what factors cancel out. The factor that cancels represents the **removable discontinuity**. There is a hole at (-1, 15). The vertical asymptote occurs at x=−2 because the factor x+2 does not cancel.

Watch the following video, focusing on the parts about vertical asymptotes.



Examples

Example 1

Earlier, you were asked how asymptotes are different than holes.

Solution

Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the **domain** of a rational function.

Example 2

Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- **Zeroes** at 2 and 5
- Hole at (4, 2)

Solution

Each criteria helps build the function. The vertical asymptotes imply that the denominator has two factors that do not cancel with the numerator:

$\frac{1}{x \cdot (x-3)}$

The zeroes at 2 and 5 imply the numerator has two factors that do not cancel.

$\frac{(x-2)(x-5)}{x \cdot (x-3)}$

The hole at (4, 2) implies that there is a factor x−4 that cancels on the numerator and the denominator.

$\frac{(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$

The tricky part is that the height of the function must be 2 after the x−4 factor has been canceled and the 4 is substituted in. Currently it is $-\frac{1}{2}$.

$\frac{(4-2)(4-5)}{4 \cdot (4-3)} = -\frac{1}{2}$

In order to make the hole exist at a height of 2, you need to multiply the function by a scalar of -4.

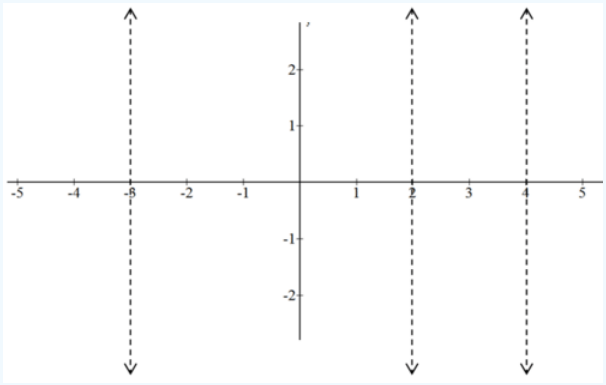
$f(x) = \frac{-4(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$

Example 3

Draw the vertical asymptotes for the following function.

$f(x) = \frac{1}{(x-4)(x-2)(x+3)}$

Solution



[Figure1]

Note that you may not know the characteristics of what the function does inside these vertical lines. You will soon learn how to use sign tests as well as techniques you've already learned to fill in the four sections that this function is divided into.

Example 4

Identify the holes and equations of the vertical asymptotes of the following rational function.

$f(x) = \frac{3(x-1)(x+2)(x-3)(x+4)}{5(x+\frac{1}{2})(2+x)(3-x)(x-8)}$

Solution

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The vertical asymptotes occur at $x = -\frac{1}{2}, x = 8$. Holes occur when x is -2 and 3. To get the height of the holes at these points, remember to cancel what can be canceled and then substitute the values. A very common mistake is to forget to cancel $\frac{x-3}{3-x} = -1$.

$$g(x) = \frac{-3(x-1)(x+4)}{5\left(x+\frac{1}{2}\right)(x-8)}$$

$$g(-2) = \frac{6}{25}$$

$$g(3) = \frac{12}{25}$$

The holes are at $\left(-2, \frac{6}{25}\right), \left(3, \frac{12}{25}\right)$.

Example 5

Identify the domain of the following function and then identify the holes and vertical asymptotes.

$f(x) = \frac{(3x-4)(1-x)(x^2+4)}{(3x-2)(x-1)}$

Solution

The domain of the function written in **interval notation** is: $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$. Note that the domain is all real numbers except for where the denominator is zero.

There are two discontinuities: one is a hole and one is a vertical asymptote. The hole occurs at (1, 5). The vertical asymptote occurs at $x = \frac{2}{3}$.

Notice that holes are identified as points while vertical asymptotes are identified as lines of the form x=a where a is some constant.

Review

1. Write a function that fits the following criteria:

- Vertical asymptotes at 1 and 4
- Zeroes at 3 and 5
- Hole at (6, 3)

2. Write a function that fits the following criteria:

- Vertical asymptotes at -2 and 2
- Zeroes at 1 and 5
- Hole at (3, -4)

3. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 1 and 2
- Hole at (8, 21)

4. Write a function that fits the following criteria:

- Vertical asymptotes at 2 and 6
- Zero at 5
- Hole at (4, 1)

5. Write a function that fits the following criteria:

- Vertical asymptote at 4
- Zeroes at 0 and 3
- Hole at (5, 10)

Give the equations of the vertical asymptotes for the following functions.

6. $f(x) = \frac{(2-x)}{(x-2)(x-4)}$
7. $g(x) = \frac{-x}{(x+1)(x-3)}$
8. $h(x) = 6 - \frac{x+2}{(x+1)(x-5)}$
9. $j(x) = \frac{10}{x-3} - \frac{x}{(x+2)(x-3)}$
10. $k(x) = 2 - \frac{(4-x)}{(x+3)(x-4)}$

Identify the holes and equations of the vertical asymptotes of the following rational functions.

11. $f(x) = \frac{3(x-1)(x+1)(x-4)(x+4)}{4(x+4)(2+x)(4-x)(x+1)}$
12. $g(x) = \frac{x(x-3)(x-8)(x-3)(x+4)}{7(x+1)(1+x)(3-x)(x-8)}$

State the domain of the following rational functions.

13. $h(x) = \frac{x(x+1)(x-3)(x+4)}{x(3-x)(x-1)}$
14. $j(x) = \frac{x^2+3x-4}{x^2-6x-16}$
15. $k(x) = \frac{2x-10}{x^3+4x^2+3x}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.9.

Vocabulary

Term	Definition
domain	The domain of a function is the set of x-values for which the function is defined.
Hole	A hole exists on the graph of a rational function at any input value that causes both the numerator and denominator of the function to be equal to zero.
Interval Notation	Interval notation is the notation [a,b), where a function is defined between a and b. Use (or) to indicate that the end value is not included and [or] to indicate that the end value is included. Never use [or] with infinity or negative infinity.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeros	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.

Image Attributions

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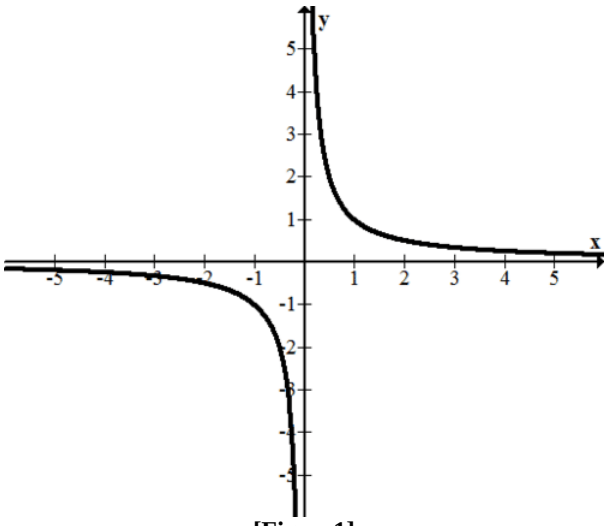
2.4.3: Horizontal Asymptotes

Horizontal Asymptotes

Vertical asymptotes describe the behavior of a function as the values of x approach a specific number. **Horizontal asymptotes** describe the behavior of a function as the values of x become infinitely large and infinitely small. Since functions cannot touch vertical asymptotes, are they not allowed to touch horizontal asymptotes either?

Finding Horizontal Asymptotes

Horizontal asymptotes are a means of describing **end behavior** of a function. End behavior essentially is a description of what happens on either side of the graph as the function continues to the right and left infinitely. When you are determining the horizontal asymptotes, it is important to consider both the right and the left hand sides, because the horizontal asymptotes will not necessarily be the same in both places. Consider the reciprocal function and note how as x goes to the right and left it flattens to the line y=0.



[Figure1]

Sometimes functions flatten out and other times functions increase or decrease without bound. There are basically three cases.

Case 1: Degree of Numerator is Less than Degree of Denominator

The first case is the function flattens out to 0 as x gets infinitely large or infinitely small. This happens when the degree of the numerator is less than the degree of the denominator. The degree is determined by the greatest exponent of x.

$$f(x) = \frac{2x^8+3x^2+100}{x^9-12}$$

One way to reason through why this makes sense is because when x is a ridiculously large number then most parts of the function hardly make any impact. The 100 for example is nothing in comparison and neither is the $3x^2$. The two important terms to compare are x^8 and x^9. The 2 isn't even important now because if x is even just a million than the x^9 will be a million times bigger than the x^8 and the 2 hardly matters again. Essentially, when x gets big enough, this function acts like $\frac{1}{2}$ which has a **horizontal asymptote** of 0.

Case 2: Degree of Numerator is Equal to the Degree of Denominator

If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is equal to the ratio of the leading coefficients.

$$f(x) = \frac{6x^4-3x^3+12x^2-9}{3x^4+144x-0.001}$$

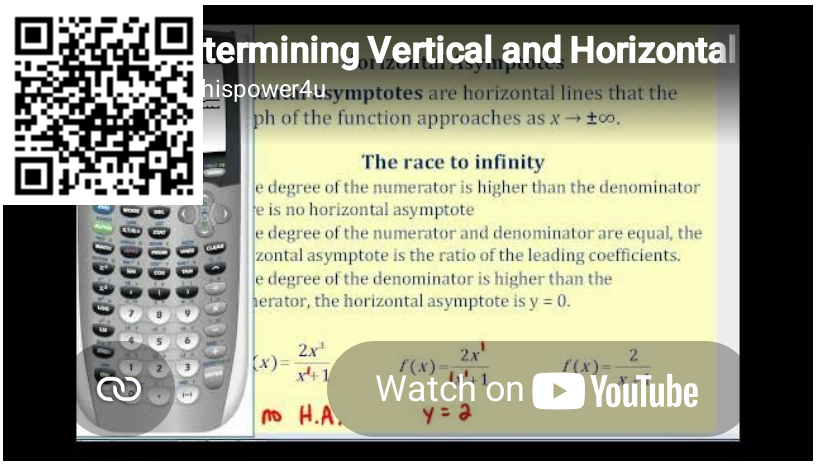
Notice how the degree of both the numerator and the denominator is 4. This means that the horizontal asymptote is $y = \frac{6}{3} = 2$. One way to reason through why this makes sense is because when x gets to be a very large number all the smaller powers will not really

make much of an impact. The biggest contributors are only the biggest powers. Then the value of the numerator will be about twice that of the denominator. As x gets even bigger, then the function will get even closer to 2.

Case 3: Degree of Numerator is Greater than the Degree of Denominator

If the degree of the numerator is greater than the degree of the denominator, there does not exist a horizontal asymptote. You must determine if the function increases or decreases without bound in both the left and right directions.

Watch the following video, focusing on the parts about horizontal asymptotes.



Examples

Example 1

Earlier, you were asked if functions are allowed to touch their horizontal asymptotes.

Solution

Functions may touch and pass through horizontal asymptotes without limit. This is a difference between vertical and horizontal asymptotes. In calculus, there are rigorous proofs to show that functions like the one in Example C do become arbitrarily close to the asymptote.

Example 2

Identify the vertical and horizontal asymptotes of the following **rational function**.

$$f(x) = \frac{(x-2)(4x+3)(x-4)}{(x-1)(4x+3)(x-6)}$$

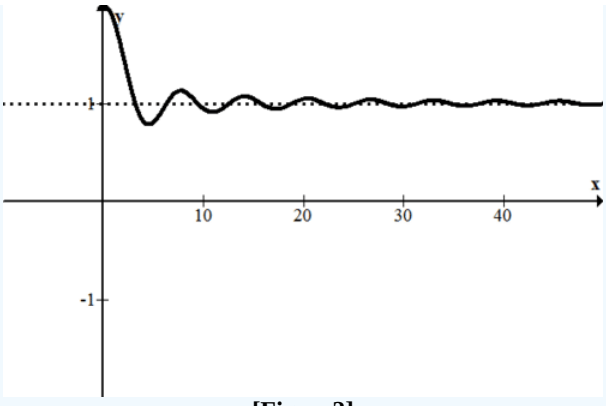
Solution

There is factor that cancels that is neither a horizontal or **vertical asymptote**. The vertical asymptotes occur at x=1 and x=6. To obtain the horizontal asymptote you could methodically multiply out each binomial, however since most of those terms do not matter, it is more efficient to first determine the relative powers of the numerator and the denominator. In this case they both happen to be 3. Next determine the coefficient of the cubic terms only. The numerator will have $4x^3$ and the denominator will have $4x^3$ and so the horizontal asymptote will occur at $y = \frac{4}{4} = 1$.

Example 3

Describe the right hand end behavior of the following function.

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[Figure2]

Solution

Notice how quickly this dampening wave function settles down. There seems to be an obvious horizontal axis on the right at y=1

Example 4

Identify the horizontal asymptotes of the following function.

$$f(x) = \frac{(x-3)(x+2)}{|(x-5)|\cdot(x-1)}$$

Solution

Add text hereFirst notice the absolute value surrounding one of the terms in the denominator. The degrees of both the numerator and the denominator will be 2 which means that the horizontal asymptote will occur at a number. As x gets infinitely large, the function is approximately:

$$f(x) = \frac{x^2}{x^2}$$

So the horizontal asymptote is y=−1 as x gets infinitely large.

On the other hand, as x gets infinitely small the function is approximately:

$$f(x) = \frac{x^2}{-x^2}$$

So the horizontal asymptote is y=−1 as x gets infinitely small.

In this case, you cannot blindly use the leading coefficient rule because the absolute value changes the sign.

Example 5

Identify the horizontal asymptotes if they exist for the following 3 functions.

1. $f(x) = \frac{3x^6-72x}{x^6+999}$
2. $h(x) = \frac{ax^4+bx^3+cx^2+dx+e}{fx^4+gx^3+hx^2}$
3. $g(x) = \frac{f(x)}{h(x)}$

Solution

1. The degrees of the numerator and the denominatro are equal so the horizontal asymptote isy=3.
2. The degrees of the numerator and the denominator are equal again so the horizontal asymptote is $y = \frac{a}{f}$
3. As x gets infinitely large, $g(x) = \frac{f(x)}{h(x)} = \frac{\frac{3x^6-72x}{x^6+999}}{\frac{ax^4+bx^3+cx^2+dx+e}{fx^4+gx^3+hx^2}} \approx \frac{3}{\frac{a}{f}} = \frac{3f}{a}$

When you study calculus, you will learn the rigorous techniques that enable you to feel more confident about results like this.

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Review

Identify the horizontal asymptotes, if they exist, for the following functions.

1. $f(x) = \frac{5x^4-2x}{x^4+32}$
2. $g(x) = \frac{3x^4-2x^6}{-x^4+2}$
3. $h(x) = \frac{3x^4-5x}{8x^3+3x^4}$
4. $j(x) = \frac{2x^3-15x}{-x^4+3}$
5. $k(x) = \frac{2x^5-3x}{5x^2+3x^4+2x-7x^5}$
6. $f(x) = \frac{ax^{14}+bx^{23}+cx^{12}+dx+e}{fx^{24}+gx^{23}+hx^{21}}$
7. $g(x) = \frac{(x-1)(x+4)}{|(x-2)|\cdot(x-1)}$
8. Write a function that fits the following criteria:
   - Vertical asymptotes at x=1 and x=4
   - **Zeroes** at 3 and 5
   - **Hole** when x=6
   - Horizontal asymptote at $y = \frac{2}{3}$
9. Write a function that fits the following criteria:
   - Vertical asymptotes at x=−2 and x=2
   - Zeroes at 1 and 5
   - Hole when x=3
   - Horizontal asymptote at y=1
10. Write a function that fits the following criteria:
    - Vertical asymptotes at x=0 and x=3
    - Zeroes at 1 and 2
    - Hole when x=8
    - Horizontal asymptote at y=2
11. Write a function that fits the following criteria:
    - Vertical asymptotes at 2 and 6
    - **Zero** at 5
    - Hole when x=4
    - Horizontal asymptote at y=0
12. Write a function that fits the following criteria:
    - Vertical asymptote at 4
    - Zeroes at 0 and 3
    - Hole at when x=5
    - No horizontal asymptotes

Identify the vertical and horizontal asymptotes of the following rational functions.

13. $f(x) = \frac{(x-5)(2x+1)(x-3)}{(x-3)(4x+5)(x-6)}$
14. $g(x) = \frac{x(x-1)(x+3)(x-5)}{3x(x-1)(4x+3)}$
15. $h(x) = \frac{(x-2)(x+3)(x-6)}{(x-4)(x+3)^2(x+2)}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

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Vocabulary

Term	Definition
End behavior	End behavior is a description of the trend of a function as input values become very large or very small, represented as the 'ends' of a graphed function.
Hole	A hole exists on the graph of a rational function at any input value that causes both the numerator and denominator of the function to be equal to zero.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.
Zero	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeros	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.

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2.4.3.1: Oblique Asymptotes of Rational Functions

Oblique Asymptotes

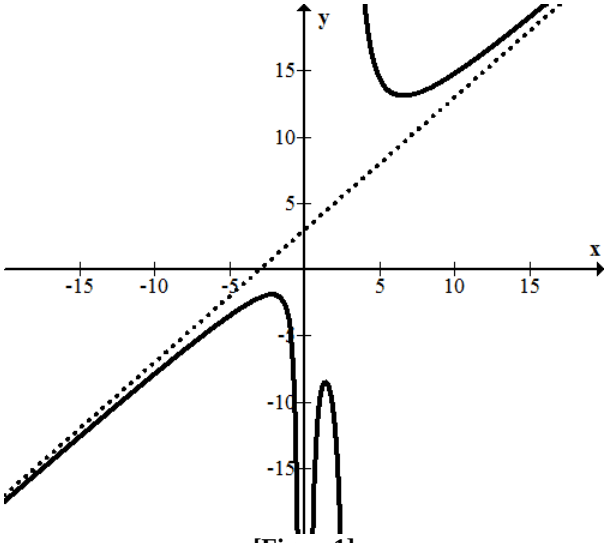
When the degree of the numerator of a rational function exceeds the degree of the denominator by one then the function has **oblique asymptotes**. In order to find these asymptotes, you need to use polynomial long division and the non-remainder portion of the function becomes the oblique asymptote. A natural question to ask is: what happens when the degree of the numerator exceeds that of the denominator by more than one?

Oblique Asymptotes

The following function is shown before and after polynomial long division is performed.

$$f(x) = \frac{x^4+3x^2+2x+14}{x^3-3x^2} = x + 3 + \frac{12x^2+2x+14}{x^3-3x^2}$$

Notice that the remainder portion will go to zero when x gets extremely large or extremely small because the power of the numerator is smaller than the power of the denominator. This means that while this function might go haywire with small absolute values of x, large absolute values of x are extremely close to the line y=x+3.



[Figure1]

Oblique asymptotes are these slanted asymptotes that show exactly how a function increases or decreases without bound. Oblique asymptotes are also called slant asymptotes.

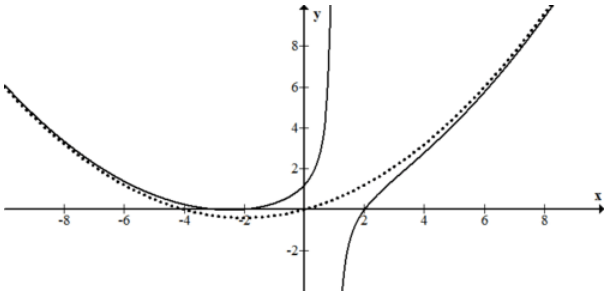
Sometimes a function will have an asymptote that does not look like a line. Take a look at the following function:

$$f(x) = \frac{(x^2-4)(x+3)}{10(x-1)}$$

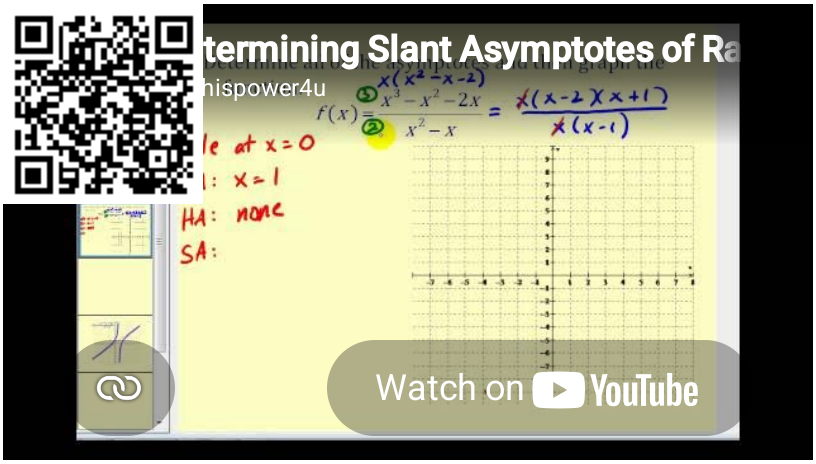
The degree of the numerator is 3 while the degree of the denominator is 1 so the slant asymptote will not be a line. However when the graph is observed, there is still a clear pattern as to how this function increases without bound as x approaches very large and very small numbers.

$$f(x) = \frac{1}{10}\left(x^2+4x\right) - \frac{12}{10(x-1)}$$

As you can see, this looks like a parabola with a remainder. This rational function has a parabola backbone. A backbone is a function that a graph tends towards. This is not technically an oblique asymptote because it is not a line.



[Figure2]



Examples

Example 1

Earlier, you were asked what happens when the power of the numerator exceeds the power of the denominator by more than one.

Solution

As seen above, when the power of numerator exceeds the power of the denominator by more than one, the function develops a backbone that be shaped like any polynomial. Oblique asymptotes are always lines.

Example 2

Find the asymptotes and intercepts of the function:

$f(x)=\frac{x^3}{x^2-4}$

Solution

The function has vertical asymptotes at x=±2.

After long division, the function becomes:

$f(x)=x+\frac{4}{x^2-4}$

This makes the oblique asymptote at y=x

Example 3

Identify the vertical and oblique asymptotes of the following rational function.

$f(x)=\frac{x^3-x^2-x-1}{(x-3)(x+4)}$

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Solution

After using polynomial long division and rewriting the function with a remainder and non-remainder portion it looks like this:

$$f(x) = x - 2 + \frac{13x-25}{x^2+x-12} = x - 2 + \frac{13x-25}{(x-3)(x+4)}$$

The oblique asymptote is y=x−2. The vertical asymptotes are at x=3 and x=−4 which are easier to observe in last form of the function because they clearly don't cancel to become holes.

Example 4

Create a function with an oblique asymptote at y=3x−1, vertical asymptotes at x=2,−4 and includes a hole where x is 7.

Solution

While there are an infinite number of functions that match these criteria, one example is:

$$f(x) = 3x - 1 + \frac{(x-7)}{(x-2)(x+4)(x-7)}$$

Example 5

Identify the backbone of the following function and explain why the function does not have an oblique asymptote.

$$f(x) = \frac{5x^5+27}{x^3}$$

Solution

While polynomial long division is possible, it is also possible to just divide each term by x^3.

$$f(x) = \frac{5x^5+27}{x^3} = \frac{5x^5}{x^3} + \frac{27}{x^3} = 5x^2 + \frac{27}{x^3}$$

The backbone of this function is the parabola $y=5x^2$. This is not an oblique asymptote because it is not a line.

Review

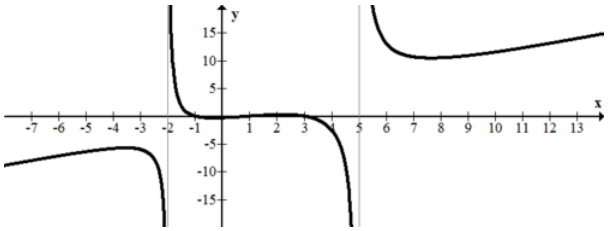
1. What is an oblique asymptote?

2. How can you tell by looking at the equation of a function if it will have an oblique asymptote or not?

3. Can a function have both an oblique asymptote and a horizontal asymptote? Explain.

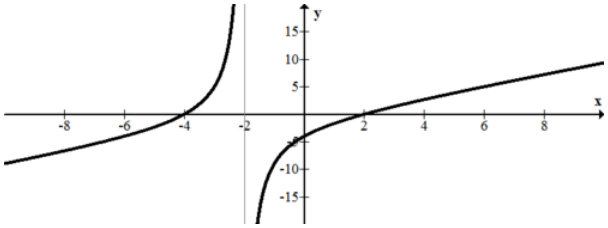
For each of the following graphs, sketch the graph and then sketch in the oblique asymptote if it exists. If it doesn't exist, explain why not.

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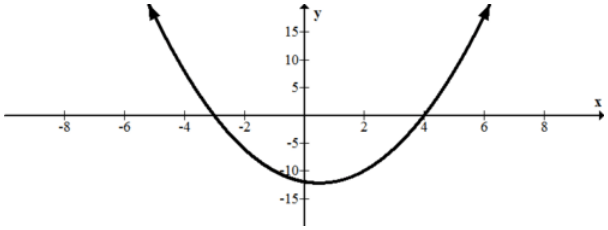


[Figure3]

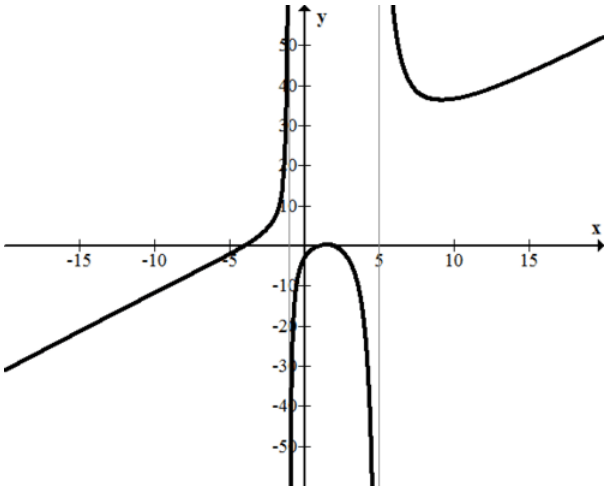
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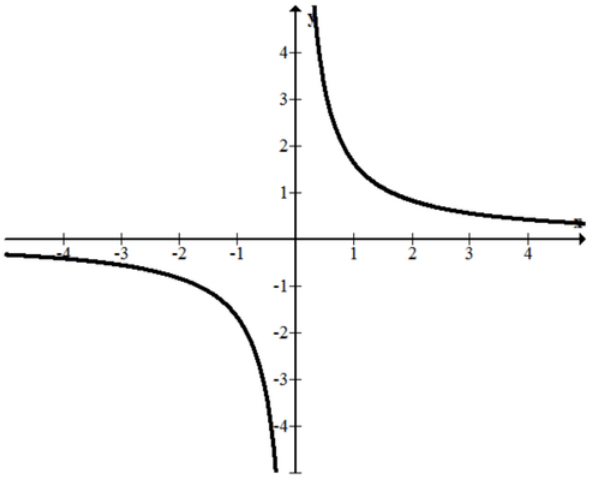
[Figure4]

6. 

[Figure5]

7. 

[Figure6]

8. 

[Figure7]

Find the equation of the oblique asymptote for each of the following rational functions. If there is not an oblique asymptote, explain why not and give an equation of the backbone of the function if one exists.

9. $f(x) = \frac{x^3-7x-6}{x^2-2x-15}$
10. $g(x) = \frac{x^3-7x-6}{x^4-3x^2-10}$
11. $h(x) = \frac{x^2+5x+6}{x^2+2x+1}$
12. $k(x) = \frac{x^4+9x^3+21x^2-x-30}{x^2+2x+1}$
13. Create a function with an oblique asymptotes at y=2x−1, a vertical asymptote at x=3 and a hole where x is 7.
14. Create a function with an oblique asymptote at y=x, vertical asymptotes at x=1,−3 and no holes.
15. Does a parabola have an oblique asymptote? What about a cubic function?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.11.

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2.4.4: Solving Rational Equations

Solving Rational Equations

The techniques for solving rational equations are extensions of techniques you already know. Recall that when there are fractions in an equation you can multiply through by the denominator to clear the fraction. The same technique helps turn rational expressions into polynomials that you already know how to solve. When you multiply by a constant there is no problem, but when you multiply through by a value that varies and could possibly be zero interesting things happen.

Since every equation is trivially true when both sides are multiplied by zero, how do you account for this when solving rational equations?

Finding Solutions to Rational Equations

The first step in solving rational equations is to transform the equation into a polynomial equation. This is accomplished by clearing the fraction which means multiplying the entire equation by the common denominator of all the rational expressions. Then you should solve using what you already know. The last thing to check once you have the solutions is that they do not make the denominators of any part of the equation equal to zero when substituted back into the original equation. If so, that solution is called **extraneous** and is a "fake" solution that was introduced when both sides of the equation were multiplied by a number that happened to be zero.

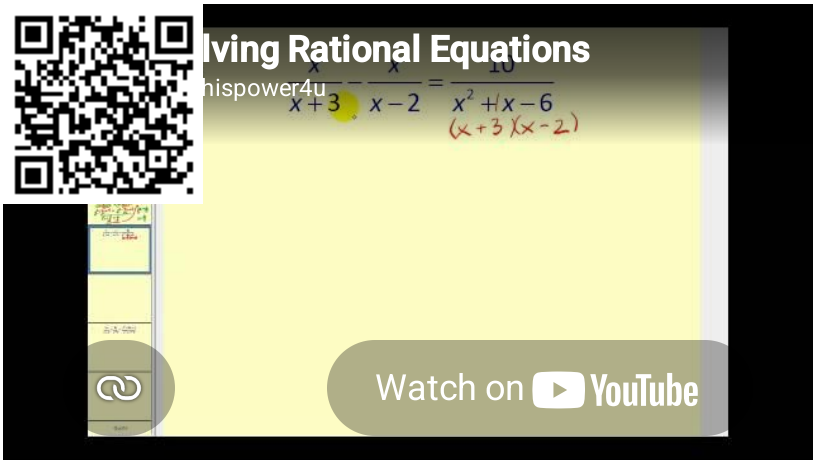
Take the following **rational equation**:

$$x - \frac{5}{x+3} = 12$$

To find the solutions of the equation, first multiply all parts of the equation by (x+3), the common denominator, and then simplify.

$$\begin{aligned} x(x+3) - 5 &= 12(x+3) \\ x^2 + 3x - 5 - 12x - 36 &= 0 \\ x^2 - 9x - 41 &= 0 \\ x &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \cdot 1 \cdot (-41)}}{2 \cdot 1} \\ x &= \frac{9 \pm 7\sqrt{5}}{2} \end{aligned}$$

The only potential extraneous solution would have been -3 since that is the number that makes the denominator of the original equation zero. Therefore, both answers are possible.



Examples

Example 1

Earlier, you were asked to account for the extra solutions introduced when both sides of an equation are multiplied by a variable.

Solution

In order to deal with the possible extra solutions, you must check each solution to see if it makes the denominator of any fraction in the original equation zero. If it does, it is called an extraneous solution.

Example 2

Solve the following rational equation

$\frac{3x}{x+4} - \frac{1}{x+2} = -\frac{2}{x^2+6x+8}$

Solution

Multiply each part of the equation by the common denominator of $x^2+6x+8=(x+2)(x+4)$.

$$\begin{aligned}(x+2)(x+4)\left[\frac{3x}{x+4} - \frac{1}{x+2}\right] &= \left[\frac{-2}{(x+2)(x+4)}\right](x+2)(x+4)\\ 3x(x+2) - (x+4) &= -2\\ 3x^2 + 6x - x - 2 &= 0\\ 3x^2 + 5x - 2 &= 0\\ (3x-1)(x+2) &= 0\\ x &= \frac{1}{3}, -2\end{aligned}$$

Note that -2 is an extraneous solution. The only actual solution is $x = \frac{1}{3}$.

Example 3

Solve the following rational equation for y.

$x = 2 + \frac{1}{2+\frac{1}{y+1}}$

This question can be done multiple ways. You can use the clearing fractions technique twice.

Solution

$$\begin{aligned}\left(2+\frac{1}{y+1}\right)x &= \left[2+\frac{1}{2+\frac{1}{y+1}}\right]\left(2+\frac{1}{y+1}\right)\\ 2x + \frac{x}{y+1} &= 2\left(2+\frac{1}{y+1}\right)+1\\ 2x + \frac{x}{y+1} &= 4 + \frac{2}{y+1} + 1\\ (y+1)\left[2x + \frac{x}{y+1}\right] &= \left[5 + \frac{2}{y+1}\right](y+1)\\ 2x(y+1) + x &= 5(y+1) + 2\\ 2xy + 2x + x &= 5y + 5 + 2\end{aligned}$$

Now just get the y variable to one side of the equation and everything else to the other side.

$$\begin{aligned}2xy - 5y &= -3x + 7\\ y(2x-5) &= -3x + 7\\ y &= \frac{-3x+7}{2x-5}\end{aligned}$$

Example 4

Solve the following rational equation.

$\frac{3x}{x-5} + 4 = x$

Solution

$$\begin{aligned} 3x + 4x - 20 &= x^2 - 5x \\ 0 &= x^2 - 12x + 20 \\ 0 &= (x-2)(x-10) \\ x &= 2, 10 \end{aligned}$$

Neither solution is extraneous.

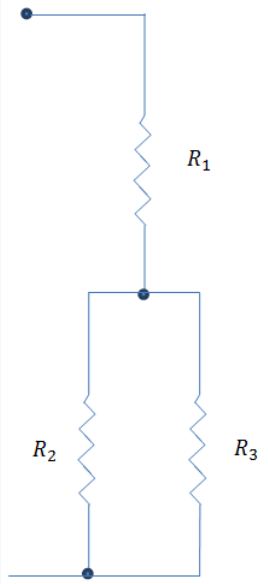
Example 5

In electrical circuits, resistance can be solved for using rational expressions. This is an electric circuit diagram with three resistors. The first resistor R_1 is run in series to the other two resistors R_2 and R_3 which are run in parallel. If the total resistance R is 100 ohms and R_1 and R_3 are each 22 ohms, what is the resistance of R_2?

Solution

The equation of value is:

$$R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$$



[Figure1]

$$\begin{aligned} R &= R_1 + \frac{R_2 R_3}{R_2 + R_3} \\ 100 &= 22 + \frac{x \cdot 22}{x + 22} \\ 78(x+22) &= 22x \\ 78x + 1716 &= 22x \\ 56x &= -1716 \\ x &= -30.65 \end{aligned}$$

A follow up question would be to ask whether or not ohms can be negative which is beyond the scope of this text.

Review

Solve the following rational equations. Identify any extraneous solutions.

1. $\frac{2x-4}{x} = \frac{16}{x}$
2. $\frac{4}{x+1} - \frac{x}{x+1} = 2$
3. $\frac{5}{x+3} + \frac{2}{x-3} = 1$
4. $\frac{3}{x-4} - \frac{5}{x+4} = 6$

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5. $\frac{x}{x+1} - \frac{6}{x+2} = 4$
6. $\frac{x}{x-4} - \frac{4}{x-4} = 8$
7. $\frac{4x}{x-2} + 3 = 1$
8. $\frac{-2x}{x+1} + 6 = -x$
9. $\frac{1}{x+2} + 1 = -2x$
10. $\frac{-6x-3}{x+1} - 3 = -4x$
11. $\frac{x+3}{x} - \frac{3}{x+3} = \frac{6}{x^2+3x}$
12. $\frac{x-4}{x} - \frac{2}{x-4} = \frac{8}{x^2-4x}$
13. $\frac{x+6}{x} - \frac{2}{x+6} = \frac{12}{x^2+6x}$
14. $\frac{x+5}{x} - \frac{3}{x+5} = \frac{15}{x^2+5x}$
15. Explain what it means for a solution to be extraneous.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.6.

Vocabulary

Term	Definition
extraneous	An extraneous solution is a solution of a simplified version of an original equation that, when checked in the original equation, is not actually a solution.
Rational Equation	A rational equation is an equation that contains a rational expression.
Rational Expression	A rational expression is a fraction with polynomials in the numerator and the denominator.

Image Attributions

1. [Figure 1]
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SECTION OVERVIEW

2.5: Forms of Inequalities

Topic hierarchy

2.5.1: Quadratic Inequalities

2.5.2: Polynomial and Rational Inequalities

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2.5.1: Quadratic Inequalities

Quadratic Inequalities

How would you express the following as a function?

You are supposed to mow your square-shaped lawn for your parents, but the mower only has part of a tank of gas. If you can mow 2500 sf per gallon, and the mower has approximately 2.5 gallons in it, what is the maximum length of one side of the lawn you can mow? If your lawn is 75 feet long, will you need more gas?

Quadratic Inequalities

Quadratic inequalities are inequalities that have one of the following forms

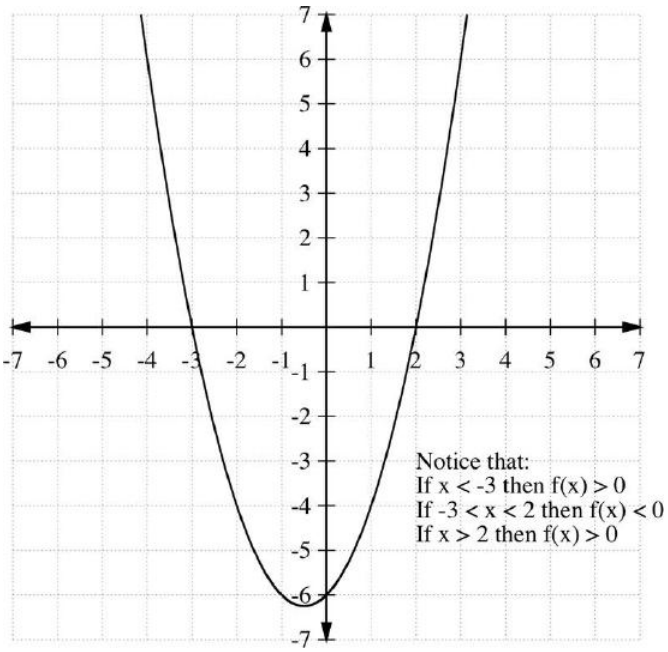
$$ax^2+bx+c>0$$

and

$$ax^2+bx+c<0$$

We can solve these inequalities by using the techniques that we have learned about solving quadratic equations. For example, consider the graph of the equation:

$$y=f(x)=x^2+x-6$$

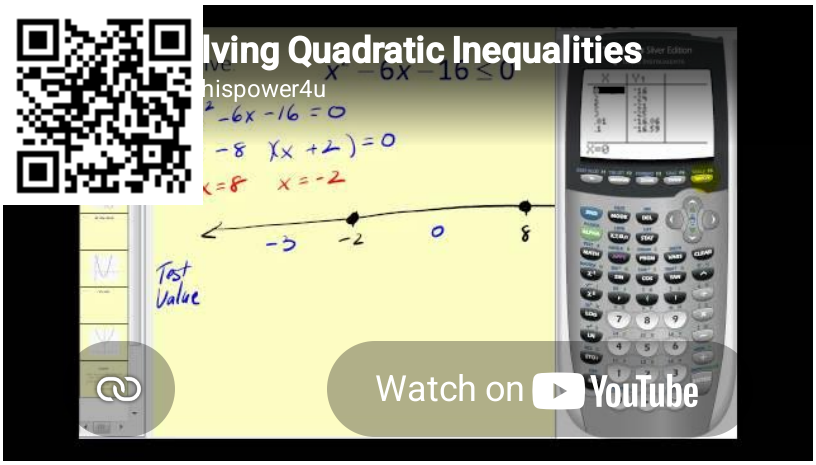


[Figure1]

Notice that the curve intersects the x−axis at -3 and 2. From graph, we notice the followings

- If $x<-3$ then $f(x)>0$
- If $-3<x<2$, then $f(x)<0$
- If $x>2$, then $f(x)>0$

Therefore, $x^2+x-6>0$ whenever $x<-3$ or $x>2$, and $x^2+x-6<0$ when $-3<x<2$.



Examples

Example 1

Earlier, you were given a question about mowing a lawn.

If you can mow 2500 sf of grass per gallon of gas, and the mower has 2.5 gallons in it, what is the maximum length of one side of the lawn you can mow?

If your lawn is 75 ft long, will you need more gas?

Solution

We know that the function $S^2<6250$ describes the possible side lengths of square shapes you could mow before running out of gas (if you're having trouble understanding why, see Example 4 for a similar process).

Solving for *S* gives:

$$\sqrt{s^2} = \sqrt{6250}$$
$$s = 79$$

With 2.5 gal of gas, you could mow a square up to apx 79 ft on each side.

You should not need more gas if the lawn is only 75 ft long on each side.

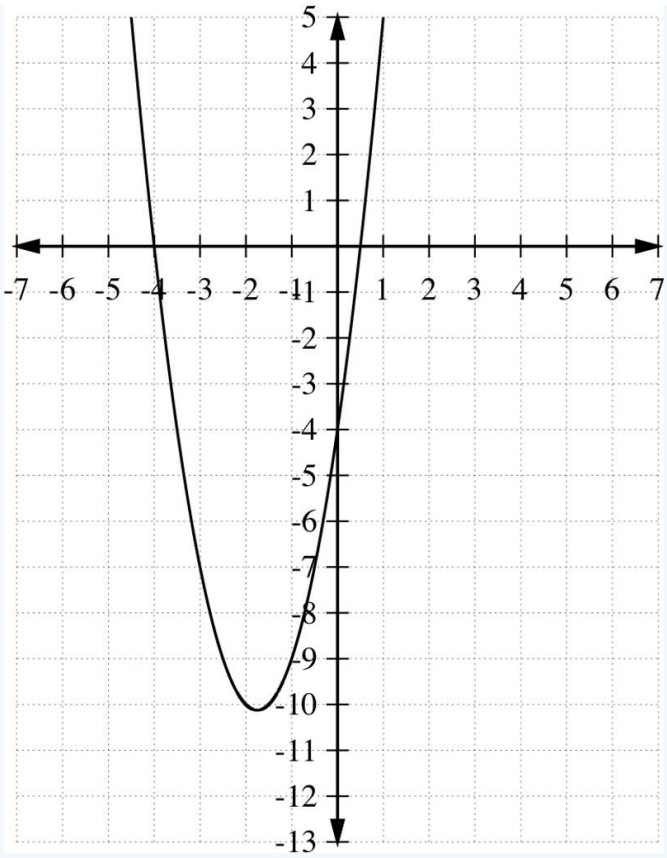
Example 2

What is the solution set of the inequality $2x^2+7x-4<0$?

Solution

It is best to graph the function $f(x)=2x^2+7x-4$ and look for the values of x such that the inequality $f(x)<0$ is true.

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Thus from graph, $2x^2+7x-4<0$ only if

$$-4 < x < \frac{1}{2}$$

So the solution set is $x \in \left(-4, \frac{1}{2}\right)$ or in set builder notation, $\left\{x \mid -4 < x < \frac{1}{2}\right\}$

Although the method of graphing to find the solution set of an inequality is easy to follow, another algebraic method can be used. The algebraic method involves finding the x−intercepts of the graph and then dividing the x−axis into intervals separated by the x−intercepts. The examples below illustrate the method.

Example 3

Find the solution set of the **quadratic inequality** $x^2+2x-8>0$ without graphing.

Solution

To find the solution set without graphing, first factor:

$x^2+2x-8=0$

$(x+4)(x-2)=0$

Recalling the **zero product rule**, we can see that the two solutions to this quadratic equation are $x=-4$ and $x=2$, thus, the x−intercepts of the function $f(x)=x^2+2x-8$ are -4 and 2.

These points divide the x−axis into three intervals: $(-\infty,-4)|(-4,2)|(2,\infty)$. We can choose a test point from each **interval**, substitute it into f(x) and see if the function is negative or positive with that value as *x*. This procedure can be simplified by making a table as shown below:

Interval	Test Point	Is x^2+2x-8 positive or negative?	Part of Solution set?
$(-\infty,-4)$	-5	+	yes
$(-4,2)$	1	−	no
$(2,+\infty)$	3	+	yes

From the table, we conclude that since $x^2+2x-8>0$ if and only if $x<-4$ and $x>2$. The solution set can also be written as:

$$x\in(-\infty,-4)\cup(2,+\infty)$$

Some problems in science involve quadratic inequalities. The example below illustrates one such application.

Example 4

A rectangle has a length 10 meters more than twice the width. Find all of the possible widths that result in the area of the rectangle not exceeding 100 squared meters.

Solution

Let w be the width of the rectangle and l its length. Given the information in the question, we can say:

$l=10+2w$

Then we can use the formula for the area of a rectangle:

$area=l\times w$

Substituting $10+2w$ in for l gives:

$area=w\times(10+2w)$

The area cannot exceed 100 m^2, so

$10w+2w^2<100$

or

$2w^2+10w-100<0$

Simplify by dividing both sides by 2:

$w^2+5w-50<0$

Factor the trinomial:

$w^2+5w-50=(w+10)(w-5)$

So the partition points are 5 and -10, which means we have three intervals. Since width cannot be negative, we can safely ignore -10. That means the maximum area is 100 m^2

$\therefore w<5.$

The width must be less than 5 meters.

Example 5

Find the solution set of the inequality $x^2\leq16$.

Solution

Set the function equal to zero:

$x^2-16=0$

Factor to find the **critical values** (points where the graph crosses the *x* axis, thereby changing signs):

$(x-4)(x+4)=0$

By the zero product rule: $x=4$ or $x=-4$

That gives us three sections on the graph:

$x\leq -4$

$-4\leq x\leq 4$

$4\leq x$

Test one sample value from each division to identify possible solution sets.

Set	$x\leq -4$	$-4\leq x\leq 4$	$4\leq x$
Test value	-5		5
f(x) true with value?	$4\cdot(-5)^2=100\not\leq 16$ ∴No	$4\cdot 0^2=0\leq 16$ ∴Yes	$4\cdot 5^2=100\not\leq 16$ ∴No

Therefore the solution set is $-4\leq x\leq 4$.

Example 6

Graph the solution set: $(x-3)(x+4)\geq 0$.

Solution

The solutions to $(x-3)(x+4)\geq 0$ can be identified with the rules for multiplying negative numbers:

Recall from Pre-Algebra that an *even* number of negatives yields a positive answer, and an *odd* number of negatives yields a negative answer.

Since $(x-3)(x+4)\geq 0$ we know we need a positive answer or zero.

Therefore either:

Case #1: $(x-3)\geq 0\rightarrow x\geq 3$ **and** $(x+4)\geq 0\rightarrow x\geq -4$

or

Case #2: $(x-3)\leq 0\rightarrow x\leq 3$ **and** $(x+4)\leq 0\rightarrow x\leq -4$

Since any number greater than 3 is already greater than -4, from Case #1 we get: $x\geq 3$

Since any number less than -4 is already less than 3, from Case #2 we get $x\leq -4$

Therefore our answer is $x\leq -4$ **or** $x\geq 3$

In set notation: $x\in(-\infty,-4]\cup[3,+\infty)$

To graph this information, we draw a line graph, and mark the values that *x* **can be**, with solid dots on the end numbers to indicate that those values are included.

Visually that is:



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" frameborder="0" height="450px" name="99198" src="https://www.ck12.org/flx/show/video/...ties-Example-2" thumbnailurl="" title="VideoObject?hash=f688e36d565e0820730d1d47d13380a6" uploaddate="2016-07-06 16:34:48" width="95%">

Review

Graph the solutions sets below on a number line:

1. $x<3$ **or** $x>4$
2. $x\geq -5$ **and** $x\geq 3$
3. $x<6$ **and** $x\geq -2$
4. $x>7$ **or** $x\geq -4$
5. $x\leq -8$ **and** $x>3$

Identify critical points, solve, and graph:

6. $x^2+9x>-14$
7. $x^2-5x\leq 50$
8. $x^2+2x\leq 48$
9. $x-\frac{20}{x}-8<0$ (hint: multiply both sides by x first)
10. $x+10\geq -\frac{21}{x}$
11. $(x+6)(x-3)>0$
12. $(x-8)(x+1)>0$
13. $x^2-x\geq 90$
14. $3x^2-23x\leq 8$
15. $x^2+x-6>0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.9.

Vocabulary

Term	Definition
Critical Values	The critical values are the x−intercepts of a quadratic function.
interval	An interval is a specific and limited part of a function.
quadratic inequality	A quadratic inequality is a quadratic expression that is specified to be greater or lesser than a given value.
Roots	The roots of a function are the values of x that make y equal to zero.
Zero Product Rule	The zero product rule states that if the product of two expressions is equal to zero, then at least one of the original expressions much be equal to zero.

Image Attributions

1. [Figure 1]
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2.5.2: Polynomial and Rational Inequalities

Polynomial and Rational Inequalities

Often it is easier to use and remember new terms when you have a 'hook' or comparison to a term you know already.

"**Polynomial inequality**" is a term generally used to refer to inequalities where the *x* variable has a degree of 3 or greater.

The prefix "poly" means 'multiple' or 'many', and the root word "nomial" means 'name' or 'term'.

Therefore a "polynomial" is literally: "many terms".

The prefix "in" means 'not', and the root word "equal" of course means 'the same'.

Therefore and "inequality" refers to things which are "not same" or "not equal".

Can you use this logic to identify the origin of some of the other terms used in this lesson?

Polynomial and Rational Inequalities

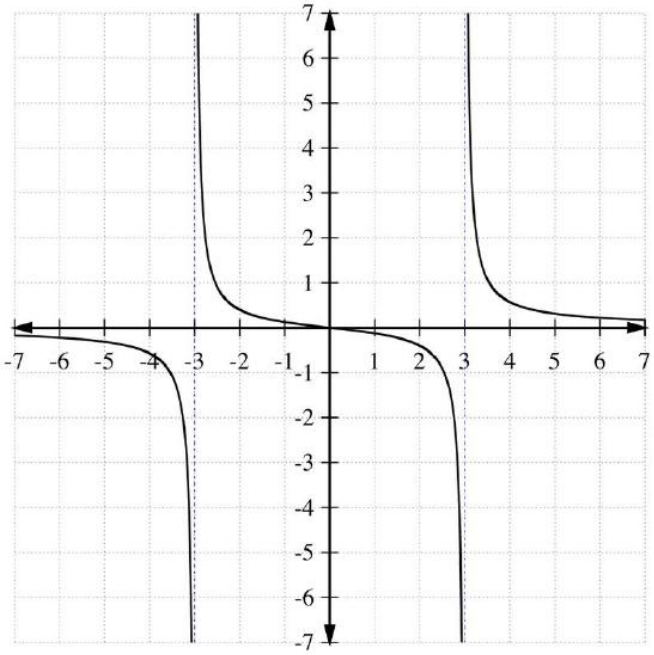
Polynomial Inequalities

Solving polynomial inequalities is very similar to solving quadratic inequalities. The basic steps are the same:

1. Set up the inequality in the form p(x)>0 (or p(x)<0,p(x)≤0, p(x)≥0)
2. Find the solutions to the equation p(x)=0.
3. Divide the number line into intervals based on the solutions to p(x)=0.
4. Use test points to find solution sets to the equation.

Rational Inequalities

There is one step added to the process of solving rational inequalities because a **rational function** can also change signs at its vertical asymptotes or at a break in the graph. For instance, look at the graph of the function $r(x) = \frac{x}{x^2-9}$ below.



If we want to solve the inequality $\frac{x}{x^2-9} > 0$, then we need to use the following critical points: x=0,x=3, and x=−3. x=0 is the solution of setting the numerator equal to 0, and this gives us the only root of the function. x=±3 are the vertical asymptotes, the x−coordinates that make the function undefined because putting in 3 or -3 for x will cause a division by zero.

Testing the intervals between each critical point to see if the values in that interval satisfy the function gives us:

Interval	Test Point	Positive/Negative?	Part of Solution Set?
$(-\infty,-3)$	-4	-	no
(-3, 0)	-2	+	yes
(0, 3)	2	-	no
$(3,+\infty)$	4	+	yes

Thus, the solutions to $\frac{x}{x^2-9} > 0$ are $x \in (-3,0) \cup (3,+\infty)$.

Examples

Example 1

Earlier, you were given a question about identifying the origins of other terms in this lesson.

Solution

How many of the terms we have used recently were you able to track down the origins of? A few are listed below, did you find others?

Bi-nomial: "two-named" or "two-terms" - from "bi", meaning 'two' and "nomial", meaning 'name' or 'term'. Quad-ratic: "related to a square" - from "quadratus", meaning 'square'. Ratio-nal: "related to a ratio" - from "ratio", meaning 'reason' (as in "to reason" or "calculate") and "-al", meaning "related to".

Example 2

Solve $x^3-3x^2+2x \geq 0$.

Solution

The polynomial is already in the correct form $p(x) \geq 0$ so we solve the equation

$x^3-3x^2+2x=0$

$x(x^2-3x+2)=0$

$x(x-2)(x-1)=0$

The zeros are at $x=0, x=1$, and $x=2$.

Interval	Test Point	Positive/Negative?	Part of Solution Set?
$(-\infty,0)$	-5	-	no
(0, 1)	$\frac{1}{2}$	+	yes
(1, 2)	$\frac{3}{2}$	-	no
$(2,+\infty)$	3	+	yes

Notice that this inequality is greater than or equal to zero, so we include the zeros in the solution set. Therefore the solutions are $x \in [0,1] \cup [2,+\infty]$.

Example 3

Solve $6x^4+5x^2<25$.

Solution

First we will change the inequality to $6x^4+5x^2-25<0$. Now, solve the equation $6x^4+5x^2-25=0$.

$6x^4+5x^2-25=0$

$(3x^2-5)(2x^2+5)=0$

The first term yields the solutions $x=\pm\sqrt{\frac{5}{3}}$ and there are no real solutions for the second term.

Interval	Test Point	Positive/Negative?	Part of Solution Set?
$\left(-\infty,-\sqrt{\frac{5}{3}}\right)$	-3	+	no
$\left(-\sqrt{\frac{5}{3}},\sqrt{\frac{5}{3}}\right)$	0	-	yes
$\left(\sqrt{\frac{5}{3}},+\infty\right)$	3	+	no

Finally, the solution set is $x\in\left(-\sqrt{\frac{5}{3}},\sqrt{\frac{5}{3}}\right)$.

Example 4

Find the solution set of the inequality

$\frac{4x-12}{3x-2}<0$

Solution

From the numerator we solve $4x-12=0$ or $x=3$. In the denominator, solve $3x-2=0$ and we find the critical point $x=frac23$.

Making the table

Interval	Test Point	Positive/Negative?	Part of Solution Set?
$\left(-\infty,\frac{2}{3}\right)$	0	+	no
$\left(\frac{2}{3},3\right)$	1	-	yes
$(3,+\infty)$	5	+	no

Therefore, the solution set includes the numbers in the interval $\left(\frac{2}{3},3\right)$. Or in set-builder notation, the solution is $\left\{x\mid\frac{2}{3}<x<3\right\}$.

Example 5

(using technology) The McNeil Surf Company makes wetsuits. For a given number of wetsuits x, McNeil's profit, in dollars, is given by the function $P(x)=-0.01x^2+25x-3000$.

a. If the manager of McNeil wants the profit to stay above $9,000, what is the minimum and maximum number of wetsuits they can manufacture to maintain that level of profit?
b. What is the maximum profit McNeil can make?
c. Can you explain why this shape might make sense for a profit function?

Solution

a. Set up the inequality

$-0.01x^2+25x-3000>9000$

$-0.01x^2+25x-12000>0$

With a calculator you can graph the function $Y_1=-0.01x^2+25x-12000$.

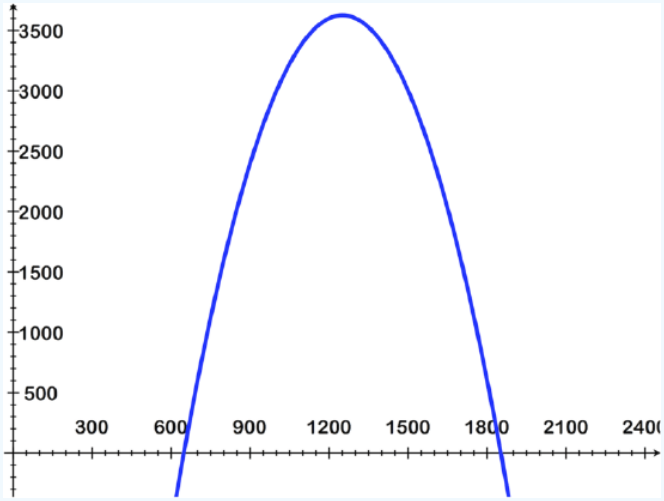
On a TI-83: use the window $[-1000,4000]\times[-5000,15000]$.

The settings are:

Xmin=−1000, Xmax=4000, Xscl=500

Ymin=−5000, Ymax=5000, Yscl=1000 xres=1

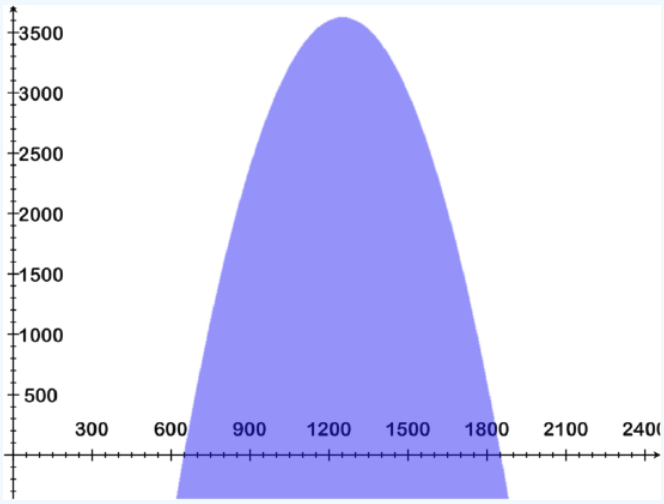
On a software grapher, the image looks like this with the window x:0→2400 and y:0→3500



Using the **CALC** menu **(2ND TRACE)**, and selecting the option **ZEROS**, we can see that the zeros of $Y_1=-0.01x^2+25x-12000$ are at x=647.920 and x=1852.080.

By inspecting the graph, we can see that the solution set to the inequality $-0.01x^2+25x-12000>0$ is $x\in(648,1852)$.

Visually that looks like:



b. Keeping the same graph open, use **CALC MAXIMUM** to solve for the maximum profit. The maximum is at (1250, 3625), indicating that the maximum profit is $3625 **above the minimum we set:** of $9000.

So the actual maximum profit is $12,625 when 1250 wetsuits are produced.

c. One possible reason the profit function might take this shape is labor costs. If McNeil wants to make a very large number of wetsuits in a short period of time, then that may require paying overtime for workers, and this could reduce the profit margin.

Example 6

For the following rational function, determine limitations on the **domain** and the asymptotes, and then sketch the graph.

$f(x) \geq \frac{2x+5}{x-1}$

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Solution

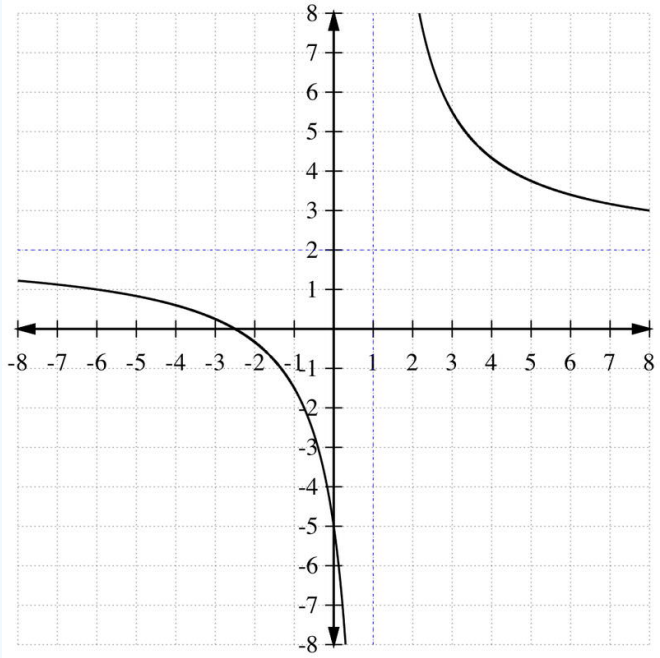
To identify the graph of the inequality $f(x) \geq \frac{2x+5}{x-1}$, first treat it as if it were the equality $f(x) \geq \frac{2x+5}{x-1}$

For $f(x) \geq \frac{2x+5}{x-1}$:

To find the critical points, identify the value(s) which make the denominator = 0: x≠1

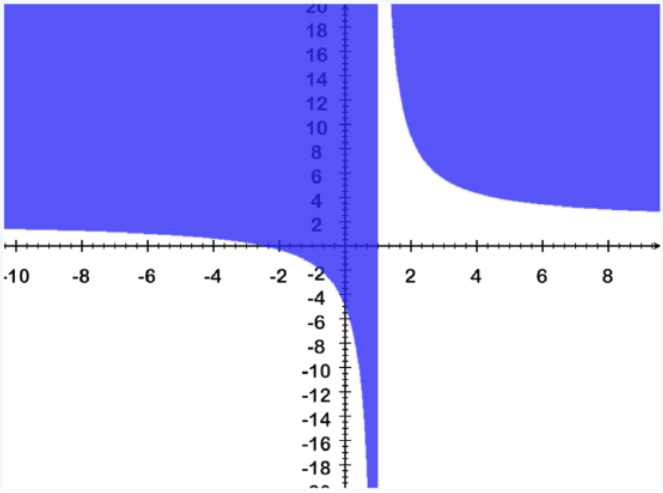
That gives us a **vertical asymptote** of x=1

The horizontal asymptote becomes apparent as *x* becomes truly huge and the "+5" and "-1" no longer matter. At that point, we have $f(x) = \frac{2x}{x} \rightarrow f(x) = 2$ So the horizontal asymptote is y=2



Now that you know the shape of the graph, simply shade the area *above* the lines, since the original function was *f*(*x*) is *greater-than* function, and leave the lines solid since it was a *greater-than or equal to*.

The final graph should look like:



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403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

```
Generated by cloudfront (CloudFront)
Request ID: dLRpVQ6vaO5GdPm80RCd1HfRz1PRxlxjoyBHvVXmP4olYQ9zpFRjhA==
```

403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

```
Generated by cloudfront (CloudFront)
Request ID: FDqI5eevidaHY9AhZ5gsJsmHGVhpr5_nz-UxSUkZeDH_4dUZfTa48g==
```

Review

Find the solution set of the following inequalities without using a calculator. Display the solution set on a number line.

1. $x^2 + 2x - 3 \leq 0$
2. $3x^2 - 7x + 2 > 0$
3. $-6x^2 - 13x + 5 \geq 0$
4. $\frac{5x-1}{x-2} > 0$
5. $\frac{1-x}{x} < 1$
6. Solve: $4x^3-4x^2-3x>0$
7. Solve: $\frac{x^4}{4} - x^2 < 0$
8. Solve: $4x^3 - 8x^2 - x + 2 \geq 0$
9. $\frac{n^3-2n^2-n+2}{n^3+3n^2+4n+12} < 0$
10. $\frac{n^3+3n^2-4n-12}{n^3-5n^2+4n-20} \leq 0$
11. $\frac{2n^3+5n^2-18n-45}{3n^3-n^2+27n-9} \geq 0$
12. $\frac{12n^3+16n^2-3n-4}{8n^3+12n^2+10n+15} > 0$

Use a calculator to solve the following inequalities. Round your answer to three places after the decimal.

13. $-9.8t^2 + 357.6t \geq 0$
14. $x^3 - 5x + 7 \leq -4x^2 + 18$
15. $\frac{x^2-2x}{x-5} > x^2 - 25$
16. Solve and graph: $f(x) > \frac{9x^2-4}{3x+2}$
17. The total resistance of two electronics components wired in parallel is given by $\frac{R_1R_2}{R_1+R_2}$ where R_1 and R_2 are the individual resistances (in Ohms) of the two components. a) If the resistance of R_1 is 20 Ohms, what is the maximum resistance of R_2 if the

total resistance must be less than 15 Ohms? b) What is the maximum theoretical resistance of this circuit? How do you know?

18. A rectangular lot of land has a length that is 7 meters more than twice its width. If the area of the lot is greater than 60 square meters, what are the possible values of the widths of the lot?

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.10.

Vocabulary

Term	Definition
domain	The domain of a function is the set of x-values for which the function is defined.
Polynomial inequality	The term polynomial inequality is generally used to describe an inequality with an x term coefficient of three or greater.
quadratic inequality	A quadratic inequality is a quadratic expression that is specified to be greater or lesser than a given value.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.
Rational Inequality	A rational inequality is a ratio of two polynomials, specified to be greater or less than a given value.
Vertical Asymptote	A vertical asymptote is a vertical line marking a specific value toward which the graph of a function may approach, but will never reach.

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SECTION OVERVIEW

2.6: Finding Zeros of Polynomials

Topic hierarchy

2.6.1: Long Division of Polynomials

2.6.2: Synthetic Division of Polynomials

2.6.3: Real Zeros of Polynomials

2.6.4: Intermediate Value Theorem

2.6.5: Fundamental Theorem of Algebra

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2.6.1: Long Division of Polynomials

Long Division and Synthetic Division

Can you divide the following polynomials?

$\frac{x^2-5x+6}{x-2}$

Long Division and Synthetic Division

Polynomial Long Division

Whenever you want to divide a polynomial by a polynomial, you can use a process called polynomial long division. This process is similar to long division for regular numbers. Look at the problem below:

$\frac{(x^2+3x+2)}{(x+1)}$

This is the same as the division problem below:

$$x + 1\overline{)x^2 + 3x + 2}$$

Step 1: Divide the first term in the numerator (x^2) by the first term in the denominator (x). Put this result above the division bar in your answer. In this case, $\frac{x^2}{x} = x$.

$$\begin{array}{r} x \\ x + 1\overline{)x^2 + 3x + 2} \end{array}$$

Step 2: Multiply the denominator (x+1) by the result from Step 1 (x), and put the new result below your numerator. Then, subtract to get your new polynomial. This is the same process as in regular number long division!

$$\begin{array}{r} x \\ x + 1\overline{)x^2 + 3x + 2} \\ \underline{x^2 + x \quad \downarrow} \\ 2x + 2 \end{array}$$

Step 3: Divide the first term in the new polynomial (2x) by the first term in the denominator (x). Put this result above the division bar in your answer. Multiply, subtract, and repeat this process until you cannot repeat it anymore.

$$\begin{array}{r} x + 2 \\ x + 1\overline{)x^2 + 3x + 2} \\ \underline{x^2 + x \quad \downarrow} \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

Therefore: $\frac{(x^2+3x+2)}{(x+1)} = (x + 2)$

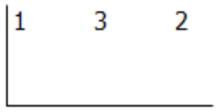
Synthetic Division

Synthetic division is another method of dividing polynomials. It is a shorthand of long division that only works when you are dividing by a polynomial of degree 1. Usually the **divisor** is in the form (x±a). In synthetic division, unlike long division, you are only concerned with the coefficients in the polynomials. Consider the same problem as above:

$$x + 1\overline{)x^2 + 3x + 2}$$

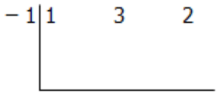
[Figure1]

Step 1: Write the coefficients in an upside down division sign.

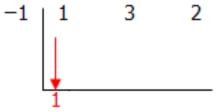


[Figure2]

Step 2: Put the opposite of the number from the divisor to the left of the division symbol. In this case, the divisor is x+1, so you will use a −1.

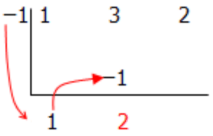


Step 3: Take your leading coefficient and bring it down below the division symbol.



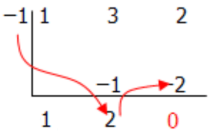
[Figure4]

Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



[Figure5]

Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



[Figure6]

The numbers below the division sign represent your coefficients. Therefore,

$\frac{(x^2+3x+2)}{(x+1)} = (x+2)$

Examples

Example 1

Earlier, you were asked to divide polynomials:

$\frac{x^2-5x+6}{x-2}$

Solution

You can use long division or synthetic division.

Long Division:

Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x} = x$.

$$\begin{array}{r} x \\ (x-2)\overline{|x^2-5x+6} \end{array}$$

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Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} x \\ (x-2)\overline{|x^2-5x+6} \\ x^2-2x \\ \hline 3x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} x-3 \\ (x-2)\overline{|x^2-5x+6} \\ x^2-2x \quad \downarrow \\ \hline -3x+6 \\ -3x+6 \\ \hline 0 \end{array}$$

Therefore: $\frac{x^2-5x+6}{x-2} = (x-3)$

Example 2

Use long division to divide:

$\frac{x^2+6x-7}{x-1}$

Solution

Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x} = x$.

$$\begin{array}{r} x \\ (x-1)\overline{|x^2+6x-7} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} x \\ (x-1)\overline{|x^2+6x-7} \\ x^2-x \\ \hline 7x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} x+7 \\ (x-1)\overline{|x^2+6x-7} \\ x^2-x \quad \downarrow \\ \hline 7x-7 \\ 7x-7 \\ \hline 0 \end{array}$$

Therefore: $\frac{x^2+6x-7}{x-1} = (x+7)$

Example 3

Use long division to divide:

$\frac{2x^2+7x+5}{2x+5}$

Solution

Step 1: Divide the first term in the numerator by the first term in the denominator; put this in your answer. Therefore $\frac{2x^2}{2x} = x$.

$$\begin{array}{r} x \\ (2x+5)\overline{|2x^2+7x+5} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} x \\ (2x+5)\overline{|2x^2+7x+5} \\ \underline{2x^2+5x} \\ 2x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} x+1 \\ (2x+5)\overline{|2x^2+7x+5} \\ \underline{2x^2+5x \quad \downarrow} \\ 2x+5 \\ \underline{2x+5} \\ 0 \end{array}$$

Therefore: $\frac{2x^2+7x+5}{2x+5} = (x+1)$

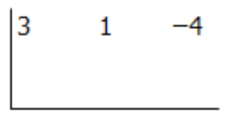
Example 4

Use synthetic division to divide:

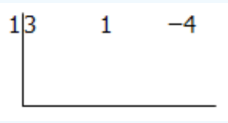
$\frac{3x^2+x-4}{x-1}$

Solution

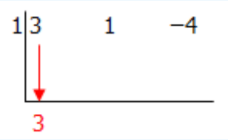
Step 1: Write the coefficients in an upside down division sign.



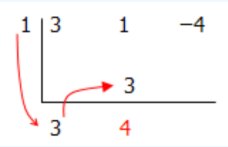
Step 2: Put the opposite of the number from the divisor to the left of the division symbol.



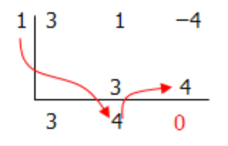
Step 3: Take your leading coefficient and bring it down below the division symbol.



Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



Therefore: $\frac{3x^2+x-4}{x-1} = (3x+4)$

Review

Use long division to divide each of the following:

1. $(x^2+7x+12)\div(x+3)$
2. $(x^2+4x+3)\div(x+3)$
3. $(a^2-4a-45)\div(a-9)$
4. $(3x^2+5x-2)\div(3x-1)$
5. $(2x^2-5x+2)\div(2x-1)$

Use synthetic division to divide each of the following:

6. $(b^2-5b+6)\div(b-3)$
7. $(x^2-6x+8)\div(x-4)$
8. $(a^2-1)\div(a+1)$
9. $(c^2-9)\div(c-3)$
10. $(5r^2+2r-3)\div(r+1)$

Divide each of the following:

11. $\frac{2x^3-7x^2-14x-5}{x-5}$
12. $\frac{9x^4-15x^3+12x^2-11x-15}{3x^3+4x+3}$
13. $\frac{6x^4+4x^3+9x^2+2x+3}{2x^2+1}$
14. $\frac{x^4+4x^3+3x^2+x+1}{x+1}$
15. $\frac{2x^3+7x^2-27x+18}{x+6}$
16. $\frac{8x^3-2x^2+7x+5}{2x+1}$
17. $\frac{3x^3-15x^2+4x-20}{x-5}$
18. $\frac{9x^3+26x^2-48x+5}{x^2+3x-5}$
19. $\frac{-x^3+13x+12}{x+3}$
20. $\frac{x^3-2x^2-5x+10}{x-2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.13.

Vocabulary

Term	Definition
Dividend	In a division problem, the dividend is the number or expression that is being divided.
divisor	In a division problem, the divisor is the number or expression that is being divided into the dividend. For example: In the expression 152÷6, 6 is the divisor and 152 is the dividend.

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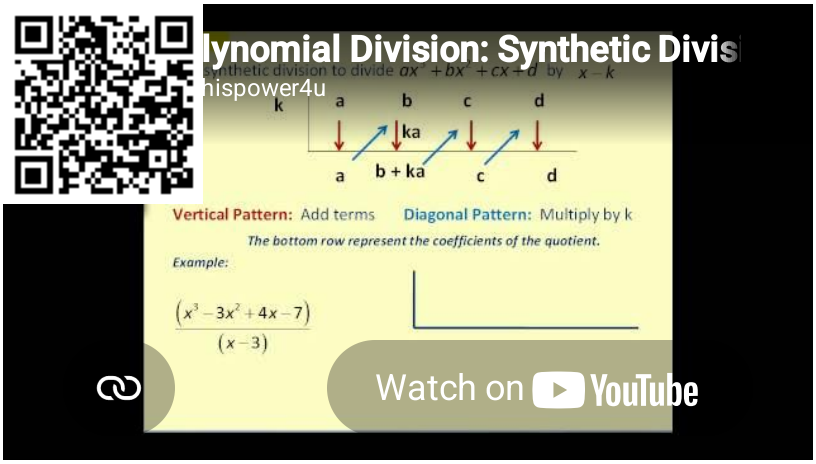
2.6.2: Synthetic Division of Polynomials

Synthetic Division of Polynomials

The volume of a rectangular prism is $2x^3+5x^2-x-6$. Determine if $2x+3$ is the length of one of the prism's sides.

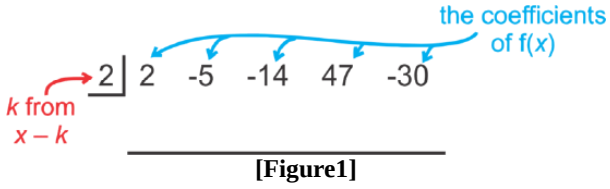
Synthetic Division

Synthetic division is an alternative to long division. It can also be used to divide a polynomial by a possible factor, $x-k$. However, synthetic division cannot be used to divide larger polynomials, like quadratics, into another polynomial.

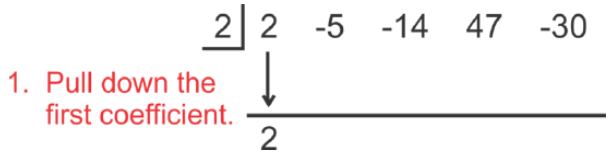


Let's use synthetic division to divide $2x^4-5x^3-14x^2+47x-30$ by $x-2$.

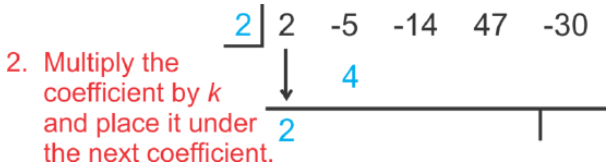
Using synthetic division, the setup is as follows:



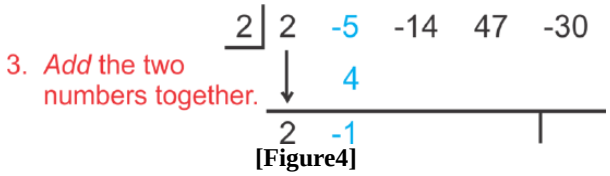
[Figure1]



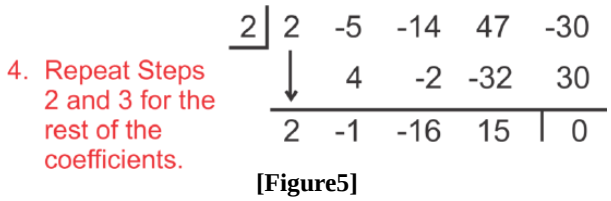
[Figure2]



[Figure3]

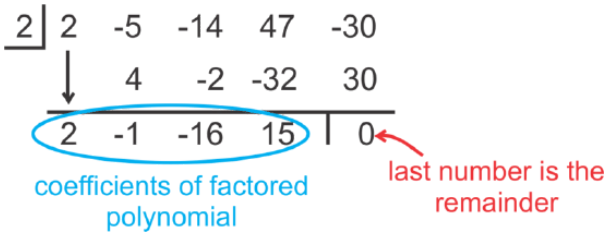


[Figure4]



[Figure5]

To “read” the answer, use the numbers as follows:



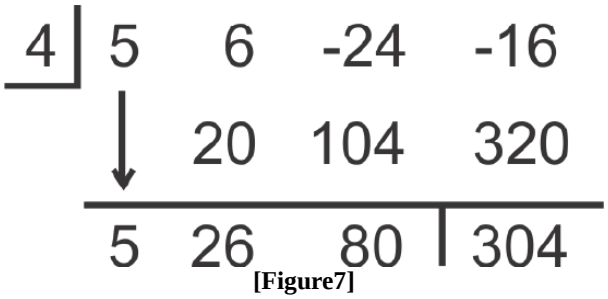
[Figure6]

Therefore, 2 is a solution, because the remainder is zero. The factored polynomial is $2x^3-x^2-16x+15$. Notice that when we synthetically divide by k, the “leftover” polynomial is one degree less than the original. We could also write

$(x-2)(2x^3-x^2-16x+15)=2x^4-5x^3-14x^2+47x-30$.

Now, let's determine if 4 is a solution to $f(x)=5x^3+6x^2-24x-16$.

Using synthetic division, we have:



[Figure7]

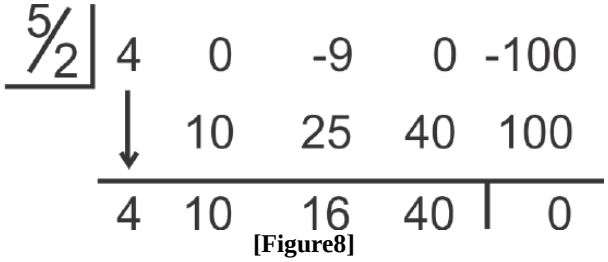
The remainder is 304, so 4 is not a solution. Notice if we substitute in x=4, also written f(4), we would have $f(4)=5(4)^3+6(4)^2-24(4)-16=304$. This leads us to the **Remainder Theorem**.

Remainder Theorem: If f(k)=r, then r is also the remainder when dividing by (x−k).

This means that if you substitute in x=k or divide by k, what comes out of f(x) is the same. r is the remainder, but it is also the corresponding y−value. Therefore, the point (k,r) would be on the graph of f(x).

Finally, let's determine if (2x−5) is a factor of $4x^4-9x^2-100$.

If you use synthetic division, the factor is not in the form (x−k). We need to solve the possible factor for zero to see what the possible solution would be. Therefore, we need to put $\frac{5}{2}$ up in the left-hand corner box. Also, not every term is represented in this polynomial. When this happens, you must put in zero placeholders. In this problem, we need zeros for the x^3−term and the x−term.



[Figure8]

This means that $\frac{5}{2}$ is a zero and its corresponding binomial, (2x−5), is a factor.

Example 1

Earlier, you were asked to determine if 2x+3 is the length of one of the prism's sides.

Solution

If $2x+3$ divides evenly into $2x^3+5x^2-x-6$ then it is the length of one of the prism's sides.

If we want to use synthetic division, notice that the factor is not in the form (x−k). Therefore, we need to solve the possible factor for zero to see what the possible solution would be. If 2x+3=0 then x= $-\frac{3}{2}$. Therefore, we need to put $-\frac{3}{2}$ up in the left-hand corner box.

-3/2	2	5	-1	-6
	↓	-3	-3	6
	2	2	-4	0

When we perform the synthetic division, we get a remainder of 0. This means that (2x+3) is a factor of the volume. Therefore, it is also the length of one of the sides of the rectangular prism.

Example 2

Divide $x^3+9x^2+12x-27$ by (x+3). Write the resulting polynomial with the remainder (if there is one).

Solution

Using synthetic division, divide by -3.

-3	1	9	12	-27
	↓	-3	-18	18
	1	6	-6	-9

[Figure9]

The answer is $x^2+6x-6-\frac{9}{x+3}$.

Example 3

Divide $2x^4-11x^3+12x^2+9x-2$ by (2x+1). Write the resulting polynomial with the remainder (if there is one).

Solution

Using synthetic division, divide by $-\frac{1}{2}$.

-1/2	2	-11	12	9	-2
	↓	-1	6	-9	0
	2	-12	18	0	-2

[Figure10]

The answer is $2x^3-12x^2+18x-\frac{2}{2x+1}$

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Example 4

Is 6 a solution for $f(x)=x^3-8x^2+72$? If so, find the real-number zeros (solutions) of the resulting polynomial.

Solution

Put a zero placeholder for the x−term. Divide by 6.



The resulting polynomial is $x^2-2x-12$. While this quadratic does not factor, we can use the Quadratic Formula to find the other roots.

$$x=\frac{2\pm\sqrt{2^2-4(1)(-12)}}{2}=\frac{2\pm\sqrt{4+48}}{2}=\frac{2\pm2\sqrt{13}}{2}=1\pm\sqrt{13}$$

The solutions to this polynomial are $6, 1+\sqrt{13}\approx 4.61$ and $1-\sqrt{13}\approx -2.61$.

Review

Use synthetic division to divide the following polynomials. Write out the remaining polynomial.

1. $(x^3+6x^2+7x+10)\div(x+2)$
2. $(4x^3-15x^2-120x-128)\div(x-8)$
3. $(4x^2-5)\div(2x+1)$
4. $(2x^4-15x^3-30x^2-20x+42)\div(x+9)$
5. $(x^3-3x^2-11x+5)\div(x-5)$
6. $(3x^5+4x^3-x-2)\div(x-1)$
7. Which of the division problems above generate no remainder? What does that mean?
8. What is the difference between a zero and a factor?
9. Find $f(-2)$ if $f(x)=2x^4-5x^3-10x^2+21x-4$.
10. Now, divide $2x^4-5x^3-10x^2+21x-4$ by $(x+2)$ synthetically. What do you notice?

Find all real zeros of the following polynomials, given one zero.

11. $12x^3+76x^2+107x-20$; -4
12. $x^3-5x^2-2x+10$; -2
13. $6x^3-17x^2+11x-2$; 2

Find all real zeros of the following polynomials, given two zeros.

14. $x^4+7x^3+6x^2-32x-32$; $-4, -1$
15. $6x^4+19x^3+11x^2-6x$; $0, -2$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.10.

Vocabulary

Term	Definition
Oblique Asymptote	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Oblique Asymptotes	An oblique asymptote is a diagonal line marking a specific range of values toward which the graph of a function may approach, but generally never reach. An oblique asymptote exists when the numerator of the function is exactly one degree greater than the denominator. An oblique asymptote may be found through long division.
Remainder Theorem	The remainder theorem states that if f(k)=r, then r is the remainder when dividing f(x) by (x−k).
Synthetic Division	Synthetic division is a shorthand version of polynomial long division where only the coefficients of the polynomial are used.

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   Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm
5. [Figure 5]
   Credit: CK-12 Foundation
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6. [Figure 6]
   Credit: CK-12 Foundation
   Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm
7. [Figure 7]
   Credit: CK-12 Foundation
   Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm
8. [Figure 8]
   Credit: CK-12 Foundation
   Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm
9. [Figure 9]
   Credit: CK-12 Foundation
   Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm

10. [Figure 10]

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Source: http://centros5.pntic.mec.es/sierrami/dematesna/demates67/opciones/sabias/Ruffinni/Ruffinni.htm

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2.6.3: Real Zeros of Polynomials

Real Zeros of Polynomials

In the real world, problems do not always easily fit into quadratic or even cubic equations. Financial models, population models, fluid activity, etc., all often require many degrees of the input variable in order to approximate the overall behavior. While it can be challenging to model some of these more complex interactions, the effort can be well worth it. Mathematical models of stocks are used constantly as a way to "look into the future" of finance and make the kinds of educated guesses that are behind some of the largest fortunes in the world.

What benefits can you think of to modeling the behavior of large populations? Can you think of other useful applications not mentioned here?

Finding Real Zeros of Polynomials

There are three theorems and a rule that we will be referring to during this lesson in order to help make the discovery of the **roots** of **polynomial** functions easier. You should review them and be prepared to refer to them often during the practice problems.

The Remainder Theorem

If a polynomial $f(x)$ of degree $n>0$ is divided by $x-c$, then the remainder R is a constant and it is equal to the value of the polynomial when c is substituted for x. That is

$f(c)=R$

The Factor Theorem

If $f(x)$ is a polynomial of degree $n>0$ and $f(c)=0$, then $x-c$ is a *factor* of the polynomial $f(x)$. Further, if $x-c$ is a factor, then c is a *zero* of f.

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The Rational Zero Theorem

Given the polynomial

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$

$a_n\neq0$ and n is a positive integer. If the coefficients are integers and $\frac{p}{q}$ is a rational zero in lowest terms, then p is a divisor of a_0 and q is a divisor of a_n.

Descartes' Rule of Signs

Given any polynomial, p(x),

1. Write it with the terms in descending order, i.e. from the highest degree term to the lowest degree term.
2. Count the number of sign changes of the terms in p(x). Call the number of sign changes n.
3. Then the number of *positive* roots of p(x) is less than or equal to n.
4. Further, the possible number of positive roots is n, n−2, n−4,…
5. To find the number of *negative* roots of p(x), write p(−x) in descending order as above (i.e. change the sign of all terms in p(x) with odd powers), and repeat the process above. Then the *maximum* number of negative roots is n.

Examples

Example 1

Earlier, you were asked if you could identify some valuable real-world uses for modeling higher-degree polynomials.

Solution

Here are a few possibilities:

- Identifying what times of the day people are most likely to want coffee (market research like this is key to running your own business)
- Predicting population growth in a particular neighborhood or area of town (useful for identifying a good location to start a small business)
- Identifying which stocks are likely to rise or fall based on weather or season
- Forecasting the weather
- Calculating the right head start to give a slower car to make a drag race exciting

There are many, many more.

Example 2

Use **synthetic division** and the remainder and factor theorems to find the quotient Q(x) and the remainder R if $f(x)=2x^3-3x^2+6$ is divided by $x-5$.

Solution

$$\begin{array}{r|rrrr} 5 & 2 & -3 & 0 & 6 \\ & \downarrow & 10 & 35 & 175 \\ \hline & 2 & 7 & 35 & 181 \end{array}$$

Hence

$2x^3-3x^2+6=(2x^2+7x+35)(x-5)+181$

Notice that the remainder is 181 and it can also be obtained if we simply substituted x=5 into f(x),

$$\begin{aligned} f(5) &= 2(5)^3-3(5)^2+6 \\ &= 250-75+6 \\ &= 181 \end{aligned}$$

Example 3

Use the rational zero theorem and synthetic division to find all the possible rational zeros of the polynomial

$f(x)=x^3-2x^2-x+2$

Solution

From the rational zero theorem, $\frac{p}{q}$ is a rational zero of the polynomial f. So p is a divisor of 2 and q is a divisor of 1. Hence, p can take the following values: -1, 1, -2, 2 and q can be either -1 or 1. Therefore, the possible values of $\frac{p}{q}$ are

$$\frac{p}{q}: -1, 1, -2, 2$$

So there are four possible zeros. Of these four, not more than three can be zeros of f because f is a polynomial with degree 3. To test which of the four possible candidates are zeros of f, we use the synthetic division. Recall from the remainder theorem if f(c)=0, then c is a zero of f. We have

$$\begin{array}{r|rrrr} 2 & 1 & -2 & -1 & 2 \\ & \downarrow & 2 & 0 & -2 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

Hence, 2 is a zero of f. Further, by the division algorithm,

$$f(x) = (x-c)Q(x) + R(x)$$
$$= (x-2)(x^2-1) + 0$$

The remaining zeros of f are simply the zeros of $Q(x)=x^2-1$ which is easier to manipulate,

$$Q(x) = x^2 - 1$$
$$= (x-1)(x+1)$$

and thus the remaining zeros are -1 and 1. Thus the rational zeros of f are -1, 1, and 2.

Example 4

Graph the polynomial function $h(x)=2x^3-9x^2+12x-5$.

Solution

Notice that the leading term is $2x^3$, where n=3 odd and $a_n=2>0$. This tells us that the end behavior will take the shape of a power function with an odd exponent.

Here, as you can see, there is no straight-forward way to find the zeros of h(x). However, with the use of the factor theorem and the synthetic division, we can find the rational roots of h(x).

First, we use the rational zero theorem and find that the possible rational zeros are

$$\frac{p}{q} : -1, 1, -2, 2, -5, 5, -\frac{5}{2}, \frac{5}{2}$$

testing all these numbers by the synthetic division,

$$\begin{array}{r|rrrr} -1 & 2 & -9 & 12 & -5 \\ & \downarrow & -2 & 11 & -23 \\ \hline & 2 & -11 & 23 & -28 \end{array}$$

-1 is not a root. Now let's test x=1.

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 12 & -5 \\ & \downarrow & -2 & -7 & 5 \\ \hline & 2 & -7 & 5 & -28 \end{array}$$

we find that 1 is a zero of h and so we can re-write h(x),

$h(x)=(2x^2-7x+5)(x-1)$

Looking at quadratic part,

$2x^2-7x+5=(2x-5)(x-1)$

and so

$h(x)=(2x-5)(x-1)^2$

Thus 1 and $\frac{5}{2}$ are the x−intercepts of h(x). The y−intercept is

h(0)=−5

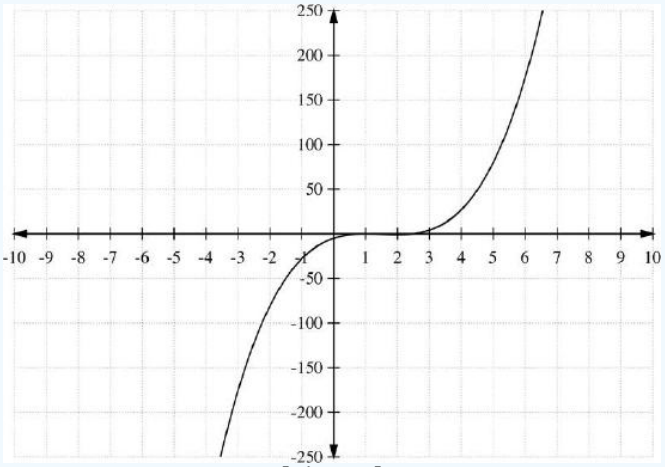
Further, the synthetic division can be also used to form a table of values for the graph of h(x):

x	−1	0	1	2	$\frac{5}{2}$	3
h(x)	−28	−5	0	-1	0	4

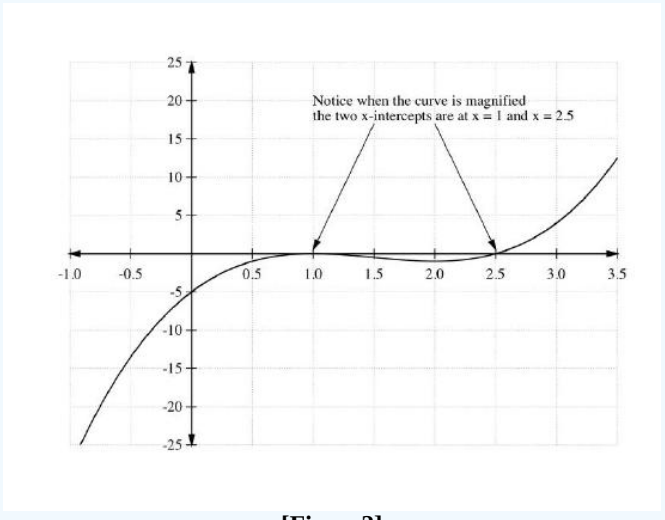
We choose test points from each interval and find g(x).

Interval	Test Value x	h(x)	Sign of h(x)	Location of points on the graph
$(-\infty, 1)$	-1	-28	-	below the x−axis
$(1, \frac{5}{2})$	$\frac{3}{2}$	$\frac{-11}{4}$	-	below the x−axis
$(\frac{5}{2}, \infty)$	3	4	+	above the x−axis

From this information, the graph of h(x) is shown in the two graphs below. Notice that the second graph is a magnification of h(x) in the vicinity of the x−axis.



[Figure1]



[Figure2]

Example 5

Use the 'rational zero' theorem and synthetic division to find all the possible rational zeros of the polynomial

$f(x)=x^3-2x^2-5x+6$

Solution

Assume $\frac{p}{q}$ is a rational zero of f. By the rational zero theorem, p is a divisor of 6 and q is a divisor of 1. Thus p and q can assume the following respective values

p: 1, −1, 2, −2, 3, −3, 6, −6

and

q: −1, 1

Therefore, the possible rational zeros will be

$\frac{p}{q}$: −1, 1, −2, 2, −3, 3, −6, 6

Notice that with these choices for p and q there could be 8·2=16 rational zeros. But, eight of them are duplicates. For example $\frac{1}{-1} = \frac{-1}{1} = -1$. The next step is to test all these values by the synthetic division (we'll let you do this on your own for practice) and we finally find that 1, −2, and 3 are zeros of f. That is

$f(x)=x^3-2x^2-5x+6$

$=(x-1)(x+2)(x-3)$

Example 6

Use Descartes' Rule of Signs to identify the possible number of positive and negative roots of

$f(x)=-2x^3+x^2-3x^5+5x-1$.

Solution

First, re-write f(x) in descending order

$f(x)=-3x^5-2x^3+x^2+5x-1$.

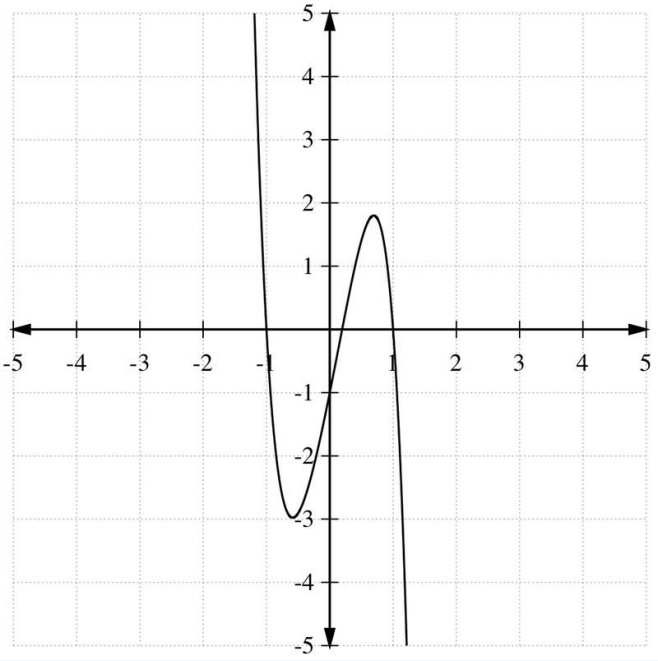
The number of sign changes of f(x) is 2, so the number of positive roots is either 2 or 0.

For the negative roots, write

$f(-x)=-3x^5+2x^3+x^2-5x-1$

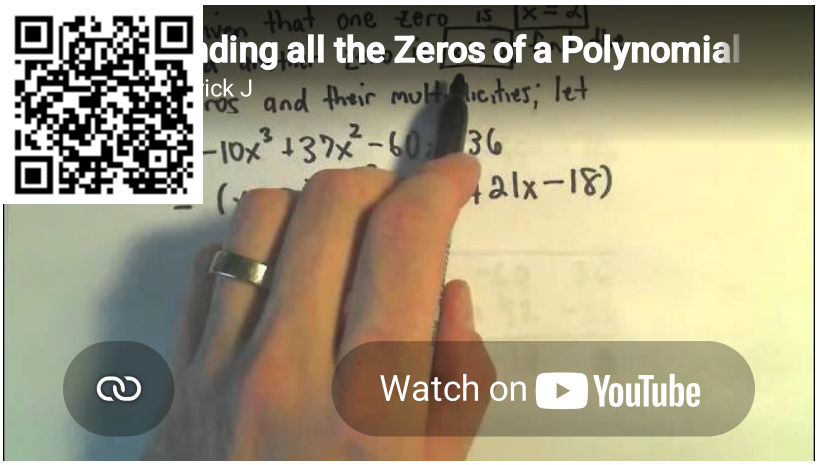
The number of sign changes of f(−x) is 2, so the maximum number of negative roots is 2.

The graph of f(x) below shows that there is one negative root and two positive roots.



[Figure3]

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Review

Questions 1 - 3: Use a) long division and b) synthetic division to perform the divisions. Express each result in the form: $f(x)=D(x)\cdot Q(x)+R$.

1. $5x^5 - 3x^4 + 2x^3 + x^2 - 7x + 3$ by $x-2$
2. $-4x^6 - 5x^3 + 3x^2 + x + 7$ by $x-1$
3. $2x^3 - 5x^2 + 5x + 11$ by $x - \frac{1}{2}$
4. Use synthetic division to find Q(x) and f(c) so that $f(x)=(x-c)Q(x)+f(c)$ if $f(x) = -3x^4 - 3x^3 + 3x^2 + 2x - 4$ and $c = -2$
5. If $f(x) = x^3 + 2x^2 - 10x + 10$, use synthetic division to determine the following: a) $f(-1)$ b) $f(-3)$ c) $f(0)$ d) $f(4)$ e) What are the factors of f(x)?
6. Find k so that $x-2$ is a factor of $f(x) = 3x^3 + 4x^2 + kx - 20$

7. Use synthetic division to determine all the zeros of the polynomials: a) $f(x) = 3x^3 - 7x^2 + 8x - 2$ b) $g(x) = 4x^4 - 4x^3 - 7x^2 + 4x + 3$
8. Graph the polynomial function $f(x) = x^3 - 2x^2 - 5x + 6$ by using synthetic division to find the x−intercepts and locate the y−intercepts.
9. Graph the polynomial function $h(x)= x^3 - 3x^2 + 4$ by using synthetic division to find the x−intercepts and locate the y−intercepts.
10. Write a 3rd degree equation of a polynomial function with the **zeroes**: 0, 2, and -5.
11. Write a 7th degree equation of a polynomial function with the zeroes: 0 (**multiplicity** 2), 2 (multiplicity 3), and -5 (multiplicity 2)
12. Write a quadratic equation which has 4 (multiplicity 2) as the zero and opens downward.
13. Write a 3rd degree polynomial function with the zeroes: -2, 2, and 6, passing through the point (3, 4)
14. Let $f(x) = 2x^3 - 5x^2 - 4x + 3$ and find the solutions: a) $f(x)=0$ b) $f(2x)=0$
15. Graph and find the solution set of the inequality $x^3 - 2x^2 - 5x + 6 \leq 0$.
16. Use the graph of $f(x)=x(x-1)(x+2)$ to find the solution set of the inequality $x(x-1)(x+2)>0$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.12.

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Vocabulary

Term	Definition
Descartes' Rule of Signs	Descartes' rule of signs is a technique for determining the number of positive and negative real roots of a polynomial.
factor theorem	The factor theorem states that if f(x) is a polynomial of degree n>0 and f(c)=0, then x−c is a factor of the polynomial f(x).
factorization theorem	The factorization theorem states that If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$, and n is a positive integer, then $f(x) = a_n(x-c_1)(x-c_2)\cdots(x-c_0)$ where the numbers ci are complex numbers.
Multiplicity	The multiplicity of a term describes the number of times the given term acts as a zero of the given function.
Polynomial	A polynomial is an expression with at least one algebraic term, but which does not indicate division by a variable or contain variables with fractional exponents.
Rational Zero Theorem	The rational zero theorem states that for a polynomial, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where an, $a_{n-1}, \cdots a_0$ are integers, the rational roots can be determined from the factors of an and a0. More specifically, if p is a factor of a0 and q is a factor of an, then all the rational factors will have the form $\pm \frac{p}{q}$.
Remainder Theorem	The remainder theorem states that if f(k)=r, then r is the remainder when dividing f(x) by (x−k).
Roots	The roots of a function are the values of *x* that make *y* equal to zero.
Synthetic Division	Synthetic division is a shorthand version of polynomial long division where only the coefficients of the polynomial are used.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeros	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.

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2.6.4: Intermediate Value Theorem

Intermediate Value Theorem

This lesson introduces two theorems: **The Intermediate Value Theorem**, and **The Bounds on Zeros Theorem**. A technical definition is given below, but what do these theorems really mean in 'ordinary language'?

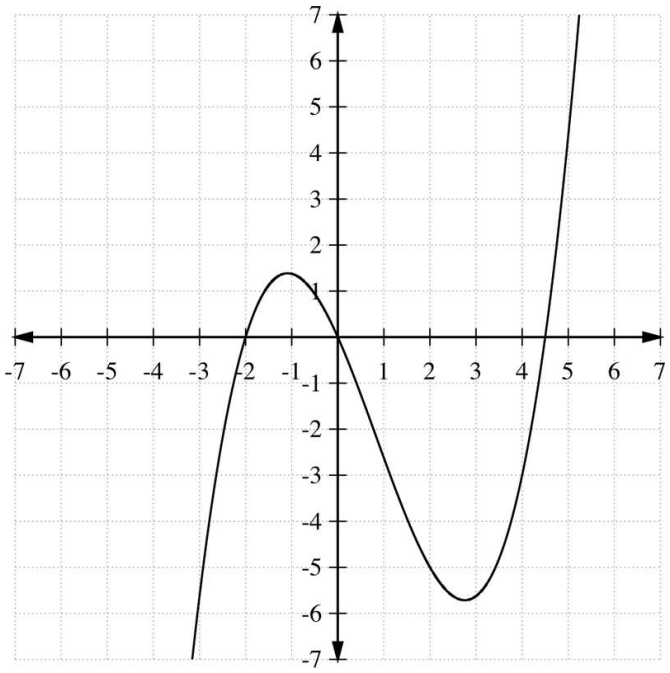
Intermediate Value Theorem

The intermediate value theorem offers one way to find roots of a **continuous function**. An informal definition of **continuous** is that a function is continuous over a certain **interval** if it has no breaks, jumps, asymptotes, or holes in that interval. Polynomial functions are continuous for all real numbers x. Rational functions are often not continuous over the set of real numbers because of asymptotes or holes in the graph, but for intervals without holes, rational functions are continuous.

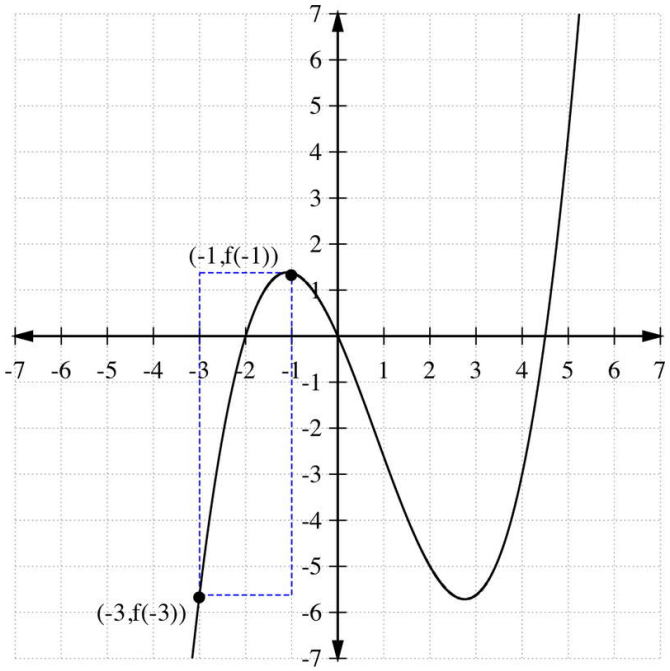
If we know a function is continuous over some interval [a,b], then we can use the **intermediate value theorem**:

If f(x) is continuous on some interval [a,b] and n is between f(a) and f(b), then there is some c∈[a,b] such that f(c)=n.

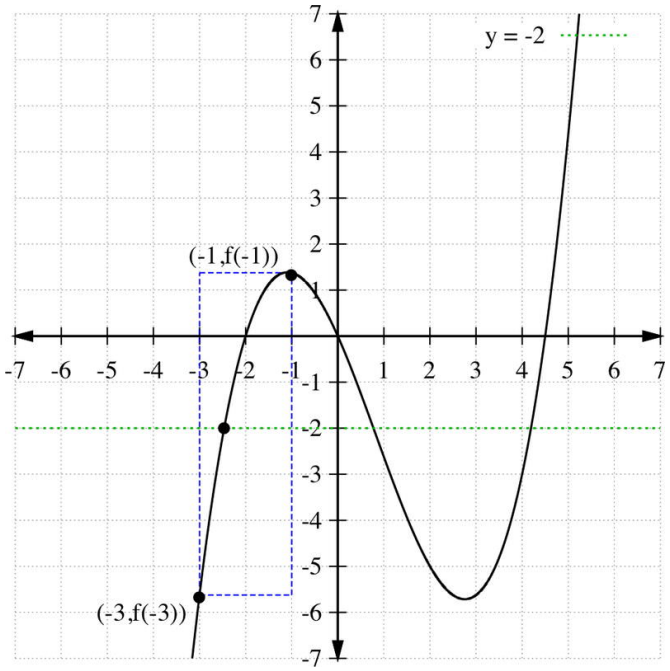
The following graphs highlight how the intermediate value theorem works. Consider the graph of the function $f(x) = \frac{1}{4}\left(x^3 - \frac{5x^2}{2} - 9x\right)$ below on the interval [-3, -1].

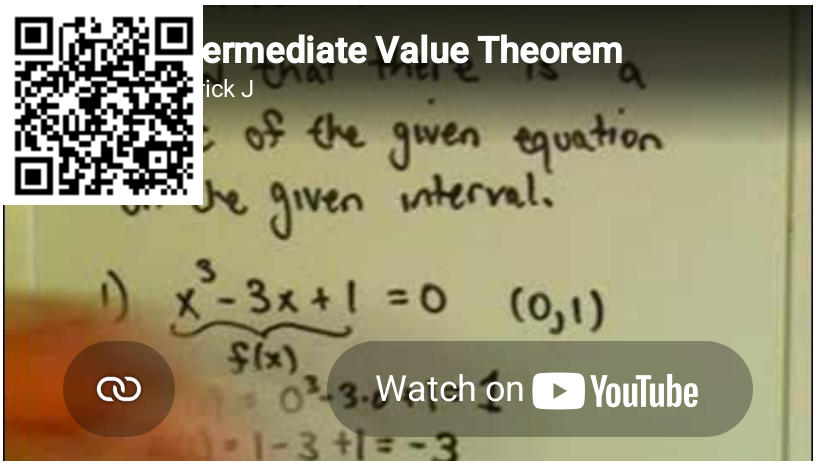


f(−3)=−5.625 and f(−1)=1.375. If we draw bounds on [-3, -1] and [f(−3), f(−1)], then we see that for any y−value between y=−5.625 and y=1.375, there must be an x value in [-3, -1] such that f(x)=y.



So, for example, if we choose $c=-2$, we know that for some $x \in [-3,-1]$, $f(x)=-2$, even though solving this by hand would be a chore!





The Bounds on Zeros Theorem is a corollary to the Intermediate Value Theorem:

Bounds on Zeros Theorem

If f is continuous on [a,b] and there is a sign change between f(a) and f(b) (that is, f(a) is positive and f(b) is negative, or vice versa), then there is a $c \in (a,b)$ such that $f(c)=0$.

The bounds on zeros theorem is a corollary to the intermediate value theorem because it is not fundamentally different from the general statement of the intermediate value theorem, just a special case where n=0.

Looking back at $f(x) = \frac{1}{4}\left(x^3 - \frac{5x^2}{2} - 9x\right)$ above, because $f(-3)<0$ and $f(-1)>0$, we know that for some $x \in [-3,-1]$, f(x) has a root. In fact, that root is at $x=-2$. and we can test that using **synthetic division** or by evaluating $f(-2)$ directly.

Approximate Zeros of Polynomial Functions

In calculus you will learn several methods for numerically approximating the roots of functions. In this section we show one elementary numerical method for finding the zeros of a polynomial which takes advantage of the Intermediate Value Theorem.

Given a continuous function g(x),

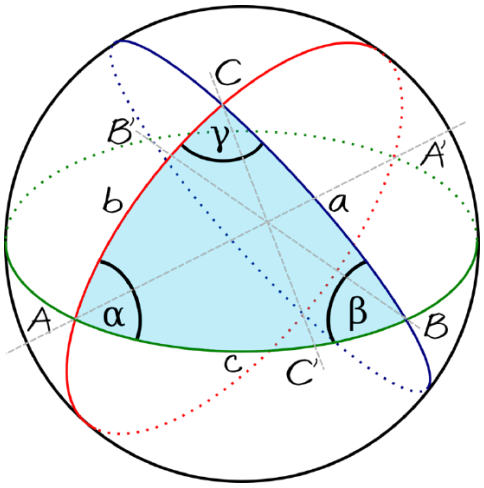
1. Find two points such that $g(a) > 0$ and $g(b) < 0$. Once you have found these two points, you can iteratively use the steps below to find the root of g(x) on the interval [a,b]. (Note, we will assume $a < b$, but the same algorithm works with minor adjustments if $b > a$)
2. Evaluate $g\left(\frac{a+b}{2}\right)$.
   a. If $g\left(\frac{a+b}{2}\right) = 0$, then the root is $x = \frac{a+b}{2}$.
   b. If $g\left(\frac{a+b}{2}\right) > 0$, replace a with $\frac{a+b}{2}$. and repeat steps 1-2 using $\left[\frac{a+b}{2}, b\right]$
   c. If $g\left(\frac{a+b}{2}\right) < 0$, replace b with $\frac{a+b}{2}$. and repeat steps 1-2 using $\left[a, \frac{a+b}{2}\right]$

This algorithm will not usually find the exact root of g(x), but it will allow you to find a reasonably small interval for the root. For example, you could repeat this process enough times so that you find an interval with $|a-b|<0.01$, and you will know the root of g(x) within a reasonably good approximation. The quality of the approximation you use (and the number of steps you use) will depend on why you are looking for the root. For most applications coming within 0.01 of the root is a reasonable approximation, but for some applications (such as building a bridge or launching a rocket) you need much more accuracy.

An Interesting Corollary of the Intermediate Value Theorem

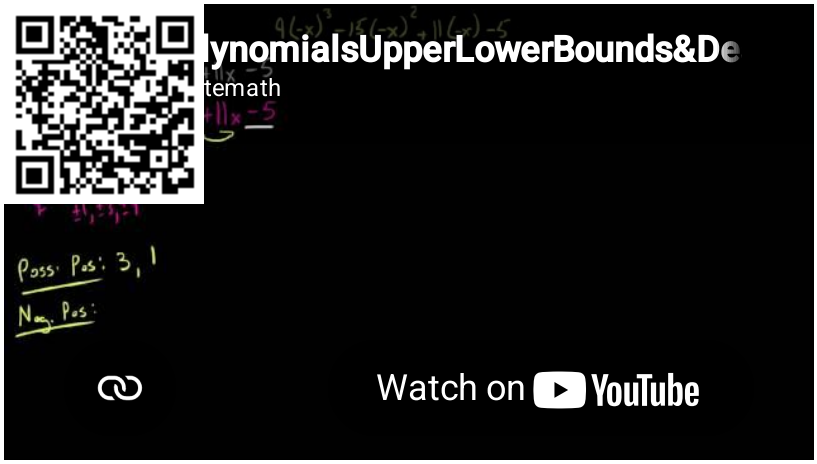
One surprising result of the Intermediate Value Theorem is that if you draw any great circle around the globe, then there must two antipodal points on that great circle that have exactly the same temperature.

Recall that a **great circle** is a path around a sphere that gives the shortest distance between any two points on the sphere. The equator is a great circle around the globe. **Antipodal** points are two points on opposite sides of the sphere. In the diagram below, B and B′ are antipodal.



For an informal proof of this result, look at the image of a sphere with three great circles above. Suppose that the temperature at B is 75∘ and the temperature B′ is 50∘. The difference between the temperature at B and at B′ is 75−50=25. Now imagine rotating the segment $\overline{BB'}$ around the blue great circle. When the segment has rotated 180 degrees (i.e. when B has rotated to where B′ is), then the difference between the temperatures at these two points is 50−75=−25. Since temperatures vary continuously, by the intermediate value theorem, there must be some point on that circle when the difference was 0, implying two antipodal points had the same temperature.

Notice that this little demonstration does not tell us which two antipodal points had the same temperature, only that there must be two such points on any great circle.



Examples

Example 1

Earlier, you were asked if you could write less formal definitions of the **Intermediate Value Theorem** and the **Bounds on Zeros Theorem**.

Solution

One possibility might be:

The Intermediate Value Theorem simply states that if you know two points on a graph, and know that the graph includes all the points between them, then any point between them must be on the graph.

The Bounds on Zeros Theorem suggests that if a graph contains all of the points between a positive and a negative value, then a zero point between those two values is on the graph.

Example 2

Show that $f(x)=-3x^3+5x$ has at least one root in the interval [1, 2].

Solution

Since f(x) is a polynomial we know that it is continuous. f(1) = 2 and f(2) = −14. Let n = 0 ∈ [−14,2]. Applying the Intermediate Value Theorem, there must exist some point c ∈ [1,2] such that f(c) = 0. This proves that f(x) has a root in [1, 2].

Example 3

The table below shows several sample values of a polynomial p(x).

x	−4	−2	0	1	4	6	8	10	15	18
p(x)	44.15	6.62	−4.12	−4.09	1.16	0	−8.74	−24.07	−49.8918	3.41

Solution

Based on the information in the table:

a. What is the minimum number of roots of p(x)?

b. What are bounds on the roots of p(x) that you identified in (a)?

Since p(x) is a polynomial we already know that it is continuous. We can use the Intermediate Value Theorem to identify roots by looking at when p(x) changes from negative to positive, or from positive to negative.

a. There are four sign changes of p(x) in the table, so at minimum, p(x) has four roots.

b. The roots are in the following intervals x ∈ [−2, 0], x ∈ [1, 4], x ∈ [15, 18], and the table also tells us that one root is at x=6.

Example 4

Show the first 5 iterations of finding the root of $h(x)=x^2-x-1$ using the starting values a=0 and b=2.

Solution

1. First we verify that there is a root between x=0 and x=2. h(0)=−1 and h(2)=1 so we know there is a root in the interval [0, 2]. Check $h\left(\frac{2+0}{2}\right)=h(1)=-1$. Since −1 < 0 we know the root is between x=1 and x=2, and we use the new interval [1, 2].
2. Now we use the interval [1, 2]. $h\left(\frac{1+2}{2}\right)=h(1.5)=-0.25$. Since −0.25 < 0, we use the interval [1.5, 2].
3. $h\left(\frac{1.5+2}{2}\right)=h(1.75)=0.31$. Since 0.31 > 0, we know that the zero is in the interval [1.5, 1.75].
4. $h\left(\frac{1.5+1.75}{2}\right)=h(1.625)=0.02$. Since 0.02 > 0, we know the root is between 1.5 and 1.625.
5. $h\left(\frac{1.5+1.625}{2}\right)=h(1.5620)=-0.12$. Since −0.12 < 0, we know the root is between 1.5620 and 1.625.

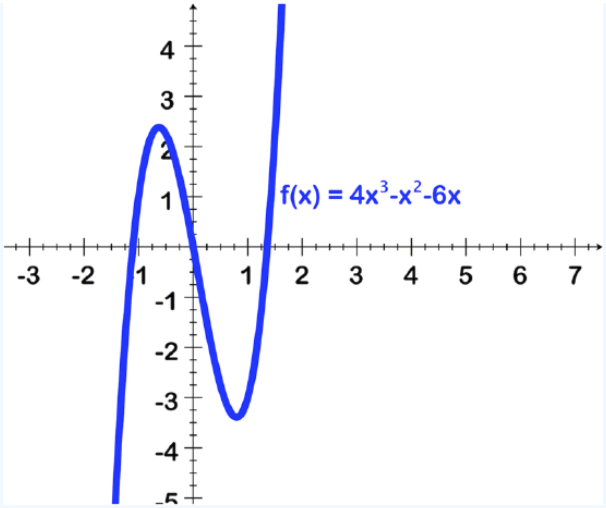
This example shows that after five iterations we have narrowed the possible location of the root to within 0.06 units. Not bad!

Example 5

Use the image of the graph below to find the following:

1. Degree of the polynomial
2. Number of real zeros and their approximate values using the graph
3. Number of imaginary zeros

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Solution

a. To identify the degree, recall that the degree of a polynomial is one number greater than the number of turns. The image shows turns at apx x = -.6 and apx x = .75, therefore this is a 3rd degree function.
b. The image shows the line crossing the *x* axis 3 times, at (apx) -1.15, at 0 and at (apx) 1.4.
c. Since this is a 3rd degree equation there are 3 possible zeros, and since there are 3 real zeros there are no imaginary ones.

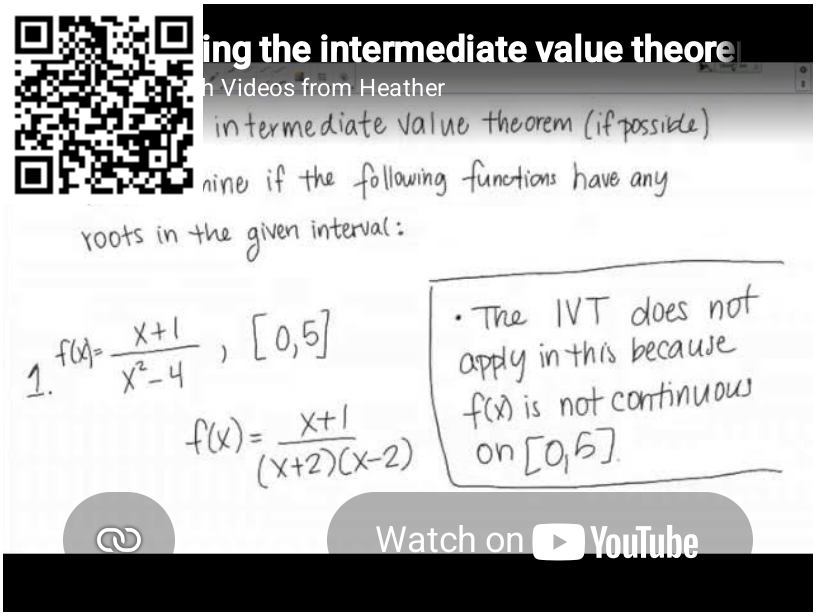
Example 6

Show that $f(x)=8x^3-5x^2-7x-5$ has at least one root in the interval [1, 2].

Solution

Since f(x) is a polynomial we know that it is continuous. f(1) = −9 and f(2) = 25. Let n = 0 ∈ [−9, 25]. Applying the Intermediate Value Theorem, there must exist some point c ∈ [1,2] such that f(c) = 0. This proves that f(x) has a root in [1, 2].

Note that since this is a 3rd degree polynomial, there are three zeros. Since there is only one real zero, the other two must be imaginary.

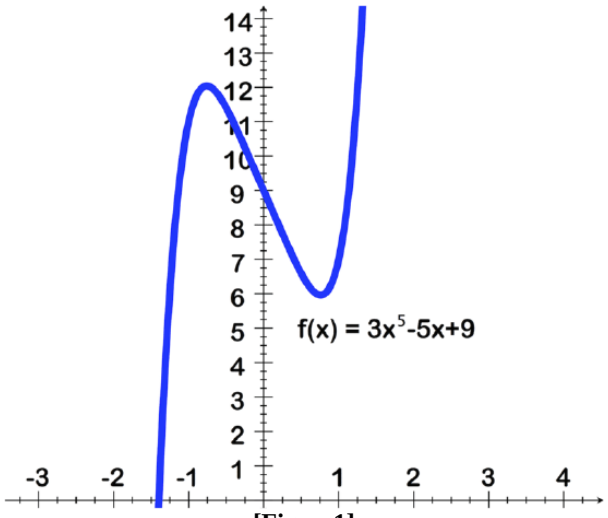


Review

For questions 1 - 6, use the image of the graph to find the following:

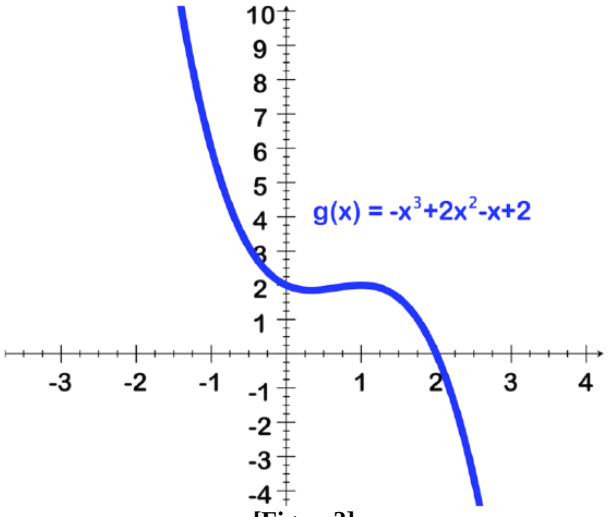
a. Leading coefficient and degree of the polynomial
b. Number of real zeros and their approximate values using the graph
c. Number of imaginary zeros

1.



[Figure1]

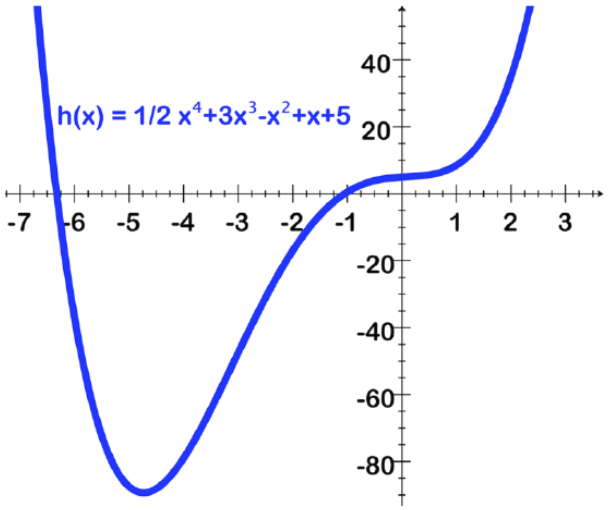
2.



[Figure2]

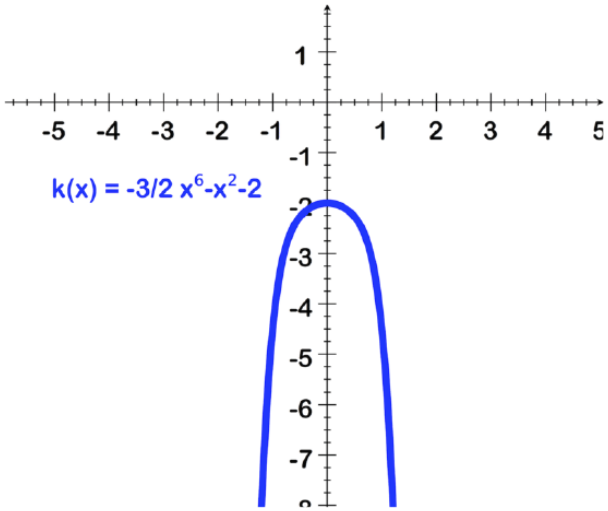
3.

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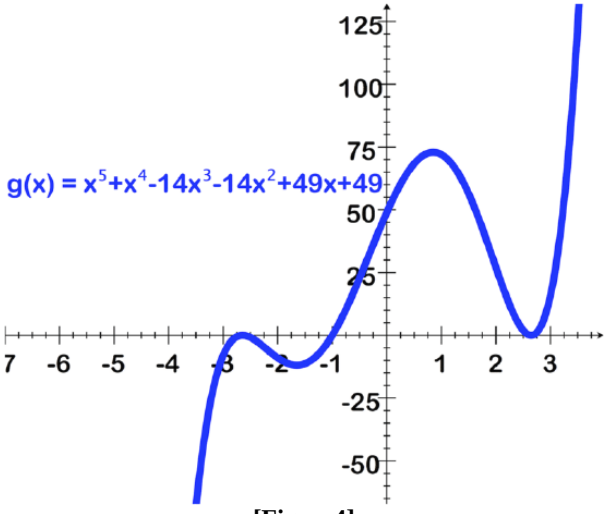


[Figure3]

4.



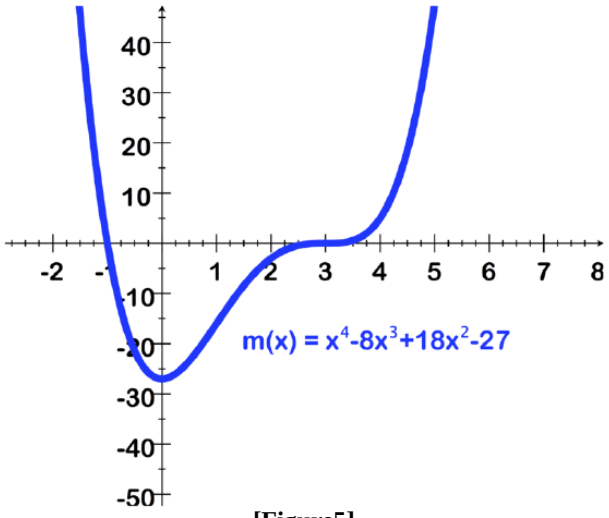
5.



[Figure4]

6.

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[Figure5]

For questions 7-12, use the intermediate value theorem to show the bounds on the zeros of each function. (Your bounds should be within the whole number)

7. $f(x)=2x^3-3x+4$
8. $g(x)=-5x^2+8x+12$
9. $h(x)=\frac{1}{2}x^4-x^3-3x^2+1$
10. $j(x)=-\frac{2}{x^2+1}+\frac{1}{2}$
11. k(x) is a polynomial and selected values of k(x) are given in the following table:

x	-3	-2	-1	0	1	2	3
k(x)	−23.5	−1	0.5	−1	.5	−1	−23.5

12. Stephen argues the function $r(x)=\frac{4x+1}{x+3.5}$ has two zeros based on the following table and an application of the Bounds on Zeros Theorem. What is faulty about Stephen's reasoning?

x	−5	−4	−3	−2	−1	0	1	2	3	4
r(x)	12.67	30	−22	−4.67	-1.20	0.29	1.11	1.64	2.0	2.27

For questions 13-15, apply the numerical algorithm five times to find a bound on the zeros of the following functions given the indicated starting values. Give a final estimate for the zero.

13. $k(x)=x^4-3x+1$ on $[0,1]$
14. $b(x)=-0.1x^5+3x^3-5x^2$ on $[1,3]$
15. $c(x)=\frac{3x^2-2}{x^4+2}$ on $[0,2]$

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.13.

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Vocabulary

Term	Definition
Bounds on Zeros Theorem	The bounds on zeros theorem states that if f is continuous on [a,b] and there is a sign change between f(a) and f(b) (that is, f(a) is positive and f(b) is negative, or vice versa), then there is a c $\in$ (a, b) such that f(c) = 0.
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous Function	A continuous function is a function without breaks or gaps. It contains an infinite, uncountable number of values.
intermediate value theorem	The intermediate value theorem states that if f(x) is continuous on some interval [a, b] and n is between f(a) and f(b), then there is some c $\in$ [a, b] such that f(c) = n.
interval	An interval is a specific and limited part of a function.
Rational Function	A rational function is any function that can be written as the ratio of two polynomial functions.
Synthetic Division	Synthetic division is a shorthand version of polynomial long division where only the coefficients of the polynomial are used.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeros	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.

Image Attributions

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2. [Figure 2]
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   Source: CK-12
3. [Figure 3]
   Credit: CK-12
   Source: CK-12
4. [Figure 4]
   Credit: CK-12
   Source: CK-12
5. [Figure 5]
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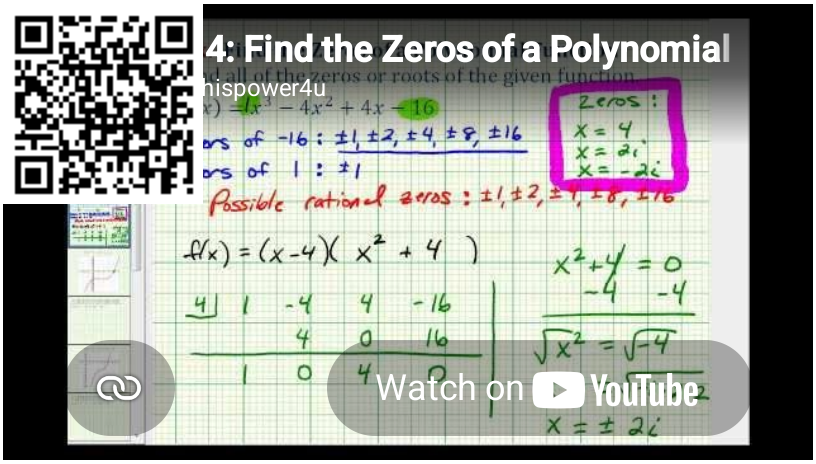
2.6.5: Fundamental Theorem of Algebra

Finding Imaginary Solutions

Louis calculates that the area of a rectangle is represented by the equation $3x^4+7x^2=2$. Did Louis calculate it right? Explain based on the degree and zeros of the function.

Imaginary Solutions

Remember, imaginary solutions always come in pairs. To find the imaginary solutions to a function, use the Quadratic Formula.



Let's solve $f(x)=3x^4-x^2-14$.

First, this quartic function can be factored just like a quadratic equation.

$g(x)=x^4+21x^2+90$

Now, because neither factor can be factored further and there is no x−term, we can set each equal to **zero** and solve.

$$\begin{array}{lll} x^2+2=0 & & 3x^2-7=0 \\ x^2=-2 & \text{and} & 3x^2=7 \\ x=\pm\sqrt{-2} \text{ or } \pm i\sqrt{2} & & x^2=\frac{7}{3} \\ & & x=\pm\sqrt{\frac{7}{3}} \text{ or } \pm\frac{\sqrt{21}}{3} \end{array}$$

Including the imaginary solutions, there are four, which is what we would expect because the degree of this function is four.

Now, let's find all the solutions of the function $g(x)=x^4+21x^2+90$.

When graphed, this function does not touch the x−axis. Therefore, all the solutions are imaginary. To solve, this function can be factored like a quadratic equation. The factors of 90 that add up to 21 are 6 and 15.

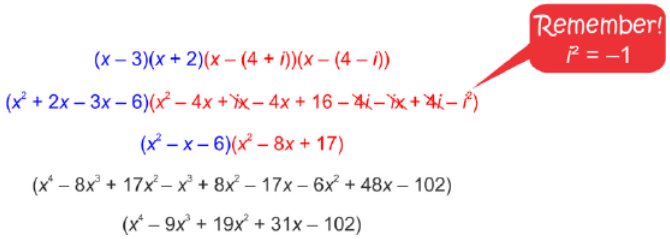
$$\begin{aligned} g(x) &= x^4+21x^2+90 \\ 0 &= \left(x^2+6\right)\left(x^2+15\right) \end{aligned}$$

Now, set each factor equal to zero and solve.

$$\begin{array}{lll} x^2+6=0 & & x^2+15=0 \\ x^2=-6 & \text{and} & x^2=-15 \\ x=\pm i\sqrt{6} & & x=\pm i\sqrt{15} \end{array}$$

Finally, let's find the function that has the solution 3, -2, and 4+i.

Notice that one of the given solutions involves an **imaginary number**. Imaginary and complex solutions always come in pairs, so 4−i is also a factor. The two factors are **complex conjugates.** Translate each solution into a factor and multiply them all together.



[Figure1]

Any multiple of this function would also have these **roots**.

For example, $2x^4-18x^3+38x^2+62x-204$ would have these roots as well.

Examples

Example 1

Earlier, you were asked to determine if Louis calculated his work correctly.

Solution

First we need to change the equation to standard form. Then we can factor it.

$$\begin{aligned} 3x^4+7x^2 &= 2 \\ 3x^4+7x^2-2 &= 0 \\ (3x^2+1)(x^2+2) &= 0 \end{aligned}$$

Solving for x we get

$$\begin{aligned} 3x^2+1 &= 0 & & & x^2+2 &= 0 \\ x^2 &= \frac{-1}{3} & \text{and} & & x^2 &= -2 \\ x &= \pm i\sqrt{\frac{1}{3}} & & & x &= \pm i\sqrt{2} \end{aligned}$$

All of the solutions are imaginary and the area of a rectangle must have real solutions. Therefore Louis did not calculate correctly.

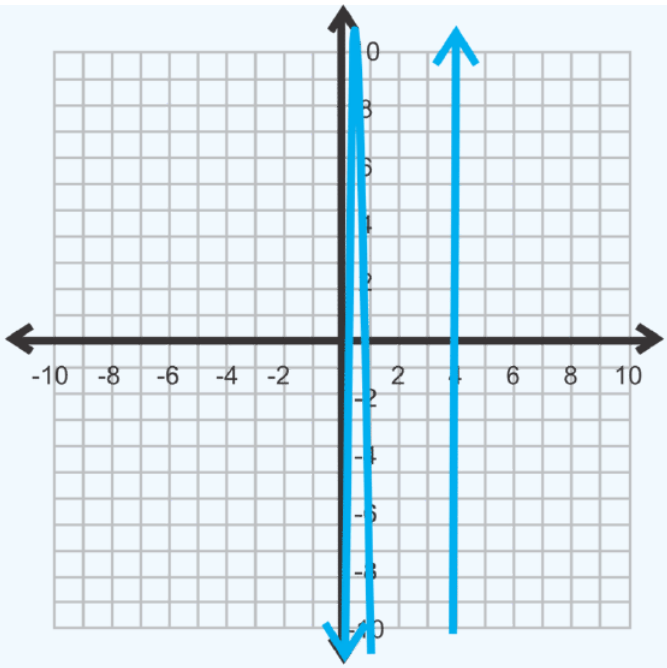
Example 2

Find all the solutions to the following function: $f(x)=25x^3-120x^2+81x-4$.

Solution

First, graph the function.

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Using the Rational Root Theorem, the possible realistic zeros could be $\frac{1}{25}$, 1, or 4. Let's try these three possibilities using synthetic division.

$$\begin{array}{r|rrrr} \frac{1}{25} & 25 & -120 & 81 & -4 \\ & \downarrow & 1 & -118.96 & -1.5184 \\ \hline & 25 & -119 & -37.96 & -5.5184 \end{array} \qquad \begin{array}{r|rrrr} 1 & 25 & -120 & 81 & -4 \\ & \downarrow & 25 & -95 & -14 \\ \hline & 25 & -95 & -14 & -18 \end{array}$$

$$\begin{array}{r|rrrr} 4 & 25 & -120 & 81 & -4 \\ & \downarrow & 100 & -80 & 4 \\ \hline & 25 & -20 & 1 & 0 \end{array}$$

Of these three possibilities, only 4 is a zero. The leftover **polynomial**, $25x^2-20x+1$ is not factorable, so we need to use the Quadratic Formula to find the last two zeros.

$$\begin{aligned} x &= \frac{20 \pm \sqrt{20^2 - 4(25)(1)}}{2(25)} \\ &= \frac{20 \pm \sqrt{400 - 100}}{50} \\ &= \frac{20 \pm 10\sqrt{3}}{50} \text{ or } \frac{2 \pm \sqrt{3}}{5} \approx 0.746 \text{ and } 0.054 \end{aligned}$$

Helpful Hint: Always find the decimal values of each zero to make sure they match up with the graph.

Example 3

Find all the solutions to the following function: $f(x)=4x^4+35x^2-9$.

Solution

$f(x)=4x^4+35x^2-9$ is factorable. $ac=-36$.

$$\begin{aligned} & 4x^4 + 35x^2 - 9 \\ & 4x^4 + 36x^2 - x^2 - 9 \\ & 4x^2(x^2+9) - 1(x^2+9) \\ & (x^2+9)(4x^2-1) \end{aligned}$$

Setting each factor equal to zero, we have:

$$\begin{array}{lll} & & 4x^2-1=0 \\ x^2+9=0 & & 4x^2=1 \\ x^2=-9 & \text{or} & x^2=\frac{1}{4} \\ x=\pm 3i & & x=\pm\frac{1}{2} \end{array}$$

Example 4

Find the equation of a function with roots $4, \sqrt{2}$ and $1-i$.

Solution

Recall that irrational and imaginary roots come in pairs. Therefore, all the roots are $4, \sqrt{2}, -\sqrt{2}, 1+i, 1-i$. Multiply all 5 roots together.

$$\begin{aligned} &(x-4)(x-\sqrt{2})(x+\sqrt{2})(x-(1+i))(x-(1-i)) \\ &(x-4)\left(x^2-2\right)\left(x^2-2x+2\right) \\ &\left(x^3-4x^2-2x+8\right)\left(x^2-2x+2\right) \\ &x^5-6x^4+8x^3-4x^2-20x+16 \end{aligned}$$

Review

Find all solutions to the following functions. Use any method.

1. f(x)=$x^4+x^3-12x^2-10x+20$
2. f(x)=$4x^3-20x^2-3x+15$
3. f(x)=$2x^4-7x^2-30$
4. f(x)=$x^3+5x^2+12x+18$
5. f(x)=$4x^4+4x^3-22x^2-8x+40$
6. f(x)=$3x^4+4x^2-15$
7. f(x)=$2x^3-6x^2+9x-27$
8. f(x)=$6x^4-7x^3-280x^2-419x+280$
9. f(x)=$9x^4+6x^3-28x^2+2x+11$
10. f(x)=$2x^5-19x^4+30x^3+97x^2-20x+150$

Find a function with the following roots.

11. $4, i$
12. $-3, -2i$
13. $\sqrt{5}, -1+i$
14. $2, \frac{1}{3}, 4-\sqrt{2}$
15. **Writing:** Write down the steps you use to find all the zeros of a polynomial function.
16. **Writing:** Why do imaginary and irrational roots always come in pairs?
17. **Challenge:** Find all the solutions to $f(x)=x^5+x^3+8x^2+8$.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 6.12.

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Vocabulary

Term	Definition
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of a+bi is a−bi. When complex conjugates are multiplied, the result is a single real number.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form a+bi.
conjugate pairs theorem	The conjugate pairs theorem states that if f(z) is a polynomial of degree n, with $n \neq 0$ and with real coefficients, and if $f(z_0)=0$, where $z_0=a+bi$, then $f(z_0^*)=0$. Where z_0^* is the complex conjugate of z_0.
fundamental theorem of algebra	The fundamental theorem of algebra states that if f(x) is a polynomial of degree $n \geq 1$, then f(x) has at least one zero in the complex number domain. In other words, there is at least one complex number c such that f(c)=0. The theorem can also be stated as follows: an nth degree polynomial with real or complex coefficients has, with multiplicity, exactly n complex roots.
Imaginary Number	An imaginary number is a number that can be written as the product of a real number and i.
Imaginary Numbers	An imaginary number is a number that can be written as the product of a real number and i.
Polynomial	A polynomial is an expression with at least one algebraic term, but which does not indicate division by a variable or contain variables with fractional exponents.
Roots	The roots of a function are the values of *x* that make *y* equal to zero.
Zero	The zeros of a function f(x) are the values of x that cause f(x) to be equal to zero.
Zeroes	The zeroes of a function f(x) are the values of x that cause f(x) to be equal to zero.

Image Attributions

1. [Figure 1]
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CHAPTER OVERVIEW

Unit 3: Exponential and Logarithmic Functions

3.1: Functions and Inverses

3.1.1: One-to-One Functions and Their Inverses

3.2: Basic Exponential Functions

3.2.1: Solving Exponential Equations

3.3: Properties of Logarithms

3.3.1: Product and Quotient Properties of Logarithms

3.3.2: Power Property of Logarithms

3.3.3: Inverse Properties of Logarithms

3.4: Logarithmic Functions

3.4.1: Graphs of Logarithmic Functions

3.4.2: Common and Natural Logarithms

3.4.3: Solving Logarithmic Equations

3.4.4: Change of Base

3.5: Exponential and Logarithmic Models

3.5.1: Exponential Models

3.5.2: Logarithmic Models

3.5.3: Simple and Compound Interest

3.5.4: The Number e

3.5.5: Logistic Functions

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SECTION OVERVIEW

3.1: Functions and Inverses

Topic hierarchy

3.1.1: One-to-One Functions and Their Inverses

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3.1.1: One-to-One Functions and Their Inverses

One-to-One Functions and Their Inverses

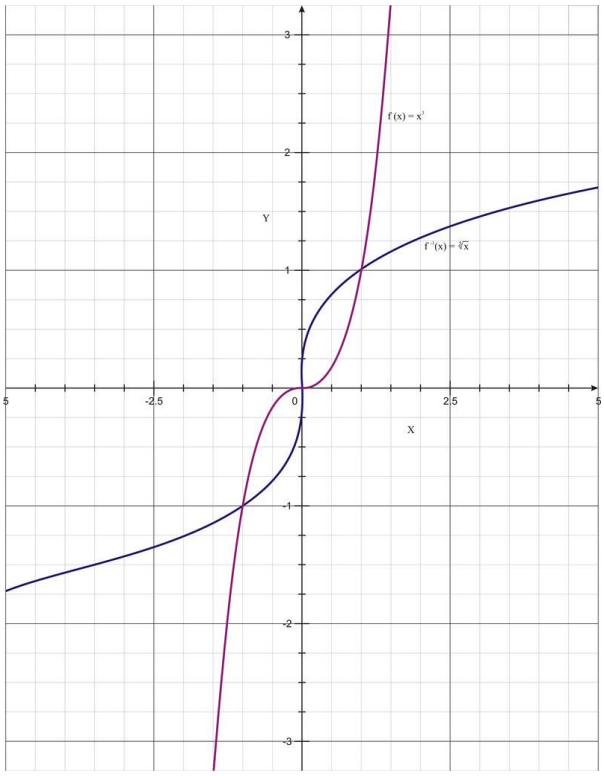
The statement "Pizza restaurants sell pizza" could be thought of as a function. It could be plotted on a graph, with different restaurants across the *x*-axis, and different foods the restaurant specializes in on the *y*-axis. Any time a pizza restaurant was input into the function, it would output "pizza" as the specialized food.

Is this pizza restaurant function a **one-to-one** function? How can we tell?

One-to-One Functions and Their Inverses

Consider the function $f(x) = x^3$, and its **inverse** $f^{-1}(x) = \sqrt[3]{x}$.

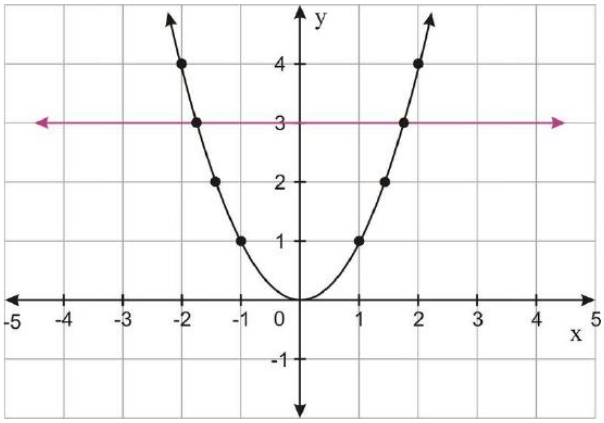
The graphs of these functions are shown below:



The function $f(x) = x^3$ is an example of a **one-to-one function,** which is defined as follows:

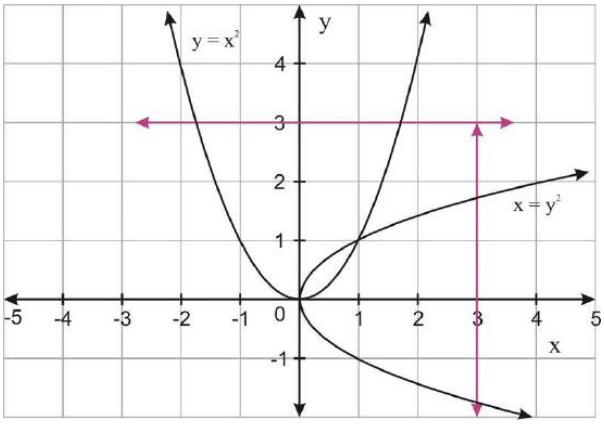
A function is **one-to-one** if and only if every element of its range corresponds to at most one element of its domain.

The function $y = x^2$, however, is not one-to-one. The graph of this function is shown below.



You may recall that you can identify a relation as a function if you can draw a vertical line anywhere through the graph, and the line touches only one point.

Notice then that if we draw a *horizontal* line through $y = x^2$, the line touches more than one point. That indicates that the inverse will *not* be a function, here is why: If we invert the function $y = x^2$, the result is a graph that is a reflection over the line $y = x$, effectively rotating the original 90 degrees. Since x and y have swapped, the new function fails the **vertical line test**.



The function $y = x^2$ is therefore *not* a one-to-one function. A function that *is* one-to-one will be **invertible**.

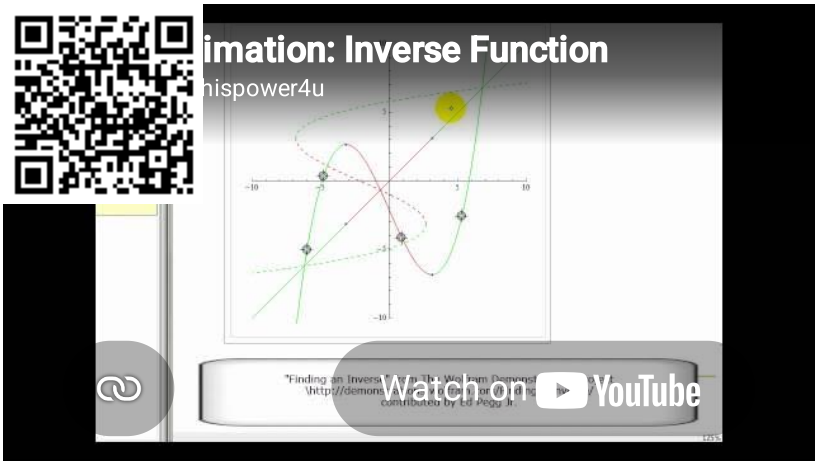
You can determine an invertible function graphically by drawing a *horizontal line* through the graph of the function, if it touches more than one point, the function is not invertible.

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Examples

Example 1

Earlier, you were given a question about a pizza function.

Solution

"Pizza restaurants sell pizza" is a function. However, it is NOT a one-to-one function.

In order to be one-to-one, it must be invertible, giving something like: "pizza sellers are pizza restaurants", and that statement must also be a function.

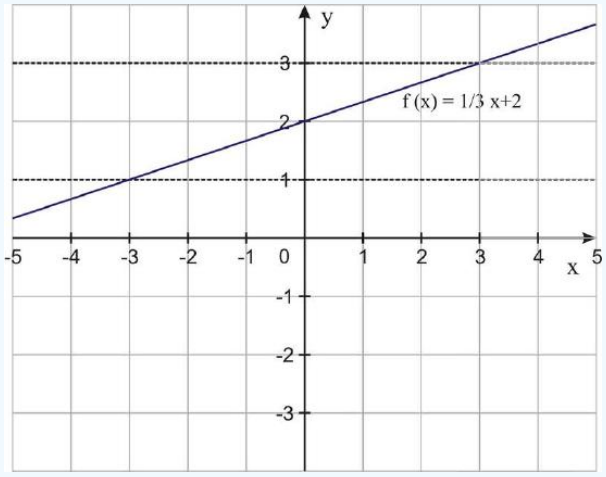
Since grocery stores sell pizza, and would therefore be among the outputs of the new function, but were not among the inputs of the original (which specified "pizza restaurants"), the functions are not invertible.

Example 2

Graph the function $f(x) = \frac{1}{3}x + 2$. Use a **horizontal line test** to verify that the function is invertible.

Solution

The graph below shows that this function is invertible. We can draw a horizontal line at any *y* value, and the line will only cross $f(x) = \frac{1}{3}x + 2$.



In sum, a one-to-one function is invertible. That is, if we invert a one-to-one function, its inverse is also a function. Now that we have established what it means for a function to be invertible, we will focus on the domain and range of inverse functions.

Example 3

State the domain and range of the following function and its inverse: *(1, 2), (2, 5), (3, 7).*

Solution

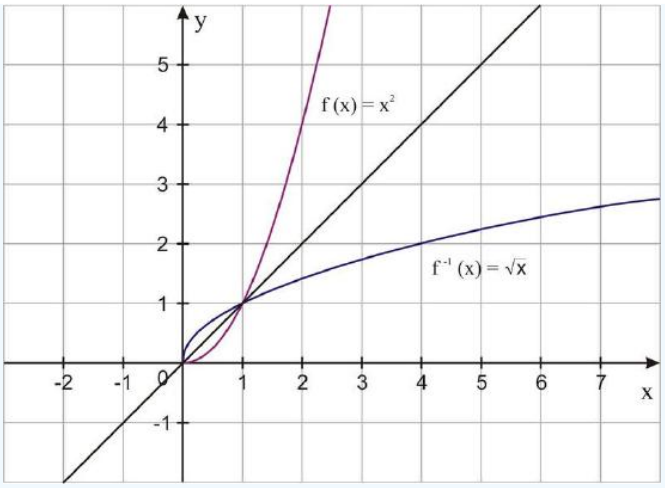
The inverse of this function is the set of points *(2, 1), (5, 2), (7, 3).*

The domain of the function is *{1, 2, 3}*. This is also the range of the inverse.

The range of the function is *{2, 5, 7}*. This is also the domain of the inverse.

The linear functions we examined previously, as well as $f(x) = x^3$, all had domain and range both equal to the set of all real numbers. Therefore the inverses also had domain and range equal to the set of all real numbers. Because the domain and range were the same for these functions, switching them maintained that relationship.

Also, as we found above, the function $y = x^2$ is not one-to-one, and hence it is not invertible. That is, if we invert it, the resulting relation is not a function. We can change this situation if we define the domain of the function in a more limited way. Let $f(x)$ be a function defined as follows: $f(x) = x^2$, with domain limited to real numbers ≥ 0. Then the inverse of the function is the square root function: $f^{-1}(x) = \sqrt{x}$.



Example 4

Define the domain for the function $f(x) = (x - 2)^2$ so that f is invertible.

Solution

The graph of this function is a parabola. We need to limit the domain to one side of the parabola. Conventionally in cases like these we choose the positive side; therefore, the domain is limited to real numbers ≥ 2.

Example 5

Is g(x)=3x−2 a one-to-one function?

Solution

Algebraic test for one-to-one functions: if $f(a) = f(b)$ implies that $a = b$, then f is one-to-one.

∴ if g(x)=3x−2 is one-to-one, then g(a)=g(b)→a=b

Test: g(a)=g(b)

3a−2=3b−2

3a=3b

a=b

∴3x−2 is one-to-one.

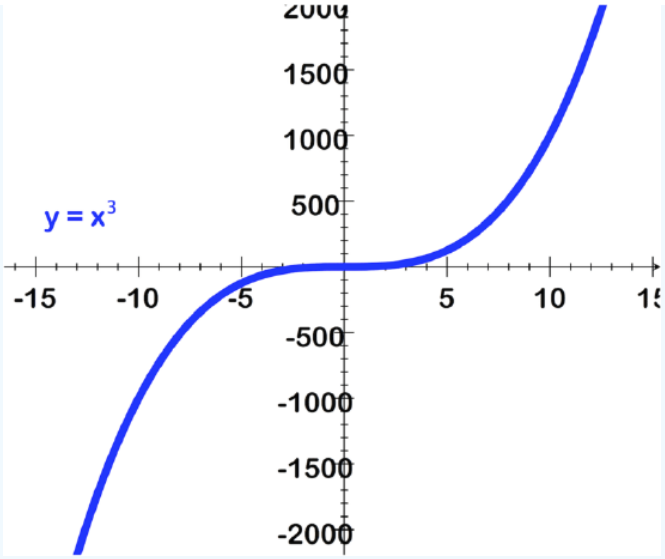
Example 6

Use the horizontal line test to see if $f(x)=x^3$ is one-to-one.

Solution

Graph the equation:

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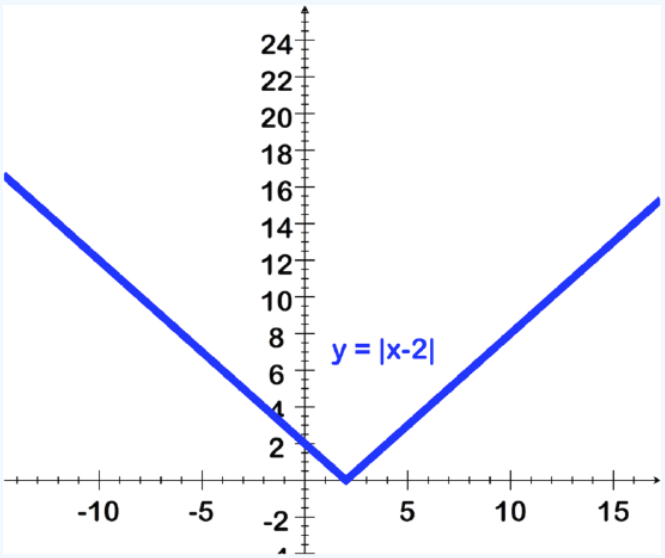
This is the parent function of the cubic function family. Each x value has one unique y-value that is not used by any other x-element. Since that is the definition of a one-to-one function, this function is one-to-one.

Example 7

Is g(x)=|x−2| one-to-one?

Solution

Graph the equation:



This absolute value function has y-values that are paired with more than one x-value, such as (4, 2) and (0, 2). This function is not one-to-one. Note that this function also fails the horizontal line test used in Example 6.

Review

1. Describe the one-to-one horizontal line test.
2. Describe the one-to-one algebraic test.

Which functions are one-to-one?

3. (3,28),(4,29),(4,30),(6,31)

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4. (4,5),(9,6),(7,8),(23,5)
5. (8,18),(33,4),(5,16),(7,19)

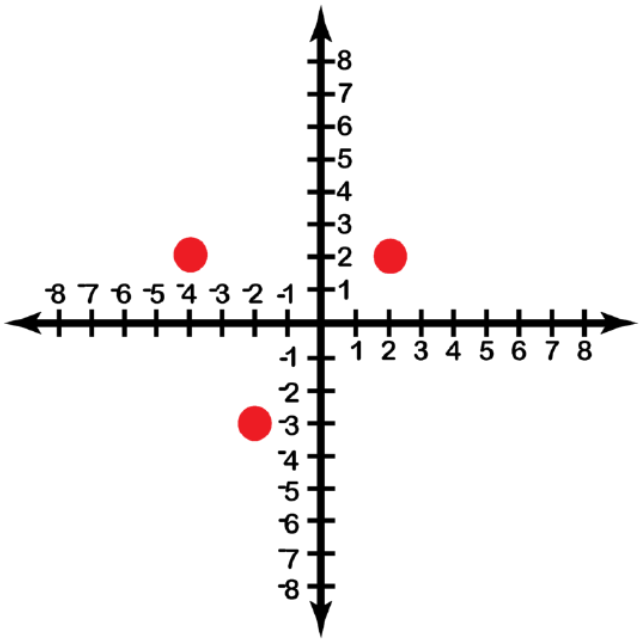
For the following to be a one-to-one function, x cannot be what values?

6. (9,12),(35,6),(7,18),(12,X)
7. (20,21)(21,14),(110,112),(X,7)

Are the following one-to-one functions?

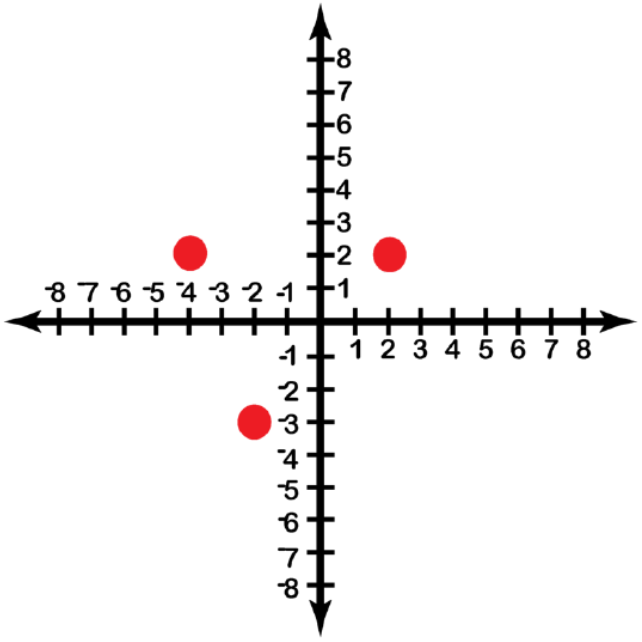
8. $f(x)=x^2$
9. $f(x)=x^3$
10. $f(x)=\frac{1}{x}$
11. $f(x)=x^n-x, n>0$
12. $x=y^2+2$

Determine if the relations below are functions, one-to one functions or neither:

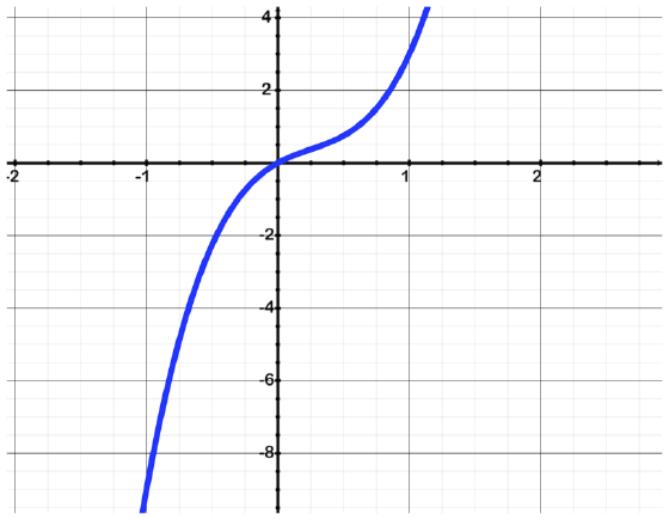


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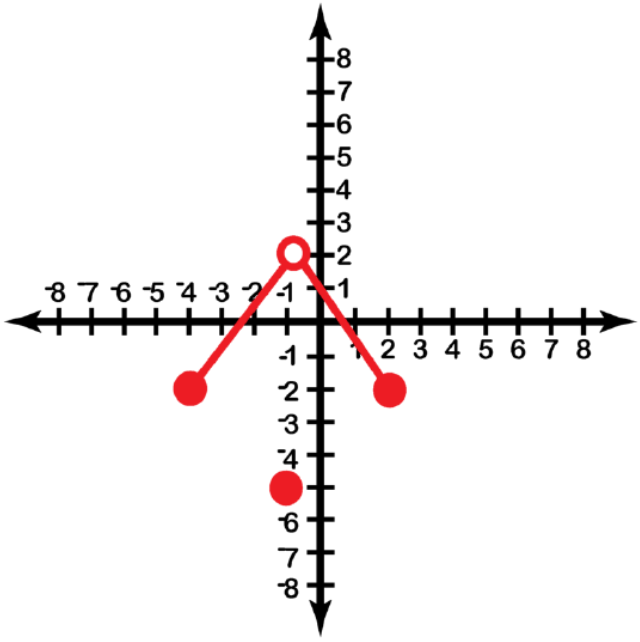
13.



14.



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15.

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.2.

Vocabulary

Term	Definition
1-1 function	A function is 1-1 if its inverse is also a function.
Horizontal Line Test	The horizontal line test says that if a horizontal line drawn anywhere through the graph of a function intersects the function in more than one location, then the function is not one-to-one and not invertible.
inverse	Inverse functions are functions that 'undo' each other. Formally: f(x) and g(x) are inverse functions if f(g(x))=g(f(x))=x.
inverse function	Inverse functions are functions that 'undo' each other. Formally f(x) and g(x) are inverse functions if f(g(x))=g(f(x))=x.
invertible	A function is invertible if it has an inverse.
One-to-one	A function is one-to-one if its inverse is also a function.
Vertical Line Test	The vertical line test says that if a vertical line drawn anywhere through the graph of a relation intersects the relation in more than one location, then the relation is *not* a function.

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SECTION OVERVIEW

3.2: Basic Exponential Functions

Topic hierarchy

3.2.1: Solving Exponential Equations

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3.2.1: Solving Exponential Equations

Exponential Equations

When you were first learning equations, you learned the rule that whatever you do to one side of an equation, you must also do to the other side so that the equation stays in balance. The basic techniques of adding, subtracting, multiplying and dividing both sides of an equation worked to solve almost all equations up until now. With logarithms, you have more tools to isolate a variable. Consider the following equation and ask yourself: why is x=3? Logically it makes sense that if the bases match, then the exponents must match as well, but how can it be shown for examples like this one?

$$1.79898^{2x}=1.79898^{6}$$

Solving Exponential Equations

A common technique for solving equations with unknown variables in exponents is to take the log of the desired base of both sides of the equation. Then, you can use properties of logs to simplify and solve the equation.

Take the following equation. To solve for t, you should first simplify the expression as much as possible and then take the natural log of both sides.

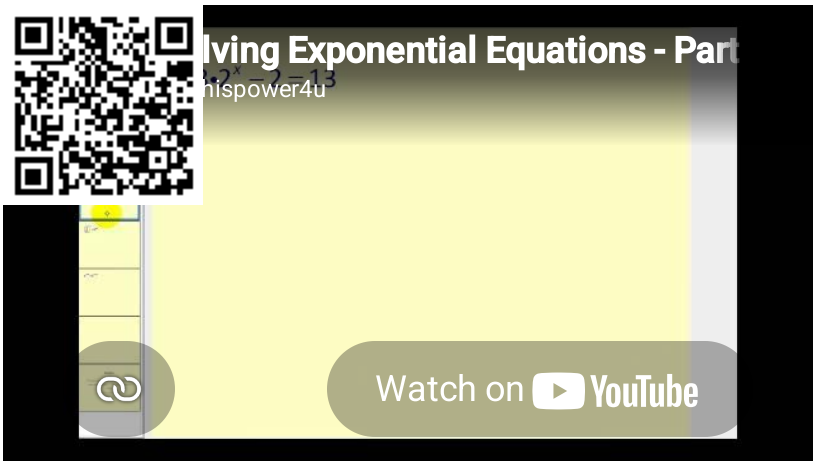
$$\begin{aligned} 9,000 &= 300 \cdot \frac{(1.06)^t - 1}{0.06} \\ 30 &= \frac{(1.06)^t - 1}{0.06} \\ 1.8 &= (1.06)^t - 1 \\ 2.8 &= 1.06^t \\ \ln 2.8 &= \ln\left(1.06^t\right) = t \cdot \ln(1.06) \\ t &= \frac{\ln 2.8}{\ln 1.06} \approx 17.67 \text{ years} \end{aligned}$$

It does not matter what base you use in this situation as long as you use the same base on both sides. Choosing natural log allows you to use a calculator to finish the problem.

Note that this type of equation is common in financial mathematics. The equation above represents the unknown amount of time it will take you to save $9,000 in a savings account if you save $300 at the end of each year in an account that earns 6% annual compound interest.

The other good base to use is base 10. When solving the following equation for x: $16^x=25$, you will need to use a calculator to get the final answer and your calculator can handle base 10 as well. First take the log of both sides. Then, use log properties and your calculator to help.

$$\begin{aligned} 16^x &= 25 \\ \log 16^x &= \log 25 \\ x \log 16 &= \log 25 \\ x &= \frac{\log 25}{\log 16} \\ x &= 1.16 \end{aligned}$$



Examples

Example 1

Earlier, you were asked how to show that if the bases match in an equation, the exponents should match. In the equation, logs can be used to reduce the equation to 2x=6.

Solution

$1.79898^{2x}=1.79898^{6}$

Take the log of both sides and use the property of exponentiation of logs to bring the exponent out front.

$$\log 1.79898^{2x} = \log 1.79898^{6}$$
$$2x \cdot \log 1.79898 = 6 \cdot \log 1.79898$$
$$2x = 6$$
$$x = 3$$

Example 2

Solve the following equation for all possible values of x: $(x+1)^{x-4}-1=0$

Solution

$(x+1)^{x-4}-1=0$

$(x+1)^{x-4}=1$

Case 1 is that x+1 is positive in which case you can take the log of both sides.

$$\log (x+1)^{(x-4)} = \log 1$$
$$(x-4) \cdot \log(x+1) = 0$$
$$x-4=0 \text{ or } \log(x+1)=0$$
$$x = 4 \text{ or } (x+1) = 10^0 = 1$$
$$x = 4, 0$$

Note that log1=0

Case 2 is that (x+1) is negative 1 and raised to an even power. This happens when x=−2.

$$(x+1)^{(x-4)} = 1$$
$$(-2+1)^{(-2-4)} - 1 = (-1)^{-6} - 1$$
$$= \frac{1}{(-1)^6} - 1$$
$$= 0$$

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The reason why this exercise is included is because you should not fall into the habit of assuming that you can take the log of both sides of an equation. It is only valid when the argument is strictly positive. For example, $\log(-2+1)^{(-2-4)}=\log(-1)$ is not possible.

Example 3

Light intensity, measured in lumens, can be described by the relationship between i for intensity and d for depth in feet as it travels at specific depths of water in a swimming pool. What is the intensity of light at 10 feet?

Solution

$\log\left(\frac{i}{12}\right) = -0.0145 \cdot d$

Given d=10, solve for i measured in lumens.

$$\begin{aligned}\log\left(\frac{i}{12}\right) &= -0.0145 \cdot d \\ \log\left(\frac{i}{12}\right) &= -0.0145 \cdot 10 \\ \log\left(\frac{i}{12}\right) &= -0.145 \\ \left(\frac{i}{12}\right) &= 10^{-0.145} \\ i &= 12 \cdot 10^{-0.145} \approx 8.594\end{aligned}$$

Example 4

Solve the following equation for all possible values of x.

$\frac{e^x-e^{-x}}{3} = 14$

Solution

First solve for e^x,

$$\begin{aligned}\frac{e^x - e^{-x}}{3} &= 14 \\ e^x\left(e^x - e^{-x}\right) &= (42)e^x \\ e^{2x} - 1 &= 42e^x \\ \left(e^x\right)^2 - 42e^x - 1 &= 0\end{aligned}$$

Let $u=e^x$.

$u^2 - 42u - 1 = 0$

$u = \frac{-(-42)\pm\sqrt{(-42)^2-4\cdot1\cdot(-1)}}{2\cdot1} = \frac{42\pm\sqrt{1768}}{2} \approx 42.023796, -0.0237960$

Note that the negative result is extraneous because ex must be greater than zero, so you only proceed in solving for x for one result.

$$\begin{aligned}e^x &\approx 42.023796 \\ x &\approx \ln 42.023796 \approx 3.738\end{aligned}$$

Example 5

Solve the following equation for all possible values of x: $(\log_2 x)^2 - \log 2x^7 = -12$.

Solution

In calculus it is common to use a small substitution to simplify the problem and then substitute back later. In this case let $u=\log_2 x$. Notice that this is a quadratic problem.

$$\begin{aligned}(\log_2 x)^2 - 7\log_2 x + 12 &= 0 \\ u^2 - 7u + 12 &= 0 \\ (u-3)(u-4) &= 0 \\ u &= 3, 4\end{aligned}$$

Now substitute back and solve for x in each case.

$$\log_2 x = 3 \leftrightarrow 2^3 = x = 8$$
$$\log_2 x = 4 \leftrightarrow 2^4 = x = 16$$

Review

Solve each equation for x. Round each answer to three decimal places.

1. $4^x=6$
2. $5^x=2$
3. $12^{4x}=1020$
4. $7^{3x}=2400$
5. $2^{x+1}-5=22$
6. $5x+12^x=5x+7$
7. $2^{x+1}=2^{2x+3}$
8. $3^{x+3}=9^{x+1}$
9. $2^{x+4}=5^x$
10. $13 \cdot 8^{0.2x}=546$
11. $b^x=c+a$
12. $32^x=0.94-.12$

Solve each log equation by using log properties and rewriting as an exponential equation.

13. $\log_3 x+\log_3 5=2$
14. $2\log x=\log 8+\log 5-\log 10$
15. $\log_9 x= \frac{3}{2}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.6.

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SECTION OVERVIEW

3.3: Properties of Logarithms

Topic hierarchy

3.3.1: Product and Quotient Properties of Logarithms

3.3.2: Power Property of Logarithms

3.3.3: Inverse Properties of Logarithms

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3.3.1: Product and Quotient Properties of Logarithms

Product and Quotient Properties of Logarithms

Your friend Robbie works as a server at a pizza parlor. You and two of your friends go to the restaurant and order a pizza. You ask Robbie to bring you separate checks so you can split the cost of the pizza. Instead of bringing you three checks, Robbie brings you one with the total $\log_3 162 - \log_3 2$. "This is how much each of you owes," he says as he drops the bill on the table. How much do each of you owe?

Product and Quotient Properties of Logarithms

Just like exponents, logarithms have special properties, or shortcuts, that can be applied when simplifying expressions. In this lesson, we will address two of these properties.

Let's simplify $\log_b x + \log_b y$.

First, notice that these logs have the same base. If they do not, then the properties do not apply.

$\log_b x = m$ and $\log_b y = n$, then $b^m = x$ and $b^n = y$.

Now, multiply the latter two equations together.

$$b^m \cdot b^n = xy$$
$$b^{m+n} = xy$$

Recall, that when two exponents with the same base are multiplied, we can add the exponents. Now, reapply the logarithm to this equation.

$$b^{m+n} = xy \rightarrow \log_b xy = m + n$$

Recall that $m = \log_b x$ and $n = \log_b y$, therefore $\log_b xy = \log_b x + \log_b y$.

This is the **Product Property of Logarithms**.

Now, let's expand $\log_{12} 4y$.

Applying the Product Property from the previous problem, we have:

$\log_{12} 4y = \log_{12} 4 + \log_{12} y$

Finally, let's simplify $\log_3 15 - \log_3 5$.

As you might expect, the **Quotient Property of Logarithms** is $\log_b \frac{x}{y} = \log_b x - \log_b y$ (proof in the Review section). Therefore, the answer is:

$$\begin{aligned} \log_3 15 - \log_3 5 &= \log_3 \frac{15}{5} \\ &= \log_3 3 \\ &= 1 \end{aligned}$$

Examples

Example 1

Earlier, you were asked to find the amount that each of you owes.

Solution

If you rewrite $\log_3 162 - \log_3 2$ as $\log_3 \frac{162}{2}$, you get $\log_3 81$.

$3^4 = 81$ so you each owe \$4.

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Example 2

Simplify the following expression: $\log_7 8 + \log_7 x^2 + \log_7 3y$.

Solution

Combine all the logs together using the Product Property.

$\log_7 8 + \log_7 x^2 + \log_7 3y = \log_7 8x^2 3y$

$= \log_7 24x^2 y$

Example 3

Simplify the following expression: $\log y - \log 20 + \log 8x$.

Solution

Use both the Product and Quotient Property to condense.

$$\log y - \log 20 + \log 8x = \log \frac{y}{20} \cdot 8x$$
$$= \log \frac{2xy}{5}$$

Example 4

Simplify the following expression: $\log_2 32 - \log_2 z$.

Solution

Be careful; you do not have to use either rule here, just the definition of a logarithm.

$\log_2 32 - \log_2 z = 5 - \log_2 z$

Example 5

Simplify the following expression: $\log_8 \frac{16x}{y^2}$.

Solution

When expanding a log, do the division first and then break the numerator apart further.

$$\log_8 \frac{16x}{y^2} = \log_8 16x - \log_8 y^2$$
$$= \log_8 16 + \log_8 x - \log_8 y^2$$
$$= \frac{4}{3} + \log_8 x - \log_8 y^2$$

To determine $\log_8 16$, use the definition and powers of 2:

$8^n = 16 \rightarrow 2^{3n} = 2^4 \rightarrow 3n = 4 \rightarrow n = \frac{4}{3}$

Review

Simplify the following logarithmic expressions.

1. $\log_3 6 + \log_3 y - \log_3 4$
2. $\log 12 - \log x + \log y^2$
3. $\log_6 x^2 - \log_6 x - \log_6 y$
4. $\ln 8 + \ln 6 - \ln 12$
5. $\ln 7 - \ln 14 + \ln 10$
6. $\log_{11} 22 + \log_{11} 5 - \log_{11} 55$

Expand the following logarithmic functions.

7. $\log_6 (5x)$
8. $\log_3 (abc)$
9. $\log\left(\frac{a^2}{b}\right)$
10. $\log_9\left(\frac{xy}{5}\right)$
11. $\log\left(\frac{2x}{y}\right)$
12. $\log\left(\frac{8x^2}{15}\right)$
13. $\log_4\left(\frac{5}{9y}\right)$
14. Write an algebraic proof of the Quotient Property. Start with the expression $\log_a x - \log_a y$ and the equations $\log_a x = m$ and $\log_a y = n$ in your proof. Refer to the proof of the Product Property in the first practice problem as a guide for your proof.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.8.

Vocabulary

Term	Definition
Product Property of Logarithms	The product property of logarithms states that as long as $b \neq 1$, then $\log_b xy = \log_b x + \log_b y$
Quotient Property of Logarithms	The quotient property of logarithms states that as long as $b \neq 1$, then $\log_b \frac{x}{y} = \log_b x - \log_b y$.

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3.3.2: Power Property of Logarithms

Power Property of Logarithms

The hypotenuse of a right triangle has a length of $\log_3 27^8$. How long is the triangle's hypotenuse?

Power Property

The last property of logs is the **Power Property**.

$\log_b x=y$

Using the definition of a log, we have $b^y=x$. Now, raise both sides to the n power.

$$(b^y)^n = x^n$$
$$b^{ny} = x^n$$

Let's convert this back to a log with base b, $\log_b x^n = ny$. Substituting for y, we have $\log_b x^n = n \log_b x$.

Therefore, the Power Property says that if there is an exponent within a logarithm, we can pull it out in front of the logarithm.

Let's use the Power Property to expand the following logarithms.

1. To expand this log, we need to use the Product Property and the Power Property.

$$\log_6 17x^5 = \log_6 17 + \log_6 x^5$$
$$= \log_6 17 + 5 \log_6 x$$

2. $\ln \left(\frac{2x}{y^3}\right)^4$ We will need to use all three properties to expand this problem. Because the expression within the natural log is in parenthesis, start with moving the 4th power to the front of the log.

$$\ln \left(\frac{2x}{y^3}\right)^4 = 4 \ln \frac{2x}{y^3}$$
$$= 4 \left(\ln 2x - \ln y^3\right)$$
$$= 4(\ln 2 + \ln x - 3 \ln y)$$
$$= 4 \ln 2 + 4 \ln x - 12 \ln y$$

Depending on how your teacher would like your answer, you can evaluate 4 ln2 ≈ 2.77, making the final answer 2.77 + 4 lnx − 12 ln y.

Now, let's condense log 9 − 4 log 5 − 4 log x + 2 log 7 + 2 log y.

This is the opposite of the previous two problems. Start with the Power Property.

log 9 − 4 log 5 − 4 log x + 2 log 7 + 2 log y

$\log 9 - \log 5^4 - \log x^4 + \log 7^2 + \log y^2$

Now, start changing things to division and multiplication within one log.

$\log \frac{9 \cdot 7^2 y^2}{5^4 x^4}$

Lastly, combine like terms.

$\log \frac{441 y^2}{625 x^4}$

Examples

Example 1

Earlier, you were asked to find the length of the triangle's hypotenuse.

Solution

We can rewrite $\log_3 27^8$ and $8 \log_3 27$ and solve.

$8 \log_3 27$

$=8 \cdot 3$

$=24$

Therefore, the triangle's hypotenuse is 24 units long.

Example 2

Expand the following expression: $\ln x^3$.

Solution

The only thing to do here is apply the Power Property: 3 lnx.

Example 3

Expand the following expression: $\log_{16} \frac{x^2 y}{32 z^5}$.

Solution

Let's start with using the Quotient Property.

$$\log_{16} \frac{x^2 y}{32 z^5} = \log_{16} x^2 y - \log_{16} 32 z^5$$

Now, apply the Product Property, followed by the Power Property.

$$= \log_{16} x^2 + \log_{16} y - (\log_{16} 32 + \log_{16} z^5)$$
$$= 2\log_{16} x + \log_{16} y - \tfrac{5}{4} - 5\log_{16} z$$

Simplify $\log_{16} 32 \rightarrow 16^n = 32 \rightarrow 2^{4n} = 2^5$ and solve for n. Also, notice that we put parenthesis around the second log once it was expanded to ensure that the z^5 would also be subtracted (because it was in the denominator of the original expression).

Example 4

Expand the following expression: $\log (5c^4)^2$.

Solution

For this problem, you will need to apply the Power Property twice.

$$\begin{aligned} \log (5c^4)^2 &= 2\log 5c^4 \\ &= 2(\log 5 + \log c^4) \\ &= 2(\log 5 + 4\log c) \\ &= 2\log 5 + 8\log c \end{aligned}$$

Important Note: You can write this particular log several different ways. Equivalent logs are: $\log 25 + 8\log c, \log 25 + \log c^8$ and $\log 25c^8$. Because of these properties, there are several different ways to write one logarithm.

Example 5

Condense into one log: $\ln 5 - 7\ln x^4 + 2\ln y$.

Solution

To condense this expression into one log, you will need to use all three properties.

$$\begin{aligned} \ln 5 - 7\ln x^4 + 2\ln y &= \ln 5 - \ln x^{28} + \ln y^2 \\ &= \ln \frac{5y^2}{x^{28}} \end{aligned}$$

Important Note: If the problem was $\ln 5 - (7\ln x^4 + 2\ln y)$, then the answer would have been $\ln \frac{5}{x^{28}y^2}$. But, because there are no parentheses, the y^2 is in the numerator.

Review

Expand the following logarithmic expressions.

1. $\log_7 y^2$
2. $\log_{12} 5z^2$
3. $\log_4 (9x)^3$
4. $\log \left(\frac{3x}{y}\right)^2$
5. $\log_8 \frac{x^3 y^2}{z^4}$
6. $\log_5 \left(\frac{25x^4}{y}\right)^2$
7. $\ln \left(\frac{6x}{y^3}\right)^{-2}$
8. $\ln \left(\frac{e^5 x^{-2}}{y^3}\right)^6$

Condense the following logarithmic expressions.

9. $6\log x$
10. $2\log_6 x + 5\log_6 y$
11. $3(\log x - \log y)$
12. $\frac{1}{2}\log(x+1) - 3\log y$
13. $4\log_2 y + \frac{1}{3}\log_2 x^3$
14. $\frac{1}{5}[10\log_2(x-3) + \log_2 32 - \log_2 y]$
15. $4\left[\frac{1}{2}\log_3 y - \frac{1}{3}\log_3 x - \log_3 z\right]$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.9.

Vocabulary

Term	Definition
Power Property	The power property for logarithms states that as long as $b \neq 1$, then $\log_b x^n = n \log_b x$.

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3.3.3: Inverse Properties of Logarithms

Inverse Properties of Logarithmic Functions

If you continue to study mathematics into college, you may take a course called Differential Equations. There you will learn that the solution to the differential equation y′=y is the general function $y=Ce^x$. What is the inverse of this function?

Inverse Properties of Logarithms

By the definition of a logarithm, it is the inverse of an exponent. Therefore, a logarithmic function is the inverse of an exponential function. Recall what it means to be an inverse of a function. When two inverses are composed, they equal x. Therefore, if $f(x) = b^x$ and $g(x) = \log_b x$, then:

$$f \circ g = b^{\log_b x} = x \text{ and } g \circ f = \log_b b^x = x$$

These are called the **Inverse Properties of Logarithms**.

Let's solve the following problems. We will use the Inverse Properties of Logarithms.

1. Find $10^{\log 56}$.

   Using the first property, we see that the bases cancel each other out. $10^{\log 56} = 56$

   $e^{\ln 6} \cdot e^{\ln 2}$

   Here, e and the natural log cancel out and we are left with 6·2=12.

2. Find $\log_4 16^x$ We will use the second property here. Also, rewrite 16 as 4^2.

   $$\log_4 16^x = \log_4 (4^2)^x = \log_4 4^{2x} = 2x$$

3. Find the inverse of $f(x) = 2e^{x-1}$.

   Change $f(x)$ to y. Then, switch x and y.

   $$y = 2e^{x-1}$$
   $$x = 2e^{y-1}$$

   Now, we need to isolate the exponent and take the logarithm of both sides. First divide by 2.

   $$\frac{x}{2} = e^{y-1}$$
   $$\ln\left(\frac{x}{2}\right) = \ln e^{y-1}$$

   Recall the Inverse Properties of Logarithms from earlier in this concept. $\log_b b^x = x$; applying this to the right side of our equation, we have $\ln e^{y-1} = y - 1$. Solve for y.

   $$\ln\left(\frac{x}{2}\right) = y - 1$$
   $$\ln\left(\frac{x}{2}\right) + 1 = y$$

   Therefore, $\ln\left(\frac{x}{2}\right) + 1$ is the inverse of $2e^{y-1}$.

Examples

Example 1

Earlier, you were asked to find the inverse of $y = Ce^x$.

Solution

Switch *x* and *y* in the function $y = Ce^x$ and then solve for *y*.

$$x = Ce^y$$
$$\frac{x}{C} = e^y$$
$$\ln \frac{x}{C} = \ln (e^y)$$
$$\ln \frac{x}{C} = y$$

Therefore, the inverse of $y = Ce^x$ is $y = \ln \frac{x}{C}$.

Example 2

Simplify $5^{\log_5 6x}$.

Solution

Using the first inverse property, the log and the base cancel out, leaving $6x$ as the answer.

$$5^{\log_5 6x} = 6x$$

Example 3

Simplify $\log_9 81^{x+2}$.

Solution

Using the second inverse property and changing 81 into 9^2 we have:

$$\begin{aligned} \log_9 81^{x+2} &= \log_9 9^{2(x+2)} \\ &= 2(x+2) \\ &= 2x+4 \end{aligned}$$

Example 4

Find the inverse of $f(x) = 4^{x+2} - 5$.

Solution

$$\begin{aligned} f(x) &= 4^{x+2} - 5 \\ y &= 4^{x+2} - 5 \\ x &= 4^{y+2} - 5 \\ x+5 &= 4^{y+2} \\ \log_4(x+5) &= y+2 \\ \log_4(x+5) - 2 &= y \end{aligned}$$

Review

Use the Inverse Properties of Logarithms to simplify the following expressions.

1. $\log_3 27^x$
2. $\log_5 \left(\frac{1}{5}\right)^x$
3. $\log_2 \left(\frac{1}{32}\right)^x$
4. $10^{\log(x+3)}$
5. $\log_6 36^{(x-1)}$
6. $9^{\log_9(3x)}$
7. $e^{\ln(x-7)}$
8. $\log \left(\frac{1}{100}\right)^{3x}$
9. $\ln e^{(5x-3)}$

Find the inverse of each of the following exponential functions.

10. $y = 3e^{x+2}$
11. $f(x) = \frac{1}{5}e^{\frac{x}{7}}$
12. $y = 2 + e^{2x-3}$
13. $f(x) = 7^{\frac{3}{x}+1-5}$
14. $y = 2(6)^{\frac{x-5}{2}}$
15. $f(x) = \frac{1}{3}(8)^{\frac{x}{2}-5}$

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Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.6.

Vocabulary

Term	Definition
Inverse Properties of Logarithms	The inverse properties of logarithms are $\log_b b^x = x$ and $b^{\log_b x} = x, b \neq 1$.

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SECTION OVERVIEW

3.4: Logarithmic Functions

Topic hierarchy

3.4.1: Graphs of Logarithmic Functions

3.4.2: Common and Natural Logarithms

3.4.3: Solving Logarithmic Equations

3.4.4: Change of Base

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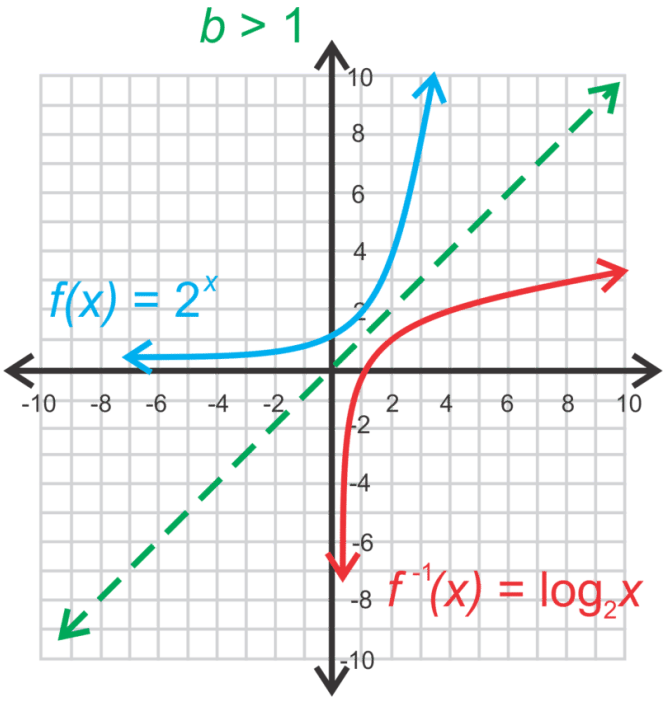
3.4.1: Graphs of Logarithmic Functions

Graphing Logarithmic Functions

Your math homework assignment is to find out which quadrants the graph of the function f(x)=4ln(x+3) falls in. On the way home, your best friend tells you, "This is the easiest homework assignment ever! All logarithmic functions fall in Quadrants I and IV." You're not so sure, so you go home and graph the function as instructed. Your graph falls in Quadrant I as your friend thought, but instead of Quadrant IV, it also falls in Quadrants II and III. Which one of you is correct?

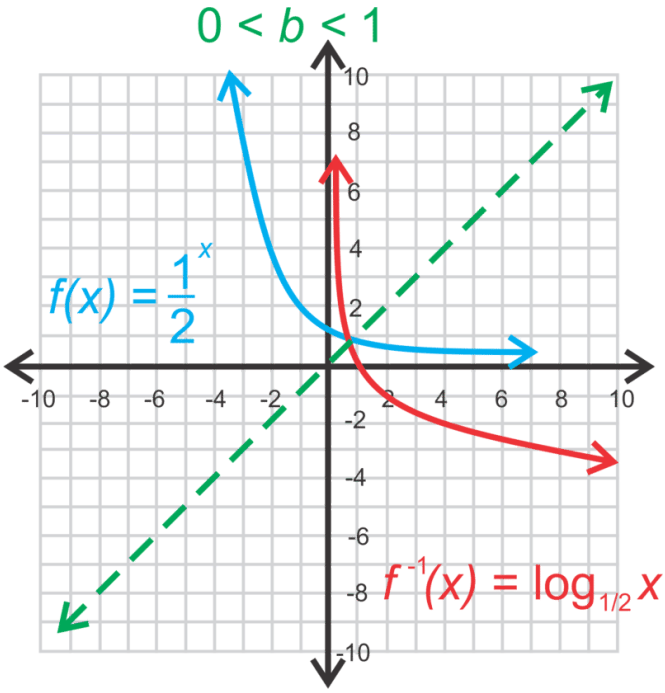
Graphing Logarithmic Functions

Now that we are more comfortable with using these functions as inverses, let's use this idea to graph a logarithmic function. Recall that functions are inverses of each other when they are mirror images over the line y=x. Therefore, if we reflect $y=b^x$ over y=x, then we will get the graph of $y=\log_b x$.



[Figure1]

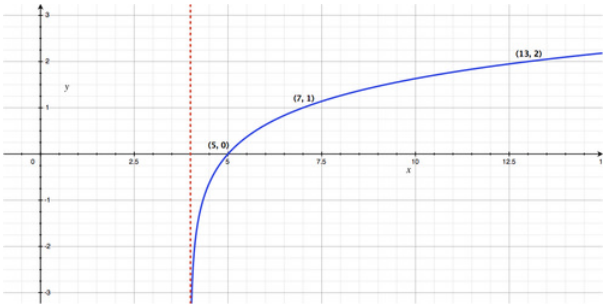
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[Figure2]

Recall that an exponential function has a horizontal asymptote. Because the logarithm is its inverse, it will have a *vertical* asymptote. The general form of a logarithmic function is $f(x)=a\log_b(x-h)+k$ and the vertical asymptote is $x=h$. The domain is $x>h$ and the range is all real numbers. Lastly, if $b>1$, the graph moves *up* to the right. If $0<b<1$, the graph moves *down* to the right.

Let's graph $y=\log_3(x-4)$ and state the domain and range.



To graph a logarithmic function without a calculator, start by drawing the vertical asymptote, at $x=4$. We know the graph is going to have the general shape of the first function above. Plot a few points, such as (5, 0), (7, 1), and (13, 2) and connect.

The domain is $x>4$ and the range is all real numbers.

Now, let's determine if (16, 1) is on $y=\log(x-6)$.

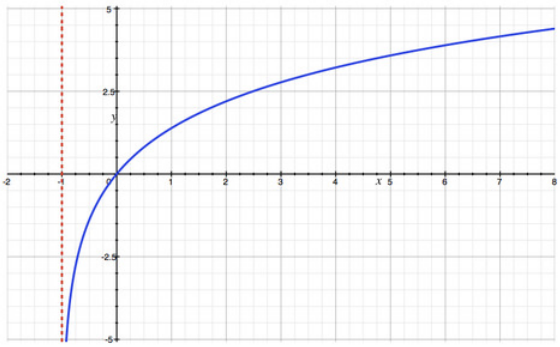
Plug in the point to the equation to see if it holds true.

$$1 = \log(16 - 6)$$
$$1 = \log 10$$
$$1 = 1$$

Yes, this is true, so (16, 1) is on the graph.

Finally, let's graph $f(x)=2\ln(x+1)$.

To graph a natural log, we can use a graphing calculator. Press Y= and enter in the function, $Y=2\ln(x+1)$, **GRAPH**.



Examples

Example 1

Earlier, you were asked to determine if your friend was correct.

Solution

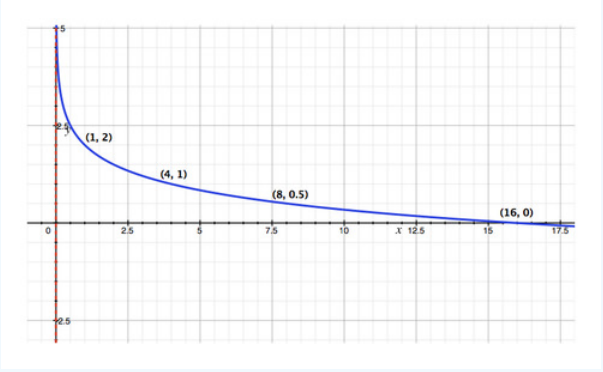
The vertical asymptote of the function f(x)=4ln(x+3) is x=−3. Since x will approach −3 but never quite reach it, x can assume some negative values. Hence, the function will fall in Quadrants II and III. Therefore, you are correct and your friend is wrong.

Example 2

Graph $y = \log_{\frac{1}{4}} x + 2$ in an appropriate window.

Solution

First, there is a vertical asymptote at x=0. Now, determine a few easy points, points where the log is easy to find; such as (1, 2), (4, 1), (8, 0.5), and (16, 0).



To graph a logarithmic function using a TI-83/84, enter the function into Y= and use the Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$. The keystrokes would be: $Y = \frac{\log(x)}{\log(\frac{1}{4})} + 2$, **GRAPH**

To see a table of values, press 2nd → **GRAPH**.

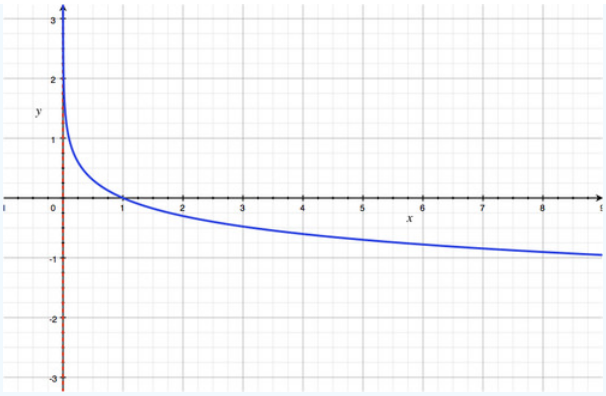
Example 3

Graph y=−logx using a graphing calculator. Find the domain and range.

Solution

The keystrokes are Y=−log(x), **GRAPH**.

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The domain is x>0 and the range is all real numbers.

Example 4

Is (-2, 1) on the graph of $f(x) = \log_{\frac{1}{2}}(x+4)$?

Solution

Plug (-2, 1) into $f(x) = \log_{\frac{1}{2}}(x+4)$ to see if the equation holds true.

$$1 = \log_{\frac{1}{2}}(-2+4)$$
$$1 = \log_{\frac{1}{2}} 2$$
$$1 \neq -1$$

Therefore, (-2, 1) is not on the graph. However, (-2, -1) is.

Review

Graph the following logarithmic functions without using a calculator. State the equation of the asymptote, the domain and the range of each function.

1. $y = \log_5 x$
2. $y = \log_2(x+1)$
3. $y = \log(x) - 4$
4. $y = \log_{\frac{1}{3}}(x-1) + 3$
5. $y = -\log_{\frac{1}{2}}(x+3) - 5$
6. $y = \log_4(2-x) + 2$

Graph the following logarithmic functions using your graphing calculator.

7. $y = \ln(x+6) - 1$
8. $y = -\ln(x-1) + 2$
9. $y = \log(1-x) + 3$
10. $y = \log(x+2) - 4$
11. How would you graph $y = \log_4 x$ on the graphing calculator? Graph the function.
12. Graph $y = \log_{\frac{3}{4}} x$ on the graphing calculator.
13. Is (3, 8) on the graph of $y = \log_3(2x-3) + 7$?
14. Is (9, -2) on the graph of $y = \log_{\frac{1}{4}}(x-5)$?
15. Is (4, 5) on the graph of $y = 5\log_2(8-x)$?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 8.7.

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Vocabulary

Term	Definition
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
operation	Operations are actions performed on variables, constants, or expressions. Common operations are addition, subtraction, multiplication, and division.

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3.4.2: Common and Natural Logarithms

Common and Natural Logarithms

By now, you know that $\log_2 64=x$ can be solved if you recognize that $2^6=64$. What about numbers that aren't so 'clean'? There aren't many people who could calculate the answer to $\log_7 247=x$ in their head! It would be great to use a calculator, but most only have two log functions: base 10 and base ***e***.

Is there a way to convert from one base to another, so we can use a calculator?

Common and Natural Logarithms

Although a log function can have any positive number as a base, there are really only two bases that are commonly used in the real world. Both may be written without a base noted, like: logx, so you may need to use the context to decide which is appropriate.

The **common log** is a log with base 10. It is used to define pH, earthquake magnitude, and sound decibel levels, among many many other common real-world values.

The **natural log**, sometimes written ln(x), is a log with base *e*. The **transcendental number** *e* is approximately 2.71828 and is used in any number of calculations involving constant growth in chemistry, physics, biology, finance, etc..

Using a Calculator for logs

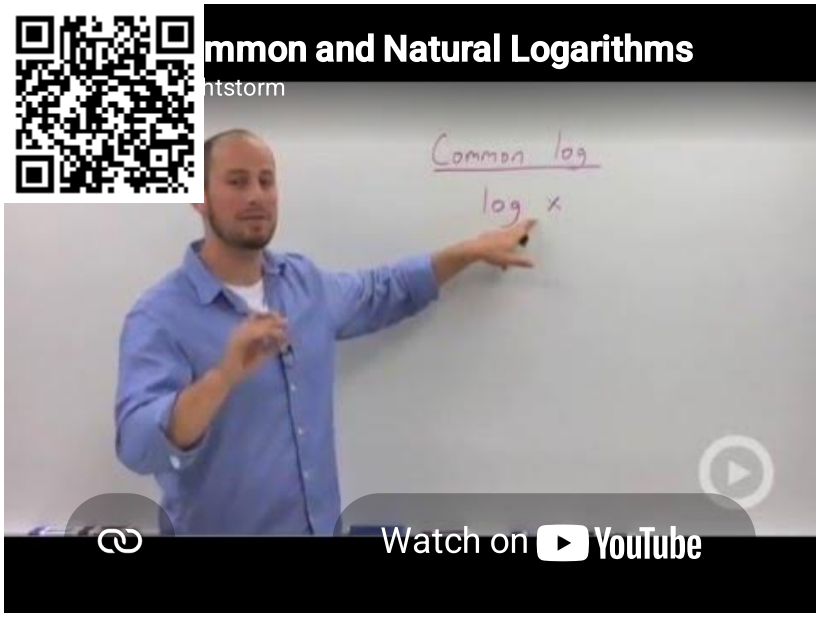
You may have noticed that the common log and the natural log are the only log buttons on your calculator. We can use either the common log or the natural log to find the values of logs with other bases.

The equation $\log_b x = \frac{\log x}{\log b}$ is called the change of base formula, and may be used to convert to **common log** or **natural log**.

You may also see the change of base formula as $\log_b x = \frac{\ln x}{\ln b}$, which is the same formula specifying a conversion to the **natural log**.

Using the change of base formula, we can find the common log (or the natural log) equivalent of any other base so that we can use a calculator to find the value of an expression.

Consider $\log_3 35$. If we use the **change of base formula** to convert to base 10, and then the log button on a calculator, we find that $\log_3 35 = \frac{\log 35}{\log 3} = 3.23621727$.



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Examples

Example 1

Earlier, you were asked to solve the following problem: $\log_7 247 = x$

Solution

Using the change of base formula: $\log_7 247 = \frac{\log 247}{\log 7}$

Using a calculator to find the common logs of 247 and 7, we get (approximately) $\frac{2.4}{.8} = 2.8313$.

We can verify with $7^{2.8313} = 247$

$\therefore \log_7 247 = 2.8313$

Example 2

Evaluate each log.

Remember that logx (with no base specified) commonly refers to $\log_{10}$x.

a. $\log 1$
b. $\log 10$
c. $\log \sqrt{10}$

Solution

a. $\log 1 = 0$ because $10^0 = 1$
b. $\log 10 = 1$ because $10^1 = 10$

c. $\log\sqrt{10} = \frac{1}{2}$ because $\sqrt{10} = 10^{1/2}$

Example 3

For each log value, determine two integers between which the log value should lie. Then use a calculator to find the value of the log.

a. $\log 50$
b. $\log 818$

Solution

a. The value of this log should be between 1 and 2, as $10^1 = 10$, and $10^2 = 100$.

Using a calculator, you should find that log 50 ≈ 1.698970004.

b. The value of this log should be between 2 and 3, as $10^2 = 100$, and $10^3 = 1000$.

Using a calculator, you should find that log 818 ≈ 2.912753304.

Example 4

Estimate the value, and then use the change of base formula to find the value of $\log_2 17$.

Solution

$\log_2 17$ is close to 4 because 2^4=16 and 2^5=32.

Using the change of base formula, we have $\log_2 17 = \frac{\log 17}{\log 2}$.

Using a calculator, you should find that the approximate value of this expression is 4.087462841.

Example 5

Find the value of each natural log.

a. $\ln 100$
b. $\ln\sqrt{e}$

Solution

a. $\ln 100$ is between 4 and 5. You can estimate this by rounding e up to 3, and considering powers of 3:

3^{4}=81 \text { and } 3^{5}=243

Using a calculator, you should find that $\ln 100 = 4.605171086$

b. Recall that a square root is the same as an exponent of 1/2.

Therefore $\ln\sqrt{e} = \ln\left(e^{1/2}\right) = 1/2$

Example 6

Solve the equation: $5^x = 3 \cdot 7^x$

Solution

To solve: $3^x\left(2^{3x}\right) = 7\left(5^x\right)$

$3^x\left(2^3\right)^x = 7\left(5^x\right)$: Rule of exponents $(x^y)^z = x^{yz}$

$3^x\left(8^x\right) = 7\left(5^x\right) \rightarrow 24^x = 7\left(5^x\right)$: By multiplication

$\left(\frac{24}{5}\right)^x = 7$: Divide both sides by 5^x

$\log\left(\frac{24}{5}\right)^x = \log 7$: Take the log of both sides

$x \log\left(\frac{24}{5}\right) = \log 7$: Using $\log x^y = y \log x$

$x = \frac{\log 7}{\log \frac{24}{5}}$: Divide both sides by $\log\left(\frac{24}{5}\right)$

$x = 1.24$: With a calculator

Example 7

Find the value: $\ln 6 + \ln 7$

Solution

Use a calculator to find the values:

$\ln 6 = 1.79175$ and $\ln 7 = 1.94591$

1.79175 + 1.94591 = 3.73766

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Review

1. What is a **common logarithm**? Where are common logs most commonly used?
2. What is a **natural logarithm**? Where are natural logs commonly used?

Evaluate each expression:

3. $\log \frac{17^4}{5}$
4. $\log 7\left(4^3\right)$

Convert to a common logarithm and evaluate:

5. $\log_6 832$

6. $\log_{11} 47$
7. $\log_3 9$

Convert to a natural logarithm and evaluate:

8. $\log_7 91$
9. $\log_5 256$
10. $\log_9 0.712$

Find the values of the natural logarithms:

11. $\ln 56$
12. $\ln 2000$
13. $\ln 950.1$
14. $\ln .9$

Convert the natural logs to exponential form, and solve.

15. If $\ln e = x$ and $e^x = e$ then $x = ?$
16. If $\ln e^5$ then $x = ?$
17. If $\ln e^a = x$ then $x = ?$
18. If $\ln e^{-3} = x$ then $x = ?$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.8.

Vocabulary

Term	Definition
e	e is an irrational number that is approximately equal to 2.71828. As $n \to \infty, \left(1 + \frac{1}{n}\right)^n \to e$.
Change of Base Formula	Let b, x, and y be positive numbers, b≠1 and y≠1. Then, $\log_y x = \frac{\log_b x}{\log_b y}$. More specifically, $\log_y x = \frac{\log x}{\log y}$ and $\log_y x = \frac{\ln x}{\ln y}$, so that expressions can be evaluated using a calculator.
Common Log	A common logarithm is a log with base 10. The log is usually written without the base.
Common Logarithm	A common logarithm is a log with base 10. The log is usually written without the base.
e	e is an irrational number that is approximately equal to 2.71828. As $n \to \infty, \left(1 + \frac{1}{n}\right)^n \to e$.
Natural Log	A natural logarithm is a log with base e. The natural logarithm is written as ln.
Natural Logarithm	A natural logarithm is a log with base e. The natural logarithm is written as ln.
Transcendental Number	A transcendental number is a number that is not the root of any rational polynomial function. Examples include e and π.

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3.4.3: Solving Logarithmic Equations

Solving Logarithmic Equations

This section addresses how to solve different types of logarithmic equations. It explores how to condense expanded equations and how to convert logarithmic equations into exponential ones. In addition, the section demonstrates how to check the solutions to logarithmic equations to be sure they are correct and not extraneous.

Warm-Up

Solutions to equations can be understood graphically as the location where two graphs intersect. Use the interactive below to visualize the solutions to logarithmic equations. Later in this section you will learn how to find these solutions algebraically.

*Interactive

Work it Out 1

Here is a logarithmic equation: $\log_2(x+3)=6$.

1. How many processes can be used to solve this equation?
2. What are the different processes? What is the value of x?
3. Can these processes be applied to solving any logarithmic equation? If not, why might one process be applicable for some logarithmic equations and not others?
4. How do you check to make sure your answer is correct?

Discussion

Recall that a logarithmic equation is the inverse of an exponential equation. For example, if $\log_b(x+y)=z$, then you can apply the inverse property $b^{\log_b x}=x$ to cancel out the log. Therefore, the equation $\log_b(x+y)=z$ can be rewritten as $b^z=x+y$. How can you apply this property to solve the equation in this problem?

This video provides a brief overview of the process for solving logarithmic equations.

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```

Example 1

Solve the following logarithmic equation using two different processes, and check your answer to make sure your solution is not extraneous.

$\log_2(x+5)=9$

Solution

There are two different ways to solve this equation. The first is to use the definition of a logarithm.

$$\begin{aligned}\log_2(x+5)&=9\\ 2^9&=x+5\\ 512&=x+5\\ 507&=x\end{aligned}$$

The second way to solve this equation is to put everything into the exponent of 2, and then use the inverse property.

$$\begin{aligned}2^{\log_2(x+5)}&=2^9\\ x+5&=512\\ x&=507\end{aligned}$$

To check, plug in 507 for x.

$\log_2(507+5)=9 \rightarrow \log_2 512=9$

Work it Out 2

Calista was asked to solve the following logarithmic equation: $\log 5x+\log(x-1)=2$ Her work is shown below.

a. For each step, describe in writing what she is doing and why she is doing it. In other words, narrate the process she is using to solve the equation.

b. What solutions would she obtain by using this process?

Step 1: $\log 5x+\log(x-1)=2$

Step 2: $\log[5x(x-1)]=2$

Step 3: $\log(5x^2-5x)=2$

Step 4: $10^{\log(5x^2-5x)} = 10^2$

Step 5: $5x^2-5x=100$

Step 6: $5x^2-5x-100=0$

Step 7: $x^2-x-20=0$

Step 8: $(x-5)(x+4)=0$

Discussion

In Step 1, Calista wrote the equation which she likely noticed includes several terms. In Step 2 she condenses the equation. Why did she do this? What did she do to condense the equation? In Step 4 she's changed the equation into an exponential one. Why did she feel this was necessary to do? How did she go from Step 3 to Step 4?

CK-12 INTERACTIVE

The following interactive will provide you with additional practice exploring the process for solving logarithmic equations.

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Example 2

Start with the equation from above: $\log 5x + \log(x-1) = 2$. Find the two solutions that Calista would have calculated, and check both answers.

Solution

The two solutions to this problem are $x=5$ and $x=-4$. Now check to confirm these solutions are correct by substituting them back into the original equation.

$$\log 5(5) + \log(5-1) = 2 \qquad \log 5(-4) + \log((-4)-1) = 2$$
$$\log 25 + \log 4 = 2 \qquad \log(-20) + \log(-5) = 2$$
$$\log 100 = 2$$

In the step on the right, log(−20) + log(−5) = 2, both logs have negative values, and you cannot take the log of a negative number. Therefore -4 is considered an **extraneous solution**, or in other words, it is not a solution. 5 is the only solution.

Work it Out 3

Terrance was working to solve this equation: $\log 10x^2 - \log x = 3$, but isn't sure how to get started. What would be the first step he would take, and why? What solution would he eventually find?

Discussion

Recall the subtraction/division property for logs. When subtracting one logarithmic term from another with the same bases, you can pull out the log function and combine the remaining part of each term into a fraction or division problem. Perform this step for Terrance, and then solve the rest of the equation.

CK-12 INTERACTIVE

The following interactive will help you explore the relationship between exponential, radical, and logarithmic equations.

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Work it Out 4

Start with the equation $y=\log_2(x)$

1. Graph the equation.
2. What is the value of x when y=4? Show this on the graph and solve algebraically.
3. What are the **domain** and **range** of this function? What is the **vertical asymptote**?

Discussion

You should use a graphing utility to visualize the curve of this equation. Predict what would look different about a graph as the base got larger, and then check your prediction by graphing additional equations.

Example 3

Given the equation $y=\log_5 x+2$

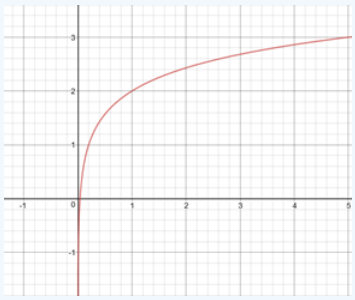
a. Graph the equation.

b. Determine the value of x when y=2.

c. What is the domain and range of the function?

d. Determine the equation for the vertical asymptote.

Solution

a. 

b. When y=2 then x=1.

c. Domain: $(0,\infty)$; Range: $(-\infty,\infty)$.

d. Vertical Asymptote: x=0 .

Review

Use properties of logarithms and a calculator to solve the following equations for x. Round answers to three decimal places and check for extraneous solutions.

1. $\log_2 x = 15$
2. $\log_{12} x = 2.5$
3. $\log_9(x-5) = 2$
4. $\log_7(2x+3) = 3$
5. $4\log_3 3x - \log_3 x = 5$
6. $\log(x+5) + \log x = \log 14$
7. $3\log_3(x-5) = 3$
8. $\frac{2}{3}\log_3 x = 2$
9. $5\log\frac{x}{2} - 3\log\frac{1}{x} = \log 8$
10. $2\log_6 x + 1 = \log_6(5x+4)$
11. $2\log_{\frac{1}{2}} x + 2 = \log_{\frac{1}{2}}(x+3)$
12. $3\log_{\frac{2}{3}} x - \log_{\frac{2}{3}} 27 = \log_{\frac{2}{3}} 8$

Work it Out

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3.4.4: Change of Base

Change of Base

While it is possible to change bases by always going back to exponential form, it is more efficient to find out how to change the base of logarithms in general. Since there are only base e and base 10 logarithms on most calculators, how would you evaluate an expression like $\log_3 12$?

Changing the Base of Logarithms

The **change of base property** states:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

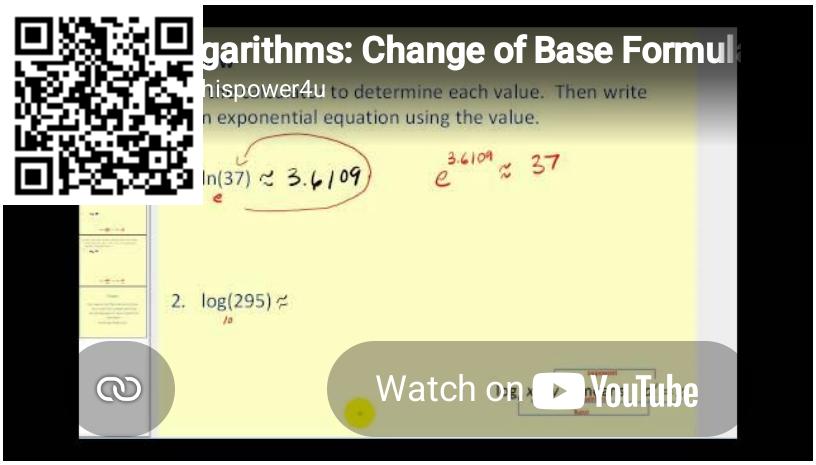
You can derive this formula by converting $\log_b x$ to exponential form and then taking the **log** base x of both sides. This is shown below.

$$\begin{aligned} \log_b x &= y \\ b^y &= x \\ \log_a b^y &= \log_a x \\ y \log_a b &= \log_a x \\ y &= \frac{\log_a x}{\log_a b} \end{aligned}$$

Therefore, $\log_b x = \frac{\log_a x}{\log_a b}$.

If you were to evaluate log34 using your calculator, you may need to use the **change of base formula** since some calculators only have base 10 or base e . The result would be:

$$\log_3 4 = \frac{\log_{10} 4}{\log_{10} 3} = \frac{\ln 4}{\ln 3} \approx 1.262$$



Examples

Example 1

Earlier, you were asked how to use a calculator to evaluate an expression like $\log_3 12$.

Solution

In order to evaluate an expression like $\log_3 12$ you have some options on your calculator:

$$\frac{\ln 12}{\ln 3} = \frac{\log 12}{\log 3} \approx 2.26$$

Some graphing calculators also have another option. Press the **MATH** followed by the **A** buttons and enter $\log_3 12$.

Example 2

Prove the following log identity.

$$\log_a b = \frac{1}{\log_b a}$$

Solution

$$\log_a b = \frac{\log_x b}{\log_x a} = \frac{1}{\frac{\log_x a}{\log_x b}} = \frac{1}{\log_b a}$$

Example 3

Simplify to an exact result: $(\log_4 5) \cdot (\log_3 4) \cdot (\log_5 81) \cdot (\log_5 25)$

Solution

$$\frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 3^4}{\log 5} \cdot \frac{\log 5^2}{\log 5} = \frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{4 \cdot \log 3}{\log 5} \cdot \frac{2 \cdot \log 5}{\log 5} = 4 \cdot 2 = 8$$

Example 4

Evaluate: $\log_2 48 - \log_4 36$

Solution

$$\begin{aligned}
\log_2 48 - \log_4 36 &= \frac{\log 48}{\log 2} - \frac{\log 36}{\log 4} \\
&= \frac{\log 48}{\log 2} - \frac{\log 6^2}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{2 \cdot \log 6}{2 \cdot \log 2} \\
&= \frac{\log 48 - \log 6}{\log 2} \\
&= \frac{\log\left(\frac{48}{6}\right)}{\log 2} \\
&= \frac{\log 8}{\log 2} \\
&= \frac{\log 2^3}{\log 2} \\
&= \frac{3 \cdot \log 2}{\log 2} \\
&= 3
\end{aligned}$$

Example 5

Given $\log_3 5 \approx 1.465$ find $\log_{25} 27$ without using a log button on the calculator.

Solution

$$\log_{25} 27 = \frac{\log 3^3}{\log 5^2} = \frac{3}{2} \cdot \frac{1}{\left(\frac{\log 5}{\log 3}\right)} = \frac{3}{2} \cdot \frac{1}{\log_3 5} \approx \frac{3}{2} \cdot \frac{1}{1.465} = 1.0239$$

Review

Evaluate each expression by changing the base and using your calculator.

1. $\log_6 15$

2. $\log_9 12$

3. $\log_5 25$

Evaluate each expression.

4. $\log_8(\log_4(\log_3 81))$

5. $\log_2 3 \cdot \log_3 4 \cdot \log_6 16 \cdot \log_4 6$

6. $\log 125 \cdot \log_9 4 \cdot \log_4 81 \cdot \log_5 10$

7. $\log_5(5^{\log_5 125})$

8. $\log(\log_6(\log_2 64))$

9. $10^{\log_{100} 9}$

10. $(\log_4 x)(\log_x 16)$

11. $\log_{49} 49^5$

12. $3\log_{24} 24^8$

13. $4^{\log_2 3}$

Prove the following properties of logarithms.

14. $(\log_a b)(\log_b c)=\log_a c$

15. $(\log_a b)(\log_b c)(\log_c d)=\log_a d$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.5.

Vocabulary

Term	Definition
Change of Base Formula	Let b, x, and y be positive numbers, $b \neq 1$ and $y \neq 1$. Then $\log_y x = \frac{\log_b x}{\log_b y}$. More specifically, $\log_y x = \frac{\log x}{\log y}$ and $\log_y x = \frac{\ln x}{\ln y}$, so that expressions can be evaluated using a calculator.
log	"log" is the shorthand term for 'the logarithm of', as in "$\log_b n$" means "the logarithm, base b, of n."
Logarithm	A logarithm is the inverse of an exponential function and is written $\log_b a=x$ such that $b^x=a$.

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SECTION OVERVIEW

3.5: Exponential and Logarithmic Models

Topic hierarchy

3.5.1: Exponential Models

3.5.2: Logarithmic Models

3.5.3: Simple and Compound Interest

3.5.4: The Number e

3.5.5: Logistic Functions

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3.5.1: Exponential Models

Exponential Models

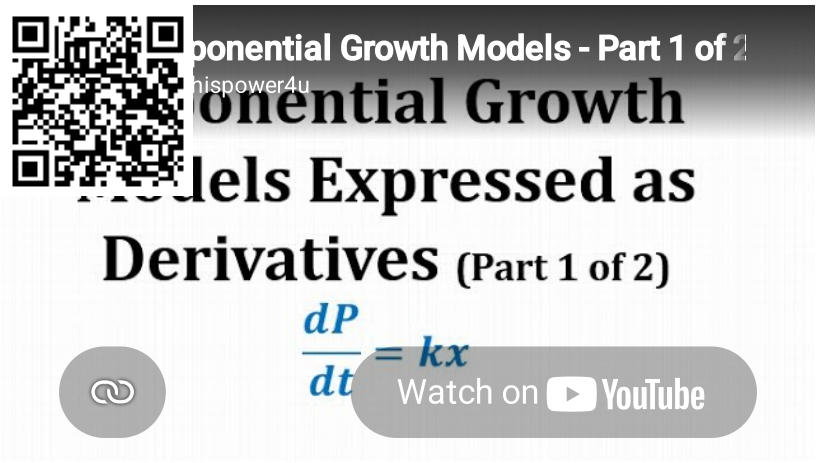
Suppose you were evaluating a particular site as a possible future location for your new car stereo store, "Rock *can* Roll". You know that in order to be successful, the store must be located in a town with a population of at least 100,000. You also know that you have saved up enough to carry you through the first two years of becoming established, so the town can start out a little under the minimum population as long as it will hit 100,000 by year three.

The town you like best has a current population of 89,000, and is growing at a rate of 6% per year. Is it a big enough town for your store to be successful?

Exponential Models

Exponential growth can be a bit surprising, as it can seem to be rather slow at first. At some point though, an exponential function will (sometimes rather suddenly) begin to increase very rapidly.

Population growth can often be modeled with an exponential function (assuming population grows as a percentage of the current population, i.e. 8% per year).



Examples

Example 1

Earlier, you were asked a question about a possible location for the new car stereo store.

Solution

You need to find a town that will have a minimum population of 100,000 by the third year from now. The town you are considering has a population of 89,000, with a yearly growth rate of 6%. Will it work?

Final population is equal to initial population multiplied by the growth rate once each year.

That indicates that final population is: $[(P_i \cdot \text{growth}) \cdot \text{growth}] \cdot \text{growth}\ldots$ etc. Where P_i is initial population.

Using *r* for growth **r**ate, and *x* for years passed, this simplifies to the exponential function:

$P(f)=P_i \cdot r^x$

In our town, the population after *x* years would be: $P(x)=89{,}000 \cdot (1.06)^x$

The beginning of the 3rd year will occur after 2 years have passed, substituting 2 in for *x* gives:

$P(2)=89{,}000 \cdot (1.06)^2$

$\therefore P(2)=100{,}000.4$

The projected population is 100,000 (and 4/10... someone is pregnant!) in the 3rd year, just big enough.

Example 2

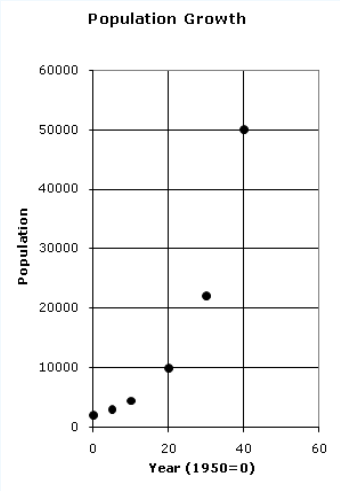
The population of a small town was 2,000 in the year 1950. The population increased over time, as shown by the values in the table below.

What is the number of people added to the population yearly? Why is this question more complex than it seems?

Year (1950 = 0)	Population
0	2000
5	2980
10	4450
20	9900
30	22,000
40	50,000

Solution

If you plot these data points, you will see that the growth pattern is non-linear:



The population does not continue to increase by the same number of people each year, it rather increases by a percentage of the population at the end of each year, an exponential function.

Example 3

Use a graphing calculator to find a function of the form $y = a(b^x)$ that fits the data in the table.

Year (1950 = 0)	Population
0	2000
5	2980
10	4450
20	9900
30	22,000
40	50,000

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Using a TI-83/84 graphing calculator to find an exponential function that best fits a set of data.

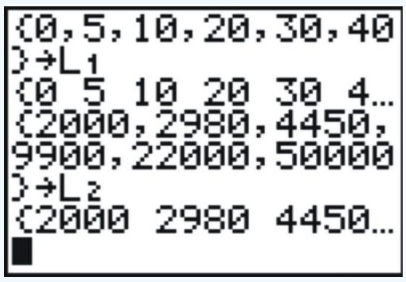
Solution

1. Entering the data

   Data must be entered into "lists". The calculator has six named lists, L1, L2, ... L6. We will enter the *x* values in L1 and the *y* values in L2. One way to do this is shown below:

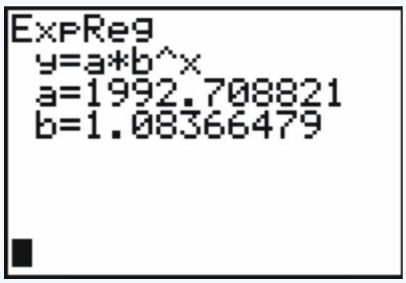
   Press <TI font_2nd> [{] and then enter the numbers separated by commas, and close by pressing the following: <TI font_2nd> [{] <TI font_STO> <TI font_2nd> [L1].

   The top three lines of the figure below show the entry into list L1, followed by the entry of the *y* values into list L2.



   Now press <TI font_STAT>, and move to the right to the CALC menu. Scroll down to option 10, **ExpReg.** Press <TI font_ENTER>, and you will return to the home screen. You should see **ExpReg** on the screen. As long as the numbers are in L1 and L2, the calculator will proceed to find an exponential function to fit the data you listed in List L1 and List L2. You should see on the home screen the values for *a* and *b* in the exponential function (See figure below).

   Therefore the function $y = 1992.7(1.0837)^x$ is an approximate model for the data.



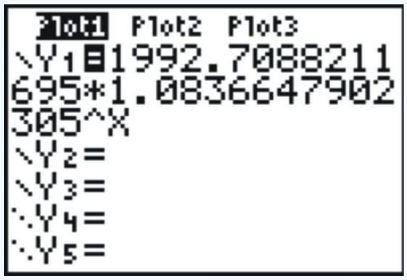
2. Plotting the data and the equation

   To view plots of the data points and the equation on the same screen, do the following.

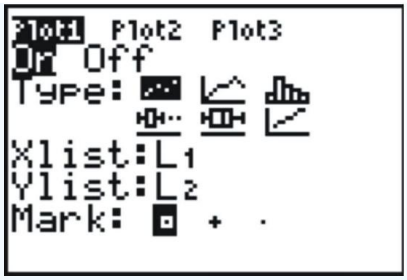
   a. First, press <TI font_Y=> and clear any equations.

      You can type in the equation above, or to get the equation from the calculator, do the following:

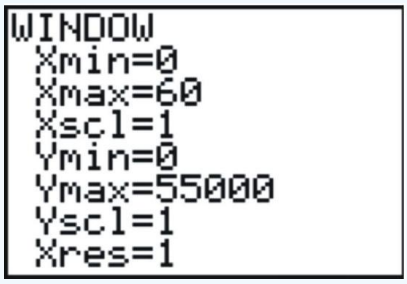
   b. Enter the above rounded-off equation in Y1, or use the following procedure to get the full equation from the calculator: put the cursor in Y1, press <TI font_VARS>, 5, EQ, and 1. This should place the equation in Y1 (see figure below).



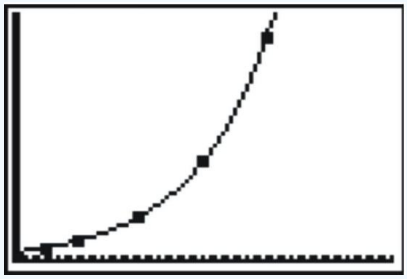
   c. Now press <TI font_2nd>[STAT PLOT] and complete the items as shown in the figure below.



d. Now set your window. (Hint: use the range of the data to choose the window – the figure below shows our choices.)

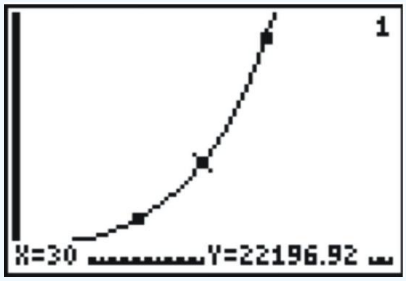


e. Press <TI font GRAPH> and you will to see the function and the data points as shown in the figure below.



3. Comparing the real data with the modeled results

It looks as if the data points lie on the function. However, using the TRACE function you can determine how close the modeled points are to the real data. Press <TI font_TRACE> to enter the TRACE mode. Then press the right arrow to move from one data point to another. Do this until you land on the point with value Y=22000. To see the corresponding modeled value, press the up or down arrow. See the figure below. The modeled value is approximately 22197, which is quite close to the actual data. You can verify any of the other data points using the same method.



Example 4

Tiny Town, CO, currently (year 2012) has a population of 26 people, but it is growing at a rate of 17% per year.

Recall from the lesson that the simplified function for population growth is $P_f=P_i \cdot r^t$ Where "P_f" is final population, "P_i" is initial (starting) population, "r" is growth rate, and "t" is time (in years).

a. What is the *growth factor* for Tiny Town?
b. What will the population be in 2030?

Solution

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a. The growth factor is 1.17, since the population each year is the entire population from the year before: 1·P added to the new population: .17·P.
b. The population in 2030 will be apx 375:

$P_f=26\cdot1.17^{17}$

$P_f=26\cdot14.426$

$P_f=375$

Example 5

Abbi invests $4000 in a savings account with an APR of 6.5% compounded yearly.

Abbi's money can be calculated with the same type of formula as above: $A=P\cdot r^t$ Where "*A*" is final Amount, "'P*" is principal (starting money), "*r*" is growth rate (interest), and "*t*" is time (in years).*

a. How much will she have after 2 years?
b. How much will she have after 15 years?
c. How many years will it take to reach $50,000?

Solution

a. After 2 years, Abbi will have apx $4500

$A=\$4000\cdot1.065^2$

$A=\$4000\cdot1.134$

$A=\$4536$

b. After 15 years, Abbi will have apx $10,300 $A=\$4000\cdot1.065^{15}$

$A=\$4000\cdot2.5718$

$A=\$10287.20$

c. To calculate how long it will take to reach $50,000, we use the formula with A=$50,000 and x (in the **exponent**) is the number of years.

$\$50,000=\$4000\cdot1.065^x$

$12.5=1.065^x$: Divide both sides by $4000

$\log12.5=\log1.065^x$: Take the log of both sides

$\log12.5=x\log1.065$: Using $\log x^y=y\log x$ from prior lesson

$\frac{\log 12.5}{\log 1.065}=x$: Divide both sides by $\log 1.065$

$\frac{1.096}{.0273}=x$: With a calculator

$40.14=x$: With a calculator

It will take just over 40 years for Abbi's initial $4000 to become $50,000 at 6.5% interest compounded yearly.

Example 6

Brandon bought a new car for $30,000. It wasn't until he drove away that his friend Kyle mentioned that the car was going to depreciate at a rate of 50% per year!

The formula for calculating decay is again very similar: $V_f=V_i\cdot r^t$ Where "V_f" is final value, "V_i" is initial value, "*r*" is decay factor (depreciation rate), and "*t*" is time (in years).

a. What is the decay factor of the car?
b. How much will the car be worth in 5 years?
c. Using your calculation from "b", apx how long will it take before the car is only worth $100?

Solution

a. The decay rate is simply 1−.5=.5 since the car's value decays at a rate of 50% per year.
b. In 5 years, the car will be worth apx

$V_f=\$30{,}000\cdot.5^5$

$V_f=\$30{,}000\cdot.03125$

$V_f=\$937.50$ OUCH!

c. Using your calculation from "b", apx how long will it take before the car is only worth $100?

If the car loses 1/2 of its value each year, and it is worth apx $1000 after 5 years:

Year 6 = $1000·.5=$500

Year 7 = $500·.5=$250

Year 8 = $250·.5=$125

It will take only about 8 years before the car is only worth $100. Brandon **may** have made a questionable purchase!

It will take only about 8 years before the car is only worth $100. Brandon **may** have made a questionable purchase!

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```

Review

Calculate the following values using: A=P·rt

Assume all rates are x% per year, compounded yearly unless specified otherwise

1. What is the value of a $5000 investment after 5 years at a rate of 5% ?
2. What is the value of a $15000 investment after 3 years at a rate of 8% ?
3. What is the value of a $3500 investment after 12 years at a rate of 2% ?
4. What is the value of a $7550 investment after 7 years at a rate of 4.3% ?
5. What is the value of a $42,340 investment after 13 years at a rate of 5.034% ?

For questions 6-10, calculate:

a. The growth factor
b. The final population

6. If a population starts at 5,000 people in 1995, and increases at a rate of 7% per year, what is the population in 2032?
7. If a population starts at 15,000 people in 2000, and increases at a rate of 3% per year, what is the population in 2027?
8. If a population starts at 25,500 people in 1900, and increases at a rate of 2% per year, what is the population in 2008?
9. If a population starts at 87,432 people in 1940, and increases at a rate of 4.3% per year, what is the population in 2040?
10. If a population starts at 126,352 people in 1776, and increases at a rate of 1.067% per year, what is the population in 2012?

For questions 11-15, calculate:

1. The decay factor (recall that decay factor = 1 - %decay as a decimal)
2. The final value, using $V_f=V_i \cdot r^t$ from the lesson.

11. A car is worth $4000, and loses value at a rate of 12% per year, what will it be worth in 5 years?
12. A boat is purchased for $14,000, and loses value at a rate of 16% per year, what will it be worth in 7 years?
13. A car is purchased for $40,500, and loses value at a rate of 21% per year, what will it be worth in 4 years?
14. A motorcycle is worth $9350, and loses value at a rate of 6.5% per year, what will it be worth in 3.5 years?
15. A plane is purchased for $342,137, and loses value at a rate of 4.67% per year, what will it be worth in 13 years?

For questions 16-20, calculate the number of years required before the value reaches the specified total, using $A_f=A_i \cdot r^t$ and beginning with A_f= final amount, and x (in the exponent) as the number of years.

16. How many years before a population of 5,000 reaches at least 8,000 at a growth rate of 6%?
17. How many years before a value of $4,000 reaches at least $7,000 at a growth rate of 4%?
18. How many years before a value of $12,000 reaches at least $25,000 at a growth rate of 12%?
19. How many years before a population of 15,500 reaches at least 46,000 at a growth rate of 8.5%?
20. How many years before the value of a car currently worth $52,138 depreciates to at least $8,000 at a depreciation rate of 14.7% ?

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.9.

Vocabulary

Term	Definition
Exponent	Exponents are used to describe the number of times that a term is multiplied by itself.
exponential model	An exponential model is a function reflecting a quantity that grows or decays at a rate proportional to its current value.

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3.5.2: Logarithmic Models

Logarithmic Models

In prior lessons, you used an **exponential model** to predict the population of a town based on a constant growth rate such as 6% per year.

In the real world however, populations often do *not* just grow continuously and without limit. A town originally founded near a convenient water source may grow very quickly at first, but the expansion will slow dramatically as houses and businesses run out of room near the water source, and need to begin transporting water further and further away.

How can a situation like this be modeled with an equation?

Logarithmic Models

In a prior lesson, we considered the solutions of simple log equations. Now we return to that topic and explore some more complex examples. Solving more complicated log equations can be less difficult than you might think, by using our knowledge of log properties.

For example, consider the equation $\log_2 (x) + \log_2 (x - 2) = 3$. We can solve this equation using a log property.

$\log_2 (x) + \log_2 (x - 2) = 3$	
$\log_2 (x(x - 2)) = 3$	$\log_b x + \log_b y = \log_b(xy)$
$\log_2 (x^2 - 2x) = 3 \Rightarrow$	write the equation in exponential form.
$2^3 = x^2 - 2x$	
$x^2 - 2x - 8 = 0$	Solve the resulting quadratic
$(x - 4)(x + 2) = 0$	
$x = -2, 4$	

The resulting quadratic has two solutions. However, only $x = 4$ is a solution to our original equation, as $\log_2(-2)$ is undefined. We refer to $x = -2$ as an **extraneous solution.**

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Examples

Example 1

Earlier, you were asked how a situation could be modeled with an equation.

Solution

A population that increases continuously at a constant rate may be modeled with an exponential function.

A population that increases rapidly and then levels off may be modeled with a logarithmic function.

Example 2

Solve each equation.

a. $\log (x + 2) + \log 3 = 2$
b. $\ln (x + 2) - \ln (x) = 1$

Solution

a.

$\log(3(x+2)) = 2$	$\log_b x + \log_b y = \log_b (xy)$
$\log(3x+6) = 2$	Simplify the expression $3(x+2)$
$10^2 = 3x + 6$	Write the log expression in exponential form
$100 = 3x + 6$	
$3x = 94$	Solve the linear equation
$x = 94/3$	

b.

$\ln\left(\frac{x+2}{x}\right) = 1$	$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$
$e^1 = \frac{x+2}{x}$	Write the log expression in exponential form.
ex=x+2	Multiply both sides by x.
ex−x=2	Factor out x.
x(e−1)=2	Isolate x.
$x = \frac{2}{e-1}$	

The solution above is an *exact* solution. If we want a decimal approximation, we can use a calculator to find that $x \approx 1.16$. We can also use a graphing calculator to find an approximate solution. Consider again the equation $\ln(x + 2) - \ln(x) = 1$. We can solve this equation by solving a system:

$$\begin{cases} y = \ln(x+2) - \ln(x) \\ y = 1 \end{cases}$$

If you graph the system on your graphing calculator, you should see that the curve and the horizontal line intersection at one point. Using the **INTERSECT** function on the **CALC** menu (press <2nd>[CALC]), you should find that the x coordinate of the intersection point is approximately 1.16. This method will allow you to find approximate solutions for more complicated log equations.

Example 3

Use a graphing calculator to solve each equation:

a. $\log(5 - x) + 1 = \log x$
b. $\log_2(3x + 8) + 1 = \log_3(10 - x)$

Solution

a. The graphs of $y = \log(5 - x) + 1$ and $y = \log x$ intersect at $x \approx 4.5454545$.

Therefore the solution of the equation is $x \approx 4.54$.

b. First, in order to graph the equations, you must rewrite them in terms of a **common log** or a **natural log**. The resulting equations are: $y = \frac{\log(3x+8)}{\log 2} + 1$ **and** $y = \frac{\log(10-x)}{\log 3}$ The graphs of these functions intersect at $x \approx -1.87$. This value is the approximate solution to the equation.

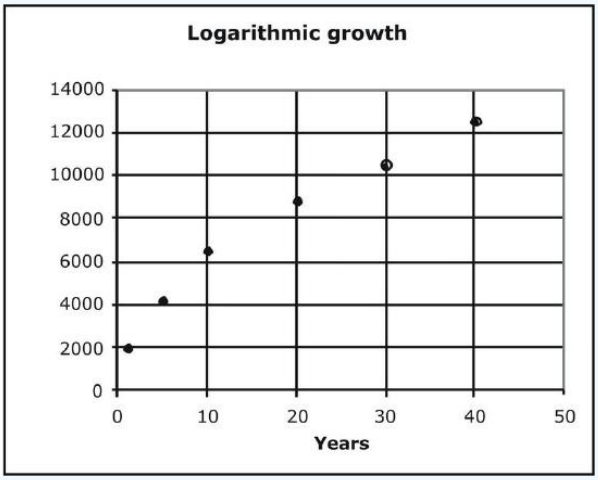
Example 4

Consider population growth:

Year	Population
1	2000
5	4200
10	6500
20	8800
30	10500
40	12500

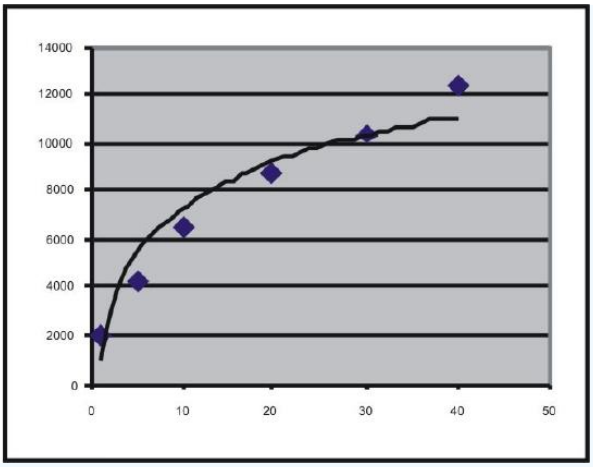
Solution

If we plot this data, we see that the growth is not quite linear, and it is not exponential either.



We can find a logarithmic function to model this data. First enter the data in the table in L1 and L2. Then press STAT to get to the CALC menu. This time choose option 9. You should get the function $y = 930.4954615 + 2780.218173 \ln x$. If you view the graph and the data points together, as described in the Technology Note above, you will see that the graph of the function does not touch the data points, but models the general trend of the data.

Note about technology: you can also do this using an Excel spreadsheet. Enter the data in a worksheet, and create a scatterplot by inserting a chart. After you create the chart, from the chart menu, choose “add trendline.” You will then be able to choose the type of function. Note that if you want to use a logarithmic function, the domain of your data set must be positive numbers. The chart menu will actually not allow you to choose a logarithmic trendline if your data include zero or negative x values. See below:



Example 5)

Solve for x : log2x−log2(x−4)=12.

Solution

To solve $\log_2 x - \log_2(x-4) = 12$:

$\log_2 \frac{x}{x-4} = 12$: Using $\log_x y - \log_x z = \log_x \frac{y}{z}$

$2^{12} = \frac{x}{x-4}$: Write in exponential form

$4,096 = \frac{x}{x-4}$: With a calculator

4,096x−16384=x : Multiply both sides by x−4

4,095x=16,384 : Simplify

x=4 : Divide

Example 6

Biologists use the formula n=k·logA to estimate the number of species n that live in a given area A by multiplying by a constant k which changes by location. If a particular rain forest has a constant k of 943 how many species would be estimated to live in an area of 950km^2?

Solution

To find the number of species in an area of 950km^2:

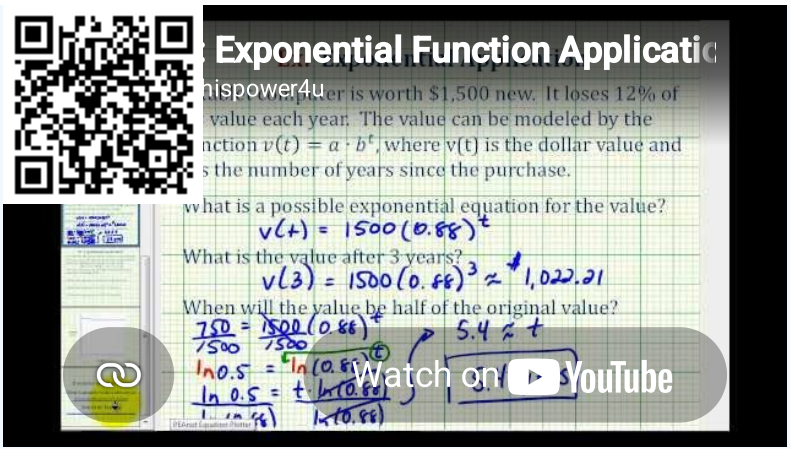
n=943·log950 : Substitute the given k and A values

n=943·2.977 : With a calculator

n=2,807

Therefore 2,807 species would likely live in the area.

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Review

Express 1-7 in exponential form:

1. $\log_{12} \frac{1}{1728} = -3$
2. $\log_{216} 6 = \frac{1}{3}$
3. $\log_{\frac{1}{3}} \frac{1}{9} = 3$
4. $\log_{\frac{1}{4}} \frac{1}{16} = 2$
5. $\log_5 125 = 3$
6. $\log_{15} 225 = 2$
7. $\log_{25} 5 = \frac{1}{2}$

For questions 8-13, solve for x.

8. $\log_x 64 = 2$
9. $\log_3 6561 = x$
10. $\log_5 x = 4$
11. $\log_x 27 = 3$
12. $\log_2 x = 6$
13. $\log_4 64 = x$

For questions 14-19, solve for x.

14. $4\log\left(\frac{x}{5}\right) + \log\left(\frac{625}{4}\right) = 2\log x$
15. $\log_5 z + \frac{\log_5 125}{\log_5 x} = \frac{7}{2}$
16. $\log p = \frac{2 - \log p}{\log p}$
17. $2\log x - 2\log(x+1) = 0$
18. $\log(25 - z^3) - 3\log(4 - z) = 0$
19. $\frac{\log(35 - y^3)}{\log(5 - y)} = 3$

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.10.

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Vocabulary

Term	Definition
Common Log	A common logarithm is a log with base 10. The log is usually written without the base.
Common Logarithm	A common logarithm is a log with base 10. The log is usually written without the base.
exponential model	An exponential model is a function reflecting a quantity that grows or decays at a rate proportional to its current value.
Extraneous Solution	An extraneous solution is a solution of a simplified version of an original equation that, when checked in the original equation, is not actually a solution.
Logarithmic functions	Logarithmic functions are the inverses of exponential functions. Recall that $\log_b n=a$ is equivalent to $b^a=n$.
Natural Log	A natural logarithm is a log with base e. The natural logarithm is written as ln.
Natural Logarithm	A natural logarithm is a log with base e. The natural logarithm is written as ln.

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3.5.3: Simple and Compound Interest

Simple and Compound Interest

Suppose you are re-negotiating an allowance with your parents. Currently you are given $25 per week, but it is the first of June, and you have started mowing the lawn and taking out the trash every week, and you think your allowance should be increased.

Your father considers the situation and makes you the following offer:

"I tell you what, son. I will give you three options for your allowance, you tell me which you would like"

"Option A: You keep the $25 per week"

"Option B: You take $15 this week, then $16 next week, and so on. I'll continue adding $1 per week until New Year's."

"Option C: I'll give you 1 penny this week, and then double your allowance each week until the first of October, then keep it at that **rate**."

Which option would you choose?

Simple and Compound Interest

Simple interest is interest which accrues based only on the **principal** of an investment or loan. The simple interest is calculated as a percent of the principal.

Simple Interest: $i=p\cdot r\cdot t$.

Variable *i* is interest, *p* represents the principal amount, *r* represents the interest rate, and *t* represents the amount of time the interest has been accruing. For example, say you borrow $2,000 from a family member, and you insist on repaying with interest. You agree to pay 5% interest, and to pay the money back in 3 years.

The interest you will owe will be $2000(0.05)(3) = \$300$. This means that when you repay your loan, you will pay $2300. Note that the interest you pay after 3 years is not 5% of the original loan, but 15%, as you paid 5% of $2000 each year for 3 years.

Now let's consider an example in which interest is compounded. Say that you invest $2000 in a bank account, and it earns 5% interest annually. How much is in the account after 3 years?

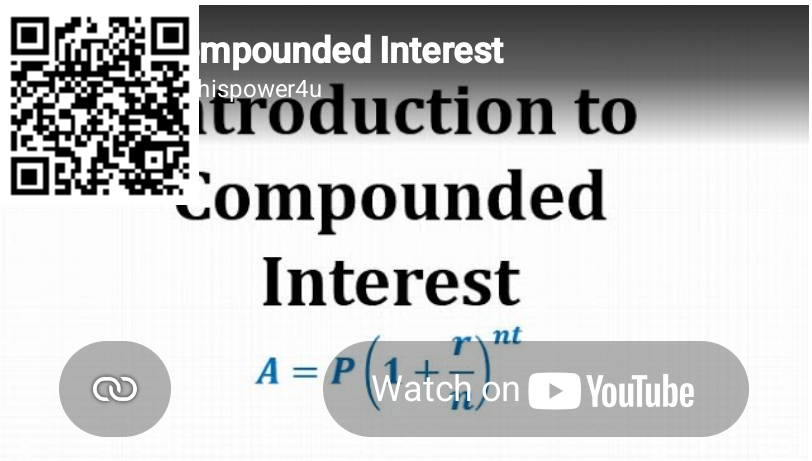
Compound interest: $A(t)=p\cdot(1+r)^t$

Here, $A(t)$ is the *A*mount in the account after a given *t*ime in years, *p*rincipal is the initial investment, and *r*ate is the interest rate. Note that we use (1+r) instead of just r, so we can find the entire amount in the account, not just the interest paid.

$A(t)=2000\cdot(1.05)^3$

After three years, you will have $2315.25 in the account, which means that you will have earned $315.25 in interest.

Compounding results in more interest because the principal on which the interest is calculated increased each year. Another way to look at it is that compounding creates more interest because you are earning interest on interest, and not just on the principal.



Examples

Example 1

Earlier, you were asked which allowance option you would choose.

Solution

Assume you want to make the most money possible by the end of the year. Assume also that there are 24 weeks left.

Option A = 25·24=\$600 *total*

Option B = 15+16+17...+39=\$609 *total*

Option C (assuming 16 weeks until Oct.) = $1\cdot(2^{16})$=\$655.36 *each week* after Oct 1.

It is entirely possible that dear old dad didn't take exponential growth seriously enough, he may need a second job!

Example 2

Use the formula for compound interest to determine the amount of money in an investment after 20 years, if you invest \$2000, and the interest rate is 5% compounded annually.

Solution

The investment will be worth \$5306.60.

$A(t) = P(1 + r)^t$

$A(20) = 2000(1.05)^{20}$

A(20) = \$5306.60

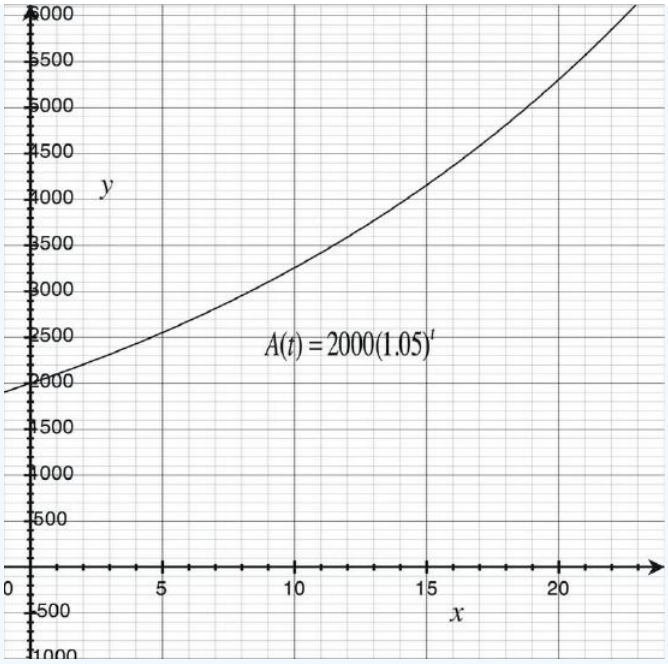
Example 3

How long will it take for \$2000, invested at 5% compounded annually, to reach \$7,000?

Solution

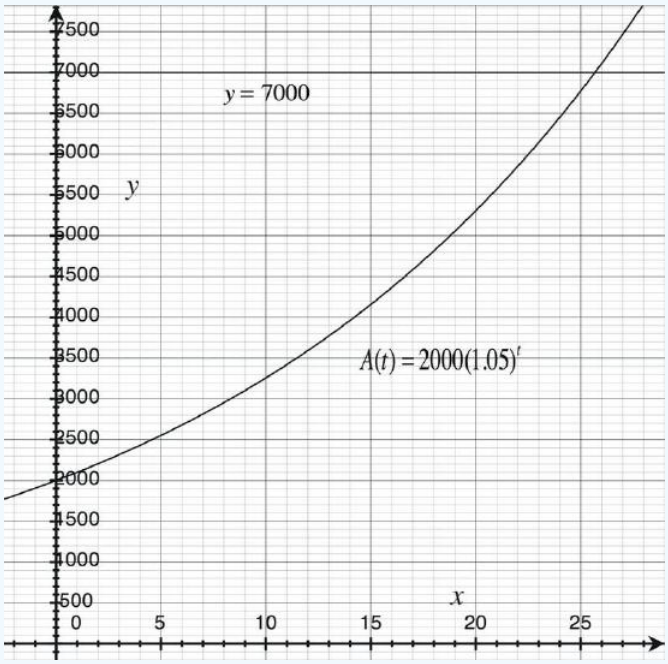
If we graph the function $A(t) = 2000(1.05)^t$, we can see the values for any number of years.

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If you graph this function using a graphing calculator, you can determine the value of the investment by tracing along the function, or by pressing <TRACE> on your graphing calculator and then entering an x value.

You can also choose an investment value you would like to reach, and then determine the number of years it would take to reach that amount. Find the intersection of the exponential function with the line $y = 7000$.



[Figure1]

You can see here that the line and the curve intersect at a little less than $x = 26$. Therefore it would take almost 26 years for the investment to reach $7000.

Example 4

What is the value of an investment after 20 years, if you invest $2000, and the interest rate is 5% compounded ***continuously***?

Solution

The more often interest is compounded, the more it increases, but there is a limit. Each time you increase the number of compoundings, you decrease the fraction of the annual interest that is applied to each compounding. Eventually, the differences become so small as to be negligible. This is known as **continuous compounding**.

The function $A(t) = Pe^{rt}$ is the formula we use to calculate the amount of money when interest is continuously compounded, rather than interest that is compounded at discrete intervals, such as monthly or quarterly.

$A(t) = Pe^{\text{rt}}$

$A(20) = 2000e^{.05(20)}$

$A(20) = 2000e^{1}$

$A(20) = \$5436.56$

Example 5

Compare the values of the investments shown in the table. If everything else is held constant, how does the compounding influence the value of the investment?

	Principal	r	n	t
a.	$4,000	.05	1 (annual)	8
b.	$4,000	.05	4 (quarterly)	8
c.	$4,000	.05	12 (monthly)	8
d.	$4,000	.05	365 (daily)	8
e.	$4,000	.05	8760 (hourly)	8

Solution

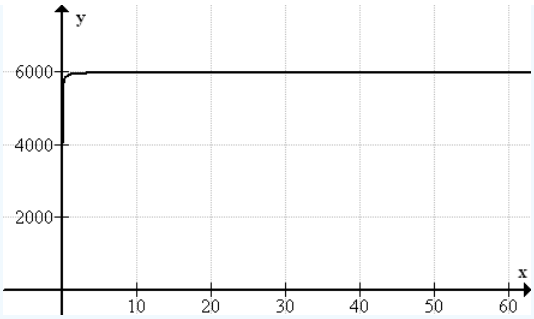
Use the compound interest formula. For this example, the n is the quantity that changes:

$$A(8) = 4000\left(1 + \frac{.05}{n}\right)^{8n}$$

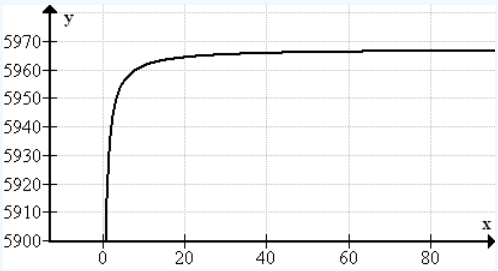
	Principal	r	n	t	A
a.	$4,000	.05	1 (annual)	8	$5909.82
b.	$4,000	.05	4 (quarterly)	8	$5952.52
c.	$4,000	.05	12 (monthly)	8	$5962.34
d.	$4,000	.05	365 (daily)	8	$5967.14
e.	$4,000	.05	8760 (hourly)	8	$5967.29

A graph of the function $f(x) = 4000\left(1 + \frac{.05}{x}\right)^{8x}$ is shown below:

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The graph seems to indicate that the function has a horizontal asymptote at $6000. However, if we zoom in, we can see that the horizontal asymptote is closer to 5967.



[Figure2]

What does this mean? This means that for the investment of $4000, at 5% interest, for 8 years, compounding more and more frequently will never result in more than about $5968.00.

Example 6

Determine the value of each investment.

a. You invest $5000 in an account that gives 6% interest, compounded monthly. How much money do you have after 10 years?

b. You invest $10,000 in an account that gives 2.5% interest, compounded quarterly. How much money do you have after 10 years?

Solution

a. $5000, invested for 10 years at 6% interest, compounded monthly.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(10) = 5000\left(1 + \frac{.06}{12}\right)^{12 \cdot 10}$$

$$A(10) = 5000(1.005)^{120}$$

A(10)=$9096.98

b. $10000, invested for 10 years at 2.5% interest, compounded quarterly.

Quarterly compounding means that interest is compounded four times per year. So in the equation, n = 4.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A(10) = 6000\left(1 + \frac{.025}{4}\right)^{4 \cdot 10}$$

$$A(10) = 6000(1.00625)^{40}$$

$$A(10) = \$12,830.30$$

In each example, the value of the investment after 10 years depends on three quantities: the principal of the investment, the number of compoundings per year, and the interest rate.

Example 7

How long will it take $2000 to grow to $25,000 at a 5% interest rate?

Solution

It will take about 50 years:

$A(t) = Pe^{rt}$	
$25{,}000 = 2000e^{.05(t)}$	
$12.5 = e^{.05(t)}$	Divide both sides by 2000
$\ln 12.5 = \ln e^{.05(t)}$	Take the ln of both sides
$\ln 12.5 = .05t \ln e$	Use the power property of logs
$\ln 12.5 = .05t \times 1$	$\ln e = 1$
$\ln 12.5 = 0.5t$	Isolate t
$t = \frac{\ln 12.5}{.05} \approx 50.5$	



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Review

1. What is the formula for figuring simple interest?
2. What is the formula for figuring compound interest?
3. If someone invested $4500.00, how much would they have earned after 4 years, at a simple interest rate of 2%?
4. Kyle opened up a savings account in July. He deposited $900.00. The bank pays a simple interest rate of 5% annually. What is Kyle's balance at the end of 4 years?
5. After having an account for 6 years, how much money does Roberta have in the account, if her original deposit was $11,000, and her bank's yearly simple interest rate is 8.4%?
6. Tom called his bank today to check on his savings account balance. he was surprised to find a balance of $6600, when he started the account with just $5000.00 8 years ago. Based on this data, what percentage rate has the bank been paying on the account?
7. Julie opened a 4% interest account with a bank that compounds the interest quarterly. If Julie were to deposit $3000.00 into the account at the beginning of the year, how much could she expect to have at the end of the year?
8. Susan has had a saving account for a few years now. The bank has been paying her simple interest at a rate of 5%. She has earned $45.00 on her initial deposit of $300.00. How long has she had the account?
9. What is the balance on a deposit of $818.00 earning 5% interest compounded semiannually for 5 years?
10. Karen made a decent investment. After 4 years she had $3250.00 in her account and expects to have $16,250, after another 4 years. Her savings account is a compounding interest account. How much was her original deposit?
11. What is the yearly simple interest rate that Ken earns, if after only three months he earned $16.00 on an initial $800.00 deposit?

12. Write an expression that correctly represents the balance on an account after 7 years, if the account was compounded yearly at a rate of 5%, with an initial balance of $1000.00
13. Caryl gives each of his three kids $3000.00 each, and they each use it to open up saving accounts at three different banks. Georgia, his oldest, is earning 3% annually at her bank. Kirk earns 7% annually at his bank. Lottie's bank is paying her an annual rate of 4%. At the end of 6 years show much will each of them have in their respective accounts.
14. Kathy receives an inheritance check for $3000.00 and decides to put it in a saving account so she can send her daughter to college when she gets older. After looking she finds an account that pays compounding interest annually at a rate of $14%. The balance on the account can be represented by a function, where x is the time in years. Write a function, and then use it to determine how much will be in the account at the end of 7 years.
15. Stan is late on his car payment. The finance company charges 3% interest per month it is late. His monthly payment is $300.00. What is the total amount he will owe if he pays the August first bill October first? (assuming he was able to make his September bill on - time)

Today, you get your first credit card. It charges 12.49% interest on all purchases and compounds that interest monthly. Within one day you max out the credit limit of $1,200.00.

16. If you pay the monthly accrued interest plus $50.00 towards the initial $1,200 amount every month, how much will you still owe at the end of the first 12 months?
17. How much will you have paid in total at the end of the year?

You are preparing for retirement. You invest $10,000 for 5 years, in an account that compounds monthly at 12% per year. However, unless this money is in an IRA or other tax-free vehicle, with zero inflation, you also have an annual tax payment of 30% on the earned interest.

18. How much will you have in 5 years?
19. Now take into account that the money loses 3% spending value per year due to inflation, how much is what you have saved really worth at the end of the 5 years?

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.11.

Vocabulary

Term	Definition
Accrue	Accrue means "increase in amount or value over time." If interest accrues on a bank account, you will have more money in your account. If interest accrues on a loan, you will owe more money to your lender.
Compound interest	Compound interest refers to interest earned on the total amount at the time it is compounded, including previously earned interest.
Continuous compounding	Continuous compounding refers to a loan or investment with interest that is compounded constantly, rather than on a specific schedule. It is equivalent to infinitely many but infinitely small compounding periods.
Principal	The principal is the amount of the original loan or original deposit.
Rate	The rate is the percentage at which interest accrues.

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1. [Figure 1]
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2. [Figure 2]
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3.5.4: The Number e

The Transcendental Number e and Natural Log

The Transcendental Number e

This section introduces the transcendental number e *which is a special number that is considered a natural base. In addition to exploring how to simplify and solve equations with* e*, this section also investigates how it can be used to understand compounding interest and real life situations with exponential growth and decay.*

Warm-Up

Compounding interest monthly, annually or over any other discrete interval can easily be modeled with exponential functions. Use the interactive below to explore an exponential model that compounds annually. Later in this section you will see how the number e can be used to model interest rates that are compounded continuously.

Interactive*

Investigation

Work it Out 1

Gianna opens a savings account with $1,000 and it accrues interest *monthly* at a rate of 5%. What is the balance in the account after the first month? After 2 months? After 1 year? After 2 years? Can you write an equation that would allow you to determine the account balance after m months?

Look at the process you've used to calculate the value after the second month. You started with $1,000 which is called the **Principal**, and multiplied it by 1.05 (which is the same as adding 5% of the value of the Principal back to the Principal). The second month you multiplied by 1.05 again, and will continue doing so for as many months as you're trying to calculate. This repeated multiplication suggests the equation will contain an exponent.

Discussion

After 1 month, her bank account will have the original $1,000 plus 5% interest ($50.00), or $1,050.00. After the second month she will have the monthly starting balance ($1,050.00) plus 5% of that ($52.50), or a total of $1,102.50. You can continue this process to calculate her balance after 1 or 2 years. Is there a shortcut? In other words, what equation would allow you to more quickly determine her balance?

Work it Out 2

Start with the equation: $\left(1+\frac{1}{n}\right)^{n}$.

a. Construct a table that shows what happens to the overall value of this expression as n gets larger. Round each decimal to the nearest 4 decimal places.
b. Based on your table, does it seem like the overall value is approaching a certain number? If so, what do you think this number is?
c. Estimate the value of the expression when n=100 and when n=1,000.
d. Explain how you made these estimates.
e. As n approaches infinity, what does the value of the expression approach?

Discussion

The start to a table is below. Fill in the missing values, and continue to add others.

n	$(1+\frac{1}{n})^n$
1	$(1+\frac{1}{1})^1=2$
2	$(1+\frac{1}{2})^2=2.25$
3	$(1+\frac{1}{3})^3=$
10	
25	

What do you notice about the values as n gets larger? How can you use this table to predict the value of the expression when n=100 and n=1,000?

As n approaches positive infinity, $(1+\frac{1}{n})^n$ approaches 2.718281828459.

You can see that the value of this expression will never reach 3, but rather the 2.718... number above. This number has a special name: e.

The Number e

e is the number that $(1+\frac{1}{n})^n$ approaches as $n\to\infty$. e is an irrational number that is approximated with 2.718281828459.

e is also called the **natural number** (or base), or the **Euler number**, named after the Swiss mathematician Leonhard Euler who popularized the use of the letter e for the constant.

Graphing & Simplifying

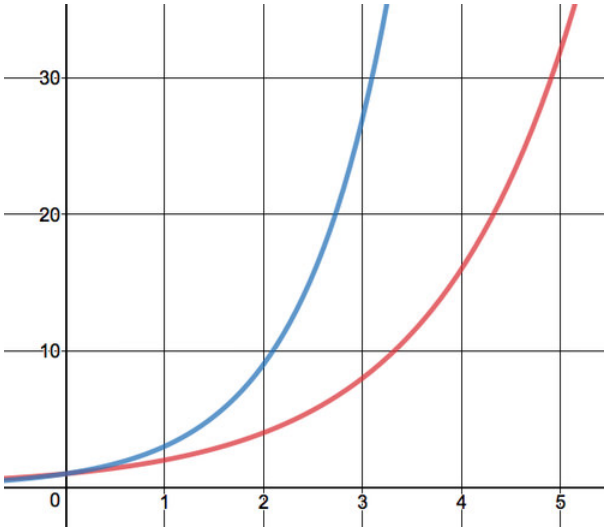
Work it Out 3

Graph the equations $y=2^x$ and $y=3^x$ on the same axis using a graphing utility.

a. Drawing on what you learned above, what would you expect the graph of $y=e^x$ to look like?
b. What is the asymptote?
c. What is the y-intercept?
d. What is the domain and range?

Discussion

The graph of the two given equations is below. Given the value of e stated above, what do you think the graph of $y=e^x$ would look like, and where would it be on this graph?



What do you think the asymptote for $y=e^x$ would be given what is happening to the graph as it approaches the x-axis?

What is the value of any number raised to the 0 power? How will this help you determine the y-intercept?

How can you use this graph to determine the range (the y-values) and the domain (the x-values)?

Example 1

Simplify $e^2 \cdot e^4$.

Solution

The Laws of Exponents apply with e as a base. Because these bases are the same, you can just add the exponents. The answer is e^6.

Using a Calculator with Logs

The value e has another common purpose: finding a log on a calculator.

You may have noticed that most calculators only have two log functions, LOG and LN.

- The LOG function is actually the 'common log', log10x, and it is used in many real-world scenarios such as PH, earthquake magnitude, and sound pressure.
- The LN function is the 'natural log', logex, and is used in many, many calculations involving constant growth in finance and the sciences.

In order to use these functions with other logs, first change the existing base to either 10 or e with the **change of base formula**:

$\log_b x = \frac{\log x}{\log b}$ OR $\log_b x = \frac{\ln x}{\ln b}$

The change of base formula allows any base to be evaluated as base 10 or base e.

Example 2

Use a calculator and the change of base formula to determine the value of $\log_7 247$.

Solution

First, apply the change of base formula to convert to LN or LOG (the process is the same, it just depends on the function you wish to use on the calculator):

$\log_7 247 = \frac{\ln 7}{\ln 247}$

Then use the LN function to find the natural log of 7 and 247:

$\frac{\ln 7}{\ln 247} = \frac{2.4}{0.8} \approx 2.831$

Work it Out 4

Evaluate each log. Remember that $\log x = \log_{10} x$. Use a calculator as needed.

1. $\log 1$
2. $\log 100$
3. $\ln 100$
4. $\log_3 29$
5. $\log_9 0.518$

Solving Natural log Equations

Work it Out 5

Solve the following logarithmic equation and check your answer.

$3\ln(-x) - 5 = 10$

Discussion

First, add 5 to both sides and then divide by 3 to isolate the natural log. Recall that the inverse of the natural log is the natural number. Therefore, everything needs to be put into the exponent of e in order to get rid of the log.

Example 3

Rewrite the following expression under a single log.

$\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$

Solution

Apply the properties of logarithms:

$$\begin{aligned} \ln e - \ln 4x + 2\left(e^{\ln x} \cdot \ln 5\right) &= \ln\left(\frac{e}{4x}\right) + 2x \cdot \ln 5 \\ &= \ln\left(\frac{e}{4x}\right) + \ln\left(5^{2x}\right) \\ &= \ln\left(\frac{e \cdot 5^{2x}}{4x}\right) \end{aligned}$$

Example 4

Solve: $\ln(x-1) - \ln(x+1) = 8$.

Solution

Start by condensing the left-hand side using the **Quotient Rule**. Because this problem involves natural logs, you'll need to put everything in the exponent of e.

$$\begin{aligned} \ln(x-1) - \ln(x+1) &= 8 \\ \ln\left(\frac{x-1}{x+1}\right) &= 8 \\ \frac{x-1}{x+1} &= e^8 \\ x-1 &= (x+1)e^8 \\ x-1 &= xe^8 + e^8 \\ x - xe^8 &= 1 + e^8 \\ x\left(1-e^8\right) &= 1 + e^8 \\ x &= \frac{1+e^8}{1-e^8} \approx -1.0007 \end{aligned}$$

The solution is approximately -1.0007, which you can check by plugging it back into the equation. You will end up with $\ln(-1.0007-1)-\ln(-1.0007+1)=8$, and because you cannot take the log of a negative number, there is no solution for this equation. If you graph the left side and the right side, you can also see that there is no solution.

Continuous Exponential Growth & Decay

Example 5

The interest on a sum of money that compounds continuously can be calculated with the formula $I=Pe^{rt}-P$, where P is the amount invested (the principal), r is the interest rate, and t is the amount of time the money is invested. If you invest $1,000 in a bank account that pays 2.5% interest compounded continuously and you leave the money in that account for 4 years, how much interest will you earn?

Solution

Plug the given values into the equation $I=Pe^{rt}-P$ and solve for I.

$$I = Pe^{rt} - P$$
$$I = 1000 \cdot e^{0.025 \cdot 4} - 1000$$
$$I = 1000 \cdot e^{0.1} - 1000$$
$$I = 1000 \cdot 1.10517 - 1000$$
$$I = 1105.17 - 1000$$
$$I = 105.17$$

Therefore, at the end of 4 years, you will have earned $105.17 in interest.

Work it Out 6

Gianna opens a savings account with $1,000 and it accrues interest continuously at a rate of 5%. What is the balance in the account after 6 years?

Discussion

This example builds off of Active Learning 1. Here, the interest compounds **continuously**, which is different from word problems that involve interest compounding monthly, quarterly, annually, etc. (in other words, at defined intervals). The equation changes slightly, from $A = P\left(1+\frac{r}{n}\right)^{nt}$ to $A = Pe^{rt}$, without n, because there is no longer any interval.

What is the equation for this scenario? How can you use the equation to determine the balance in the account after 6 years?

Work it Out 7

Determine if $y = \frac{1}{2}e^{x}$ represents exponential growth, decay, or neither.

Discussion

Recall that to be exponential growth, the base must be greater than one. To be exponential decay, the base must be between zero and one.

Therefore, is this a growth equation, a decay equation, or neither? Why?

Work it Out 8

The rate of radioactive decay of radium is modeled by $R=Pe^{-0.00043t}$, where R is the amount (in grams) of radium present after t years and P is the initial amount (also in grams). If there is 698.9 grams of radium present after 5,000 years, what was the initial amount?

Discussion

How can you use the formula given in the problem to write an equation? What steps are needed to solve the equation for P?

* Interactive

Review

Determine if the following functions are exponential growth, decay or neither. Give a reason for your answer.

1. $y = \frac{4}{3}e^{x}$
2. $y = \left(\frac{1}{e}\right)^{x} + 2$

Simplify the following expressions with e.

3. $e^{-3} \cdot e^{12}$
4. $\frac{5e^{-4}}{e^{3}}$
5. $\left(\frac{4e^{4}}{3e^{-2}e^{3}}\right)^{-2}$

Solve the following word problems.

The population of Springfield is growing exponentially. The growth can be modeled by the function $P=Ie^{0.055t}$, where P represents the projected population, I represents the current population of 100,000 in 2012 and t represents the number of years after 2012.

6. Graph this equation.
7. To the nearest person, what will the population be in 2022?
8. In what year will the population double in size if this growth rate continues?

The value of Steve's car decreases in value according to the exponential decay function: $V=Pe^{-0.12t}$, where V is the current value of the vehicle, t is the number of years Steve has owned the car and P is the purchase price of the car, $25,000.

9. To the nearest dollar, what will the value of Steve's car be in 2 years?
10. To the nearest dollar, what will the value be in 10 years?

Naya invests $5,000 in an account which accrues interest *monthly* at a rate of 2%.

11. Write an exponential growth function to model the value of her investment after t years.
12. How much total interest does Naya earn in the first eight months to the nearest dollar?
13. How much money, to the nearest dollar, is in the account after 3 years?

Malcolm invests $7,500 in an account which accrues interest *continuously* at a rate of 4.5%.

14. Write an exponential growth function to model the value of his investment after t years.
15. How much interest does Malcolm earn in the first six months to the nearest dollar?
16. How much money, to the nearest dollar, is in the account after 8 years?

Work it Out

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3.5.5: Logistic Functions

Logistic Functions

Exponential growth increases without bound. This is reasonable for some situations; however, for populations there is usually some type of upper bound. This can be caused by limitations on food, space or other scarce resources. The effect of this limiting upper bound is a curve that grows exponentially at first and then slows down and hardly grows at all. This type of growth is called **logistic growth**. What are some other situations in which logistic growth would be an appropriate model?

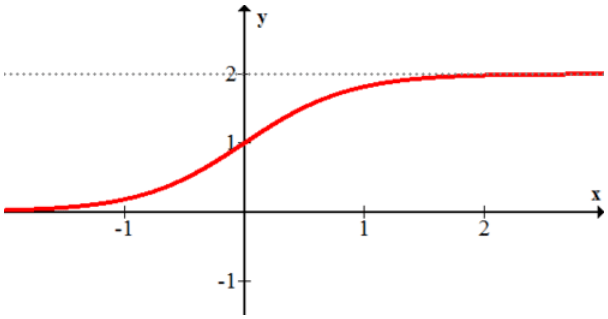
Logistic Functions

Logistic growth can be described with a logistic equation. The **logistic equation** is of the form:

$f(x) = \frac{c}{1+a \cdot b^x}$

The following **logistic function** has a carrying capacity of 2 which can be directly observed from its graph.

$f(x) = \frac{2}{1+0.1^x}$



[Figure1]

An important note about the logistic function is that it has an **inflection point**. From the previous graph you can observe that at the point (0, 1) the graph transitions from curving up (concave up) to curving down (concave down). This change in curvature will be studied more in calculus, but for now it is important to know that the inflection point occurs halfway between the carrying capacity and the x axis.



Examples

Example 1

Earlier, you were asked what situations the **logistic model** is appropriate for.

Solution

The logistic model is appropriate whenever the total count has an upper limit and the initial growth is exponential. Examples are the spread of rumors and disease in a limited population and the growth of bacteria or human population when resources are limited.

Example 2

A rumor is spreading at a school that has a total student population of 1200. Four people know the rumor when it starts and three days later three hundred people know the rumor. About how many people at the school know the rumor by the fourth day?

Solution

In a limited population, the count of people who know a rumor is an example of a situation that can be modeled using the logistic function. The population is 1200 so this will be the carrying capacity.

Identifying information: c=1200; (0,4); (3,300). First, use the point (0, 4) to solve for a.

$$\frac{1200}{1+a\cdot b^0}=4$$
$$\frac{1200}{1+a}=4$$
$$\frac{1200}{4}=1+a$$
$$a=299$$

Next, use the point (3, 300) to solve for b.

$$\frac{1200}{1+299\cdot b^3}=300$$
$$4=1+299b^3$$
$$\frac{3}{299}=b^3$$
$$0.21568\approx b$$

The modeling equation at x=4:

$$f(x)=\frac{1200}{1+299\cdot 0.21568^x}\rightarrow f(4)\approx 729 \text{ people}$$

A similar growth pattern will exist with any kind of infectious disease that spreads quickly and can only infect a person or animal once.

Example 3

A special kind of algae is grown in giant clear plastic tanks and can be harvested to make biofuel. The algae are given plenty of food, water and sunlight to grow rapidly and the only limiting resource is space in the tank. The algae are harvested when 95% of the tank is full leaving the tank 5% full of algae to reproduce and refill the tank. Currently the time between harvests is twenty days and the payoff is 90% harvest. Would you recommend a more optimal harvest schedule?

Solution

Identify known quantities and model the growth of the algae.

Known quantities: (0,0.05);(20,0.95);c=1 or 100%

$$0.05 = \frac{1}{1 + a \cdot b^0}$$
$$1 + a = \frac{1}{0.05}$$
$$a = 19$$
$$0.95 = \frac{1}{1 + 19 \cdot b^{20}}$$
$$1 + 19 \cdot b^{20} = \frac{1}{0.95}$$
$$b^{20} = \frac{\left(\frac{1}{0.95} - 1\right)}{19}$$
$$b \approx 0.74495$$

The model for the algae growth is:

$f(x) = \frac{1}{1+19 \cdot (0.74495)^x}$

The question asks about optimal harvest schedule. Currently the harvest is 90% per 20 day or a unit rate of 4.5% per day. If you shorten the time between harvests where the algae are growing the most efficiently, then potentially this unit rate might be higher. Suppose you leave 15% of the algae in the tank and harvest when it reaches 85%. How much time will that take to yield 70%?

$$0.15 = \frac{1}{1 + 19 \cdot (0.74495)^x}$$
$$x_1 \approx 4.10897$$
$$0.85 = \frac{1}{1 + 19 \cdot (0.74495)^x}$$
$$x_2 \approx 15.8914$$
$$x_2 - x_1 \approx 15.8914 - 4.10897 \approx 11.78$$

It takes about 12 days for the batches to yield 70% harvest which is a unit rate of about 6% per day. This is a significant increase in efficiency. A harvest schedule that maximizes the time where the logistic curve is steepest creates the fastest overall algae growth.

Example 4

Determine the logistic model given c=12 and the points (0, 9) and (1, 11).

Solution

The two points give two equations, and the logistic model has two variables. Use these points to solve for a and b.

$$9 = \frac{12}{1 + a \cdot b^0}$$
$$1 + a = \frac{12}{9}$$
$$a = \frac{1}{3}$$
$$11 = \frac{12}{1 + \left(\frac{1}{3}\right) \cdot b^1}$$
$$1 + \left(\frac{1}{3}\right) \cdot b = \frac{12}{11}$$
$$b = 0.\overline{27} = \frac{3}{11}$$

Thus the approximate model is:

$f(x) = \frac{12}{1+\left(\frac{1}{3}\right) \cdot \left(\frac{3}{11}\right)^x}$

Example 5

Determine the logistic model given c=7 and the points (0, 2) and (3, 5).

Solution

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The two points give two equations, and the logistic model has two variables. Use these two points to solve for a and b.

$$\begin{aligned} 2 &= \frac{7}{1+a} \\ 1+a &= \frac{7}{2} \\ a &= 2.5 \\ 5 &= \frac{7}{1+(2.5)\cdot b^3} \\ 1+(2.5)\cdot b^3 &= \frac{7}{5} \\ b^3 &= 0.16 \\ b &\approx 0.5429 \end{aligned}$$

Thus the approximate model is:

$f(x) = \frac{7}{1+(2.5)\cdot(0.5429)^x}$

Review

For 1-5, determine the logistic model given the carrying capacity and two points.

1. c=12;(0,5);(1,7)

2. c=200;(0,150);(5,180)

3. c=1500;(0,150);(10,1000)

4. c=1000000;(0,100000);(−40,20000)

5. c=30000000;(−60,10000);(0,8000000)

For 6-8, use the logistic function $f(x) = \frac{32}{1+3e^{-x}}$.

6. What is the carrying capacity of the function?

7. What is the y-intercept of the function?

8. Use your answers to 6 and 7 along with at least two points on the graph to make a sketch of the function.

For 9-11, use the logistic function $g(x) = \frac{25}{1+4\cdot 0.2^x}$.

9. What is the carrying capacity of the function?

10. What is the y-intercept of the function?

11. Use your answers to 9 and 10 along with at least two points on the graph to make a sketch of the function.

For 12-14, use the logistic function $h(x) = \frac{4}{1+2\cdot 0.68^x}$.

13. What is the y-intercept of the function?

14. Use your answers to 12 and 13 along with at least two points on the graph to make a sketch of the function.

15. Give an example of a logistic function that is decreasing (models decay). In general, how can you tell from the equation if the logistic function is increasing or decreasing?

Review (Answers)

To see the Review answers, open this PDF file and look for section 3.7.

Vocabulary

Term	Definition
carrying capacity	Carrying capacity is the maximum sustainable population that the environmental factors will support. In other words, it is the population limit.
logistic function	A logistic function is one that grows or decays rapidly for a period of time and then levels out. It takes the form $f(x) = \frac{c}{1+a \cdot b^x}$.
logistic model	A logistic model is used to represent a function that grows or decays rapidly for a period of time and then levels out.

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CHAPTER OVERVIEW

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4.1: Logarithmic Functions

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4.1.1: Graphs of Logarithmic Functions

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When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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4.1.2: Common and Natural Logarithms

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4.1.3: Solving Logarithmic Equations

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4.2: Exponential and Logarithmic Models

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4.2.1: Exponential Models

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4.2.2: Logarithmic Models

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4.2.3: Simple and Compound Interest

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4.2.4: The Number e

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4.2.5: Logistic Functions

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4.3: Exponential and Logarithmic Functions

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4.3.1: Basic Exponential Functions

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When to use this feature?

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4.3.1.1: Solving Exponential Equations

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When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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SECTION OVERVIEW

4.3.2: Properties of Logarithms

Topic hierarchy

4.3.2.1: Product and Quotient Properties of Logarithms

4.3.2.2: Power Property of Logarithms

4.3.2.3: Inverse Properties of Logarithms

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4.3.2.1: Product and Quotient Properties of Logarithms

Feature

This feature does ...

What are the benefits of this feature?

When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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4.3.2.2: Power Property of Logarithms

Feature

This feature does ...

What are the benefits of this feature?

When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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4.3.2.3: Inverse Properties of Logarithms

Feature

This feature does ...

What are the benefits of this feature?

When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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SECTION OVERVIEW

4.3.3: Functions and Inverses

Topic hierarchy

4.3.3.1: One-to-One Functions and Their Inverses

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4.3.3.1: One-to-One Functions and Their Inverses

Feature

This feature does ...

What are the benefits of this feature?

When you use this feature, you gain ...

When to use this feature?

Use this feature to ...

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SECTION OVERVIEW

4.4: Polar Equations

Topic hierarchy

4.4.1: Polar Coordinates

4.4.2: Polar and Cartesian Transformation

4.4.3: Systems of Polar Equations

4.4.4: Polar Equations of Conics

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4.4.1: Polar Coordinates

Polar Coordinates

Everyone has dreamed of flying at one time or another. Not only would there be much less traffic to worry about, but directions would be so much simpler!

Walking or driving: "Go East 2 blocks, turn left, then North 6 blocks. Wait for the train. Turn right, East 3 more blocks, careful of the cow! Turn left, go North 4 more blocks and park."

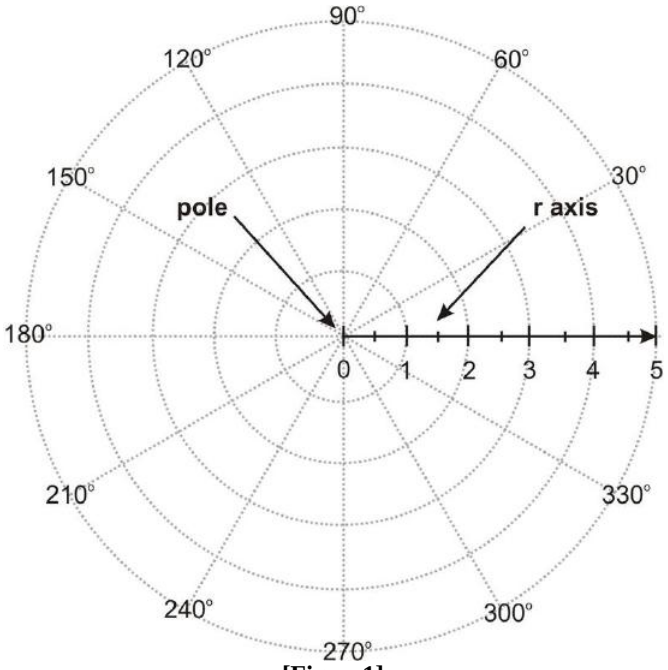
Flying: "Fly 30 deg East of North for a little less than 11 and 1/4 blocks. Land."

Nice daydream, but what does it have to do with polar coordinates?

Polar Coordinates

The **polar coordinate system** is an alternative to the **Cartesian coordinate system** you have used in the past to graph functions. The polar coordinate system is specialized for visualizing and manipulating angles.

Angles are identified by travelling counter-clockwise around the circular graph from the 0 deg line, or *r-axis* (where the + *x* axis would be) to a specified angle.



[Figure1]

To plot a specific point, first go along the *r*-axis by *r* units. Then, rotate counterclockwise by the given angle, commonly represented "θ". Be careful to use the correct units for the angle measure (either radians or degrees).

Radians

Usually polar plots are done with radians (especially if they include trigonometric functions), but sometimes degrees are used.

A **radian** is the angle formed between the *r* axis and a **polar axis** drawn to meet a section of the circumference that is the same length as the radius of a circle.

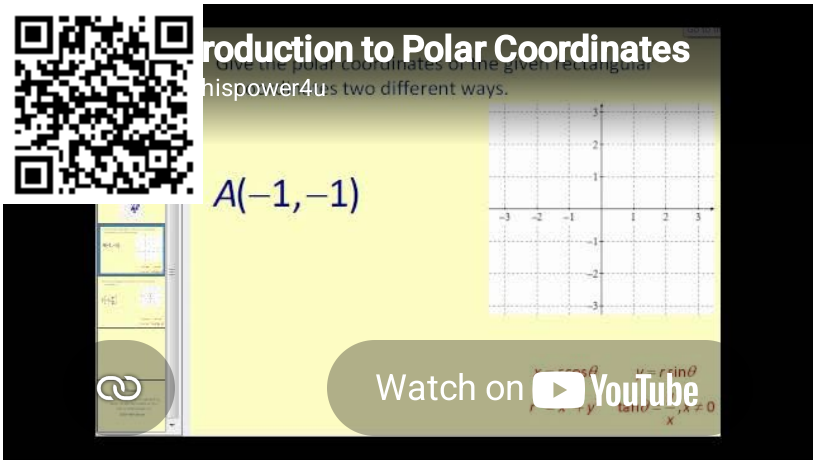
Given that the circumference of a circle is 2π·r, and since *r* is the radius, that means there are 2π radians in a complete circle, and 1π radians in 1/2 of a circle.

If 1/2 of a circle is π radians, and is 180 deg, that means that there are $\frac{180}{\pi}$ degrees in each radian.

That translates to approximately 57.3 degrees = 1 radian.

Graphing Using Technology

Polar equations can be graphed using a graphing calculator: With the graphing calculator- go to **MODE**. There select **RADIAN** for the angle measure and **POL** (for Polar) on the **FUNC** (function)line. When Y = is pressed, note that the equation has changed from y = to r = . There input the polar equation. After pressing graph, if you can't see the full graph, adjust *x*- and *y*- max/min, etc in **WINDOW.**



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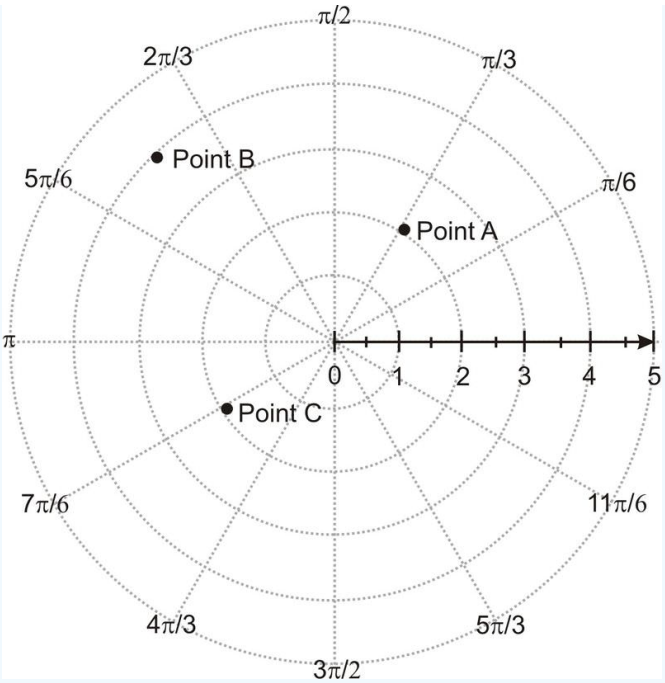
Examples

Example 1

Plot the points on a polar coordinate graph: Point A $\left(2, \frac{\pi}{3}\right)$, Point B (4, 135°), and Point C $\left(-2, \frac{\pi}{6}\right)$

Solution

Below is the **pole**, polar axis and the points A, B and C.



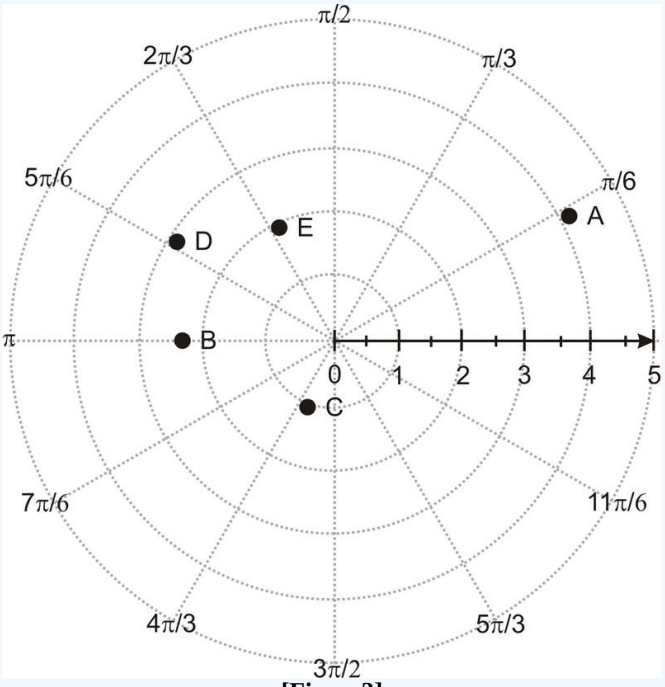
[Figure2]

Example 2

Plot the following points.

a. (4, 30°)
b. (2.5, π)
c. $\left(-1, \frac{\pi}{3}\right)$
d. $\left(3, \frac{5\pi}{6}\right)$
e. (−2, 300°)

Solution



[Figure3]

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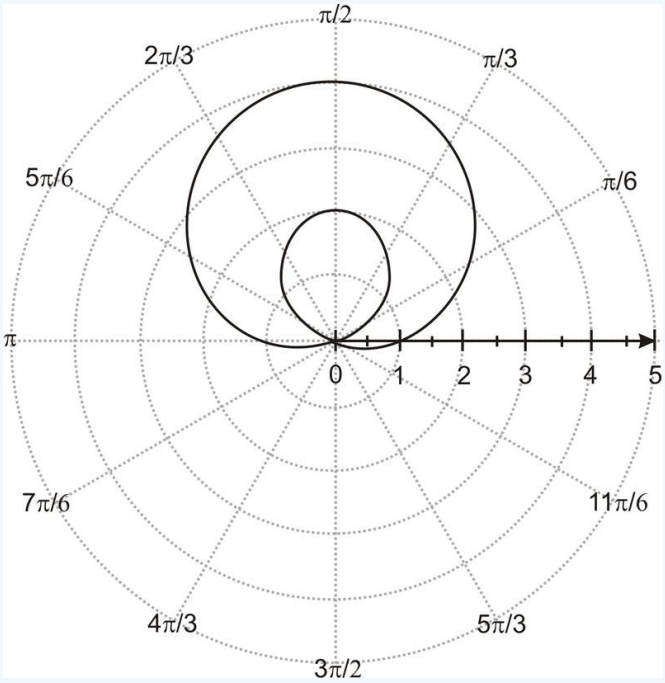
Example 3

Use a graphing calculator or plotting program to plot the following equations.

a. r=1+3sinθ
b. r=1+2cosθ

Solution

a.



b. Review the steps above under **graphing using technology** if you are having trouble.

Example 4

Convert from radians to degrees.

Recall that $\pi rad = 180^\circ$ and $1\text{rad} = \frac{180}{\pi} \approx 57.3^\circ$

a. $\frac{\pi}{2}$
b. 5.17
c. $\frac{3\pi}{2}$

Solution

a. If $\pi rad = 180^\circ$ then $\frac{\pi}{2} rad = 90^\circ$
b. If $1 rad \approx 57.3^\circ$ then $5.17 rad \approx 296^\circ$
c. If $\pi rad = 180^\circ$ then $\frac{3\pi}{2} rad = 270^\circ$

Example 5

Convert from degrees to radians.

Recall that $\frac{180^\circ}{\pi} = 57.3^\circ \approx 1\text{rad}$.

a. 251°
b. 360°
c. 327°

Solution

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a. If $57.3^\circ \approx 1rad$ then $251^\circ \approx 4.38\text{rad} \approx 1.4\pi\text{rad}$
b. If $57.3^\circ \approx 1rad$ then $360^\circ \approx 6.28rad$
c. If $57.3^\circ \approx 1rad$ then $\frac{327^\circ}{57.3^\circ} \approx 5.71rad$

Example 6

Convert from degrees to radians, answer in terms of π.

Recall that 2πrad=360o and therefore πrad=180o.

a. 90o
b. 270o
c. 45o

Solution

a. If $\pi rad = 180^\circ$ then $\frac{\pi}{2}rad = 90^\circ$
b. If $\pi rad = 180^\circ$ and $\frac{\pi}{2}rad = 90^\circ$ then $1\frac{1}{2}\pi rad \rightarrow \frac{3}{2}\pi \rightarrow \frac{3\pi}{2}rad = 270^\circ$
c. If $\frac{\pi}{2}rad = 90^\circ$ then $\frac{\pi}{4}rad = 45^\circ$

Review

1. Why can a point on the plane not be labeled using a unique ordered pair (r, θ).
2. Explain how to graph (r, θ) if r<0 and/or θ>360.

Graph each point on the polar plane.

3. $\text{A}\,(6, 145^\circ)$
4. $\text{B}(-2, \frac{13\pi}{6})$
5. $\text{C}(\frac{7}{4}, -210^\circ)$
6. $\text{D}(5, \frac{\pi}{2})$
7. $\text{E}(3.5, \frac{-\pi}{8})$

Name two other pairs of polar coordinates for each point.

8. $(1.5, 170^\circ)$
9. $(-5, \frac{\pi}{-3})$
10. $(3, 305^\circ)$

Graph each polar equation.

11. $r = 3$
12. $\theta = \frac{\pi}{5}$
13. $r = 15.5$
14. $r = 1.5$
15. $\theta = -175^\circ$

Find the distance between the given points.

16. $P_1\,(5, \frac{\pi}{2})$ and $P_2\,(7, \frac{3\pi}{9})$
17. $P_1\,(1.3, -52^o)$ and $P_2\,(-13.6, -162^\circ)$
18. $P_1\,(3, 250^\circ)$ and $P_2\,(7, 90^\circ)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.1.

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Vocabulary

Term	Definition
π	π (Pi) is the ratio of the circumference of a circle to its diameter. It is an irrational number that is approximately equal to 3.14.
Cartesian coordinate system	The Cartesian plane is a grid formed by a horizontal number line and a vertical number line that cross at the (0, 0) point, called the origin.
polar axis	The polar axis is a ray drawn from the pole at the 0° angle on a polar graph.
polar coordinate system	The polar coordinate system is a special coordinate system in which the location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
pole	The pole is the center point on a polar graph.
radian	A radian is a unit of angle that is equal to the angle created at the center of a circle whose arc is equal in length to the radius.

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4.4.2: Polar and Cartesian Transformation

Polar and Cartesian Transformation

You will see during this lesson that points can be converted from **rectangular form** to **polar form** with a little algebra and trigonometry.

Can the equation of a shape be converted also? How about a circle, for instance?

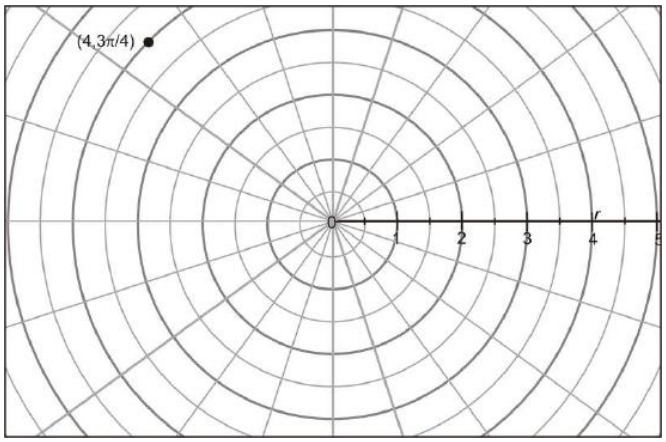
Polar and Cartesian Transformation

Polar Form to Rectangular Form

Sometimes a problem will be given with coordinates in polar form but rectangular form may be needed.

To transform the polar point $(4, \frac{3\pi}{4})$ into **rectangular coordinates**: first identify (r, θ).

$r = 4$ and $\theta = \frac{3\pi}{4}$



Second, draw a vertical line from the point to the polar axis (the horizontal axis). The distance from the pole to where the line you just drew intersects the polar axis is the x value, and the length of the line segment from the point to the polar axis is the y value.

These distances can be calculated using trigonometry:

$x = r \cos \theta$ and $y = r \sin \theta$

$x = 4\cos\frac{3\pi}{4}$ and $y = 4\sin\frac{3\pi}{4}$ or $x = -2\sqrt{2} \quad y = 2\sqrt{2}$

$(4, \frac{3\pi}{4})$ in **polar coordinates** is equivalent to $x = -2\sqrt{2} \quad y = 2\sqrt{2}$

Rectangular Form to Polar Form

Going from rectangular coordinates to polar coordinates is also possible, but it takes a bit more work. Suppose we want to find the polar coordinates of the rectangular point (2, 2). To begin doing this operation, the distance that the point (2, 2) is from the origin (the radius, *r*) can be found by

$r = \sqrt{x^2 + y^2}$
$r = \sqrt{2^2 + 2^2}$
$r = \sqrt{8} = 2\sqrt{2}$

The angle that the line segment between the point and the origin can be found by

$\tan\theta = \frac{y}{x}$
$\tan\theta = \frac{2}{2}$
$\tan\theta = 1$
$\theta = \tan^{-1} 1$
$\theta = \frac{\pi}{4}$

Since this point is in the first **quadrant** (both the x and y coordinate are positive) the angle must be 45° or $\frac{\pi}{4}$ radians. It is also possible that when $\tan\theta = 1$ the angle can be in the third quadrant, or $\frac{5\pi}{4}$ radians. But this angle will not satisfy the conditions of the problem, since a third quadrant angle must have both x and y negative.

Note

when using $\tan\theta = \frac{y}{x}$ to find the measure of θ you should consider, at first, the quotient $\tan\theta = \left|\frac{y}{x}\right|$ and find the first quadrant angle that satisfies this condition. This angle will be called the reference angle, denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle must be given the signs of x and y.

Examples

Example 1

Earlier, you were asked if the equation of a circle could be converted from rectangular form to polar form.

Solution

Equation of a circle: $x^2 + y^2 = k^2$ is the equation of a circle with a radius of k in rectangular coordinates.

The equation of a circle is extremely simple in polar form. In fact, a circle on a polar graph is analogous to a horizontal line on a rectangular graph!

You can transform this equation to polar form by substituting the polar values for x, y. Recall $x = r\cos\theta$ and $y = r\sin\theta$.

$(r\cos\theta)^2 + (r\sin\theta)^2 = k^2$,

square the terms: $r^2\cos^2\theta + r^2\sin^2\theta = k^2$,

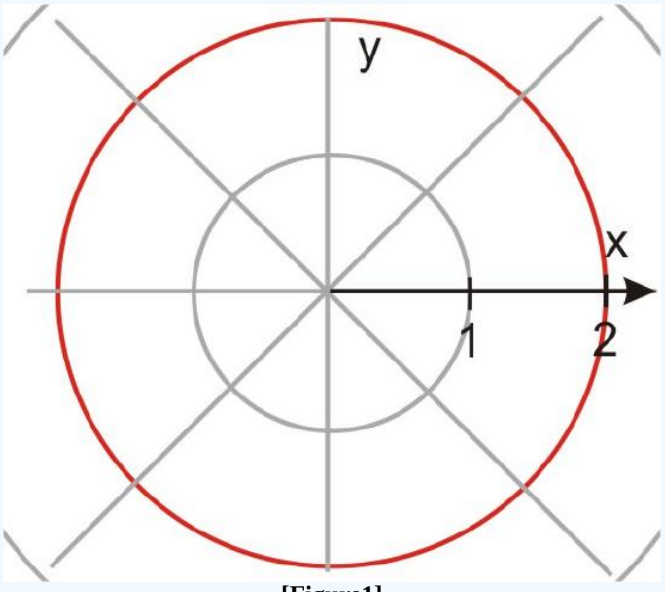
factor the r^2 from both terms on the left: $r^2(\cos^2\theta + \sin^2\theta) = k^2$

recall the identity: $\cos^2\theta + \sin^2\theta = 1$

$r^2 = k^2$

Therefore: r=±k is an equation for a circle in polar units.

When r is equal to a constant, the polar graph is a circle.

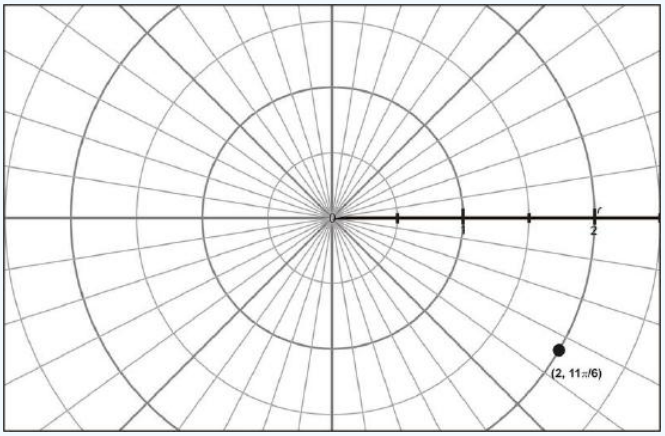


[Figure1]

Example 2

Transform the polar coordinates $\left(2, \frac{11\pi}{6}\right)$ to rectangular form.

Solution



$r = 2$ and $\theta = \frac{11\pi}{6}$

$x = r\cos\theta$ and $y = r\sin\theta$

$x = 2\cos\frac{11\pi}{6}$ and $y = 2\sin\frac{11\pi}{6}$ or $x = 3\sqrt{2}$ $y = -1$

$\left(2, \frac{11\pi}{6}\right)$ is equivalent to $(3\sqrt{2}, -1)$ or in decimal form, approximately (4.342,−1).

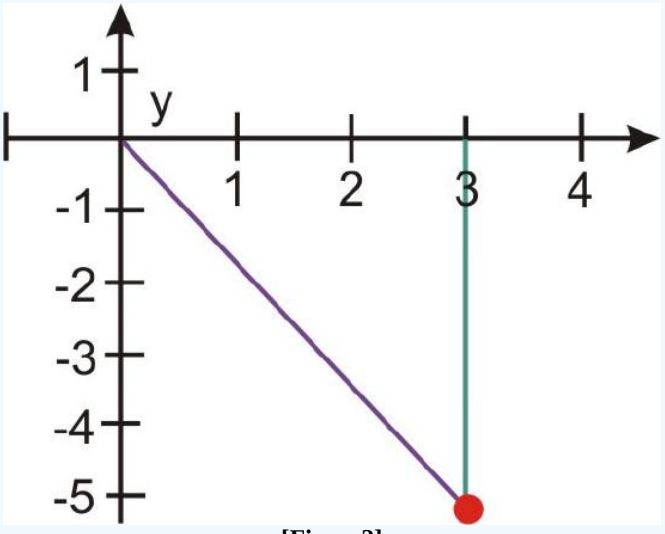
Example 3

Find the polar coordinates for $(3, -3\sqrt{3})$.

Solution

$x = 3$ and $y = -3\sqrt{3}$

Draw a right triangle in standard form. Find the distance the point is from the origin and the angle the line segment that represents this distance makes with the +x axis:



[Figure2]

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$r = \sqrt{3^2 + (-3\sqrt{3})^2}$
$= \sqrt{9 + 27}$
$= \sqrt{36}$
$= 6$

And for the angle,

$\tan\theta_{ref} = \left|\frac{(-3\sqrt{3})}{3}\right|$
$\tan\theta_{ref} = \sqrt{3}$
$\theta_{ref} = \tan^{-1}\sqrt{3}$
$\theta_{ref} = \frac{\pi}{3}$

So, $\theta_{ref} = \frac{\pi}{3}$ and we can look at the signs of *x* and *y* -- (+, -) -- to see that $\theta = \frac{5\pi}{3}$ since it is a 4th quadrant angle.

The rectangular point $(3, -3\sqrt{3})$ is equivalent to the polar point $(6, \frac{5\pi}{3})$.

Recall that when solving for θ, we used

$\tan\theta = \left|\frac{(-3\sqrt{3})}{3}\right|$ or $\tan\theta = \sqrt{3}$

We found

$\theta = \frac{5\pi}{3}$. BUT, θ could also be $\theta = \frac{2\pi}{3}$. You must examine the signs of each coordinate to see that the angle must be in the fourth quadrant in rectangular units or between $\frac{3\pi}{2}$ and 2π in polar units. Of the two possible angles for θ, only $\frac{5\pi}{3}$ is valid. Note that when you use $\tan^{-1}$ on a calculator you will always get an answer in the range $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Example 4

Convert the following rectangular coordinates to polar coordinates.

a. $(3, 3\sqrt{3})$
b. $(-2, 2)$

Convert the following polar coordinates to rectangular coordinates.

c. $(4, \frac{2\pi}{3})$
d. $(-1, \frac{5\pi}{6})$

Solution

a. $(6, 66°)$
b. $(2\sqrt{2}, 225°)$
c. $(-2, 2\sqrt{3})$
d. $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

Example 5

Express the equation in rectangular form: r=6cosθ.

Solution

$r^2=6r\cos\theta$: multiply both sides by r

$x^2+y^2=6x$: Using $x^2+y^2=r^2$ and $x=r\cos\theta$

$\therefore x^2+y^2=6x$ is the equation in rectangular form.

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Example 6

Express the equation in rectangular form: r=6.

Solution

This one is easy:

r=6 is the polar form of the equation for a circle

$r^2=6^2$: square both sides

$x^2+y^2=36$: Using $x^2+y^2=r^2$ and simplifying

$\therefore x^2+y^2=36$ is the equation in rectangular form.

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Review

1. How is the point with polar coordinates (5,π) represented in rectangular coordinates?

Plot each point below in polar coordinates (r, θ). Then write the rectangular coordinates (x, y) for the point.

2. $(3, 60°)$
3. $\left(-10, \frac{\pi}{3}\right)$
4. $(15, \pi)$

The rectangular coordinates (x, y) are given. For each question: a) Find two pairs of polar coordinates (r, θ), one with r > 0 and the other with r < 0. b) Express θ in radians, and round to the nearest hundredth.

5. $(5, -5)$
6. $(0, 10)$
7. $(-8, 6)$

Transform each polar equation to an equation using rectangular coordinates. Identify the graph, and give a rough sketch or description of the sketch.

8. $\theta = \frac{\pi}{10}$
9. $r = 8$
10. $r \sin\theta = 7$
11. $r \cos\theta = -3$

Transform each rectangular equation to an equation using polar coordinates. Identify the graph, and give a rough sketch or description of the sketch.

12. $x^2 + y^2 - 2x = 0$

13. $y = \sqrt{3}x$
14. $y = -5$
15. $xy = 15$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.2.

Vocabulary

Term	Definition
Tangent	The tangent of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the side adjacent to the given angle.
cosine	The cosine of an angle in a right triangle is a value found by dividing the length of the side adjacent the given angle by the length of the hypotenuse.
polar coordinates	Polar coordinates describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
Quadrants	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
rectangular coordinates	A point is written using rectangular coordinates if it is written in terms of x and y and can be graphed on the Cartesian plane.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.

Image Attributions

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2. [Figure 2]
   Source: https://www.flickr.com/photos/massdep/4472895387/in/photolist-7PfKXZ-UhtQTD-UhtSXZ-VjvsCF-UVVbX1-6jUJqS-68McDj-6TYvuN-6TYozN-az5Ck6-az8gqE-6TYdzs-6jUGF1-6TXXkb-az5CGz-6TTVQi-az5C6z-6TTYbT-6jUGVS-6jQxen-6TU9ZK-az5Dp4-8hLpDn-UVVadu-az8ga3-6jUHV9-az8h8u-az8hnQ-6jUGzb-6RbWLA-6jUH8h-az8fSA-az8fYC-

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4.4.3: Systems of Polar Equations

Systems of Polar Equations

You likely recall that when you graph multiple equations on the same line, you usually end up with locations on the graph where they intersect (unless you are graphing parallel lines!).

The same is true when graphing equations in polar form, and/or on a polar graph. When you graph the intersection of multiple polar equations, you treat them just as you would rectangular equations, graph both and find the areas that are true for both equations.

Systems of Polar Equations

Polar equations can be graphed using **polar coordinates**. Graphing two polar equations on the same set of axes may result in having point(s) of intersection.

All points on a polar graph are coordinates that make the equation valid. The coordinates of point(s) of intersection when substituted into each equation will make both of the equations valid.

One method to find point(s) of intersection for two polar graphs is by setting the equations equal to each other.

Call the first equation r_1 and the second equation r_2.

Points of intersection are when $r_1 = r_2$, so set the equations equal and then solve the resulting trigonometric equation.

Examples

Example 1

Find the intersection of $r_1 = 3\sin\theta$ and $r_2 = \sqrt{3}\cos\theta$

Solution

Set the equations equal to each other: $3\sin\theta = \sqrt{3}\cos\theta$

divide both sides of the equation by cos θ and 3: $\frac{3\sin\theta}{3\cos\theta} = \frac{\sqrt{3}\cos\theta}{3\cos\theta}$

Simplify: $\frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{3}$

Use the identity: $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta = \frac{\sqrt{3}}{3}$

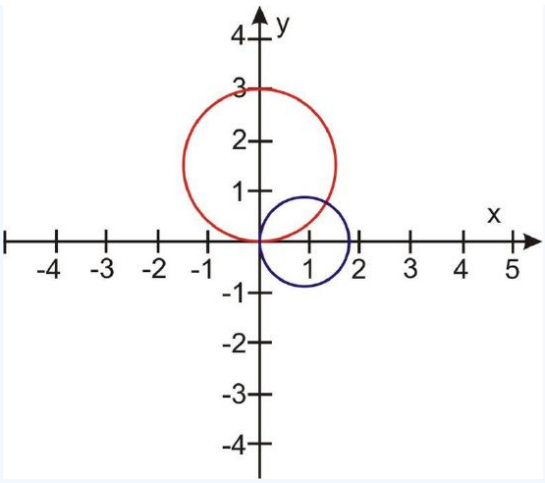
$\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

substitute $\frac{\pi}{6}$ in either equation to obtain $r = 1.5$

substitute $\frac{7\pi}{6}$ in either equation to obtain -1.5

NOTE: the coordinates $\left(1.5, \frac{\pi}{6}\right)$ and $\left(-1.5, \frac{7\pi}{6}\right)$ represent the SAME polar point so there is only one solution to this equation.

Are we done? If we look at the graphs of r_1 and r_2, we can see that there is another point of intersection:



when θ = 0, r_1 = 3 sin θ = 3 sin(0) = 0

That means r_1 = 3 sin θ goes through the **pole** (0, 0).

For r_2: when $\theta = \frac{\pi}{2}$, r_2 = 0 that is $r_2 = \sqrt{3}\cos\theta$ goes through the point $(0, \frac{\pi}{2})$.

Therefore, both graphs go through the pole and the pole is a point of intersection.

The pole was NOT revealed as a point of intersection using the first step! (Why? Hint: How many ways are there to represent the pole in polar coordinates?) This shows us that after you use algebraic methods to find intersections at points other than the pole, you should also check for intersections at the pole.

Example 2

Find the point(s) of intersection for the two graphs: $r_1 = 1$ and $r_2 = 2\sin 2\theta$.

Solution

Set $r_1 = r_2$ and solve:

$$1 = 2\sin 2\theta$$
$$\frac{1}{2} = \sin 2\theta$$

Use a substitution $\alpha = 2\theta$ to solve

$$\frac{1}{2} = \sin\alpha$$
$$\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$$

Since $\alpha = 2\theta$, solving for θ gives us

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

But, recall that θ has the range $0 \le \theta \le 2\pi$. Since we solved with $0 \le \alpha \le 2\pi$ we actually need to consider values of θ with $0 \le \theta \le 4\pi$. Why? Recall that sin(2θ) has two cycles between 0 and 2π, and so we add two more solutions,

$$\alpha = \frac{13\pi}{6}, \frac{17\pi}{6}$$

and since $\alpha = 2\theta$,

$$\theta = \frac{13\pi}{12}, \frac{17\pi}{12}$$

Finally, we need to consider solutions when r = -1 because r = 1 and r = -1 are the same polar equation. So, solving

$$-\frac{1}{2} = \sin\alpha$$

$$\alpha = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Again, using $\alpha = 2\theta$ and adding solutions for the repetition gives us four more solutions,

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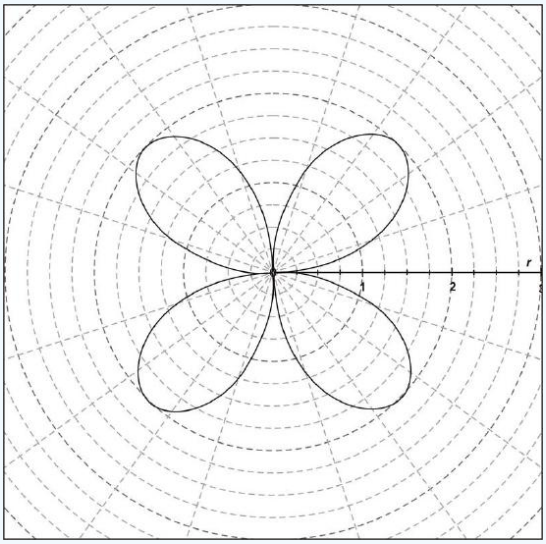
$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}$

So in total, there are eight solutions to this set of equations.

NOTE: Recall that solving trigonometric equations where the angle is θ requires looking at all potential values between 0 and 2π. When the angle is 2θ as it is in this case, be sure to look for all potential values between 0 and 4π. When the angle is 3θ as it is in this case, be sure to look for all potential values between 0 and 6π., and so on.

Since r_1 cannot equal 0, the pole is not on its graph and not a point of intersection.

The graph reveals eight points of intersection which were found earlier.



Example 3

Find point(s) of intersection, if any exist, for the following pair of equations: $r_1 = 2$ and $r_2 = \sec\theta$.

Solution

Here we will use a table of values for each function, solving by **quadrant**. Recall that the period of sec θ is 2π.

For the first quadrant:

θ (angle)	0	π/6	π/4	π/3	π/2
r_1 (distance)	2	2	2	2	2
r_2	1	1.15	1.4	2	und

For the second quadrant:

θ (angle)	2π/3	3π/4	5π/6	π
r_1 (distance)	2	2	2	2
r_2	-2	-1.4	-1.15	-1

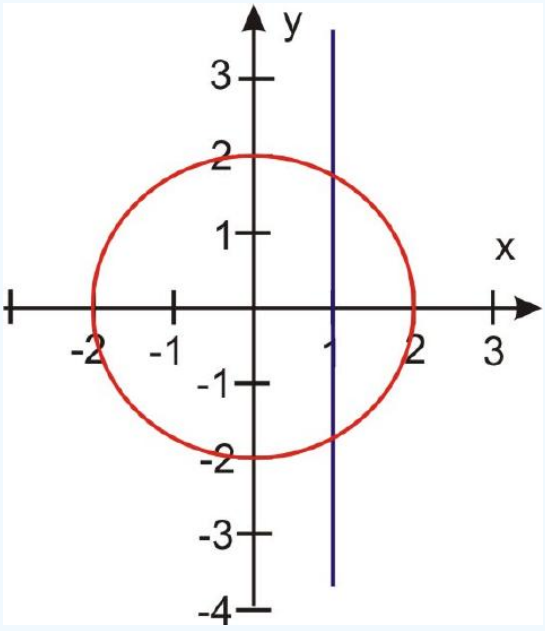
For the third quadrant:

θ (angle)	7π/6	5π/4	4π/3	3π/2
r_1 (distance)	2	2	2	2
r_2	-1.15	-1.4	-2	und

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For the fourth quadrant:

θ (angle)	5π/3	7π/4	11π/6	2π
r_1 (distance)	2	2	2	2
r_2	2	1.4	1.15	1



Note that the 3rd and 4th quadrant repeat 1st and 2nd quadrant values.

Observe in the table of values that (2, π/3) and (2, 5π/3) are the points of intersection. Look at the curves- the first equation yields a circle while the second yields a line. The maximum number of intersecting points for a line and a circle is 2. The two points have been found.

Example 4

Find the point(s) of intersection for this pair of Polar equations: $r = 2 + 4 \sin\theta$ and $\theta = 60°$.

Solution

The equation $\theta = 60^o$ is a line making a 60° angle with the r axis.

Make a table of values for $r = 2 + 4 \sin \theta$

For the first quadrant:

θ (angle)	0	30	45	60	90
R (distance)	2	4	4.83	5.46	6

For the second quadrant:

θ (angle)	120	135	150	180
R (distance)	5.46	4.83	4	2

For the third quadrant:

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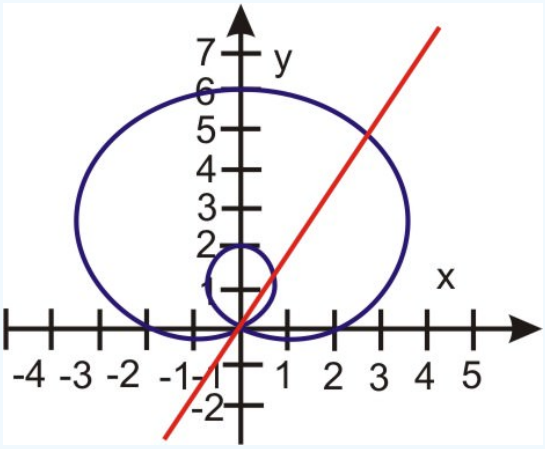
θ (angle)	210	225	240	270
R (distance)	0	-.83	-1.46	-2

For the fourth quadrant:

θ (angle)	300	315	330	360
R (distance)	-1.46	-.83	0	2

Notice there are two solutions in the table., (60, 5.46) and (240, -1.46) = (60, 1.46). Recall that when $r < 0$, you plot a point (r, θ), by rotating 180° (*or* π).

Finally, we need to check the pole: $r = 2 + 4 \sin \theta$ passes through the pole for $\theta = 330°$, and $\theta = 60°$ also passes through the pole. Thus, the third point of intersection is (0, 0).



Example 5

Find the point(s) of intersection for this pair of Polar equations: $r_1 = 2 \cos\theta$ and $r_2 = 1$.

Solution

Make a table:

For the first quadrant:

θ (angle)	0	π/6	π/4	π/3	π/2
r_1 (distance)	2	$\sqrt{3}$	$\sqrt{2}$	1	0
r_2	1	1	1	1	1

For the second quadrant:

θ (angle)	2π/3	3π/4	5π/6	π
r_1 (distance)	-1	$-\sqrt{2}$	$-\sqrt{3}$	-2
r_2	1	1	1	1

For the third quadrant:

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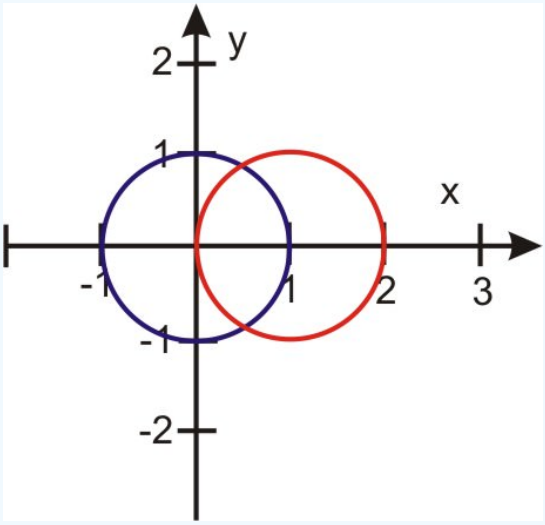
θ (angle)	7π/6	5π/4	4π/3	3π/2
r_1 (distance)	$-\sqrt{3}$	$-\sqrt{2}$	-1	0
r_2	1	1	1	1

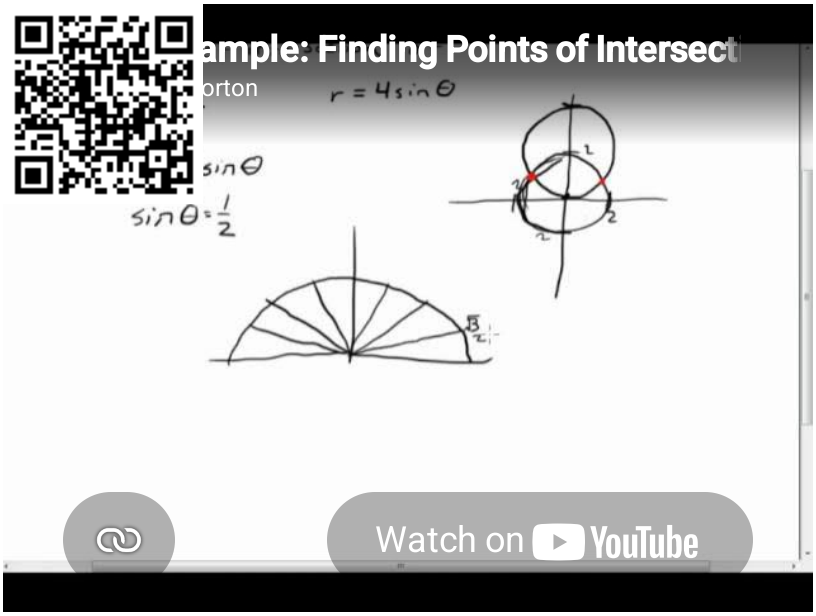
For the fourth quadrant:

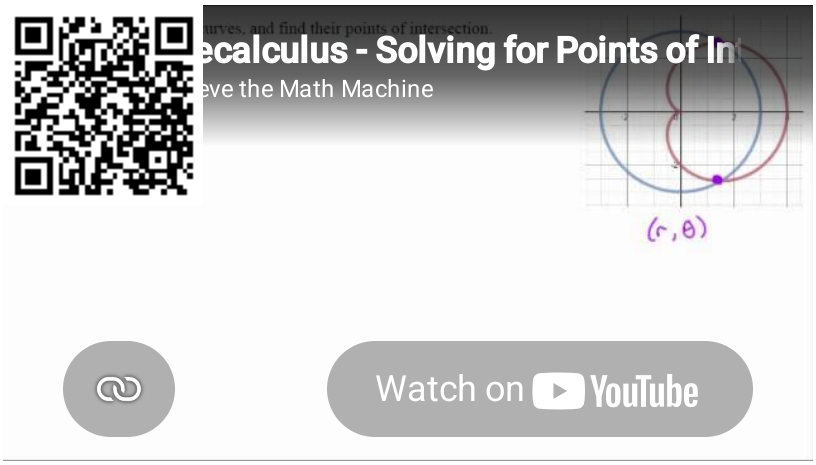
θ (angle)	5π/3	7π/4	11π/6	2π
r_1 (distance)	1	$\sqrt{2}$	$\sqrt{3}$	2
r_2	1	1	1	1

So the unique solutions are at $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$. There are also two repeated solutions in this set (can you find them?).

Here is a graph showing the two solutions:

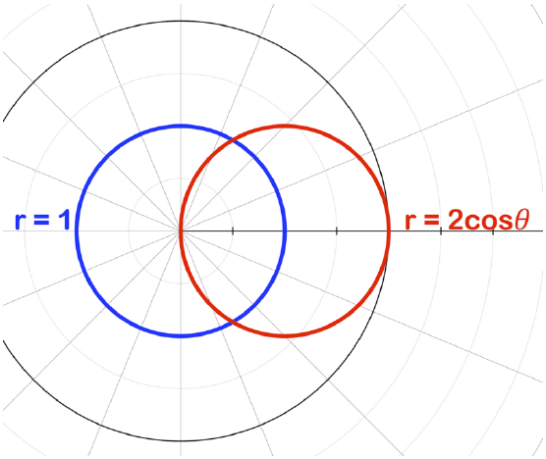






Review

The graphs of r=1 and r=2cosθ are shown below.



1. How many times do they intersect?
2. In which quadrants do they intersect?
3. At what points do the intersections occur?

Based on the image below and the following information: the intersection of the graphs of r=cosθ and r=1−cosθ

4. Identify how many times they intersect?
5. At what points do the intersections occur?

Find the points of intersection of the following pairs of curves.

6. r=2 r=2cosθ
7. r=sin2θ r=2sinθ
8. r=2+2sinθ r=2−2cosθ
9. r=3cosθ r=2−cosθ

Find the point(s) of intersection for each system of equations. Graph to verify your solution.

10. $r_1 = \csc\theta \; r_2 = 2\sin\theta$
11. $r_1 = \cos\theta \; r_2 = 1 + \sin\theta$
12. $r_1 = \sin\theta \; r_2 = \sin 2\theta$
13. $r_1 = -4\sin\theta \; r_2 = -4\cos\theta$
14. $r_1 = 1 - 2\sin\theta \; r_2 = \sqrt{9\cos(\theta)}$
15. $r_1 = 1 - \cos\theta \; r_2 = 4\cos(3\theta)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.3.

Vocabulary

Term	Definition
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Points of intersection	Points of intersection are locations where two different equations have the same solutions.
polar coordinates	Polar coordinates describe locations on a grid using the polar coordinate system. The location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
pole	The pole is the center point on a polar graph.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.

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4.4.4: Polar Equations of Conics

Polar Equations of Conics

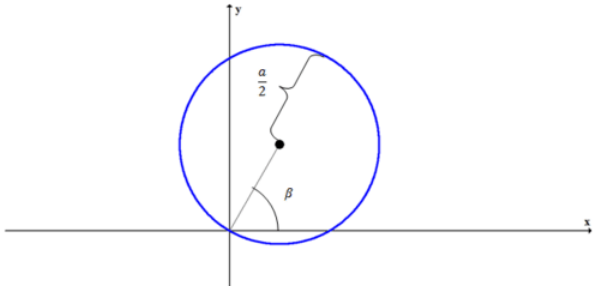
Polar coordinates allow you to extend your knowledge of conics in a new context. Calculators are an excellent tool for graphing polar conics. What settings do you need to know in order to properly use your calculator?

Polar Equations of Conics

Polar equations refer to the radius r as a function of the angle θ. There are a few typical polar equations you should be able to recognize and graph directly from their **polar form**.

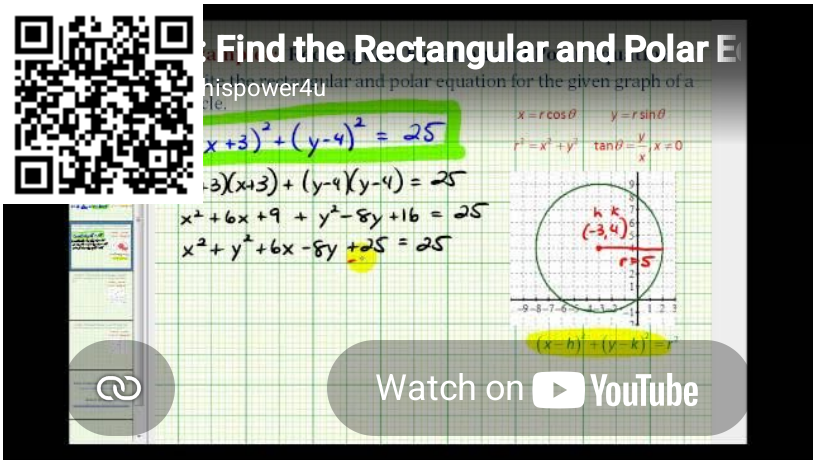
The following polar function is a circle of radius $\frac{a}{2}$ passing through the origin with a center at angle β.

$r=a\cdot\cos(\theta-\beta)$



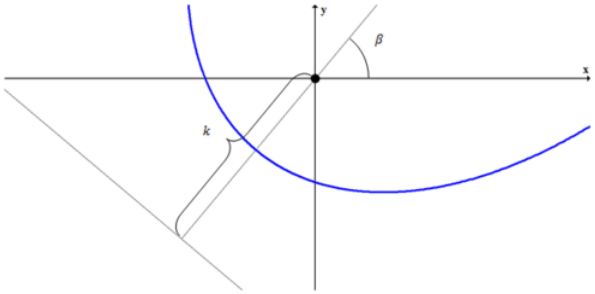
[Figure1]

There are other ways of representing a circle like this using cofunction identities and coterminal angles.



Ellipses, parabolas and hyperbolas have a common general polar equation. Just like with the circle, there are other ways of representing these relations using cofunction and coterminal angles; however, this general form is easiest to use because each **parameter** can be immediately interpreted in a graph. A **parameter** is a constant in a general equation that takes on a specific value in a specific equation.

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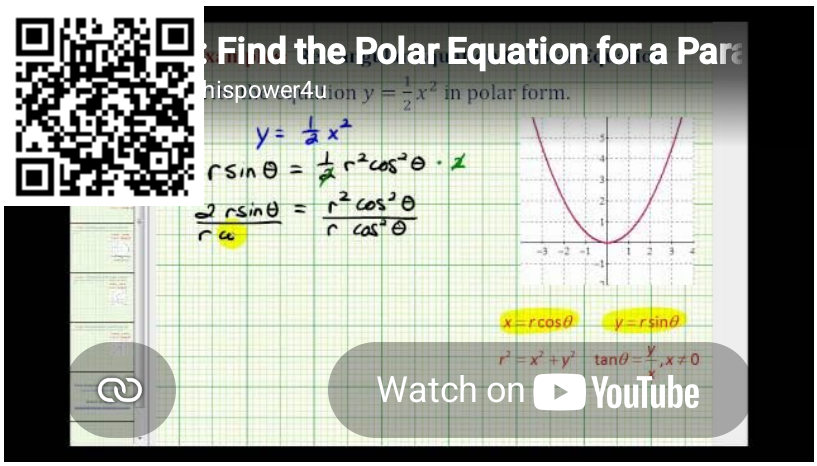


[Figure2]

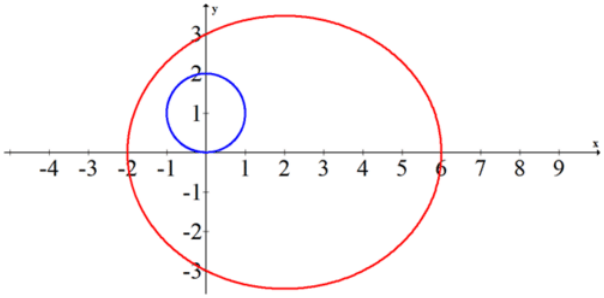
One of the focus points of a **conic** written in this way is always at the **pole** (the origin). The angle β indicates the angle towards the center if the conic is an ellipse, the opening direction if the conic is a parabola and the angle away from the center if the conic is a hyperbola. The **eccentricity** e should tell you what conic it is. The constant k is the distance from the focus at the pole to the nearest directrix. This directrix lies in the opposite direction indicated by β.

There are many opportunities for questions involving partial information with polar conics. A few relationships that are often useful for solving these questions are:

- $e = \frac{c}{a} = \frac{\overline{PF}}{\overline{PD}} \rightarrow \overline{PF} = e \cdot \overrightarrow{PD}$
- Ellipses: $k = \frac{a^2}{c} - c$
- Hyperbolas: $k = c - \frac{a^2}{c}$



A great way to discover new types of graphs in polar coordinates is to experiment on your own with your calculator. Try to come up with equations and graphs that look similar to the following two polar functions.



[Figure3]

The circle in blue has a center at 90∘ and has a diameter of 2. Its equation is

$r = 2\cos(\theta - 90°)$.

The red ellipse appears to have center at (2, 0) with a=4 and c=2. This means the eccentricity is $e = \frac{1}{2}$. In order to write the equation in polar form you still need to find k.

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$k = \frac{a^2}{c} - c = \frac{4^2}{2} - 2 = 8 - 2 = 6$

Thus the equation for the ellipse is:

$r = \frac{6 \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \cos(\theta)}$

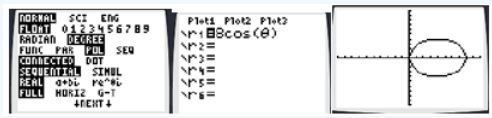
Examples

Example 1

Earlier, you were asked about how to use your calculator to graph polar equations.

Solution

Most calculators have a polar coordinate mode. On the TI-84, the mode can be switched to polar in the mode menu. This changes the graphing features. You can choose to be in radians or degrees and graphs will look the same. When you graph a circle of the form r=8·cosθ. you should see the following on your calculator.



[Figure4]

When you go to the window setting you should notice that in addition to X_{min}, X_{max} there are new settings called θ_{min}, θ_{max} and θ_{step}.

If θmin and θmax do not span an entire period, you may end up missing part of your polar graph.

The θ_{step} controls how accurate the graph should be. If you put θ_{step} at a low number like 0.1 the graph will plot extremely slowly because the calculator is doing 3600 cosine calculations. On the other hand if θ_{step}=30 then the calculator will do fewer calculations producing a rough circle, but probably not accurate enough for your purposes.



[Figure5]

Example 2

Identify the center, foci, vertices and equations of the directrix lines for the following conic:

$r = \frac{20}{4 - 5 \cdot \cos\left(\theta - \frac{3\pi}{4}\right)}$

Solution

First the polar equation needs to be in graphing form. This means that the denominator needs to look like $1 - e \cdot \cos(\theta - \beta)$.

$$r = \frac{20}{4 - 5 \cdot \cos\left(\theta - \frac{3\pi}{4}\right)} \cdot \frac{\frac{1}{4}}{\frac{1}{4}} = \frac{5}{1 - \frac{5}{4} \cdot \cos\left(\theta - \frac{3\pi}{4}\right)} = \frac{4 \cdot \frac{5}{4}}{1 - \frac{5}{4} \cdot \cos\left(\theta - \frac{3\pi}{4}\right)}$$

$$e = \frac{5}{4}, \quad k = 4, \quad \beta = \frac{3\pi}{4} = 135^\circ$$

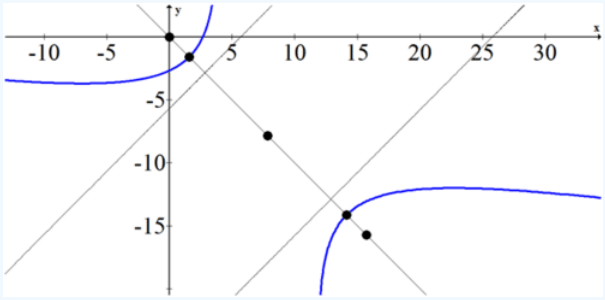
Using this information and the relationships you were reminded of in the guidance, you can set up a system and solve for a and c.

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$$4 = c - \frac{a^2}{c}$$
$$\frac{5}{4} = \frac{c}{a} \rightarrow \frac{4}{5} = \frac{a}{c} \rightarrow \frac{4c}{5} = a$$
$$4 = c - \left(\frac{4c}{5}\right)^2 \cdot \frac{1}{c}$$
$$4 = c - \frac{16c^2}{25c}$$
$$4 = \frac{9c}{25}$$
$$\frac{100}{9} = c$$
$$\frac{80}{9} = a$$

The center is the point $\left(\frac{100}{9}, \frac{7\pi}{4}\right)$ which is much more convenient to write in polar coordinates. The closest directrix is the line $r = 4 \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. The other directrix is the line $r = \left(2 \cdot \frac{100}{9} - 4\right) \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. One focus is at the pole, the other focus is the point $\left(\frac{200}{9}, \frac{7\pi}{4}\right)$. The vertices are at the center plus or minus a in the same angle:

Putting all this information together, the graph of the conic is:



[Figure6]

Example 3

Convert the following conic from polar form to **rectangular form**.

$r = \frac{3}{2 - \cos\theta}$

Solution

There are many ways to convert from polar form to rectangular form. You should become comfortable with the algebra.

$$r = \frac{3}{2 - \cos\theta}$$
$$r(2 - \cos\theta) = 3$$
$$2r - r \cdot \cos\theta = 3$$
$$2r = 3 + r \cdot \cos\theta = 3 + y$$
$$4r^2 = 9 + 6y + y^2$$
$$4\left(x^2 + y^2\right) = 9 + 6y + y^2$$
$$4x^2 + 4y^2 = 9 + 6y + y^2$$
$$4x^2 + 3y^3 + 6y = 9$$
$$4x^2 + 3\left(y^2 + 2y + 1\right) = 9 + 3$$
$$4x^2 + 3(y+1)^2 = 12$$
$$\frac{x^2}{3} + \frac{(y+1)^2}{4} = 1$$

Example 4

Graph the following conic.

$r = \frac{3}{2 - \cos(\theta - 30°)}$

Solution

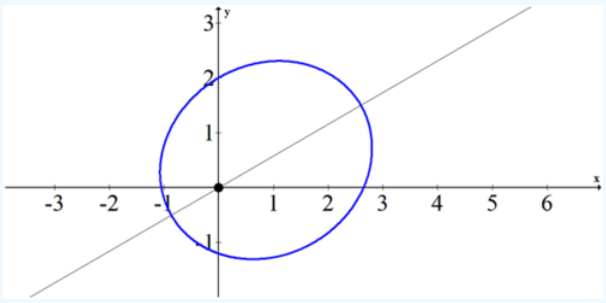
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Convert to the standard conic form.

$$r = \frac{3}{2 - \cos(\theta - 30°)}$$

$$r = \frac{3}{2 - \cos(\theta - 30°)} \cdot \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{\frac{3}{2}}{1 - \frac{1}{2} \cdot \cos(\theta - 30°)} = \frac{3 \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \cos(\theta - 30°)}$$

$$k = 3, \quad e = \frac{1}{2}, \quad \beta = 30°$$



[Figure7]

Example 5

Translate the following conic to polar form.

$$(x-3)^2 + (y+4)^2 = 25$$

Solution

Expand the original equation and then translate to polar coordinates:

$$(x-3)^2 + (y+4)^2 = 25$$
$$x^2 - 6x + 9 + y^2 + 8y + 16 = 25$$
$$r^2 - 6x + 8y = 0$$
$$r^2 - 6r \cdot \cos\theta + 8r \cdot \sin\theta = 0$$
$$r - 6\cos\theta + 8\sin\theta = 0$$
$$r = 6\cos\theta - 8\sin\theta$$

Review

Convert the following conics from polar form to rectangular form. Then, identify the conic.

1. $r = \frac{5}{3 - \cos\theta}$
2. $r = \frac{4}{2 - \cos\theta}$
3. $r = \frac{2}{2 - \cos\theta}$
4. $r = \frac{3}{2 - 4\cos\theta}$
5. $r = 5\cos(\theta)$

Graph the following conics.

6. $r = \frac{5}{4 - 2\cos(\theta - 90°)}$
7. $r = \frac{5}{3 - 7\cos(\theta - 60°)}$
8. $r = \frac{3}{3 - 3\cos(\theta - 30°)}$
9. $r = \frac{1}{2 - \cos(\theta - 60°)}$
10. $r = \frac{3}{6 - 3\cos(\theta - 45°)}$

Translate the following conics to polar form.

11. $\frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$
12. $(x-5)^2 + (y+12)^2 = 169$
13. $x^2 + (y+1)^2 = 1$

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14. $x^2 + (y+1)^2 = 1$
15. $-3x^2 - 4x + y^2 - 1 = 0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 10.2.

Vocabulary

Term	Definition
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Eccentricity	The eccentricity of a conic section is a measure of how much the conic section deviates from being circular. The eccentricity of circles is 0, the eccentricity of ellipses is between 0 and 1, the eccentricity of parabolas is 1, and the eccentricity of hyperbolas is greater than 1. For ellipses and hyperbolas, $e = \frac{c}{a}$.
parameter	A parameter is a variable in a general equation that takes on a specific value in order to create a specific equation.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
pole	The pole is the center point on a polar graph.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.

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SECTION OVERVIEW

4.5: Imaginary Numbers and Complex Numbers

Topic hierarchy

4.5.1: Imaginary Numbers

4.5.2: Complex Numbers

4.5.3: Quadratic Formula and Complex Sums

4.5.4: Products and Quotients of Complex Numbers

4.5.5: Polar Form of Complex Numbers

4.5.6: Product and Quotient Theorems

4.5.7: Powers and Roots of Complex Numbers

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4.5.1: Imaginary Numbers

Imaginary Numbers

Have you ever had an imaginary pet? Some people have, particularly as young children.

Wouldn't you be surprised if you and your real friend left your two imaginary puppies alone together, and you came back to find a real puppy?

It's a silly thought, so what does it have to do with **imaginary numbers**?

Imaginary Numbers

What is the square root of -1?

You may recall running into roots of negatives in algebra, when attempting to solve equations like $x^2 + 4 = 0$.

Since there are no real numbers that can be squared to equal -4, this equation has no real solution. Enter the imaginary constant: "**i**".

The definition of "i" : $i = \sqrt{-1}$

The use of the word *imaginary* does not mean these numbers are useless. For a long period in the history of mathematics, it was thought that the square root of a negative number was in fact only within the mathematical imagination, without real-world significance hence, imaginary. That has changed. Mathematicians now consider the imaginary numbers as another set of numbers that have real significance, but do not fit on what is called the number line, and engineers, scientists, and others solve real world problems using combinations of real and imaginary numbers (called complex numbers) every day.

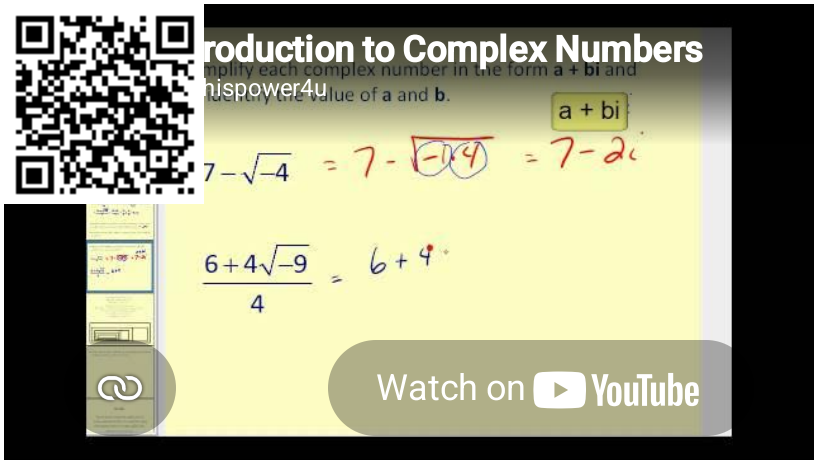
Imaginary values such as $\sqrt{-16}$ can be simplified by simplifying the radical into $\sqrt{16} \cdot \sqrt{-1}$, yielding: $4\sqrt{-1}$ or $4i$.

The uses of *i* become more apparent when you begin working with increased powers of *i*, as you will see in the examples below.

Complex Numbers

When you combine imaginary numbers with real numbers, you get complex numbers:

Complex numbers are of the form $a + bi$, where a is a real number, b is a real number, and i is the imaginary constant $\sqrt{-1}$.





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Examples

Example 1

Earlier, you were given an analogy about imaginary pets.

Solution

Do you see its application?

Two imaginary puppies creating a real puppy is an oddly effective metaphor for the behavior of the powers of *i*.

One *i* is imaginary, but two *i's* multiply to be a real number. In fact, *every* even power of *i* results in a real number!

Example 2

Simplify $\sqrt{-5}$.

Solution

$$\sqrt{-5} = \sqrt{(-1) \cdot (5)}$$
$$= \sqrt{-1}\sqrt{5}$$
$$= i\sqrt{5}$$

Example 3

Simplify $\sqrt{-72}$.

Solution

$$\sqrt{-72} = \sqrt{(-1) \cdot (72)}$$
$$= \sqrt{-1}\sqrt{72}$$
$$= i\sqrt{72}$$

But, we're not done yet! Since 72=36·2

$$i\sqrt{72} = i\sqrt{36}\sqrt{2}$$
$$= i(6)\sqrt{2}$$
$$= 6i\sqrt{2}$$

Example 4

Strange things happen when the imaginary constant *i* is multiplied by itself different numbers of times.

a. What is i^2?
b. What is i^3?

c. What is i^4?

Solution

a. i^2 is the same as $(\sqrt{-1})^2$. When you square a square root, they cancel and you are left with the number originally inside the radical, in this case -1

$\therefore i^2 = -1$

b. i^3 is the same thing as $i^2 \cdot i$, which is $-1 \cdot i$ or $-i$

$\therefore i^3 = -i$

c. $i^4 = i^2 \cdot i^2$ which is $-1 \cdot -1$

$\therefore i^4 = 1$

Example 5

Simplify the following radical: $\sqrt{108 - 140}$.

Solution

$\sqrt{-32}$: Subtract within the parenthesis

$\sqrt{32 \cdot -1}$: Rewrite -32 as $-1 \cdot 32$

$\sqrt{32} \cdot \sqrt{-1}$: Rewrite as a product of radicals

$\sqrt{32} \cdot i$: Substitute $\sqrt{-1} \rightarrow i$

$\sqrt{16 \cdot 2} \cdot i$: Factor 32

$4i\sqrt{2}$: Simplify $\sqrt{16}$

Example 6

Multiply the imaginary numbers: $4i \cdot 3i$.

Solution

$4 \cdot 3 \cdot i \cdot i$: Using the commutative law for multiplication

$12 \cdot i^2$: Simplify

$12 \cdot -1$: Recall $i^2 = -1$

-12

Example 7

Multiply the imaginary numbers: $\sqrt{4i^2} \cdot \sqrt{12i}$.

Solution

$\sqrt{4} \cdot \sqrt{i^2} \cdot \sqrt{4} \cdot \sqrt{3} \cdot i$: Factor

$2 \cdot i \cdot 2 \cdot \sqrt{3} \cdot i$: Simplify the roots

$4\sqrt{3} \cdot i^2$: Collect terms and simplify

$4\sqrt{3} \cdot -1$: Recall $i^2 = -1$

$-4\sqrt{3}$

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Review

Simplify:

1. $\sqrt{-49}$
2. $\sqrt{-81}$
3. $\sqrt{-324}$
4. $\sqrt{-121}$
5. $-\sqrt{-16}$
6. $-\sqrt{-1}$
7. $\sqrt{-1.21}$

Simplify:

8. i^8
9. i^{12}
10. i^3
11. $24i^{20}$
12. i^{225}
13. i^{1024}

Multiply:

14. $i^4 \cdot i^{11}$
15. $5i^6 \cdot 5i^8$
16. $3\sqrt{-75} \cdot 5\sqrt{-3}$
17. $2\sqrt{-12} \cdot 6\sqrt{-27}$
18. $-4\sqrt{-10} \cdot 5\sqrt{-3} \cdot 6\sqrt{-18}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.4.

Vocabulary

Term	Definition
i	i is an imaginary number. $i = \sqrt{-1}$.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a + bi$.
i	i is an imaginary number. $i = \sqrt{-1}$.
Imaginary Number	An imaginary number is a number that can be written as the product of a real number and i.
Imaginary Numbers	An imaginary number is a number that can be written as the product of a real number and i.

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4.5.2: Complex Numbers

Defining Complex Numbers

The coldest possible temperature, known as *absolute zero* is almost –460 degrees Fahrenheit. What is the square root of this number?

Complex Numbers

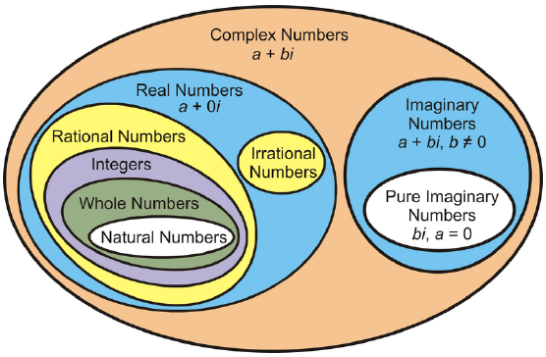
Before this concept, all numbers have been real numbers. $2, -5, \sqrt{11}$, and $\frac{1}{3}$ are all examples of real numbers. With what we have previously learned, we cannot find $\sqrt{-25}$ because you cannot take the square root of a negative number. There is no **real number** that, when multiplied by itself, equals -25. Let's simplify $\sqrt{-25}$.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1}$$

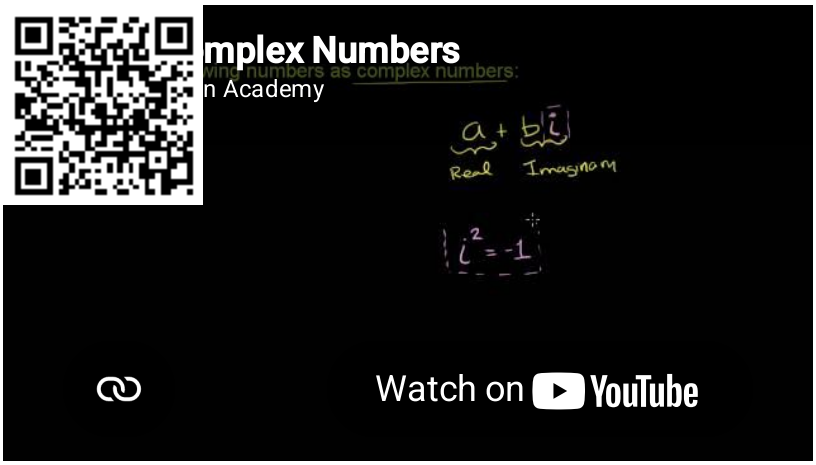
In order to take the square root of a negative number we are going to assign $\sqrt{-1}$ a variable, i. i represents an **imaginary number**. Now, we can use i to take the square root of a negative number.

$$\sqrt{-25} = \sqrt{25 \cdot -1} = 5\sqrt{-1} = 5i$$

All **complex numbers** have the form a+bi, where a and b are real numbers. a is the **real part** of the complex number and b is the **imaginary part**. If b=0, then a is left and the number is a **real number.** If a=0, then the number is only bi and called a **pure imaginary number**. If b≠0 and a≠0, the number will be an imaginary number.



[Figure1]



Let's find $\sqrt{-162}$.

First pull out the i. Then, simplify $\sqrt{162}$.

$$\sqrt{-162} = \sqrt{-1} \cdot \sqrt{162} = i\sqrt{162} = i\sqrt{81 \cdot 2} = 9i\sqrt{2}$$

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Powers of i

In addition to now being able to take the square root of a negative number, i also has some interesting properties. Try to find $i^2, i^3,$ and i^4.

Step 1: Write out i^2 and simplify. $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{-1}^2 = -1$

Step 2: Write out i^3 and simplify. $i^3 = i^2 \cdot i = -1 \cdot i = -i$

Step 3: Write out i^4 and simplify. $i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$

Step 4: Write out i^5 and simplify. $i^5 = i^4 \cdot i = 1 \cdot i = i$

Step 5: Write out i^6 and simplify. $i^6 = i^4 \cdot i^2 = 1 \cdot -1 = -1$

Step 6: Do you see a pattern? Describe it and try to find i^{19}.

You should see that the powers of i repeat every 4 powers. So, all the powers that are divisible by 4 will be equal to 1. To find i^{19}, divide 19 by 4 and determine the remainder. That will tell you what power it is the same as.

$$i^{19} = i^{16} \cdot i^3 = 1 \cdot i^3 = -i$$

Now, let's find the following powers of i.

a. i^{32}

32 is divisible by 4, so $i^{32} = 1$.

b. i^{50}

50÷4=12, with a remainder of 2. Therefore, $i^{50} = i^2 = -1$.

c. i^7

7÷4=1, with a remainder of 3. Therefore, $i^7 = i^3 = -i$

Finally, let's simplify the following complex expressions.

a. $(6 - 4i) + (5 + 8i)$

$(6 - 4i) + (5 + 8i) = 6 - 4i + 5 + 8i = 11 + 4i$

b. $9 - (4 + i) + (2 - 7i)$

$9 - (4 + i) + (2 - 7i) = 9 - 4 - i + 2 - 7i = 7 - 8i$

To add or subtract complex numbers, you need to combine like terms. Be careful with negatives and properly distributing them. Your answer should always be in **standard form**, which is $a + bi$.

Examples

Example 1

Earlier, you were asked to find the square root of -460 degrees.

Solution

We're looking for $\sqrt{-460}$.

First we need to pull out the i. Then, we need to simplify $\sqrt{460}$.

$\sqrt{-460} = \sqrt{-1} \cdot \sqrt{460} = i\sqrt{460} = i\sqrt{4 \cdot 115} = 2i\sqrt{115}$

Example 2

Simplify $\sqrt{-49}$.

Solution

Rewrite $\sqrt{-49}$ in terms of i and simplify the radical.

$\sqrt{-49} = i\sqrt{49} = 7i$

Example 3

Simplify $\sqrt{-125}$.

Solution

Rewrite $\sqrt{-125}$ in terms of i and simplify the radical.

$\sqrt{-125} = i\sqrt{125} = i\sqrt{25 \cdot 5} = 5i\sqrt{5}$

Example 4

Simplify i^{210}.

Solution

210÷4=52, with a remainder of 2. Therefore, $i^{210} = i^2 = -1$.

Example 5

Simplify $(8 - 3i) - (12 - i)$.

Solution

Distribute the negative and combine like terms.

$(8 - 3i) - (12 - i) = 8 - 3i - 12 + i = -4 - 2i$

Review

Simplify each expression and write in standard form.

1. $\sqrt{-9}$
2. $\sqrt{-242}$
3. $6\sqrt{-45}$
4. $-12i\sqrt{98}$
5. $\sqrt{-32} \cdot \sqrt{-27}$
6. $7i\sqrt{-126}$
7. i^8
8. $16i^{22}$
9. $-9i^{65}$
10. i^{365}
11. $2i^{91}$
12. $\sqrt{-\frac{16}{80}}$
13. $(11 - 5i) + (6 - 7i)$
14. $(14 + 2i) - (20 + 9i)$
15. $(8 - i) - (3 + 4i) + 15i$
16. $-10i - (1 - 4i)$

17. $(0.2+1.5i)-(-0.6+i)$
18. $6+(18-i)-(2+12i)$
19. $-i+(19+22i)-(8-14i)$
20. $18-(4+6i)+(17-9i)+24i$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 5.8.

Vocabulary

Term	Definition
i	i is an imaginary number. $i=\sqrt{-1}$.
Absolute Value	The absolute value of a number is the distance the number is from zero. The absolute value of a complex number is the distance from the complex number on the complex plane to the origin.
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of $a+bi$ is $a-bi$. When complex conjugates are multiplied, the result is a single real number.
i	i is an imaginary number. $i=\sqrt{-1}$.
Imaginary Number	An imaginary number is a number that can be written as the product of a real number and i.
imaginary part	The imaginary part of a complex number $a+bi$ is bi.
Pure Imaginary Numbers	The pure imaginary numbers are the subset of complex numbers without real parts, only bi.
Real Number	A real number is a number that can be plotted on a number line. Real numbers include all rational and irrational numbers.
real part	The real part of a complex number $a+bi$ is a.
rectangular coordinates	A point is written using rectangular coordinates if it is written in terms of x and y and can be graphed on the Cartesian plane.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.
standard form	The standard form of a complex number is $a+bi$ where a and b are real numbers.

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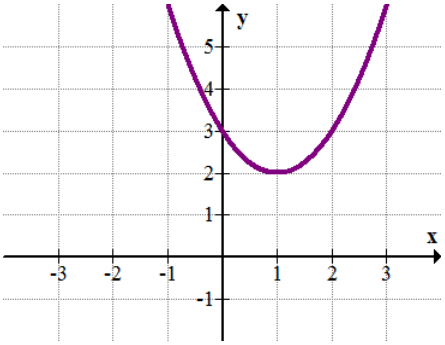
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4.5.3: Quadratic Formula and Complex Sums

Complex Roots of Quadratic Functions

The quadratic function $y = x^2 - 2x + 3$ (shown below) does not intersect the x-axis and therefore has no real roots. What are the complex roots of the function?



Complex Roots of Quadratic Functions

Recall that the imaginary number, i, is a number whose square is –1:

$i^2 = -1$ and $i = \sqrt{-1}$

The sum of a **real number** and an imaginary number is called a **complex number**. Examples of complex numbers are $5 + 4i$ and $3 - 2i$. All complex numbers can be written in the form $a + bi$ where a and b are real numbers. Two important points:

- The set of real numbers is a subset of the set of complex numbers where $b = 0$.

  Examples of real numbers are $2, 7, \frac{1}{2}, -4.2$.

- The set of **imaginary numbers** is a subset of the set of complex numbers where $a = 0$.

  Examples of imaginary numbers are $i, -4i, \sqrt{2}i$.

This means that the set of complex numbers includes real numbers, imaginary numbers, and combinations of real and imaginary numbers.

When a quadratic function does not intersect the x-axis, it has complex roots. When solving for the roots of a function algebraically using the **quadratic formula**, you will end up with a negative under the square root symbol. With your knowledge of complex numbers, you can still state the complex roots of a function just like you would state the real roots of a function.

Let's solve the quadratic equation: $m^2-2m+5=0$

You can use the quadratic formula to solve. For this quadratic equation, $a = 1, b = -2, c = 5$.

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$$

$$m = \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$m = \frac{2 \pm \sqrt{-16}}{2} \qquad \sqrt{-16} = \sqrt{16} \times i = 4i$$

$$m = \frac{2 \pm 4i}{2}$$

$$m = 1 \pm 2i$$

$$m = 1 + 2i \text{ or } m = 1 - 2i$$

There are no real solutions to the equation. The solutions to the quadratic equation are $1 + 2i$ and $1 - 2i$.

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Now, let's solve the following equation by rewriting it as a quadratic and using the quadratic formula:

$\frac{3}{e+3} - \frac{2}{e+2} = 1$

To rewrite as a quadratic equation, multiply each term by $(e+3)(e+2)$.

$\frac{3}{e+3}(e+3)(e+2) - \frac{2}{e+2}(e+3)(e+2) = 1(e+3)(e+2)$
$3(e+2) - 2(e+3) = (e+3)(e+2)$

Expand and simplify.

$3e + 6 - 2e - 6 = e^2 + 2e + 3e + 6$
$e^2 + 4e + 6 = 0$

Solve using the quadratic formula. For this quadratic equation, $a = 1, b = 4, c = 6$.

$e = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$e = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)}$
$e = \frac{-4 \pm \sqrt{16 - 24}}{2}$
$e = \frac{-4 \pm \sqrt{-8}}{2}$ $\qquad \sqrt{-8} = \sqrt{8} \times i = \sqrt{4 \cdot 2} \times i = 2i\sqrt{2}$
$e = \frac{-4 \pm 2i\sqrt{2}}{2}$
$e = -2 \pm i\sqrt{2}$
$e = -2 + i\sqrt{2}$ or $e = -2 - i\sqrt{2}$

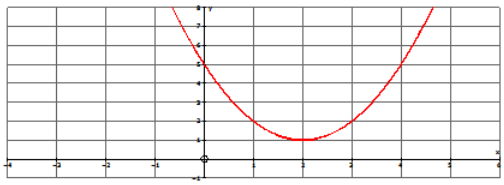
There are no real solutions to the equation. The solutions to the equation are

$-2 + i\sqrt{2}$ and $-2 - i\sqrt{2}$

Finally, let's sketch the graph of the following quadratic function. What are the roots of this function?

$y = x^2 - 4x + 5$

Use your calculator or a table to make a sketch of the function. You should get the following:



As you can see, the quadratic function has no x-intercepts; therefore, the function has no real roots. To find the roots (which will be complex), you must use the quadratic formula.

For this quadratic function, $a = 1, b = -4, c = 5$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$
$x = \frac{4 \pm \sqrt{16 - 20}}{2}$
$x = \frac{4 \pm \sqrt{-4}}{2}$ $\qquad \sqrt{-4} = \sqrt{4} \times i = 2i$
$x = \frac{4 \pm 2i}{2}$
$x = 2 \pm i$
$x = 2 + i$ or $x = 2 - i$

The complex roots of the quadratic function are $2 + i$ and $2 - i$.

Examples

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Example 1

Earlier, you were asked to find the complex roots of $y = x^2 - 2x + 3$.

Solution

To find the complex roots of the function $y = x^2 - 2x + 3$, you must use the quadratic formula.

For this quadratic function, $a = 1, b = -2, c = 3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$
$$x = \frac{2 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \times i = 2\sqrt{2}i$$
$$x = \frac{2 \pm 2\sqrt{2}i}{2}$$
$$x = 1 \pm \sqrt{2}i$$

Example 2

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$2n^2 + n = -4$$

Solution

Set the equation equal to zero.

$$2n^2 + n + 4 = 0$$

Solve using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$n = \frac{-(1) \pm \sqrt{(1)^2 - 4(2)(4)}}{2(2)}$$
$$n = \frac{-1 \pm \sqrt{1 - 32}}{4}$$
$$n = \frac{-1 \pm \sqrt{-31}}{4}$$
$$n = \frac{-1 \pm i\sqrt{31}}{4}$$

Example 4.5.3.1

Solve the following quadratic equation. Express all solutions in simplest radical form.

$$m^2 + (m+1)^2 + (m+2)^2 = -1$$

Solution

Expand and simplify.

$$m^2 + (m+1)(m+1) + (m+2)(m+2) = -1$$
$$m^2 + m^2 + m + m + 1 + m^2 + 2m + 2m + 4 = -1$$
$$3m^2 + 6m + 5 = -1$$

Write the equation in general form.

$$3m^2 + 6m + 6 = 0$$

Divide by 3 to simplify the equation.

$$\frac{3m^2}{3} + \frac{6m}{3} + \frac{6}{3} = \frac{0}{3}$$
$$m^2 + 2m + 2 = 0$$

Solve using the quadratic formula:

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$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$m = \frac{-(2) \pm \sqrt{(2)^2-4(1)(2)}}{2(1)}$

$m = \frac{-2 \pm \sqrt{4-8}}{2}$

$m = \frac{-2 \pm \sqrt{-4}}{2}$

$m = \frac{-2 \pm 2i}{2}$

$m = -1 \pm i$

Example 4

Is it possible for a quadratic function to have exactly one **complex root**?

Solution

No, even in higher degree polynomials, complex roots will always come in pairs. Consider when you use the quadratic formula-- if you have a negative under the square root symbol, both the + version and the - version of the two answers will end up being complex.

Review

1. If a quadratic function has 2 x-intercepts, how many complex roots does it have? Explain.
2. If a quadratic function has no x-intercepts, how many complex roots does it have? Explain.
3. If a quadratic function has 1 x-intercept, how many complex roots does it have? Explain.
4. If you want to know whether a function has complex roots, which part of the quadratic formula is it important to focus on?
5. You solve a quadratic equation and get 2 complex solutions. How can you check your solutions?
6. In general, you can attempt to solve a quadratic equation by graphing, factoring, completing the square, or using the quadratic formula. If a quadratic equation has complex solutions, what methods do you have for solving the equation?

Solve the following quadratic equations. Express all solutions in simplest radical form.

7. $x^2 + x + 1 = 0$
8. $5y^2 - 8y = -6$
9. $2m^2 - 12m + 19 = 0$
10. $-3x^2 - 2x = 2$
11. $2x^2 + 4x = -11$
12. $-x^2 + x - 23 = 0$
13. $-3x^2 + 2x = 14$
14. $x^2 + 5 = -x$
15. $\frac{1}{2}d^2 + 4d = -12$

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.7.

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Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.
complex root	A complex root is a complex number that, when used as an input (x) value of a function, results in an output (y) value of zero.
Imaginary Numbers	An imaginary number is a number that can be written as the product of a real number and i.
Quadratic Formula	The quadratic formula states that for any quadratic equation in the form $ax^2+bx+c=0, x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.
Real Number	A real number is a number that can be plotted on a number line. Real numbers include all rational and irrational numbers.

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4.5.4: Products and Quotients of Complex Numbers

Multiplying and Dividing Complex Numbers

Mr. Marchez draws a triangle on the board. He labels the height $(2+3i)$ and the base $(2-4i)$. "Find the area of the triangle," he says. (Recall that the area of a triangle is $A=\frac{1}{2}bh$, b is the length of the base and h is the length of the height.)

Multiplying and Dividing Complex Numbers

When multiplying complex numbers, FOIL the two numbers together and then combine like terms. At the end, there will be an i^2 term. Recall that $i^2=-1$ and continue to simplify.





Let's simplify the following expressions.

1. Simplify $6i(1-4i)$.

   Distribute the $6i$ to both parts inside the parenthesis.

   Substitute $i^2=-1$ and simplify further.

   $$=6i-24(-1)$$
   $$=24+6i$$

   Remember to always put the real part first.

   $(5-2i)(3+8i)$

   FOIL the two terms together.

   $$(5-2i)(3+8i)=15+40i-6i-16i^2$$
   $$=15+34i-16i^2$$

Substitute $i^2 = -1$ and simplify further.

$$= 15 + 34i - 16(-1)$$
$$= 15 + 34i + 16$$
$$= 31 + 34i$$

Dividing complex numbers is a bit more complicated. Similar to irrational numbers, complex numbers cannot be in the denominator of a fraction. To get rid of the **complex number** in the denominator, we need to multiply by the **complex conjugate**. If a complex number has the form $a + bi$, then its complex conjugate is $a - bi$. For example, the complex conjugate of $-6 + 5i$ would be $-6 - 5i$. Therefore, rather than dividing complex numbers, we multiply by the complex conjugate.

2. Simplify $\frac{8-3i}{6i}$.

In the case of dividing by a pure imaginary number, you only need to multiply the top and bottom by that number. Then, use multiplication to simplify.

$$\frac{8-3i}{6i} \cdot \frac{6i}{6i} = \frac{48i - 18i^2}{36i^2}$$
$$= \frac{18 + 48i}{-36}$$
$$= \frac{18}{-36} + \frac{48}{-36}i$$
$$= -\frac{1}{2} - \frac{4}{3}i$$

When the complex number contains fractions, write the number in standard form, keeping the real and imaginary parts separate. Reduce both fractions separately.

3. Simplify $\frac{3-5i}{2+9i}$.

Now we are dividing by $2 + 9i$, so we will need to multiply the top and bottom by the complex conjugate, $2 - 9i$.

$$\frac{3-5i}{2+9i} \cdot \frac{2-9i}{2-9i} = \frac{6 - 27i - 10i + 45i^2}{4 - 18i + 18i - 81i^2}$$
$$= \frac{6 - 37i - 45}{4 + 81}$$
$$= \frac{-39 - 37i}{85}$$
$$= -\frac{39}{85} - \frac{37}{85}i$$

Notice, by multiplying by the complex conjugate, the denominator becomes a real number and you can split the fraction into its real and imaginary parts.

In the previous three problems above, we substituted $i^2 = -1$ to simplify the fraction further.

Your final answer should never have any power of i greater than 1.

Examples

Example 1

Earlier, you were asked to find the area of the triangle.

Solution

The area of the triangle is $\frac{(2+3i)(2-4i)}{2}$ so FOIL the two terms together and divide by 2.

$$(2 + 3i)(2 - 4i) = 4 - 8i + 6i - 12i^2$$
$$= 4 - 2i - 12i^2$$

Substitute $i^2 = -1$ and simplify further.

$$= 4 - 2i - 12(-1)$$
$$= 4 - 2i + 12$$
$$= 16 - 2i$$

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Now divide this product by 2.

$\frac{16-2i}{2} = 8 - i$

Therefore the area of the triangle is $8 - i$.

Example 2

What is the complex conjugate of $7 - 5i$?

Solution

$7 + 5i$

Example 3

Simplify the following complex expression: $(7 - 4i)(6 + 2i)$.

Solution

FOIL the two expressions.

$$\begin{aligned}(7-4i)(6+2i) &= 42 + 14i - 24i - 8i^2 \\ &= 42 - 10i + 8 \\ &= 50 - 10i\end{aligned}$$

Example 4

Simplify the following complex expression: $\frac{10-i}{5i}$.

Solution

Multiply the numerator and denominator by $5i$

$$\begin{aligned}\frac{10-i}{5i} \cdot \frac{5i}{5i} &= \frac{50i - 5i^2}{25i^2} \\ &= \frac{5 + 50i}{-25} \\ &= \frac{5}{-25} + \frac{50}{-25}i \\ &= -\frac{1}{5} - 2i\end{aligned}$$

Example 5

Simplify the following complex expression: $\frac{8+i}{6-4i}$.

Solution

Multiply the numerator and denominator by the complex conjugate, $6 + 4i$.

$$\begin{aligned}\frac{8+i}{6-4i} \cdot \frac{6+4i}{6+4i} &= \frac{48 + 32i + 6i + 4i^2}{36 + 24i - 24i - 16i^2} \\ &= \frac{48 + 38i - 4}{36 + 16} \\ &= \frac{44 + 38i}{52} \\ &= \frac{44}{52} + \frac{38}{52}i \\ &= \frac{11}{13} + \frac{19}{26}i\end{aligned}$$

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Review

Simplify the following expressions. Write your answers in standard form.

1. $i(2-7i)$
2. $8i(6+3i)$
3. $-2i(11-4i)$
4. $(9+i)(8-12i)$
5. $(4+5i)(3+16i)$
6. $(1-i)(2-4i)$
7. $4i(2-3i)(7+3i)$
8. $(8-5i)(8+5i)$
9. $\frac{4+9i}{3i}$
10. $\frac{4+9i}{3i}$
11. $\frac{7+12i}{-5i}$
12. $\frac{4-2i}{6-6i}$
13. $\frac{2-i}{2+i}$
14. $\frac{10+8i}{2+4i}$
15. $\frac{14+9i}{7-20i}$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 5.9.

Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form $a+bi$.

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4.5.5: Polar Form of Complex Numbers

Polar Form of Complex Numbers

Complex numbers can be graphed on a polar graph just like real numbers can. You will discover during this lesson that there are actually a few different ways of doing this.

Polar Form of Complex Numbers

You have learned that rectangular graphs can be put into polar form, and that points in rectangular coordinates can be plotted in the **polar coordinate system**. In this section you will learn how to do the same process with complex numbers.

There are three common forms of complex numbers that you will see when graphing:

a. In the standard form of: $z = a + bi$, a **complex number** z can be graphed using rectangular coordinates (a, b). **'a'** represents the x - coordinate, while **'b'** represents the y - coordinate.

b. The polar form: (r, θ) which we explored in a previous lesson, can also be used to graph a complex number. Recall that you can use x and y to convert between rectangular and polar forms with: $r = \sqrt{x^2 + y^2}$ and $\tan\theta_{ref} = \left|\frac{y}{x}\right|$. Unfortunately, there is a problem with using a conversion from rectangular form to polar form like:

$$a + bi \rightarrow (r, \theta)$$

or

$$-1 - i\sqrt{3} \rightarrow \left(2, \tfrac{4\pi}{3}\right)$$

The problem is that we have lost the i. So, in order to "keep track" of the imaginary part, we can use another form.

c. The third form is **trigonometric form**. It is often abbreviated as ***rcisθ***, short for: z = **r**(**c**osθ + **is**inθ), and will be used quite often as you progress. This form comes from the substitutions:

$x = r \cos\theta$ and $y = r \sin\theta$.

Using this fact, and sample values of 2 for r and $\frac{\pi}{3}$ for θ, we can write

$$z = -1 - i\sqrt{3} = 2\cos\tfrac{4\pi}{3} + 2i\sin\tfrac{4\pi}{3}$$

Finally, factoring the 2, we get: $z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

Summary of Forms

The complex number: $z = -1 - \sqrt{3}i$, the rectangular point $(-1, -\sqrt{3})$, the polar point: $\left(2, \frac{4\pi}{3}\right)$, and $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ or $2\,\mathrm{cis}\left(\frac{4\pi}{3}\right)$ all represent the same number.

Steps for Conversion

To convert from polar to rectangular form, the distance that the point (2, 2) is from the origin can be found by

$$d = \sqrt{x^2 + y^2} \text{ or } \sqrt{2^2 + 2^2} d = \sqrt{8} \text{ or } 2\sqrt{2}$$

The **reference angle** (i.e. the corresponding angle in the first **quadrant**) that the line segment between the point and the origin can be found by

$$\tan\theta_{ref} = \left|\frac{y}{x}\right|$$

for $z = 2 + 2i$,

$$\tan\theta_{ref} = \frac{2}{2}$$

$$\tan\theta_{ref} = 1.$$

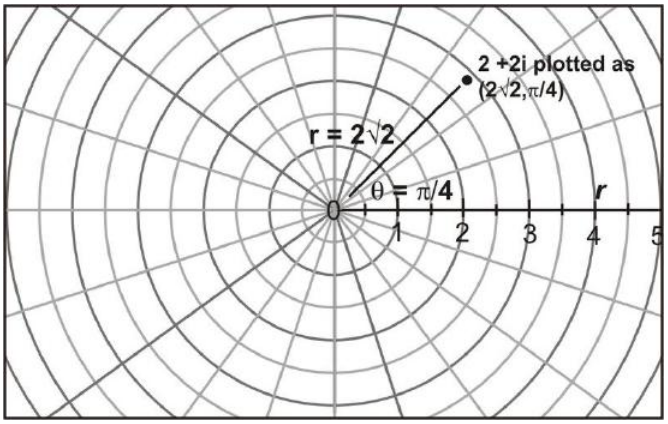
Since this point is in the first quadrant (both the x and y coordinate are positive) the angle must be 45° or $\frac{\pi}{4}$ radians.

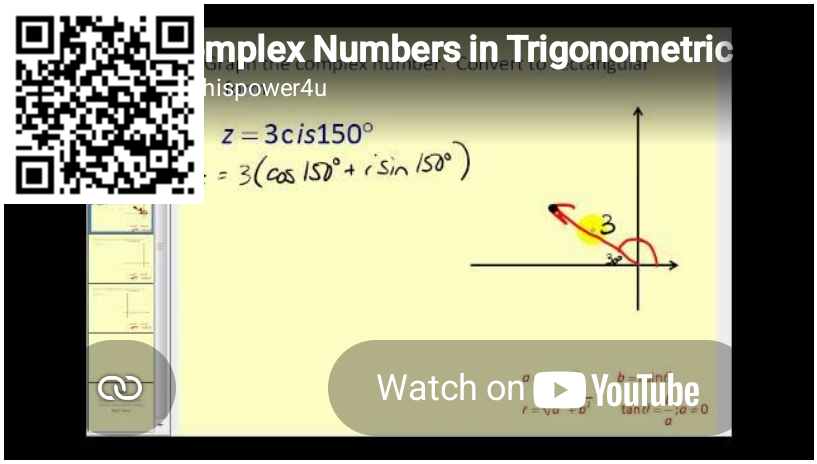
It is also possible that when $\tan\theta = 1$ the angle can be in the third quadrant or $\frac{5\pi}{4}$ radians. But this angle will not satisfy the conditions of the problem, since a third quadrant angle must have both x and y as negatives.

Note

When using $\tan\theta = \frac{y}{x}$, you should first consider, the quotient $\left|\frac{y}{x}\right|$ and find the first quadrant angle that satisfies this condition. This angle will be called the **reference angle,** denoted θ_{ref}. Find the actual angle by analyzing which quadrant the angle must be given the *x* and *y* signs.

The complex number 2 + 2*i* or (2, 2) in rectangular form has polar coordinates $\left(2\sqrt{2}, \frac{\pi}{4}\right)$





Examples

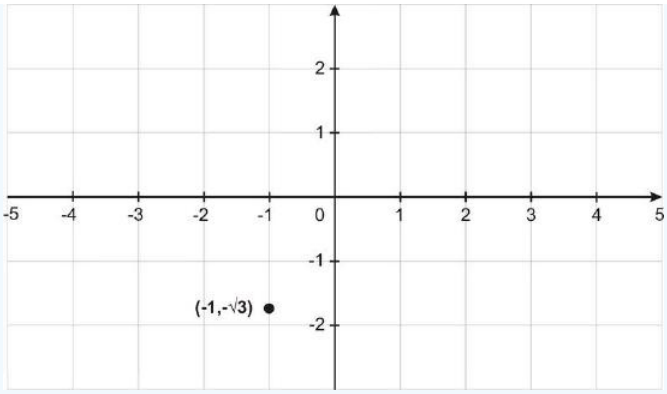
Example 1

Graph in polar form: $z = -1 - i\sqrt{3}$.

Solution

Here is what it looks like in the rectangular coordinate system:

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In polar form, we find r with

$$r = \sqrt{a^2 + b^2}$$
$$= \sqrt{(-1)^2 + (-\sqrt{3})^2}$$
$$= \sqrt{1 + 3}$$
$$= \sqrt{4}$$
$$= 2$$

and to find θ,

$$\tan\theta_{ref} = \left|\frac{-\sqrt{3}}{-1}\right|$$
$$\tan\theta_{ref} = \sqrt{3}$$
$$\theta_{ref} = \tan^{-1}\sqrt{3}$$
$$\theta_{ref} = \frac{\pi}{3}$$

Since this angle is in the 4th quadrant, $\theta = \frac{4\pi}{3}$.

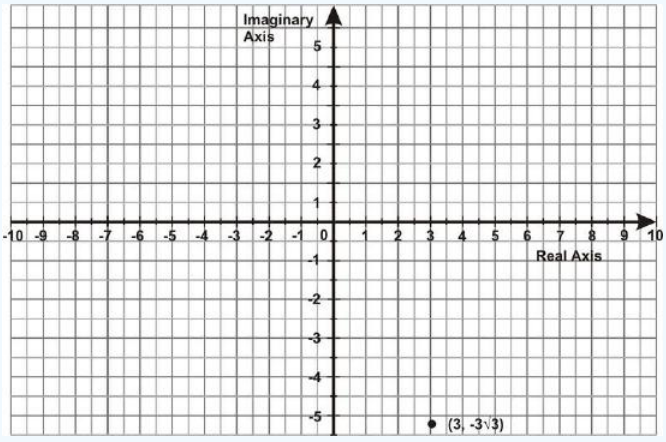
Example 2

Find the polar coordinates that represent the complex number $z = 3 - 3\sqrt{3}i$.

Solution

$a = 3$ and $b = -3\sqrt{3}$: the rectangular coordinates of the point are $(3, -3\sqrt{3})$.

Now, draw a right triangle in standard form. Find the distance the point is from the origin and the angle the line segment that represents this distance makes with the +x axis:



We know $a = 3, b = -3\sqrt{3}$

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$r = \sqrt{3^2 + (-3\sqrt{3})^2}$
$= \sqrt{9 + 27}$
$= \sqrt{36}$
$= 6$

And for the angle,

$\tan\theta_{ref} = \left|\frac{(-3\sqrt{3})}{3}\right|$
$\tan\theta_{ref} = \sqrt{3}$
$\theta_{ref} = \frac{\pi}{3}$

But, since it is a 4th quadrant angle

$\theta = \frac{5\pi}{3}$

The rectangular point $(3, -3\sqrt{3}i)$ is equivalent to the polar point $\left(6, \frac{5\pi}{3}\right)$.

In rcisθ form, $(3, -3\sqrt{3}i)$ is $6\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

Example 3

Convert the following complex numbers into polar form, use a TI-84 equivalent graphing calculator:

a. $\sqrt{3} - i$
b. $9\sqrt{3} + 9i$

Solution

On the TI-84: go to **[ANGLE]** (or **[2nd]** function) **[APPS]**. Scroll down to 5 or "R-Pr(" and press **[Enter]** . Next, enter the rectangular coordinates and close the parenthesis. Press **[Enter]**, the "r" value appears. Scroll down to 6R-Pθ, and the polar angle appears in decimal radian form.

Note: Also under the **[ANGLE]** menu, commands 7 and 8 allow transformation from polar form to rectangular form.

Example 4

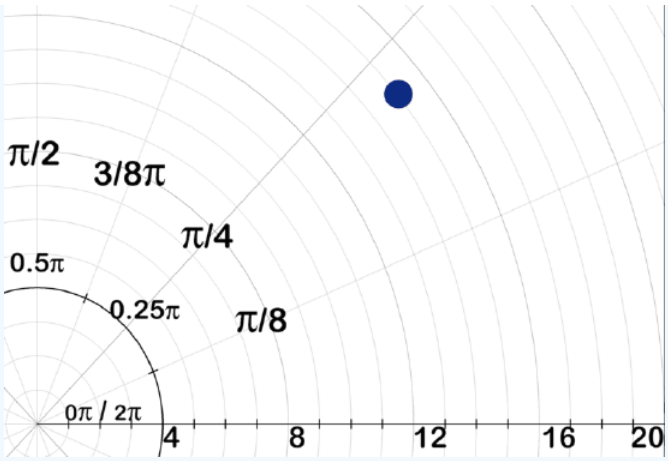
Plot the complex number $z = 12 + 9i$.

a. What is needed in order to plot this point on the polar plane?
b. How could the r-value be determined?
c. What is the r for this point?
d. How could θ be determined?
e. What is θ for this point?
f. What would $z = 12 + 9i$ look like on the polar plane?

Solution

a. First, we will need to know r and θ.
b. The r value is the hypotenuse of a triangle with two other sides, A=12 and B=9. It can be determined with the Pythagorean theorem: $A^2+B^2=C^2$.
c. The r value for this point is $\sqrt{144 + 81} \rightarrow \sqrt{225} = 15$.
d. θ can be calculated using either $\sin\theta = \frac{9}{15}$ or $\cos\theta = \frac{12}{15}$.
e. For this point, $\sin\theta = \frac{3}{5} \rightarrow 37°$ or $\cos\theta = \frac{4}{5} \rightarrow 37°$.
f. $z = 12 + 9i$ looks like the image below when plotted on a polar plane.

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Example 5

What quadrant does $z = -3 + 2i$ occur in when graphed?

Solution

The point $z = -3 + 2i$ occurs 3 units to the *left* and 2 units *up*, placing it in Quadrant II.

Example 6

What are the coordinates of $z = -3 + 2i$ in polar form and trigonometric form?

Solution

To identify the coordinates of $z = -3 + 2i$ in polar form and trigonometric form:

$r = \sqrt{(-3^2) + (2^2)} \rightarrow \sqrt{13}$ First find r

$\sin\theta = \frac{2}{\sqrt{13}} \rightarrow 33.7°$ Second, find θ

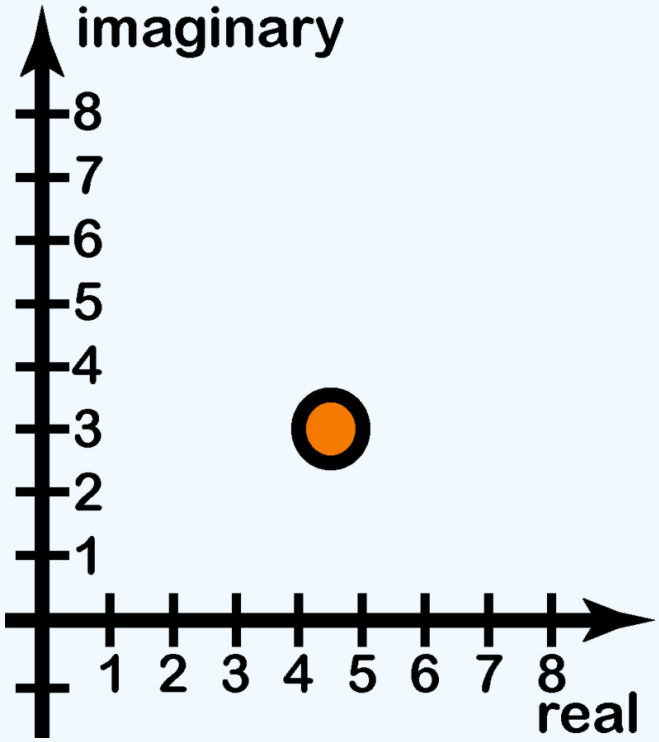
$\therefore \left[\sqrt{13}, 33.7°\right]$ are the coordinates in polar form.

\therefore \operatorname{rcis} \sqrt{13}\left(\frac{\pi}{5}\right) are the coordinates in rcis form

Example 7

What would be the polar coordinates of the point graphed below?

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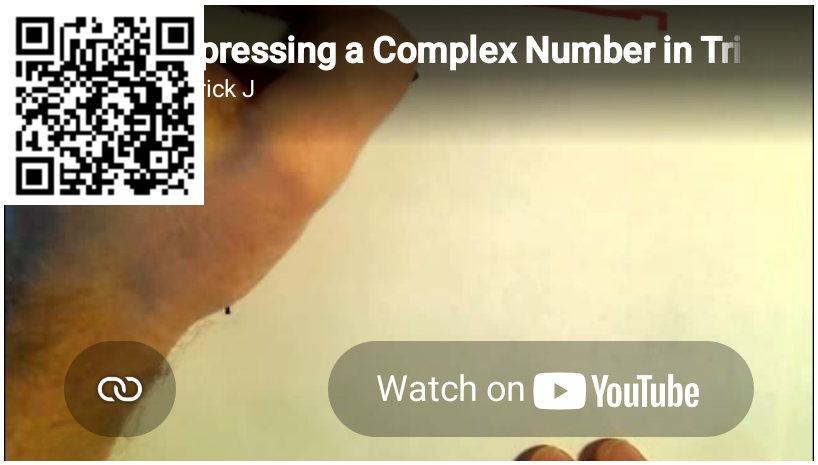
Solution

The rectangular coordinates are $(4.5, 3i)$ therefore the complex number would be $z = 4.5 + 3i$

r=5.4 Using the Pythagorean Theorem as in Q #3

$\theta = 33.75°$ Using $\sin = \frac{opp}{hyp}$ as in Q #3

∴[5.4,33.65°] is the point in polar form $\therefore \text{rcis}\, 5.4\left(\frac{\pi}{5}\right)$ are the coordinates in $rcis$ form



Review

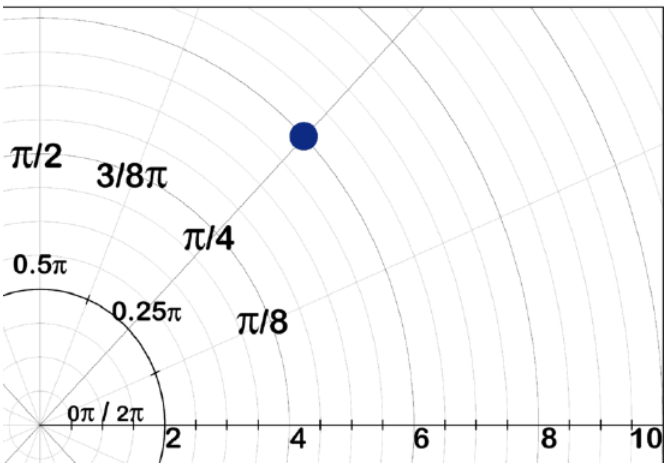
Plot each complex number in the complex plane. Find its polar form, $[r, \theta]$ and give the argument θ in degrees.

1. a. $1 + i$
   b. i
   c. $(1 + i)i$
2. a. -2

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b. $3i$
c. $(-2)(3i)$

3. a. $1+i$
   b. $1-i$
   c. $(1+i)(1-i)$
4. a. $1+i\sqrt{3}$
   b. $\sqrt{3}-i$
   c. $(1+i\sqrt{3})(\sqrt{3}-i)$
5. What are the rectangular coordinates for the point graphed below?



Compute and convert to $rcis$ form.

6. $\frac{-2-2i}{1-i}$
7. $1+i^6$
8. $\frac{\sqrt{3}}{2}+\frac{1}{2}i^{10}$

Change to polar form.

9. $-3-2i$
10. $2\sqrt{3}-2i$

Change to rectangular form.

11. $15\,(\cos 120^\circ + i\sin 120^\circ)$
12. $12\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
13. For the complex number in standard form $x+iy$ find:
    a. Polar form
    b. Trigonometric form

    (Hint: Recall that $x=r\cos\theta$ and $y=r\sin\theta$)

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.8.

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Vocabulary

Term	Definition
rcisθ	rcisθ is shorthand for the expression rcosθ + risinθ.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form a+bi.
polar coordinate system	The polar coordinate system is a special coordinate system in which the location of each point is determined by its distance from the pole and its angle with respect to the polar axis.
polar form	The polar form of a point or a curve is given in terms of r and θ and is graphed on the polar plane.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
Reference Angle	A reference angle is the angle formed between the terminal side of the angle and the closest of either the positive or negative x-axis.
trigonometric form	To write a complex number in trigonometric form means to write it in the form rcosθ + risinθ. rcisθ is shorthand for this expression.
trigonometric polar form	To write a complex number in trigonometric form means to write it in the form rcosθ + risinθ. rcisθ is shorthand for this expression.

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4.5.6: Product and Quotient Theorems

Product and Quotient Theorems

Complex numbers are found in real world calculations involving: quantum mechanics, signal analysis, fluid dynamics, control theory, and many other fields.

In electrical engineering, complex numbers are used for calculations involving *impedance* (the resistance to electric flow in a circuit).

Electrical engineers are familiar with the formula:

$$V = V_0 e^{j\omega t} = V_0(\cos \omega t + j \sin \omega t)$$

by comparing it to the similar expression below, which is explored in this lesson, can you identify the variable *j*?

$$r_2 (\cos \theta_2 + i \sin \theta_2)$$

Product and Quotient Theorems

The Product Theorem

Since complex numbers can be transformed to polar form, the multiplication of complex numbers can also be done in polar form. Suppose we know $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

To multiply the two complex numbers in polar form:

$$\begin{aligned} & z_1 \cdot z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) \cdot r_2 (\cos \theta_2 + i \sin \theta_2) \\ & = r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\ & = r_1 r_2 \cdot (\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2) \\ & = r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ & = r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2) \\ & = r_1 r_2 ([\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] + i [\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2]) \end{aligned}$$

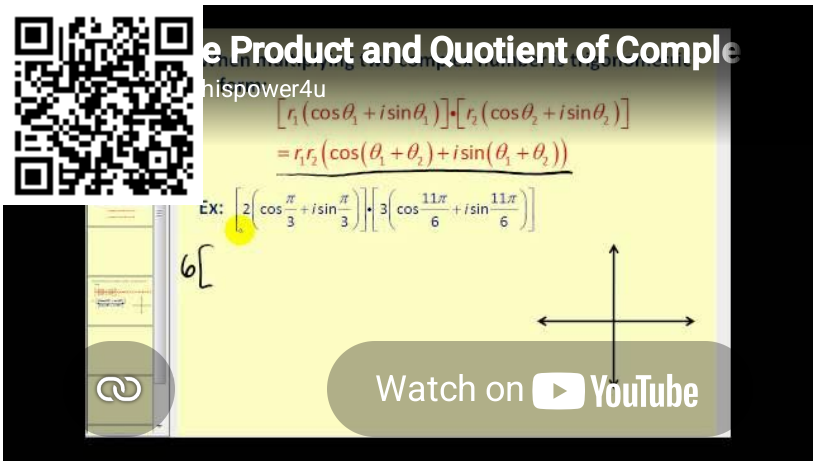
(Use $i^2 = -1$, gather like terms, factor out i, substitute the angle sum formulas for both sine and cosine)

$$z_1 \cdot z_2 = r_1 r_2 \operatorname{cis} [(\theta_1 + \theta_2)]$$

This last equation states that the product of two complex numbers in polar form can be obtained by multiplying the polar r values of each of the complex numbers and then multiplying that value by cis of the sum of each of the two angles of the individual complex numbers. This is more concise than the **rectangular form** for multiplication of complex numbers.

The Quotient Theorem

Dividing complex numbers in polar form can be shown using a similar proof that was used to show multiplication of complex numbers. Here we omit the proof and give the result. For $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \times \operatorname{cis} [\theta_1 - \theta_2]$



Examples

Example 1

Earlier, you were asked to identify the variable *j* in the following formula:

$$V = V_0 e^{j\omega t} = V_0(\cos\omega t + j\sin\omega t)$$

Solution

In electrical calculations, the letter *I* is commonly used to denote *current*, therefore imaginary numbers are identified with a *j*.

Note the similar usage of *i* in r(cosθ+isinθ).

Example 2

Multiply $z_1 \cdot z_2$ where $z_1 = 2 + 2i$ and $z_2 = 1 - \sqrt{3}i$.

Solution

For z_1,

$$r_1 = \sqrt{2^2 + 2^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

and

$$\tan\theta_1 = \frac{2}{2}$$
$$\tan\theta_1 = 1$$
$$\theta_1 = \frac{\pi}{4}$$

Note that θ_1 is in the first quadrant since *a*, and *b* > 0.

For z_2,

$$r_2 = \sqrt{1^2 + (-\sqrt{3})^2}$$
$$= \sqrt{1+3}$$
$$= \sqrt{4}$$
$$= 2$$

and

$$\tan\theta_2 = \frac{-\sqrt{3}}{1}$$
$$\theta_2 = \frac{5\pi}{3}$$

Now we can use the formula $z_1 \cdot z_2 = r_1 \cdot r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Substituting gives:

$z_1 \cdot z_2 = 2\sqrt{2} \times 2 \operatorname{cis}\left[\frac{\pi}{4} + \frac{5\pi}{3}\right]$
$= 4\sqrt{2} \operatorname{cis}\left[\frac{23\pi}{12}\right]$

So we have

$z_1 \cdot z_2 = 4\sqrt{2}\left(\cos\frac{23\pi}{12} + i \sin\frac{23\pi}{12}\right)$

Re-writing in approximate decimal form:

$5.656(0.966 - 0.259i)$
$5.46 - 1.46i$

If the problem was done using only rectangular units then

$z_1 \times z_2 = (2 + 2i)(1 - \sqrt{3}i)$ or
$= 2 - 2\sqrt{3}i + 2i - 2\sqrt{3}i^2$

Gathering like terms and using $i^2 = -1$

$= (2 + 2\sqrt{3}) - (2\sqrt{3} + 2)i$

or

$5.46 - 1.46i$

Example 2

Using polar multiplication, find the product $(6 - 2\sqrt{3}i)(4 + 4\sqrt{3}i)$.

Solution

Let $z_1 = 6 - 2\sqrt{3}i$ and $z_2 = 4 + 4\sqrt{3}i$

$r_1 = \sqrt{(6)^2 - (2\sqrt{3})^2}$ and $r_2 = \sqrt{(4)^2 + (4\sqrt{3})^2}$
$r_1 = \sqrt{36 + 12} = \sqrt{48} = 4\sqrt{3}$ and $r_2 = \sqrt{16 + 48} = \sqrt{64} = 8$

For θ_1, first find $\tan\theta_{ref} = \left|\frac{y}{x}\right|$

$\tan\theta_{ref} = \frac{(2\sqrt{3})}{6}$
$\tan\theta_{ref} = \frac{\sqrt{3}}{3}$
$\theta_{ref} = \frac{\pi}{6}$

Since $x > 0$ and $y < 0$ we know that θ_1 is in the in the 4th quadrant:

$\theta_1 = \frac{11\pi}{6}$

For θ_2,

$\tan\theta_{ref} = \frac{(4\sqrt{3})}{4}$
$\tan\theta_{ref} = \sqrt{3}$
$\theta_{ref} = \frac{\pi}{3}$

Since θ_2 is in the first quadrant,

$\theta_2 = \frac{\pi}{3}$

Using polar multiplication,

$z_1 \times z_2 = 4\sqrt{3} \times 8\left(\operatorname{cis}\left[\frac{11\pi}{6} + \frac{\pi}{3}\right]\right)$
$z_1 \times z_2 = 32\sqrt{3}\left(\operatorname{cis}\left[\frac{13\pi}{6}\right]\right)$

subtracting 2π from the augment:

z_{1} \times z_{2}=32 \sqrt{3}\left(\operatorname{cis}\left[\frac{\pi}{6}\right]\right)

or in expanded form: $32\sqrt{3}\left(\cos\left[\frac{\pi}{6}\right]+i\sin\left[\frac{\pi}{6}\right]\right)$

In decimal form this becomes: $55.426(0.866+0.500i)$ or $48+27.713i$

Check:

$(6-2\sqrt{3}i)(4+4\sqrt{3}i)=24+24\sqrt{3}i-8\sqrt{3}i-24i^2$
$=24+16\sqrt{3}i+24$
$=48+27.713i$

Example 3

Using polar division, find the quotient of $\frac{z_1}{z_2}$ given that $z_1=5-5i$ and $z_2=-2\sqrt{3}-2i$.

Solution

For $z_1: r_1=\sqrt{5^2+(-5)^2}$ and $\tan\theta_1=\frac{-5}{5}$, so $\theta_1=\frac{7\pi}{4}\left(4^{th}\text{ quadrant}\right)$

For $z_2: r_2=\sqrt{(-2\sqrt{3})^2+(-2)^2}$ or $\sqrt{16}=4$ and $\tan\theta_2=\frac{-2}{(-2\sqrt{3})}$, so $\theta_2=\frac{7\pi}{6}\left(3^{rd}\text{ quadrant}\right)$

Using the formula, $\frac{z_1}{z_2}=\frac{r_1}{r_2}\times\operatorname{cis}\left[\theta_1-\theta_2\right]$ or

$=\frac{5\sqrt{2}}{4}\times\operatorname{cis}\left[\frac{7\pi}{4}-\frac{7\pi}{6}\right]$
$=\frac{5\sqrt{2}}{4}\times\operatorname{cis}\left[\frac{7\pi}{12}\right]$
$=\frac{5\sqrt{2}}{4}\left[\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right]$
$=1.768[-0.259+(0.966)i]$
$=-0.458+1.708i$

Check by using the **complex conjugate** to do the division in rectangular form:

$\frac{5-5i}{-2\sqrt{3}-2i}\cdot\frac{-2\sqrt{3}+2i}{-2\sqrt{3}+2i}=\frac{-10\sqrt{3}+10i+10\sqrt{3}i-10i^2}{(-2\sqrt{3})^2-(2i)^2}$
$=\frac{-10\sqrt{3}+10i+10\sqrt{3}i+10}{12+4}$
$=\frac{(-10\sqrt{3}+10)+(10+10\sqrt{3})i}{16}$
$=\frac{(-17.3+10)+(10+17.3)i}{16}$
$=\frac{(-7.3)+(27.3)i}{16}$ or
$-0.456+1.706i$

The two radically different approaches yield the same answer. The small difference between the two answers is a result of decimal rounding.

Example 5

Find the product: $\left(7\left(\frac{\pi}{6}\right)\right)\cdot\left(5\left(\frac{-\pi}{4}\right)\right)$.

Solution

This one is easier than it looks: Recall $z_1\cdot z_2=r_1\cdot r_2\operatorname{cis}(\theta_1+\theta_2)$.

$r_1\cdot r_2\rightarrow 7\cdot 5=35$..... By substitution and multplication

$\theta_1+\theta_2\rightarrow\left(\frac{\pi}{6}\right)+\left(\frac{-\pi}{4}\right)$..... Substitute

$\left(\frac{2\pi}{12}\right)+\left(\frac{-3\pi}{12}\right)$..... Find common denominators

$\left(\frac{-\pi}{12}\right)$..... Simplify

$\therefore 35\operatorname{cis}\left(\frac{-\pi}{12}\right)$ is the product

Example 6

Find the quotient: $\frac{1+2i}{2-i}$.

Solution

First, find the quotient by polar multiplication:

$r_1 = \sqrt{(1)^2 + (2)^2} = \sqrt{5} \qquad r_2 = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

$\tan\theta_1 = \frac{2}{1}$

$\tan\theta_1 = 2$

$\theta_{ref} = 1.107 \text{ radians}$

since the angle is in the 1st quadrant

$\theta_1 = 1.107$ radians

for θ_2,

$\tan\theta_2 = \frac{-1}{2}$

$\tan\theta_{ref} = \frac{1}{2}$

$\theta_{ref} = 0.464 \text{ radians}$

since θ_2 is in the 4th quadrant, between $4.712 \left(\text{ or } \frac{3\pi}{2} \right)$ and 6.282 radians (or 2π)

$\theta_2 = 5.820$ *radians*

Finally, using the division formula,

$\frac{z_1}{z_2} = \frac{\sqrt{5}}{\sqrt{5}}[\text{cis}(1.107 - 5.820)]$

$\frac{z_1}{z_2} = [\text{cis}(-4.713)]$

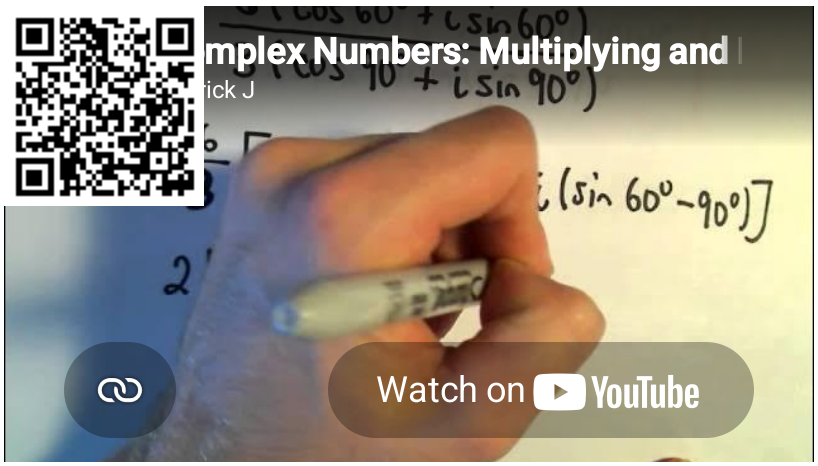
$\frac{z_1}{z_2} = [\cos(-4.713) + i\sin(-4.713)]$

$\frac{z_1}{z_2} = [\cos(1.570) + i\sin(1.570)]$

$\text{If we assume that } \frac{\pi}{2} = 1.570, \text{ then}$

$\approx \frac{z_1}{z_2} = \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$

$\frac{z_1}{z_2} = 0 + 1i = i$



Review

1. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$
2. $2\,\text{cis}(40) \cdot 4\,\text{cis}(20)$
3. Multiply: $2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) \cdot 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

4. $\frac{2\,\text{cis}(80)}{6\,\text{cis}(200)}$
5. Divide: $3\,\text{cis}\,(130°) \div 4\,\text{cis}\,(270°)$

If $z_1 = 7\left(\frac{\pi}{6}\right)$ and $z_2 = 5\left(\frac{-\pi}{4}\right)$ find:

6. $z_1 \cdot z_2$
7. $\left(\frac{z_1}{z_2}\right)$
8. $\left(\frac{z_2}{z_1}\right)$

If $z_1 = 8\left(\frac{\pi}{3}\right)$ and $z_2 = 5\left(\frac{\pi}{6}\right)$ find:

9. $z_1 z_2$
10. $\left(\frac{z_1}{z_2}\right)$
11. $\left(\frac{z_2}{z_1}\right)$
12. $(z_1)^2$
13. $(z_2)^3$

Find the products.

14. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$
15. $2\,(\cos 40° + i \sin 40°) \cdot 4\,(\cos 20° + i \sin 20°)$
16. $2\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right) \cdot 2\left(\cos\frac{\pi}{10} + i \sin\frac{\pi}{10}\right)$

Find the quotients.

17. $2\,(\cos 80° + i \sin 80°) \div 6\,(\cos 200° + i \sin 200°)$
18. $3\,\text{cis}\,(130°) \div 4\,\text{cis}\,(270°)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 4.9.

Vocabulary

Term	Definition
Complex Conjugate	Complex conjugates are pairs of complex binomials. The complex conjugate of a+bi is a−bi. When complex conjugates are multiplied, the result is a single real number.
complex number	A complex number is the sum of a real number and an imaginary number, written in the form a+bi.
rectangular form	The rectangular form of a point or a curve is given in terms of x and y and is graphed on the Cartesian plane.

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4.5.7: Powers and Roots of Complex Numbers

Powers and Roots of Complex Numbers

Manually calculating (simplifying) a statement such as: $(14-17i)^5$ or $\sqrt[4]{(3-2i)}$ in present (rectangular) form would be a very intensive process at best.

Fortunately you will learn in this lesson that there is an alternative: **De Moivre's theorem**. De Moivre's theorem is really the only practical method for finding the powers or roots of a **complex number**, but there is a catch...

What must be done to a complex number before De Moivre's theorem can be utilized?

Powers and Roots of Complex Numbers

Powers of Complex Numbers

How do we raise a complex number to a power? Let's start with an example:

$$(-4-4i)^3 = (-4-4i)\cdot(-4-4i)\cdot(-4-4i)$$

In rectangular form, this can get very complex. What about in r cis θ form?

$$(-4-4i) = 4\sqrt{2}\,\text{cis}\left(\tfrac{5\pi}{4}\right)$$

and using our multiplication rule from the previous section,

$$(-4-4i)^3 = (4\sqrt{2})^3\,\text{cis}\left(\tfrac{15\pi}{4}\right)$$

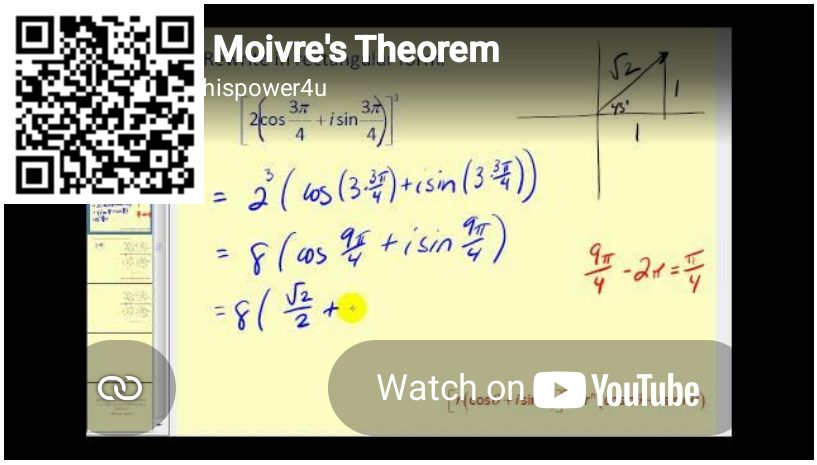
Notice, $(a+bi)^3 = r^3\ \text{cis}\ 3\theta$

In words: Raise the *r*-value to the same degree as the complex number is raised and then multiply that by *cis* of the angle multiplied by the number of the degree.

Reflecting on the example above, we can identify **De Moivre's Theorem**:

> **Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in rcisθ form. If n is a positive integer, z^n is $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$**

It should be clear that the polar form provides a much faster result for raising a complex number to a power than doing the problem in rectangular form.



Roots of Complex Numbers

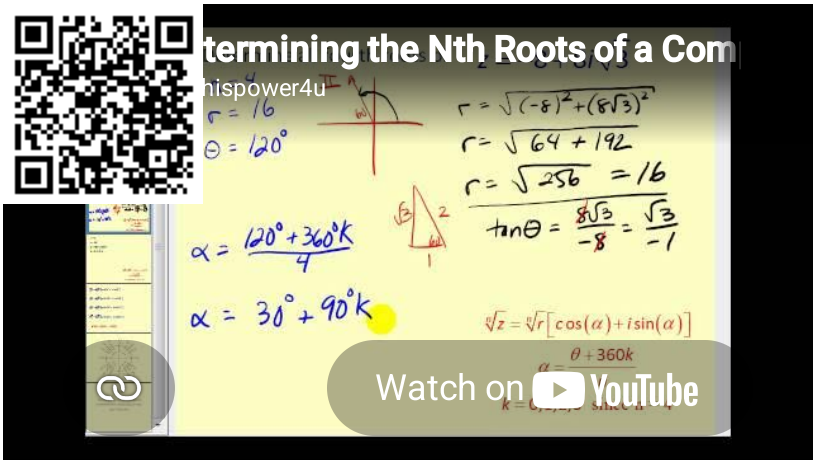
You probably noticed long ago that when an new operation is presented in mathematics, the inverse operation often follows. That is generally because the inverse operation is often procedurally similar, and it makes good sense to learn both at the same time.

This is no exception:

The inverse operation of finding a power for a number is to find a root of the same number.

a. Recall from algebra that any root can be written as $x^{1/n}$
b. Given that the formula for De Moivre's theorem also works for fractional powers, the same formula can be used for finding roots:

$$z^{1/n} = (a+bi)^{1/n} = r^{1/n}\,\text{cis}\left(\frac{\theta}{n}\right)$$



Examples

Example 1

Earlier, you were asked what should be done to a complex number before you can use De Moivre's theorem on it.

Solution

A complex number operation written in rectangular form, such as: $(13-4i)^3$ must be converted to polar form before utilizing De Moivre's theorem.

Example 2

Find the value of $(1+\sqrt{3}i)^4$.

Solution

$$r = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$$

$$\tan\theta_{ref} = \frac{\sqrt{3}}{1}$$

and θ is in the 1st quadrant, so

$$\theta = \frac{\pi}{3}$$

Using our equation from above:

$$z^4 = r^4\,\text{cis}\,4\theta$$
$$z^4 = (2)^4\,\text{cis}\,4\frac{\pi}{3}$$

Expanding cis form:

$$z^4 = 16\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$$
$$= 16((-0.5) - 0.866i)$$

Finally we have

$$z^4 = -8 - 13.856i$$

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Example 3

Find $\sqrt{1+i}$.

Solution

First, rewriting in exponential form: $(1+i)^{1/2}$

And now in polar form:

$\sqrt{1+i} = \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{1/2}$

Expanding cis form,

$= \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^{1/2}$

Using the formula:

$= \left(2^{1/2}\right)^{1/2}\left(\cos\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) + i\sin\left(\frac{1}{2}\cdot\frac{\pi}{4}\right)\right)$
$= 2^{1/4}\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)$

In decimal form, we get

$= 1.189(0.924 + 0.383i)$
$= 1.099 + 0.455i$

To check, we will multiply the result by itself in rectangular form:

$(1.099 + 0.455i)\cdot(1.099 + 0.455i) = 1.099^2 + 1.099(0.455i) + 1.099(0.455i) + (0.455i)^2$
$= 1.208 + 0.500i + 0.500i + 0.208i^2$
$= 1.208 + i - 0.208$ or
$= 1 + i$

Example 4

Find the value of $x : x^3 = (1 - \sqrt{3}i)$

Solution

First we put $1 - \sqrt{3}i$ in polar form.

Use $x = 1, y = -\sqrt{3}$ to obtain $r = 2, \theta = \frac{5\pi}{3}$

let $z = (1 - \sqrt{3}i)$ in rectangular form

$z = 2\,\text{cis}\left(\frac{5\pi}{3}\right)$ in polar form

$x = (1 - \sqrt{3}i)^{1/3}$

$x = \left[2\,\text{cis}\left(\frac{5\pi}{3}\right)\right]^{1/3}$

Use De Moivre's theorem to find the first solution:

$x_1 = 2^{1/3}\,\text{cis}\left(\frac{5\pi/3}{3}\right)$ or $2^{1/3}\,\text{cis}\left(\frac{5\pi}{9}\right)$

Leave answer in cis form to find the remaining solutions:

n = 3 which means that the 3 solutions are $\frac{2\pi}{3}$ radians apart or

$x_2 = 2^{1/3}\,\text{cis}\left(\frac{5\pi}{9} + \frac{2\pi}{3}\right)$ and $x_3 = 2^{1/3}\,\text{cis}\left(\frac{5\pi}{9} + \frac{2\pi}{3} + \frac{2\pi}{3}\right)$

Note

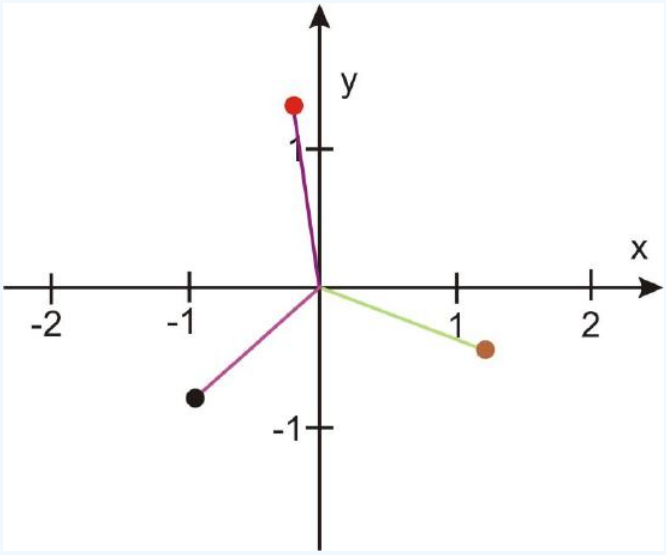
It is not necessary to add $\frac{2\pi}{3}$ again. Adding $\frac{2\pi}{3}$ three times equals 2π. That would result in rotating around a full circle and to start where it all began- that is the first solution.

The three solutions are:

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$x_1 = 2^{1/3}\,\text{cis}\left(\frac{5\pi}{9}\right)$
$x_2 = 2^{1/3}\,\text{cis}\left(\frac{11\pi}{9}\right)$
$x_3 = 2^{1/3}\,\text{cis}\left(\frac{17\pi}{9}\right)$

Each of these solutions, when graphed will be $\frac{2\pi}{3}$ apart.



Check any one of these solutions to see if the results are confirmed.

Checking the second solution:

$x_2 = 2^{1/3}\,\text{cis}\left(\frac{11\pi}{9}\right)$
$= 1.260\left[\cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)\right]$
$= 1.260[-0.766 - 0.643i]$
$= -0.965 - 0.810i$
Does $(-0.965 - 0.810i)^3$ or $(-0.965 - 0.810i)(-0.965 - 0.810i)(-0.965 - 0.810i)$
$= (1 - \sqrt{3}i)$?

Example 5

What are the two square roots of i?

Solution

Let $z = \sqrt{0 + i}$.

$r = 1, \theta = \pi/2$ or $z = \left[1 \times \text{cis}\,\frac{\pi}{2}\right]^{1/2}$ Utilizing De Moivre's theorem:

$z_1 = \left[1 \times \text{cis}\,\frac{\pi}{4}\right]$ or $z_2 = \left[1 \times \text{cis}\,\frac{5\pi}{4}\right]$

$z_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ or $z_2 = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_1 = 0.707 + 0.707i$ or $z_2 = -0.707 - 0.707i$

Check for z_1 solution: $(0.707 + 0.707i)^2 = i$?

$0.500 + 0.500i + 0.500i + 0.500i^2 = 0.500 + i + 0.500(-1)$ or i

Example 6

Calculate $\sqrt[4]{(1 + 0i)}$. What are the four fourth roots of 1?

Solution

Let $z = 1$ or $z = 1 + 0i$. Then the problem becomes find $z^{1/4} = (1 + 0i)^{1/4}$.

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Since $r = 1\theta = 0, z^{1/4} = [1 \times \text{cis}\, 0]^{1/4}$ with $z_1 = 1^{1/4}\left(\cos\frac{0}{4} + i\sin\frac{0}{4}\right)$ or $1(1+0)$ or 1

That root is not a surprise. Now use De Moivre's to find the other roots:

$z_2 = 1^{1/4}\left[\cos\left(0+\frac{\pi}{2}\right) + i\sin\left(0+\frac{\pi}{2}\right)\right]$ Since there are 4 roots, dividing 2π by 4 yields 0.5π or $0 + i$ or just i $z_3 = 1^{1/4}\left[\cos\left(0+\frac{2\pi}{2}\right) + i\sin\left(0+\frac{2\pi}{2}\right)\right]$ which yields $z_3 = -1$

Finally, $z_4 = 1^{1/4}\left[\cos\left(0+\frac{3\pi}{2}\right) + i\sin\left(0+\frac{3\pi}{2}\right)\right]$ or $z_4 = -i$

The four fourth roots of 1 are 1, i, -1 and -i.

Example 7

Calculate $(\sqrt{3}+i)^7$.

Solution

To calculate $(\sqrt{3}+i)^7$ start by converting to *rcis* form.

First, find *r*. Recall $r = \sqrt{\sqrt{3}^2 + 1^2}$.

$r = \sqrt{3+1}$
$r = 2$

If $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$ then $\theta = 30°$ and is in quadrant I. Now that we have trigonometric form, the rest is easy:

$(\sqrt{3}+i)^7 = [2(\cos 30° + i\sin 30°)]^7$..... Write the original problem in *rcis* form

$2^7[(\cos(7\cdot 30°) + i\sin(7\cdot 30°)]$..... De Moivre's theorem

$128\left[-\frac{\sqrt{3}}{2} + \frac{-1}{2}i\right]$..... Simplify

$(\sqrt{3}+i)^7 = -64\sqrt{3} - 64i$..... Simplify again

$\therefore (\sqrt{3}+i)^7 = -64\sqrt{3} - 64i$

Review

Perform the indicated operation on these complex numbers:

1. Divide: $\frac{2+3i}{1-i}$
2. Multiply: $(-6-i)(-6+i)$
3. Multiply: $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^2$
4. Find the product using polar form: $(2+2i)(\sqrt{3}-i)$
5. Multiply: $2(\cos 40° + i\sin 40°)\cdot 4(\cos 20° + i\sin 20°)$
6. Multiply: $2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\cdot 2\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$
7. Divide: $2(\cos 80° + i\sin 80°) \div 6(\cos 200° + i\sin 200°)$
8. Divide: $3\,\text{cis}(130°) \div 4\,\text{cis}(270°)$

Use De Moivre's theorem.

9. $[3(\cos 80° + i\sin 80°)]^3$
10. $\left[\sqrt{2}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)\right]^4$
11. $(\sqrt{3}-i)^6$
12. Identify the 3 complex cube roots of $1+i$
13. Identify the 4 complex fourth roots of $-16i$
14. Identify the five complex fifth roots of i

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Review (Answers)

To see the Review answers, open this PDF file and look for section 4.10.

Vocabulary

Term	Definition
complex number	A complex number is the sum of a real number and an imaginary number, written in the form a+bi.
De Moivre's Theorem	De Moivre's theorem is the only practical manual method for identifying the powers or roots of complex numbers. The theorem states that if $z=r(\cos\theta+i\sin\theta)$ is a complex number in rcisθ form and n is a positive integer, then $z^n=r^n(\cos(n\theta)+i\sin(n\theta))$.

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CHAPTER OVERVIEW

Unit 5: Vector Analysis

5.1: Two- and Three- Dimension Positions

5.1.1: Positions and Midpoints in Two Dimensions

5.1.2: Three-Dimensional Positions

5.2: Vector Calculations

5.2.1: Dot Products

5.2.2: Scalar Projections

5.2.3: Cross Products

5.2.4: Vector Projection

5.3: Planes in Space

5.3.1: Distance Between a Point and a Plane

5.3.2: Vector Direction

5.4: Vector Equations and Applications

5.4.1: Vector Equation of a Line

5.4.2: Vector Analysis Applications

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SECTION OVERVIEW

5.1: Two- and Three- Dimension Positions

Topic hierarchy

5.1.1: Positions and Midpoints in Two Dimensions

5.1.2: Three-Dimensional Positions

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5.1.1: Positions and Midpoints in Two Dimensions

Positions and Midpoints in Two Dimensions

A digital graphic artist is designing a new company logo. Currently she is sketching a sample on graph paper to show her client, and needs to find the center of the text she has drawn so she can line up the artwork properly.

If she knows that the first letter starts 10 boxes up and 5 boxes over from the bottom-left corner of the page, and the last letter ends 12 boxes up and 32 boxes over from the same corner, how could she find the center?

Positions and Midpoints in Two Dimensions

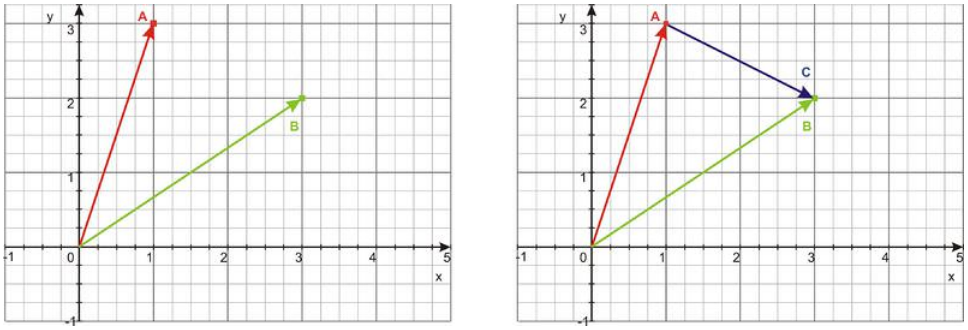
Using the Coordinate System Efficiently

In the past, as you used the coordinate system for graphing in algebra and geometry, you probably became very familiar with the *x* / *y* axes running left/right and up/down on the page, with *0* in the center. In more advanced mathematics, and in physics or other motion studies, you will find that it is often much simpler to instead move the graph to line up with one vector than to align all of the vectors you are calculating with the standard orientation.

By aligning one of multiple vectors with either the *x* or *y* axis, and/or setting the origin of your graph at the start of a vector, you minimize the complexity of your calculations.

Vectors Between Two Points

A ***displacement vector*** represents motion beginning at one point and ending at another. In the diagram below, vector C starts at point A and ends at point B. This means that $\vec{A}+\vec{C}=\vec{B}$ and $\vec{C}=\vec{B}-\vec{A}$. In this case, $\vec{A}=\langle 1,3\rangle$ and $\vec{B}=\langle 3,2\rangle$, therefore $\vec{C}=\langle (3-1),(2-3)\rangle=\langle 2,-1\rangle$ matching what we can see in the diagram.



Vector to a Point Between Two Points

Computer graphics artists frequently need to know the location of a point which lies midway between two other points. Once we know the position vectors for two discrete points, we can determine the **midpoint** between them using their coordinates. Specifically, the midpoint between points A and B is the "average" of the two positions, therefore the coordinates of the midpoint are given by $x_{mp}=\frac{1}{2}(x_A+x_B)$, $y_{mp}=\frac{1}{2}(y_A+y_B)$ and $z_{mp}=\frac{1}{2}(z_A+z_B)$ and the **position vector** for the midpoint can be written as $P_{mp}=\langle x_{mp},y_{mp},z_{mp}\rangle$. The vector from any other point to this midpoint can then be calculated using the method described in our discussion of displacement vectors.

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Examples

Example 1

Earlier, you were given a problem about finding the center of text on a sheet of graph paper.

Solution

If she knows that the first letter starts 10 boxes up and 5 boxes over from the bottom-left corner of the page, and the last letter ends 12 boxes up and 32 boxes over from the same corner, how could she find the center?

This is a midpoint question, so the *x* coordinate calculation is:

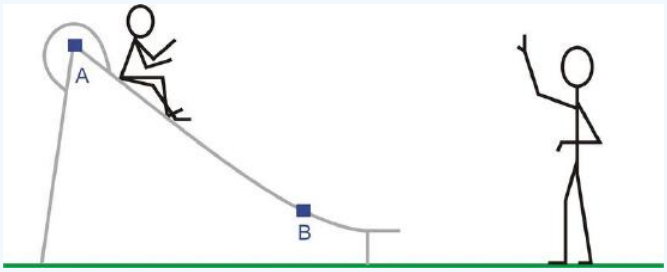
$x_{mp} = \frac{1}{2}(x_A + x_B) = \frac{1}{2}(5 + 32) = 18.5$

and the y-coordinate of the midpoint is given by: $y_{mp} = \frac{1}{2}(y_A + y_B) = \frac{1}{2}(10 + 12) = 11$

The coordinate of the center is: 18.5, 11.

Example 2

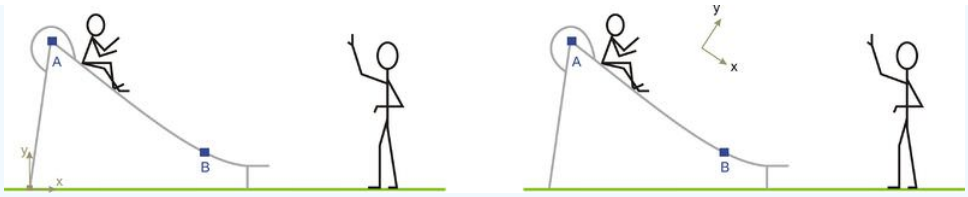
The motion of an object along an inclined plane is a very common problem in introductory physics. The diagram below shows one such situation.



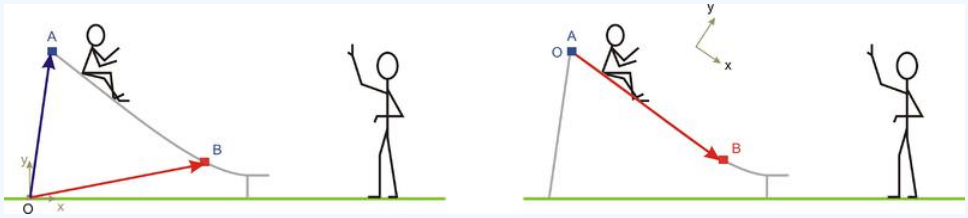
Stickman Beauford has taken his niece Brynna to the park and waves to her as she plays on the slide. Choose two coordinate systems that could be used to describe Brynna's motion and identify the position vectors for points A and B in both coordinate systems.

Solution

If we want to describe Brynna's motion as she moves from point A to point B along the slide, we could use a standard horizontal and vertical coordinate system with the origin at the base of the slide's ladder, but then the vector describing her motion would have components in both the x and y directions. Our mathematical description of her motion can be greatly simplified if we choose point A to be the origin and if we rotate the coordinate system such that the x-axis is parallel to the slide and the y-axis is perpendicular to the slide. Now Brynna's motion from point A to point B is only along the x-axis. Note, other choices of origin are possible.

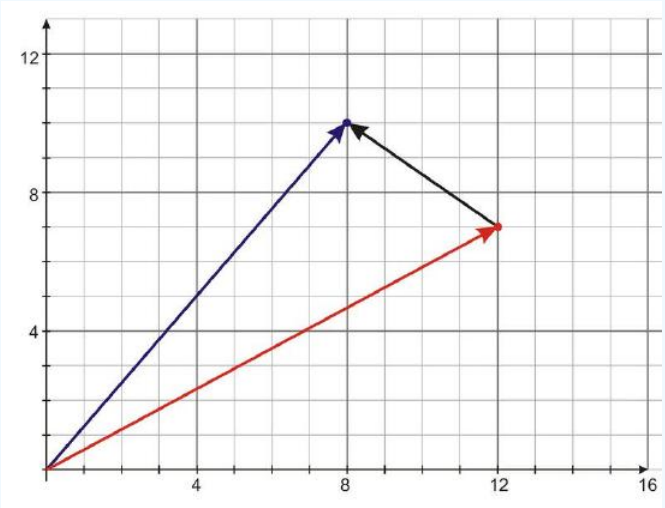


Once we have identified an origin and coordinate axes for of our reference frames, we can use vector notation to identify the location of points A and B. The position vector for point A is the vector starting at the origin and ending at point A, $\overrightarrow{OA}$. For the standard coordinate system shown on the left above, the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are shown on the left below. For the rotated coordinate system, shown on the right above, the position vector $\overrightarrow{OA} = 0$ and $\overrightarrow{OB} = \overrightarrow{AB}$.



Example 3

Determine the coordinates and magnitude of the vector, **D**, beginning at the point $\overrightarrow{P_1} = \langle 12, 7 \rangle$ and ending at $\overrightarrow{P_2} = \langle 8, 10 \rangle$.



Solution

The **displacement vector D** is the difference between the two position vectors:.

$D = \langle P_{2x} - P_{1x}, P_{2y} - P_{1y} \rangle = \langle 8 - 12, 10 - 7 \rangle = \langle -4, 3 \rangle$.

The magnitude of the vector, **D**, can be found using the Pythagorean theorem:

$|\vec{D}| = \sqrt{(-4)^2 + (3)^2} = \sqrt{25} = 5$

Example 4

Determine the position vector identifying the midpoint between points $\overrightarrow{P_1} = \langle 12, 7 \rangle$ and $\overrightarrow{P_2} = \langle 8, 10 \rangle$.

Solution

Since these two points are located in the x-y plane, the x-coordinate of the midpoint is given by

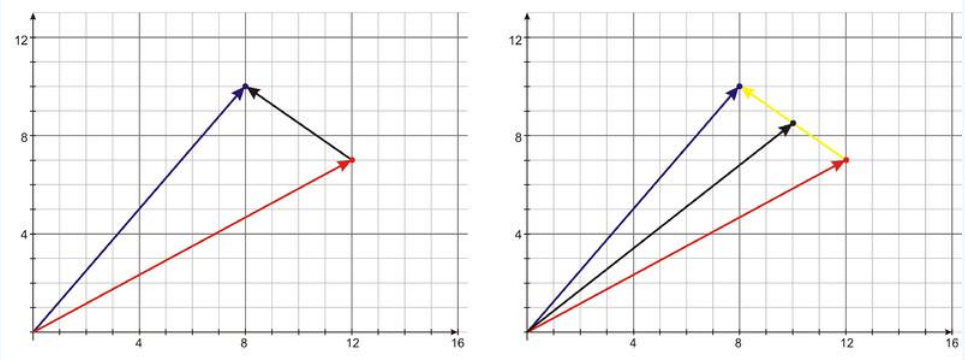
$x_{mp} = \frac{1}{2}(x_A + x_B) = \frac{1}{2}(12 + 8) = 10$

and the y-coordinate of the midpoint is given by

$y_{mp} = \frac{1}{2}(y_A + y_B) = \frac{1}{2}(7 + 10) = 8.5$

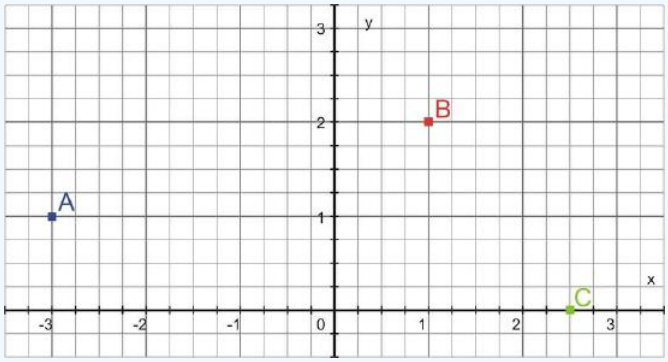
Therefore the position vector for this midpoint can be written as

$P_{mp} = \langle 10, 8.5 \rangle$



Example 5

Identify the position vectors for the three points shown on the grid below.



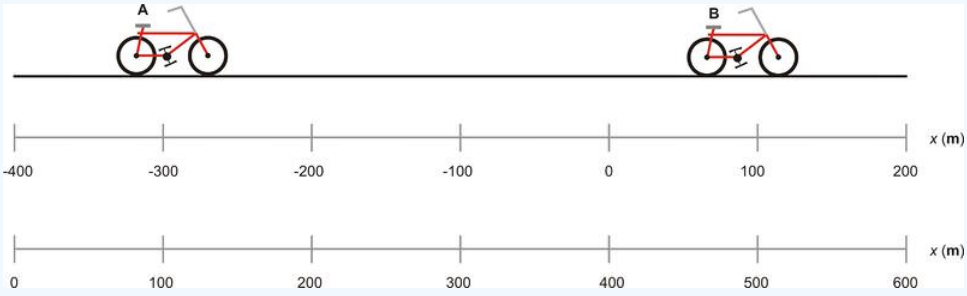
Solution

The position vectors begin at the origin, (0, 0) and end at each point:

$\overrightarrow{OA} = \langle -3, 1 \rangle, \overrightarrow{OB} = \langle 1, 2 \rangle$ and: $\overrightarrow{OC} = \langle 2.5, 0 \rangle$

Example 6

The diagram shows two positions of a bicycle as it moves along a long straight road. Two possible coordinate systems for the motion are shown below. Determine the position vectors in each of the two coordinate systems for the bicycle at points A and B. Then determine the displacement vector from A to B in each case. (Not drawn to scale.)



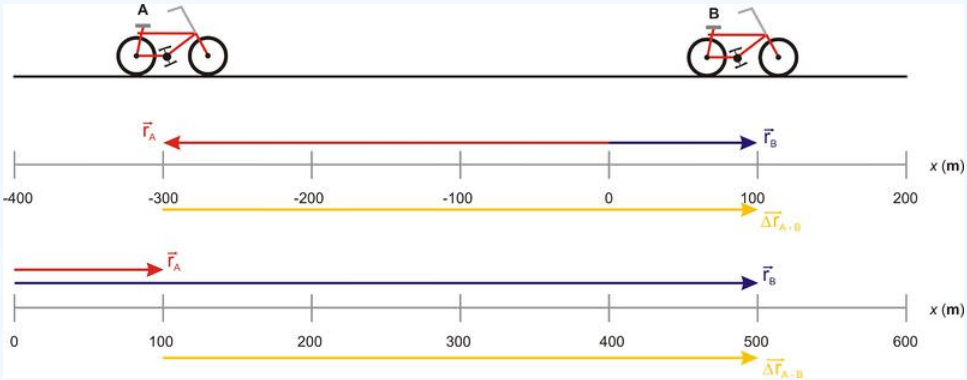
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Solution

The diagram shows two positions of a bicycle as it moves along a long straight road. Two possible coordinate systems for the motion are shown below. Determine the position vectors in each of the two coordinate systems for the bicycle at points A and B. Then determine the displacement vector from A to B in each case.

For the upper coordinate system, the position vector of the bicycle at point A is given by $\vec{r_A} = \langle -300m, 0, 0 \rangle$ and that at point B is given by $\vec{r_B} = \langle 100m, 0, 0 \rangle$. This gives a displacement of $\overrightarrow{\Delta r_{A-B}} = \langle (100m - (-300m)), (0-0), (0-0) \rangle = \langle 400m, 0, 0 \rangle$.

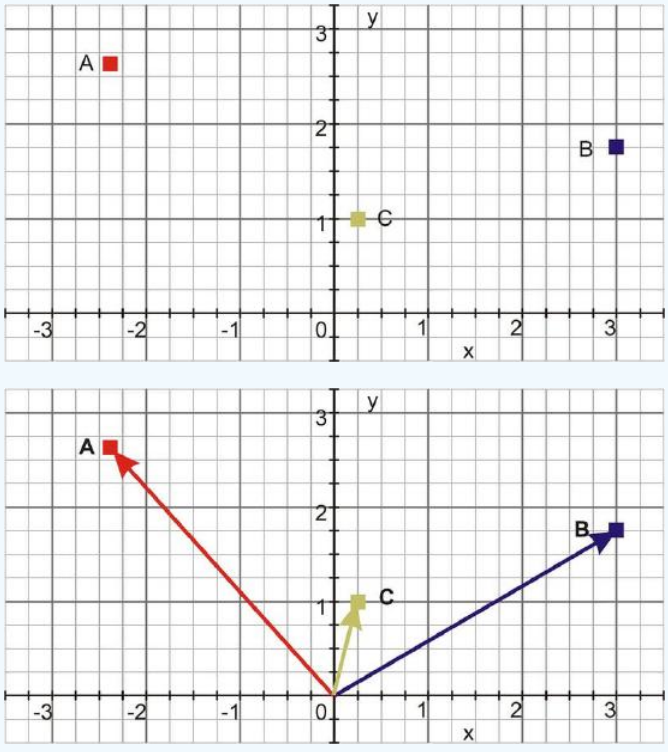
For the upper coordinate system, the position vector of the bicycle at point A is given by $\vec{r_A} = \langle 100m, 0, 0 \rangle$ and that at point B is given by $\vec{r_B} = \langle 500m, 0, 0 \rangle$. This gives a displacement of $\overrightarrow{\Delta r_{A-B}} = \langle (500m - 100m), (0-0), (0-0) \rangle = \langle 400m, 0, 0 \rangle$.



The position vectors for the bicycle at point A are shown in red and the position vectors for point B are shown in blue. The displacement vector between points A and B is shown in gold. As you can see, the position vectors representing this motion depend on the choice of coordinate system, but the displacement vector is independent of the coordinate system. No matter how we define the origin, the bike moves 400 m in the +*x* direction and does not move in the *y* or z direction.

Example 7

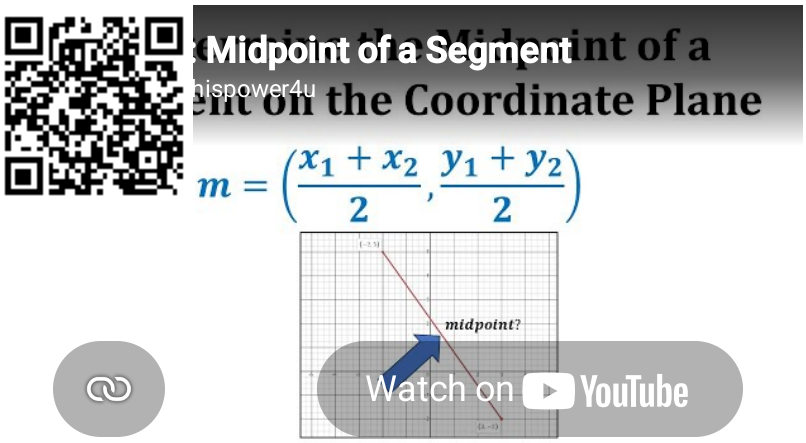
Identify the position vectors for the three points shown in the diagram below.



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Solution

$\vec{r_A} = \langle -2.63, 2.63, 0 \rangle, \vec{r_B} = \langle 3, 1.75, 0 \rangle, \vec{r_C} = \langle 0.25, 1, 0 \rangle$



Review

1. What is a displacement vector used for?

Determine the coordinates and magnitude of the displacement vector, D, beginning at the point $\vec{P_1}$ and ending at $\vec{P_2}$.

2. $overrightarrow{P_1} = \langle 25, 3 \rangle$ and ending at $\vec{P_2} = \langle 8, 11 \rangle$
3. $\vec{P_1} = \langle 5, 3 \rangle$ and ending at $\vec{P_2} = \langle 7, 9 \rangle$
4. $\vec{P_1} = \langle 21, 18 \rangle$ and ending at $\vec{P_2} = \langle 4, 15 \rangle$
5. $\vec{P_1} = \langle 8, 5 \rangle$ and ending at $\vec{P_2} = \langle 5, 8 \rangle$
6. $\vec{P_1} = \langle 16, 25 \rangle$ and ending at $\vec{P_2} = \langle 9, 11 \rangle$
7. $\vec{P_1} = \langle 14, 3 \rangle$ and ending at $\vec{P_2} = \langle 23, 20 \rangle$
8. $\vec{P_1} = \langle 11, 4 \rangle$ and ending at $\vec{P_2} = \langle 15, 11 \rangle$
9. $\vec{P_1} = \langle 23, 13 \rangle$ and ending at $\vec{P_2} = \langle 1, 17 \rangle$

Determine the position vector identifying the midpoint between points $\vec{P_1}$ and $\vec{P_2}$

10. $\vec{P_1} = \langle 17, 6 \rangle$ and $\vec{P_2} = \langle 18, 12 \rangle$
11. $\vec{P_1} = \langle 2, 5 \rangle$ and $\vec{P_2} = \langle 1, 9 \rangle$
12. $\vec{P_1} = \langle 24, 7 \rangle$ and $\vec{P_2} = \langle 21, 10 \rangle$
13. $\vec{P_1} = \langle 12, 9 \rangle$ and $\vec{P_2} = \langle 2, 20 \rangle$
14. $\vec{P_1} = \langle 15, 17 \rangle$ and $\vec{P_2} = \langle 18, 1 \rangle$
15. $\vec{P_1} = \langle 22, 14 \rangle$ and $\vec{P_2} = \langle 23, 8 \rangle$
16. $\vec{P_1} = \langle 1, 7 \rangle$ and $\vec{P_2} = \langle 14, 21 \rangle$
17. $\vec{P_1} = \langle 3, 9 \rangle$ and $\vec{P_2} = \langle 8, 1 \rangle$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.2.

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Vocabulary

Term	Definition
Displacement vector	A displacement vector models the movement between one point and another on a coordinate plane.
midpoint	The midpoint of two vectors is the location in the center of their endpoints.
position vector	A position vector describes the straight-line travel between a starting point (usually the origin) and the location of a second point on a coordinate plane.

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5.1.2: Three-Dimensional Positions

Three-Dimensional Positions

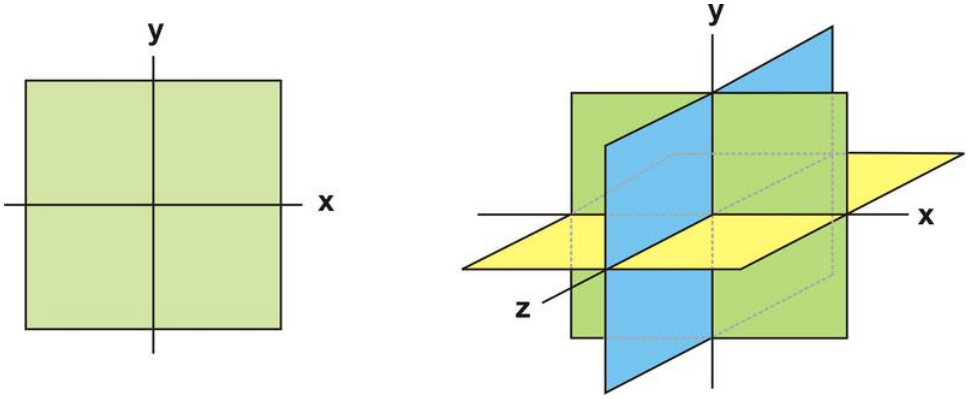
Brian just entered the museum. Looking at the "You are here" map, he sees that his favorite exhibit, "Dinosaurs in Motion!" has been moved to the third floor, and is in the fourth room on the right.

Brian knows the museum well, so he knows the entrance is the first room on the left on the first floor, and is already familiar with the museum layout of 8 rooms on each side of the main hall on each floor, with the stairs at the front of the building.

How could the location of the dinosaur exhibit be described in 3-dimensional coordinates? What is the displacement of the dinosaur room from his position at the front entrance?

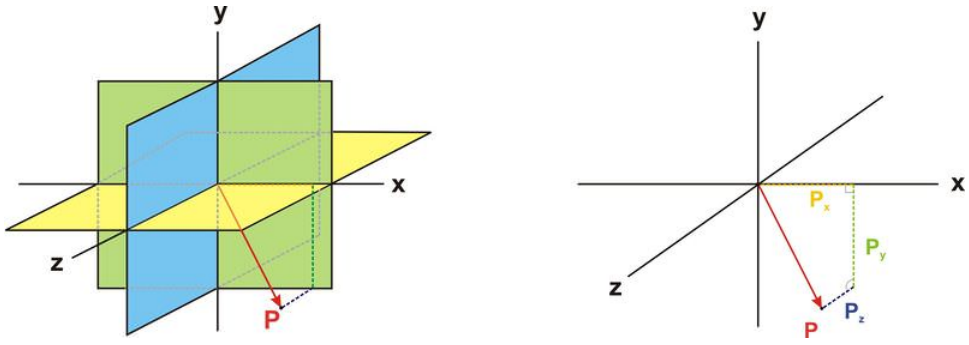
Three-Dimensional Positions

The rectangular (or Cartesian) coordinate system is used to describe a plane divided into four **quadrants**, as shown below left. (Note, the colored squares are used to help you visualize the space, remember that the coordinate planes actually extend outward toward infinity.)



The Cartesian coordinate system use to describe three-dimensional space consists of an origin and six open axes, +**z** and –**z** are perpendicular to the **x**-**y** plane. These axes define three planes which divide the space into eight parts knowns as **octants** as shown above right. Think of these planes as cutting space three ways: left to right, top to bottom, and front to back.

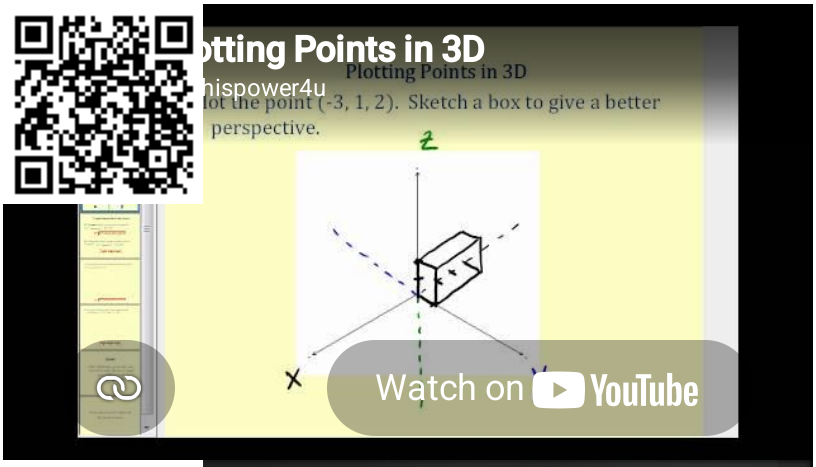
By convention, we number the four quadrants of the x-y plane in this way: points in **quadrant** 1 have +x and +y coordinates, those in quadrant 2 have –x and +y, those in quadrant 3 have –x and –y, and those in quadrant 4 have +x and –y. There is currently no standardized numbering system for the octants in three-dimensional space, although most people identify the region with +x, +y, and +z as the first **octant**. The method used to identify the octants is to indicate verbally the portion of space they occupy. For example, the first octant could also be identified as (top, front, right).

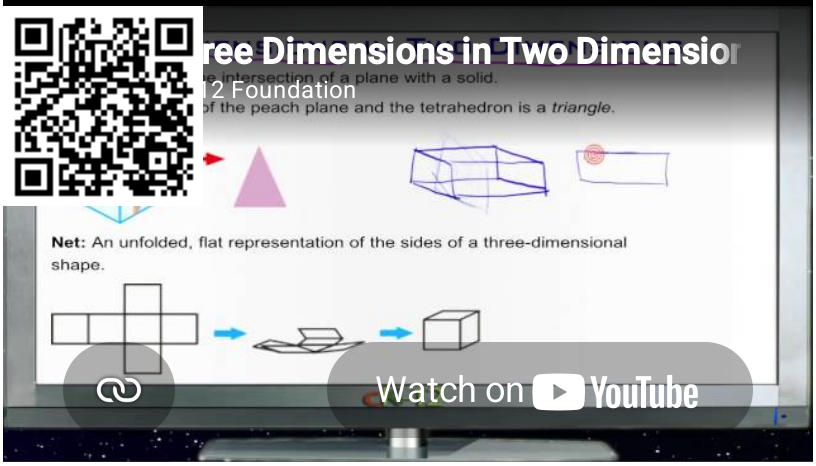


Position vectors in 3D space are still represented by arrows that begin at the origin and end at the point in question. The diagram above shows a point, P, located in the front, lower, right octant. The three components of the **position vector** (P_x, P_y, and P_z) are shown in the diagram. According to the Pythagorean Theorem, the magnitude of the position vector is given by:

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$$|\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2}$$





Examples

Example 1

Earlier, you were asked a question about Brian, who wants to describe the location of the dinosaur exhibit in 3-dimensions.

Solution

Brian knows the museum well, so he knows the entrance is the first room on the left on the first floor, and is already familiar with the museum layout of 8 rooms on each side of the main hall on each floor, with the stairs at the front of the building. How could the location of the dinosaur exhibit be described in 3-dimensional coordinates? What is the displacement of the dinosaur room from his position at the front entrance?

A coordinate system can be assumed to use any units you wish, in this case the location information is given in units of "rooms" and "floors". Let the x axis represent the horizontal left and right-hand sides of the main halls, y can be the vertical axis, and z can be the front-back axis of rooms down each hall.

Let Brian's starting position at the entrance represent (0,0,0) in three dimensional rectangular coordinates. "Dinosaurs in Motion!" is on the third floor, and is in the fourth room on the right.

(2,3,4) would represent the dinosaur exhibit room.

Since we were clever, and oriented our coordinate system on the entrance, the vector displacement is the same as the room's location: ⟨2rooms,3floors,4rooms⟩.

Example 2

Darnell was driving home from a football game in a nearby town when he swerved to avoid a deer which had run onto the road. Fortunately for Darnell, he was able to avoid hitting the deer. Unfortunately, his car ended up in the ditch beside the road. When he was unable to remove the car from the ditch by himself, he walked across a nearby field to the Tucker family farm to ask for help. Darnell later looked at a topographical map and identified his trip across the field. He traveled 300 yards south and 750 yards west from where he left his car. The map showed that he also walked uphill from an altitude of 800 feet to an altitude of 850 feet above sea level. If we treat the location of Darnell's car as the origin of coordinates, what the position vector of the Tucker farm?

Solution

Define a coordinate system where **x** = E, **y** = N, and **z** = up. Since Darnell walked south and west from the car, the **x** and **y** coordinates of the farm are both negative. If we measure all of the distances in feet (1 yard = 3 feet), the farm's position vector can be written as

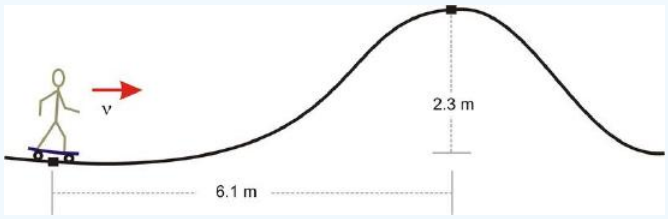
$\vec{P} = \langle P_{\text{east}}, P_{\text{north}}, P_{up} \rangle = \langle -2250, -900, 50 \rangle$.

Note in this example that Darnell's walking distance was given in yards, while the elevation change was given in feet. You need to watch out for these small changes when you are solving real-life problems.

Example 3

Zeke is enjoying an afternoon at the local skate-park. The diagram below shows his starting position and his ending position at the highest point on the new hill.

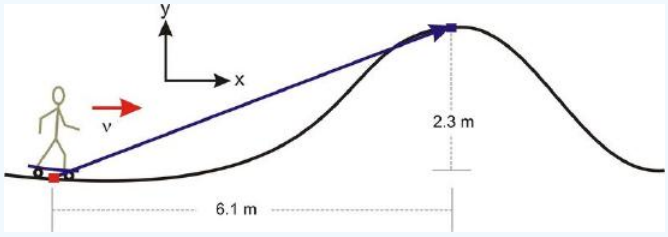
Choose two different coordinate systems which could describe this system. Find Zeke's initial and final position vectors in each of the two coordinate systems. Then identify the displacement vector from his starting position to his final position at the top of the hill.



Solution

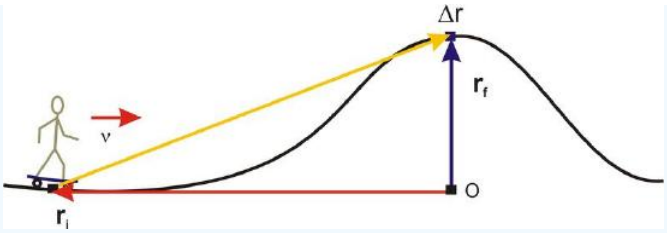
One possible origin of coordinates is located at Zeke's starting position. In this case, the initial position vector is given by $\vec{r}_i = \langle 0m, 0m, 0m \rangle$ and his final position is given by $\vec{r_f} = \langle 6.1m, 2.3m, 0m \rangle$. Zeke's displacement is the difference between these two vectors,

$\overrightarrow{\Delta r} = \vec{r}_f - \vec{r}_i = \langle 6.1m, 2.3m, 0m \rangle - \langle 0m, 0m, 0m \rangle = \langle 6.1m, 2.3m, 0m \rangle$



Another possible origin of coordinates is at the point marked O in the diagram below. In this case his original position is given by $\vec{r_i} = \langle -6.1m, 0m, 0m \rangle$ and his final position is given by $\vec{r_f} = \langle 0m, 2.3m, 0m \rangle$. Zeke's displacement is the difference between these two vectors,

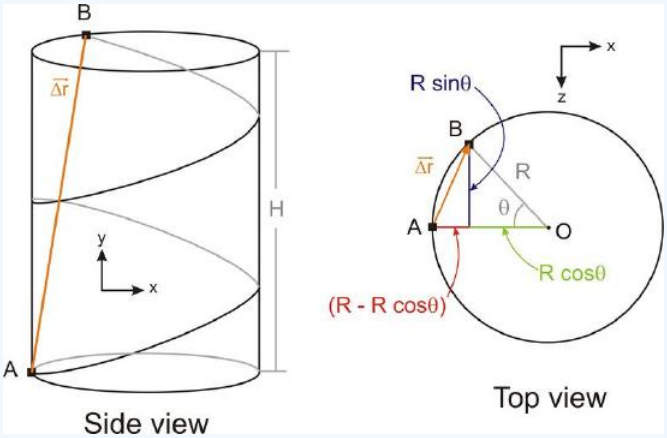
$\overrightarrow{\Delta r} = \vec{r}_f - \vec{r}_i = \langle 0m, 2.3m, 0m \rangle - \langle -6.1m, 0m, 0m \rangle = \langle 6.1m, 2.3m, 0m \rangle$



Note that the position vectors representing this motion depend on the choice of coordinate system, but the displacement vector is independent of the coordinate system.

Example 4

An architecture student designs a spiral staircase, a model of which is shown below. The staircase winds its way around a cylinder of radius 3.5 m and height 11 m. The staircase makes 1 7/8 turns progressing counter-clockwise from its beginning at point A to its end point at B. Using an origin of coordinates at the bottom center of the staircase, determine the position vectors of points A and B. Then find the displacement vector between the two points.



Solution

As you can see in the top view of the diagram, point A is directly to the left of the origin, therefore the position vector for point A is given by $\vec{r_A} = \langle -R, 0, 0\rangle = \langle -3.5m, 0, 0\rangle$.

Point B is located 11m above point A and 7/8 of one turn counter-clockwise is equal to 1/8 of one turn clockwise, so θ = 45°. We can also use the geometry of the system to determine the *x* and *z* coordinates of point B:

$$\vec{r_B} = \langle -R\cos\theta, R\sin\theta, H\rangle = \langle (-3.5m)\cos 45°, (3.5m)\sin 45°, 11m\rangle$$
$$= \langle -2.475m, 2.475m, 11m\rangle$$

The displacement vector between these two points is the vector which obeys the equation:

$$\overrightarrow{\Delta r} = \vec{r_B} - \vec{r_A}$$
$$\overrightarrow{\Delta r} = \langle -2.475m, 2.475m, 11m\rangle - \langle -3.5m, 0m, 0m\rangle$$
$$\overrightarrow{\Delta r} = \langle (-2.475m - (-3.5m)), (2.475m - 0m), (11m - 0m)\rangle = \langle 1.025m, 2.475m, 11m\rangle$$

Example 5

Identify the **midpoint** between points *P* = (3.7, 8.4, -2.1) and *Q* = (5.5, -1.9, -8.6).

Solution

To find the midpoint between two points, determine the average of the two positions.

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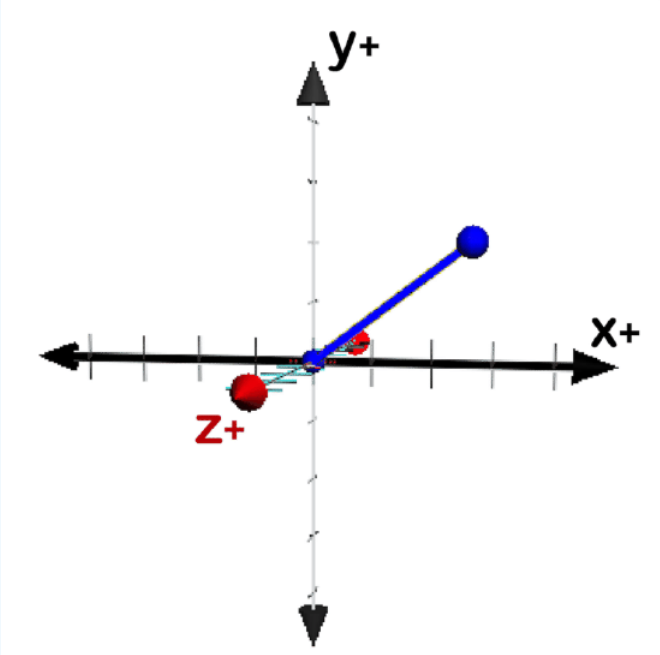
$M = \left(\frac{1}{2}(3.7 + 5.5), \frac{1}{2}(8.4 + (-1.9)), \frac{1}{2}(-2.1 + (-8.6))\right) = \left(\frac{1}{2}(9.2), \frac{1}{2}(6.5)\right), \frac{1}{2}(-10.7))$
$M = (4.6, 3.25, -5.35)$

Example 6

Draw the three dimensional vector $\vec{A} = 3\hat{i} + 2\hat{j} + 5\hat{k}$

Solution

I, j and k are unit vectors respectively in the positive direction of the x, y, and z axes.



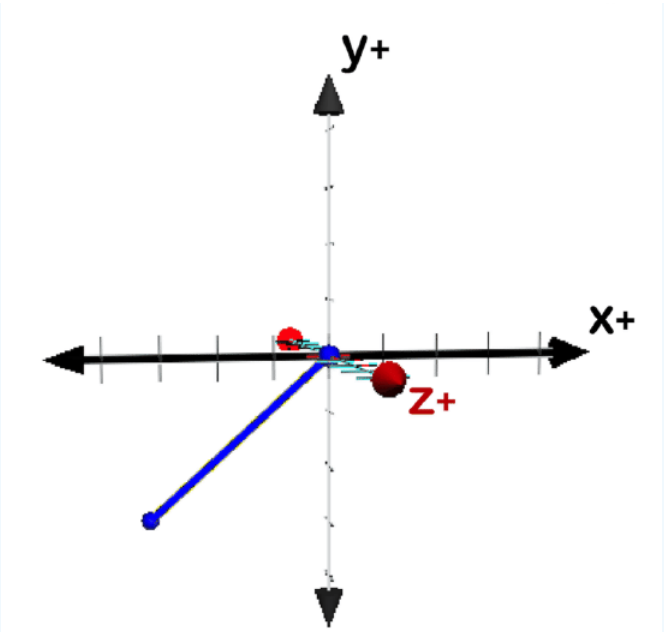
Example 7

Draw the three dimensional vector $\vec{A} = -3\hat{i} + -4\hat{j} - 5\hat{k}$

Solution

I, j and k are unit vectors respectively in the negative direction of the x, y, and z axes.

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Review

Given: initial and final position vectors the coordinate system. Identify the displacement vector or midpoint from the initial to the final position.

1. What is the displacement or difference between these two vectors? The initial position vector $\vec{r_i} = \langle 0m, 0m, 0m \rangle$ and the final position vector $\vec{r_f} = \langle 16.1\ \text{m}, 7.5\ \text{m}, 3\ \text{m} \rangle$.
2. What is the displacement or difference between these two vectors? The initial position vector $\vec{r_i} = \langle 0mi, 0mi, 0mi \rangle$ and the final position vector $\vec{r_f} = \langle 9.4mi, 12.5mi, 6.6\text{mi} \rangle$.
3. What is the displacement or difference between these two vectors? The initial position vector $\vec{r_i} = \langle 5km, 3km, 8km \rangle$ and the final position vector $\vec{r_f} = \langle 10km, 20km, 19km \rangle$.
4. What is the displacement or difference between these two vectors? The initial position vector $\vec{r}_i = \langle 1cm, 3cm, 1cm \rangle$ and the final position vector $\vec{r}_f = \langle 5.1cm, 2cm, 5cm \rangle$.
5. What is the displacement or difference between these two vectors? The initial position vector $\vec{r_i} = \langle 5.6mm, 10.2mm, 2.2mm \rangle$ and the final position vector $\vec{r_f} = \langle 20.4mm, 31.1mm, 1.1mm \rangle$.
6. What is the displacement or difference between these two vectors? The initial position vector $\vec{r}_i = \langle 1in, 2in, 3in \rangle$ and the final position vector $\vec{r}_f = \langle 4in, 5in, 3in \rangle$.

Identify the midpoint between points A and B.

7. $\text{A} = \langle 0m, 0m, 0m \rangle$ and $\text{B} = \langle 16.1m, 7.5m, 3m \rangle$
8. $\text{A} = \langle 0mi, 0mi, 0mi \rangle$ and $\text{B} = \langle 9.4mi, 12.5mi, 6.6mi \rangle$
9. $\text{A} = \langle 5km, 3km, 8km \rangle$ and $\text{B} = \langle 10km, 20km, 19km \rangle$
10. $\text{A} = \langle 1cm, 3cm, 1cm \rangle$ and $\text{B} = \langle 5.1cm, 2cm, 5cm \rangle$
11. $\text{A} = \langle 5.6mm, 10.2mm, 2.2mm \rangle$ and $\text{B} = \langle 20.4m, 31.1mm, 1.1mm \rangle$
12. $\text{A} = \langle 1in, 2in, 3in \rangle$ and $\text{B} = \langle 4in, 5in, 3in \rangle$

Draw the three dimensional vector.

13. $\vec{A} = 3\hat{i} + 2\hat{j} + 5\hat{k}$
14. $\vec{A} = 5\hat{i} - 3\hat{j} + 1\hat{k}$
15. $\vec{A} = 7\hat{i} + 1\hat{j} - 4\hat{k}$
16. $\vec{A} = -7\hat{i} - 4\hat{j} + 8\hat{k}$

17. $\vec{A} = -4\hat{i} + 7\hat{j} - 3\hat{k}$
18. $\vec{A} = -3\hat{i} - 2\hat{j} - 5\hat{k}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.3.

Vocabulary

Term	Definition
midpoint	The midpoint of two vectors is the location in the center of their endpoints.
octant	An octant is any one of the eight "corners" of the three-dimensional rectangular coordinate system.
position vector	A position vector describes the straight-line travel between a starting point (usually the origin) and the location of a second point on a coordinate plane.
quadrant	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.
Quadrants	A quadrant is one-fourth of the coordinate plane. The four quadrants are numbered using Roman Numerals I, II, III, and IV, starting in the top-right, and increasing counter-clockwise.

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SECTION OVERVIEW

5.2: Vector Calculations

Topic hierarchy

5.2.1: Dot Products

5.2.2: Scalar Projections

5.2.3: Cross Products

5.2.4: Vector Projection

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5.2.1: Dot Products

Dot Product and Angle Between Two Vectors

While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. You can study more about the cross product between two vectors when you take Linear Algebra. The second type of product is the **dot product** between two vectors which results in a regular number. Other names for the dot product include **inner product** and **scalar product**. This number represents how much of one vector goes in the direction of the other. In one sense, it indicates how much the two vectors agree with each other. This concept will focus on the dot product between two vectors.

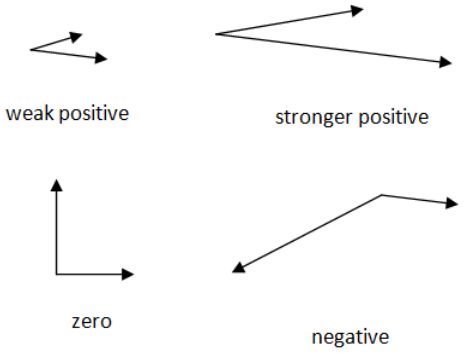
What is the dot product between <−1,1> and <4,4>? What does the result mean?

Properties of the Dot Product

The dot product is defined as: $u \cdot v =< u_1, u_2 > \cdot < v_1, v_2 >= u_1 v_1 + u_2 v_2$

This procedure states that you multiply the corresponding values and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product.



Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than 90°.

There are a many important properties related to the dot product. The two most important are 1) what happens when a vector has a dot product with itself and 2) what is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = |v|^2$
- v and u are perpendicular if and only if $v \cdot u = 0$

The commutative property, $u \cdot v = v \cdot u$, holds for the dot product between two vectors. The following proof is for two dimensional vectors although it holds for any dimensional vectors.

Start with the vectors in component form.

$u =< u_1, u_2 >$
$v =< v_1, v_2 >$

Then apply the definition of dot product and rearrange the terms. The commutative property is already known for regular numbers so we can use that.

$$\begin{aligned} u \cdot v &= \langle u_1, u_2 > \cdot < v_1, v_2 > \\ &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \\ &=< v_1, v_2 > \cdot < u_1, u_2 > \\ &= v \cdot u \end{aligned}$$

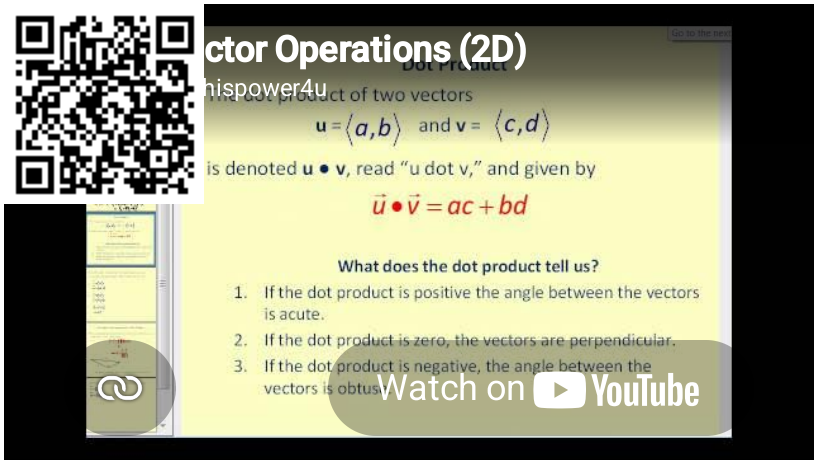
The distributive property, u·(v+w)=uv+uw, also holds under the dot product**.** The following proof will work with two dimensional vectors although the property does hold in general.

$$\begin{aligned} u =< u_1, u_2 >, v =< v_1, v_2 >, w = \langle w_1, w_2 \rangle \\ u \cdot (v + w) &= u \cdot (< v_1, v_2 > + < w_1, w_2 >) \\ &= u \cdot < v_1 + w_1, v_2 + w_2 > \\ &=< u_1, u_2 > \cdot < v_1 + w_1, v_2 + w_2 > \\ &= u_1 (v_1 + w_1) + u_2 (v_2 + w_2) \\ &= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 \\ &= u_1 v_1 + u_2 v_2 + u_1 w_1 + u_2 w_2 \\ &= u \cdot v + v \cdot w \end{aligned}$$

The dot product can help you determine the angle between two vectors using the following formula. Notice that in the numerator the dot product is required because each term is a vector. In the denominator only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

Watch the portion of this video focusing on the dot product:



Examples

Example 1

Earlier, you were asked to find the dot product between the two vectors <−1,1> and <4,4>.

Solution

It can be computed as:

(−1)(4)+1(4)=−4+4=0

The result of zero makes sense because these two vectors are perpendicular to each other.

Example 2

Find the dot product between the following vectors: <3,1>·<5,−4>

Solution

u=<3,5>

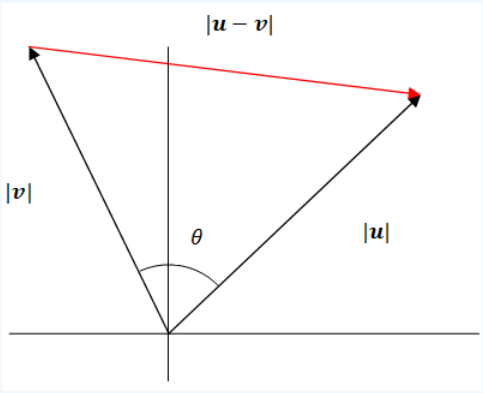
Example 3

Prove the angle between two vectors formula:

$\cos\theta = \frac{u \cdot v}{|u||v|}$

Solution

Start with the law of cosines.



$$
\begin{aligned}
|u-v|^2 &= |v|^2 + |u|^2 - 2|v||u|\cos\theta \\
(u-v)\cdot(u-v) &= \\
u\cdot u - 2u\cdot v + v\cdot v &= \\
|u|^2 - 2u\cdot v + |v|^2 &= \\
-2u\cdot v &= -2|v||u|\cos\theta \\
\frac{u\cdot v}{|u||v|} &= \cos\theta
\end{aligned}
$$

Example 4

Find the dot product between the following vectors.

$(4i - 2j)\cdot(3i - 8j)$

Solution

The standard unit vectors can be written as component vectors.

$< 4, -2 > \cdot < 3, -8 >= 12 + (-2)(-8) = 12 + 16 = 28$

Example 5

What is the angle between $u =< 3, 5 >$ and $v =< 2, 8 >$?

Solution

Use the angle between two vectors formula.

$$
\begin{aligned}
u =< 3,5 > \text{ and } v &=< 2,8 > \\
\frac{u\cdot v}{|u||v|} &= \cos\theta \\
\frac{< 3,5 > \cdot < 2,8 >}{\sqrt{34}\cdot\sqrt{68}} &= \cos\theta \\
\frac{6+40}{\sqrt{34}\cdot\sqrt{68}} &= \cos\theta \\
\cos^{-1}\left(\frac{46}{\sqrt{34}\cdot\sqrt{68}}\right) &= \theta \\
16.93^\circ &\approx \theta
\end{aligned}
$$

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Review

Find the dot product for each of the following pairs of vectors.

1. <2,6>·<−3,5>
2. <5,−1>·<4,4>
3. <−3,−4>·<2,2>
4. <3,1>·<6,3>
5. <−1,4>·<2,9>

Find the angle between each pair of vectors below.

6. <2,6>·<−3,5>
7. <5,−1>·<4,4>
8. <−3,−4>·<2,2>
9. <3,1>·<6,3>
10. <−1,4>·<2,9>
11. What is $v \cdot v$?
12. How can you use the dot product to find the magnitude of a vector?
13. What is $0 \cdot v$?
14. Show that $(cu) \cdot v = u \cdot (cv)$ where c is a constant.
15. Show that <2,3> is perpendicular to <1.5,−1>.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.4.

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5.2.2: Scalar Projections

Scalar Projections

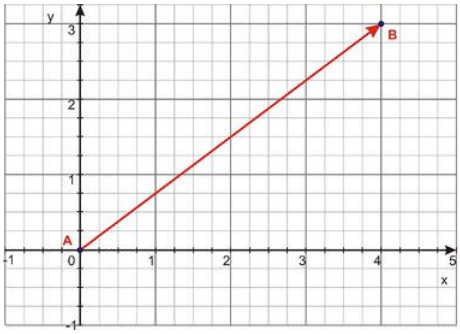
As an engineer, or a pilot, or racecar driver, or even a chef, it can often be useful to determine the result of different amounts and directions of force applied to a particular action to decide on the best use of added strength, altitude, speed, or heat in order to achieve the optimum result.

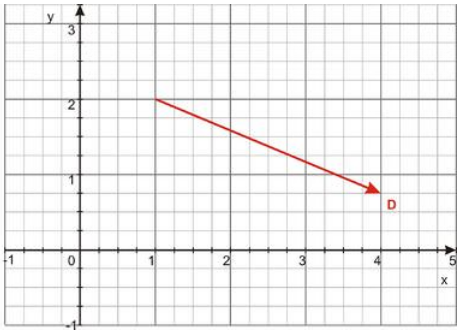
This lesson is about **scalar projections**, which allow you to do the things above and a whole host of others.

Scalar Projections

A **scalar projection** allows you to investigate the result of different "lengths" of one vector on an overall study. The projection of a vector onto a particular direction is, in effect, the result of applying a specific value in that direction. Recall that the **dot product** of a vector is a scalar quantity describing only the magnitude of a particular vector. A **scalar projection** is given by the dot product of a vector with a **unit vector** for that direction.

For example, the component notations for the vectors shown below are AB=⟨4,3⟩ and D=⟨3,−1.25⟩.





The scalar projection of vector $\boldsymbol{AB}$ onto $\hat{x}$ is given by

$$\overrightarrow{AB} \times \hat{x} = (4 \cdot 1) + (3 \cdot 0) + (0 \cdot 0) = 4$$

The scalar projection of vector $\boldsymbol{AB}$ onto $\hat{y}$ is given by

$$\overrightarrow{AB} \times \hat{y} = (4 \cdot 0) + (3 \cdot 1) + (0 \cdot 0) = 3$$

And the scalar projection of vector $\boldsymbol{AB}$ onto $\hat{z}$ is given by

$$\overrightarrow{AB} \times \hat{z} = (4 \cdot 0) + (3 \cdot 0) + (0 \cdot 1) = 0$$

The scalar projections of $\boldsymbol{AB}$ onto the x and y directions are non-zero numbers because the vector is located in the x-y plane. The scalar projection of $\boldsymbol{AB}$ onto the z direction is equal to zero, because the z direction is perpendicular to $\boldsymbol{AB}$.

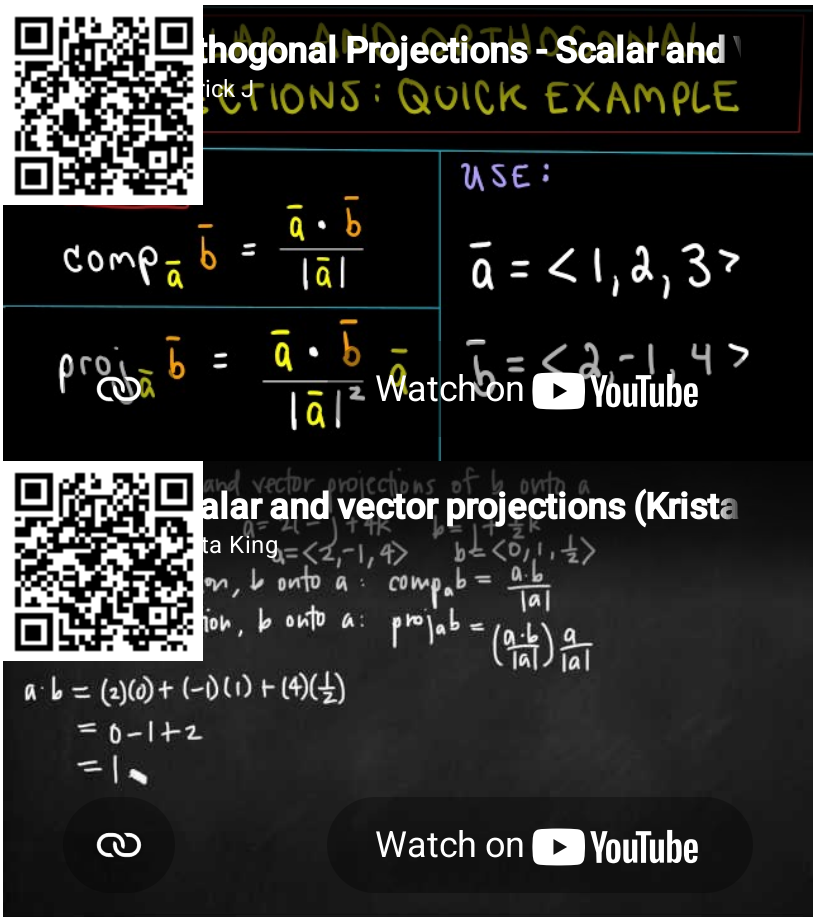
Vector Projections

The **vector projection** of a vector onto a given direction has a magnitude equal to the scalar projection. The direction of the vector projection is the same as the unit vector of that given direction. Recall that when a vector $\vec{v}$ is multiplied by a scalar s, its components are given by

$$\overrightarrow{sv} = \langle sv_x, sv_y, sv_z \rangle$$

To calculate the vector projection of $\boldsymbol{AB}$ onto the direction of vector $\boldsymbol{D}$, use the scalar projection calculated in the previous example and the unit vector $\vec{D}$.

$$(\overrightarrow{AB} \times \vec{D})\vec{D} = \langle (1.23 \cdot 0.923), (1.23 \cdot -0.385) \rangle = \langle 1.135, -0.474 \rangle$$



Examples

Example 1

The diagram below shows both vectors ***AB*** and ***D*** together on the same grid. Determine the scalar projection of vector ***AB*** onto the direction of vector ***D***.

Solution

To find the scalar projection onto the direction of another vector we need to know the unit vector in the direction of vector ***D***.

First, the components of $\vec{D}$ are

$$\vec{D} = \langle 3, -1.25 \rangle$$

Now the magnitude of $\vec{D}$ is

$$|D| = \sqrt{(D_x)^2 + (D_y)^2} = \sqrt{3^2 + (-1.25)^2} = \sqrt{9 + 1.5625} = \sqrt{10.5625} = 3.25$$

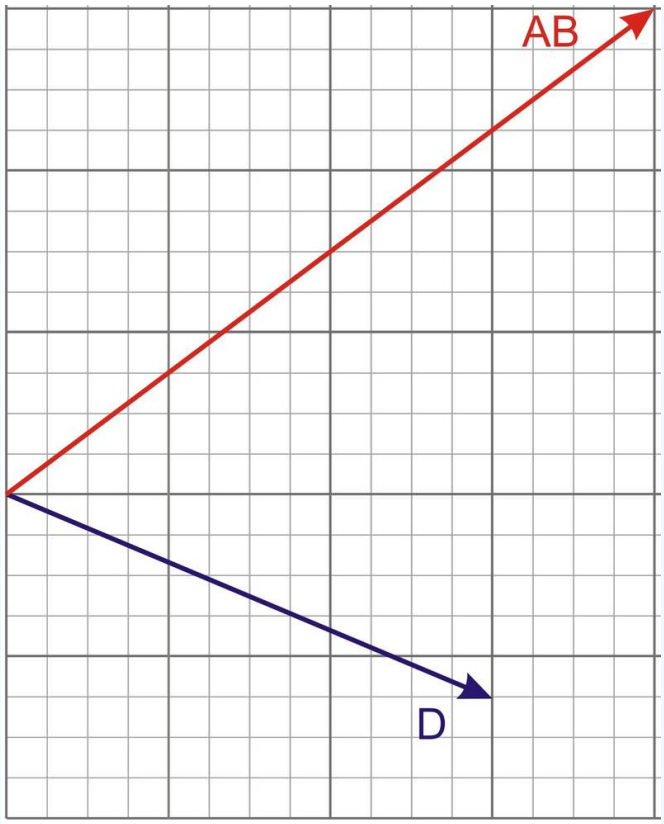
Finally, the direction vector of $\vec{D}$ is

$$\vec{D} = \frac{\vec{D}}{|D|} = \frac{3\hat{x} + (-1.25)\hat{y}}{3.25} = \frac{3}{3.25}\hat{x} + \frac{-1.25}{3.25}\hat{y}$$

$$\vec{D} = \langle 0.923, -0.385 \rangle$$

Now we can use the dot-product to calculate the scalar projection of ***AB*** onto the direction of vector ***D***.

$$\overrightarrow{AB} \times \vec{D} = (3 \cdot 0.923) + (4 \cdot -0.385) + (0 \cdot 0) = 2.769 + (-1.54) = 1.23$$



Example 2

Determine the scalar projection of the vector $\vec{R} = \langle 27, 39, 52 \rangle$ onto the direction of $\vec{T} = \langle 44, 26, 17 \rangle$.

Solution

The scalar projection of one vector onto the direction of the other is the dot product of the first vector with the unit vector representing the direction of the second vector. To calculate the scalar projection, we need to determine the unit vector in the direction of vector $\vec{T} = \langle 44, 26, 17 \rangle$. Remember that a unit vector is equal to the ratio of the vector and its magnitude, therefore we first need to calculate the length of vector $\vec{T}$

$$|\vec{T}| = \sqrt{T_x^2 + T_y^2 + T_z^2} = \sqrt{(44)^2 + (26)^2 + (17)^2} = \sqrt{1936 + 676 + 289}$$

$$= \sqrt{2901} = 53.86$$

$$\vec{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{\langle 44,26,17 \rangle}{53.86} = \left\langle \frac{44}{53.86}, \frac{26}{53.86}, \frac{17}{53.86} \right\rangle = \langle 0.8169, 0.4827, 0.3156 \rangle$$

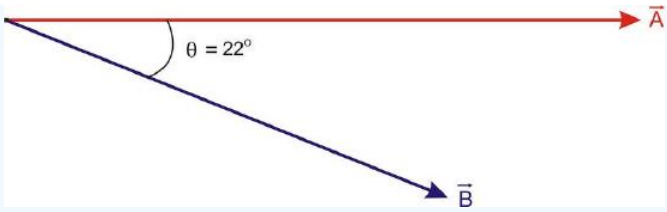
Now we can calculate the scalar projection of $\vec{R}$ onto $\vec{T}$ by calculating the dot product

$$\vec{R} \times \vec{T} = (27 \cdot 0.8169) + (39 \cdot 0.4827) + (52 \cdot 0.3156) = 22.0563 + 18.6253 + 16.4112 = 57.0928$$

Example 3

Determine the vector projection of $\vec{A}$ onto the direction of $\vec{B}$ and the vector projection of $\vec{B}$ onto the direction of $\vec{A} \cdot \vec{A} = 7cm \text{ @ } 0°$ and $\vec{B} = 4cm \text{ @ } -22°$.

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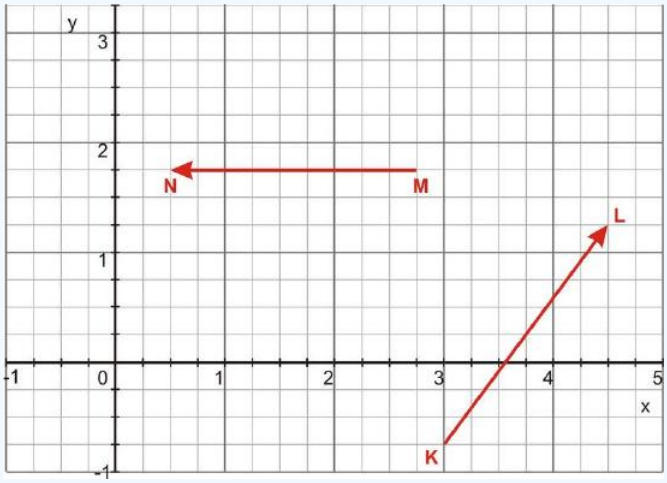
Solution

The vector projection of one vector onto the direction of another vector is given by $(\vec{A} \times \vec{B})\vec{B}$, where $\vec{B}$ is the unit vector in the direction of $\vec{B}$. Since it is a unit vector $\vec{B}$ has a magnitude of 1 and has the same direction as $\vec{B}, \vec{B} = 1 \text{ @ } -22°$. Therefore, $(\vec{A} \times \vec{B})\vec{B} = (|\vec{A}\|\vec{B}| \cos\theta)\vec{B} = ((7)(1)\cos 22) \text{ @ } -22° = 6.49 \text{ @ } -22°$

The vector projection of one vector onto the direction of another vector is given by $(\vec{B} \times \vec{A})\vec{A}$, where $\vec{A}$ is the unit vector in the direction of $\vec{A}$. Since it is a unit vector $\vec{A}$ has a magnitude of 1 and has the same direction as $\vec{A}, \vec{A} = 1 \text{ @ } 0°$. Therefore, $(\vec{B} \times \vec{A})\vec{A} = (|\vec{B}||\vec{A}| \cos\theta)\vec{A} = ((4)(1)\cos 22) \text{ @ } 0° = 3.71 \text{ @ } 0°$

Example 4

Determine the vector projection of vector $\overrightarrow{MN}$ onto the vector $\overrightarrow{KL}$.



Solution

The vector progression of one vector onto a second vector is the multiplication of the dot products of the two vectors and the unit vector defining the direction of the second vector. In this case, $(\overrightarrow{MN} \times \overrightarrow{KL})\overrightarrow{KL}$. First we need to identify the components of the two vectors by using the information given on the graph. In this case, $\overrightarrow{MN} = \langle -2.25, 0.0 \rangle$ and $\overrightarrow{KL} = \langle 1.5, 2, 0 \rangle$. Then we need to determine the dot product of the two vectors.

$$\overrightarrow{MN} \times \overrightarrow{KL} = (MN)_x(KL)_x + (MN)_y(KL)_y + (MN)_z(KL)_z$$
$$= (2.25)(1.5) + (2)(0) + (0)(0) = 3.375$$

We also need to determine the unit vector in the direction of $\overrightarrow{KL}$. Remember that a unit vector is equal to the ratio of the vector and its magnitude, therefore we first need to calculate the length of vector $\overrightarrow{KL}$.

$$|\overrightarrow{KL}| = \sqrt{(KL)_x^2 + (KL)_y^2 + (KL)_z^2} = \sqrt{(1.5)^2 + (2)^2 + (0)^2} = \sqrt{2.25 + 4 + 0}$$
$$= \sqrt{6.25} = 2.5$$
$$\overrightarrow{KL} = \frac{\overrightarrow{KL}}{|\overrightarrow{KL}|} = \frac{\langle 1.5, 2.0, 0 \rangle}{2.5} = \left\langle \frac{1.5}{2.5}, \frac{2}{2.5}, \frac{0}{2.5} \right\rangle = \langle 0.6, 0.8, 0 \rangle$$

Lastly, we multiply the dot product of the two vectors by this unit vector,

$(\overrightarrow{MN} \times \overrightarrow{KL})\overrightarrow{KL} = (3.375)\langle 0.6, 0.8, 0\rangle = \langle 2.025, 2.7, 0\rangle$

Review

1. Calculate the result of a scalar multiple of 6 on the vector $\langle 1, 19\rangle$
2. Calculate the result of a scalar multiple of (-8) on the vector $\langle 7, 8\rangle$
3. Calculate the result of a scalar multiple of 15 on the vector $\langle -7, 3\rangle$
4. Calculate the result of a scalar multiple of (-11) on the vector $\langle -9, 20\rangle$
5. Calculate the result of a scalar multiple of 16 on the vector $\langle 9, 10\rangle$
6. Given Vector $A = \langle -4, 6\rangle$ and Vector $B = \langle 9, 15\rangle$. What is the projection of A onto B?
7. What is the projection of $\langle 2, 10\rangle$ onto $\langle 1, 3\rangle$?
8. Given Vector $C = \langle 1, 2\rangle$ and Vector $D = \langle 1, 11\rangle$. What is the projection of C onto D?
9. Given Vector $E = \langle -5, 3\rangle$ and Vector $F = \langle -1, 18\rangle$. What is the projection of E onto F?
10. What is the projection of $\langle -2, 1\rangle$ onto $\langle -5, 6\rangle$?
11. Given Vector $\mathrm{H} = \langle 8i + 11j\rangle$ and Vector $I = \langle -2i + 15j\rangle$ What is the projection of H onto I?
12. What is the projection of $\langle -1, 6\rangle$ onto $\langle 8, 16\rangle$?
13. What is the projection of $\langle -1, 8\rangle$ onto $\langle 7, 8\rangle$?
14. Given Vector $\mathrm{J} = \langle -4i + 8j\rangle$ and Vector $K = \langle -8i + 10j\rangle$. What is the projection of J onto K?
15. Given Vector $L = \langle -4, 3\rangle$ and Vector $M = \langle 7, 17\rangle$. What is the projection of L onto M?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.5.

Vocabulary

Term	Definition
dot product	The dot product is also known as inner product or scalar product. The two forms of the dot product are $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$ and $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$.
scalar projection	The scalar projection of a vector onto another vector is equal to the length of the projection of the first vector onto the second vector.
unit vector	A unit vector is a vector with a magnitude of one.

Image Attributions

1. [Figure 1]
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5.2.3: Cross Products

Cross Products

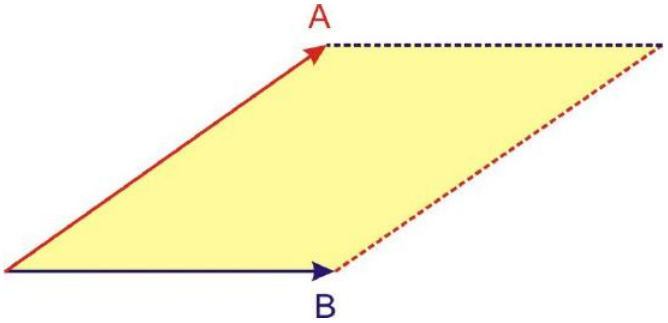
Cross products are related to **dot products** in a number of ways. Both are vector calculations, both are related not only to the magnitude of each vector, but also to the relative directions of both vectors.

Dot products in a sense calculate the *joined* force of two vectors in a given direction, and so are greatest when the vectors are parallel. Cross products, however, are greatest when the vectors are perpendicular... what then do they calculate?

Cross Products

Whereas a **dot product** of two vectors produces a scalar value; the **cross product** of the same two vectors produces a vector quantity having a direction perpendicular to the original two vectors.

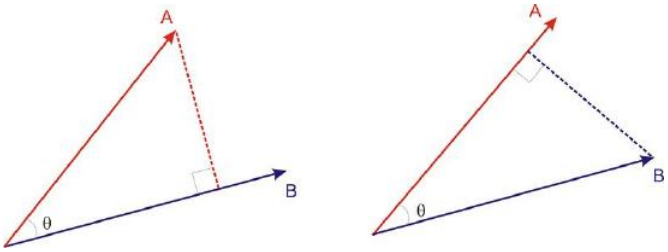
The cross product of two vector quantities is another vector whose magnitude varies as the angle between the two original vectors changes. The cross product is sometimes referred to as the **vector product** of two vectors. The magnitude of the cross product represents the area of the parallelogram whose sides are defined by the two vectors, as shown in the figure below. Therefore, the maximum value for the cross product occurs when the two vectors are perpendicular to one another, but when the two vectors are parallel to one another the magnitude of the cross product is equal to zero.



The algebraic form of the cross product equation is more complicated than that for the dot product. For two 3D vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \times \vec{B} = \langle (A_2B_3 - A_3B_2), (A_3B_1 - A_1B_3), (A_1B_2 - A_2B_1) \rangle$$

Another way to describe the process is to say that the cross product is the multiplication of one vector by the component of the other vector which is perpendicular to the first vector. In the diagram below are two vectors, ***A*** and ***B***. A perpendicular line has been drawn radially outward from ***B*** towards ***A*** to create a right triangle with ***A*** as the hypotenuse.

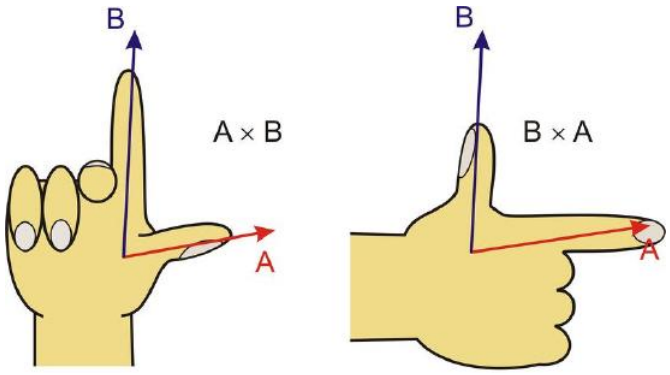


The component of $\vec{A}$ which is perpendicular to $\vec{B}$ is given by $A \sin \theta$ so the magnitude of the cross product can be written as $|\vec{A} \times \vec{B}| = \vec{A}(\vec{B} \sin \theta) = |\vec{A}||\vec{B}| \sin \theta$

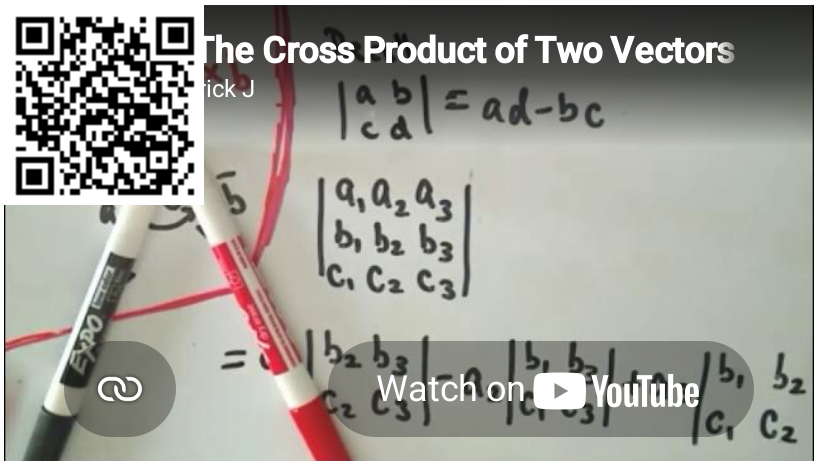
The direction of the cross product is perpendicular to the plane defined by the two crossed vectors. For example, the cross product of two vectors in the x-y plane will be parallel to the z-axis. This still leaves two possible directions for the cross product, though: either $+\hat{z}$ or $-\hat{z}$.

We use a **right-hand-rule** to indicate the direction of the cross product. Position the thumb and index finger of your right hand with the first vector along your thumb and the second vector along your index finger. Your middle finger, when extended

perpendicular to your palm, will indicate the direction of the cross product of the two vectors.



As you can see in the diagram above, $\vec{A} \times \vec{B}$ is along $+\hat{z}$ (coming up out of the page) while $\vec{B} \times \vec{A}$ is along $-\hat{z}$ (going down into the page) and $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

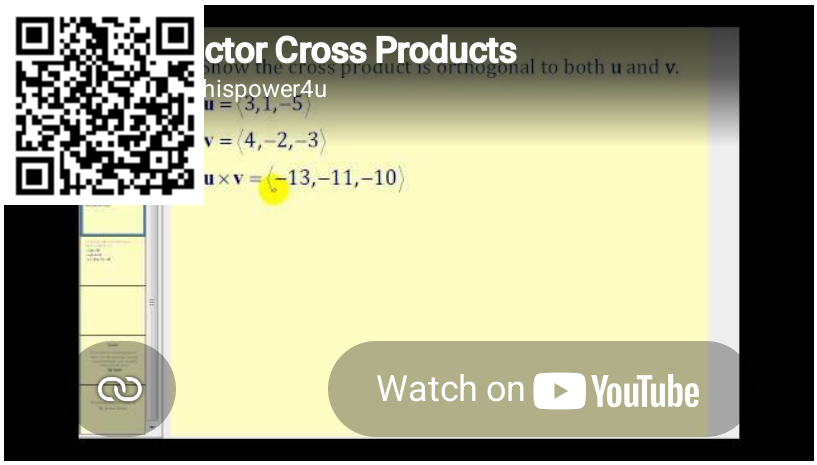


The Normal Vector

We can use the cross product and the definition of the **unit vector** to determine the direction which is perpendicular to a plane.

In general, we can define a normal vector, $\hat{n}$, which has a unity magnitude (i.e. magnitude equal to one) and which is perpendicular to a plane occupied by a pair of vectors, **U** and **V**.

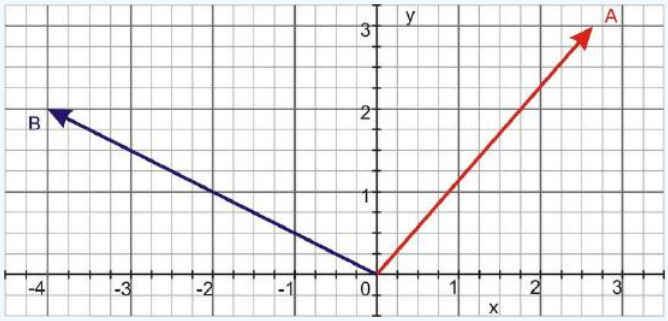
$\hat{n} = \frac{\vec{U} \times \vec{V}}{|\vec{U} \times \vec{V}|}$



Examples

Example 1

Calculate the cross product of the two vectors shown below.



Solution

Use the components of the two vectors to determine the cross product.

$$\vec{A} \times \vec{B} = \langle (A_y B_z - A_z B_y), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \rangle$$

Since these two vectors are both in the x-y plane, their own z-components are both equal to 0 and the vector product will be parallel to the z axis.

$$\vec{A} \times \vec{B} = \langle [(3 \cdot 0) - (0 \cdot 2)], [(0 \cdot -4) - (2.5 \cdot 0)], [(2.5 \cdot 2) - (3 \cdot -4)] \rangle$$
$$\vec{A} \times \vec{B} = \langle [(0) - (0)], [(0) - (0)], [(5) - (-12)] \rangle = \langle 0, 0, (5 + 12) \rangle = \langle 0, 0, 17 \rangle$$

We can check our answer using the sine version of the cross product, but first we need to know the angle between the two vectors. We can use the dot product to find θ. First use the components to find the dot product.

$$\vec{A} \times \vec{B} = A_x B_x + A_y B_y + A_z B_z = (2.5 * -4) + (3 * 2) + (0 * 0) = -10 + 6 + 0 = -4$$

Then find the magnitudes of the two vectors:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{2.5^2 + 3^2 + 0^2} = \sqrt{6.25 + 9 + 0} = \sqrt{15.25}$$
$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(-4)^2 + 2^2 + 0^2} = \sqrt{16 + 4 + 0} = \sqrt{20}$$

Then use these magnitudes with the cosine version of the dot product to find θ.

$$\vec{A} \times \vec{B} = |A||B| \cos\theta$$
$$-4 = \sqrt{15.25}\sqrt{20} \cos\theta$$
$$\cos\theta = \frac{-4}{\sqrt{305}} \approx \frac{-4}{17.5} = -0.229$$
$$\theta = 103^\circ$$

Now use the sine of this angle and the two magnitudes to determine the cross product:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$
$$|\vec{A} \times \vec{B}| = \sqrt{15.25}\sqrt{20} \sin 103^\circ = \sqrt{305} \sin 103^\circ = 17$$

This is the same answer that we obtained from the component notation, which is good. We use the right-hand rule to determine the direction of the vector product. If you place your thumb along vector **A** and your forefinger along vector **B**, your middle finger will point along $+\hat{z}$ and $|\vec{A} \times \vec{B}| = \langle 0, 0, 17 \rangle$

Example 2

The diagram shows two vectors **A** and **B** which define a plane passing through the origin. Use these two vectors to determine the normal vector to this plane. $\vec{A} = \langle 3, 0, 4 \rangle$ and $\vec{B} = \langle 5, 10, 0 \rangle$

Solution

The normal vector is defined by

$\hat{n} = \frac{\vec{U} \times \vec{V}}{|\vec{U} \times \vec{V}|}$

In this case, we obtain

$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

Use the component version of the cross-product equation to find the components of $\vec{A} \times \vec{B}$

$\vec{A} \times \vec{B} = \langle (A_y B_z - A_z B_y), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \rangle$

$\vec{A} \times \vec{B} = \langle [(0 \cdot 0) - (4 \cdot 10)], [(4 \cdot 5) - (3 \cdot 0)], [(3 \cdot 10) - (0 \cdot 5)] \rangle$

$\vec{A} \times \vec{B} = \langle (0 - 40), (20 - 0), (30 - 0) \rangle = \langle -40, 20, 30 \rangle$

Next, calculate the magnitude of the cross product, $|\vec{A} \times \vec{B}|$

$|\vec{A} \times \vec{B}| = \sqrt{(-40)^2 + 20^2 + 30^2} = \sqrt{1600 + 400 + 900} = \sqrt{2900} = 53.8516$

$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{\langle -40, 20, 30 \rangle}{53.9} = \left\langle \frac{-40}{53.9}, \frac{20}{53.9}, \frac{30}{53.9} \right\rangle = \langle -0.743, 0.371, 0.557 \rangle$

Example 3

Determine the cross product $\vec{F} \times \vec{r}$ for the two vectors $\vec{F} = \langle 2, 3, 4 \rangle$ and $\vec{r} = \langle 7, 6, 5 \rangle$. Then use the cross product to determine the angle between the two vectors.

Solution

One of the two ways to determine the magnitude of the cross product of two vectors uses the components of the two vectors:

$\vec{F} \times \vec{r} = \langle (F_y r_z - F_z r_y), (F_z r_x - F_x r_z), (F_x r_y - F_y r_x) \rangle$

$\vec{F} \times \vec{r} = \langle (3 \cdot 5 - 4 \cdot 6), (4 \cdot 7 - 2 \cdot 5), (2 \cdot 6 - 3 \cdot 7) \rangle = \langle (15 - 24), (28 - 10), (12 - 21) \rangle$

$\vec{F} \times \vec{r} = \langle -9, 18, -9 \rangle$

Now we can use the cross product and the second definition of the cross product to determine the angle between the two vectors.

$|\vec{F} \times \vec{r}| = |F||r| \sin\theta$

We need to calculate the magnitudes of the vectors and of the cross product.

$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2} = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29} = 5.385$

$|\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} = \sqrt{7^2 + 6^2 + 5^2} = \sqrt{49 + 36 + 25} = \sqrt{110} = 10.488$

$|\vec{F} \times \vec{r}| = \sqrt{(-9)^2 + 18^2 + (-9)^2} = \sqrt{81 + 324 + 81} = \sqrt{486} = 22.0454$

$\sin\theta = \frac{|\vec{F} \times \vec{r}|}{|F||r|} = \frac{22.0454}{(5.385)(10.488)} = 0.390$

$\theta = \sin^{-1}(0.390) = 22.98^\circ$

We can use the dot product of the two vectors to check our solution.

$\vec{F} \times \vec{r} = |\vec{F}||\vec{r}| \cos\theta$

$\vec{F} \times \vec{r} = F_x r_x + F_y r_y + F_z r_z = 2 * 7 + 3 * 6 + 4 * 5 = 14 + 18 + 20 = 52$

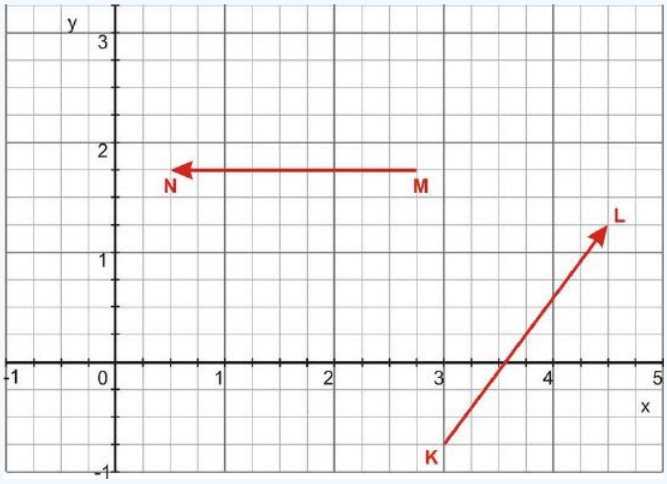
$\cos\theta = \frac{\vec{F} \times \vec{r}}{|\vec{F}||\vec{r}|} = \frac{52}{(5.385)(10.488)} = 0.920714$

$\theta = \cos^{-1}(0.920714) = 22.97$

This answer matches our value from the cross product to within rounding variations.

Example 4

Determine the magnitude of the cross product of the two vectors shown below.



Solution

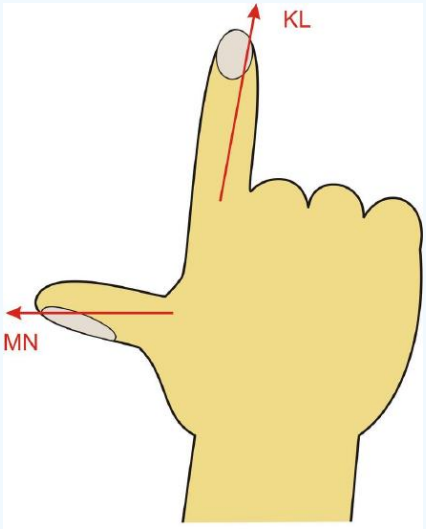
First we need to identify the components of the two vectors by using the information given on the graph. In this case, $\overrightarrow{MN} = \langle -2.25, 0, 0 \rangle$ and $\overrightarrow{KL} = \langle 1.5, 2, 0 \rangle$

$$\overrightarrow{MN} \times \overrightarrow{KL} = \langle (MN_y KL_z - MN_z KL_y), (MN_z KL_x - MN_x KL_z), (MN_x KL_y - MN_y KL_x) \rangle$$

$$\overrightarrow{MN} \times \overrightarrow{KL} = \langle (0 \cdot 0 - 0 \cdot 2), (0 \cdot 1.5 - (-2.25) \cdot 0), ((-2.25) \cdot 2 - 0 \cdot 1.5) \rangle$$

$$\overrightarrow{MN} \times \overrightarrow{KL} = \langle 0 - 0, 0 - 0, -4.5 - 0 \rangle = \langle 0, 0, -4.5 \rangle$$

As we can see by the components, this vector has a magnitude of 4.5 units and lies in the –z direction. We can also use the right-hand-rule to see the direction of the cross product. As shown in the figure below, if we align the right thumb with vector MN and the right fore-finger with vector KL, the palm and extended middle-finger point in the –z direction.



Example 5

A plane passing through the origin is defined by the two vectors, $\vec{W} = \langle 4, 5, 2 \rangle$ and $\vec{L} = \langle 8, 1, 9 \rangle$. Determine the equation of a unit vector representing a direction perpendicular to this plane.

Solution

To solve this problem we need to use the definition of the normal vector $\hat{n} = \frac{\vec{W} \times \vec{L}}{|\vec{W} \times \vec{L}|}$, the component form of the definition of the cross product,

$\vec{W} \times \vec{L} = \langle (W_y L_z - W_z L_y), (W_z L_x - W_x L_z), (W_x L_y - W_y L_x) \rangle$. In this case, we obtain

$\vec{W} \times \vec{L} = \langle (5 \cdot 9 - 2 \cdot 1), (2 \cdot 8 - 4 \cdot 9), (4 \cdot 1 - 5 \cdot 8) \rangle$

$\vec{W} \times \vec{L} = \langle (45 - 2), (16 - 36), (4 - 40) \rangle = \langle 43, -20, -36 \rangle$

We also need to know the magnitude of this cross product

$|\vec{W} \times \vec{L}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(43)^2 + (-20)^2 + (-36)^2} = \sqrt{1849 + 400 + 1296} =$
$\sqrt{3545} = 59.54$

Now we can determine the normal vector

$\hat{n} = \frac{\vec{W} \times \vec{L}}{|\vec{W} \times \vec{L}|} = \frac{\langle 43, -20, -36 \rangle}{59.54} = \left\langle \frac{43}{59.54}, \frac{-20}{59.54}, \frac{-36}{59.54} \right\rangle = \langle 0.7222, -0.3359, -0.6046 \rangle$

Example 6

Determine the area of a parallelogram whose sides are defined by the vectors $\vec{w} = \langle 85, 89, 91 \rangle$ and $\vec{h} = \langle 67, 70, 88 \rangle$, lengths measured in centimeters.

Solution

The area of the parallelogram whose sides are defined by a pair of vectors is equal to the magnitude of the cross product of the two vectors, $|\vec{w} \times \vec{h}|$. First we need to find the cross product of the two vectors:

$\vec{w} \times \vec{h} = \langle (w_y h_z - w_z h_y), (w_z h_x - w_x h_z), (w_x h_y - w_y h_x) \rangle$

$\vec{w} \times \vec{h} = \langle (89 \cdot 88 - 91 \cdot 70), (91 \cdot 67 - 85 \cdot 88), (85 \cdot 70 - 89 \cdot 67) \rangle$

$\vec{w} \times \vec{h} = \langle (7832 - 6370), (6097 - 7480), (5950 - 5963) \rangle = \langle 1462, -1383, -13 \rangle$

$|\vec{w} \times \vec{h}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{1462^2 + (-1383)^2 + (13)^2} = \sqrt{4050302} \approx 2012.5$

Since the lengths of the two vectors were measured in centimeters, the area of the parallelogram is 2013 cm^2 measured to the nearest square centimeter.

Example 7

Determine the cross product of the two vectors $\vec{f} = \langle 3, 13, 11 \rangle$ and $\vec{g} = \langle 9, 6, 15 \rangle$

Solution

$\vec{f} \times \vec{g} = \langle (f_y g_z - f_z g_y), (f_z g_x - f_x g_z), (f_x g_y - f_y g_x) \rangle$

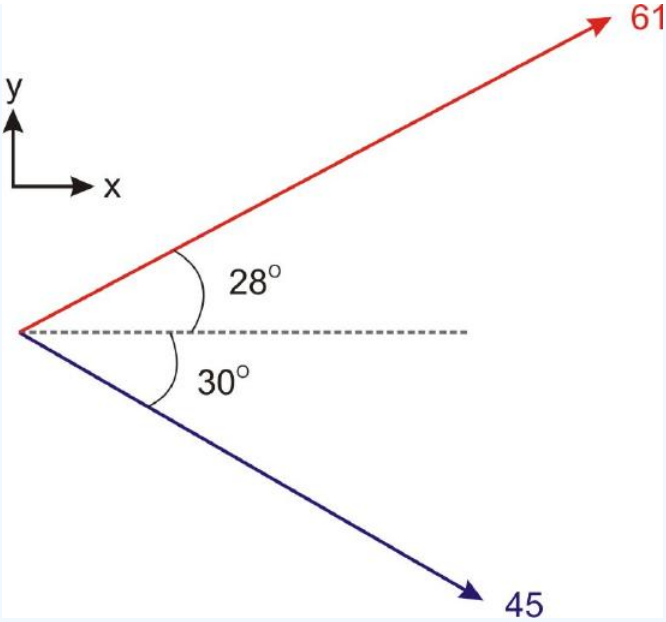
$\vec{f} \times \vec{g} = \langle (13 \cdot 15 - 11 \cdot 6), (11 \cdot 9 - 3 \cdot 15), (3 \cdot 6 - 13 \cdot 9) \rangle$

$\vec{f} \times \vec{g} = \langle (195 - 66), (99 - 45), (18 - 117) \rangle = \langle 129, 54, -99 \rangle$

Example 8

Determine the magnitude of the cross-product of these two vectors.

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Solution

Since we know the magnitudes of the two vectors and the angle between them, we can use the angle-version of the cross-product equation to determine the magnitude of the cross-product:

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta = (61)(45)\sin 58 = 2328$

Since these two vectors lie in the x-y plane, the direction of the cross-product will be parallel to the z-axis.

Review

Calculate the cross products:

1. Vectors $c = -6i + 2j + 3k$ and $a = -6i + 2j + 13k$
2. Vectors $v = \langle -1, 4, 7 \rangle$ and $u = \langle -5, 10, 3 \rangle$
3. Vectors $f = -6i + 8j - 6k$ and $s = -3i + 15j + 19k$
4. Vectors $\mathrm{j} = -3i + 15j - 4k$ and $\mathrm{t} = 7i + 10j + 6k$
5. Vectors $r = \langle 3, 13, -1 \rangle$ and $v = \langle 7, 6, 1 \rangle$
6. Vectors $e = \langle -1, 8, -3 \rangle$ and $a = \langle -2, 1, 19 \rangle$
7. Vectors $\mathrm{j} = -3i + 17j + 6k$ and $h = 8i + 9j + 7k$
8. Vectors $a = 9i + 10j + 9k$ and $g = -5i + 19j + 15k$
9. Vectors $\mathrm{j} = 4i + 18j - 8k$ and $m = 2i + j + 19k$
10. What is the cross product of $\langle -2, 1, -2 \rangle$ and $\langle 5, 6, 9 \rangle$?
11. Find a vector orthogonal to both $\langle 1, 20, 2 \rangle$ and $\langle 4, 2, 3 \rangle$
12. Vectors $y = 5i + 6j + 6k$ and $f = -4i + 9j + 3k$. What is the area of the parallelogram formed by having y and f as adjacent sides?
13. What is the area of the parallelogram formed by having $\langle -2, 1, 7 \rangle$ and $\langle 5, 7, 16 \rangle$ as adjacent sides?
14. What is the cross product between $\langle 8, 6, 8 \rangle$ and $\langle 8, 6, 8 \rangle$?
15. Vectors $g = -6i + 9j - 7k$ and $y = -24i + 36j - 28k$ What is the cross product between g and y?
16. A boat is sailing on a bearing of 89° east of north at 564 feet per min. A tail wind is adding to the plane's velocity and blowing 78° west of north at 25 feet per min. Determine the actual speed of the plane in mph.
17. A plane is flying on a bearing of 77° east of south at 606 mph. A tail wind is adding to the plane's velocity and blowing 33° west of south at 80 mph. Determine the direction of the plane.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.6.

Vocabulary

Term	Definition
Cross product	The cross product of two vectors is a third vector that is perpendicular to both of the original vectors.
dot product	The dot product is also known as inner product or scalar product. The two forms of the dot product are $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$ and $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$
normal vector	A normal vector is a vector that is perpendicular to a given surface or plane. A unit normal vector is a normal vector with a magnitude of one.
Right-hand-rule	The right-hand-rule is used to indicate the direction of a cross product. Position the thumb and index finger of your right hand with the first vector along your thumb and the second vector along your index finger. Your middle finger, when extended perpendicular to your palm, will indicate the direction of the cross product of the two vectors.
unit vector	A unit vector is a vector with a magnitude of one.

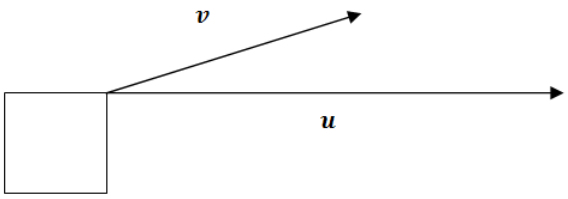
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5.2.4: Vector Projection

Vector Projection

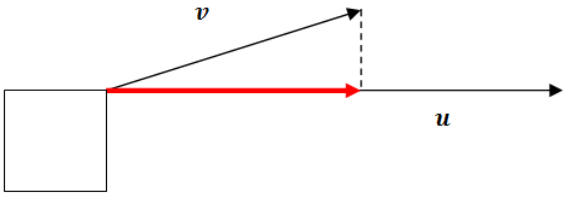
Projecting one vector onto another explicitly answers the question: “how much of one vector goes in the direction of the other vector?” The dot product is useful because it produces a scalar quantity that helps to answer this question. In this concept, you will produce an actual vector not just a scalar.

Why is **vector projection** useful when considering pulling a box in the direction of v instead of horizontally in the direction of u?



Projections

Consider the question from above.



The definition of **vector projection** for the indicated red vector is the called $\text{proj}_u v$. When you read $\text{proj}_u v$ you should say “the vector projection of v onto u.” This implies that the new vector is going in the direction of u. The vector projection is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector. The parallel vector is the vector projection.

Conceptually, this means that if someone was pulling the box at an angle and strength of vector v then some of their energy would be wasted pulling the box up and some of the energy would actually contribute to pulling the box horizontally.

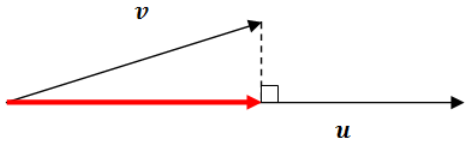
The definition of **scalar projection** is simply the length of the vector projection. When the scalar projection is positive it means that the angle between the two vectors is less than $90\circ$. When the scalar projection is negative it means that the two vectors are heading in opposite directions.

The vector projection formula can be written two ways. The version on the left is most simplified, but the version on the right makes the most sense conceptually. A.

$$\text{proj}_u v = \left(\frac{v \cdot u}{|u|^2}\right) u = \left(\frac{v \cdot u}{|u|}\right) \frac{u}{|u|}$$

The proof of the **vector projection formula** is as follows:

Given two vectors u, v, what is $\text{proj}_u v$?



First note that the projected vector in red will go in the direction of u. This means that it will be a product of the **unit vector** $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). In order to find the scalar projection, note the right triangle, the unknown

angle θ between the two vectors and the cosine ratio.

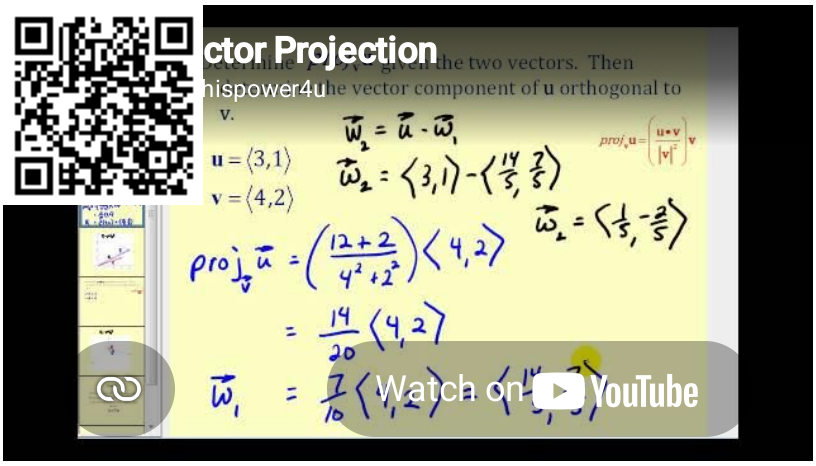
$$\cos\theta = \frac{\text{scalar projection}}{|v|}$$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now just substitute and simplify to find the length of the scalar projection.

$$\cos\theta = \frac{\text{scalar projection}}{|v|}$$
$$\frac{u \cdot v}{|u||v|} = \frac{\text{scalar projection}}{|v|}$$
$$\frac{u \cdot v}{|u|} = \text{scalar projection}$$

Now you have the length of the vector projection and the direction you want it to go:

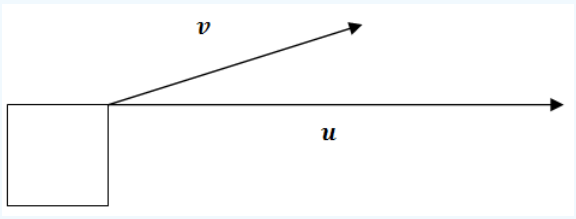
$$\text{proj}_u\, v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$$



Examples

Example 1

Earlier, you were asked why vector projection useful when considering pulling a box in the direction of instead of horizontally in the direction of u. Vector projection is useful in physics applications involving force and work.



Solution

When the box is pulled by vector v some of the force is wasted pulling up against gravity. In real life this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

Example 2

Find the scalar projection of vector $v = < 3, 4 >$ onto vector $u = < 5, -12 >$.

Solution

As noted earlier, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u \cdot v}{|u|}\right)$ where u is the vector being projected onto.

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$\frac{u \cdot v}{|u|} = \frac{<5,-12> \cdot <3,4>}{13} = \frac{15-48}{13} = -\frac{33}{13}$

Example 3

Find the vector projection of vector $v = < 3, 4 >$ onto vector $u = < 5, -12 >$

Solution

Since the scalar projection has already been found in Example 2, you should multiply the scalar by the "onto" unit vector.

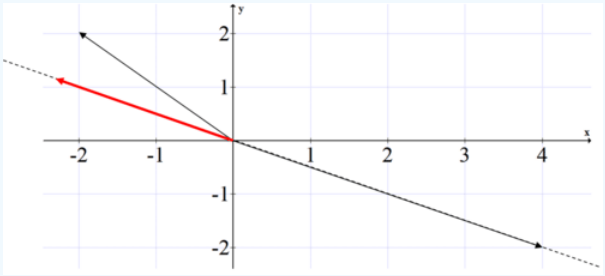
$-\frac{33}{13} < \frac{5}{13}, -\frac{12}{13} > = < -\frac{165}{169}, \frac{396}{169} >$

Example 4

Sketch the vector $< -2, -2 >$ and $< 4, -2 >$. Explain using a sketch why a negative scalar projection of $< -2, -2 >$ onto $< 4, -2 >$ makes sense.

Solution

First plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $< -2, 2 >$ goes in the opposite direction of the vector $< 4, -2 >$. This will create a vector projection going in the opposite direction of $< 4, -2 >$.

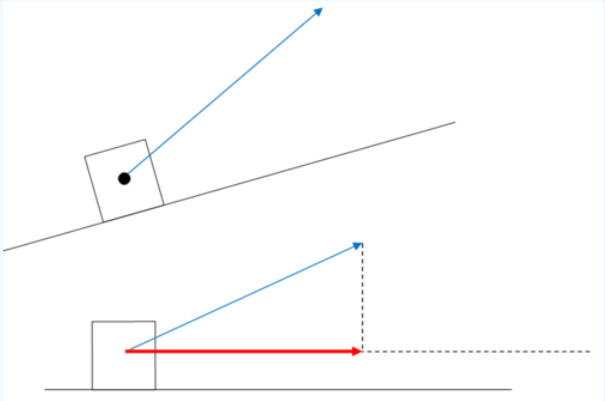


Example 5

A father is pulling his daughter up a hill. The hill has a 20º incline. The daughter is on a sled that sits on the ground and has a rope that the father pulls with a force of 100 lb as he walks. The rope makes a 39º angle with the slope. What is the effective force that the father exerts moving his daughter and the sled up the hill?

Solution

The box represents the girl and the sled. The blue arrow indicates the father's 100 lb force. Notice that the question asks for simply the amount of force which means scalar projection. Since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.



The components of the father's force vector is $100 < \cos 39°, \sin 39° >$ and the "onto" vector is any vector horizontally to the right. Since we are only looking for the length of the horizontal component and you already have the angle between the two

vectors, the scalar projection is:

$100 \cdot \cos 39^\circ \approx 77.1 \text{lb}$

Review

1. Sketch vectors $< 2, 4 >$ and $< 2, 1 >$.
2. What is the vector projection of $< 2, 4 >$ onto $< 2, 1 >$? Sketch the projection.
3. Sketch vectors $< -2, 1 >$ and $< -1, 3 >$.
4. What is the vector projection of $< -1, 3 >$ onto $< -2, 1 >$? Sketch the projection.
5. Sketch vectors $< 6, 2 >$ and $< 8, 1 >$.
6. What is the vector projection of $< 6, 2 >$ onto $< 8, 1 >$? Sketch the projection.
7. Sketch vectors $< 1, 7 >$ and $< 6, 3 >$.
8. What is the vector projection of $< 1, 7 >$ onto $< 6, 3 >$? Sketch the projection.
9. A box is on the side of a hill inclined at 30°. The weight of the box is 40 pounds. What is the magnitude of the force required to keep the box from sliding down the hill?
10. Sarah is on a sled on the side of a hill inclined at 60°. The weight of Sarah and the sled is 125 pounds. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill.
11. A 1780 pound car is parked on a street that makes an angle of 15° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.
12. A 1900 pound car is parked on a street that makes an angle of 10° with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.
13. A 30 pound force that makes an angle of 32° with an inclined plane is pulling a box up the plane. The inclined plane makes a 20° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?
14. A 22 pound force that makes an angle of 12° with an inclined plane is pulling a box up the plane. The inclined plane makes a 25° angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?
15. Anne pulls a wagon on a horizontal surface with a force of 50 pounds. The handle of the wagon makes an angle of 30° with the ground. What is the magnitude of the effective force pulling the wagon?

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.5.

Vocabulary

Term	Definition
scalar projection	The scalar projection of a vector onto another vector is equal to the length of the projection of the first vector onto the second vector.
unit vector	A unit vector is a vector with a magnitude of one.
vector projection	The vector projection of a vector onto a given direction is a vector with a magnitude equal to the scalar projection. The direction of the vector projection is the same as the unit vector of that given direction.

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1. [Figure 1]
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SECTION OVERVIEW

5.3: Planes in Space

Topic hierarchy

5.3.1: Distance Between a Point and a Plane

5.3.2: Vector Direction

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5.3.1: Distance Between a Point and a Plane

Distance Between a Point and a Plane

The distance between a specific point and a **plane** is important to a number of different activities.

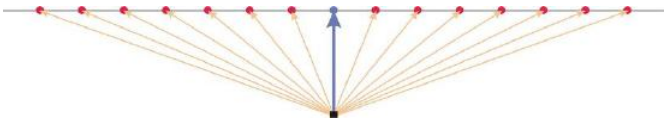
For instance, a bungee jumping tower would not be very safe if the distance to the ground were not measured at the point directly under the tower, since any angle away from straight down would make the distance measure further and lead to a cord too long!

A computer game programmer needs to know how to calculate the distance between the location of a character on the screen and the walls around it to tell the game how to identify when a projectile hits a target, or when the character hits a wall.

Distance Between a Point and a Plane

Identifying the Point Closest to the Origin

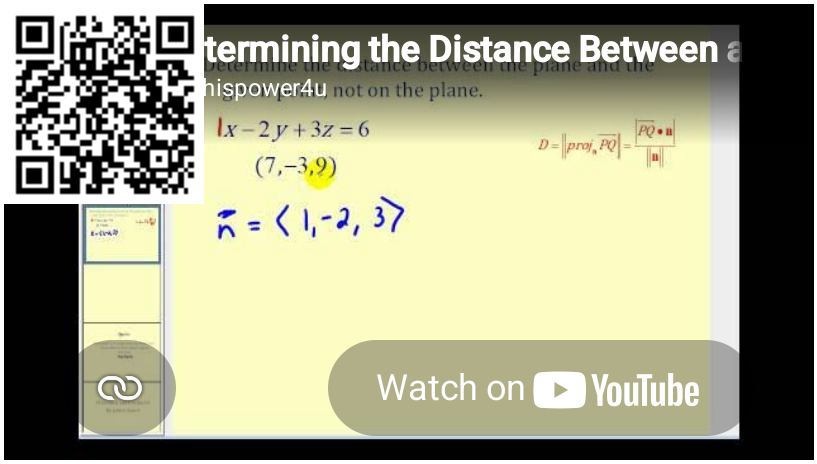
No matter what the orientation of a plane, there will always be one point located closer to the origin than any other point on the plane. This means that the **position vector** for this point is shorter than any other point in the plane. The diagram below shows a two-dimensional projection of a plane, in grey, near a point not on the plane, in black. The position vectors to a variety of points are shown in the diagram. The position vector marked in blue is shorter than the position vectors for the other points. This shortest vector is **perpendicular** to the plane. You can also see that the blue line is the vector projection of any orange vector onto the perpendicular direction.

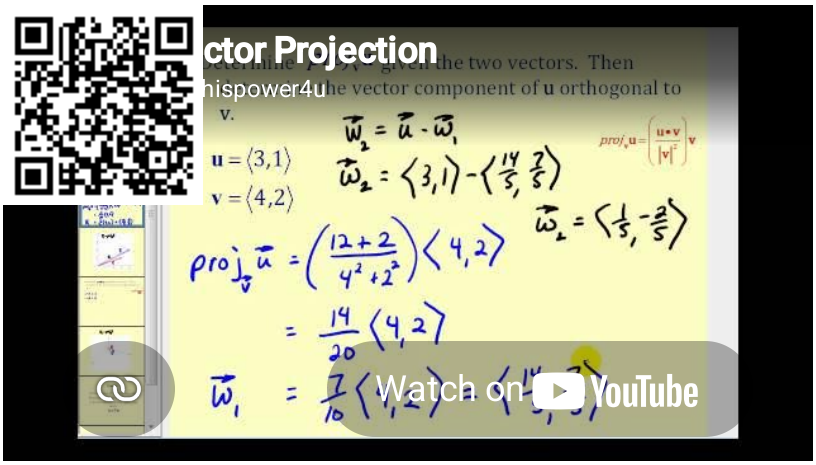


This **orthogonality** (i.e. being perpendicular) is useful for us because it means that the position vector for this special point is parallel to the normal vector. Therefore, if we know the equation for a normal vector and the position vector for any point on the plane, we can determine the location of the point on the plane closest to the origin by finding the projection of the given point's position vector onto the normal direction.

The Dihedral Angle

The angle between two planes is called the *dihedral angle*. The angle between two planes is the same as the angle between their normal vectors. If we want to determine the dihedral angle between two planes, we identify normal vectors to the two planes, then we can use the dot-product of the two normal vectors to determine the angle between the two normals which is also the two planes. Recall $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \cos\theta$.





Examples

Example 1

The three points $P=(3,7,2), Q=(1,4,3)$, and $R=(2,3,4)$ define a plane. Determine the point on the plane which is closest to the origin.

Solution

First find the vectors between two pairs of the points.

$$\overrightarrow{PQ}=\langle(Q_x-P_x),(Q_y-P_y),(Q_z-P_z)\rangle=\langle(1-3),(4-7),(3-2)\rangle=\langle-2,-3,1\rangle$$
$$\overrightarrow{PR}=\langle(R_x-P_x),(R_y-P_y),(R_z-P_z)\rangle=\langle(2-3),(3-7),(4-2)\rangle=\langle-1,-4,2\rangle$$

The cross product of these two vectors is normal to the plane.

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle(PQ_yPR_z-PQ_zPR_y),(PQ_zPR_x-PQ_xPR_z),(PQ_xPR_y-PQ_yPR_x)\rangle$$
$$\overrightarrow{PQ}\times\overrightarrow{PR}=\langle[(-3\cdot 2)-(1\cdot -4)],[(1\cdot -1)-(-2\cdot 2)],[(-2\cdot -4)-(-1\cdot -3)]\rangle$$
$$\vec{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\langle[(-6)-(-4)],[(-1)-(-4)],[(8)-(3)]\rangle=\langle-2,3,5\rangle$$

The point on the plane which is nearest to the origin can be found by determining the projection of the position vector of either of these three points onto the normal vector. Remember that the vector projection of one vector onto the direction of another, is given by the dot-product of the first vector onto the unit vector defining the direction of the second vector: $(\vec{A}\times\vec{B})\vec{B}$.

Since we know three points on the plane, we can use one of them to solve the problem. Let's start with point P. The vector projection of $\vec{P}$ onto $\hat{n}$ is given by $(\vec{P}\times\hat{n})\hat{n}$, so first we need to determine the unit vector $\hat{n}$ which is given by

$$\hat{n}=\frac{\hat{n}}{|\vec{n}|}=\frac{\langle n_x,n_y,n_z\rangle}{\sqrt{n_x^2+n_y^2+n_z^2}}=\frac{\langle-2,3,5\rangle}{\sqrt{(-2)^2+3^2+(5)^2}}=\frac{\langle-2,3,5\rangle}{\sqrt{38}}=\langle-0.32,0.49,0.81\rangle$$
$$\vec{P}\times\hat{n}=P_x\hat{n}_x+P_y\hat{n}_y+P_z\hat{n}_z=(3)(-0.32)+(7)(0.49)+(2)(0.81)=$$
$$-0.96+3.43+1.62=4.09$$
$$(\vec{P}\times\hat{n})\hat{n}=(4.09)\langle-0.32,0.49,0.81\rangle=\langle-1.3088,2.0041,3.3129\rangle$$

Therefore, the point on the plane closest to the origin is (-1.3088, 2.0041, 3.3129).

Example 2

The three points $P=(3,7,2), Q=(1,4,3)$, and $R=(2,3,4)$ define a plane. Determine the dihedral angle between this plane and the x-y plane.

Solution

As we saw in the example above, these three points define a plane which has a normal vector $\vec{n}=\langle-2,3,5\rangle$

The normal to the x-y plane is the unit vector $\hat{z} = \langle 0, 0, 1 \rangle$. To find the angle between these two vectors we use the fact that $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|$ and that $\vec{A} \times \vec{B} = A_x B_x + A_y B_y + A_z B_z$

First find a numerical value for the dot product:

$$\vec{n} \times \hat{z} = n_x z_x + n_y z_y + n_z z_z = (-2 \cdot 0) + (3 \cdot 0) + (5 \cdot 1) = 5$$
$$|\vec{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2} = \sqrt{(-2)^2 + (3)^2 + (5)^2} = \sqrt{4 + 9 + 25} = \sqrt{38}$$
$$|\hat{z}| = \sqrt{z_x^2 + z_y^2 + z_z^2} = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Then find the cosine version of the dot product:

$$\vec{n} \times \hat{z} = \sqrt{38} \cos\theta$$

Now equate the two and solve for the angle, θ

$$\vec{n} \times \hat{z} = 5 = \sqrt{38} \cos\theta$$
$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{38}}\right) = 62.5^\circ$$

Example 3

Determine the dihedral angle between the two planes $12x + 23y + 14z - 5 = 0$ and $7x + 3y + z + 12 = 0$.

Solution

The dihedral angle is defined as the angle between two planes. This angle is also equal to the angle between the normals to the two planes. In two of the previous problems we determined the unit vectors which are perpendicular to these two planes $\vec{n_1} = \left\langle \frac{12}{29.5}, \frac{23}{29.5}, \frac{14}{29.5} \right\rangle$ and $\vec{n_2} = \left\langle \frac{7}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{1}{\sqrt{59}}, \right\rangle$. We can then use the dot-product of these two normal vectors to determine the angle between the two. The dot-product is defined as $\vec{A} \times \vec{B} = A_x B_x + A_y B_y + A_z B_z + ...$ and as $\vec{A} \times \vec{B} = |A||B| \cos\theta$. First, we need to find the component version of the dot product and the magnitudes of the two normal vectors.

$$\vec{n_1} \times \vec{n_2} = \overrightarrow{n_{1x} n_{2x}} + \overrightarrow{n_{1y} n_{2y}} + \overrightarrow{n_{1z} n_{2z}} = \frac{12 \cdot 7}{29.5\sqrt{59}} + \frac{23 \cdot 3}{29 \cdot 5\sqrt{59}} + \frac{14 \cdot 1}{29.5\sqrt{59}}$$
$$\vec{n_1} \times \vec{n_2} = \frac{12 \cdot 7}{29.5\sqrt{59}} + \frac{23 \cdot 3}{29.5\sqrt{59}} + \frac{14 \cdot 1}{29.5\sqrt{59}} = \frac{119}{226.6} + \frac{69}{226.6} + \frac{14}{226.6} = \frac{202}{226.6} = 0.891$$

Since these two vectors are unit-vectors, their magnitudes are both equal to 1.

$$\cos\theta = \frac{\vec{n_1} \times \vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|} = \frac{0.891}{(1)(1)} = 0.891$$
$$\theta = \cos^{-1} 0.891 = 27.0^\circ$$

Example 4

Determine the angle between the plane $2x - 5y + 8 - 10 = 0$ and the y-z plane.

Solution

The dihedral angle is defined as the angle between the two planes and is also equal to the angle between the two normal unit vectors. In this case, we already know the normal unit vector for the y-z plane, $\hat{x} = \langle 1, 0, 0 \rangle$. We still need to determine, however, the unit vector for the plane $2x - 5y + 8z - 10 = 0$.

Comparing this equation to $n_x x + n_y y + n_z z + d = 0$, we can see that $\vec{n} = \langle 2, -5, 8 \rangle$.

Now we can use the definition of the unit vector

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\langle n_x, n_y, n_z \rangle}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{\langle 2, -5, 8 \rangle}{\sqrt{2^2 + (-5)^2 + 8^2}} = \frac{\langle 2, -5, 8 \rangle}{\sqrt{4 + 25 + 64}} = \frac{\langle 2, -5, 8 \rangle}{9.64} = \left\langle \frac{2}{9.64}, \frac{-5}{9.64}, \frac{8}{9.64} \right\rangle$$

The angle between the two planes is equal to the angle between the two normal vectors.

We can then use the dot-product of these two normal vectors to determine the angle between the two. The dot-product is defined as $\vec{A} \times \vec{B} = A_x B_x + A_y B_y + A_z B_z + ...$ and as $\vec{A} \times \vec{B} = |A||B| \cos\theta$. First, we need to find the component version of the dot product and the magnitudes of the two normal vectors.

$$\vec{n_1} \times \vec{n_2} = \vec{n_{1x}}\vec{n_{2x}} + \vec{n_{1y}}\vec{n_{2y}} + \vec{n_{1z}}\vec{n_{2z}} = \frac{2 \cdot 1}{9.64} + \frac{-5 \cdot 0}{9.64} + \frac{8 \cdot 0}{9.64} = \frac{2}{9.64} = 0.2074$$

Since these two vectors are unit-vectors, their magnitudes are both equal to 1.

$$\cos\theta = \frac{\vec{n_1} \times \vec{n_2}}{\left|\vec{n_1}\right|\left|\vec{n_2}\right|} = \frac{0.2074}{(1)(1)} = 0.2074$$

$$\theta = \cos^{-1} 0.2074 = 16.18°$$

Example 5

The three points $\vec{P} = \langle -2, 3, 4 \rangle, \vec{Q} = \langle 5, -6, 7 \rangle$, and $\vec{R} = \langle 8, 9, -1 \rangle$ identify a plane. Determine the point on the plane which is closest to the origin.

Solution

The point on the plane nearest to the origin can be found by determining the projection of the position vector of one of these three points onto the normal vector. Remember that the vector projection of one vector onto the direction of another is given by the dot-product of the first vector onto the unit vector defining the direction of the second vector: $(\vec{P} \times \hat{n})\hat{n}$.

We can use the position vectors for the three points to determine two vectors within the plane. Once we have those two vectors, their cross-product will define the direction normal to the plane. First find the two equations in the plane:

$$\vec{A} = \vec{Q} - \vec{P} = \langle 5, -6, 7 \rangle - \langle -2, 3, 4 \rangle = \langle 7, -9, 3 \rangle$$

$$\vec{B} = \vec{R} - \vec{P} = \langle 8, 9, -1 \rangle - \langle -2, 3, 4 \rangle = \langle 10, 6, -5 \rangle$$

Now determine the cross product of the two vectors

$$\vec{n} = \vec{A} \times \vec{B} = \langle (A_y B_z - A_z B_y), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \rangle$$

$$\vec{n} = \vec{A} \times \vec{B} = \langle (45 - 18), (30 - (-35)), (42 + 90) \rangle$$

$$\vec{n} = \vec{A} \times \vec{B} = \langle 27, 65, 132 \rangle$$

Now we need to determine the unit vector associated with this normal vector

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\langle n_x, n_y, n_z \rangle}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{\langle 27, 65, 132 \rangle}{\sqrt{27^2 + (65)^2 + (132)^2}} = \frac{\langle 27, 65, 132 \rangle}{\sqrt{22378}}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \langle 0.181, 0.435, 0.882 \rangle$$

Now we determine the vector progression of one of the three initial position vectors onto the direction of this normal unit-vector: $(\vec{P} \times \hat{n})\hat{n}$. Remember that the dot product is given by

$$\vec{A} \times \vec{B} = A_x B_x + A_y B_y + A_z B_z + ...$$

$$(\vec{P} \times \hat{n})\hat{n} = (-2(0.181) + 3(0.435) + 4(0.882))\langle 0.181, 0.435, 0.882 \rangle$$

$$(\vec{P} \times \hat{n})\hat{n} = (4.471)\langle 0.181, 0.435, 0.882 \rangle$$

$$(\vec{P} \times \hat{n})\hat{n} = (4.471)\langle 0.181, 0.435, 0.882 \rangle = \langle 0.809, 1.945, 3.943 \rangle$$

Example 7

Determine the point on the plane $7x + 3y + z + 12 = 0$ which is closest to the origin.

Solution

The point on the plane nearest to the origin can be found by determining the projection of the position vector of any point on the plane onto the normal vector. The vector projection of one vector onto the direction of another is given by the dot-product of the first vector onto the unit vector defining the direction of the second vector: $(\vec{P} \times \hat{n})\hat{n}$.

In this case, we can determine a normal vector using the equation of the plane. Comparing $7x + 3y + z + 12 = 0$ to the generic equation $n_x x + n_y y + n_z z + d = 0$, we can see that $\vec{n} = \langle 7, 3, 1 \rangle$ and

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\langle n_x, n_y, n_z \rangle}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \frac{\langle 7, 3, 1 \rangle}{\sqrt{(7)^2 + (3)^2 + (1)^2}} = \frac{\langle 7, 3, 1 \rangle}{\sqrt{49 + 9 + 1}} = \frac{\langle 7, 3, 1 \rangle}{\sqrt{59}} = \left\langle \frac{7}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{1}{\sqrt{59}} \right\rangle$$

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We also need to know the location of a point on the plane. If we write the equation of the plane in intercept form, we can determine the position vector for the x-, y-, and z-intercepts of the plane.

The equation $1 = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$ must be true for all points on a plane. Therefore, we should first rearrange $7x + 3y + z + 12 = 0$ into the form $1 = \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$.

$7x + 3y + z = -12$ becomes $\frac{7}{-12}x + \frac{3}{-12}y + \frac{1}{-12}z = 1$

Therefore, $a = \frac{-12}{7}, b = \frac{-12}{3} = -4$, and $c = \frac{-12}{1} = -12$ and the position vectors of the three intercepts are $\vec{A} = \langle -1.714, 0, 0 \rangle, \vec{B} = \langle 0, -4, 0 \rangle$, and $\vec{C} = \langle 0, 0, -12 \rangle$.

To complete the problem, compute the dot product.

$$(\vec{B} \times \hat{n})\hat{n} = (B_x\hat{n}_x + B_y\hat{n}_y + B_z\hat{n}_z)\,\hat{n} = \left(0\left(\frac{7}{\sqrt{59}}\right) - 4\left(\frac{3}{\sqrt{59}}\right) + 0\left(\frac{1}{\sqrt{59}}\right)\right)\left\langle \frac{7}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{1}{\sqrt{59}} \right\rangle$$

$$(\vec{B} \times \hat{n})\hat{n} = \frac{-12}{\sqrt{59}}\left\langle \frac{7}{\sqrt{59}}, \frac{3}{\sqrt{59}}, \frac{1}{\sqrt{59}} \right\rangle = \left\langle \frac{-84}{59}, \frac{-36}{59}, \frac{-12}{59} \right\rangle = \langle -1.424, -0.610, -0.203 \rangle$$

Review

The three points define a plane. Determine the point on the plane which is closest to the origin.

1. $P = (3, 6, 9), Q = (9, 6, 3)$ and $R = (6, -9, 9)$
2. $P = (1, -7, 2), Q = (4, 2, 9)$ and $R = (3, -5, 1)$
3. $P = (3, 8, 10), Q = (-2, 5, 8)$ and $R = (7, 4, 8)$
4. $P = (9, -1, 4), Q = (6, 2, -8)$ and $R = (12, 9, 10)$
5. $P = (5, 8, -9), Q = (-5, 3, 9)$ and $R = (10, 4, -6)$

Determine the dihedral angle between each of the planes in questions 1-5 and the x-y plane. Use the $|\vec{n}|$ you calculated for each plane and recall that the normal to the x-y plane is the unit vector $\hat{z} = \langle 0, 0, 1 \rangle$.

6. $P = (3, 6, 9), Q = (9, 6, 3)$ and $R = (6, -9, 9)$
7. $P = (1, -7, 2), Q = (4, 2, 9)$ and $R = (3, -5, 1)$
8. $P = (3, 8, 10), Q = (-2, 5, 8)$ and $R = (7, 4, 8)$
9. $P = (9, -1, 4), Q = (6, 2, -8)$ and $R = (12, 9, 10)$
10. $P = (5, 8, -9), Q = (-5, 3, 9)$ and $R = (10, 4, -6)$

Determine the dihedral angle between the two planes.

11. $9x + 17y - 4z - 7 = 0$ and $-17x + 24y + 14z + 2 = 0$
12. $2x - 4y + 10z - 11 = 0$ and $2x - 9y + 4z + 12 = 0$
13. $-7x + 20y + 6z + 4 = 0$ and $-19x - 3y + z + 5 = 0$
14. $5x - 8y + 20z - 5 = 0$ and $6x + y + 19z - 7 = 0$
15. $14x + 11y - 5z - 16 = 0$ and $11x - 13y + 8z + 4 = 0$
16. $-10x + 9y + 2z + 8 = 0$ and $21x + 7y - 4z + 15 = 0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.8.

Vocabulary

Term	Definition
Dihedral Angle	A dihedral angle is an angle between two planes in three dimensional space.
Origin	The origin is the point of intersection of the x and y axes on the Cartesian plane. The coordinates of the origin are (0, 0).
orthogonality	To be orthogonal is to be perpendicular.
Perpendicular	Perpendicular lines are lines that intersect at a 90° angle. The product of the slopes of two perpendicular lines is -1.
plane	A plane is a flat, two-dimensional surface. It can be conceptualized as a sheet of paper of infinite area.
position vector	A position vector describes the straight-line travel between a starting point (usually the origin) and the location of a second point on a coordinate plane.

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5.3.2: Vector Direction

Vector Direction

Alyssa loves physics, and it fascinates her that there are calculations for all of the physical interactions and motions in the world around her. On Friday, Alyssa went to the county fair with her boyfriend Kurt, who decided it would be romantic if he could win her a giant stuffed animal in the can knockdown game. Unfortunately for Kurt, he still had not knocked anything over approximately 45 mins and $35 later.

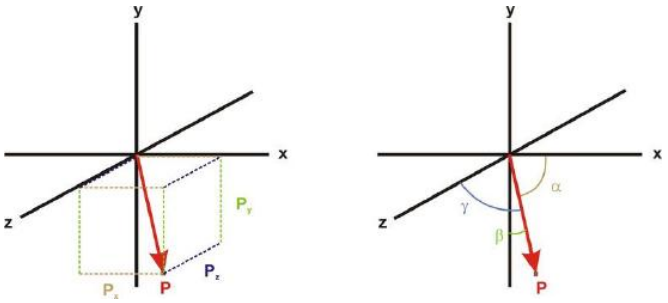
Alyssa decides to give it a try herself, and realizes that the center of the cans is a single position in space. If Alyssa knows her own throwing height, and the distance that the cans are stacked behind the throwing line, what else would she need to know to calculate the vectors involved with a straight-line path for the ball to take? How would she calculate them?

Vector Direction

Recall that the equation for a vector is given by

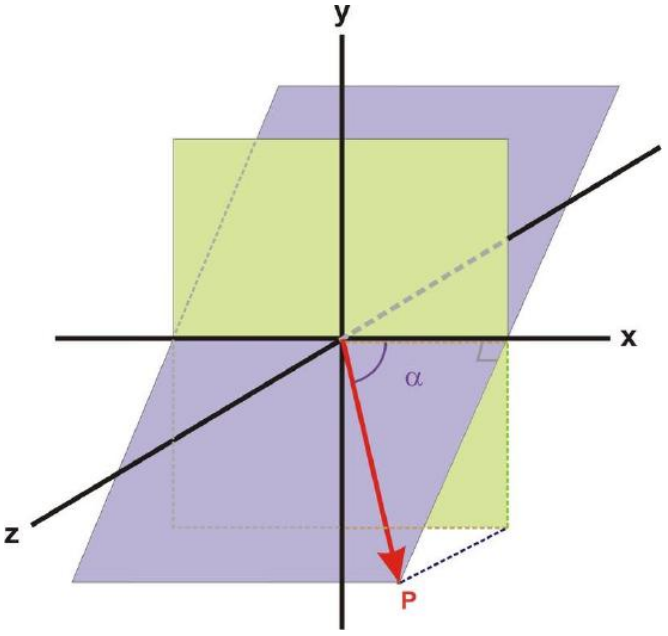
$$\vec{p} = \langle P_x, P_y, P_z \rangle$$

where P_x, P_y, and P_z are the x, y, and z coordinates of the vector obtained by projecting the vector onto the x, y, and z axes as shown below to the left.



The image above to the right, shows the angles between the **position vector**, $\vec{P}$, and the three axes: α is the angle between $\vec{P}$ and the x-axis, β is the angle between $\vec{P}$ and the y-axis, and γ is the angle between $\vec{P}$ and the z-axis.

The position vector, $\vec{P}$, and the **unit vector**, $\hat{x}$, define a plane shown in lavender below.



The direction angle, α, is the angle between $\vec{P}$ and $\hat{x}$ in the plane defined by the two vectors.

The other plane shown in the diagram is the X-Y plane, which was included in the diagram to help you visualize the orientation of the plane defined by the vectors.

In our discussion of the **dot product**, we saw that the dot product of two vectors can be given by $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \cos\theta$

Therefore, we can calculate the angle between $\vec{P}$ and the unit vector $\hat{x}$.

$$\alpha = \cos^{-1} \frac{\vec{P} \times \hat{x}}{|\vec{P}|}$$

Similarly, the **direction angles** β and γ can be calculated using the equations

$$\beta = \cos^{-1} \frac{\vec{P} \times \hat{y}}{|\vec{P}|} \text{ and } \gamma = \cos^{-1} \frac{\vec{P} \times \hat{z}}{|\vec{P}|}$$

In some applications, such as astronomy and applied optics, the **direction cosines** are used at least as frequently as the directional angles themselves.

$$\cos\alpha = \frac{\vec{P} \times \hat{x}}{|\vec{P}|}, \cos\beta \frac{\vec{P} \times \hat{y}}{|\vec{P}|}, \text{ and } \cos\gamma = \frac{\vec{P} \times \hat{z}}{|\vec{P}|}$$

Pythagorean Property of Direction Cosines

An interesting property of direction cosines can be seen if we write the equations for the direction cosines in terms of the components of the position vector, P_x, P_y, and P_z and using the definition of the vector magnitude $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

For example,

$$\cos\alpha = \frac{P_x}{\sqrt{P_x^2 + P_y^2 + P_z^2}}$$

The other two directional cosines can be rewritten similarly:

$$\cos\beta = \frac{P_y}{\sqrt{P_x^2 + P_y^2 + P_z^2}} \text{ and } \cos\gamma = \frac{P_z}{\sqrt{P_x^2 + P_y^2 + P_z^2}}$$

If we square both sides of all three equations and then sum them, we obtain

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{P_x^2}{P_x^2 + P_y^2 + P_z^2} + \frac{P_y^2}{P_x^2 + P_y^2 + P_z^2} + \frac{P_z^2}{P_x^2 + P_y^2 + P_z^2}$$

which simplifies to

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{P_x^2 + P_y^2 + P_z^2}{P_x^2 + P_y^2 + P_z^2} = 1$$

This is an important result because the definition of the unit vector stated that $|\hat{u}| = 1$ which also means that

$$u_x^2 + u_y^2 + u_z^2 = \cos^2 a + \cos^2\beta + \cos^2\gamma$$

and that the components of the unit vectors correspond to the direction cosines.

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Examples

Example 1

Earlier, you were given a problem about Alyssa and the can knockdown game.

If Alyssa knows her own throwing height, and the distance that the cans are stacked behind the throwing line, (a) what else would she need to know to calculate the vectors involved with a straight-line path for the ball to take? (b) How would she calculate them?

Solution

a. Alyssa needs to know the height of the cans above the ground.
b. If Alyssa assumes that the center of the cans is directly in front of her, and treats her own hand as the origin, she can calculate the vector directions as above, using 0 for x, the difference between her throwing height and that of the cans as y, and the distance that the cans are stacked beyond the throwing line as z.

Example 2

Because airplanes move in three dimensions, air-craft ground crews can use direction cosines to identify their location at any moment. Rather than an arbitrary set of orthogonal x, y, and z axes, the position of an airplane is measured relative to the east, north, and zenith directions. (Zenith means top or upward.) At a particular moment, a small plane is 297 km east, 135 km north, and 7.5 km above its home airport. What are the directional cosines and directional angles for the position vector of the plane at that moment?

Solution

If we use the standard orientation of maps in the northern hemisphere, the x-direction corresponds to east, the y-direction to north, and the z-direction to the zenith. Therefore, the position vector of the plane can be written as $\vec{P} = \langle 297, 135, 7.5 \rangle \text{km}$.

The direction cosines associated with this vector are given by

$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\alpha = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{297}{\sqrt{297^2+135^2+7.5^2}} = 0.582$$
$$\cos\beta = \frac{\vec{P}\times\hat{y}}{|\vec{P}|} = \frac{P_y}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\beta = \frac{P_y}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{135}{\sqrt{297^2+135^2+7.5^2}} = 0.811$$
$$\cos\gamma = \frac{\vec{P}\times\hat{z}}{|\vec{P}|} = \frac{P_z}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\gamma = \frac{P_z}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{7500}{\sqrt{297^2+135^2+7.5^2}} = 0.045$$

The associated direction angles are

$$a = \cos^{-1} 0.582 = 54.4^\circ$$
$$\beta = \cos^{-1} 0.165 = 35.8^\circ$$
$$y = \cos^{-1} 0.045 = 87.4^\circ$$

Example 3

In an upcoming episode of a crime drama, a swarm of special-effect insects bothers one of the investigators after the discovery of a murder victim in a drainage ditch. The animator uses position vectors to track the positions of the virtual pests with respect to an origin at the investigator's head. One such insect is located at a point 33 cm in front, 52 cm to the left, and 18 cm below the tip of the investigator's nose. What are the directional cosines for this insect?

Solution

As we look at our investigator, the +x direction is to her left, the +y direction is upward from her nose, and the +z direction is in front of her. The position vector of the midge can be written as $\vec{P} = \langle 33, 52, -18 \rangle \text{cm}$.

The direction cosines associated with this vector are given by

$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\alpha = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{33}{\sqrt{33^2+52^2+(-18)^2}} = 0.514$$
$$\cos\beta = \frac{\vec{P}\times\hat{y}}{|\vec{P}|} = \frac{P_y}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\beta = \frac{P_y}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{52}{\sqrt{33^2+52^2+(-18)^2}} = 0.810$$
$$\cos\gamma = \frac{\vec{P}\times\hat{z}}{|\vec{P}|} = \frac{P_z}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\gamma = \frac{P_z}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{-18}{\sqrt{33^2+52^2+(-18)^2}} = -0.281$$

Example 4

A local astronomer used direction cosines when programming the projector in a new planetarium dome. The projector itself sits at the center of the dome, 2.5 m above the floor. He wishes to project Mintaka, one of the stars in the belt of Orion, at a position 12 m south and 2.3 m east of the projector and 8.7 m above the floor. What is the equation of the directional unit vector that the astronomer must enter in to the projection computer?

Solution

We can use the same coordinate system that we used in example A above: $\hat{x} = \textbf{east}$, $\hat{y} = \textbf{north}$, and $\hat{z} = \textbf{upward}$. The projector itself is the origin. In such a coordinate system, the position vector for Mintaka becomes $\vec{P} = \langle 2.3, -12, (8.7 - 2.5) \rangle$

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$= \langle 2.3, -12, 6.2 \rangle$. Notice that we did not use Mintaka's position above the floor as the z-coordinate. Rather, since the projector is 2.5 m above the floor, we needed to use the difference between the ceiling height and the projector height. Use the component form of the directional cosine equation to calculate the three components of the unit vector.

$$\cos\alpha = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{2.3}{\sqrt{2.3^2+(-12)^2+6.2^2}} = 0.168$$
$$\cos\beta = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{-12}{\sqrt{2.3^2+(-12)^2+6.2^2}} = -0.876$$
$$\cos\gamma = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}} = \frac{6.2}{\sqrt{2.3^2+(-12)^2+6.2^2}} = 0.453$$

Therefore, the position unit vector is given by $\hat{u} = \langle 0.168, -0.876, 0.453 \rangle$.

Example 5

Determine the components of the position vector $\vec{P} = \langle 2.4, 5.3, 1.8 \rangle$ then determine the directional angles between this vector and the x-axis.

Solution

$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}}$$
$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{2.4}{\sqrt{(2.4)^2+(5.3)^2+(1.8)^2}} = \frac{2.4}{\sqrt{5.76+28.09+3.24}} = \frac{2.4}{\sqrt{37.09}} = 0.394$$
$$\alpha = \cos^{-1} 0.394 = 66.8$$

Example 6

Determine the direction cosines for the vector $\vec{N} = \langle 8, 3, -5 \rangle$.

Solution

$$\cos\alpha = \frac{\vec{N}\times\hat{x}}{|\vec{N}|} = \frac{N_x}{\sqrt{N_x^2+N_y^2+N_z^2}}$$
$$\cos\alpha = \frac{\vec{N}\times\hat{x}}{|\vec{N}|} = \frac{N_x}{\sqrt{N_x^2+N_y^2+N_z^2}} = \frac{8}{\sqrt{(8)^2+(3)^2+(-5)^2}} = \frac{8}{\sqrt{64+9+25}} = 0.7213$$
$$\cos\beta = \frac{\vec{N}\times\hat{y}}{|\vec{N}|} = \frac{N_y}{\sqrt{N_x^2+N_y^2+N_z^2}}$$
$$\cos\beta = \frac{\vec{N}\times\hat{y}}{|\vec{N}|} = \frac{N_y}{\sqrt{N_x^2+N_y^2+N_z^2}} = \frac{3}{\sqrt{(8)^2+(3)^2+(-5)^2}} = \frac{3}{\sqrt{64+9+25}} = 0.3030$$
$$\cos\gamma = \frac{\vec{N}\times\hat{z}}{|\vec{N}|} = \frac{N_z}{\sqrt{N_x^2+N_y^2+N_z^2}}$$
$$\cos\gamma = \frac{\vec{N}\times\hat{z}}{|\vec{N}|} = \frac{N_z}{\sqrt{N_x^2+N_y^2+N_z^2}} = \frac{-5}{\sqrt{(8)^2+(3)^2+(-5)^2}} = \frac{-5}{\sqrt{64+9+25}} = -0.5051$$

Example 7

Determine the position vector for a small plane at the moment it is 2.5 km east, 8.8 km south, and 4.1 km above its home airport. Use a coordinate system where the x-direction corresponds to east, the y-direction to north, and the z-direction to the zenith. Then determine the directional cosines used by the air-traffic control personnel to identify the plane's location.

Solution

In this coordinate system, with an origin at the home airport, the position vector is given by $\vec{r} = \langle 2.5, -8.8, 4.1 \rangle$ with units of kilometers.

$$\cos\alpha = \frac{\vec{r}\times\hat{x}}{|\vec{r}|} = \frac{r_x}{\sqrt{r_x^2+r_y^2+r_z^2}}$$

$$\cos\alpha = \frac{\vec{r}\times\hat{x}}{|\vec{r}|} = \frac{r_x}{\sqrt{r_x^2+r_y^2+r_z^2}} = \frac{2.5}{\sqrt{(2.5)^2+(-8.8)^2+(4.1)^2}} = \frac{2.5}{\sqrt{100.5}} = 0.249$$

$$\cos\beta = \frac{\vec{r}\times\hat{y}}{|\vec{r}|} = \frac{r_y}{\sqrt{r_x^2+r_y^2+r_z^2}}$$

$$\cos\beta = \frac{\vec{r}\times\hat{y}}{|\vec{r}|} = \frac{r_y}{\sqrt{r_x^2+r_y^2+r_z^2}} = \frac{-8.8}{\sqrt{(2.5)^2+(-8.8)^2+(4.1)^2}} = \frac{-8.8}{\sqrt{100.5}} = -0.878$$

$$\cos\gamma = \frac{\vec{r}\times\hat{z}}{|\vec{r}|} = \frac{r_z}{\sqrt{r_x^2+r_y^2+r_z^2}}$$

$$\cos\gamma = \frac{\vec{r}\times\hat{z}}{|\vec{r}|} = \frac{r_z}{\sqrt{r_x^2+r_y^2+r_z^2}} = \frac{4.1}{\sqrt{(2.5)^2+(-8.8)^2+(4.1)^2}} = \frac{4.1}{\sqrt{100.5}} = 0.409$$

Example 8

Use the method of direction cosines to identify the unit vector having the same direction as the position vector $\vec{R} = \langle 791, 978, 1310\rangle$

Solution

The unit vector which has the same direction as this vector has the components $\hat{u} = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$ where

$$\cos\alpha = \frac{\vec{R}\times\hat{x}}{|\vec{R}|} = \frac{R_x}{\sqrt{R_x^2+R_y^2+R_z^2}}, \cos\beta = \frac{\vec{R}\times\hat{y}}{|\vec{R}|} = \frac{R_y}{\sqrt{R_x^2+R_y^2+R_z^2}}, \text{ and}$$

$$\cos\gamma = \frac{\vec{R}\times\hat{z}}{|\vec{R}|} = \frac{R_z}{\sqrt{R_x^2+R_y^2+R_z^2}}$$

Once we find the three direction cosines, we have the components of the unit vector,

$$\cos\alpha = \frac{\vec{R}\times\hat{x}}{|\vec{R}|} = \frac{791}{\sqrt{791^2+978^2+1310^2}} = \frac{791}{\sqrt{329826}} = \frac{791}{1816.11} = 0.436$$

$$\cos\beta = \frac{\vec{R}\times\hat{y}}{|\vec{R}|} = \frac{978}{\sqrt{791^2+978^2+1310^2}} = \frac{978}{\sqrt{3298265}} = \frac{978}{1816.11} = 0.539$$

$$\cos\gamma = \frac{\vec{R}\times\hat{z}}{|\vec{R}|} = \frac{1310}{\sqrt{791^2+978^2+1310^2}} = \frac{1310}{\sqrt{3298265}} = \frac{1310}{1816.11} = 0.721$$

$$\hat{u} = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle = \langle 0.436, 0.539, 0.721\rangle$$

Example 9

Determine the direction angles between the vector $\vec{p} = \langle 25, 8, 15\rangle$ and the coordinate axes.

Solution

$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{P_x}{\sqrt{P_x^2+P_y^2+P_z^2}}$$

$$\cos\alpha = \frac{\vec{P}\times\hat{x}}{|\vec{P}|} = \frac{25}{\sqrt{(25)^2+(8)^2+(15)^2}} = \frac{25}{\sqrt{914}} = \frac{25}{30.23} = 0.827$$

$$\alpha = \cos^{-1} 0.827 = 34.2^\circ$$

$$\cos\beta = \frac{\vec{P}\times\hat{y}}{|\vec{P}|} = \frac{P_y}{\sqrt{P_x^2+P_y^2+P_z^2}}$$

$$\cos\beta = \frac{\vec{P}\times\hat{y}}{|\vec{P}|} = \frac{8}{\sqrt{(25)^2+(8)^2+(15)^2}} = \frac{8}{\sqrt{914}} = \frac{8}{30.23} = 0.265$$

$$\beta = \cos^{-1} 0.265 = 75.7^\circ$$

$$\cos\gamma = \frac{\vec{P}\times\hat{z}}{|\vec{P}|} = \frac{P_z}{\sqrt{P_x^2+P_y^2+P_z^2}}$$

$$\cos\gamma = \frac{\vec{P}\times\hat{z}}{|\vec{P}|} = \frac{15}{\sqrt{(25)^2+(8)^2+(15)^2}} = \frac{15}{\sqrt{914}} = \frac{15}{30.23} = 0.496$$

$$\gamma = \cos^{-1} 0.496 = 60.25^\circ$$

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Review

What is the vector direction in degrees of the following two dimensional vectors, assuming the positive x-axis is 0°?

1. What is the direction of $\langle -4, 8 \rangle$
2. What is the direction of $\langle -9, 20 \rangle$
3. What is the direction of $\langle 9, 20 \rangle$
4. What is the direction of $\langle 2, 18 \rangle$
5. What is the direction of $\langle 7, 5 \rangle$
6. What is the direction of $\langle 9, 16 \rangle$

Identify the direction cosines associated with the given vector.

7. $\vec{P} = \langle 42, 6, 9.5 \rangle$ a) cos α = b) cos β = c) cos γ =
8. $\vec{P} = \langle 50, 70, 40.25 \rangle$ a) cos α = b) cos β = c) cos γ =
9. $\vec{P} = \langle 75, 30, 102 \rangle$ a) cos α = b) cos β = c) cos γ =
10. $\vec{P} = \langle 145, 130, 25.75 \rangle$ a) cos α = b) cos β = c) cos γ =
11. $\vec{P} = \langle 220, 300, 175 \rangle$ a) cos α = b) cos β = c) cos γ =

Determine the direction angles between the given vector and the coordinate axes.

12. $\vec{P} = \langle 13, 30, 17 \rangle$
13. $\vec{P} = \langle 5, 3, 12 \rangle$
14. $\vec{P} = \langle 75, 130, 45 \rangle$
15. $\vec{P} = \langle 90, 30, 60 \rangle$
16. $\vec{P} = \langle 7, 18, 4 \rangle$

Solve the word problems as specified.

17. Determine the position vector for a kite at the moment it is 3.5 m east, 10.8 m south, and 30 m above the child flying it. Use a coordinate system where the x-direction corresponds to east, the y-direction to north, and the z-direction to the zenith. Then determine the directional cosines to identify the kites location.
18. Determine the position vector for a cat stuck in a tree at the moment it is 7 ft. east, 12.3 ft south, and 25 ft. above the base of the tree. Use a coordinate system where the x-direction corresponds to east, the y-direction to north, and the z-direction to the zenith. Then determine the directional cosines to identify the cats location.

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.9.

Vocabulary

Term	Definition
angle cosines	Angle notation or angle cosines describe a vector as the result of individual magnitudes and directions as measured from the axes, starting at the origin.
angle notation	Angle notation or angle cosines describe a vector as the result of individual magnitudes and directions as measured from the axes, starting at the origin.
Component notation	Component notation is used to describe vectors algebraically in terms of their x, y and, if relevant, z components.
direction angles	Direction angles (also known as direction cosines) describe a vector as the result of individual magnitudes and directions as measured from the axes, starting at the origin.
direction cosines	Direction cosines (also known as direction angles) describe a vector as the result of individual magnitudes and directions as measured from the axes, starting at the origin.
dot product	The dot product is also known as inner product or scalar product. The two forms of the dot product are $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$ and $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$.
position vector	A position vector describes the straight-line travel between a starting point (usually the origin) and the location of a second point on a coordinate plane.
unit vector	A unit vector is a vector with a magnitude of one.

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SECTION OVERVIEW

5.4: Vector Equations and Applications

Topic hierarchy

5.4.1: Vector Equation of a Line

5.4.2: Vector Analysis Applications

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5.4.1: Vector Equation of a Line

Vector Equation of a Line

Air traffic control is tracking two planes in the vicinity of their airport. At a given moment, one plane is at a location 45 km east and 120 km north of the airport at an altitude of 7.5 km. The second plane is located 63 km east and 96 km south of the airport at an altitude of 6.0 km. The first plane is flying directly toward the airport while the second plane is continuing at a constant altitude with a heading defined by the vector $\vec{h_2} = \langle 3, 4, 0 \rangle$ to land eventually at another airport to the northwest of our air traffic controllers. Do the paths of these two aircraft cross?

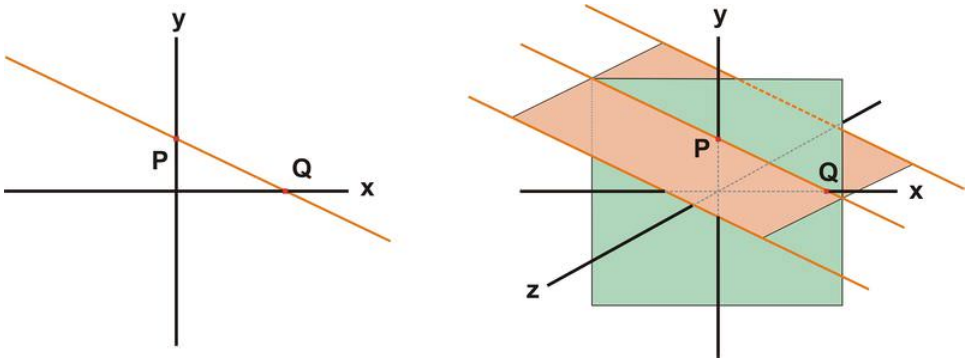
Vector Equation of a Line

You are probably very familiar with using $y = mx + b$, the slope-intercept form, as the equation of a line. While this equation works well in two-dimensional space, it is insufficient to completely define the equation of a line in higher order spaces.

Lines in Space

In a two-dimensional plane, a line can be represented by the equation $y = mx + b$, where m is the slope of the line and b is the y-intercept. In three-dimensional space, this equation can represent a line or a plane or the projection of a 3D line onto the x-y plane. For example, in the diagram below, the line on the left passes through points P and Q. The equation in 2D space for this line is $y = \frac{-P}{Q}x + P$

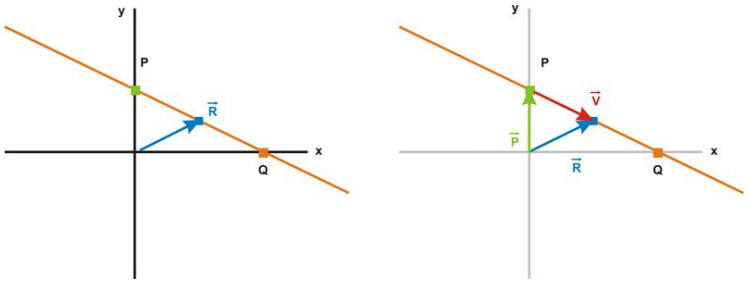
The three orange lines in the diagram at right below can also be described by this equation. All three have slope $M = \frac{-P}{Q}$ and y-intercept $B = P$. If we want to specify one of these lines in particular, we need to identify the line as the intersection of two planes. For example, the line where $y = \frac{-P}{Q}x + P$ and $z = 0$ for all points is the intersection of the orange and green planes in the diagram below right. (Remember that the orange and green planes are drawn with edges for convenience, but that they represent planes that extend outward toward infinity.)



Therefore we can specify a particular line by requiring that the equations for both planes be simultaneously true. In a previous section we developed one equation for a plane given by $d = -n_x x_o - n_y y_o - n_z z_o$. In this case $Qy + Px - QP = 0$ and $z = 0$ must both be satisfied for all points on the line passing through points P and Q.

Finding the Vector Equation of a Line

Another way to identify points on a line can be found using the vector addition method we discussed earlier in this chapter. To find the **position vector**, $\vec{r}$, for any point along a line, we can add the position vector of a point on the line which we already know and add to that a vector, $\vec{v}$, that lies on the line as shown in the diagram below.



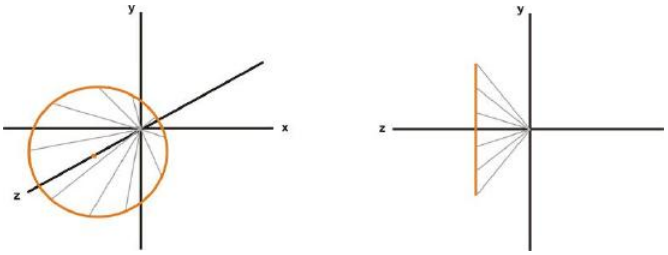
The position vector $\vec{R}$ for a point between *P* and *Q* is given by $\vec{R} = \vec{p} + \vec{v}$

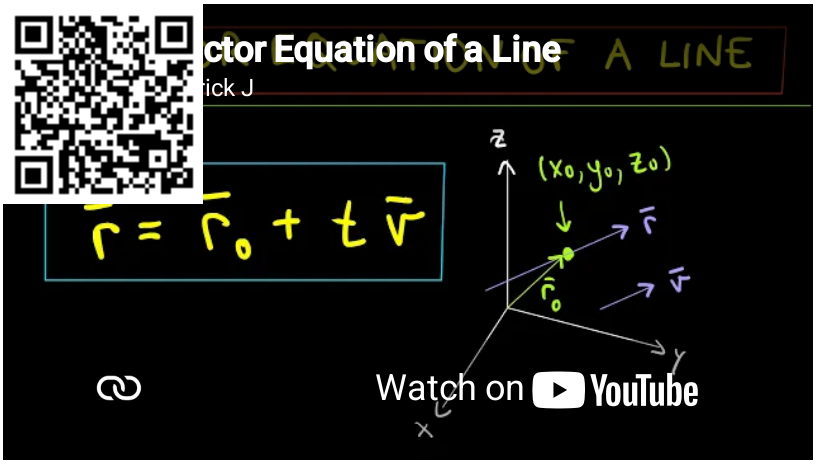
All other points on this line can be reached by traveling along the line from point *P*, therefore the position vector for any point on the line is given by $\vec{r} = \vec{p} + k\vec{v}$.

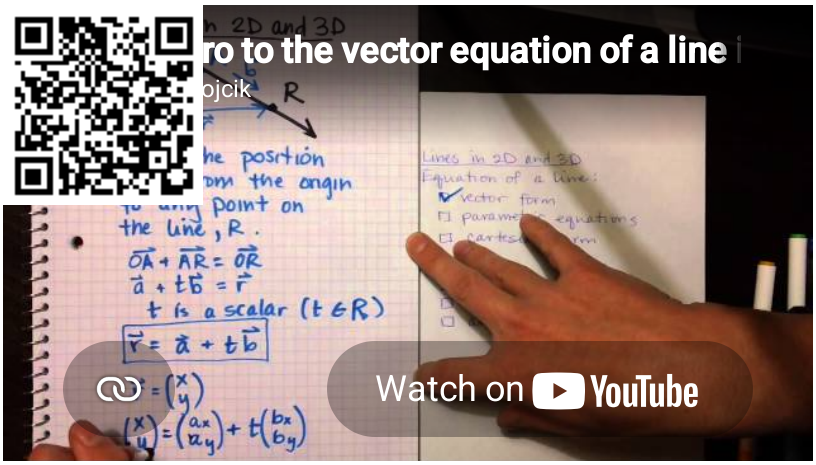
If we know the locations of two points on a line, we can determine the equation of the line. All points on a line fulfill the equation $\vec{r} = \vec{p} + k\vec{v}$, where *k* is a scalar that varies from -∞ to ∞. If we already know the position vectors for two points on the line, $\vec{p}$ and $\vec{q}$, we can use the method of vector subtraction to determine the equation of the vector, $\vec{v} = \vec{p} - \vec{q}$. Therefore, $\vec{r} = \vec{p} + k(\vec{p} - \vec{q})$, where *k* varies from -∞ to ∞.

General Vector Equations

In general, a vector equation is any function that takes any one or more variables and returns a vector. The vector equation of a line is an equation that identifies the position vector of every point along the line. This works for straight lines and for curves. For example, the vector equation $\vec{p} = \langle 3\cos\theta, 3\sin\theta, 2\rangle$ defines a circle having a radius of 3 units which sits parallel to the x-y plane at a distance of 2 units along the z-axis as shown below. The orange line is the set of points that satisfy the vector equation $\vec{p} = \langle 3\cos\theta, 3\sin\theta, 2\rangle$ while the grey lines are various individual position vectors which satisfy the vector equation.







Examples

Example 1

Earlier, you were asked a question about two aircrafts.

Air traffic control is tracking two planes in the vicinity of their airport. At a given moment, one plane is at a location 45 km east and 120 km north of the airport at an altitude of 7.5 km. The second plane is located 63 km east and 96 km south of the airport at an altitude of 6.0 km. The first plane is flying directly toward the airport while the second plane is continuing at a constant altitude with a heading defined by the vector $\vec{h_2} = \langle 3, 4, 0 \rangle$ to land eventually at another airport to the northwest of our air traffic controllers. Do the paths of these two aircraft cross?

Solution

The first thing we need to do is to determine the position vectors of the two planes. Define our airport as the origin of coordinates and define $\hat{x} =$ **east** , $\hat{y} =$ **north** , and $\hat{z} =$ **upward** . Call the position of plane #1, *P*, and the position of plane #2, *Q*. This gives $\vec{p} = \langle 45, 120, 7.5 \rangle$ and $\vec{q} = \langle 63, -96, 6.0 \rangle$.

Plane #1 is heading directly toward the airport. The vector from the position of plane #1 to the origin is given by

$$\vec{v} = \vec{o} - \vec{p} = \langle (0-45), (0-120), (0-7.5) \rangle$$
$$\vec{v} = \vec{p} - \vec{q} = \langle -45, -120, -7.5 \rangle$$

The equation of the line representing plane #1's motion then becomes

$$\vec{r_1} = \langle 45, 120, 7.5 \rangle + k_1 \langle -45, -120, -7.5 \rangle \quad \vec{r_1} = \langle 45, 120, 7.5 \rangle + k_1 \langle -45, -120, -7.5 \rangle$$

Plane #2 continues at a constant altitude with a heading $\vec{h_2} = \langle 3, 4, 0 \rangle$. The equation of the line representing plane #2's motion then becomes

$$\vec{r}_2 = \langle -63, 96, 6.0 \rangle + k_2 \langle 3, 4, 0 \rangle \quad \vec{r_2} = \langle 63, -96, 6.0 \rangle + k_2 \langle 3, 4, 0 \rangle$$
$$\vec{r_2} = \langle -63, 96, 6.0 \rangle + k_2 \langle 3, 4, 0 \rangle$$
$$\vec{r_1} = \langle 45, 120, 7.5 \rangle + k_1 \langle -45, -120, -7.5 \rangle$$

If the paths of the two planes intersect, the position of that intersection must have coordinates which satisfy both equations.

To satisfy both equations, r_{1x} must equal r_{2x}, r_{1y} must equal r_{2y}, and r_{1z} must equal r_{2z}. This means that 45 - 45k_1 = 63 + 3k_2, 120 - 120k_1 = -96 + 4k_2, and 7.5 - 7.5k_1 = 6.0 + 0k_2 must all be simultaneously true.

The first equation simplifies to 15 - 15k_1 = 21 + k$_2$ or k_2 = -6 - 15k_1. The second simplifies to 30 - 30k_1 = -24 + k_2 or k_2 = 54 - 30k_1. The third equation simplifies to 7.5 - 7.5k_1 = 6 or $k_1 = \frac{1}{5}$.

Inserting the value of k_1 from the third equation into the first two equations, we find $k_2 = -6 - 15\left(\frac{1}{5}\right) = -6 - 3 = -9$ and $k_2 = 54 - 30k_1 = 54 - 30\left(\frac{1}{5}\right) = 54 - 6 = 48$

Since $k_2 = 33$ and $k_2 = 0$ cannot be simultaneously true, the two planes' paths do not cross.

Note: There are other ways to reason through this problem. Can you find some? (Hint: Think of the geometry in 2-dimensions, or focus on the altitude as a way to simplify this problem).

Example 2

Write the equation of $y = -\frac{5}{3}x + 5$ as a vector equation.

Solution

Start by choosing two points on the line, say $(3, 0)$ and $(6, -5)$. Then $\vec{p} = \langle 3, 0 \rangle$ and $\vec{q} = \langle 6, -5 \rangle$ So,

$$\vec{p} - \vec{q} = \langle 3, 0 \rangle - \langle 6, -5 \rangle = \langle 3 - 6, 0 - (-5) \rangle = \langle -3, 5 \rangle$$

Finally, the vector equation of the line is

$$\vec{r} = \langle 3, 0 \rangle + k\langle -3, 5 \rangle = \langle 3 - 3k, 5k \rangle$$

Example 3

Write the vector equation of the line defined by the points (2, 3, 4) and (-5, 2, -1).

Solution

These two points have position vectors $\langle 2,3,4 \rangle$ and $\langle -5,2,-1 \rangle$. The vector of the line connecting the two points is given by

$$\vec{v} = \vec{p} - \vec{q} = \langle (2 - (-5)), (3 - 2), (4 - (-1)) \rangle$$
$$\hat{v} = \vec{p} - \vec{q} = \langle 7, 1, 5 \rangle$$

The equation of the line then becomes

$$\vec{r} = \langle 2, 3, 4 \rangle + k\langle 7, 1, 5 \rangle \quad \vec{r} = \langle 2, 3, 4 \rangle + k\langle 7, 1, 5 \rangle$$

Example 4

Determine the equation for the line defined by the points P = (6, 7, 5) and Q = (3, 2, 11). Then find the position vector for a point, R, half-way between these two points.

Solution

$$\vec{v} = \vec{p} - \vec{q} = \langle (6 - 3), (7 - 2), (5 - 11) \rangle$$
$$\hat{v} = \vec{p} - \vec{q} = \langle 3, 5, -6 \rangle$$

The equation of the line then becomes

$$\vec{r} = \langle 6, 7, 5 \rangle + k\langle 3, 5, -6 \rangle \quad \vec{r} = \langle 6, 7, 5 \rangle + k\langle 3, 5, -6 \rangle$$

Since the vector $\vec{v}$ points from P, the value of k for a particular point gives us some information about the location of that point. If $0 < k < 1$, the point lies on the line between P and Q. If $k < 0$, the point is not between P and Q and is closer to P than Q. If $k > 1$, the point lies on the line between P and Q. If $k < 0$, the point is not between P and Q and is closer to P than Q. The point halfway between the two points has $k = \frac{1}{2}$.

$$\vec{R} = \langle 6, 7, 5 \rangle + \tfrac{1}{2}\langle 3, 5, -6 \rangle$$
$$\vec{R} = \langle (6 + 1.5), (7 + 2.5), (5 + (-3)) \rangle = \langle 7.5, 9.5, 2 \rangle$$

Example 5

Are the two vectors $\vec{D} = \langle 1, -1, 4 \rangle + d\langle 1, -1, 1 \rangle$ and $\vec{F} = \langle 2, 4, 7 \rangle + f\langle 2, 1, 3 \rangle$ **skew lines** or do they intersect one another?

Solution

If the two vectors intersect, there must be a point identified by position vector $\vec{p}$ which satisfies the equations of both lines. In other words, we must be able to find values for d and f such that $\vec{D} = \vec{F}$ or $\langle 1, -1, 4\rangle + d\langle 1, -1, 1\rangle = \langle 2, 4, 7\rangle + f\langle 2, 1, 3\rangle$.

Each of the three components of vectors $\vec{D}$ and $\vec{F}$ must independently be equal if $\vec{D} = \vec{F}$. This means that 1 + d = 2 + 2f, -1 - d = 4 + f, and 4 + d = 7 + 3f Combining the first two equations gives: 1 + d - 1 - d = 2 + 2f + 4 + f which when simplified becomes 0 = 6 + 3f or f = -2. If we substitute this back into the first equation we obtain, 1 + d = 2 + 2(-2) which when simplified becomes d = -3. If the two lines cross, f = -2 and d = -3 should satisfy each of the component equations.

1 + d = 2 + 2f becomes 1 + (-3) = 2 + 2(-2) or -2 = -2

-1 - d = 4 + f becomes -1 - (-3) = 4 + (-2) or +2 = +2

4 + d = 7 + 3f becomes 4 + (-3) = 7 + 3(-2) or +1 = +1

All three equations are equally satisfied, so the two lines do intersect and the point of intersection is (-2, 2, 1).

Example 6

In physics, the motion of an object traveling at a constant speed is described by the equation $\vec{s_t} = \vec{s_i} + \vec{v}t$ where s_i is the initial position, s is the position at some later time t, and v is the velocity of the object. Write the vector equation which returns the set of position vectors $\vec{s}$ for an object having an initial position $\vec{s_i} = \langle 2, 3, 4\rangle$ and a velocity of $\vec{v} = \langle 1, 1, -2\rangle$ and determine the object's location at t = 10s.

Solution

$$\vec{s_t} = \vec{s_i} + \vec{v}t = \langle 2, 3, 4\rangle + t\langle 1, 1, -2\rangle = \langle 2 + t, 3 + t, 4 - 2t\rangle$$

$$\text{At } t = 10s,\ \vec{s_{10}} = \langle 2 + 10, 3 + 10, 4 - 2(10)\rangle = \langle 12, 13, -16\rangle$$

Example 7

An object has a position of $\vec{s_i} = \langle 3, 3, 6\rangle$ at t = 0 and a velocity of $\vec{v} = \langle 10, 7, 3\rangle$. Use the vector equation $\vec{s} = \vec{s_i} + \vec{v}t$ to determine the distance traveled by the object between t = 3s and t = 5s. Distance measured in meters.

Solution

The vector equation describing the motion of the object is

$$\vec{s} = \vec{s}_i + \vec{v}t = \langle 3, 3, 6\rangle + t\langle 10, 7, 3\rangle = \langle 3 + 10t, 3 + 7t, 6 + 3t\rangle$$

The object's position at t = 3s is obtained from the vector equation:

$$\vec{s_3} = \vec{s_i} + \vec{v}t = \langle 3, 3, 6\rangle + (3)\langle 10, 7, 3\rangle = \langle 3 + 10(3), 3 + 7(3), 6 + 3(3)\rangle = \langle 33, 24, 15\rangle$$

The object's position at t = 5s is obtained from the vector equation:

$$\vec{s_5} = \vec{s_i} + \vec{v}t = \langle 3, 3, 6\rangle + (5)\langle 10, 7, 3\rangle = \langle 3 + 10(5), 3 + 7(5), 6 + 3(5)\rangle = \langle 53, 38, 21\rangle$$

The distance traveled between these two points is the magnitude of the vector starting at (33, 24, 15) and ending at (53, 38, 21).

$$\overrightarrow{\triangle s} = \vec{s_5} - \vec{s_3} = \langle 53, 38, 21\rangle - \langle 33, 24, 15\rangle = \langle 20, 14, 6\rangle$$

Now we can use the Pythagorean theorem to determine the magnitude of the vector.

$$\left|\overrightarrow{\triangle s}\right| = \sqrt{a^2 + b^2 + c^2} = \sqrt{(20)^2 + (14)^2 + (6)^2} = 25.1 \text{ meters}$$

Example 8

Determine the vector equation of the straight line defined by the points (2, 2, 2) and (1, 3, 5).

Solution

These two points have position vectors $\vec{p} = \langle 2, 2, 2\rangle$ and $\vec{q} = \langle 1, 3, 5\rangle$. The vector of the line connecting the two points is given by

$\vec{v} = \vec{p} - \vec{q} = \langle (2-1), (2-3), (2-5) \rangle = \langle 1, -1, -3 \rangle$

The equation of the line, $\vec{r} = \vec{p} + k\vec{v}$, then becomes

$\vec{r} = \langle 2, 2, 2 \rangle + k\langle 1, -1, -3 \rangle$ $\vec{r} = \langle 2, 2, 2 \rangle + k\langle 1, -1, -3 \rangle$

Example 9

Do the two lines $\vec{R} = \langle 4, 4, -2 \rangle + r\langle -3, 7, 2 \rangle$ and $\vec{K} = \langle 9, -8, 7 \rangle + k\langle -2, 1, 3 \rangle$ intersect?

Solution

If the two vectors intersect, there must be a point identified by position vector $\vec{p}$ which satisfies the equations of both lines. In other words, we must be able to find values for *r* and *k* such that

$\vec{R} = \vec{K}$ or $\langle 4, 4, -2 \rangle + r\langle -3, 7, 2 \rangle = \langle 9, -8, 7 \rangle + k\langle -2, 1, 3 \rangle$.

Each of the three components of vectors $\vec{R}$ and $\vec{K}$ must independently be equal if $\vec{R} = \vec{K}$. This means that 4 - 3*r* = 9 - 2*k*, 4 + 7*r* = -8 + *k*, and -2 + 2*r* = 7 + 3*k*

Solving the second equation for *k* gives 12 + 7*r* = *k*. Now substitute this into one of the other two equations: 4 - 3*r* = 9 - 2(12 + 7*r*) or 4 - 3*r* = 9 - 24 - 14*r* which simplifies to 19 = -11*r* or $r = \frac{-19}{11} = -1.727$. Substitute this value into one of the other equations: -2 + 2(-1.727) = 7 + 3*k* which becomes -2 - 3.454 = 7 + 3*k*. Solving for *k* gives *k* = -4.151. If the two lines cross, *r* = -1.727 and *k* = -4.151 should satisfy each of the component equations.

4 - 3*r* = 9 - 2*k* becomes 4 - 3 (-1.727) = 9 - 2(-4.151) or -1.181 = 17.302 Since even this first equation does not hold true, these two lines are skew and do not intersect.

Review

Write the vector equation of the line defined by the the following points:

1. (2,−2,5) and (1,5,4)
2. (2,−9,5) and (8,4,−6)
3. (15,3,−3) and (4,−3,9)
4. (−1,−1,7) and (3,11,8)
5. (1,−3,2) and (−5,3,−1)
6. (25,17,42) and (−16,12,23)

Determine if the two vectors are skew lines or if they intersect each other.

7. $\vec{D} = \langle 2, 3, 1 \rangle + d\langle 5, 3, 6 \rangle$ and $\vec{F} = \langle -5, -3, -5 \rangle + f\langle 15, 3, -3 \rangle$
8. $\vec{D} = \langle 3, 4, 7 \rangle + d\langle 3, 3, 2 \rangle$ and $\vec{F} = \langle -2, 11, 7 \rangle + f\langle -2, 11, 7 \rangle$
9. $\vec{D} = \langle 15, 3, -3 \rangle + d\langle 3, 11, 8 \rangle$ and $\vec{F} = \langle 5, 3, 6 \rangle + f\langle 1, -4, 6 \rangle$
10. $\vec{D} = \langle 13, -1, 6 \rangle + d\langle -5, 4, 12 \rangle$ and $\vec{F} = \langle 6, 9, 0 \rangle + f\langle 21, 0, 14 \rangle$

Identify the position vector for the midpoint of each line (you already found the vector equations in problems 1 through 4).

11. (2,−2,5) and (1,5,4)
12. (2,−9,5) and (8,4,−6)
13. (15,3,−3) and (4,−3,9)
14. (−1,−1,7) and (3,11,8)

Use the vector equation, and the following values for t: t = 3s and t = 5s. Distance measured in meters.

15. An object has a position of $\vec{s_i} = \langle 3, -5, 1 \rangle$ at t = 0 and a velocity of $\vec{v} = \langle 2, 7, -3 \rangle$
16. An object has a position of $\vec{s_i} = \langle 1, -4, -8 \rangle$ at t = 0 and a velocity of $\vec{v} = \langle 9, -7, 3 \rangle$
17. An object has a position of $\vec{s_i} = \langle -1, 2, -3 \rangle$ at t = 0 and a velocity of $\vec{v} = \langle 3, -1, -2 \rangle$

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.10.

Vocabulary

Term	Definition
position vector	A position vector describes the straight-line travel between a starting point (usually the origin) and the location of a second point on a coordinate plane.
Skew lines	Skew lines are lines that are in different planes and never intersect.
vector equation	A vector equation is any function that takes any one or more variables and returns a vector.

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5.4.2: Vector Analysis Applications

Vector Analysis Applications

How much **work** is done by a 10kg ball rolling down a 25° slope for 10 yards?

How can the force which propels a proton through a magnetic field be calculated?

How could you calculate the **torque** exerted by children on a see-saw?

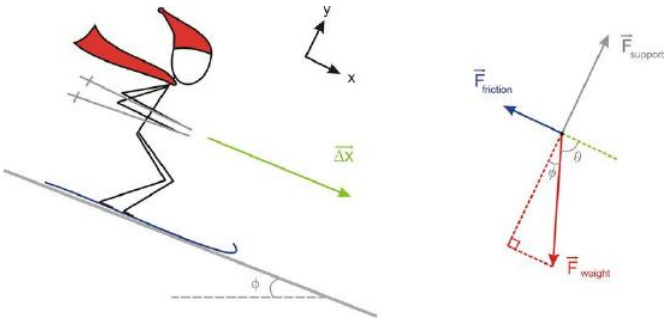
This lesson is all about real-world uses of vector analysis.

Vector Analysis Applications

Vectors can be used by air-traffic controllers when tracking planes, by meteorologists when describing wind conditions, and by computer programmers when they are designing virtual worlds. In this section, we will present three applications of vectors which are commonly used in the study of physics: work, torque, and magnetic force.

Work

In physics, the term **work** is used to describe energy that is added to or removed from an object or system when a force is applied to it. From experiment, it has been determined that work is maximized when the applied force is parallel to the motion of the object and that no work is done when the force is applied perpendicular to the motion. Therefore, the work done by a force can be described by the dot product of the force vector and the displacement vector. For example, several forces act on the skier in the diagram below.



Sven's weight pulls downward toward the center of the earth, the snow support's Sven by pushing upward on his skis perpendicular to the slope, and the friction between Sven's skis and the snow points in the opposite direction from his motion. The work done by each of these forces can be determined using the dot product of the force and the displacement vector $\overrightarrow{\triangle x}$.

$$W = \overrightarrow{F} \times \overrightarrow{\triangle x} = F(\triangle x)\cos\theta$$

$$W_{\text{friction}} = \overrightarrow{F_{\text{friction}}} \times \overrightarrow{\triangle x} = (F_{\text{friction}})\,(\triangle x)\cos 180^\circ = -F_{\text{friction}}\,\triangle x$$

$$W_{\text{support}} = \overrightarrow{F_{\text{support}}} \times \overrightarrow{\triangle x} = (F_{\text{support}})(\triangle x)\cos 90^\circ = 0$$

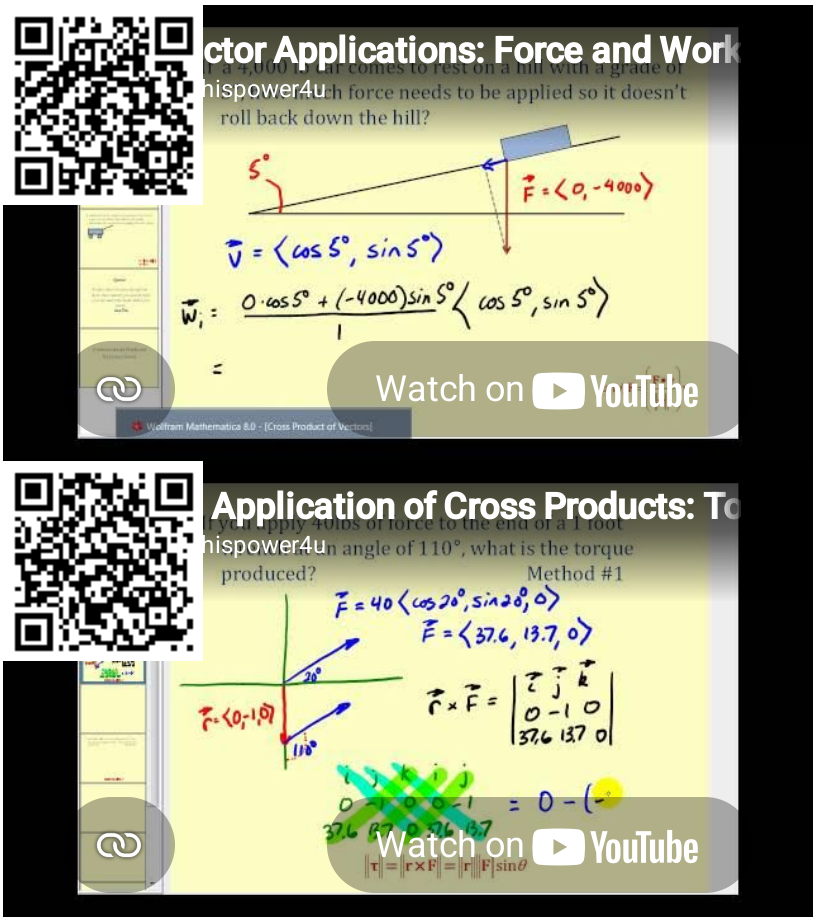
$$W_{\text{weight}} = \overrightarrow{F_{\text{weight}}} \times \overrightarrow{\triangle x} = (F_{\text{weight}})\,(\triangle x)\cos\theta = (F_{\text{weight}})\,(\triangle x)\sin\phi$$

Magnetic Force

The force that a magnetic field exerts on a charged particle is strongest when the particle moves perpendicular to the field and the magnetic force on the particle is equal to zero when it moves parallel to the field. Therefore the magnetic force can be described using the cross-product of the field strength vector and the particle's velocity vector: $\vec{F} = q\vec{v} \times \vec{B}$ where $\vec{F}$ is the force on the particle, q is the charge of the particle, $\vec{v}$ is the velocity of the particle, and $\vec{B}$ is the vector representing the magnetic field. If the velocity is measured in m/s and if the magnetic field is measured in tesla, the force will be measured in **newtons**, the metric base-unit of force.

Torque

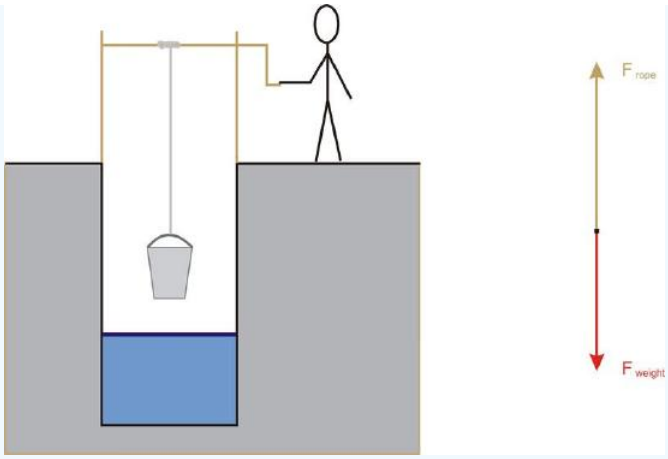
When you lift a baseball off a table-top, you are exerting a force that moves the object as a whole. When you apply a force to a doorknob, you cause the door to rotate on its hinges. Scientists use the term **torque** to describe the force-like property that affects the rotation of an object. The torque can be described using the cross-product of the force vector and the lever arm, a vector pointing radially outward from the axis of rotation to the point where the force is applied to the object: $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{\tau}$ is the torque, $\vec{r}$ is the lever arm, and $\vec{F}$ is the applied force.



Examples

Example 1

Xiao turns a crank to lower a bucket of water into a well. Determine the total work done on the bucket if the weight of the bucket is 15 N and the tension force in the rope is 13 N. The bucket rises a distance of 4.5 m while he is cranking.



Solution

The work done by a force acting on an object is described by the dot product of the force vector and the displacement vector: $W = \vec{F} \times \overrightarrow{\Delta x} = F(\Delta x)\cos\theta$. In this case, the rope does negative work on the bucket because the motion and the force are in opposite directions. If the force is measured in newtons and the displacement in meters, the work is measured in **Joules**.

$$W_{\text{rope}} = \overrightarrow{F_{\text{rope}}} \times \overrightarrow{\Delta x} = F_{\text{rope}}\,(\Delta x)\cos\theta = (13N)(4.5m)\cos 180^\circ = -58.5J$$

The weight force does positive work on the bucket because the motion and the force are in the same direction.

$$W_{\text{weight}} = \overrightarrow{F_{\text{weight}}} \times \overrightarrow{\Delta x} = F_{\text{weight}}\,(\Delta x)\cos\theta = (15N)(4.5m)\cos 0^\circ = 67.5J$$

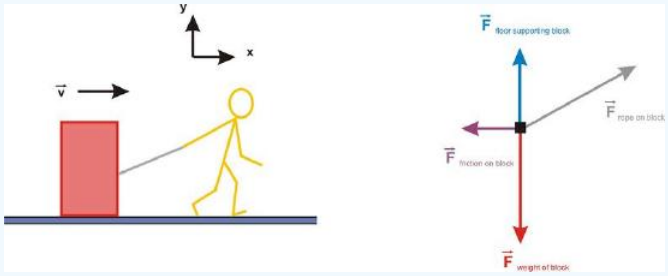
The total work is the sum of the two individual amounts of work.

$$W_{\text{total}} = W_{\text{rope}} + W_{\text{weight}} = -58.5 + 67.5J = 9.0J$$

A total of 9.0 J of work is done on the bucket as the bucket moves downward into the well.

Example 2

The diagram below shows Sanjay pulling a large crate across the floor. The four forces which act on the crate during this process are shown in the diagram below. Which of the four forces exert non-zero force on the crate?



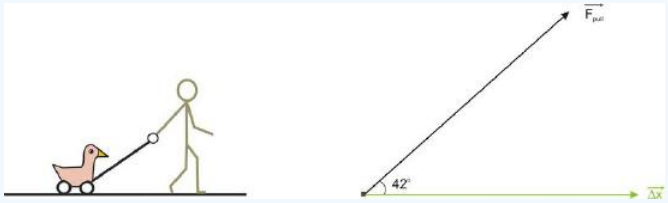
Solution

The dot product is defined by $\vec{A} \times \vec{B} = |A||B|\cos\theta$ therefore only forces which have at least some component parallel to the motion will do non-zero work on the object. The angle θ between the displacement and forces perpendicular to the motion is 90° so $\vec{A} \times \vec{B} = |A||B|\cos\theta = 0$. The force from the floor and the weight of the crate do no work, since both of these forces are perpendicular to the motion of the crate. The rope does positive work on the crate since the force of the rope on the crate has a non-zero x-component. The friction does negative work on the crate since it is in the opposite direction from the displacement.

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Example 3

De'andra is pulling her toy duck (mass 0.75 kg) at a constant speed of 3.0 m/s. The string she uses to pull the duck makes an angle of 42° above the horizontal and De'andra keeps a constant tension in the string of 2.0 N. What is the amount of work done by the tension force when the duck is pulled forward a distance of 2.8 m?



Solution

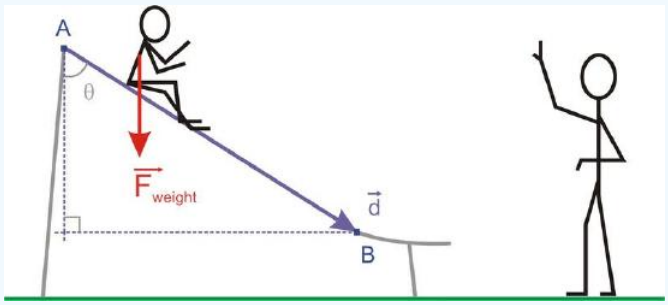
The work done by De'andra on the duck depends on the force she uses to pull the duck and on the distance the duck moves while she pulls. It also depends on the angle between the pulling force and the displacement vector.

$$\overrightarrow{F_{\text{pull}}} \times \vec{x} = \left|\overrightarrow{F_{\text{pull}}}\right| |\vec{x}| \cos\theta = (2.0N)(2.8m)\cos 42^\circ = 4.16\text{Nm}$$

The N represents "newton", the unit of force. The m represents "meters," the unit of displacement. (1.0 N)(1.0 m) = 1.0 J where J represents "joules," the unit of work and energy.

Example 4

Beuford has once again taken Brynna to the park to play on the slide. If Brynna has a weight of 25 kg and if the slide has a 30° incline above the horizontal, what work is done by her weight as she slides down the 3.5 m incline? Remember that the weight force in newtons is equal to the product of the mass and the acceleration of gravity, 9.8 m/s^2.



Solution

If the slide is inclined at 30° above the horizontal, then θ = 60° from the vertical. The work done by a force on an object is given by the dot product of the force and the displacement of the object. Here $\overrightarrow{F_{\text{weight}}} = mg = (25\text{ kg})(9.8\text{ m/s}^2) = 245\text{ N}$. Therefore, $W = \overrightarrow{F_{\text{weight}}} \times \vec{d} = \left|\overrightarrow{F_{\text{weight}}}\right| |\vec{d}| \cos\theta = (245N)(3.5m)\cos 60^\circ = 428.75J$

Example 5

The scientists at Fermi Lab in Chicago, IL use magnetic fields to direct beams of protons during their explorations of the submicroscopic structure of atoms and quarks. Both magnetic field strength and the velocity of the protons are vector quantities. Determine the force on a proton moving northward at 4.2 × 10^6 m/s through a magnetic field of 2.5 T oriented from east to west. (Note: protons are very tiny and are therefore able to move VERY fast.)

Solution

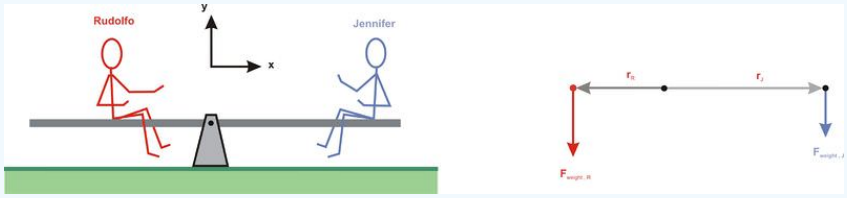
Define a coordinate system where eastward is the +z direction, northward is the +y direction, and upward is the +z direction. In this coordinate system, $\vec{v} = \langle 0, 4.2 \times 10^6, 0\rangle$ and $\vec{B} = \langle -2.5, 0, 0\rangle$. Therefore, $|\vec{F}| = q|\vec{v} \times \vec{B}| = q(|\vec{v}||\vec{B}|\sin\theta)$. Since the velocity is northward and the magnetic field is westward, the angle between the two vectors is 90°.

$$|\vec{F}| = q|\vec{v} \times \vec{B}| = q(|\vec{v}||\vec{B}|\sin\theta) = \left(1.6 \times 10^{-19}\right)\left(\left(4.2 \times 10^{6}\right)(2.5)\sin 90°\right) = 16.8 \times 10^{-13} N$$

Using the **right-hand-rule**, we can determine the direction of the force on the proton. If you point your thumb northward along the velocity vector and your fore-finger westward along the magnetic field vector, your palm and your extended middle-finger point upward. Therefore the force which the magnetic field exerts on the proton is in the +z direction:
$|\vec{F}| = q\vec{v} \times \vec{B} = \langle 0, 0, 16.8 \times 10^{-13} N \rangle$.

Example 6

If two children, Rudolfo (weight = 210 N) and Jennifer (weight = 175 N), sit on either end of a see-saw as shown below. Rudolfo is 1.0 m from the pivot and Jennifer is 1.4 m from the pivot. What torques are exerted by the children on the see-saw?



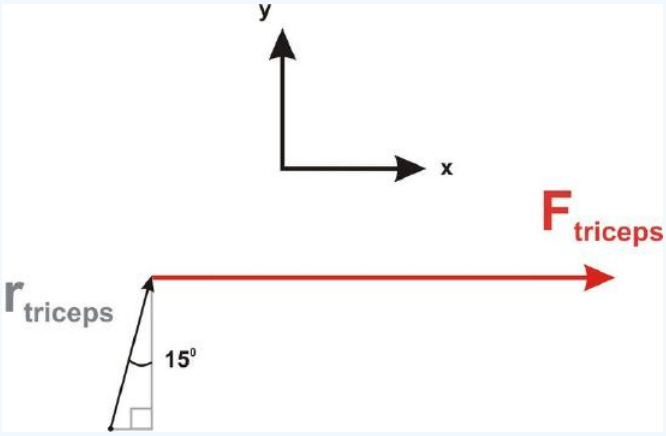
Solution

The coordinate system has been defined such that the weight-force vectors are parallel to the y-axis and the lever-arm vectors are parallel to the x-axis. First determine the component form of each vector equation and then use the component version of the cross-product equation to determine the torque exerted by each child.

$$\vec{\tau_R} = \vec{r_R} \times \overrightarrow{F_{\text{weight},R}} = \langle -1.0\ \text{m}, 0, 0\rangle \times \langle 0, -210N, 0\rangle$$
$$\vec{r} \times \vec{F} = \langle (r_y F_z - r_z F_y), (r_z F_x - r_x F_z), (r_x F_y - r_y F_x)\rangle$$
$$\vec{r_R} \times \overrightarrow{F_R} = \langle((0*0) - (0*-210)), ((0*0) - (-1*0)), ((-1*-210) - (0*0))\rangle = \langle 0, 0, 210mN\rangle$$
$$\vec{\tau_J} = \vec{r_J} \times \overrightarrow{F_{\text{weight, J}}} = \langle 1.4m, 0, 0\rangle \times \langle 0, -170N, 0\rangle$$
$$\vec{r} \times \vec{F} = \langle (r_y F_z - r_z F_y), (r_z F_x - r_x F_z), (r_x F_y - r_y F_x)\rangle$$
$$\vec{r_J} \times \overrightarrow{F_J} = \langle((0*0) - (0*-170)), ((0*0) - (1.4*0)), ((1.4*-170) - (0*0))\rangle = \langle 0, 0, -238mN\rangle$$

Example 7

Exercise scientists and physical therapists use torque to analyze various exercises such as the triceps exercise shown in the diagram below. Here, the triceps muscle exerts a force on the elbow-end of the forearm, affecting the rotation of the forearm. If the triceps exerts a force of 17 N, what torque is applied to the forearm?



Solution

Since the forearm is positioned at an angle of 15° to the vertical, the angle between the two vectors is 90° – 15° = 75°. The magnitude of the lever-arm vector is the distance from the elbow-pivot to the point where the triceps pulls on the bone, $|\vec{r}| = 2.5\ \text{cm}$. Similarly, the magnitude of the force vector is the strength of the force, $|\vec{F}| = 17N$. Since we know the magnitudes of both vectors and the angle between them, we can use the angle-version of the cross-product equation to determine the magnitude of the torque.

$$|\vec{\tau}| = rF\sin\theta = (2.5\ \text{cm})(17N)\sin 75 = 41.05\ \text{cm}\cdot\text{N}$$

Review

1. A fishing boat sails due west at a constant 15 km/h. A current of 3 km/h is running due south. What is the velocity of the boat relative the sea floor?
2. Two bulldozers pull on a large tree to move it out of the way of a road. One bulldozer pulls 3000N west and the other bulldozer 2500N in a southerly direction because of the terrain. Determine the resultant force acting on the tree. If the tree has a mass of 1000 kg calculate the acceleration of the tree due to the resultant force acting on it.
3. Three coplanar horizontal forces each of magnitude 10 N act on a body of mass 5 kg as shown below. Determine the magnitude of the net force acting on the body and the magnitude of the resultant acceleration.
4. If vector A = 5 N north and vector B = 10 N east, find the resultant of vector A – vector B.
5. A submarine is travelling at 20 km/h due east. Shortly after, it is travelling due north at 15 km/h. Calculate the change in velocity.
6. An fighter jet has a true airspeed of 1000 km/h due east. There is a cross wind blowing 60 degrees east of south at 100 km/h. Calculate the velocity of the jet relative to the ground.
7. A pallet of mass 10 kg sits on a horizontal surface, a force is exerted on the pallet by means of a chain inclined at 60° to the horizontal. If the tension in the chain is 150 N and the frictional force between the pallet and the horizontal surface is 55 N, will the pallet move under these conditions? Explain. If the pallet does move determine the size of the acceleration with which it moves.
8. A broken down car with a total mass of 300 kg is pulled by a single tow line attached to a tow-truck with a mass of 1000 kg. If the drag on the truck and the car is one-tenth of their respective weights, and the total forward force exerted by the tow-truck is 5130 kg force (ie 5130 x 9.8 N), find the magnitude of: a. the total force resisting the forward motion of the truck and car; b. the acceleration of the truck/car system; c. the unbalanced accelerating force on the broken down car; d. the tension in the towing line.
9. A fighter pilot with a mass of MA = 80 kg sits in the cock-pit with his back horizontal to the ground. His jet is moving vertically with acceleration a. If the acceleration due to gravity on the pilot is g = 9.8 ms-2, write a mathematical expression for and calculate the value of the reaction force R between the pilot and the back of his seat in the jet when: a. a = 0; b. a = 8 ms-2 upwards; c. a = 8 ms-2 downwards.
10. A plane is flying 691mph headed 13° east of north. A tail wind is blowing 67° west of north at 92 mph. Determine the actual speed of the plane relative the ground.
11. A bird is headed 40° east of north at 67 mph. A tail wind is blowing 45° west of south at 68 mph. Determine the direction of the bird.
12. A UFO is gliding along at 516 mph, headed 32° east of south. Determine the direction of the UFO if a tail wind is blowing 42° west of north at 29 mph.
13. A rocket is flying 63° west of north at 575 mph. Determine the actual direction of the rocket if a tail wind is blowing 22° west of south at 78 mph.
14. What is the resultant direction of a speed boat racing along at 66.5mph headed 80° west of south at 611 mph, if a tail wind is pushing it toward a direction 89° east of south at 4.5 mph?
15. Several evil villains are holding Spiderman in place with ropes. If Flat-Nose Frankie is pulling at a constant force of 19 pounds, 44° to the horizontal, Green-Toe Gary is pulling at a constant force of 29 pounds along 31° to the horizontal, and Jelly-Knee Jennifyr is pulling at a constant force of 26 pounds at an angle of 33°, How hard and in what direction is Orange-Chin Oswald pulling if Spiderman is stuck in place?

Review (Answers)

To see the Review answers, open this PDF file and look for section 5.11.

Vocabulary

Term	Definition
Joules	Joules are a standard unit of energy. One joule is equivalent to one Newton-meter.
Newtons	Newtons are a standard unit of work.
Right-hand-rule	The right-hand-rule is used to indicate the direction of a cross product. Position the thumb and index finger of your right hand with the first vector along your thumb and the second vector along your index finger. Your middle finger, when extended perpendicular to your palm, will indicate the direction of the cross product of the two vectors.
torque	The term torque is used to describe the force-like property that affects the rotation of an object.
Work	Work is an expression of how much energy it takes to move an object over a distance. Work is defined as force times displacement, and is measured in Joules.

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CHAPTER OVERVIEW

Unit 6: Conic Sections

6.1: Ellipses

6.1.1: Ellipses Centered at the Origin

6.1.2: Equation of an Ellipse

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SECTION OVERVIEW

6.1: Ellipses

Topic hierarchy

6.1.1: Ellipses Centered at the Origin

6.1.2: Equation of an Ellipse

6.1.3: Ellipses Not Centered at the Origin

6.1.4: Focal Property of Ellipses

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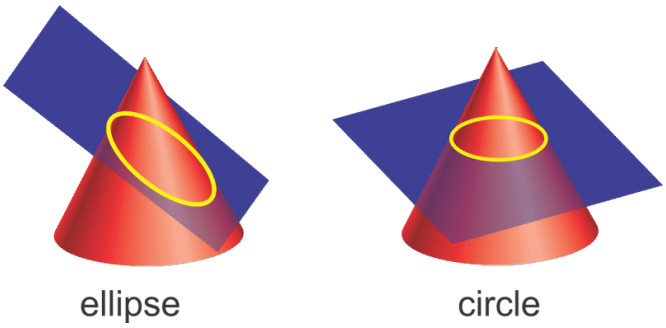
6.1.1: Ellipses Centered at the Origin

Ellipses Centered at the Origin

Your homework assignment is to draw the ellipse $16x^2 + 4y^2 = 144$. Where will the foci of your graph be located?

Ellipses Centered at the Origin

The third conic section is an ellipse. Recall that a circle is when a plane sliced through a cone and that plane is parallel to the base of the cone. An ellipse is formed when that plane is not parallel to the base. Therefore, a circle is actually a more specific version of an ellipse.



By definition, an **ellipse** is the set of all points such that the sum of the distances from two fixed points, called **foci** (the plural of **focus**), is constant.

Drawing an Ellipse

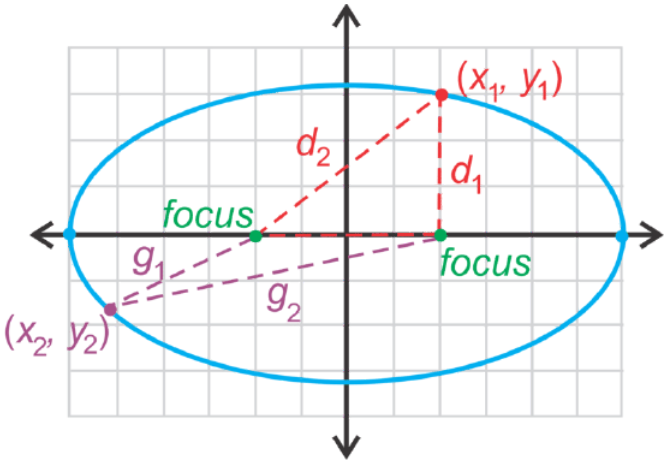
We will use the definition of an ellipse to draw an ellipse.

Step 1: On a piece of graph paper, draw a set of axes and plot (−2, 0) and (2, 0). These will be the foci.

Step 2: From the definition, we can conclude a point (x, y) is on an ellipse if the sum of the distances is always constant. In the picture, $d_1 + d_2 = r$ and $g_1 + g_2 = r$.

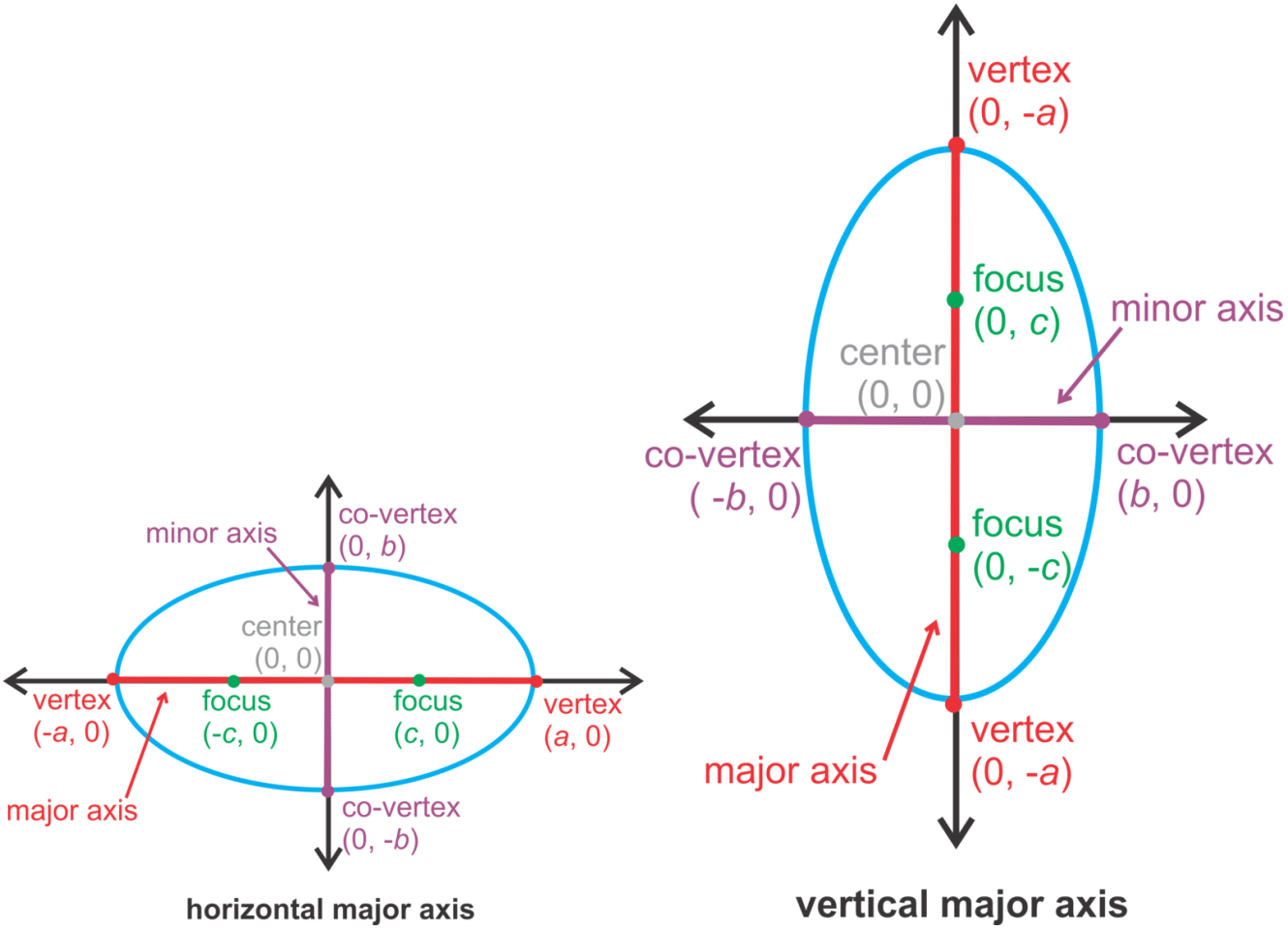
Step 3: Determine how far apart the foci are. Then, find d_1 and d_2.

Step 4: Determine if the point (−2, 3) is on the ellipse.



In this concept, the **center** of an ellipse will be (0,0). An ellipse can have either a vertical or horizontal orientation (see below). There are always two foci and they are on the **major axis**. The major axis is the longer of the two axes that pass through the center of an ellipse. Also on the major axis are the **vertices**, which its endpoints and are the furthest two points away from each other on

an ellipse. The shorter axis that passes through the center is called the **minor axis**, with endpoints called **co-vertices**. The midpoint of both axes is the center.



Equation of an Ellipse, Centered at the Origin

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	HORIZONTAL major axis is the x-axis with length 2a. minor axis is the y-axis with length 2b.
$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$	VERTICAL major axis is the y-axis with length 2a. minor axis is the x-axis with length 2b.

Other Important Facts

- a is ALWAYS greater than b. If they are equal, we have a circle.
- The foci, vertices, and co-vertices relate through a version of the Pythagorean Theorem: $c^2 = a^2 - b^2$

Let's find the vertices, co-vertices, and foci of $\frac{x^2}{64}+\frac{y^2}{25}=1$. Then, let's graph the ellipse.

First, we need to determine if this is a horizontal or vertical ellipse. Because 64 > 25, we know that the ellipse will be horizontal. Therefore, $a^2=64$ making $a=\sqrt{64}=8$ and $b^2=25$, making $b=\sqrt{25}=5$. Using the pictures above, the vertices will be $(8,0)$ and $(-8,0)$ and the co-vertices will be $(0,5)$ and $(0,-5)$.

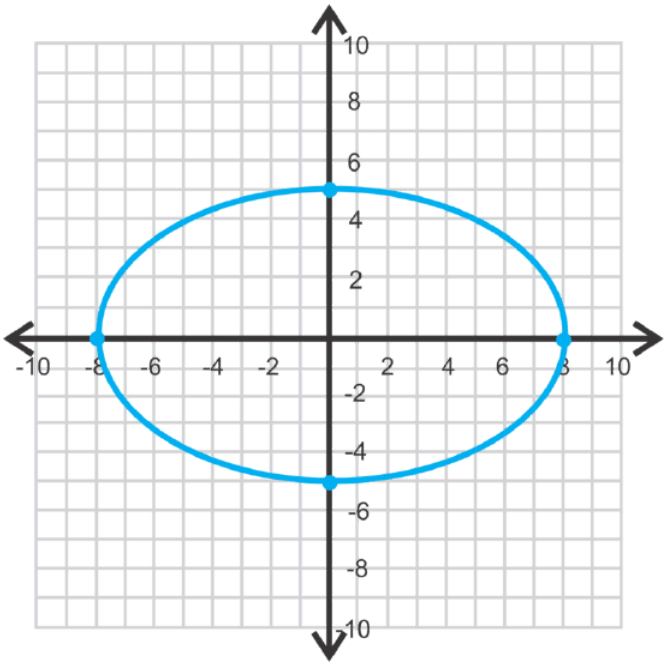
To find the foci, we need to use the equation $c^2=a^2-b^2$ and solve for c.

$c^2=64-25=39$

$c=\sqrt{39}$

The foci are $(\sqrt{39}, 0)$ and $(-\sqrt{39}, 0)$.

To graph the ellipse, plot the vertices and co-vertices and connect the four points to make the closed curve.

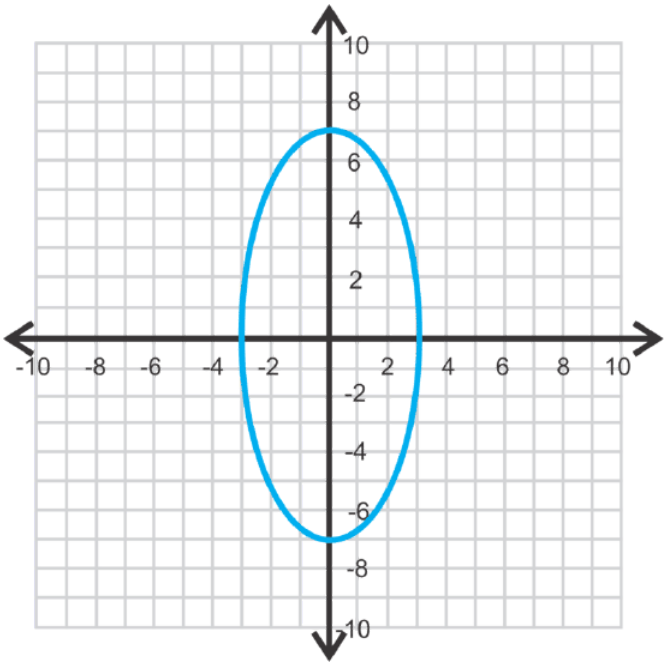


Now, let's graph $49x^2 + 9y^2 = 441$ and identify the foci.

This equation is not in standard form. To rewrite it in standard form, the right side of the equation must be 1. Divide everything by 441.

$$\frac{49x^2}{441} + \frac{9y^2}{441} = \frac{441}{441}$$
$$\frac{x^2}{9} + \frac{y^2}{49} = 1$$

Now, we can see that this is a vertical ellipse, where $b = 3$ and $a = 7$.



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To find the foci, use $c^2 = a^2 - b^2$.

$c^2 = 49 - 9 = 40$

$c = \sqrt{40} = 2\sqrt{10}$

The foci are $(0, 2\sqrt{10})$ and $(0, -2\sqrt{10})$.

Finally, let's write equations for the ellipses with the given characteristics below and centered at the origin.

In either part, you may wish to draw the ellipse to help with the orientation.

a. vertex: (−6, 0), co-vertex: (0, 4)

We can conclude that $a = 6$ and $b = 4$. The ellipse is horizontal, because the larger value, a, is the x-value of the vertex. The equation is $\frac{x^2}{36} + \frac{y^2}{16} = 1$.

b. vertex: (0, 9), focus: (0, −5)

We know that $a = 9$ and $c = 5$ and that the ellipse is vertical. Solve for b using $c^2 = a^2 - b^2$

$5^2 = 9^2 - b^2$

$25 = 81 - b^2$

$b^2 = 56 \rightarrow b = 2\sqrt{14}$

The equation is $\frac{x^2}{56} + \frac{y^2}{81} = 1$

Examples

Example 1

Earlier, you were asked to determine where the foci of your graph would be located.

Solution

This equation is not in standard form. To rewrite it in standard form, the right side of the equation must be 1. Divide everything by 144.

$$\frac{16x^2}{144} + \frac{4y^2}{144} = \frac{144}{144}$$

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

Now, we can see that this is a vertical ellipse, where $b = 3$ and $a = 6$.

To find the foci, use $c^2 = a^2 - b^2$.

$c^2 = 36 - 9 = 27$

$c = \sqrt{27} = 3\sqrt{3}$

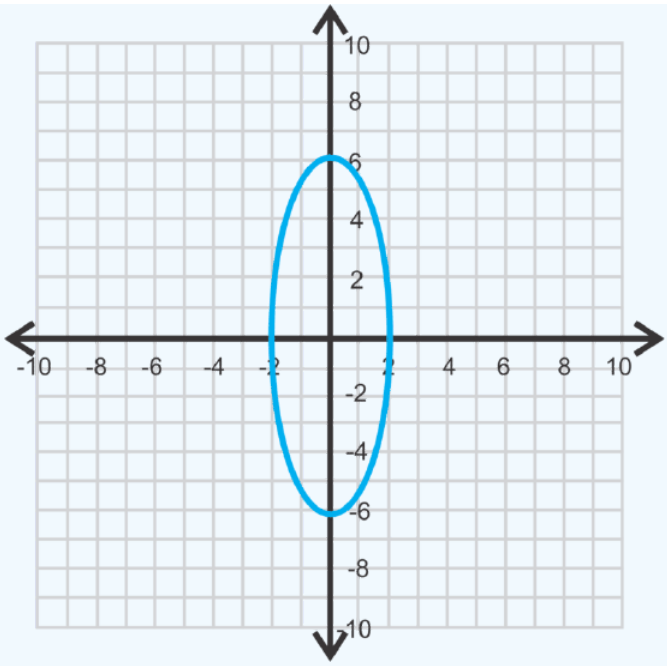
The foci are therefore $(0, 3\sqrt{3})$ and $(0, -3\sqrt{3})$.

Example 2

Find the vertices, co-vertices, and foci of $\frac{x^2}{4} + \frac{y^2}{36} = 1$. Then, graph the equation.

Solution

Because the larger number is under y^2, the ellipse is vertical. Therefore, $a = 6$ and b^2. Use $c^2 = a^2 - b^2$ to find c.



$c^2 = 6^2 - 2^2 = 36 - 4 = 32$

$c = \sqrt{32} = 4\sqrt{2}$

vertices: $(0, 6)$ and $(0, -6)$

co-vertices: $(2, 0)$ and $(-2, 0)$

foci: $(0, 4\sqrt{2})$ and $(0, -4\sqrt{2})$

Example 3

Graph $49x^2 + 64y^2 = 3136$ and find the foci.

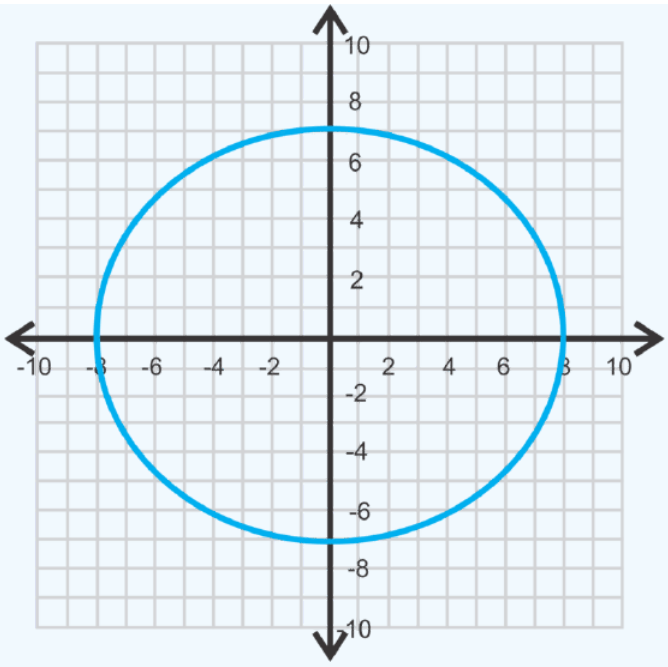
Solution

Rewrite $49x^2 + 64y^2 = 3136$ in standard form.

$$\frac{49x^2}{3136} + \frac{64y^2}{3136} = \frac{3136}{3136}$$

$$\frac{x^2}{64} + \frac{y^2}{49} = 1$$

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This ellipse is horizontal with $a = 8$ and $b = 7$. Find c.

$c^2 = 64 - 49 = 15$

$c = \sqrt{15}$

The foci are $(-\sqrt{15}, 0)$ and $(\sqrt{15}, 0)$.

Example 4

Find the equation of the ellipse with co-vertex $(0, -7)$, focus $(15, 0)$ and centered at the origin.

Solution

Because the co-vertex is $(0, -7)$, $b = 7$ and the ellipse is horizontal. From the foci, we know that $c = 15$. Find a.

$15^2 = a^2 - 7^2$

$a^2 = 225 + 49 = 274$ **The equation is** $\frac{x^2}{274} + \frac{y^2}{49} = 1$

$a = \sqrt{274}$

Review

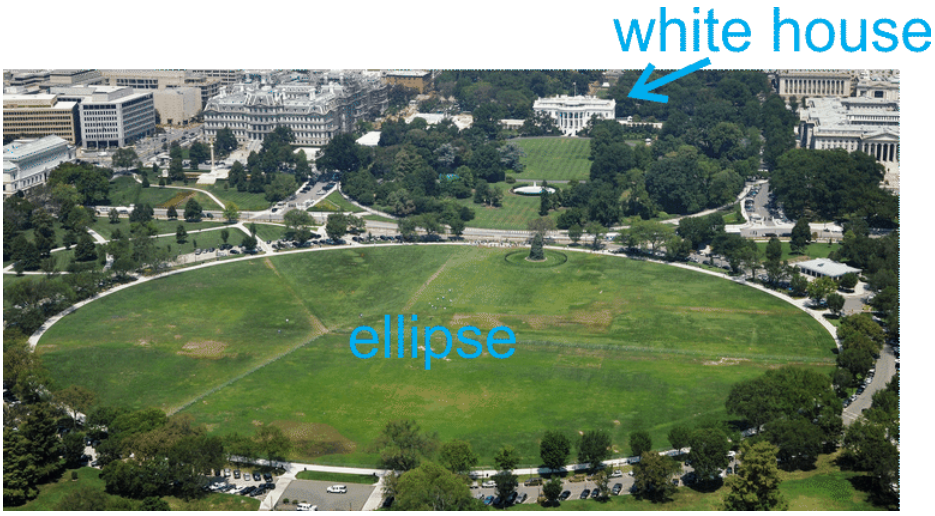
Find the vertices, co-vertices, and foci of each ellipse below. Then, graph.

1. $\frac{x^2}{9} + \frac{y^2}{16} = 1$
2. $4x^2 + 25y^2 = 100$
3. $\frac{x^2}{64} + y^2 = 1$
4. $81x^2 + 100y^2 = 8100$
5. $\frac{x^2}{49} + \frac{y^2}{16} = 1$
6. $121x^2 + 9y^2 = 1089$

Find the equation of the ellipse, centered at the origin, with the given information.

7. vertex: (−3, 0) co-vertex: (0, 1)
8. co-vertex: (7, 0) major axis: 18 units
9. vertex: (0, 5) minor axis: 4 units
10. vertex: (0, 6) co-vertex: (−2, 0)

11. co-vertex: (17, 0) focus: (0, 17)
12. vertex: (4, 0) focus: (−3, 0)
13. co-vertex: (−6, 0) focus: (0, 5)
14. focus: (0, −9) minor axis: 16 units
15. **Real Life Application** A portion of the backyard of the White House is called The Ellipse. The major axis is 1058 feet and the minor axis is 903 feet. Find the equation of the horizontal ellipse, assuming it is centered at the origin.



Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.5.

Vocabulary

Term	Definition
Equation of an Ellipse	If the center of an ellipse is (0, 0), the equation of the ellipse is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$.
Major Axis	The major axis of an ellipse is the longest diameter of the ellipse.

Image Attributions

1. [Figure 1]
   Credit: Ad Meskens
   Source: https://commons.wikimedia.org/wiki/File:Illu_pancrease.jpg;https://commons.wikimedia.org/wiki/File:President%27s_Park_with_White_House.jpg

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6.1.2: Equation of an Ellipse

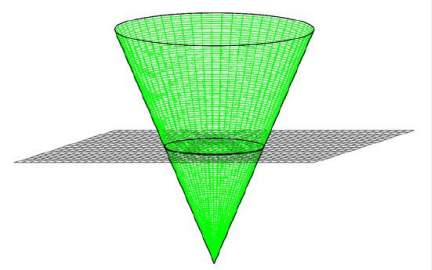
Equation of an Ellipse

Ken is having a disagreement with his friend Scott. Scott says that the track that they run on at school is not really an **ellipse**, but an oval. Ken thinks that is a silly distinction, since ovals and **ellipses** are the same thing. Who is correct?

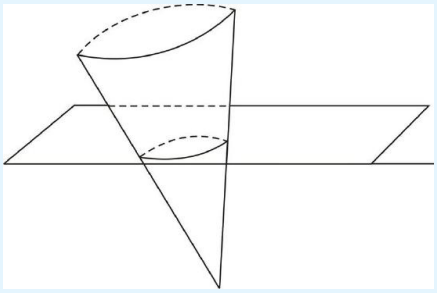
Equation of an Ellipse

When a plane "slices through" one side of a **cone**, we get either a circle or an “oval-shaped” object as shown below. It turns out that this is no ordinary oval, but something called an **ellipse**, a shape with special properties.

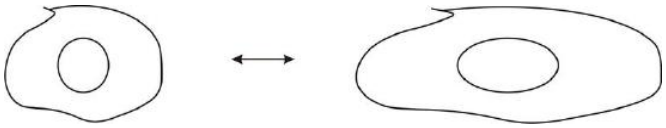
a. Cone "sliced through" by a plane 90° to the axis of the cone to form a ***circle***.



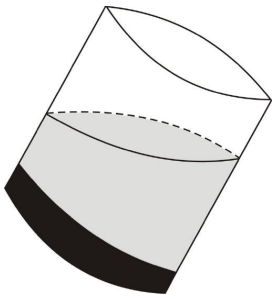
b. Cone "sliced through" by a plane at an angle to form an ***ellipse***.



Ellipses can be defined by some of their properties. One way to define an ellipse is as a “stretched out circle”. It’s the shape you would get if you sketched a circle on a deflated balloon and then stretched out the balloon evenly in two opposite directions:



It’s also the shape of the surface of water that results when you tilt a round glass:



Or an ellipse could be thought of as the shape of a circle drawn on a piece of paper when it is viewed at an angle.

Equations of Ellipses

This “stretching” can be represented algebraically. For simplicity, take the circle of radius 1 centered at the origin (0,0). The distance formula tells us that this is the set of points (x,y) that is a distance 1 unit away from the origin.

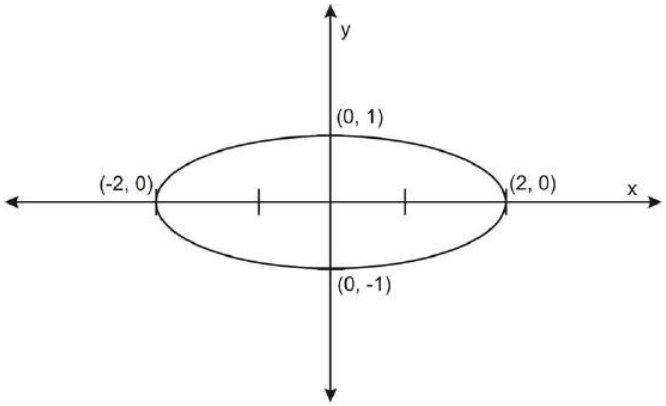
$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$1 = \sqrt{(x-0)^2 + (y-0)^2}$$
$$1 = x^2 + y^2$$

This equation, $x^2 + y^2 = 1$, can be altered to stretch the circle in the horizontal (i.e. x−axis) direction by dividing the x variable by a constant $a > 1$,

$\left(\frac{x}{a}\right)^2 + y^2 = 1$

Why does this stretch the circle horizontally? Well, the effect of dividing x by a is that for each y–value in an ordered pair (x, y) that satisfies the original equation, the corresponding x value must be multiplied by a in order for the pair to make a solution to the altered equation. So solutions (x, y) of the circle are in one-to-one correspondence with solutions (ax, y) of the altered equation, hence stretching the corresponding graph to the left and right by a factor of a.

Here is the graph of $\left(\frac{x}{2}\right)^2 + y^2 = 1$:



Generalizing the equation by allowing a stretch in the vertical direction, we get the following.

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

The factor a stretches the circle in the horizontal direction and the factor b stretches the circle in the vertical direction. If $a = b$, this is just a circle. When $a \neq b$, this equation represents an ellipse. The ellipse is stretched in the horizontal direction if $b < a$ and it is stretched in the vertical direction if $a < b$.

Often the above equation is written as follows.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

This is called the **standard form** of the equation of an ellipse, assuming that the ellipse is centered at (0, 0).

To sketch a graph of an ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, start by plotting the four axes intercepts, which are easy to find by plugging in 0 for x and then for y. Then sketch the ellipse freehand, or with a graphing program or calculator.

403 ERROR

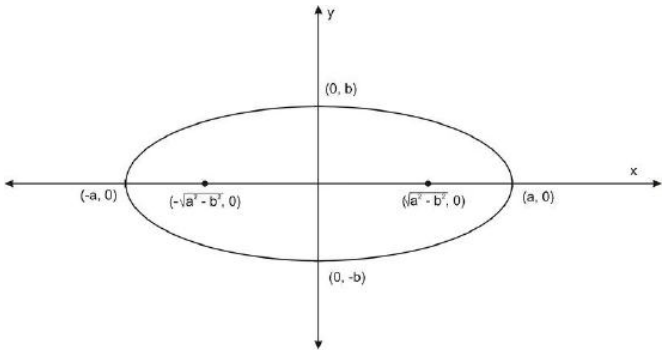
The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

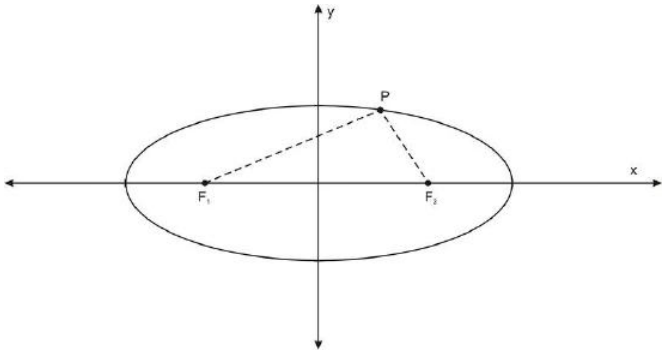
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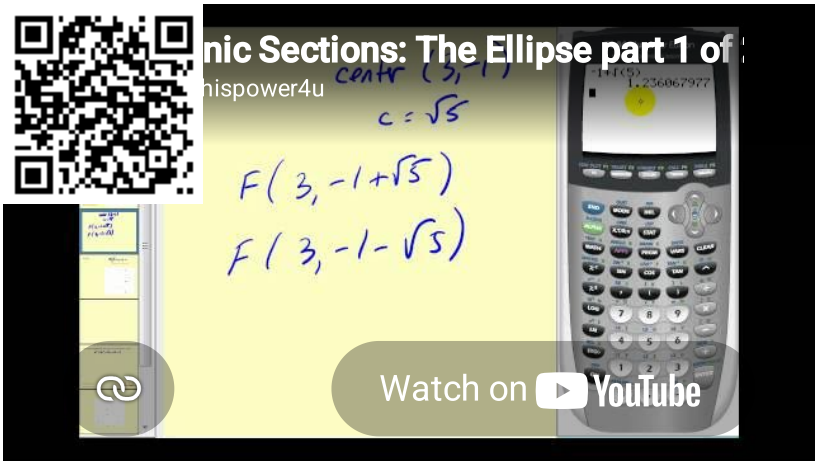
The Foci

In every ellipse there are two special points called the **foci** (foci is plural, **focus** is singular), which lie inside the ellipse and which can be used to define the shape. For an ellipse centered at (0,0) that is wider than it is tall, its **major axis** is horizontal and its foci are at $\left(\sqrt{a^2-b^2},0\right)$ and $\left(-\sqrt{a^2-b^2},0\right)$.



What is the significance of these points? Ellipses relate to these points much as a circle relates to its center. Remember a circle can be thought of as the set of points in a plane that are a certain distance from the center point, in fact, that is typically the definition of a circle. Well, the foci act like the center except that there are two of them. An ellipse is the set of points where the *sum* of the distance between each point on the ellipse and each of the two foci is a constant number. In the diagram below, for any point P on the ellipse, $F_1P + F_2P = d$, where F_1 and F_2 are the foci and d is a constant.





Examples

Example 1

Earlier, you were given a question about Ken and Scott's disagreement.

Solution

Scott is correct. Ovals are not the same as ellipses.

Example 2

Sketch the graph of $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Solution

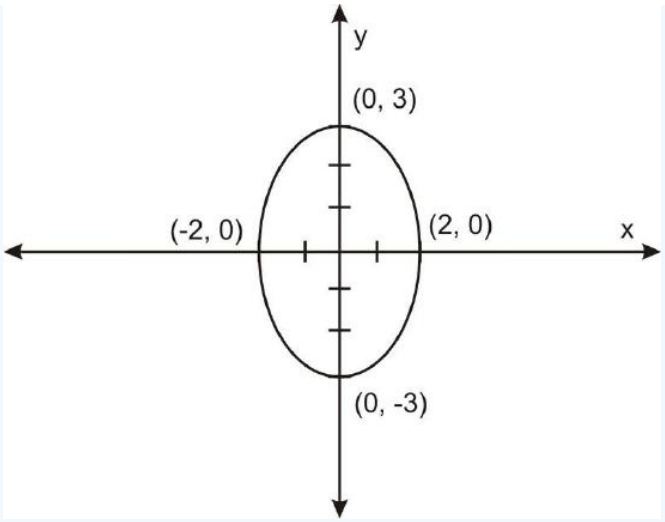
This equation can be rewritten as $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$.

To sketch, we set $x=0$ and solve for y to find the x-intercepts and set $y=0$ and solve for x to find the y– intercepts.

The four points we get are (0, 3), (0, -3), (2, 0), and (-2, 0).

Plot those points and sketch the ellipse:

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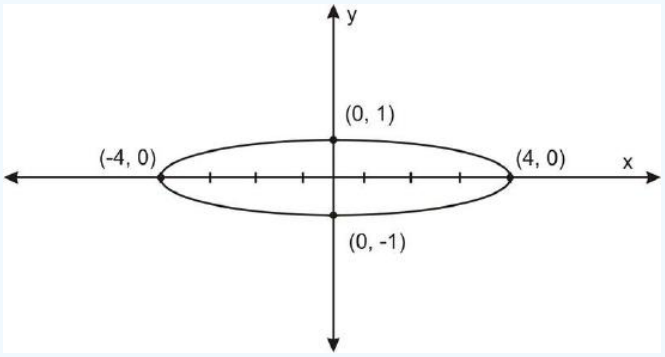


Example 3

Sketch the graph of $\frac{x^2}{16} + y^2 = 1$.

Solution

This can be rewritten as $\frac{x^2}{4^2} + \frac{y^2}{1^2} = 1$. After finding the intercepts and sketching the graph with the same process as above, we have:



The segment spanning the long direction of the ellipse is called the **major axis**, and the segment spanning the short direction of the ellipse is called the **minor axis**. So in the last example the major axis is the segment from (-4, 0) to (4, 0) and the minor axis is the segment from (0, -1) to (0, 1).

The major and minor axes are examples of what are sometimes called **reference lines**. Apollonius, the Ancient Greek mathematician who wrote an early treatise on conics, used these and other reference lines to orient **conic** sections. Though the Greeks did not use a coordinate plane to discuss geometry, these reference lines offer a framing perspective that is similar to the Cartesian plane that we use today. Apollonius' way of framing conics with reference lines was the closest mathematics came to the system of coordinate geometry that you know so well until Descartes' and Fermat's systematic work in the seventeenth century.

Example 4

What is the full range of the **eccentricity** of an ellipse? What does it look like near the extremes of this range?

Solution

Foci can be used to measure how far an ellipse is "stretched" from a circle. The symbol ε stands for the **eccentricity** of an ellipse, and it is defined by the distance between the foci divided by the length of the major axis, or $\frac{\sqrt{a^2-b^2}}{a}$ for horizontally

oriented ellipses and $\frac{\sqrt{b^2-a^2}}{b}$ for vertically oriented ellipses. Since a circle is an ellipse where $a=b$, circles have an eccentricity of 0.

The interval of possible values is $\varepsilon \in [0,1)$. At $\varepsilon = 0$, the ellipse is a circle; as the eccentricity approaches 1 it becomes more and more elongated.

Example 5

Not all equations for ellipses start off in the standard form above. For example, $25x^2+9y^2=225$ is an ellipse. Put it in the proper form and graph it.

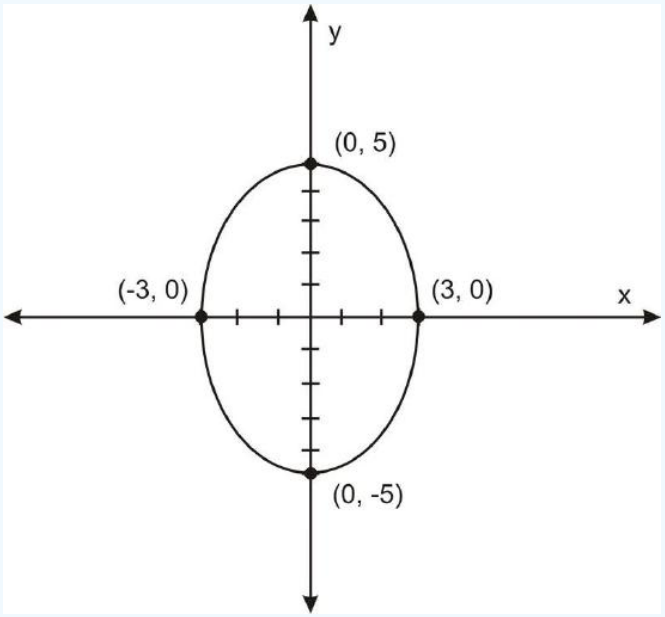
Solution

To graph $25x^2+9y^2=225$ as an ellipse:

$\frac{x^2}{9}+\frac{y^2}{25}=1$..... First, divide both sides by 225

$\frac{x^2}{3^2}+\frac{y^2}{5^2}=1$..... Find the intercepts

Graphing, we have:



Example 6

It was mentioned above that when a round glass of water is tilted, the surface of the water is an ellipse. Using our working definition of an ellipse as “stretched out circle”, explain why you think the water takes this shape.

Solution

Answers may vary, but should explain why the shape that results stretches a circle in one direction because the width of the glass is constant.

Example 7

Try to sketch this ellipse where the numbers don’t turn out to be so neat: $3x^2+4y^2=12$.

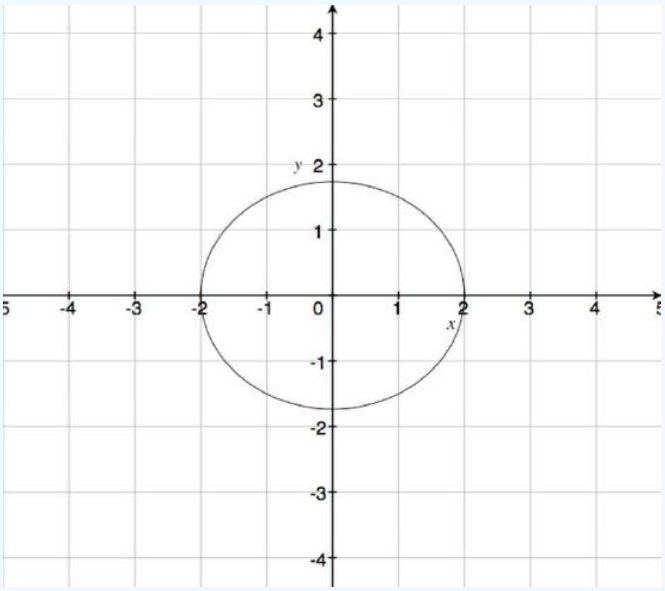
Solution

To graph $3x^2+4y^2=12$:

$\frac{x^2}{4}+\frac{y^2}{3}=1$..... First, divide both sides by 12

$\frac{x^2}{2^2} + \frac{y^2}{\sqrt{3}^2} = 1$..... Find the intercepts

Graphing, we get:



Review

Convert each equation to the standard form of an ellipse.

1. $x^2 + 4y^2 = 4$
2. $9x^2 + 16y^2 = 144$
3. $25x^2 + 36y^2 = 900$
4. $36x^2 + 9y^2 = 324$
5. $49x^2 + y^2 = 49$
6. $16x^2 + 4y^2 = 64$
7. $x^2 + 2y^2 = 2$
8. $3x^2 + 4y^2 = 12$
9. $5x^2 + 6y^2 = 30$
10. $6x^2 + 3y^2 = 18$
11. $7x^2 + y^2 = 7$
12. $4x^2 + 2y^2 = 8$

Graph the following ellipses.

13. $9x^2 + 16y^2 = 144$
14. $49x^2 + y^2 = 49$
15. $x^2 + 2y^2 = 2$
16. $7x^2 + y^2 = 7$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.1.

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Vocabulary

Term	Definition
Cone	A cone is a solid three-dimensional figure with a circular base and one vertex.
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Eccentricity	The eccentricity of a conic section is a measure of how much the conic section deviates from being circular. The eccentricity of circles is 0, the eccentricity of ellipses is between 0 and 1, the eccentricity of parabolas is 1, and the eccentricity of hyperbolas is greater than 1. For ellipses and hyperbolas, $e = \frac{c}{a}$.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Foci	The foci of an ellipse are the two points that define the ellipse. The sum of the distances from any point on the ellipse to the foci is constant.
Major Axis	The major axis of an ellipse is the longest diameter of the ellipse.
minor axis	The minor axis of an ellipse is the shortest diameter of the ellipse.
standard form	The standard form of an ellipse centered at (h, k) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

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6.1.3: Ellipses Not Centered at the Origin

Ellipses Centered at (h, k)

Your homework assignment is to draw the ellipse $16(x-2)^2+4(y+3)^2=144$. What is the vertex of your graph and where will the foci of the ellipse be located?

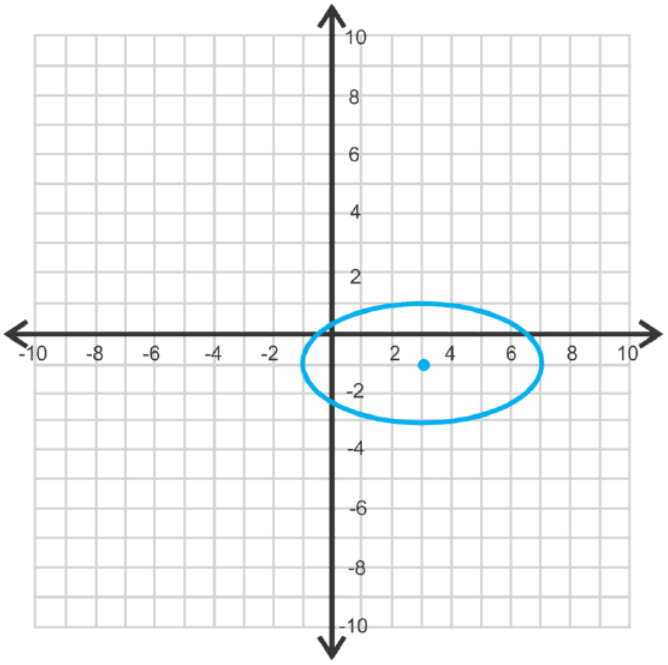
Ellipses Centered at (h,k)

An ellipse does not always have to be placed with its center at the origin. If the center is (h,k) the entire ellipse will be shifted h units to the left or right and k units up or down. The equation becomes $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$. We will address how the vertices, co-vertices, and foci change in the following problem.

Let's graph $\frac{(x-3)^2}{16}+\frac{(y+1)^2}{4}=1$. Then, we'll find the vertices, co-vertices, and foci.

First, we know this is a horizontal ellipse because $16>4$. Therefore, the center is $(3,-1)$ and $a=4$ and $b=2$. Use this information to graph the ellipse.

To graph, plot the center and then go out 4 units to the right and left and then up and down two units. This is also how you can find the vertices and co-vertices. The vertices are (3 ± 4, −1) or (7, −1) and (−1, −1). The co-vertices are (3, −1 ± 2) or (3, 1) and (3, −3).



To find the foci, we need to find c using $c^2=a^2-b^2$.

$c^2=16-4=12$
$c=2\sqrt{3}$

Therefore, the foci are $(3\pm2\sqrt{3},-1)$.

From this problem, we can create formulas for finding the vertices, co-vertices, and foci of an ellipse with center (h,k). Also, when graphing an ellipse, not centered at the origin, make sure to plot the center.

Orientation	Equation	Vertices	Co-Vertices	Foci
Horizontal	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	$(h\pm a,k)$	$(h,k\pm b)$	$(h\pm c,k)$
Vertical	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	$(h,k\pm a)$	$(h\pm b,k)$	$(h,k\pm c)$

Now, let's find the equation of the ellipse with vertices (−3, 2) and (7, 2) and co-vertex (2, −1).

These two vertices create a horizontal major axis, making the ellipse horizontal. If you are unsure, plot the given information on a set of axes. To find the center, use the midpoint formula with the vertices.

$$\left(\frac{-3+7}{2}, \frac{2+2}{2}\right) = \left(\frac{4}{2}, \frac{4}{2}\right) = (2, 2)$$

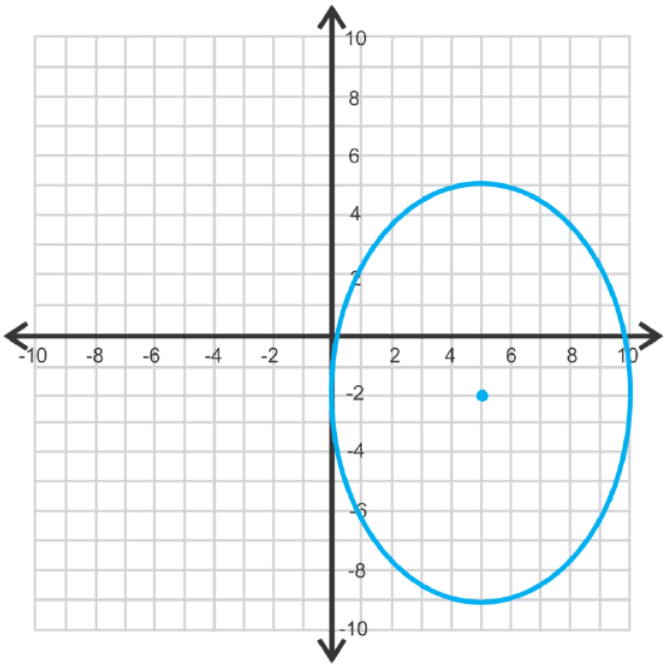
The distance from one of the vertices to the center is $a, |7-2| = 5$. The distance from the co-vertex to the center is $b, |-1-2| = 3$. Therefore, the equation is $\frac{(x-2)^2}{5^2} + \frac{(y-2)^2}{3^2} = 1$ or $\frac{(x-2)^2}{25} + \frac{(y-2)^2}{9} = 1$.

Finally, let's graph $49(x-5)^2 + 25(y+2)^2 = 1225$ and find the foci.

First we have to get this into standard form, like the equations above. To make the right side 1, we need to divide everything by 1225.

$$\frac{49(x-5)^2}{1225} + \frac{25(y+2)^2}{1225} = \frac{1225}{1225}$$
$$\frac{(x-5)^2}{25} + \frac{(y+2)^2}{49} = 1$$

Now, we know that the ellipse will be vertical because 25 < 49. a=7, b=5 and the center is (5,−2).



To find the foci, we first need to find c by using $c^2 = a^2 - b^2$.

$$c^2 = 49 - 25 = 24$$
$$c = \sqrt{24} = 2\sqrt{6}$$

The foci are $(5, -2 \pm 2\sqrt{6})$ or $(5, -6.9)$ and $(5, 2.9)$.

Examples

Example 1

Earlier, you were asked to find the vertex of your graph and to determine where the foci of the ellipse will be located.

Solution

We first need to get our equation in the form of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. So we divide both sides by 144.

$$\frac{16(x-2)^2}{144} + \frac{4(y+3)^2}{144} = \frac{144}{144}$$
$$\frac{(x-2)^2}{9} + \frac{(y+3)^2}{36}$$

Now we can see that $h = 2$ and $3 = -k$ or $k = -3$. Therefore the origin is $(2, -3)$.

Because $9 < 36$, we know this is a vertical ellipse. To find the foci, use $c^2 = a^2 - b^2$.

$$c^2 = 36 - 9 = 27$$
$$c = \sqrt{27} = 3\sqrt{3}$$

The foci are therefore $(2, -3 + 3\sqrt{3})$ and $(2, -3 - 3\sqrt{3})$

Example 2

Find the center, vertices, co-vertices and foci of $\frac{(x+4)^2}{81} + \frac{(y-7)^2}{16} = 1$.

Solution

The center is $(-4, 7), a = \sqrt{81} = 9$ and $b = \sqrt{16} = 4$, making the ellipse horizontal. The vertices are $(-4 \pm 9, 7)$ or $(-13, 7)$ and $(5, 7)$. The co-vertices are $(-4, 7 \pm 4)$ or $(-4, 3)$ and $(-4, 11)$. Use $c^2 = a^2 - b^2$ to find c.

$$c^2 = 81 - 16 = 65$$
$$c = \sqrt{65}$$

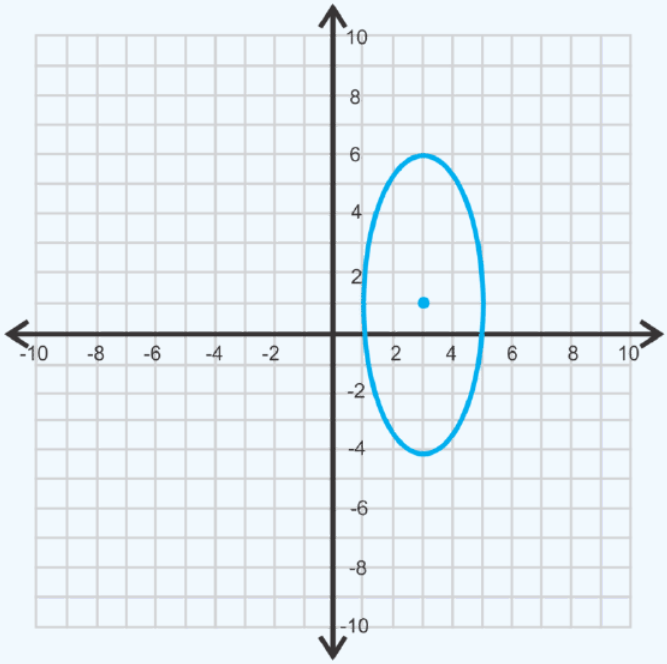
The foci are $(-4 - \sqrt{65}, 7)$ and $(-4 + \sqrt{65}, 7)$.

Example 3

Graph $25(x-3)^2 + 4(y-1)^2 = 100$ and find the foci.

Solution

Change this equation to standard form in order to graph.



$$\frac{25(x-3)^2}{100} + \frac{4(y-1)^2}{100} = \frac{100}{100}$$
$$\frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$$

center: $(3, 1), b = 2, a = 5$

Find the foci.

$$c^2 = 25 - 4 = 21$$
$$c = \sqrt{21}$$

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The foci are $(3, 1+\sqrt{21})$ and $(3, 1-\sqrt{21})$.

Example 4

Find the equation of the ellipse with co-vertices (−3,−6) and (5,−6) and focus (1,−2).

Solution

The co-vertices (−3,−6) and (5,−6) are the endpoints of the minor axis. It is |−3−5|=8 units long, making b=4. The midpoint between the co-vertices is the center.

$\left(\frac{-3+5}{2}, -6\right) = \left(\frac{2}{2}, -6\right) = (1, -6)$

The focus is (1,−2) and the distance between it and the center is 4 units, or c. Find a.

$16 = a^2 - 16$
$32 = a^2$
$a = \sqrt{32} = 4\sqrt{2}$

The equation of the ellipse is $\frac{(x-1)^2}{16} + \frac{(y+6)^2}{32} = 1$.

Review

Find the center, vertices, co-vertices, and foci of each ellipse below.

1. $\frac{(x+5)^2}{25} + \frac{(y+1)^2}{36} = 1$
2. $(x+2)^2 + 16(y-6)^2 = 16$
3. $\frac{(x-2)^2}{9} + \frac{(y-3)^2}{49} = 1$
4. $25x^2 + 64(y-6)^2 = 1600$
5. $(x-8)^2 + \frac{(y-4)^2}{9} = 1$
6. $81(x+4)^2 + 4(y+5)^2 = 324$
7. Graph the ellipse in #1.
8. Graph the ellipse in #2.
9. Graph the ellipse in #4.
10. Graph the ellipse in #5.

Using the information below, find the equation of each ellipse.

11. vertices: (−2, −3) and (8, −3) co-vertex: (3, −5)
12. vertices: (5, 6) and (5, −12) focus: (5, −7)
13. co-vertices: (0, 4) and (14, 4) focus: (7, 1)
14. foci: (−11, −4) and (1, −4) vertex: (−12, −4)
15. **Extension** Rewrite the equation of the ellipse, $36x^2 + 25y^2 - 72x + 200y - 464 = 0$ in standard form, by completing the square for both the x and y terms.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.6.

Image Attributions

1. [Figure 1]
   Credit: Ad Meskens
   Source:
   https://commons.wikimedia.org/wiki/File:Illu_pancrease.jpg;https://commons.wikimedia.org/wiki/File:President%27s_Park_with_White_House.jpg

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6.1.4: Focal Property of Ellipses

Focal Property of Ellipses

In 1602, the astronomer Johannes Kepler was working on a problem for his boss Tycho Brahe. He was continuing a centuries-old study trying to devise a formula that could calculate the orbits of the planets, specifically Mars in this case. He worked for years on the problem with the most advanced technology of the time, but was never satisfied with the results as they were never truly accurate or reliable.

To simplify his calculations, he hit upon the concept that a planet sweeps out equal areas of space (in a pie-slice shaped figure with the sun at the tip) in equal amounts of time, despite differences in the distance of the planet from the sun.

For the next three years he attempted to design the perfect equation to describe an orbit which would fit this and other observations he was using.

Finally, in 1605, he realized that he had the answer. His realization/discovery would firmly place him as one of the most famous scientists of all time.

What did he realize?

Focal Property of Ellipses

Ellipses Not Centered at the Origin

To find an equation for ellipses centered around another point, say (h, k), simply replace x with $x - h$ and y with $y - k$. This will shift all the points of the **ellipse** to the right h units (or left if $h < 0$) and to up k units (or down if $k < 0$). So the general form for a horizontally- or vertically-oriented ellipse is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

It is centered about the point (h, k). If $b < a$, the ellipse is horizontally oriented and has **foci** $\left(h + \sqrt{a^2 - b^2}, k\right)$ and $\left(h - \sqrt{a^2 - b^2}, k\right)$ on its horizontal **major axis**. If $a < b$, it is vertically oriented and has foci $\left(h, k + \sqrt{a^2 - b^2}\right)$ and $\left(h, k - \sqrt{a^2 - b^2}\right)$ on its vertical major axis.

Examples

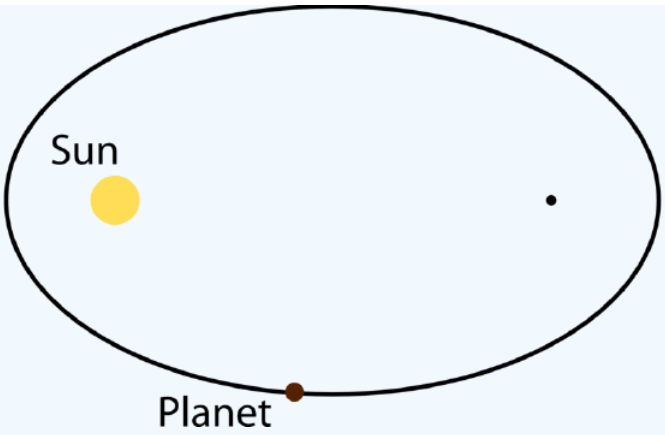
Example 1

Earlier, you were asked a question about the problem that astronomer Johannes Kepler was working on for his boss, Tycho Brahe.

Solution

Kepler's realization was that, even though he had deliberately avoided them for a very long time because they were so simple, **ellipses** were the perfect shape to make all of his calculations come together.

When a planet orbits the sun (or when any object orbits any other), it takes an elliptical path and the sun lies at one of the two foci of the ellipse. Kepler's laws regarding planetary motion are accurate enough to produce modern computations which are still used to predict the motion of artificial satellites today.



Example 2

Explain why subtracting h from the x–term and k from the y–term in the equation for an ellipse shifts the ellipse h horizontally and k vertically.

Solution

If (x, y) is a solution to $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

then $(x + h, y + k)$ is a solution to $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

This produces a graph that is shifted horizontally by h and vertically by k.

Example 3

Graph the equation $4x^2 + 8x + 9y^2 - 36y + 4 = 0$.

Solution

We need to get the equation into the form of general equation above. The first step is to group all the x terms and y terms, factor our the leading coefficients of x^2 and y^2, and move the constants to the other side of the equation:

$$4\left(x^2 + 2x\right) + 9\left(y^2 - 4y\right) = -4$$

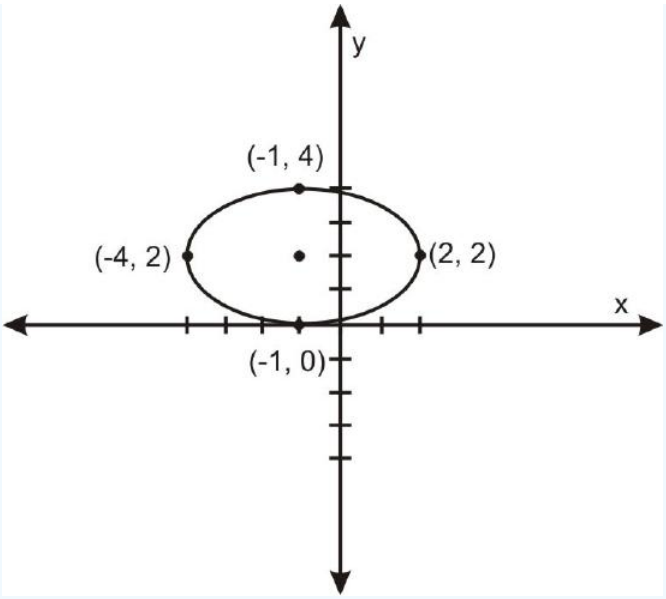
Now, we "complete the square" by adding the appropriate terms to the x expressions and the y expressions to make a perfect square.

$$4\left(x^2 + 2x + 1\right) + 9\left(y^2 - 4y + 4\right) = -4 + 4 + 36$$

Now we factor and divide by the coefficients to get:

$$\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

And there we have it. Once it's in this form, we see this is an ellipse is centered around the point (-1,2), it has a horizontal major axis of length 3 and a vertical **minor axis** of length 2, and from this we can make a sketch of the ellipse:



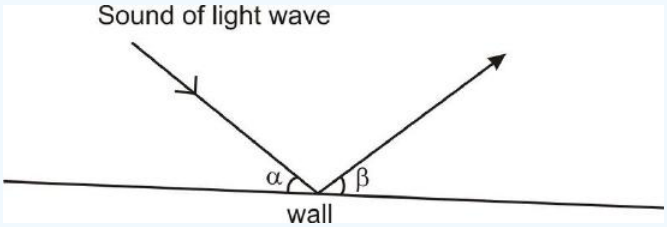
Example 4

The National Statuary Hall in the United States Capital Building is an example of an ellipse-shaped room, sometimes called an "echo room", which provide an interesting application to a property of ellipses. If a person whispers very quietly at one of the foci, the sound echoes in a way such that a person at the other focus can often hear them very clearly. Rumor has it that John Quincy Adams took advantage of this property to eavesdrop on conversations in this room.

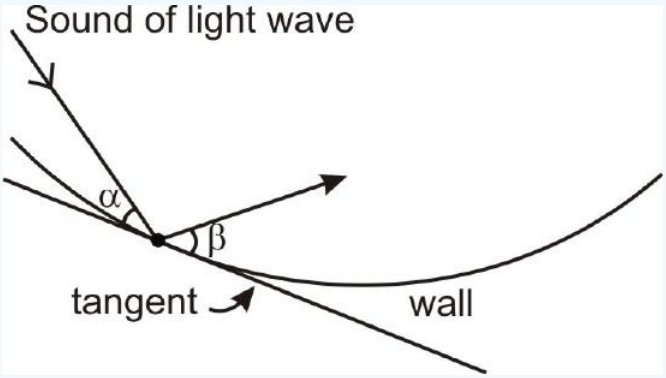
How do *Echo Rooms* work? What does the elliptical shape of the room have to do with it?

Solution

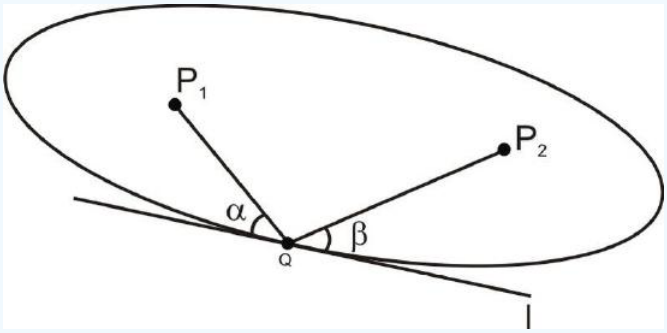
The property of ellipses that makes echo rooms work is called the "optical property." So why echoes, if this is an optical property? Well, light rays and sound waves bounce around in similar ways. In particular, they both bounce off walls at equal angles. In the diagram below, $\alpha = \beta$.



For a curved wall, they bounce at equal angles to the tangent line at that point:



So the "optical property" of ellipses is that lines between a point on the ellipse and the two foci form equal angles to the tangent at that point, or in other words, whispers coming from one foci bounce directly to the other foci. In the diagram below, for each Q on the ellipse, $\angle\alpha \cong \angle\beta$.



Example 5

Though planets take an elliptical path around the sun, these ellipses often have a very low **eccentricity**, meaning they are close to being circles. The diagram above exaggerates the elliptical shape of a planet's orbit. The Earth's orbit has an eccentricity of 0.0167. Its minimum distance from the sun is 146 million km. What is its maximum distance from the sun? If the sun's diameter is 1.4 million kilometers, do both foci of the Earth's orbit lie within the sun?

Solution

Recall that the eccentricity of an ellipse is $\varepsilon = \frac{\sqrt{a^2-b^2}}{a}$

Assume that the orbit of the sun is an ellipse centered at (0,0). Then we can use the distance from the origin to the focus $\sqrt{a^2-b^2}$ to set up the equations $146 + 146 + 2\sqrt{a^2-b^2} = 2a$ and $0.167 = \frac{\sqrt{a^2-b^2}}{a}$. Solving we get $a = 175.270, b = 175.245$ and the distance from (0, 0) to the foci, c=2.927 (all units are in millions of km). Finally the maximum distance from the earth to the sun is approximately 152 million km. From Kepler's law, we know one of the foci of its orbit is at the center of the sun. The other foci is 2(2.927)=5.854 million kilometers away, so it is outside the sun (but not by very far!)

Example 6

What is the sum of the distances to the foci of the points on a vertically-oriented ellipse?

Solution

2b

Example 7

Try to graph the ellipse: $3x^2 - 12x + 5y^2 + 10y - 3 = 0$. What goes wrong?

Solution

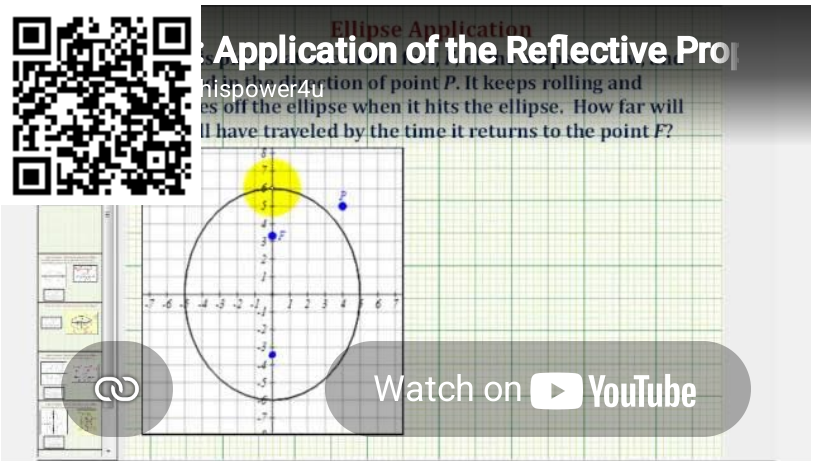
After **completing the square**, we have the sum of positive numbers equaling a negative number. This is an impossibility, so the equation has no solutions.

Example 8

Graph the ellipse (plot points): $5x^2 - 15x - 2y^2 + 8y - 50 = 0$. What is different here? Explain what you think the graph of this equation might look like.

Solution

After completing the square, the x term and the y term are opposite signs. If you plot some points you will see that the graph has two disconnected sections. This class of **conic** sections will be discussed in the lesson on **hyperbolas**.



Review

Graph the following more advanced ellipses.

1. $(x)^2 + \frac{(y+2)^2}{9} = 1$
2. $\frac{(x+2)^2}{4} + (y+2)^2 = 1$
3. $\frac{(x+2)^2}{9} + (y)^2 = 1$
4. $\frac{(x-1)^2}{9} + \frac{(y-4)^2}{16} = 1$
5. $4(x-1)^2 + 9(y)^2 = 36$
6. $(x+1)^2 + \frac{(y-1)^2}{9} = 1$
7. $(x-4)^2 + \frac{(y+3)^2}{16} = 1$
8. $\frac{(x)^2}{16} + \frac{(y)^2}{9} = 1$
9. $4(x+1)^2 + (y-3)^2 = 16$
10. $(x+4)^2 + \frac{(y-1)^2}{16} = 1$

Graph the following special-case ellipses.

11. $(x+4)^2 + (y+2)^2 = 4$
12. $(x+2)^2 + (y)^2 = 16$
13. $(x-3)^2 + (y-2)^2 = 1$
14. $(x+2)^2 + (y-1)^2 = 9$
15. $(x-3)^2 + (y-1)^2 = 9$
16. What do these ellipses have in common?

Answer the following word problems.

17. While the elliptical paths of planets are ellipses that are closely approximated by circles, comets and asteroids often have orbits that are ellipses with very high eccentricity. Halley's comet has an eccentricity of 0.967, and comes within 54.6 million miles of the sun at its closest point, or "perihelion". What is the furthest point it reaches from the sun?
18. Calculate the *area* of an ellipse with the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (Hint: use a geometric argument starting with the area of a circle.)
19. Design the largest possible echo room with the following constraints: You would like to spy on someone who will be 3 m from the tip of the ellipse. The room cannot be more than 100 m wide in any direction. How far from the person you're spying on will you be standing?
20. No matter what the orientation of a stick, if you trace out the path that the shadow of the tip makes on a flat surface, you will find it is an ellipse. Describe why this is true.

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Review (Answers)

To see the Review answers, open this PDF file and look for section 6.2.

Resources



" frameborder="0" height="450px" name="92213" src="https://www.ck12.org/flx/show/video/...icity-Overview" thumbnailurl="" title="VideoObject?hash=30acd2f0d8272541c1ce4d3f07362e1a" uploaddate="2016-07-12 20:22:16" width="80%">

Vocabulary

Term	Definition
Completing the Square	Completing the square is a common method for rewriting quadratics. It refers to making a perfect square trinomial by adding the square of 1/2 of the coefficient of the x term.
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Eccentricity	The eccentricity of a conic section is a measure of how much the conic section deviates from being circular. The eccentricity of circles is 0, the eccentricity of ellipses is between 0 and 1, the eccentricity of parabolas is 1, and the eccentricity of hyperbolas is greater than 1. For ellipses and hyperbolas, $e = \frac{c}{a}$.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Foci	The foci of an ellipse are the two points that define the ellipse. The sum of the distances from any point on the ellipse to the foci is constant.
Major Axis	The major axis of an ellipse is the longest diameter of the ellipse.
minor axis	The minor axis of an ellipse is the shortest diameter of the ellipse.
standard form	The standard form of an ellipse centered at (h, k) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

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SECTION OVERVIEW

6.2: Parabolas

Topic hierarchy

6.2.1: Parabolas with Vertex at the Origin

6.2.2: Parabolas and the Distance Formula

6.2.3: Parabolas with any Vertex

6.2.4: Parabolas and Analytic Geometry

6.2.5: Applications of Parabolas

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6.2.1: Parabolas with Vertex at the Origin

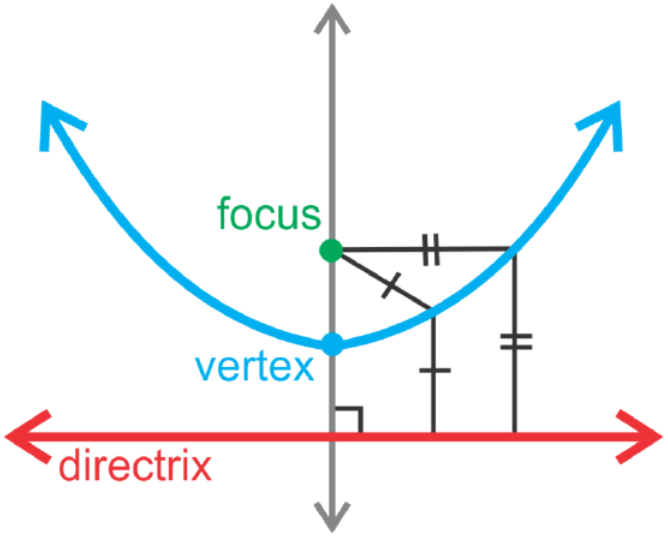
Parabolas with Vertex at the Origin

The area of a square is represented by the equation $y = 9x^2$. What are the **focus** and **directrix** of this equation?

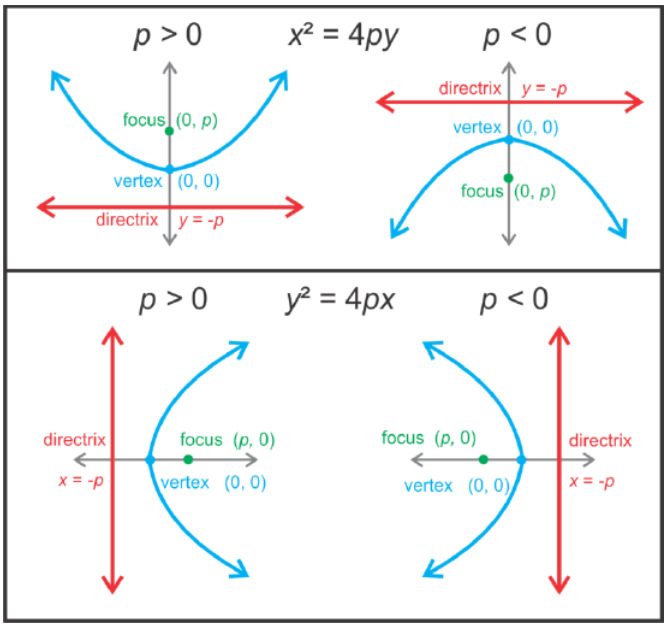
Parabolas with Vertex at the Origin

You already know that the graph of a **parabola** has the parent graph $y = x^2$, with a vertex of (0, 0) and an axis of symmetry of $x = 0$. A parabola can also be defined in a different way. It has a property such that any point on it is equidistant from another point, called the **focus**, and a line called the **directrix**.

The focus is on the axis of symmetry and the vertex is halfway between it and the directrix. The directrix is perpendicular to the axis of symmetry.



Until now, we have been used to seeing the equation of a parabola like $y = ax^2$. In this concept, we will rewrite the equation to look like $x^2 = 4py$ where p is used to find the focus and directrix. We will also draw the parabola with a horizontal orientation, such that the equation will be $y^2 = 4px$.



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Notice, that when the parabola opens to the left or right, the y is squared. In this concept, the vertex will be (0, 0).

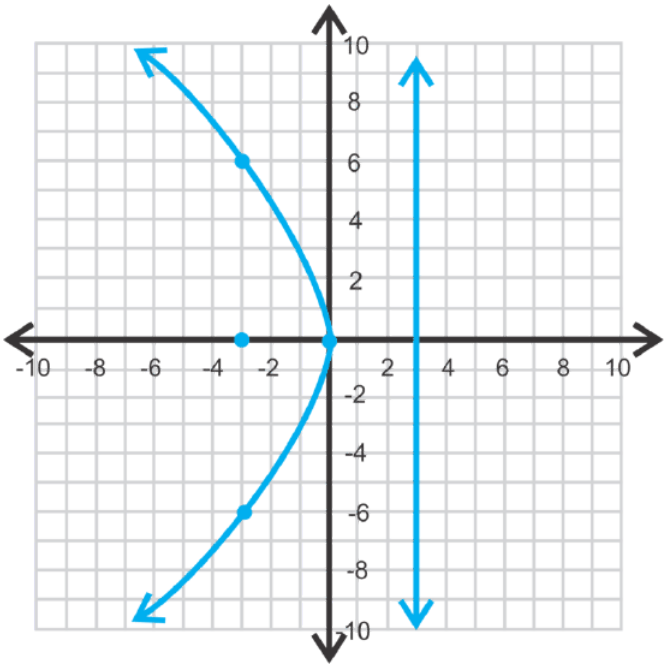
Let's analyze the equation $y^2 = -12x$. We'll find the focus, directrix, and determine if the function opens up, down, to the left or right. Then, we'll graph the parabola.

To find the focus and directrix, we need to find p. We can set $-12 = 4p$ and solve for p.

$$\begin{aligned} -12 &= 4p \\ -3 &= p \end{aligned}$$

Because y is squared, we know that the parabola opens to the left or right. Because p is negative, we know it is going to open to the left, towards the negative side of the x-axis. Using the pictures above, this parabola is like the second one under $y^2 = 4px$. Therefore, the focus is (−3, 0) and the directrix is x=3. To graph the parabola, plot the vertex, focus, directrix, and sketch the curve. Find at least one or two points on the curve to make sure your sketch is accurate. For example, because (−3, 6) is on the parabola, then (−3, −6) is also on the parabola because it is symmetrical.

Notice that the points (−3, 6) and (−3, −6) are equidistant from the focus and the directrix. They are both 6 units from each.

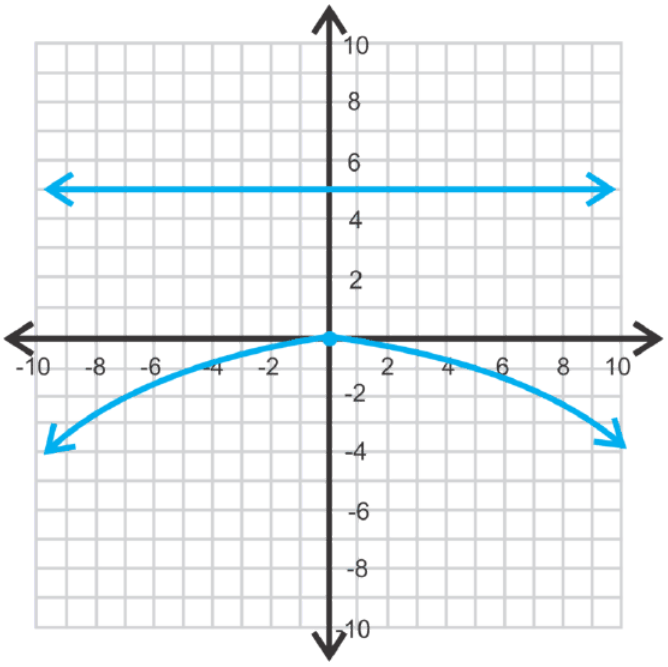


The focus of a parabola is $\left(0, \frac{1}{2}\right)$. Let's find the equation of the parabola.

Because the p value is the y-value and positive, this parabola is going to open up. So, the general equation is $x^2 = 4py$. Plugging in $\frac{1}{2}$ for p, we have $x^2 = 4 \cdot \frac{1}{2}y$ or $x^2 = 2y$.

Now, let's find the equation of the parabola below.

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The equation of the directrix is $y = 5$, which means that $p = -5$ and the general equation will be $x^2 = 4py$. Plugging in -5 for p, we have $x^2 = -20y$.

Examples

Example 1

Earlier, you were asked to find the focus and directrix of the equation $y = 9x^2$.

Solution

To find the focus and directrix, we need to solve for x^2 and then find p.

$$\begin{aligned} y &= 9x^2 \\ \tfrac{1}{9}y &= x^2 \end{aligned}$$

We can now set $\frac{1}{9} = 4p$ and solve for p.

$$\begin{aligned} \tfrac{1}{9} &= 4p \\ \tfrac{1}{36} &= p \end{aligned}$$

Therefore, the focus is $\left(0, \frac{1}{36}\right)$ and the directrix is $y = -\frac{1}{36}$.

Example 2

Determine if the parabola $x^2 = -2y$ opens up, down, left or right.

Solution

Down; p is negative and x is squared.

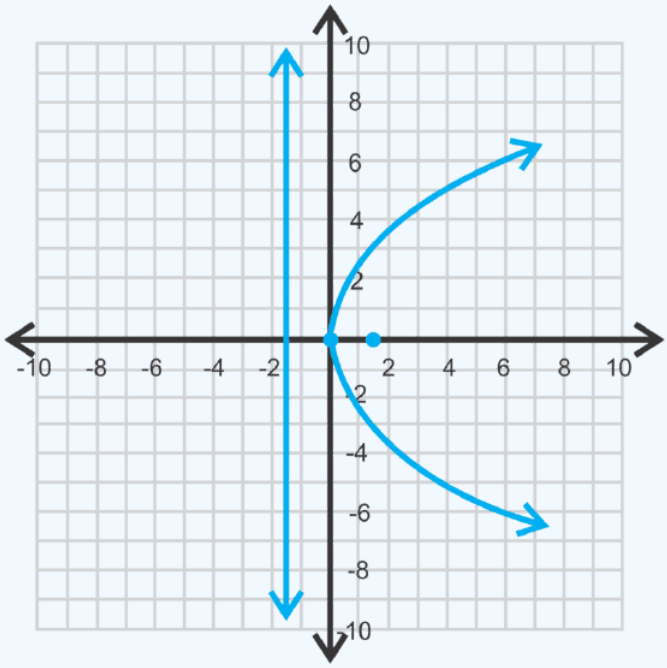
Example 3

Find the focus and directrix of $y^2 = 6x$. Then, graph the parabola.

Solution

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Solving for p, we have $4p = 6 \rightarrow p = \frac{3}{2}$. Because y is squared and p is positive, the parabola will open to the right. The focus is $\left(\frac{3}{2}, 0\right)$ and the directrix is $x = -\frac{3}{2}$.



Example 4

Find the equation of the parabola with directrix $x = -\frac{3}{8}$.

Solution

If the directrix is negative and vertical (x=), we know that the equation is going to be $y^2 = 4px$ and the parabola will open to the right, making p positive; $p = \frac{3}{8}$. Therefore, the equation will be $y^2 = 4 \cdot \frac{3}{8} \cdot x \rightarrow y^2 = \frac{3}{2}x$.

Review

Determine if the parabola opens to the left, right, up or down.

1. $x^2 = 4y$
2. $y^2 = -\frac{1}{2}x$
3. $x^2 = -y$

Find the focus and directrix of the following parabolas.

4. $x^2 = -2y$
5. $y^2 = \frac{1}{4}x$
6. $y^2 = -5x$

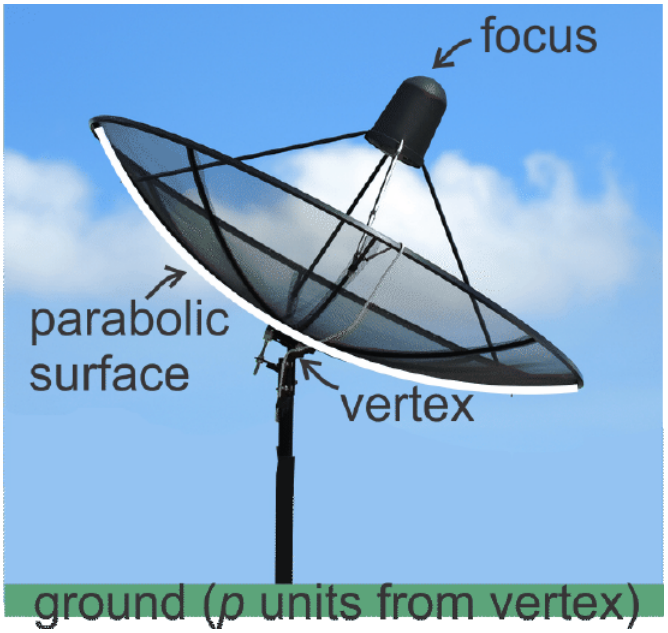
Graph the following parabolas. Identify the focus and directrix as well.

7. $x^2 = 8y$
8. $y^2 = \frac{1}{2}x$
9. $x^2 = -3y$

Find the equation of the parabola given that the vertex is (0, 0) and the focus or directrix.

10. focus: (4, 0)
11. directrix: x=10
12. focus: $\left(0, \frac{7}{2}\right)$

13. You have seen that earlier the basic parabolic equation was $y = ax^2$. Now, we write $x^2 = 4py$. Rewrite p in terms of a and determine how they affect each other.
14. **Challenge** Use the distance formula, $d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$, to prove that the point (4, 2) is on the parabola $x^2 = 8y$.
15. **Real World Application** A satellite dish is a 3-dimensional parabola used to retrieve sound, TV, or other waves. Assuming the vertex is (0, 0), where would the focus have to be on a satellite dish that is 4 feet wide and 9 inches deep? You may assume the parabola has a vertical orientation (opens up).



Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.1.

Vocabulary

Term	Definition
directrix	The directrix of a parabola is the line that the parabola seems to curve away from. All points on a parabola are equidistant from the focus of the parabola and the directrix of the parabola.
focus	The focus of a parabola is the point that "anchors" a parabola. Any point on the parabola is exactly the same distance from the focus as from the directrix.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.

Image Attributions

1. [Figure 1]
   Credit: Holly Fischer
   Source: https://commons.wikimedia.org/wiki/File:Three_Main_Layers_of_the_Eye.png
   License: CC BY-NC 3.0

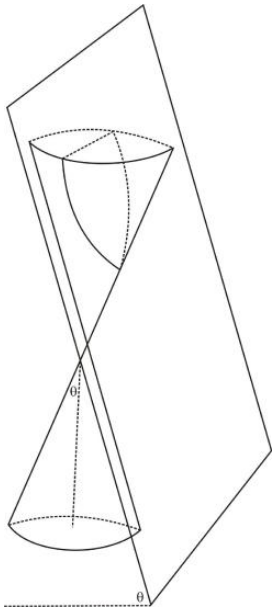
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6.2.2: Parabolas and the Distance Formula

Parabolas and the Distance Formula

In previous lessons on **conic sections**, we discussed both the circle and the **ellipse**, which each result from "slicing" a cone clear through from left to right. In this lesson, we will discuss the shape formed when we slice through only one side of the cone, creating a bowl-shaped figure called a **parabola**.

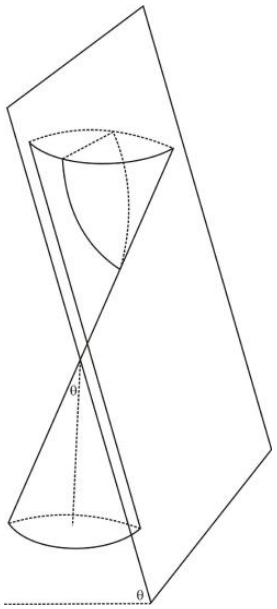
Consider the "hourglass" figure we used in the definitions of the circle and ellipse, created by connecting two infinite cones at their tips. What limitation would there be on the angle of the slice we would take out of one of the cones, if we wanted to only get a parabola (not get an ellipse, and not hit the other cone in any way)?



Parabolas and the Distance Formula

We've examined **ellipses** and circles, the two cases when a plane intersects only one side of the cone and creates a finite cross-section. Is it possible for a plane to intersect only one side of the cone, but create an infinite cross-section?

It turns out that this is possible if and only if the plane is parallel to one of the lines making up the cone. Or, in other words, the angle between the plane and the horizon, is equal to the angle formed by a side of the cone and the horizontal plane.

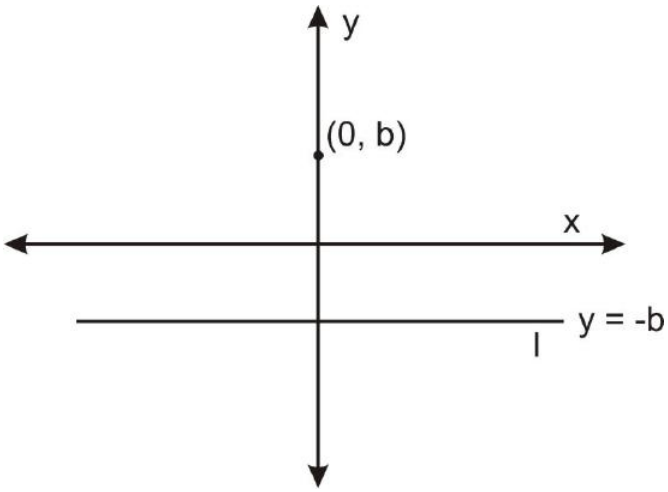


In the image above, if you tilt the plane a little bit to the left it will cut off a finite ellipse (possibly a very large one if you only tilt it a little.) Tilt the plane to the right and it will intersect both sides of the cone, making a two-part conic section called a hyperbola, which will be discussed in the next section.

When the plane is parallel to from the side of the cone, the infinite shape that results from the intersection of the plane and the cone is called a parabola. Like the ellipse, it has a number of interesting geometric properties.

The equation of a parabola is simpler than that of the ellipse. There are a couple of methods of deriving the equation of a parabola, in this lesson we explore the **distance formula**.

This first method arises directly from the **focus-directrix** property discussed in the previous section. Suppose we have a line and a point not on that line in a plane, and we want to find the equation of the set of points in the plane that is equidistant to these two objects. Without losing any generality, we can orient the line horizontally and the point on the y−axis, with the origin halfway between them. Since the parabola is the set of points equidistant from the line and the point, The parabola passes through the origin, (0, 0). The picture below shows this configuration. The point directly between the directrix and the focus (the origin in this case) is called the **vertex** of the parabola. Suppose the focus is located at (0, b). Then the directrix must be y=−b.



Thus, the parabola is the set of points (x, y) equidistant from the line $y = -b$ and the focus point $(0, b)$. The distance to the line is the vertical segment from (x, y) down to $(0, -b)$, which has length $y - (-b) = y + b$. The distance from (x, y) to the focus $(0, b)$ is $\text{distance} = \sqrt{(x-0)^2 + (y-b)^2}$ by the distance formula. So the equation of the parabola is the set of points where these two distances equal.

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$$y + b = \sqrt{(x-0)^2 + (y-b)^2}$$

Since distances are always positive, we can square both sides without losing any information, obtaining the following.

$$\begin{aligned} y^2 + 2by + b^2 &= x^2 + y^2 - 2by + b^2 \\ 2by &= x^2 - 2by \\ 4by &= x^2 \\ y &= \frac{1}{4b}x^2 \end{aligned}$$

But b was chosen arbitrarily and could have been any positive number, and for any positive number, a, there always exists a number b such that $a = \frac{1}{4b}$, so we can rewrite this equation as:

$$y = ax^2$$

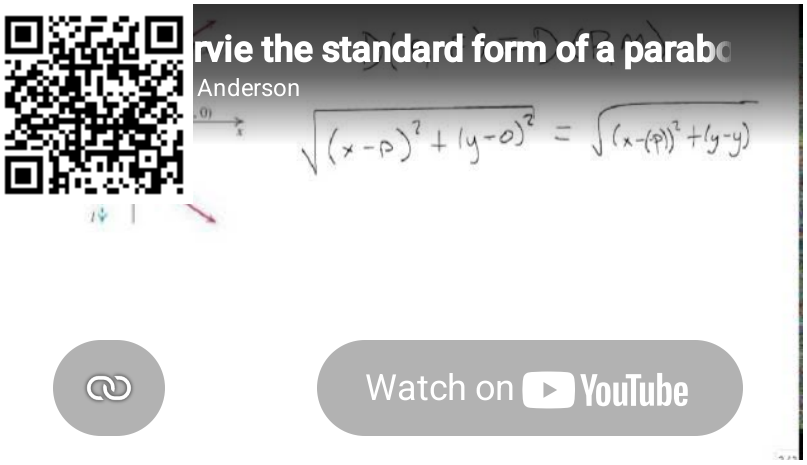
where a is any constant.

This is the general form of a parabola with a horizontal directrix, with a focus lying above it, and with a vertex at the origin. If a is negative, the parabola is reflected about the x−axis, resulting in a parabola with a horizontal directrix, with a focus lying below it, and with a vertex at the origin. The equation can be shifted horizontally or vertically by moving the vertex, resulting in the general form of a parabola with a horizontal directrix and passing through a vertex of (h, k):

$$y - k = a(x-h)^2$$

Switching x and y, the equation for a parabola with a vertical directrix and with a vertex at (h, k) is:

$$x - h = a(y-k)^2$$



Examples

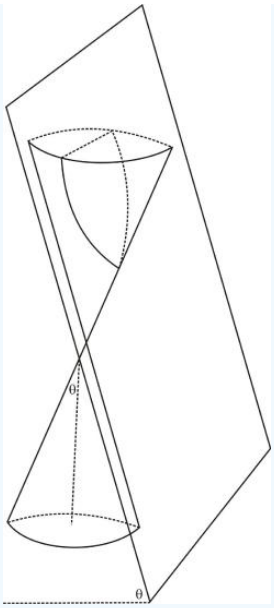
Example 1

Earlier, you were asked a question about the limitation on the angle of the slice taken out of one of the cones if you only wanted a parabola.

Solution

In order to only get a parabola cross section, the slice must be taken at the same angle as the side of the cone, that way the edge of the slice runs parallel to the edge of the cone and never intersects it at either top or bottom.

This can be seen by a close look at the image from above:



Example 2

Sketch a graph of the parabola $y = 3x^2 + 12x + 17$.

Solution

First, we need to factor out the coefficient of the x^2 term and complete the square:

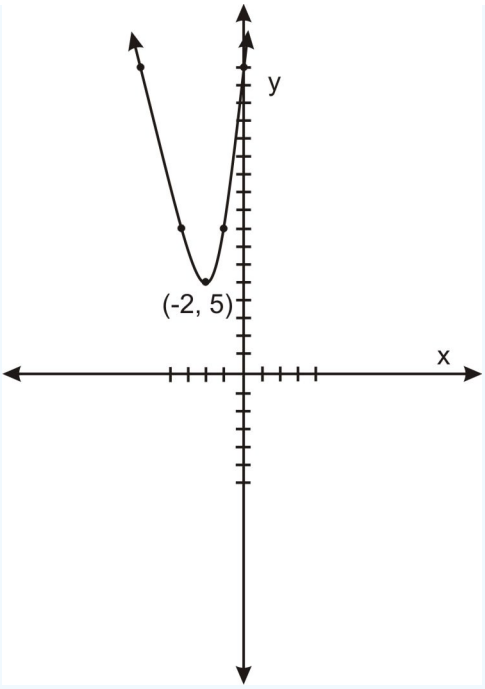
$y = 3\left(x^2 + 4x\right) + 17$
$y = 3\left(x^2 + 4x + 4\right) + 17 - 12$
$y = 3(x + 2)^2 + 5$

Now we write it in the form $y - k = a(x - h)^2$:

$y - 5 = 3(x + 2)^2$

So the vertex is at (−2, 5) and plotting a few x−values on either side of x=−2, we can draw the following sketch of the parabola:

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Example 3

Sketch a graph of the following parabola: $y = 2x^2 - 2x - 3$.

Solution

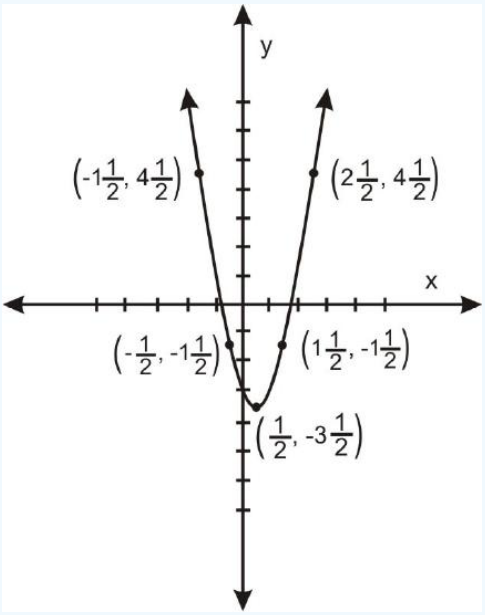
Factor out the 2: $y = 2(x^2 - x) - 3$

Complete the square: $y + \frac{1}{2} = 2\left(x^2 - x + \frac{1}{4}\right) - 3$

Add 3 to both sides and factor the trinomial: $y + 3\frac{1}{2} = 2\left(x - \frac{1}{2}\right)^2$

The vertex (h, k) is: $\left(\frac{1}{2}, -3\frac{1}{2}\right)$

Plot a couple of points to get:



Example 4

Sketch a graph of the following parabola: $3x^2 + 12x + 11 - y = 0$.

Solution

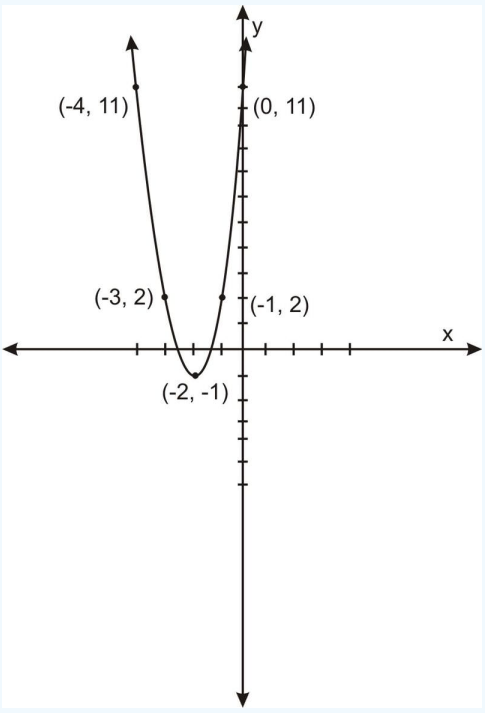
Factor out the 3 and move y and 11: $3\left(x^2 + 4x\right) = y - 11$

Complete the square: $3\left(x^2 + 4x + 4\right) = y - 11 + 12$

Factor the trinomial and collect like terms: $3(x + 2)^2 = y + 1$

The vertex (h, k) is at: $(-2, -1)$

Plot a couple of points to get:



Example 5

Identify which of the following equations are parabolas:

a. $y - 5x + x^2 = 3$
b. $x^2 - 3y^2 + 3y - 2x + 15 = 0$
c. $x - 6y^2 + 20x - 100 = 0$

Solution

Recall that a parabola has a squared input term **only**:

Since **b.** $x^2 - 3y^2 + 3y - 2x + 15 = 0$ is squared on the input and **output** (in other words, both x and y are squared), it is not a parabola.

a. $y - 5x + x^2 = 3$ and **c.** $x - 6y^2 + 20x - 100 = 0$ are both parabolas.

By the way, after completing the square, equation a factors as:

$(x - 1)^2 - 3\left(y - \frac{1}{2}\right)^2 + 14\frac{3}{4} = 0$.

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Example 6

Calculate the distance between (3, 4) and (9, 5).

Solution

To calculate the distance between (3, 4) and (9, 5), use the distance formula

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(9-3)^2 + (5-4)^2} \ldots\ldots \text{Substitute}$$

$$\text{distance} = \sqrt{(6)^2 + (1)^2} \ldots\ldots \text{Simplify}$$

$$\text{distance} = \sqrt{36+1} \rightarrow \sqrt{37}$$

$\therefore \sqrt{37}$ is the distance between (3, 4) and (9, 5).

Example 7

Calculate the distance between (-2, 7) and (11, 23).

Solution

To calculate the distance between (-2, 7) and (11, 23), use the distance formula

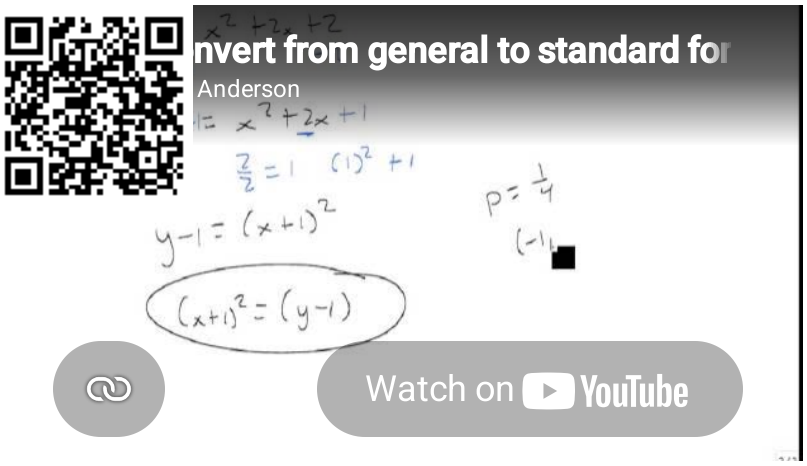
$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{distance} = \sqrt{(11-(-2))^2 + (23-7)^2} \ldots\ldots \text{Substitute}$$

$$\text{distance} = \sqrt{(13)^2 + (16)^2} \ldots\ldots \text{Simplify}$$

$$\text{distance} = \sqrt{169+256} \rightarrow \sqrt{425} \rightarrow 5\sqrt{17}$$

$\therefore 5\sqrt{17}$ is the distance between (-2, 7) and (11, 23).



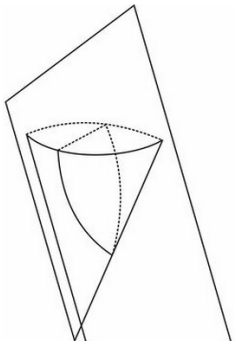
Review

Graph the following:

1. $y = -3(x-1)^2 + 2$
2. $y = 3(x-1)^2 - 1$
3. $x = -2(y+2)^2 - 1$
4. $y = 3(x+4)^2 - 1$
5. $y = (x-3)^2$
6. $x = -3(y)^2 + 1$
7. $x = -(y-3)^2 - 4$

8. $x = -3(y-4)^2 + 4$
9. $y = 2(x+3)^2 - 3$
10. $x = (y-1)^2 - 2$

For questions 11-20, imagine a limited cone (not infinitely tall), as pictured below. Assume the two coordinates listed represent the intersection of a parabolic curve and the top of the cone. If the top surface of the cone were represented by the *x*-axis, then the two coordinates could be considered the *x*-intercepts of the equation of the parabola. Find the distance between the points, and where required, the coordinates of the points.



11. Coordinates: (-20, -17) and (6, -1)
12. Coordinates: (-1, -5) and (6, -13)
13. Coordinates: (1, 2) and (5, -5)
14. Coordinates: (13, 12) and (15, 6)
15. Coordinates: (3, 9) and (6, -14)
16. $-25x^2 + 15x + 10 = 0$
17. $-24x^2 + 22x - 4 = 0$
18. $4x^2 - 24x + 32 = 0$
19. $24x^2 + 54x + 27 = 0$
20. $12x^2 + 25x + 12 = 0$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.3.

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Vocabulary

Term	Definition
Congruent	Congruent figures are identical in size, shape and measure.
directrix	The directrix of a parabola is the line that the parabola seems to curve away from. All points on a parabola are equidistant from the focus of the parabola and the directrix of the parabola.
Distance Formula	The distance between two points (x_1, y_1) and (x_2, y_2) can be defined as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
focus	The focus of a parabola is the point that "anchors" a parabola. Any point on the parabola is exactly the same distance from the focus as from the directrix.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.
Vertex	The vertex of a parabola is the highest or lowest point on the graph of a parabola. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.

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6.2.3: Parabolas with any Vertex

Parabolas with Vertex at (h, k)

Your homework assignment is to find the focus of the parabola $(x+4)^2=-12(y-5)$. You say the focus is (−4, 5). Banu says the focus is (0, −3). Carlos says the focus is (−4, 2). Which one of you is correct?

Parabolas with Vertex at (h, k)

You have already learned that parabolas don't always have their vertex at (0, 0). In this concept, we will address parabolas where the vertex is (h, k), learn how to find the focus, directrix and graph.

Recall that the equation of a parabola is $x^2=4py$ or $y^2=4px$ and the vertex is on the origin. Also, recall that the **vertex form** of a parabola is $y=a(x-h)^2+k$. Combining the two, we can find the vertex form for conics.

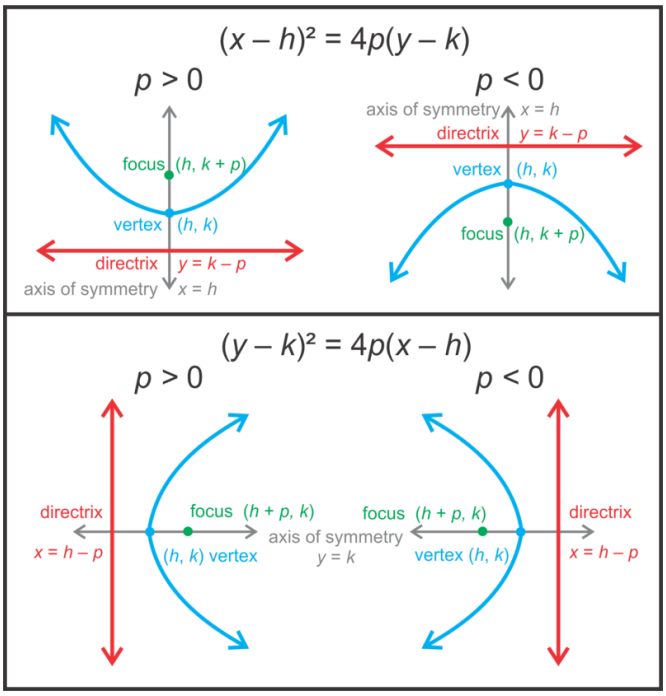
$y=a(x-h)^2+k$ and $x^2=4py$ — Solve the first for $(x-h)^2$

$(x-h)^2=\frac{1}{a}(y-k)$ — We found that $4p=\frac{1}{a}$

$(x-h)^2=4p(y-k)$

If the parabola is horizontal, then the equation will be $(y-k)^2=4p(x-h)$. Notice, that even though the orientation is changed, the h and k values remain with the x and y values, respectively.

Finding the focus and directrix are a little more complicated. Use the extended table below to help you find these values.



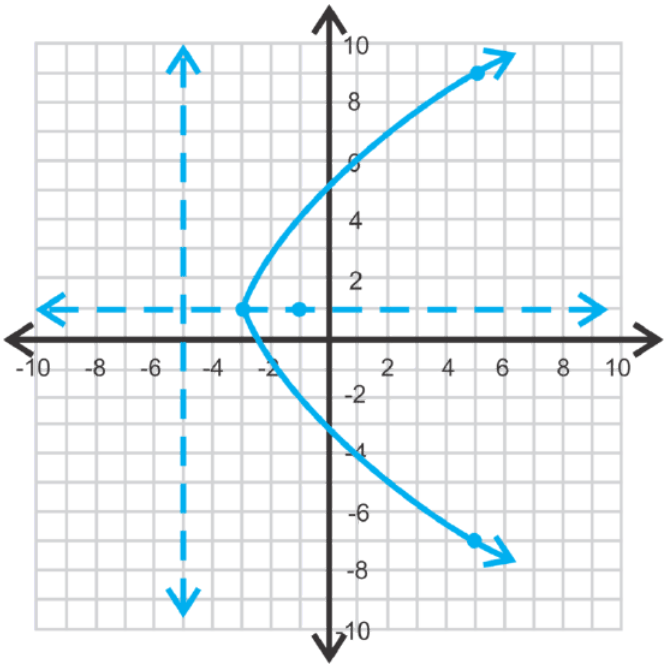
Notice that the way we find the focus and directrix does not change whether p is positive or negative.

Let's analyze the equation $(y-1)^2=8(x+3)$. We'll find the vertex, axis of symmetry, focus, and directrix. Then, we'll determine if the function opens up, down, left or right.

First, because y is squared, we know that the parabola will open to the left or right. We can conclude that the parabola will open to the *right* because 8 is positive, meaning that p is positive. Next, find the vertex. Using the general equation, $(y-k)^2=4p(x-h)$, the vertex is (−3, 1) and the axis of symmetry is $y=1$. Setting $4p=8$, we have that $p=2$. Adding p to the x-value of the vertex, we get the focus, (−1, 1). Subtracting p from the x-value of the vertex, we get the directrix, $x=-5$.

Let's graph the parabola from the problem above. Plot the vertex, axis of symmetry, focus, and directrix.

First, plot all the critical values we found earlier. Then, determine a set of symmetrical points that are on the parabola to make sure your curve is correct. If $x = 5$, then y is either -7 or 9. This means that the points (5, −7) and (5, 9) are both on the parabola.



It is important to note that parabolas with a horizontal orientation are not functions because they do not pass the vertical line test.

The vertex of a parabola is (−2, 4) and the directrix is y=7. Let's find the equation of the parabola.

First, let's determine the orientation of this parabola. Because the directrix is horizontal, we know that the parabola will open up or down (see table/pictures above). We also know that the directrix is *above* the vertex, making the parabola open down and p will be negative (plot this on an x−y plane if you are unsure).

To find p, we can use the vertex, (h, k) and the equation for a horizontal directrix, $y = k - p$.

$$\begin{aligned} 7 &= 4 - p \\ 3 &= -p \qquad \textbf{Remember, } p \textbf{ is negative because of the downward orientation of the parabola.} \\ -3 &= p \end{aligned}$$

Now, using the general form, $(x - h)^2 = 4p(y - k)$, we can find the equation of this parabola.

$$\begin{aligned} (x - (-2))^2 &= 4(-3)(y - 4) \\ (x + 2)^2 &= -12(y - 4) \end{aligned}$$

Examples

Example 1

Earlier, you were asked to determine which student is correct.

Solution

This parabola is of the form $(x - h)^2 = 4p(y - k)$. From the table earlier in this lesson, we can see that the focus of a parabola of this form is $(h, k + p)$. So now we have to find *h*, *k*, and *p*.

If we compare $(x + 4)^2 = -12(y - 5)$ to $(x - h)^2 = 4p(y - k)$, we see that:

1. $4 = -h$ or $h = -4$
2. $-12 = 4p$ or $p = -3$
3. $5 = k$

From these facts we can find $k + p = 5 + (-3) = 2$.

Therefore, the focus of the parabola is $(-4, 2)$ and Carlos is correct.

Example 2

Find the vertex, focus, axis of symmetry and directrix of $(x+5)^2 = 2(y+2)$.

Solution

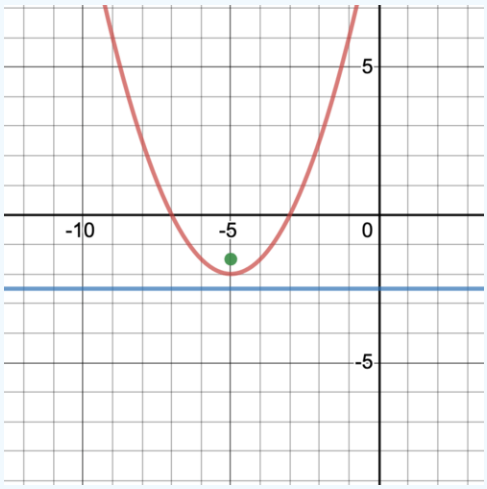
The vertex is $(-5, -2)$ and the parabola opens up because p is positive and x is squared.

$4p = 2$, making $p = \frac{1}{2}$. The focus is $(-5, -2+2)$ or $(-5, 0)$, the axis of symmetry is $x = -5$, and the directrix is $y = -2 - \frac{1}{2}$ or $y = -2\frac{1}{2}$.

Example 3

Graph the parabola from Example 2.

Solution



Example 4

Find the equation of the parabola with vertex (−5, −1) and focus (−8, −1).

Solution

The vertex is (−5, −1), so $h = -5$ and $k = -1$. The focus is (−8, −1), meaning that that parabola will be horizontal. We know this because the y-values of the vertex and focus are both -1. Therefore, p is added or subtracted to h.

$(h+p, k) \rightarrow (-8, -1)$ we can infer that $h + p = -8 \rightarrow -5 + p = -8$ and $p = -3$

Therefore, the equation is $(y-(-1))^2 = 4(-3)(x-(-5)) \rightarrow (y+1)^2 = -12(x+5)$

Review

Find the vertex, focus, axis of symmetry, and directrix of the parabolas below.

1. $(x+1)^2 = -3(y-6)$
2. $(x-3)^2 = y-7$
3. $(y+2)^2 = 8(x+1)$
4. $y^2 = -10(x-3)$
5. $(x+6)^2 = 4(y+8)$
6. $(y-5)^2 = -\frac{1}{2}x$

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7. Graph the parabola from #1.
8. Graph the parabola from #2.
9. Graph the parabola from #4.
10. Graph the parabola from #5.

Find the equation of the parabola given the vertex and either the focus or directrix.

11. vertex: (2, −1), focus: (2, −4)
12. vertex: (−3, 6), directrix: x = 2
13. vertex: (6, 10), directrix: y = 9.5
14. **Challenge** focus: (−1, −2), directrix: x = 3
15. **Extension** Rewrite the equation of the parabola, $x^2 - 8x + 2y + 22 = 0$, in standard form by completing the square. Then, find the vertex.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.2.

Vocabulary

Term	Definition
Vertex form	The vertex form of a parabola is $(x-h)^2 = 4p(y-k)$ or $(y-k)^2 = 4p(x-h)$ where (h, k) is the vertex.

Image Attributions

1. [Figure 1]
   Credit: Holly Fischer
   Source: https://commons.wikimedia.org/wiki/File:Three_Main_Layers_of_the_Eye.png

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6.2.4: Parabolas and Analytic Geometry

Parabolas and Analytic Geometry

Brandyn is graphing parabolas as a part of his homework assignment. He is familiar with the standard form of a **parabola**: $y = ax^2$, and the longer form: $y - k = a(x - h)^2$.

On his third problem, he runs into a snag. He has simplified the equation significantly, and has been trying to get it to fit the standard form, but he keeps coming up with this: $x - 4 = 3(y - 3)^2$. He can't understand why the *y* term is the squared one, instead of the *x* term.

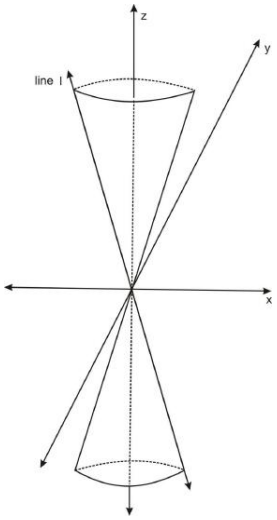
What is going on here?

Parabolas and Analytic Geometry

This is our second lesson on parabolas. In the initial lesson, we explored the parabola using the distance formula, and touched on the use of the **focus** and **directrix**. In this lesson, we first examine parabolas from the "analytic geometry" point of view, and then work a few examples with the focus and directrix of a parabola.

Finding the Equation of a Parabola Using Analytic Geometry

Consider a cone oriented in space as pictured below:



If the cone opens at an angle such that at any point its radius to height ratio is a, then the cone could be defined as the set of points such that the distance from the z−axis is a times the z−coordinate. Or, in other words, the set of points (x, y, z) satisfying:

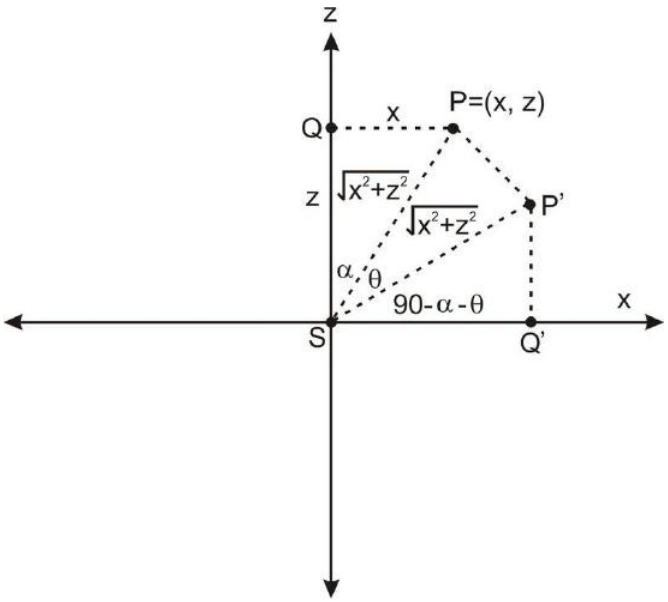
$$\sqrt{(x-0)^2 + (y-0)^2} = az$$

or

$$x^2 + y^2 = a^2z^2$$

This equation works for negative values of x, y, and z, giving the general equation for a two-sided cone.

To consider the intersection of this cone with a plane that is parallel to the line marked l marked in the diagram above, it is most convenient to rotate the entire cone about the y axis until the left side of the cone is vertical, then intersect it with a vertical plane perpendicular to the x−axis. Such a rotation leaves the y−variable unchanged. To see what it does to the x and z variables, let's see what happens to the point (x, z) on the xz−plane when it is rotated by an angle of θ.



In the above diagram, $P(x, z)$ is rotated by an angle of θ to the point P'. We have marked the side lengths $QP = x$ and $SQ = z$. By the Pythagorean theorem, $SP = \sqrt{x^2 + z^2}$. We also have $SP' = \sqrt{x^2 + z^2}$, since rotation leaves the distance from the origin unchanged. To find the x−coordinates of our rotated point P', we can use the fact that

$\cos(90 - \alpha - \theta) = \frac{SQ'}{\sqrt{x^2+z^2}}$. But by properties of cosine we have:

$$\cos(90 - \alpha - \theta) = \sin(\alpha + \theta),$$

and substituting with the sine addition formula gives us:

$$\frac{SQ'}{\sqrt{x^2+z^2}} = \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta),$$

which we can use our diagram to change to:

$$\frac{SQ'}{\sqrt{x^2+z^2}} = \frac{x}{\sqrt{x^2+z^2}}\cos(\theta) + \frac{z}{\sqrt{x^2+z^2}}\sin(\theta)$$

which simplifies to:

$$SQ' = x\cos(\theta) + z\sin(\theta)$$

To find the x−coordinates of our rotated point P', we can use the fact that

$\sin(90 - \alpha - \theta) = \frac{P'Q'}{\sqrt{x^2+z^2}}$. But by properties of sine we have:

$$\sin(90 - \alpha - \theta) = \cos(\alpha + \theta)$$

and substituting with the cosine addition formula gives us:

$$\frac{P'Q'}{\sqrt{x^2+z^2}} = \cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta),$$

which we can use our diagram to change to:

$$\frac{P'Q'}{\sqrt{x^2+z^2}} = \frac{z}{\sqrt{x^2+z^2}}\cos(\theta) - \frac{x}{\sqrt{x^2+z^2}}\sin(\theta)$$

which simplifies to:

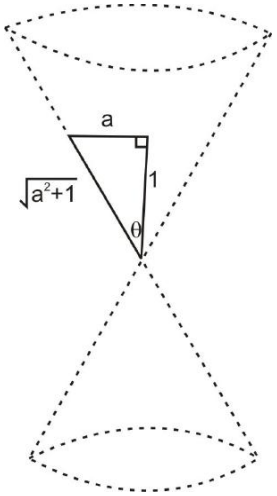
$$P'Q' = z\cos(\theta) - x\sin(\theta)$$

Looking back at the picture, this means that the coordinates of P' are $(x\cos(\theta) + z\sin(\theta), z\cos(\theta) - x\sin(\theta))$. In other words, in rotating from P to P', the x−coordinate changes to $x\cos(\theta) + z\sin(\theta)$ and the z− coordinate changes to $z\cos(\theta) - x\sin(\theta)$.

If this rotation happens to every point on the cone, we can substitute $x\cos(\theta) + z\sin(\theta)$ for x and $z\cos(\theta) - x\sin(\theta)$ for z into our equation of the cone, resulting in a new equation for the cone after rotating by θ.

$$(x\cos(\theta)+z\sin(\theta))^2+y^2=a^2(z\cos(\theta)-x\sin(\theta))^2$$
$$x^2\cos^2(\theta)+2xz\cos(\theta)\sin(\theta)+z^2\sin^2(\theta)+y^2=a^2\left(x^2\sin^2(\theta)-2xz\cos(\theta)\sin(\theta)+z^2\cos^2(\theta)\right)$$
$$x^2\cos^2(\theta)+2xz\cos(\theta)\sin(\theta)+z^2\sin^2(\theta)+y^2=a^2x^2\sin^2(\theta)-2a^2xz\cos(\theta)\sin(\theta)+a^2z^2\cos(\theta))$$
$$x^2\cos^2(\theta)+2xz\cos(\theta)\sin(\theta)+z^2\sin^2(\theta)+y^2=a^2x^2\sin^2(\theta)-2a^2xz\cos(\theta)\sin(\theta)+a^2z^2\cos(\theta))$$

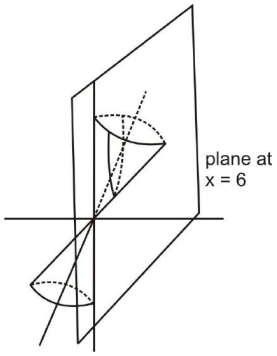
Now in the case of the tilted cone, we want to tilt the cone such that the left side becomes vertical. Since the factor a determines how tilted the cone is, we can see from the triangle below that $\sin(\theta)=\frac{a}{\sqrt{1+a^2}}$ and $\cos(\theta)=\frac{1}{\sqrt{1+a^2}}$.



So the equation becomes:

$$x^2\frac{1}{1+a^2}+2xz\frac{a}{1+a^2}+z^2\frac{a^2}{1+a^2}+y^2=a^2x^2\frac{a^2}{1+a^2}-2a^2xz\frac{a}{1+a^2}+a^2z^2\frac{1}{1+a^2}$$
$$x^2+2xza+z^2a^2+y^2\left(1+a^2\right)=a^4x^2-2a^3xz+a^2z^2$$
$$x^2+2xza+y^2\left(1+a^2\right)=a^4x^2-2a^3xz$$

Now that we have tilted our cone, to take a cross section that is parallel to the left side of the cone, we can simply cut it with a vertical plane. The equation of a vertical plane going through $(b,0,0)$ and perpendicular to the x−axis is $x=b$. Therefore, setting x equal to the constant in the equation above will give us the intersection of the tilted cone and a plane parallel to one side of the cone. b. Here is a picture of the rotation and the cross-section, which lies in an xz−plane.



Setting x equal to the constant b, we have:

$$b^2+2abz+y^2\left(1+a^2\right)=a^4b^2-2a^3bz$$
$$z\left(2ab+2a^3b\right)=-y^2\left(1+a^2\right)+a^4b^2-b^2$$
$$z=\left(\frac{-1-a^2}{2ab+2a^3b}\right)y^2+\left(a^4b^2-b^2\right)$$

Although this coefficient and constant term seem complicated, a and b can be chosen so that the coefficient of the y^2 term can be equal to any number (you will explore this fact in an exercise). The constant term can be ignored since any parabola can be shifted vertically by any amount.

So the general form of a parabola is:

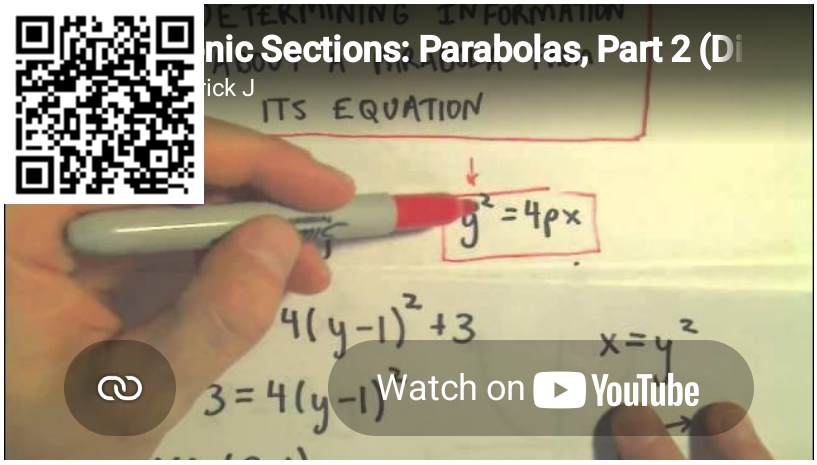
$$z = Ay^2$$

where A is any constant.

Or, using the more standard $x-$ and $y-$ coordinates the form of a parabola is

$$y = ax^2$$

As before, this equation can be adapted to produce the shifted and horizontally oriented forms.



Examples

Example 1

Earlier, you were asked a question about Brandyn, who is unsure why he has a y^2 term in his standard form equation instead of an x^2 term.

Solution

Brandyn keeps coming up with a y^2 term because this is a sideways parabola.

Example 2

In the proof above, we greatly simplified the formula near the end by substituting *A* for

$$\frac{-1-a^2}{2ab+2a^3b}$$

Solution

Explain why this was permissible by showing that for any A there exist constants a and b such that $A = \frac{-1-a^2}{2ab+2a^3b}$

Solving for b in terms of A and a, we have:

$$\begin{aligned} A\left(2ab+2a^3b\right) &= -1-a^2 \\ 2Aab\left(1+a^2\right) &= -\left(1+a^2\right) \\ 2Aab &= -1 \\ 2Aab &= -1 \\ b &= -\frac{1}{2Aa} \end{aligned}$$

So we can set $b = 2Aa$ and the relationship will hold.

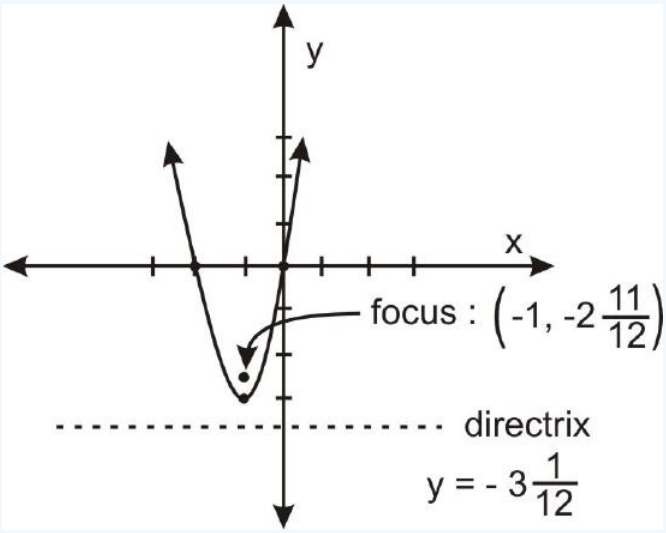
Example 3

Draw a sketch of the following parabola. Also identify its directrix and focus. $3x^2+6x-y=0$

Solution

Factor and complete the square to get: $3(x+1)=y+3$. The **vertex** is at: $(-1,-3)$.

The focus is $\left(-1,-2\frac{11}{12}\right)$ The directrix is the line $y=-3\frac{1}{12}$



Example 4

Find the equation for a parabola with directrix $y=-2$ and focus $(3,8)$.

Solution

The vertex is vertically midway between the focus and directrix: $\frac{-2+8}{2}=3$, the same horizontally as the focus: $x=3$ and therefore at: $(3,3)$.

Substituting those values into the formula gives:

$y-3=\frac{1}{20}(x-3)^2$

Example 5

Find the equation for a parabola with directrix $y=3$ and focus $(2,-1)$.

Solution

Using the vertex form of a parabola $(y-k)^2=4a(x-h)$:

Recall that the vertex y-value k is the midpoint of the directrix and the focus on the line perpendicular to the directrix and crossing the focus. Therefore the y-value of the vertex is 1

Recall that the vertex x-value h is the same as the focus, therefore the vertex x-value is 2

Finally, recall that a the distance from the vertex to the focus or from the vertex to the directrix (which are the same): $\therefore a=2$

Substituting gives: $(y-1)^2=8(x-2)\rightarrow y^2+1=8x-16\rightarrow y^2=8x-17$

Example 6

Describe the shape of a parabola as it relates to a cone or double cone.

Solution

The shape of a parabola as it relates to a cone or double cone, is that a parabola represents the revealed shape when a hollow cone is sliced through at an angle equal to the side of the cone. Particularly clear with a double cone is the fact that slicing through at a steeper angle will result in two curves (a hyperbola) and a shallower angle will result in an ellipse.

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Example 7

Sketch the following parabola and identify the directrix and focus: $4x^2 - 3x + y = 7$.

Solution

$y = ax^2 + bx + c$..... Recall the standard form of a parabola

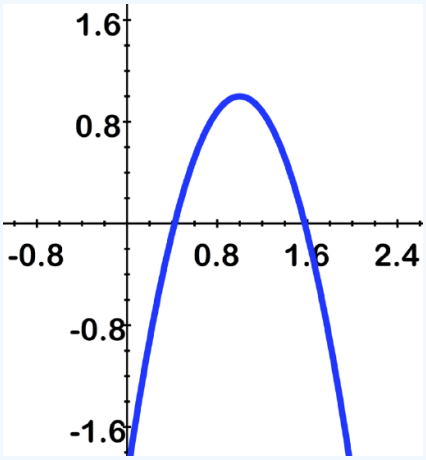
$a = -3|b = 6|c = -2$..... Extract a, b, c

$x = \frac{-6}{2(-3)} \rightarrow x = 1$..... The x-coordinate of the vertex $= \frac{-b}{2a}$

$y = (-3)(1) + 6(1) - 2$..... Substitute the calculated x-value to solve for y

$y = 1$..... The vertex = (1,1)

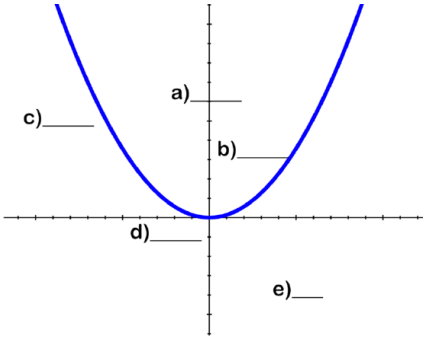
$x = 1 \pm \frac{1}{\sqrt{3}}$..... Identify the x-intercepts using the Quadratic Formula



Review

Use the image to identify the parts of the parabola:

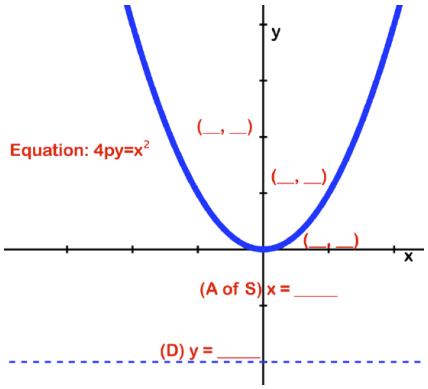
1. The Focus
2. The Vertex
3. The Focal Radius
4. The Directrix
5. The Parabola



Use the image *and given equation* of the parabola to identify the following:

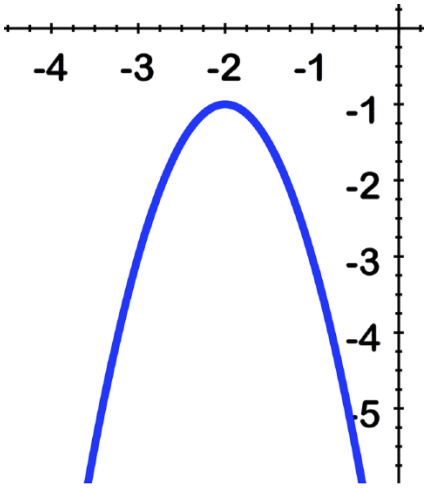
6. The coordinates of the focus
7. The equation of the directrix
8. The length of the focal radius

9. The equation of the **axis of symmetry**
10. The coordinates of the vertex

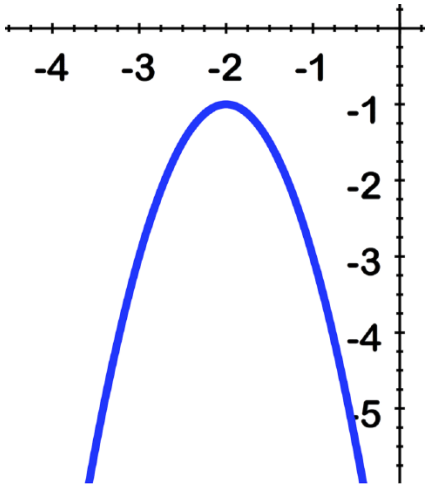


11. Find the equation for a parabola with directrix: x=2 and focus: (0, −2)
12. Find the equation for a parabola with vertex: (5, −2) and directrix: y=−5
13. Find the equation for a parabola with focus:(3, 5) and vertex:(3, 1)

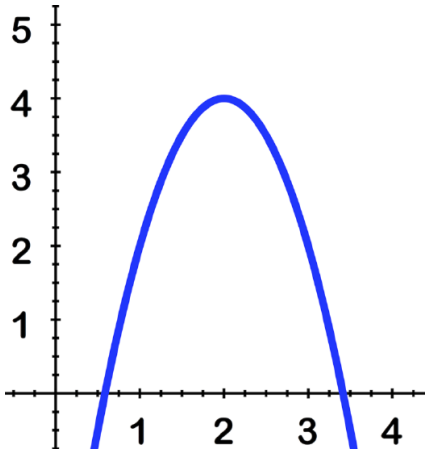
Use the image to identify the vertex, axis of symmetry and equation of the parabola:



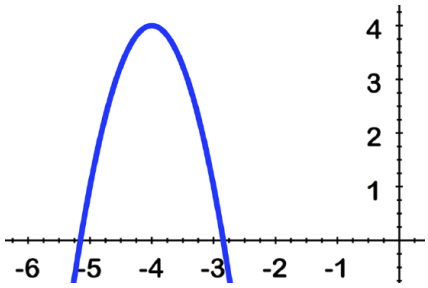
14.



15.



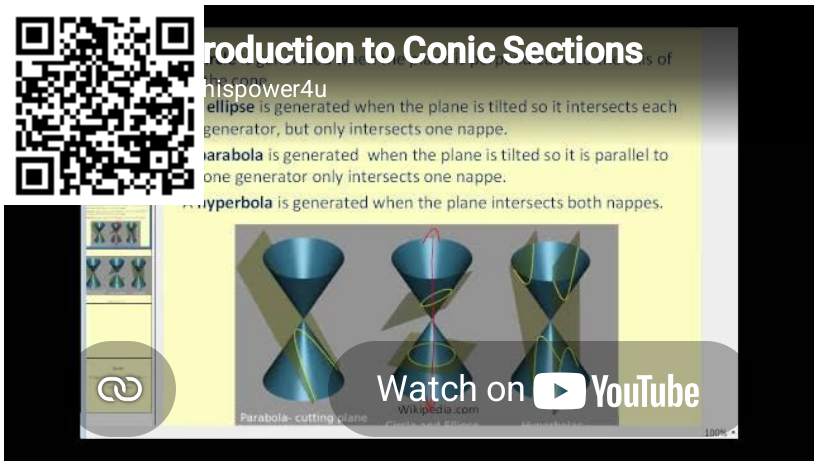
16.



Review (Answers)

To see the Review answers, open this PDF file and look for section 6.4.

Resources



Vocabulary

Term	Definition
Analytic Geometry	Analytic geometry is a branch of mathematics concerned with modeling and exploring shapes by using algebraic formulas and a coordinate system.
axis of symmetry	The axis of symmetry of a parabola is a vertical line that passes through the vertex of the parabola. The parabola is symmetrical about this line.
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
directrix	The directrix of a parabola is the line that the parabola seems to curve away from. All points on a parabola are equidistant from the focus of the parabola and the directrix of the parabola.
focus	The focus of a parabola is the point that "anchors" a parabola. Any point on the parabola is exactly the same distance from the focus as from the directrix.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.

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6.2.5: Applications of Parabolas

Applications of Parabolas

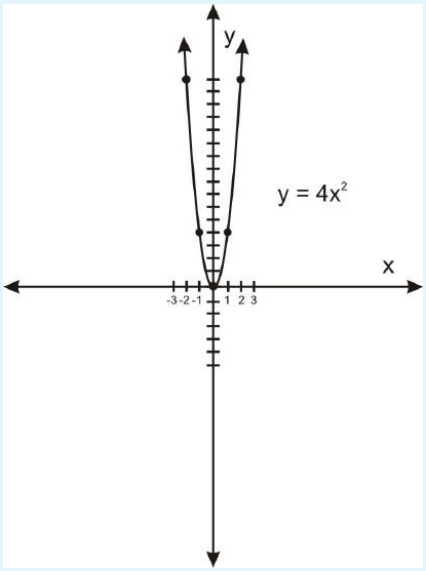
Kelly is nearly done with the chapter on parabolas in her math class. She has become pretty familiar with the general shape of a **parabola**, and has started noticing the use of parabolas all around her. One thing that she hasn't figured out yet is why satellite-TV dishes are 3D parabolas. She is sure it isn't a coincidence, but doesn't know what it is about a parabolic shape that is important.

Do you know?

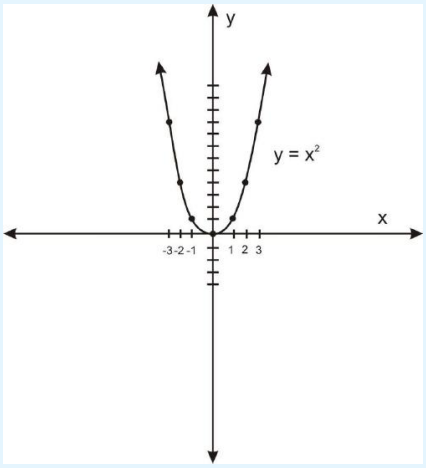
Applications of Parabolas

There is a very interesting property of parabolas. This is the fact that **all parabolas have the same shape**. Or, in the language of geometry, any two parabolas are similar to one another. This means that **any parabola can be scaled in or out to produce another parabola of exactly the same shape.** This may come across as surprising, because parabolas where x^2 has a large coefficient certainly look much "steeper" than parabolas with a small coefficient when examined over the same domain, as shown in the graphs below.

a. Tall and skinny:



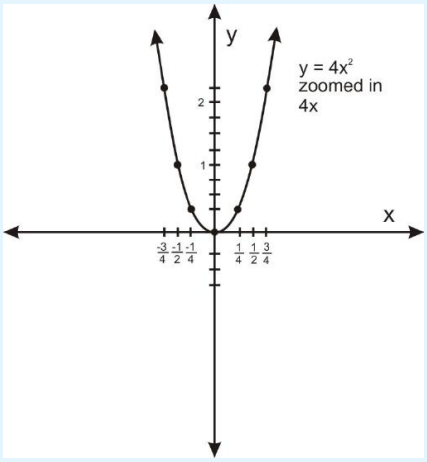
b. Short and wide:



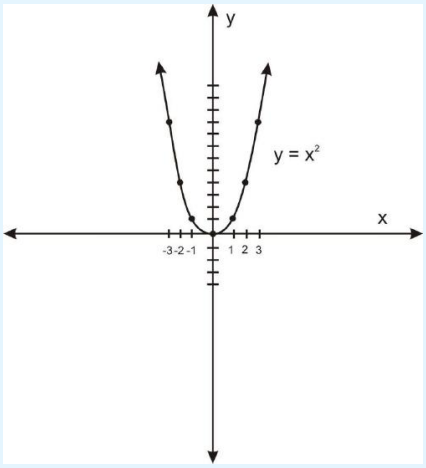
But when one of the parabolas is scaled appropriately, these parabolas are identical:

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a. Tall and skinny - zoomed in by 4x:



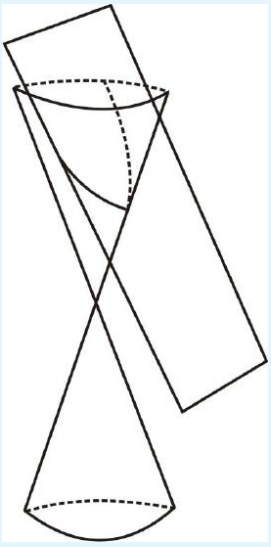
b. Short and wide:



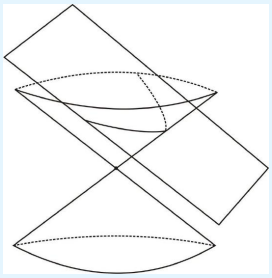
This fact about parabolas can be seen from an algebra standpoint given the fact that all parabolas are generated from a line and a point not on that line. This configuration of generating objects, a line and a point, is always the same shape. Any other line and point looks exactly the same—simply zoom in or out until the line and point are the same distance from one another. So the shapes that any two such configurations generate must also be the same shape.

From an analytic geometry standpoint, there were two factors that might affect the shape of the parabola. The first is the distance between the cutting plane and the apex of the cone. But cones have the same proportions at any scale, so no matter what this distance, the picture can be reduced or enlarged, affecting this distance but not the shape of the cone or plane. So this parameter does not actually change the shape of the conic section that results. The other factor is the shape of the actual cone. This is its steepness, defined by the angle at the apex, or equivalently by the ratio between the radius and the height at any point. This is a bit trickier. It's not at all obvious that short, squat cones and tall, skinny cones would produce parabolas of the same shape.

a. Tall and skinny:



b. Short and wide:



According to what we found, any parabola produced by slicing any cone resulted in an equation of this form:

$$y = ax^2$$

We want to show that if we generate two such parabolas, that they actually have the same shape. So suppose we use two cone constructions and come up with these parabolas:

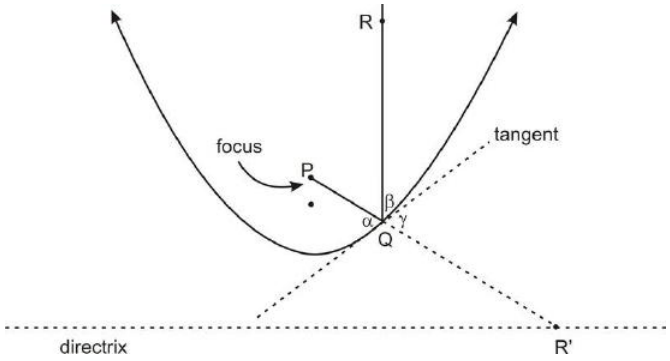
$y = a_1x^2$ and $y = a_2x^2$. We want to show there is some scale factor, call it f, that shrinks or enlarges $y = a_1x^2$ into $y = a_2x^2$. To keep a shape the same, the scale factor needs to affect both the $x-$ and $y-$ variables. So we need to find an f such that $(fy) = a_1(fx)^2$ is equivalent to $y = a_2x^2$. The first equation can be written $y = (a_1f)\,x^2$, which is equivalent to the second equation when $a_1f = a_2$, or when $f = \frac{a_2}{a_1}$. Such an f always exists for non-negative numbers a_1 and a_2. So the parabolas are indeed the same shape. If a is less than zero, then the parabola can be reflected vertically to produce a parabola of the same shape and with positive coefficient a.

Parabola Applications

Burning Mirrors

Diocles (~240−180 BCE) was a mathematician from Ancient Greece about whom we know very little. However, we know enough from a few scant documents that he thought about an important application of parabolas. It comes from what is sometimes called the "optical property" of parabolas. Remember the optical property of **ellipses**: lines from one focus "bounce off" the side of the **ellipse** to hit the other focus.

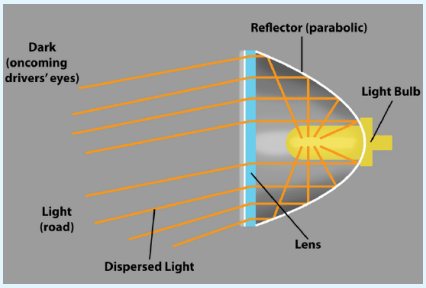
For parabolas, since parabolas have only one focus, the directrix plays a role. For the parabola, the optical property is that lines perpendicular to the directrix "bounce off" the parabola and converge at the focus. Or, alternatively, lines from the focus "bounce off" off the parabola and continue perpendicular to the directrix. As with the ellipse, "bouncing off" means that the two lines meet the parabola at equal angles to the tangent.



In the above diagram, the optical property states that □ α≅□ β. The proof is similar to the proof of the optical property of ellipses. In the above diagram P is the focus and Q is a point on the parabola. Let R′ be the point on the directrix that is obtained by extending PQ. Then PR′, the straight line, is clearly the shortest distance between P and R′ that passes through the tangent line. Let R be on the line lying directly above Q such that QR=QR′. The R can be thought of as R′, reflected across the tangent line. Then □ α≅□ γ (vertical angles) and □ γ≅□ β (reflected angles), and so □ α≅□ β (transitive property).

The optical property has some interesting applications. Diocles described one potential application in his document "On Burning Mirrors". He envisioned a parabolic-shaped mirror (basically a parabola rotated about its line of symmetry) which would collect light from the sun and focus it on the focal point, creating enough of a concentration of light to start a fire at that point. Some claim that Archimedes attempted to make such a contraption with copper plates to fight the Romans in Syracuse.

Headlights: The optical property is also responsible for parabola-shaped **unidirectional** lights, such as car headlights. If a bulb is placed at the focus of a parabolic mirror, the light rays reflect off the mirror parallel to each other, making a focused beam of light.



Cassegrain Telescopes: Satellite telescopes take advantage of the optical property of parabolas to collect as much light from a distant star as possible. The dish of the satellite below is parabolic in shape and reflects light to the point in the middle.





Examples

Example 1

Earlier, you were asked a question about why satellite-TV dishes are 3D parabolas.

Solution

The satellite dish is a 3D parabola so that all of the signal it collects over a wide area will be concentrated at the focus of the parabola, significantly increasing the reception.

Example 2

Explain why not all ellipses are similar the way parabolas are. While enlarging or shrinking doesn't work to make two ellipses identical, how can you change the view of two ellipses that have different shapes so that they look the same?

Solution

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The eccentricity of ellipses defines the shape, so when the eccentricity is different for two ellipses, the ellipses are not similar to one another. Viewing one of the ellipses at an angle, however, changes the perceived eccentricity of that ellipse, and the angle can be chosen to match the perceived eccentricity to the eccentricity of the other ellipse, producing an image that is similar to the other ellipse.

Example 3

If nothing was used to deflect light before it entered a "**burning mirror**," where would the sun have to be in relationship to you and the place you want to start a fire? Why is this a constraint? Design a way to circumvent this constraint.

Solution

The fire-locale must lie on the segment between you and the sun. This is a problem because to start a ground fire, you would have to wait until evening when the sun is low in the sky so you could aim your lens at the ground, unfortunately the sun is not as bright in the evening so you would lose a significant amount of power. A lens or mirror that changes the angle of the suns rays could help you work around this constraint.

Example 4

In the above diagram of a car headlight, the lens directs the beams of light downwards, to keep them out of the eyes of oncoming drivers, if that was the only purpose of the lens, alternatively the lens could be omitted and the headlight could just be angled down slightly. But there is another purpose to the lens. What is it?

Solution

The lens also expands the array of light which is why it is called “dispersed light.” Without the lens, the headlight would only illuminate a strip the width of the headlight itself, which would not be very useful for driving.

Example 5

A satellite dish antenna is to be constructed in the shape of a paraboloid. The paraboloid is formed by rotating the parabola with focus at the point (25, 0) and directrix $x = -25$ about the x-axis, where x and y are in inches. The diameter of the antenna is to be 80 inches.

a. Find the equation of the parabola and the domain of x.
b. Sketch the graph of the parabola, showing the location of the focus.
c. A receiver is to be placed at the focus. The designer has warned that a user or installer should take care, the receiver would hit the ground and could be damaged if the antenna were placed "face-down". Determine algebraically whether this observation is correct.

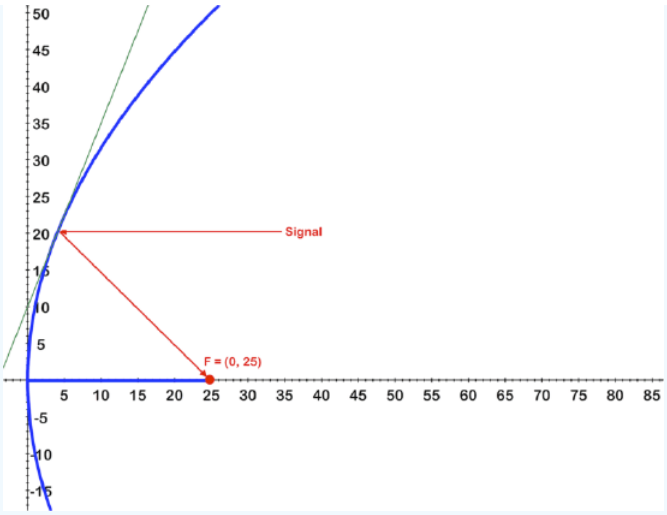
Solution

a. $x = \frac{1}{100}y^2$

The domain is the positive real number set.

b. Sketch the graph of the parabola, showing the location of the focus.

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c. Yes, as discovered in part a and shown in the graph of part b, the focus is at x = 25, while the dish only extends to x = 16.

Review

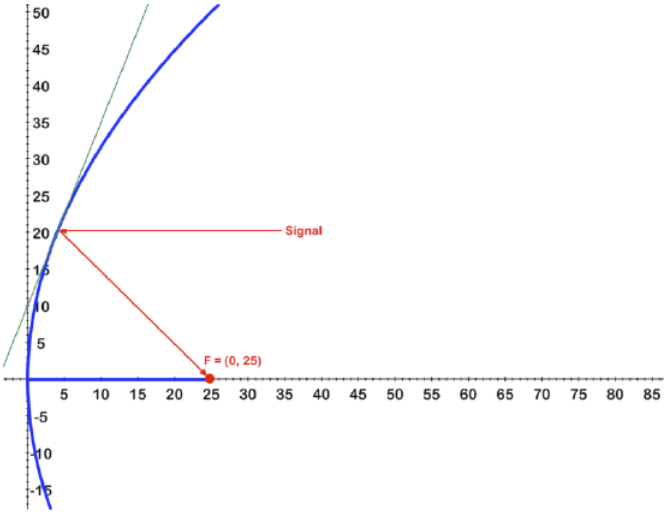
1. football with an initial vertical velocity of 96 feet per second. After how many seconds will the ball reach its maximum height above sea level? What is the maximum height?
2. A shot-gun is discharged vertically upward at a height of 3 feet above the ground. If the bullet has an initial muzzle velocity of 200 feet per second, what maximum height will it reach before it starts to fall to the ground? a) 628 feet b) 1,878 feet c) 20.87 feet d) 199.33 feet
3. An over-zealous golfer hits a flop shot with a sand wedge to get out of the corner of a sand trip with an initial vertical velocity of 45 feet per second. What is the maximum height that the golf ball will reach? a) 45 feet b) 13.19 feet c) 36.64 feet d) 95.26 feet
4. You are standing on the top of a 1680 ft tall hill, and throw a small ball upwards. At every second, you measure the distance of the ball from the ground. Exactly t seconds after you throw the object, its height, (measured in feet) is $h(t) = -16t^2 + 256t + 1680$ a) Find $h(3)$ the ball position 3 seconds after you throw it. b) How much does the object travel during the two seconds between 5 seconds and 7 seconds? c) How long does it take for the object to reach a height of 2640 ft? d) How long does it take for the object to hit the ground?
5. A student participating in a game of kick ball kicked the ball with an initial vertical velocity of 32 feet per second. Its height above the earth in feet is given by $s(t) = -16t^2 + 32t$ where t is time in seconds. What is the maximum height reached by the ball? HINT - Graph the parabola (you only need to worry about the section where $0 \le t \le 2$ or rewrite the equation to find the vertex.
6. An arch over the entrance to an enchanted trail has a parabolic shape, the arch has a height of 25 feet and it is 30 feet between the support pillars. Find an equation that models the arch, using the x-axis to represent the ground of the park. State the focus and directrix.
7. A satellite dish has a parabolic shape with a diameter of 80 meters. The collected tv signals are focused on a single point, called the "focal" point, which is the focus of the paraboloid (the cross-section of the parabola). If the focal length is 45 meters, find the depth of the dish, rounded to one decimal place.
8. When new highways go in, they are often designed with parabolic surfaces which allow water to drain off. A new highway is being laid, it is 32 feet wide and is .4 feet higher in the center of the highway then on the sides. a) Find an equation if the parabola that models the highway surface (assume that the origin is at the center of the highway) b) How far from the center of the highway is the surface of the road .1 feet lower than in the middle?
9. The internal distance of the sketch of an arch is 8cm and the height of the arch is 9cm. Assuming the scale 1cm = 2m, work out a formula which calculates the actual height of the inside edge of this structure (y), in metres at any horizontal distance x measured from the origin point which is the floor at the center of the arch.

A new bridge has been built for foot traffic cross a river. The two towers on either end of the bridge are 50 feet high and 300 feet away from each other. The supporting cables (2) are connected at the top of the towers and hang in a curve that forms the shape of a parabola. There are vertical cables that connect the walkway to the supporting cables. These cables connect every 15 feet from the walkway up to the supporting cables. At the center of the bridge, the parabola is 5 feet above the walkway.

Specialty Cable Company back east sells cable for $52.75 per 10 feet with a shipping charge of $300.00 for the entire order. Cables R US, on the West Coast sells cable for $432.90 per 100 feet with a shipping charge of $350.00 for the entire order. Cable had to be purchased in either 10-foot lengths from SCC or in 100-foot lengths from CRUS. Once purchased, the cables can be cut or welded together.

10. Write an equation for the parabola that represents each of the support cables.
11. Determine the number of vertical cables needed.
12. Determine the length of each of the vertical cables.
13. How much does it cost to purchase the needed materials from each company?

Look at the image. It shows the parabolic cross section of a satellite dish antenna. The equation for the parabola is $x = 0.01y^2$



14. What is the focus of the parabola as shown in the image?
15. A radio signal comes in at y = 20 as marked in red. Identify where the ray strikes the parabola, give coordinates.
16. Identify the angle measure between the incoming ray and a line between the strike point and the focus.
17. An incoming ray and its reflected ray make angles of equal measure with a line tangent to the curved surface. Use the angle you just measured between the signal and the reflected line to calculate the measures of the angles between the incoming signal and the tangent to the point of impact and from the tangent to the reflected ray. Are they the same? Measure with a protractor to confirm your calculations.
18. Find the equation for the tangent line. Use angle C to find its slope.
19. Randomly choose another incoming ray. Calculate the angles as before. Is the incoming ray tangent to the parabola at the point where the incoming ray strikes the graph? How do you know?
20. What can you say about the direction a reflected ray takes whenever the incoming ray is parallel to the axis of the parabola? Can you see why the name "focus" is used and why television satellite antennas and other listening devices are made in the shape of a paraboloid?

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.5.

https://k12.libretexts.org/@go/page/14756

Resources



Vocabulary

Term	Definition
burning mirror	The burning mirror is a highly reflective parabolic dish used to focus sunlight onto a single point, the focus of the parabola.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.
parabolic dish	A parabolic dish is a surface generated by a parabola being revolved about its axis of symmetry.
Unidirectional	Unidirectional means focusing in one (uni-) specific direction.

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SECTION OVERVIEW

6.3: Hyperbolas

Topic hierarchy

6.3.1: Graphs of Hyperbolas Centered at the Origin

6.3.2: Equations of Hyperbolas Centered at the Origins

6.3.3: Hyperbolas with any Center

6.3.4: Hyperbola Equations and the Focal Property

6.3.5: Hyperbolas and Asymptotes

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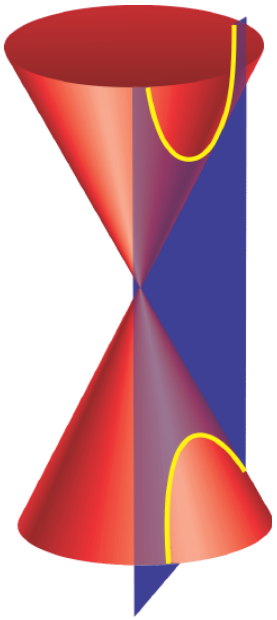
6.3.1: Graphs of Hyperbolas Centered at the Origin

Graphing Hyperbolas Centered at the Origin

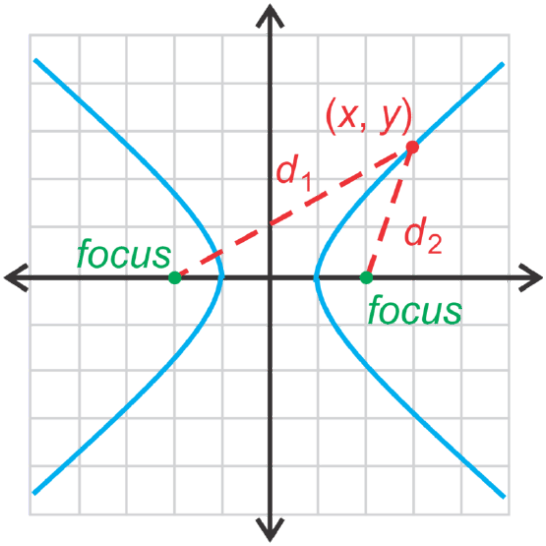
Your homework assignment is to graph the hyperbola $9y^2 - 4x^2 = 36$. What are the asymptotes and foci of your graph?

Graphing Hyperbolas

We know that the resulting graph of a rational function is a hyperbola with two **branches**. A hyperbola is also a conic section. To create a hyperbola, you would slice a plane through two inverted cones, such that the plane is perpendicular to the bases of the cones.



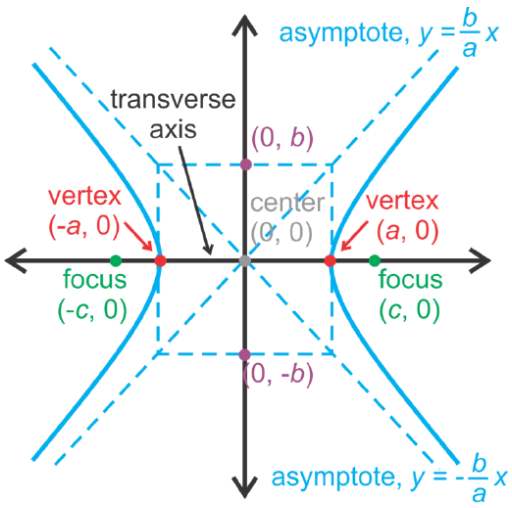
By the conic definition, a **hyperbola** is the set of all points such that the of the differences of the distances from the **foci** is constant.



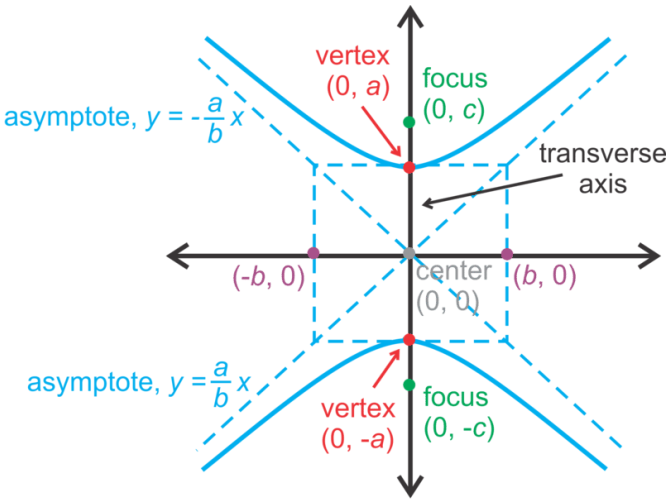
Using the picture, any point, (x, y) on a hyperbola has the property, $d_1 - d_2 = P$, where P is a constant.

Comparing this to the ellipse, where $d_1 + d_2 = P$ and the equation was $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

For a hyperbola, then, the equation will be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$. Notice in the vertical orientation of a hyperbola, the y^2 term is first. Just like with an ellipse, there are two **vertices**, on the hyperbola. Here, they are the two points that are closest to each other on the graph. The line through the vertices and foci is called the **transverse axis**. Its midpoint is the **center** of the hyperbola. In this concept, the center will be the origin. There will always be two branches for any hyperbola and two asymptotes.



horizontal transverse axis



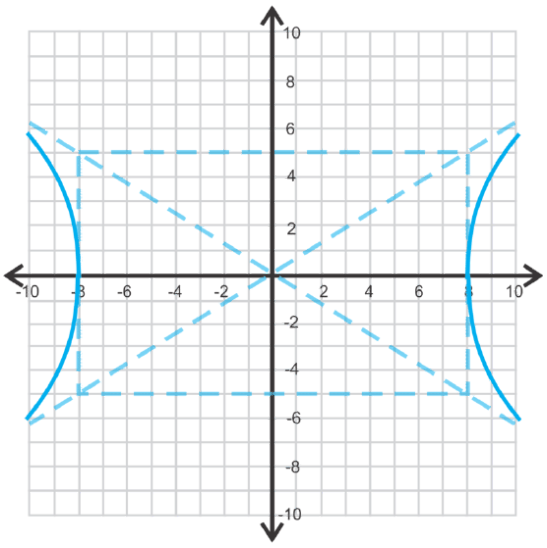
vertical transverse axis

Let's graph $\frac{x^2}{64} - \frac{y^2}{25} = 1$ and then find the vertices, foci, and asymptotes.

First, this hyperbola has a horizontal transverse axis because the x^2 term is first. Also, with hyperbolas, the a and b term stay in place, but the x and y terms switch. a ***is not*** always greater than b.

Therefore, $a = \sqrt{64} = 8$ and $b = \sqrt{25} = 5$. To graph this hyperbola, go out 8 units to the left and right of the center and 5 units up and down to make a rectangle. The diagonals of this rectangle are the asymptotes.

Draw the hyperbola branches with the vertices on the transverse axis and the rectangle. Sketch the branches to get close to the asymptotes, but not touch them.



The vertices are $(\pm 8, 0)$ and the asymptotes are $y = \pm\frac{5}{8}x$ (see pictures above. To find the foci, we use the Pythagorean Theorem, $c^2 = a^2 + b^2$ because the foci are *further away* from the center than the vertices.

$$c^2 = 64 + 25 = 89$$
$$c = \sqrt{89}$$

The foci are $(\pm\sqrt{89}, 0)$.

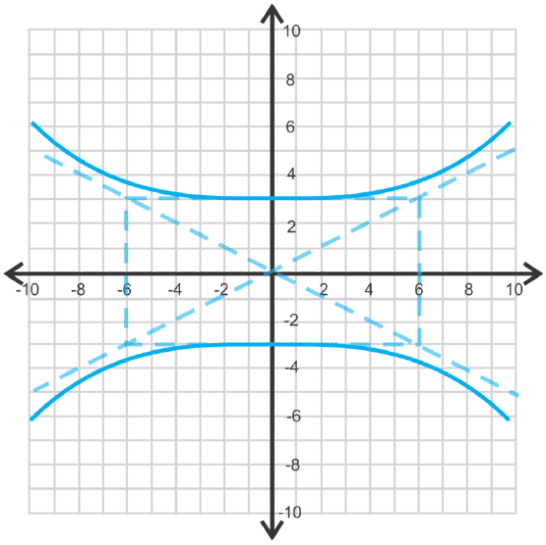
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Now, let's graph $36y^2 - 9x^2 = 324$ and identify the foci.

This equation is not in standard form. To rewrite it in standard form, the right side of the equation must be 1. Divide everything by 324.

$$\frac{36y^2}{324} - \frac{9x^2}{324} = \frac{324}{324}$$
$$\frac{y^2}{9} - \frac{x^2}{36} = 1$$

Now, we can see that this is a vertical hyperbola, where $a = 3$ and $b = 6$. Draw the rectangle, asymptotes, and plot the vertices on the y-axis.



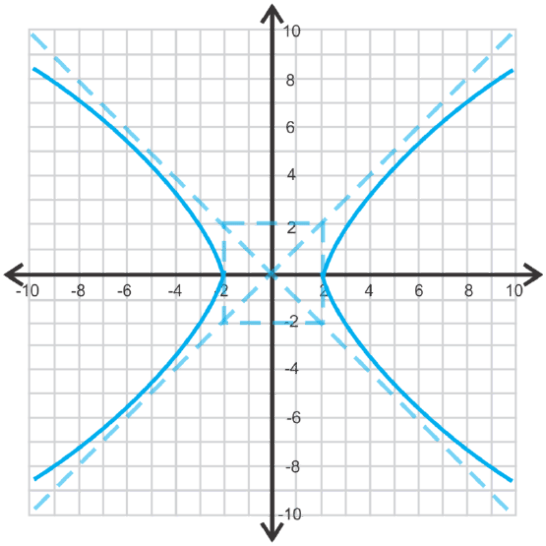
To find the foci, use $c^2 = a^2 + b^2$.

$$c^2 = 36 + 9 = 45$$
$$c = \sqrt{45} = 3\sqrt{5}$$

The foci are $(0, 3\sqrt{5})$ and $(0, -3\sqrt{5})$

Finally, let's graph $\frac{x^2}{4} - \frac{y^2}{4} = 1$ and identify the asymptotes.

This will be a horizontal hyperbola, because the x-term is first. a and b will both be 2 because $\sqrt{4} = 2$. Draw the square and diagonals to form the asymptotes.



The asymptotes are $y = \pm\frac{2}{2}x$ or $y = x$ and $y = -x$.

Important Note: The asymptotes and square are *not* a part of the function. They are included in graphing a hyperbola because it makes it easier to do so.

Also, when graphing hyperbolas, we are *sketching* each branch. We did not make a table of values to find certain points and then connect. You can do this, but using the square or rectangle with the asymptotes produces a pretty accurate graph and is much simpler.

Examples

Example 1

Earlier, you were asked to find the asymptotes and foci of your graph.

Solution

First, we need to get the equation in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so divide by 36.

$$\begin{aligned} 9y^2 - 4x^2 &= 36 \\ \frac{9y^2}{36} - \frac{4x^2}{36} &= \frac{36}{36} \\ \frac{y^2}{4} - \frac{x^2}{9} &= 1 \end{aligned}$$

Now we can see that $a^2 = 4$ and $b^2 = 9$, so $a = 2$ and $b = 3$. Also, because the y-term comes first, the hyperbola is vertically oriented. Therefore, the asymptotes are $y = -\frac{a}{b}x$ and $y = \frac{a}{b}x$.

Substituting for a and b, we get $y = -\frac{2}{3}x$ and $y = \frac{2}{3}x$.

Finally, to find the foci, use $c^2 = a^2 + b^2$.

$$\begin{aligned} c^2 &= 4 + 9 = 13 \\ c &= \sqrt{13} \end{aligned}$$

The foci are $(0, \sqrt{13})$ and $(0, -\sqrt{13})$.

Example 2

Find the vertices, foci, and asymptotes of $y^2 - \frac{x^2}{25} = 1$.

Solution

First, let's rewrite the equation like this: $\frac{y^2}{1} - \frac{x^2}{25} = 1$. We know that the transverse axis is vertical because the y-term is first, making $a = 1$ and $b = 5$. Therefore, the vertices are $(0, -1)$ and $(0, 1)$. The asymptotes are $y = \frac{1}{5}x$ and $y = -\frac{1}{5}x$. Lastly, let's find the foci using $c^2 = a^2 + b^2$.

$$\begin{aligned} c^2 &= 1 + 25 = 26 \\ c &= \sqrt{26} \end{aligned}$$

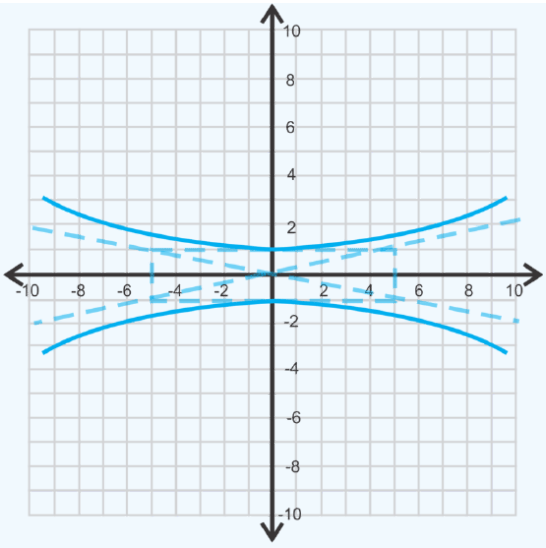
The foci are $(0, -\sqrt{26})$ and $(0, \sqrt{26})$

Example 3

Graph Example 2.

Solution

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Example 4

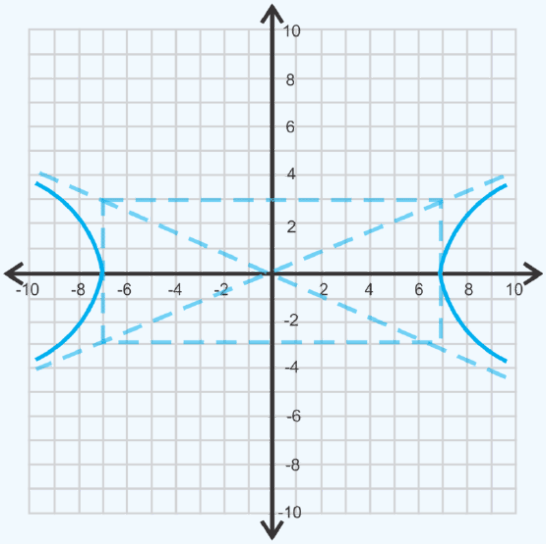
Graph $9x^2 - 49y^2 = 411$.

Solution

Rewrite the equation so that the right side is equal to 1. Divide everything by 441.

$$\frac{9x^2}{441} - \frac{49y^2}{441} = \frac{441}{441}$$
$$\frac{x^2}{49} - \frac{y^2}{9} = 1$$

$a = 9$ and $b = 6$ with a horizontal transverse axis.



Review

Find the vertices, asymptotes, and foci of each hyperbola below.

1. $\frac{x^2}{9} - \frac{y^2}{16} = 1$
2. $4y^2 - 25x^2 = 100$
3. $\frac{x^2}{81} - \frac{y^2}{64} = 1$

4. $x^2 - y^2 = 16$
5. $\frac{y^2}{49} - \frac{x^2}{25} = 1$
6. $121y^2 - 9x^2 = 1089$
7. $y^2 - x^2 = 1$
8. $\frac{x^2}{64} - \frac{y^2}{4} = 1$
9. $\frac{y^2}{4} - \frac{x^2}{64} = 1$
10. Graph #1.
11. Graph #2.
12. Graph #8.
13. Graph #9.
14. **Writing** Compare the hyperbolas from #8 and #9. How are they the same? How are they different? What do you know about the asymptotes and foci?
15. **Critical Thinking** Compare the equations $\frac{x^2}{25} - \frac{y^2}{9} = 1$ and $\frac{x^2}{25} + \frac{y^2}{9} = 1$. Graph them on the same axes and find their foci.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.7.

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6.3.2: Equations of Hyperbolas Centered at the Origins

Writing the Equation of a Hyperbola, Centered at the Origin

You are asked to solve a riddle. You are given the following pieces of information to help you.

1. I am a hyperbola centered at the origin.
2. My vertex is (0, −2).
3. One of my foci is (0, −3).

What is my equation?

The Equation of a Hyperbola

In this concept, we are going to work backwards and find the equation of hyperbolas, given certain pieces of information. For this entire concept, the hyperbola will be centered at the origin.

Let's find the equation of the hyperbola, centered at the origin, with a vertex of (−4, 0) and focus of (−6, 0).

Because the vertex and focus are on the x-axis , we know that the transverse axis is horizontal.

Therefore, the equation will be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. From the vertex, we know that $a = 4$ and $c = 6$. Let's solve for b^2 using $c^2 = a^2 + b^2$.

$$6^2 = 4^2 + b^2$$
$$36 = 16 + b^2 \rightarrow b^2 = 20$$

The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{20} = 1$.

Now, let's find the equation of the hyperbola, centered at the origin, with an asymptote of $y = \frac{2}{3}x$ and vertex of $(0, 12)$.

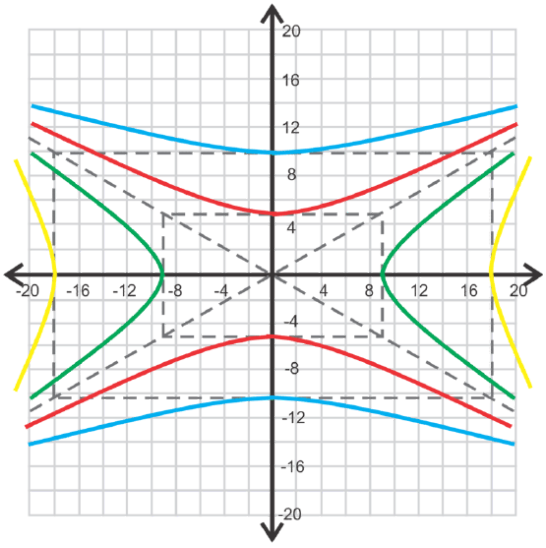
We know that $a = 12$, making the transverse axis is vertical and the general equation of the asymptote $y = \frac{a}{b}x$. Therefore, $\frac{2}{3} = \frac{12}{b}$, making $b = 18$. Therefore, the equation of the hyperbola is $\frac{y^2}{144} - \frac{x^2}{324} = 1$.

In this problem, we showed that the slope of the asymptote can be reduced to something that is not always $\frac{a}{b}$, but $c\left(\frac{m}{n}\right) = \frac{a}{b}$, where c is some constant that we can reduce the fraction by.

Finally, let's find the equations of two hyperbolas with an asymptote of $y = -\frac{5}{9}x$.

This asymptote can be for either a vertical or horizontal hyperbola. $-\frac{5}{9}$ can also be a reduced fraction of $\frac{a}{b}$, like in the previous problem. For example, the asymptote $y = -\frac{10}{18}x$ reduces to $y = -\frac{5}{9}x$.

If the hyperbola is horizontal, then the equation of the asymptote is $y = -\frac{b}{a}x$ and that would make $a = 9$ and $b = 5$ and the equation would be $\frac{x^2}{81} - \frac{y^2}{25} = 1$. If the equation is vertical, then the asymptote is $y = -\frac{a}{b}x$ and $a = 5$ and $b = 9$. The equation would be $\frac{y^2}{25} - \frac{x^2}{81} = 1$. If the slope is reduced from a larger fraction, we could also have $\frac{x^2}{324} - \frac{y^2}{100} = 1$ or $\frac{y^2}{100} - \frac{x^2}{324} = 1$ as a possible answer.



There are infinitely many hyperbolic equations with this asymptote.

Examples

Example 1

Earlier, you were asked to find the equation of the parabola given the riddle.

Solution

Because the vertex and focus are on the y-axis , we know that the transverse axis is vertical.

Therefore, the equation will be $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

From the vertex and the focus, we know that $a = 2$ and $c = 3$. Let's solve for b^2 using $c^2 = a^2 + b^2$.

$$3^2 = 2^2 + b^2$$
$$9 = 4 + b^2 \rightarrow b^2 = 5$$

Therefore, the equation of the hyperbola will be $\frac{y^2}{4} - \frac{x^2}{5} = 1$.

Find the equation of the following hyperbolas, centered at the origin, with the given information.

Example 2

vertex: (0, 2)

focus: (0, 5)

Solution

The vertex is on the y-axis, so this is a vertical hyperbola with $a = 2$. $c = 5$, so we need to find b^2.

$$c^2 = a^2 + b^2$$
$$25 = 4 + b^2 \rightarrow b^2 = 21$$

The equation of the hyperbola is $\frac{y^2}{4} - \frac{x^2}{21} = 1$.

Example 3

asymptote: $y = x$

vertex: $(4, 0)$

Solution

Rewriting the slope of $y = x$, we get $y = \frac{1}{1}x$. So, we know that a and b are in a ratio of 1:1.

Because the vertex is $(4, 0)$, we know that $a = 4$ and that the hyperbola is horizontal. Because a and b are in a ratio of 1:1, b has to equal 4 as well. The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{16} = 1$.

Example 4

Find the equations of two hyperbolas, centered at the origin, with different a and b values and an asymptote of $y = \frac{3}{4}x$.

Solution

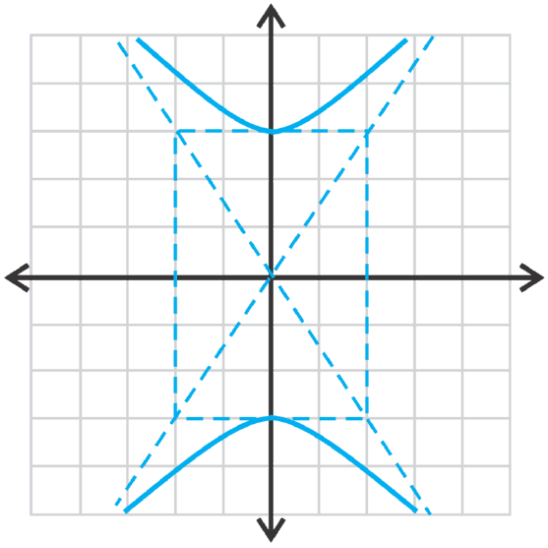
One possibility is that $b = 3$ and $a = 4$ making the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$. A second possibility could be that a and b are a multiply of the ratio 4:3. So, $a = 8$ and $b = 6$, making the equation $\frac{x^2}{64} - \frac{y^2}{36} = 1$.

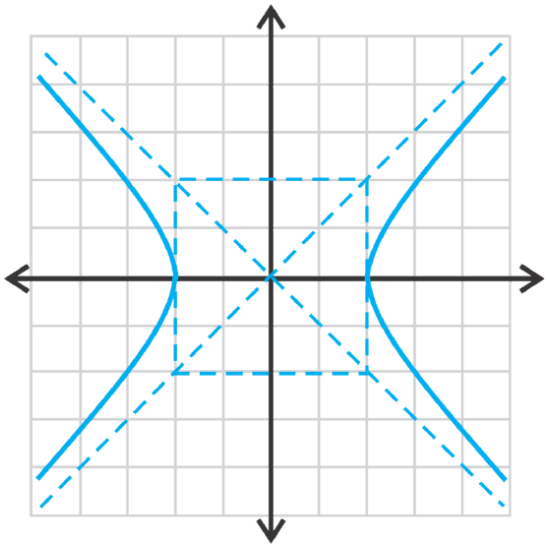
Review

Find the equation of the hyperbola, centered at the origin, given the information below.

1. vertex: $(-2, 0)$, focus: $(-5, 0)$
2. vertex: $(4, 0)$, focus: $(7, 0)$
3. $b = 8$, focus: $(-15, 0)$
4. vertex: $(-6, 0)$, asymptote: $y = \frac{4}{3}x$
5. $b = 6$, focus: $(0, 11)$
6. vertex: $(0, 5)$, asymptote: $y = x$
7. asymptote: $y = -\frac{1}{2}x$, vertex: $(6, 0)$
8. asymptote: $y = 3x$, $b = 9$, vertical transverse axis
9. vertex: $(0, 8)$, focus: $(0, 6\sqrt{2})$
10. Find the equation of two hyperbolas such that they have the same a and b values, the equation of an asymptote is $y = \frac{4}{5}x$, and centered at the origin.
11. Find the equation of two hyperbolas such that they have different a and b values, both horizontal, the equation of an asymptote is $y = -\frac{2}{3}x$, and centered at the origin.
12. Find the equation of two hyperbolas such that they have different a and b values, both vertical, the equation of an asymptote is $y = 6x$, and centered at the origin.
13. Find the equation of two hyperbolas such that they have the same a and b values, the equation of an asymptote is $y = -\frac{10}{7}x$, and centered at the origin.

Find the equation of the hyperbolas below.

14. 

15. 

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.8.

Image Attributions

1. [Figure 1]
   Credit: Ben Salter
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This page titled 6.3.2: Equations of Hyperbolas Centered at the Origins is shared under a CK-12 license and was authored, remixed, and/or curated by CK12 via source content that was edited to the style and standards of the LibreTexts platform.

6.3.3: Hyperbolas with any Center

Hyperbolas Centered at (h, k)

Your homework assignment is to graph the hyperbola $9(y+2)^2-4(x-3)^2=36$. What are the vertices of your graph?

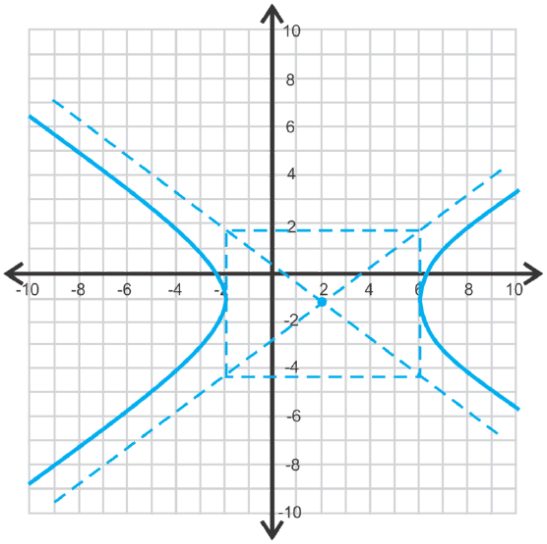
Hyperbolas Centered at (h,k)

Just like you have learned previously, a hyperbola does not always have to be placed with its center at the origin. If the center is (h,k) the entire ellipse will be shifted h units to the left or right and k units up or down. The equation becomes $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$. We will address how the vertices, co-vertices, and foci change in the next problem.

Let's graph $\frac{(x-2)^2}{16}-\frac{(y+1)^2}{9}=1$. Then, let's find the vertices, foci, and asymptotes.

First, we know this is a horizontal hyperbola because the x term is first. Therefore, the center is $(2,-1)$ and $a=4$ and $b=3$. Use this information to graph the hyperbola.

To graph, plot the center and then go out 4 units to the right and left and then up and down 3 units. Draw the box and asymptotes.



This is also how you can find the vertices. The vertices are $(2\pm4,-1)$ or $(6,-1)$ and $(-2,-1)$.

To find the foci, we need to find c using $c^2=a^2+b^2$.

$$\begin{aligned} c^2 &= 16+9=25 \\ c &= 5 \end{aligned}$$

Therefore, the foci are $(2\pm5,-1)$ or $(7,-1)$ and $(-3,-1)$.

To find the asymptotes, we have to do a little work to find the y-intercepts. We know that the slope is $\pm\frac{b}{a}$ or $\pm\frac{3}{4}$ and they pass through the center. Let's write each asymptote in point-slope form using the center and each slope.

$y-1=\frac{3}{4}(x+2)$ and $y-1=-\frac{3}{4}(x+2)$

Simplifying each equation, the asymptotes are $y=\frac{3}{4}x-\frac{5}{2}$ and $y=-\frac{3}{4}x+\frac{1}{2}$.

From this problem, we can create formulas for finding the vertices, foci, and asymptotes of a hyperbola with center (h,k). Also, when graphing a hyperbola, not centered at the origin, make sure to plot the center.

Orientation	Equation	Vertices	Foci	Asymptotes
Horizontal	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$(h\pm a,k)$	$(h\pm c,k)$	$y-k=\pm\frac{b}{a}(x-h)$
Vertical	$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$	$(h,k\pm a)$	$(h,k\pm c)$	$y-k=\pm\frac{a}{b}(x-h)$

Now, let's find the equation of the hyperbola with vertices $(-3, 2)$ and $(7, 2)$ and focus $(-5, 2)$.

These two vertices create a horizontal transverse axis, making the hyperbola horizontal. If you are unsure, plot the given information on a set of axes. To find the center, use the midpoint formula with the vertices.

$$\left(\frac{-3+7}{2}, \frac{2+2}{2}\right) = \left(\frac{4}{2}, \frac{4}{2}\right) = (2, 2)$$

The distance from one of the vertices to the center is $a, |7 - 2| = 5$. The distance from the center to the given focus is $c, |-5 - 2| = 7$. Use a and c to solve for b.

$$7^2 = 5^2 + b^2$$
$$b^2 = 24 \rightarrow b = 2\sqrt{6}$$

Therefore, the equation is $\frac{(x-2)^2}{25} - \frac{(y-2)^2}{24} = 1$.

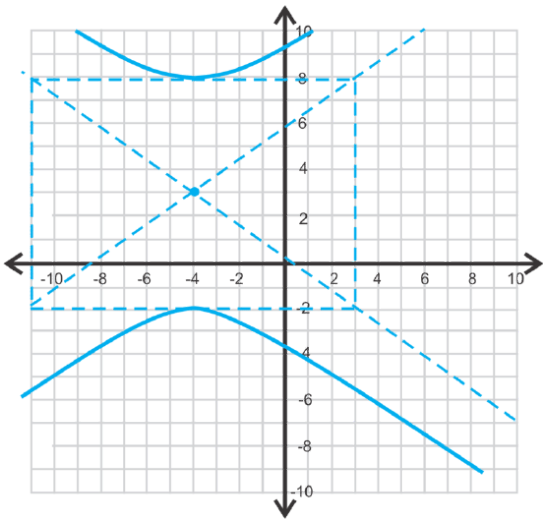
Finally, let's graph $49(y-3)^2 - 25(x+4)^2 = 1225$ and find the foci.

First we have to get the equation into standard form, like the equations above. To make the right side 1, we need to divide everything by 1225.

$$\frac{49(y-3)^2}{1225} - \frac{25(x+4)^2}{1225} = \frac{1225}{1225}$$
$$\frac{(y-3)^2}{25} - \frac{(x+4)^2}{49} = 1$$

Now, we know that the hyperbola will be vertical because the y-term is first.

$a = 5$, $b = 7$ and the center is $(-4, 3)$.



To find the foci, we first need to find c by using $c^2 = a^2 + b^2$.

$$c^2 = 49 + 25 = 74$$
$$c = \sqrt{74}$$

The foci are $(-4, 3 \pm \sqrt{74})$ or $(-4, 11.6)$ and $(-4, -5.6)$.

Examples

Example 1

Earlier, you were asked to identify the vertices of the graph of the hyperbola defined by $9(y+2)^2 - 4(x-3)^2 = 36$.

Solution

First we need to get the equation in standard form $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$, so we divide by 36.

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$$9(y+2)^2 - 4(x-3)^2 = 36$$
$$\frac{9(y+2)^2}{36} - \frac{4(x-3)^2}{36} = \frac{36}{36}$$
$$\frac{(y+2)^2}{4} - \frac{(x-3)^2}{9} = 1$$

Because the *y*-term is first, we can now see that the vertices are $(h, k \pm a)(3, -2 \pm 2)$. That is, $(3, 0)$ and $(3, -4)$

Example 2

Find the center, vertices, foci, and asymptotes of $\frac{(y-1)^2}{81} - \frac{(x+5)^2}{16} = 1$.

Solution

The center is $(-5, 1)$, $a = \sqrt{81} = 9$ and $b = \sqrt{16} = 4$, and the hyperbola is horizontal because the y-term is first. The vertices are $(-5, 1 \pm 9)$ or $(-5, 10)$ and $(-5, -8)$. Use $c^2 = a^2 + b^2$ to find c.

$$c^2 = 81 + 16 = 97$$
$$c = \sqrt{97}$$

The foci are $(-5, 1 + \sqrt{97})$ and $(-5, 1 - \sqrt{97})$.

The asymptotes are $y - 1 = \pm\frac{9}{4}(x + 5)$ or $y = \frac{9}{4}x + 12\frac{1}{4}$ and $y = -\frac{9}{4}x - 10\frac{1}{4}$.

Example 3

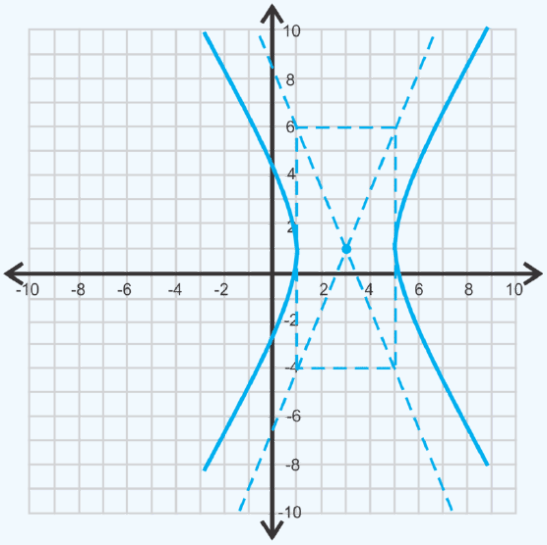
Graph $25(x-3)^2 - 4(y-1)^2 = 100$ and find the foci.

Solution

Change this equation to standard form in order to graph.

$$\frac{25(x-3)^2}{100} - \frac{4(y-1)^2}{100} = \frac{100}{100}$$
$$\frac{(x-3)^2}{4} - \frac{(y-1)^2}{25} = 1$$

center: $(3, 1), a = 2, b = 5$



Find the foci.

$$c^2 = 25 + 4 = 29$$
$$c = \sqrt{29}$$

The foci are $(3, 1 + \sqrt{29})$ and $(3, 1 - \sqrt{29})$.

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Example 4

Find the equation of the hyperbola with vertices $(-6,-3)$ and $(-6,5)$ and focus $(-6,7)$.

Solution

The vertices are $(-6,-3)$ and $(-6,5)$ and the focus is $(-6,7)$. The transverse axis is going to be vertical because the x-value does not change between these three points. The distance between the vertices is $|-3-5|=8$ units, making $a=4$. The midpoint between the vertices is the center.

$(-6, \frac{-3+5}{2}) = (-6, \frac{2}{2}) = (-6,1)$

The focus is $(-6,7)$ and the distance between it and the center is 6 units, or c. Find b.

$36 = b^2 + 16$
$20 = b^2$
$b = \sqrt{20} = 2\sqrt{5}$

The equation of the hyperbola is $\frac{(y-1)^2}{16} - \frac{(x+6)^2}{20} = 1$.

Review

Find the center, vertices, foci, and asymptotes of each hyperbola below.

1. $\frac{(x+5)^2}{25} - \frac{(y+1)^2}{36} = 1$
2. $(y+2)^2 - 16(x-6)^2 = 16$
3. $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{49} = 1$
4. $25x^2 - 64(y-6)^2 = 1600$
5. $(x-8)^2 - \frac{(y-4)^2}{9} = 1$
6. $81(y+4)^2 - 4(x+5)^2 = 324$
7. Graph the hyperbola in #1.
8. Graph the hyperbola in #2.
9. Graph the hyperbola in #5.
10. Graph the hyperbola in #6.

Using the information below, find the equation of each hyperbola.

11. vertices: $(-2,-3)$ and $(8,-3) b=7$
12. vertices: $(5,6)$ and $(5,-12)$ focus: $(5,-15)$
13. asymptote: $y+3 = \frac{4}{9}(x+1)$ horizontal transverse axis
14. foci: $(-11,-4)$ and $(1,-4)$ vertex: $(-8,-4)$
15. **Extension** Rewrite the equation of the hyperbola,
16. $x^2 - 4y^2 + 490x - 16y + 1013 = 0$ in standard form, by completing the square for both the x and y terms.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.9.

Image Attributions

1. [Figure 1]
   Credit: Ben Salter
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6.3.4: Hyperbola Equations and the Focal Property

Hyperbola Equations and the Focal Property

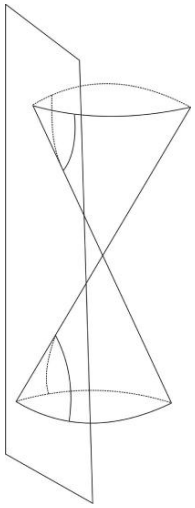
Adrian and Evan were discussing the math class they just completed. The class *focused* on **hyperbolas** (pun intended!), and reviewed the properties of hyperbolas.

Adrian thinks that hyperbolas are very similar to the parabolas that they studied last week, and believes that the shapes are really the same. Evan thinks that it does not matter what the *parts* of the **hyperbola** look like, since it is the complete shape that they are studying.

Who is correct?

Hyperbola Equations and the Focal Property

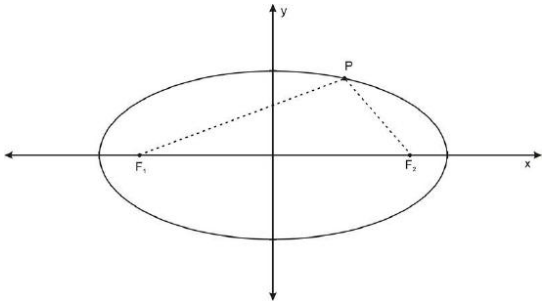
Compared to parabolas and **ellipses**, hyperbolas may seem... messy! Not only is it an infinite shape, but there are two pieces that aren't even connected! Hyperbolas are **conic** sections though: when the plane slices through two parts of the cone, the two infinite "U"-shaped parts are together called a **hyperbola**.



In this section we will see that this sprawling shape actually has some beautiful properties that make it as noble as its cousins.

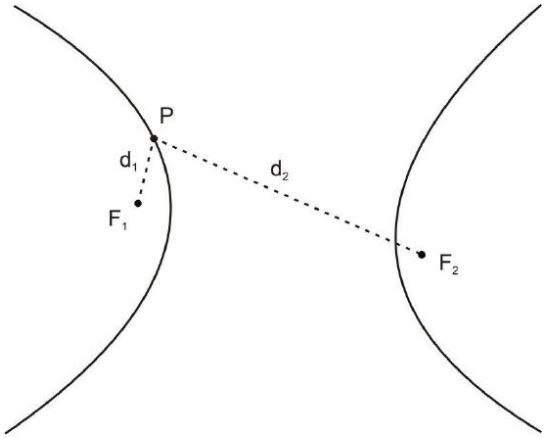
The Focal Property

Even though this shape seems much harder to conceive of than an **ellipse**, the hyperbola has a defining focal property that is as simple as the ellipse's. Remember, an ellipse has two foci and the shape can be defined as the set of points in a plane whose distances to these two foci have a fixed sum.



Hyperbolas also have two foci, and they can be defined as the set of points in a plane whose distances to these two points have the same *difference*. So in the picture below, for every point P on the hyperbola, $|d_2 - d_1| = C$ for some constant C.

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The general form for a hyperbola that opens upwards and downwards and whose foci lie on the y−axis is:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Switching x and y we have hyperbolas that open rightwards and leftwards and whose foci lie on the x−axis.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

For a hyperbola that is centered around the point (h,k) we have the shifted equations:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

for a hyperbola opening up and down, and

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

for a hyperbola opening left and right.

Examples

Example 1

Earlier, you were asked whether Evan or Adrian is correct.

Solution

Evan and Adrian are both correct in their own ways. Adrian is correct that a hyperbola is just two parabolas in opposite directions, which becomes clear when you consider that a **parabola** is created by slicing a single cone, and a hyperbola by slicing two identical cones at the same time. Evan is correct that although a hyperbola is constructed of two parabolas, it is the complete shape that they are studying, and many or most of the formulas and definitions that they will consider only apply to the complete shape.

Example 2

The hyperbola is infinite in size. In mathematics this is called **unbounded**, which means no circle, no matter how large, can enclose the shape. Explain why a focal property involving a *difference* results in an unbounded shape, while a focal property involving a sum results in a bounded shape.

Solution

In the case of an ellipse, we had two distances summing to a constant. Since the distances are both positive then there is a limit to the size of the numbers. In the case of hyperbolas, two very large positive numbers can have a much smaller difference, infinitely small in fact.

Example 3

Show that the following equation is a hyperbola. Graph it, and show its foci.

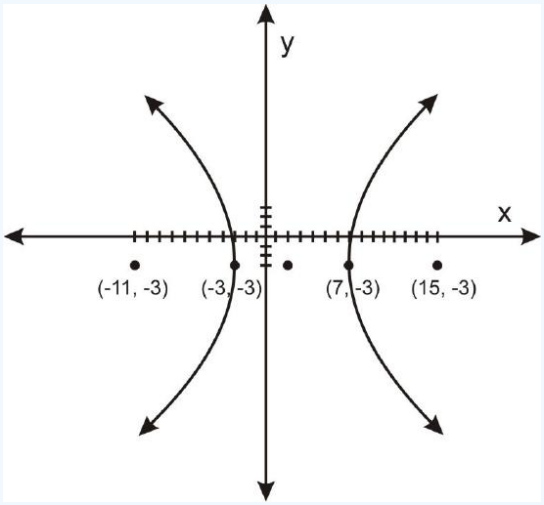
$$144x^2 - 576x - 25y^2 - 150y - 3249 = 0$$

Solution

The positive leading coefficient for the x^2 term and the negative leading coefficient for the y^2 term indicate that this is a hyperbola that is horizontally oriented. Grouping and completing the square, we have:

$$144\left(x^2 - 4x\right) - 25\left(y^2 + 6y\right) = 3249$$
$$144\left(x^2 - 4x + 4\right) - 25\left(y^2 + 6y + 9\right) = 3249 + 576 - 225$$
$$144(x-2)^2 - 25(y+3)^2 = 3600$$
$$\frac{(x-2)^2}{5^2} - \frac{(y+3)^2}{12^2} = 1$$

So our hyperbola is centered at $(2, -3)$. Its vertices are 5 units to the right and left of $(2, -3)$, or at the points $(7, -3)$ and $(-3, -3)$. It opens to the right and left from these vertices. It's foci are c units to the left and right of $(2, -3)$, where $c = \sqrt{a^2 + b^2} = \sqrt{5^2 + 12^2} = 13$. So it's foci are at $(15, -3)$ and $(-11, -3)$. Plotting a few points near $(7, -3)$ and $(-3, -3)$, the graph looks like:



Example 4

Graph the following hyperbola and mark its foci: $16x^2 + 64x - 9y^2 + 90y - 305 = 0$.

Solution

The positive leading coefficient for the term and the negative leading coefficient for the term indicate that this is a hyperbola that is horizontally oriented. Grouping and completing the square, we have:

$$16\left(x^2 + 4x + 4\right) - 9\left(y^2 - 10y + 25\right) - 305 = 64 - 225$$

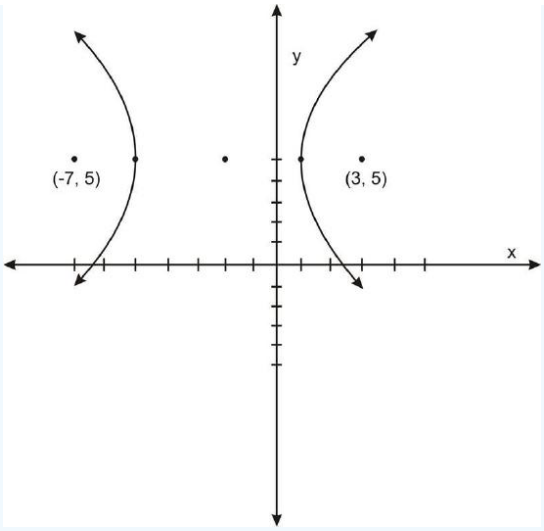
Factoring and combining like terms:

$$16(x+2)^2 - 9(y-5)^2 = 144$$

Divide both sides by 144 and rewrite *9* and *16* as 3^2 and 4^2:

$$\frac{(x+2)^2}{3^2} - \frac{(y-5)^2}{4^2} = 1$$

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Example 5

Graph the following hyperbola and mark its foci: $x^2 - 8x - y^2 - 4y = -28$.

Solution

To graph $x^2 - 8x - y^2 + 4y = -28$:

$(x-4)^2 - (y-2)^2 = -16$..... complete the square to factor

$\frac{(y-2)^2}{16} - \frac{(x-4)^2}{16} = 1$..... rewrite in standard form

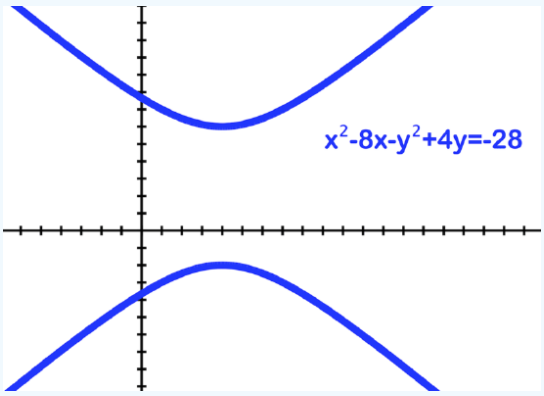
$x - 4 = 0 \rightarrow x = 4$ and $y - 2 = 0 \rightarrow y = 2$..... therefore the center is $(4, 2)$

Mark 4 units left and right of $(4, 2)$ and Mark 2 units above and below $(4, 2)$, use those four points to define the sides of a box.

Connect the corners of the box to illustrate the **asymptotes**.

Since the "y" term is the positive one, the hyperbola opens up and down.

Graph the hyperbola, it should look like this:



Example 6

Graph the following hyperbola: $x^2 - 8x - y^2 - 2y = -14$.

Solution

To graph $x^2 - 8x - y^2 - 2y = -14$:

$(x-4)^2 - (y+1)^2 = 1$..... complete the square to factor $\frac{(x-4)^2}{1} - \frac{(y+1)^2}{1} = 1$..... rewrite in standard form

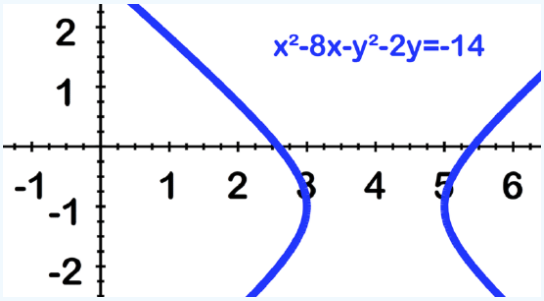
$x-4=0 \rightarrow x=4$ and $y+1=0 \rightarrow y=-1$..... therefore the center is $(4,-1)$

Mark 4 units left and right of $(4,-1)$ and Mark 1 unit above and below $(4,-1)$, use those four points to define the sides of a box.

Connect the corners of the box to illustrate the asymptotes.

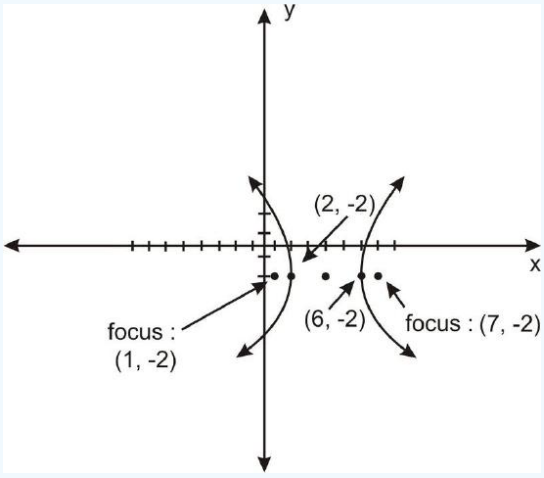
Since the "x" term is the positive one, the hyperbola opens left and right.

Graph the hyperbola, it should look like this:



Example 7

Find the equation for the following hyperbola:



Solution

$\frac{(x-4)^2}{4}-\frac{(y+2)^2}{45}=1$

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403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

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Generated by cloudfront (CloudFront)
Request ID: kkqU_MWkoVZXsIm19CfqfNp-5Ku27qTPdlUwjBWEKk1ruxTJQTuaCA==
```

403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
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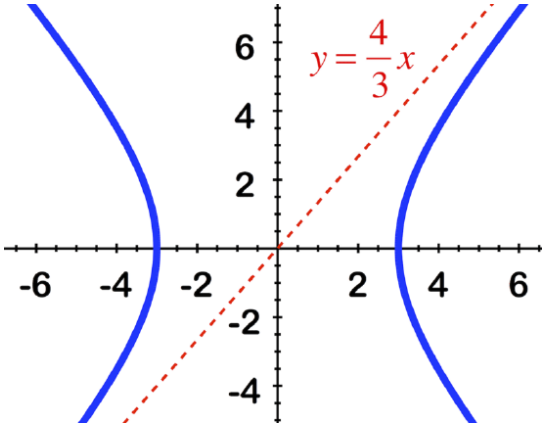
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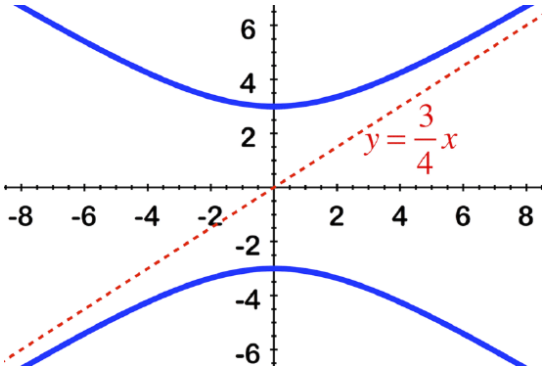
Review

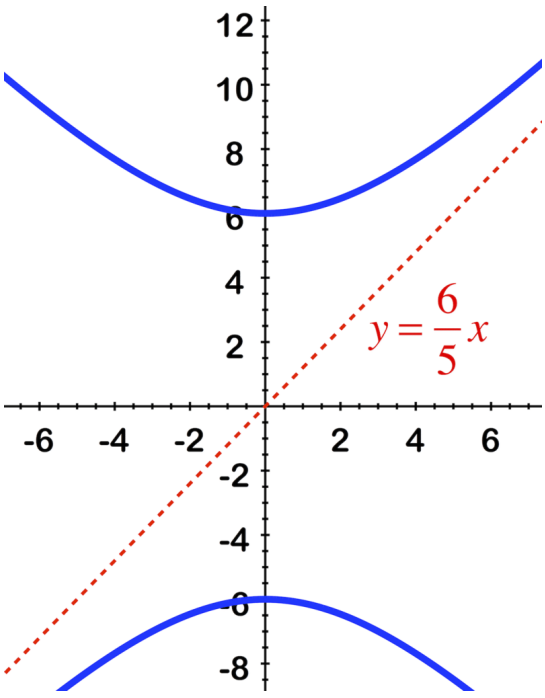
Sketch the hyperbolas.

1. $\frac{(x+4)^2}{4} - \frac{(y-1)^2}{9} = 1$
2. $\frac{(y+3)^2}{4} - \frac{(x-4)^2}{9} = 1$
3. $\frac{(y+4)^2}{16} - \frac{(x-1)^2}{4} = 1$
4. $(x-2)^2 - 4y^2 = 16$
5. $\frac{y^2}{4} - \frac{(x-1)^2}{4} = 1$
6. $\frac{(x-2)^2}{16} - \frac{(y+4)^2}{1} = 1$
7. $\frac{(x+2)^2}{9} - \frac{(y+2)^2}{16} = 1$
8. $\frac{(x+4)^2}{9} - \frac{(y-2)^2}{4} = 1$
9. Graph the hyperbola and mark its foci: $9y^2 + 18y - x^2 + 4x - 4 = 0$
10. Graph the hyperbola and mark its foci: $25x^2 + 150x - 4y^2 + 24y + 89 = 0$

Identify the equation of the hyperbola using the image.

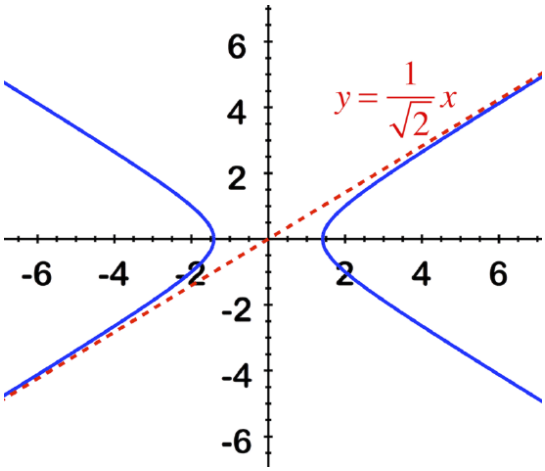
11. 

12. 

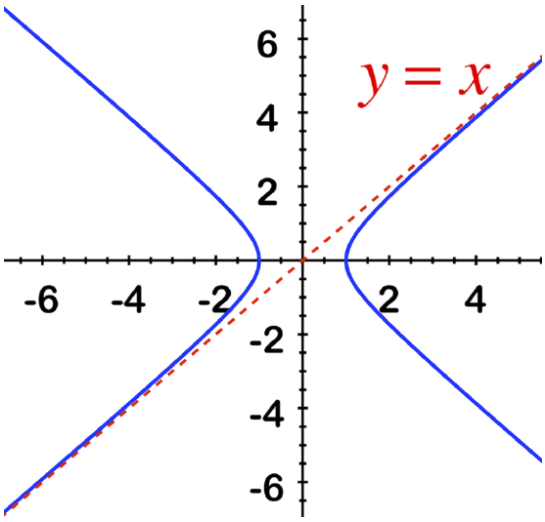
13. 

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14.



15.



Review (Answers)

To see the Review answers, open this PDF file and look for section 6.6.

Vocabulary

https://k12.libretexts.org/@go/page/14761

Term	Definition
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
hyperbola	A hyperbola is a conic section formed when the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shaped curves.
hyperbolas	A hyperbola is a conic section formed when the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shaped curves.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.
perpendicular hyperbola	A perpendicular hyperbola has asymptotes that intersect at a 90∘ angle.
unbounded	To be unbounded means to be so large that no circle, no matter how large, can enclose the shape.

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6.3.5: Hyperbolas and Asymptotes

Hyperbolas and Asymptotes

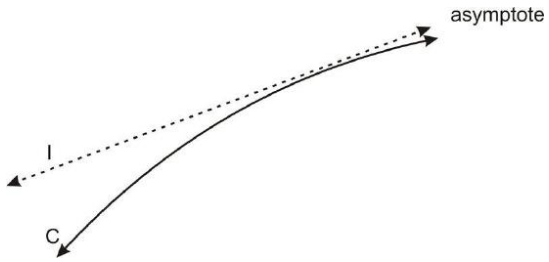
Like other **conic** sections, hyperbolas can be created by "slicing" a cone and looking at the cross-section. Unlike other conics, hyperbolas actually require 2 cones stacked on top of each other, point to point. The shape is the result of effectively creating a **parabola** out of both cones at the same time.

So the question is, should hyperbolas really be considered a shape all their own? Or are they just two parabolas graphed at the same time? Could "different" shapes be made from any of the other conic sections if two cones were used at the same time?

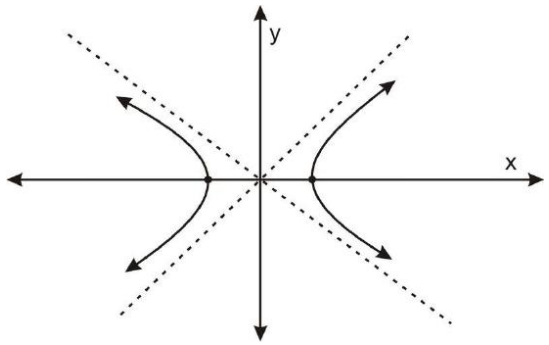
Hyperbolas and Asymptotes

In addition to their focal property, hyperbolas also have another interesting geometric property. Unlike a parabola, a **hyperbola** becomes infinitesimally close to a certain line as the x− or y−coordinates approach infinity. What we mean by “infinitesimally close?” Here we mean two things: 1) The further you go along the curve, the closer you get to the asymptote, and 2) If you name a distance, no matter how small, eventually the curve will be that close to the asymptote. Or, using the language of limits, as we go further from the vertex of the hyperbola the limit of the distance between the hyperbola and the asymptote is 0.

These lines are called **asymptotes**.



There are two asymptotes, and they cross at the point at which the hyperbola is centered:



For a hyperbola of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes are the lines:

$y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

For a hyperbola of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ the asymptotes are the lines:

$y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$.

(For a shifted hyperbola, the asymptotes shift accordingly.)

Examples

Example 1

Earlier, you were asked if hyperbolas should be considered shapes all their own.

Solution

Hyperbolas are considered different shapes, because there are specific behaviors that are unique to hyperbolas. Also, though hyperbolas are the result of dual parabolas, none of the other conics really create unique shapes with dual cones - just double figures - and in any case require multiple "slices".

Example 2

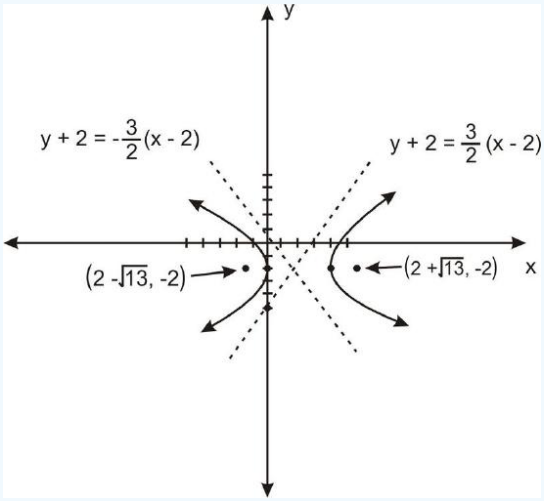
Graph the following hyperbola, drawing its foci and asymptotes and using them to create a better drawing: $9x^2 - 36x - 4y^2 - 16y - 16 = 0$

Solution

First, we put the hyperbola into the standard form:

$$\begin{aligned} 9\left(x^2 - 4x\right) - 4\left(y^2 + 4y\right) &= 16 \\ 9\left(x^2 - 4x + 4\right) - 4\left(y^2 + 4y + 4\right) &= 36 \\ \frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} &= 1 \end{aligned}$$

So $a = 2$, $b = 3$ and $c = \sqrt{4+9} = \sqrt{13}$. The hyperbola is horizontally oriented, centered at the point $(2, -2)$, with foci at $(2 + \sqrt{13}, -2)$ and $(2 - \sqrt{13}, -2)$. After taking shifting into consideration, the asymptotes are the lines: $y + 2 = \frac{3}{2}(x - 2)$ and $y + 2 = -\frac{3}{2}(x - 2)$. So graphing the vertices and a few points on either side, we see the hyperbola looks something like this:

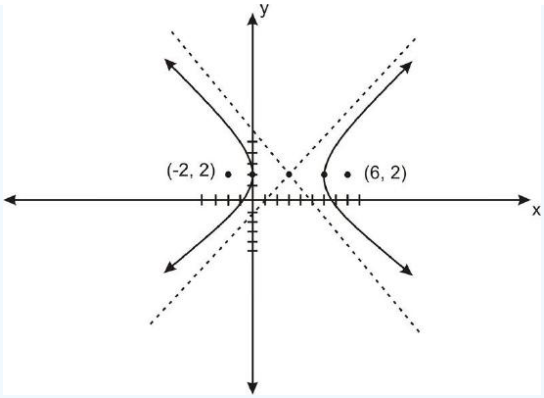


Example 3

Graph the following hyperbola, drawing its foci and asymptotes and using them to create a better drawing: $16x^2 - 96x - 9y^2 - 36y - 84 = 0$

Solution

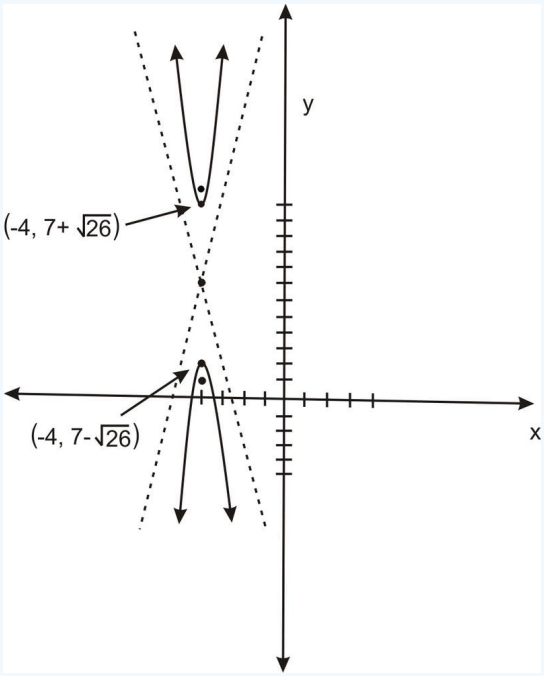
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Example 4

Graph the following hyperbola, drawing its foci and asymptotes, and use them to create a better drawing: $y^2 - 14y - 25x^2 - 200x - 376 = 0$.

Solution



Example 5

Find the equation for a hyperbola with asymptotes of slopes $\frac{5}{12}$ and $-\frac{5}{12}$, and foci at points $(2, 11)$ and $(2, 1)$.

Solution

$\frac{(y-6)^2}{25} - \frac{(x-2)^2}{144} = 1$

Example 6

A hyperbola with perpendicular asymptotes is called **perpendicular**. What does the equation of a **perpendicular hyperbola** look like?

Solution

https://k12.libretexts.org/@go/page/14762

The slopes of perpendicular lines are negative reciprocals of each other. This means that $\frac{a}{b} = \frac{b}{a}$, which, for positive a and b means $a = b$.

Example 7

Find an equation of the hyperbola with x-intercepts at $x = -7$ and $x = 5$, and foci at $(-6, 0)$ and $(4, 0)$.

Solution

The foci have the same y-coordinates, so this is a left/right hyperbola with the center, foci, and vertices on a line paralleling the x-axis.

Since it is a left/right hyperbola, the y part of the equation will be negative and equation will lead with the x^2 term (since the leading term is positive by convention and the squared term must have different signs if this is a hyperbola).: The center is midway between the foci, so the center $(h, k) = (-1, 0)$. The foci c are 5 units to either side of the center, so $c = 5 \rightarrow c^2 = 25$.

The x-intercepts are 4 units to either side of the center, and the foci are on the x-axis so the intercepts must be the vertices a $a = 4 \rightarrow a^2 = 16$.

Use the Pythagorean theorem, $a^2 + b^2 = c^2$, to get $b^2 = 25 - 16 = 9$.

Substitute the calculated values into the standard form $\frac{(x-h)^2}{a} - \frac{(y-k)^2}{b} = 1$ to get $\frac{(x+1)^2}{16} - \frac{y^2}{9} = 1$.

403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

```
Generated by cloudfront (CloudFront)
Request ID: je0AOwd4zzgdH8TFG4ntqmnStinRHE-F2uzpZUvq-xLkSLahOZ-_Iw==
```

403 ERROR

The request could not be satisfied.

Request blocked. We can't connect to the server for this app or website at this time. There might be too much traffic or a configuration error. Try again later, or contact the app or website owner.
If you provide content to customers through CloudFront, you can find steps to troubleshoot and help prevent this error by reviewing the CloudFront documentation.

```
Generated by cloudfront (CloudFront)
Request ID: KayRkK6mouxXNYung4dNearGwDJYD4WazsMuKacQT7h_chRYdsc8ug==
```

Review

Find the equations of the asymptotes of each hyperbola.

1. $\frac{(y+3)^2}{4} - (x-2)^2 = 1$
2. $\frac{y^2}{16} - (x+3)^2 = 1$
3. $\frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} = 1$
4. $\frac{(y-4)^2}{16} - \frac{(x-4)^2}{16} = 1$
5. $\frac{(x-1)^2}{1} - \frac{9(y+4)^2}{1} = 9$
6. $\frac{(y+2)^2}{16} - \frac{(x-2)^2}{1} = 1$
7. $\frac{(x-4)^2}{1} - \frac{(y+1)^2}{4} = 1$
8. $\frac{y^2}{16} - \frac{(x+1)^2}{4} = 1$
9. $\frac{(x-3)^2}{4} - \frac{(y-4)^2}{1} = 1$
10. $\frac{(x-4)^2}{4} - \frac{(y-3)^2}{1} = 1$

Graph the hyperbolas, give the equation of the asymptotes, and use the asymptotes to enhance the accuracy of your graph.

11. $\frac{(x+4)^2}{4} - \frac{(y-1)^2}{9} = 1$
12. $\frac{(y+3)^2}{4} - \frac{(x-4)^2}{9} = 1$
13. $\frac{(y+4)^2}{16} - \frac{(x-1)^2}{4} = 1$
14. $(x-2)^2 - 4y^2 = 16$
15. $\frac{y^2}{4} - \frac{(x-1)^2}{4} = 1$
16. $\frac{(x-2)^2}{16} - \frac{(y+4)^2}{1} = 1$
17. $\frac{(x+2)^2}{9} - \frac{(y+2)^2}{16} = 1$
18. $\frac{(x+4)^2}{9} - \frac{(y-2)^2}{4} = 1$

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.7.

Vocabulary

Term	Definition
Asymptotes	An asymptote is a line on the graph of a function representing a value toward which the function may approach, but does not reach (with certain exceptions).
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
hyperbola	A hyperbola is a conic section formed when the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shaped curves.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.
perpendicular hyperbola	A perpendicular hyperbola has asymptotes that intersect at a 90° angle.
unbounded	To be unbounded means to be so large that no circle, no matter how large, can enclose the shape.

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SECTION OVERVIEW

6.4: Equations of Circles

Topic hierarchy

6.4.1: Circles Centered at the Origin

6.4.2: Circles Not Centered at the Origin

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6.4.1: Circles Centered at the Origin

Circles Centered at the Origin

You draw a **circle** that is centered at the origin. You measure the **diameter** of the circle to be 32 units. Does the point $(14, 8)$ lie on the circle?

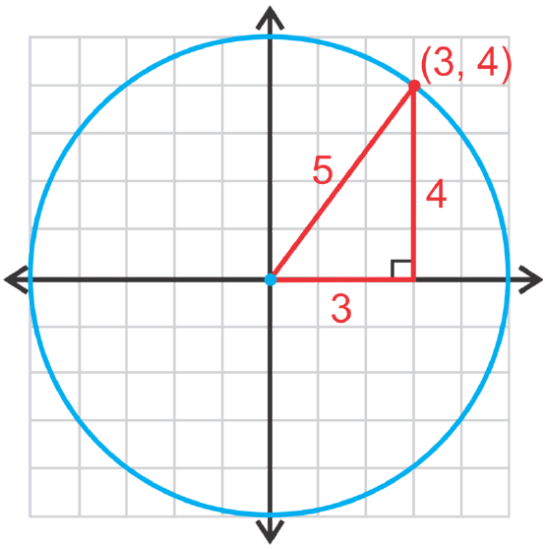
Circles Centered at the Origin

Until now, your only reference to circles was from geometry. A **circle** is the set of points that are equidistant (the **radius**) from a given point (the **center**). A line segment that passes through the center and has endpoints on the circle is a **diameter**.

Now, we will take a circle and place it on the x−y plane to see if we can find its equation. In this concept, we are going to place the center of the circle on the origin.

Finding the Equation of a Circle

Step 1: On a piece of graph paper, draw an x−y plane. Using a compass, draw a circle, centered at the origin that has a radius of 5. Find the point (3,4) on the circle and draw a right triangle with the radius as the hypotenuse.



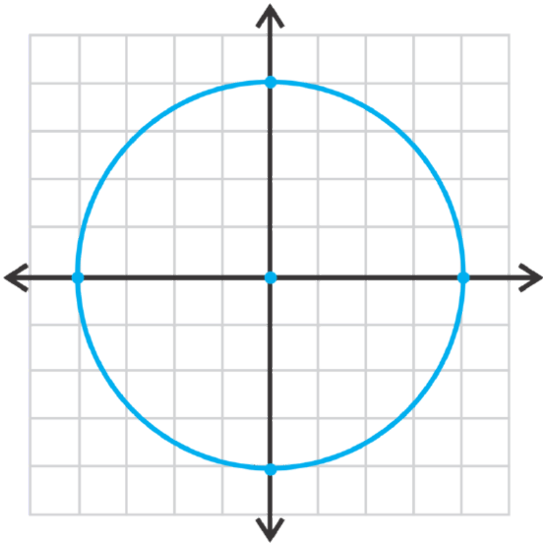
Step 2: Using the length of each side of the right triangle, show that the Pythagorean Theorem is true.

Step 3: Now, instead of using (3, 4), change the point to (x, y) so that it represents any point on the circle. Using r to represent the radius, rewrite the Pythagorean Theorem.

The **equation of a circle**, centered at the origin, is $x^2 + y^2 = r^2$, where r is the radius and (x, y) is any point on the circle.

Let's find the radius of $x^2 + y^2 = 16$ and graph.

To find the radius, we can set $16 = r^2$, making $r = 4$. r is not -4 because it is a distance and distances are always positive. To graph the circle, start at the origin and go out 4 units in each direction and connect.



Now, let's find the equation of the circle with center at the origin and passes through $(-7, -7)$.

Using the equation of the circle, we have: $(-7)^2 + (-7)^2 = r^2$. Solve for r^2.

$$\begin{aligned} (-7)^2 + (-7)^2 &= r^2 \\ 49 + 49 &= r^2 \\ 98 &= r^2 \end{aligned}$$

So, the equation is $x^2 + y^2 = 98$. The radius of the circle is $r = \sqrt{98} = 7\sqrt{2}$.

Finally, let's determine if the point $(9, -11)$ is on the circle $x^2 + y^2 = 225$.

Substitute the point in for x and y and see if it equals 225.

$$\begin{aligned} 9^2 + (-11)^2 &= 225 \\ 81 + 121 &\stackrel{?}{=} 225 \\ 202 &\neq 225 \end{aligned}$$

The point is not on the circle.

Examples

Example 1

Earlier, you were asked to determine if the point (14 ,8) lies on the circle that is centered at the origin and has a diameter of 32 units.

Solution

From this lesson, you know that the equation of a circle that is centered at the origin is $x^2 + y^2 = r^2$, where r is the radius and (x, y) is any point on the circle.

With the point $(14, 8)$, $x = 14$ and $y = 8$. We are given the diameter, but we need the radius. Recall that the radius is half the diameter, so the radius is $\frac{32}{2} = 16$.

Plug these values into the equation of the circle. If they result in a true statement, the point lies on the circle.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ 14^2 + 8^2 &\stackrel{?}{=} 16^2 \\ 196 + 64 &\stackrel{?}{=} 256 \\ 260 &\neq 256 \end{aligned}$$

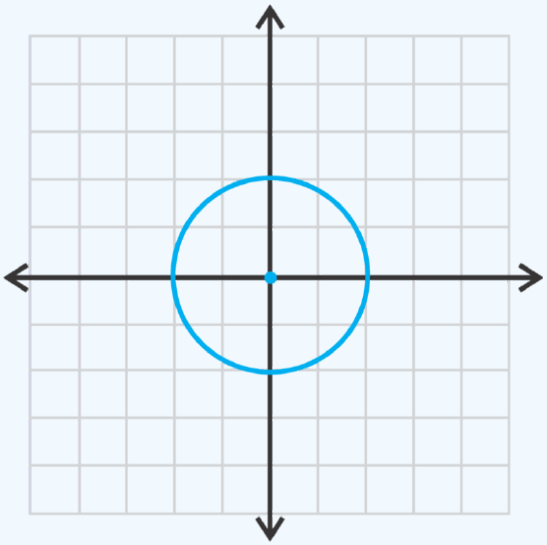
Therefore the point does not lie on the circle.

Example 2

Graph and find the radius of $x^2 + y^2 = 4$.

Solution

$r = \sqrt{4} = 2$



Example 3

Find the equation of the circle with a radius of $6\sqrt{5}$.

Solution

Plug in $6\sqrt{5}$ for r in $x^2 + y^2 = r^2$

$$x^2 + y^2 = (6\sqrt{5})^2$$
$$x^2 + y^2 = 6^2 \cdot (\sqrt{5})^2$$
$$x^2 + y^2 = 36 \cdot 5$$
$$x^2 + y^2 = 180$$

Example 4

Find the equation of the circle that passes through (5, 8).

Solution

Plug in (5, 8) for x and y, respectively.

$$5^2 + 8^2 = r^2$$
$$25 + 64 = r^2$$
$$89 = r^2$$

The equation is $x^2 + y^2 = 89$

Example 5

Determine if $(-10, 7)$ is on the circle $x^2 + y^2 = 149$.

Solution

Plug in $(-10, 7)$ to see if it is a valid equation.

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$$(-10)^2 + 7^2 = 149$$
$$100 + 49 = 149$$

Yes, the point is on the circle.

Review

Graph the following circles and find the radius.

1. $x^2 + y^2 = 9$
2. $x^2 + y^2 = 64$
3. $x^2 + y^2 = 8$
4. $x^2 + y^2 = 50$
5. $2x^2 + 2y^2 = 162$
6. $5x^2 + 5y^2 = 150$

Write the equation of the circle with the given radius and centered at the origin.

7. 14
8. 6
9. $9\sqrt{2}$

Write the equation of the circle that passes through the given point and is centered at the origin.

10. $(7, -24)$
11. $(2, 2)$
12. $(-9, -10)$

Determine if the following points are on the circle, $x^2 + y^2 = 74$.

13. $(-8, 0)$
14. $(7, -5)$
15. $(6, -6)$

Challenge In Geometry, you learned about tangent lines to a circle. Recall that the tangent line touches a circle at one point and is perpendicular to the radius at that point, called the point of tangency.

16. The equation of a circle is $x^2 + y^2 = 10$ with point of tangency $(-3, 1)$.
    a. Find the slope of the radius from the center to $(-3, 1)$.
    b. Find the perpendicular slope to (a). This is the slope of the tangent line.
    c. Use the slope from (b) and the given point to find the equation of the tangent line.
17. Repeat the steps in #16 to find the equation of the tangent line to $x^2 + y^2 = 34$ with a point of tangency of $(3, 5)$.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.3.

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Vocabulary

Term	Definition
center	The center of a circle is the point that defines the location of the circle. All points on the circle are equidistant from the center of the circle.
Circle	A circle is the set of all points at a specific distance from a given point in two dimensions.
Diameter	Diameter is the measure of the distance across the center of a circle. The diameter is equal to twice the measure of the radius.
Equation of a Circle	If the center of a circle is (0, 0), then the equation of the circle is of the form $x^2 + y^2 = r^2$, where r is the radius.
Radius	The radius of a circle is the distance from the center of the circle to the edge of the circle.

Image Attributions

1. [Figure 1]
   Credit: Tony Webster;Robert Pernett
   Source: https://commons.wikimedia.org/wiki/File%3ALas_Vegas_High_Roller_(20216869960).jpg;https://flic.kr/p/nQD1fo

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6.4.2: Circles Not Centered at the Origin

Circles Centered at (h, k)

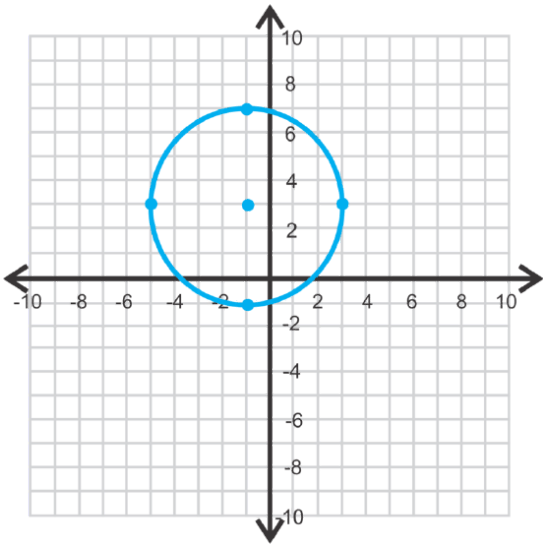
You draw a circle that is centered at (−2, −2). You measure the diameter of the circle to be 18 units. Does the point (4, 5) lie on the circle?

Circles Centered at (h,k)

When a circle is centered at the origin, the equation is $x^2 + y^2 = r^2$. If we rewrite this equation, using the center, it would look like $(x-0)^2 + (y-0)^2 = r^2$. Extending this idea to any point as the center, we would have $(x-h)^2 + (y-k)^2 = r^2$, where (h, k) is the center.

Let's find the center and radius of $(x+1)^2 + (y-3)^2 = 16$ and graph.

Using the general equation above, the center would be $(-1, 3)$ and the radius is $\sqrt{16}$ or 4. To graph, plot the center and then go out 4 units up, down, to the left, and to the right.



Now, let's find the equation of the circle with center (2,4) and radius 5.

Plug in the center and radius to the equation and simplify.

$$(x-2)^2 + (y-4)^2 = 5^2$$
$$(x-2)^2 + (y-4)^2 = 25$$

Finally, let's find the equation of the circle with center (6, −1) and (5, 2) is on the circle.

In this problem, we are not given the radius. To find the radius, we must use the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

$$\begin{aligned} r &= \sqrt{(5-6)^2 + (2-(-1))^2} \\ &= \sqrt{(-1)^2 + 3^2} \\ &= \sqrt{1+9} \\ &= \sqrt{10} \end{aligned}$$

Therefore, the equation of this circle is $(x-6)^2 + (y-(-1))^2 = (\sqrt{10})^2$ or $(x-6)^2 + (y+1)^2 = 10$.

Examples

https://k12.libretexts.org/@go/page/14765

Example 1

Earlier, you were asked to determine if the point (4, 5) lies on the circle.

Solution

In this lesson, you learned the equation of a circle that is centered somewhere other than the origin is $(x-h)^2+(y-k)^2=r^2$, where $(h,k)$is the center.

We are given that the center is $(-2,-2)$, so $h=-2$ and $k=-2$. We are also given the diameter of the circle, but we need the radius. Recall that the radius is half the diameter, so $r=\frac{18}{2}=9$.

If we plug these values into the equation for the circle, we get:

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\(x-(-2))^2+(y-(-2))^2&=9^2\\(x+2)^2+(y+2)^2&=81\end{aligned}$$

Now to find if the point $(4,5)$ lies on the circle we substitute 4 for x and 5 for y and see if the equation holds true.

$$\begin{aligned}(x+2)^2+(y+2)^2&=81\\(4+2)^2+(5+2)^2&\stackrel{?}{=}81\\6^2+7^2&\stackrel{?}{=}81\\85&\neq 81\end{aligned}$$

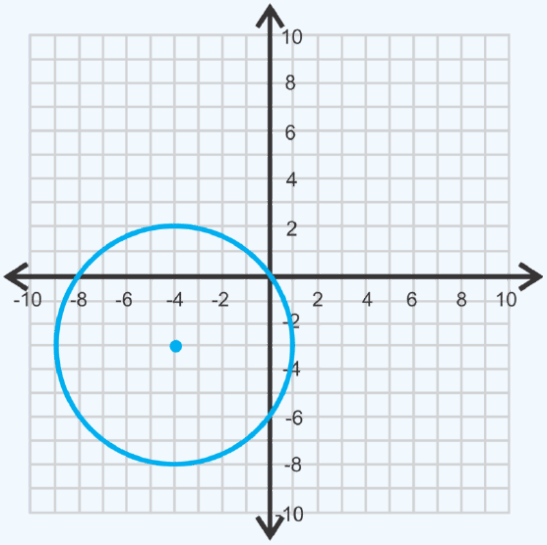
Therefore, the point does not lie on the circle.

Example 2

Graph $(x+4)^2+(y+3)^2=25$ and find the center and radius.

Solution

The center is (−4, −3) and the radius is 5.



Example 3

Find the equation of the circle with center (−8, 3) and (2, −5) is on the circle.

Solution

Use the distance formula to find the radius.

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$$r = \sqrt{(2-(-8))^2 + (-5-3)^2}$$
$$= \sqrt{10^2 + (-8)^2}$$
$$= \sqrt{100 + 64}$$
$$= \sqrt{164}$$

The equation of this circle is $(x+8)^2 + (y-3)^2 = 164$

Example 4

The endpoints of a diameter of a circle are (−3, 1) and (9, 6). Find the equation.

Solution

In this example, we are not given the center or radius. We can find the length of the diameter using the distance formula and then divide it by 2.

$$d = \sqrt{(9-(-3))^2 + (6-1)^2}$$
$$= \sqrt{12^2 + 5^2} \qquad \text{The radius is } 13 \div 2 = \frac{13}{2}$$
$$= \sqrt{144 + 25}$$
$$= \sqrt{169} = 13$$

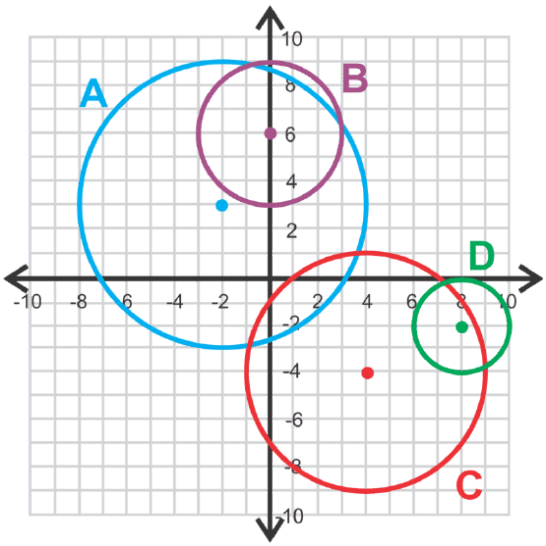
Now, we need to find the center. Use the midpoint formula with the endpoints.

$$c = \left(\frac{-3+9}{2}, \frac{1+6}{2}\right)$$
$$= \left(3, \frac{7}{2}\right)$$

Therefore, the equation is $(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{169}{4}$.

Review

For questions 1-4, match the equation with the graph.



1. $(x-8)^2 + (y+2)^2 = 4$
2. $x^2 + (y-6)^2 = 9$
3. $(x+2)^2 + (y-3)^2 = 36$
4. $(x-4)^2 + (y+4)^2 = 25$

Graph the following circles. Find the center and radius.

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5. $(x-2)^2+(y-5)^2=16$
6. $(x+4)^2+(y+3)^2=18$
7. $(x+7)^2+(y-1)^2=8$

Find the equation of the circle, given the information below.

8. center: $(-3,-3)$ radius: 7
9. center: $(-7,6)$ radius: $\sqrt{15}$
10. center: $(8,-1)$ point on circle: $(0,14)$
11. center: $(-2,-5)$ point on circle: $(3,2)$
12. diameter endpoints: $(-4,1)$ and $(6,3)$
13. diameter endpoints: $(5,-8)$ and $(11,2)$
14. Is $(-9,12)$ on the circle $(x+5)^2+(y-6)^2=54$? How do you know?
15. **Challenge** Use the following steps to find the equation of the tangent line to the circle with center $(3,-4)$ and the point of tangency $(-1,8)$.
    a. Find the slope of the radius from the center to $(-1,8)$.
    b. Find the perpendicular slope to (a). This is the slope of the tangent line.
    c. Use the slope from (b) and the given point to find the equation of the tangent line.
16. **Extension** Rewrite the equation of the circle, $x^2+y^2+4x-8y+11=0$ in **standard form** by completing the square for both the x and y terms. Then, find the center and radius.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.4.

Vocabulary

Term	Definition
Standard Form	The standard form of a circle is $(x-h)^2+(y-k)^2=r^2$, where (h,k) is the center and r is the radius.

Image Attributions

1. [Figure 1]
   Credit: Tony Webster
   Source: https://commons.wikimedia.org/wiki/File%3ALas_Vegas_High_Roller_(20216869960).jpg

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SECTION OVERVIEW

6.5: Conic Sections and Degenerate Conics

Topic hierarchy

6.5.1: Conic Sections and Dandelin Spheres

6.5.2: Classifying Conic Sections

6.5.3: Degenerate Conics

6.5.4: Solving Systems of Lines, Quadratics, and Conics

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6.5.1: Conic Sections and Dandelin Spheres

Conic Sections and Dandelin Spheres

One of the major activities of mathematicians and scientists of all kinds is learning, and attempting to prove, how and why things work.

Since Ancient Greece at the beginning of the millennium, and likely well before, the properties of **conic** sections have been studied. One that has been explored extensively is the "Focal Property", which we have referred to in nearly every lesson in this section.

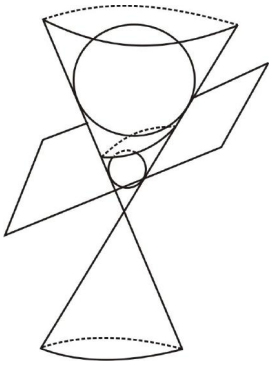
There are a number of mathematical proofs of the focal property, but trying to explain such proofs to someone who does not dedicate his or her life to mathematics is difficult. The **Dandelin Spheres** are one solution.

Conic Sections and Dandelin Spheres

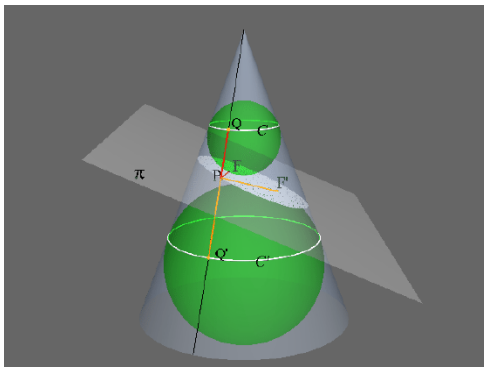
Dandelin Spheres and the Equation of an Ellipse

It wasn't until 1822 that the French mathematician Germinal Dandelin thought of this very clever construction. Dandelin found a way to find the foci and prove the focal property in one fell swoop.

Take the conic section in question. Then choose a sphere that is just the right size so that when it's dropped into the conic, it touches the intersecting plane, as well as being snug against the cone on all sides. If you prefer, you can think of the sphere as a perfectly round balloon that is blown up until it "just fits" inside the cone, still touching the plane. Then do the same on the *other* side of the plane. After we've drawn both of these spheres we have this picture:



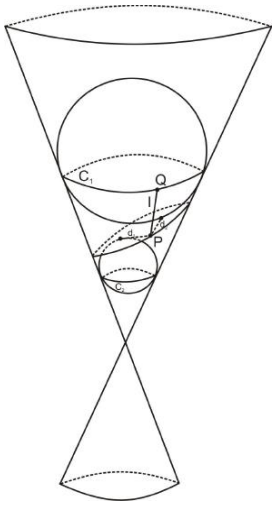
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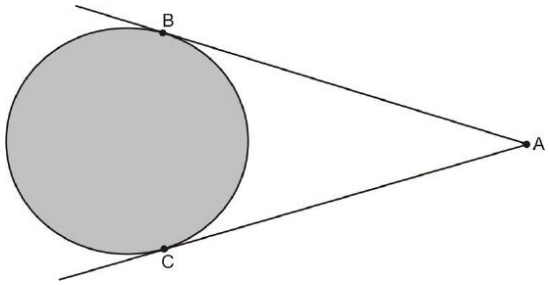
These spheres are often called "Dandelin spheres", named after their discoverer. It turns out that not only is our shape an ellipse (which, like all **ellipses** satisfies the focal property), but these spheres touch the ellipse exactly at the two foci. To see this, consider this geometric argument.

The first thing to notice is that the circles C_1 and C_2 shown on the diagram below, where each sphere lies snug against the cone, lie in parallel planes to one another. In particular, each line passing through these circles and the vertex of the cone, such as the line l drawn below, cuts off equal segments between the two circles. Let's call d the shortest distance along the cone between circles C_1 and C_2. This can also be thought of as the shortest distance between C_1 and C_2 that passes through the vertex of the cone.

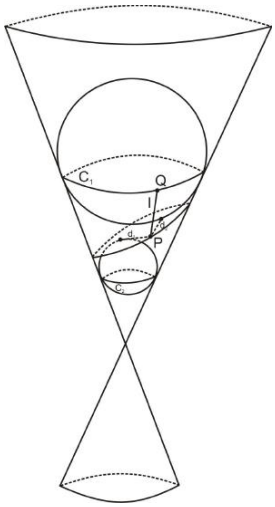
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The next thing to remember is a property of tangents to spheres that you may have learned in geometry. If two segments are drawn between a point and a sphere, and if the line containing each segment is tangent to the sphere, then the two segments are equal. In the diagram below, AB=AC. (This follows from the fact that tangents are perpendicular to the radii of a sphere and that two congruent triangles are formed in this configuration.)



Now consider the point P on the ellipse drawn below. Let $\overline{QR}$ be the segment of length d between C_1 and C_2 that passes through P. The distances between the two foci are marked d_1 and d_2. But $d_1 = RP$ and $d_2 = PQ$ by the property of tangents to spheres discussed above. So $d_1 + d_2 = RP + PQ = QR = d$. And this sum will always equal d, no matter what point P on the ellipse is chosen. So this proves the focal property of ellipses: that the sum of the distances between any point on the ellipse and the two foci is constant.

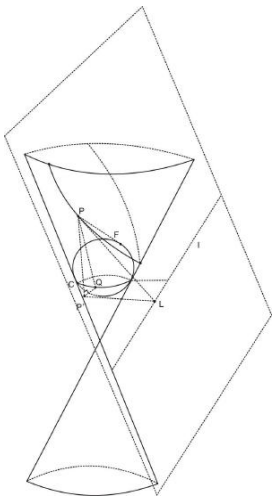


Dandelin and the Parabola

Like the ellipse, the parabola has a focal property. And, also like the ellipse, a construction similar to Dandelin's with the spheres can show us what it is. Dandelin himself didn't prove the focal property for parabolas that we are about to discuss, but Pierce Morton used a sphere construction similar to Dandelin's to prove the focal property of parabolas in 1829. We'll look at Morton's argument here.

In contrast with the argument we made for the ellipse, for a parabola we can only fit one tangent sphere inside the cone. That is, only one sphere can be tangent to both the cone and the cutting plane. In the diagram below, the sphere fits underneath the cutting plane, but there is no room for a sphere to lie on top of the cutting plane and still be tangent to the cone.

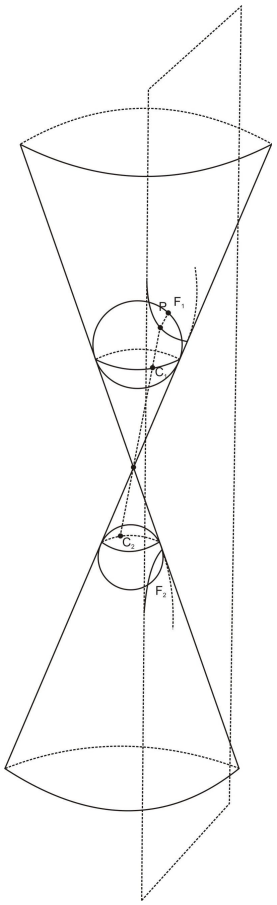
As with the ellipse, the point where the sphere intersects the plane is called a **focus**. But because there is only one sphere in this construction, and this is related to the fact that a parabola has only one focus. The other geometric object of interest is called the **directrix**. This is the line that results from the intersection between the cutting plane and the plane that contains the circle of contact between the sphere and the cone. In the diagram below, the directrix is labeled l and is found by intersecting the plane defined by circle C and the cutting plane (the planes are shown in dashed lines for clarity). Finally, we will call the angle between the planes θ.



In the above diagram, we have labeled the point where the sphere contacts the cutting plane with F, and we'll call that point the focus of the parabola. Suppose P is an arbitrarily chosen point on the parabola. Then, let Q be the point on circle c such that $\overline{PQ}$ is tangent to the sphere. In other words Q is chosen so that $\overline{PQ}$ lies on the cone itself. Let L be the point on the directrix l such that $\overline{PL}$ is perpendicular to l. Then $PF = PQ$ since both segments are tangents to the sphere from the same point P. We can also show that $PQ = PL$. This follows from the fact that the cutting plane is parallel to one side of the cone. Consider the point P' that is the projection of P onto the plane containing circle C. Then $\angle PP'Q$ and $\angle PP'L$ are both right angles by the definition of a projection. $\angle PQP'$ and $\angle PLP'$ are both equal to the angle $90 - \theta$, where θ is the angle defined above, because the cutting plane and the cone both have an angle of θ with the horizon. Since they also share a side, triangles $\triangle PQP'$ and $\triangle PLP'$ are congruent by $\mathbf{AAS}$. So the corresponding sides $\overline{PQ}$ and $\overline{PL}$ are congruent. By the transitive property we have $PF = PL$, so the distance between the point P on the parabola to the focus is the same as the distance between P and the directrix l. We have just proven the focus-directrix property of parabolas.

Dandelin Spheres and Hyperbolas

To prove the focal property of hyperbolas, we examine Dandelin's sphere construction. Unlike the construction for ellipses, which used two spheres on one side of the cone, and the sphere construction for parabolas, which used one sphere on one side of the cone, this construction uses two spheres, one on each side of the cone. As with the ellipse construction, each sphere touches the plane at one of the foci of the **hyperbola**. And as with the argument for the elliptical focal property, the argument uses the fact that tangents from a common point to a sphere are equal.



In the above diagram, suppose P is an arbitrary point on the hyperbola. We would like to examine the difference $PF_2 - PF_1$. Let C_1 be the point on the upper sphere that lies on the line between P and the vertex of the cone. Let C_2 be the point on the upper sphere when this line is extended (so P, C_1, and C_2 are all on the same line and $PC_1 + C_1C_2 = PC_2$ and the cone By the common tangent property, $PF_1 = PC_1$ and $PF_2 = PC_2$ for some points C_1 and C_2 on the circles where the spheres meet the cone. So $PF_2 - PF_1 = PC_2 - PC_1 = (PC_1 + C_1C_2) - PC_1 = C_1C_2$. But C_1C_2 is the distance along the cone between the two circles of **tangency** and is constant regardless of the choice of C_1 and C_2. So the difference $PF_2 - PF_1$ is constant.





Examples

Example 1

What would the Dandelin spheres look like if used to explore a circle?

Solution

Since the spheres touch the figure within just at the foci, and since a circle is an ellipse with both foci at the same point, the spheres would sit directly above each other, and would touch the circle at the center point on both sides.

Example 2

Conceptually speaking, why is there only one Dandelin Sphere used in the proof of a parabola?

Solution

As we learned in the proof for an ellipse, the Dandelin sphere must be tangent to both the cone, e.g., the sphere touches the cone all the way around in a circle, and the plane, where the tangency is a single point. In the case of a parabola, the sphere below the plane "fits" just fine, but a sphere above the plane would "sit on" the plane, and not touch the cone all the way around. The figure only allows for one sphere to be tangent.

Example 3

Explain why for any positive number b and a, there exists a c such that $b^2 = c^2 - a^2$. Let $c = \sqrt{a^2 + b^2}$

Solution

Since $a^2 + b^2$ is always positive for positive a and b, this number is always defined.

Geometrically, let c be the hypotenuse of a right triangle with side lengths a and b.

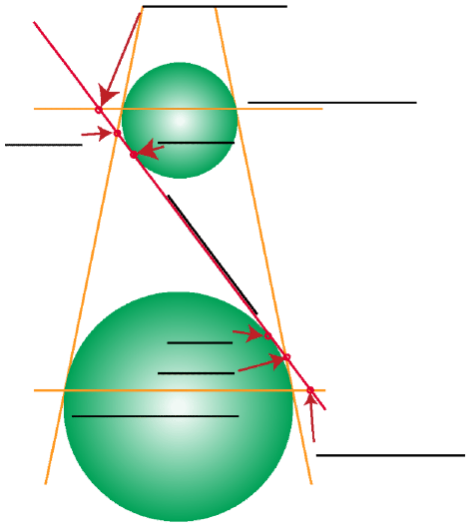
Review

1. Who was the first mathematician who conceptualized dandelin spheres? When?
2. What is proved by dandelin spheres?
3. How do you identify the foci of an ellipse using dandelin spheres?

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4. If two tangents are drawn from a single point to a sphere, what can you say about the line segments formed?
5. How do the tangents relate to a radius of a sphere?
6. Describe the focal property of ellipses.
7. How and when did Germinal Dandelin prove the focal property for parabolas?
8. What is the line that results from the intersection between the cutting plane and the plane that contains the circle of contact between the sphere and cone?
9. What is defined by the point where the sphere intersects the cutting plane?
10. Which construction uses two spheres in a single cone?
11. Which construction uses one sphere in a single cone?
12. Which construction uses two spheres and two cones?

Identify the parts listed on the diagram as specified below:



13. Directrix Line - Small Sphere
14. Directrix Line - Large Sphere
15. Focus - Small Sphere
16. Focus - Large Sphere
17. Vertex - Small Sphere
18. Vertex - Large Sphere
19. Directrix Plane - Small Sphere
20. Directrix Plane - Large Sphere
21. Cutting Plane
22. What conic section is illustrated here?

Review (Answers)

To see the Review answers, open this PDF file and look for section 6.8.

Vocabulary

Term	Definition
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
Dandelin Spheres	Dandelin spheres are spheres used to geometrically define the conic sections.
Ellipse	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
Ellipses	Ellipses are conic sections that look like elongated circles. An ellipse represents all locations in two dimensions that are the same distance from two specified points called foci.
hyperbola	A hyperbola is a conic section formed when the cutting plane intersects both sides of the cone, resulting in two infinite “U”-shaped curves.
Parabola	A parabola is the set of points that are equidistant from a fixed point on the interior of the curve, called the "'focus'", and a line on the exterior, called the "'directrix'". The directrix is vertical or horizontal, depending on the orientation of the parabola.
tangency	A tangency is a point (or set of points) which "just touch" a figure.

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6.5.2: Classifying Conic Sections

Classifying Conic Sections

You and your friends are playing Name the Conic Section. Your friend pulls a card with the equation $x^2 + 3xy = -5y^2 - 10$ written on it. What type of conic section is represented by the equation?

Classifying Conic Sections

Another way to classify a conic section when it is in the general form is to use the discriminant, like from the Quadratic Formula. The discriminant is what is underneath the radical, $b^2 - 4ac$, and we can use this to determine if the conic is a parabola, circle, ellipse, or hyperbola. If the general form of the equation is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$, then the discriminant will be $B^2 - 4AC$.

Use the table below:

$B^2 - 4AC = 0$ and $A = 0$ or $C = 0$	Parabola
$B^2 - 4AC < 0$ and $A = C$	Circle
$B^2 - 4AC < 0$ and $A \neq C$	Ellipse
$B^2 - 4AC > 0$	Hyperbola

Let's use the discriminant to determine the type of conic section for the following equations.

1. $x^2 - 4y^2 + 5x - 8y + 16 = 0$

   $A = 1$, $B = 0$, and $C = -4$

   $0^2 - 4(1)(-4) = 16$ This is a hyperbola.

2. $3x^2 + 3y^2 - 9x - 12y - 20 = 0$

   $A = 3$, $B = 0$, $C = 3$

   $0^2 - 4(3)(3) = -36$ Because $A = C$ and the discriminant is less than zero, this is a circle.

   Finally, let's use the discriminant to determine the type of conic. Then, we'll change the equation into standard form to verify our answer. We'll also find the center (or vertex, if it is a parabola).

   $x^2 + y^2 - 6x + 14y - 86 = 0$

   $A = 1$, $B = 0$, $C = 1$ This is a circle.

   $$(x^2 - 6x + 9) + (y^2 + 14y + 49) = 86 + 49 + 9$$
   $$(x - 3)^2 + (y + 7)^2 = 144$$

   The center is $(3, -7)$.

Examples

Example 1

Earlier, you were asked to determine the type of conic section represented by the equation $x^2 + 3xy = -5y^2 - 10$.

Solution

First we need to rewrite the equation is standard form.

$x^2 + 3xy = -5y^2 - 10 \quad x^2 + 3xy + 5y^2 + 10 = 0$

Now we can use the discriminant to find the type of conic section represented by the equation.

$A = 1, \ B = 3, \ C = 5$

$3^2 - 4(1)(5) = -11$ Because $A \neq C$ and the discriminant is less than zero, this equation represents an ellipse.

For Examples 2 & 3, use the discriminant to determine the type of conic.

Example 2

$2x^2 + 5y^2 - 8x + 25y + 115 = 0$

Solution

$0^2 - 4(2)(5) = -40$, this is an ellipse.

Example 3

$5y^2 - 9x - 10y - 14 = 0$

Solution

$0^2 - 4(0)(5) = 0$, this is a parabola.

Example 4

Use the discriminant to determine the type of conic. Then, change the equation into standard form to verify your answer. Find the center or vertex, if it is a parabola.

Solution

$-4x^2 + 3y^2 - 8x + 24y + 32 = 0$

$0^2 - 4(-4)(3) = 48$, this is a hyperbola. Changing it to standard form, we have:

$$\begin{aligned} (-4x^2 - 8x) + (3y^2 + 24y) &= -32 \\ -4\left(x^2 + 2x + 1\right) + 3\left(y^2 + 8y + 16\right) &= -32 + 48 - 4 \\ -4(x+1)^2 + 3(y+4)^2 &= 12 \\ \frac{-(x+1)^2}{3} + \frac{(y+4)^2}{4} &= 1 \end{aligned}$$

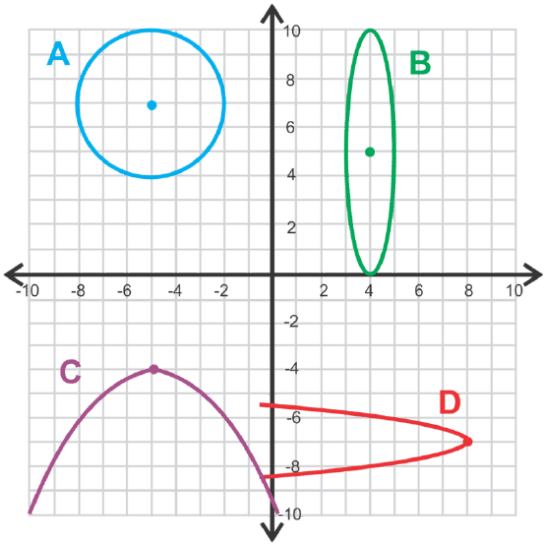
Usually, we write the negative term second, so the equation is $\frac{(y+4)^2}{4} - \frac{(x+1)^2}{3} = 1$. The center is $(-1, -4)$.

Review

Use the discriminant to determine the type of conic each equation represents.

1. $2x^2 + 2y^2 + 16x - 8y + 25 = 0$
2. $x^2 - y^2 - 2x + 5y - 12 = 0$
3. $6x^2 + y^2 - 12x + 7y + 35 = 0$
4. $3x^2 - 15x + 9y - 18 = 0$
5. $10y^2 + 6x - 40y + 253 = 0$
6. $4x^2 + 4y^2 + 32x + 48y + 465 = 0$

Match the equation with the correct graph.



7. $x^2 + 10x + 4y + 41 = 0$
8. $4y^2 + x + 56y + 188 = 0$
9. $x^2 + y^2 + 10x - 14y + 65 = 0$
10. $25x^2 + y^2 - 200x - 10y + 400 = 0$

Use the discriminant to determine the type of conic. Then, change the equation into standard form to verify your answer. Find the center or vertex, if it is a parabola.

11. $x^2 - 12x + 6y + 66 = 0$
12. $x^2 + y^2 + 2x + 2y - 2 = 0$
13. $x^2 - y^2 - 10x - 10y - 10 = 0$
14. $y^2 - 10x + 8y + 46 = 0$
15. **Find the Area of an Ellipse** Graph $x^2 + y^2 = 36$ and find its area.
    a. Then, graph $\frac{x^2}{36} + \frac{y^2}{25} = 1$ and $\frac{x^2}{25} + \frac{y^2}{36} = 1$ on the same axes.
    b. Do these ellipses have the same area? Why or why not?
    c. If the equation of the area of a circle is $A = \pi r^2$, what do you think the area of an ellipse is? Use a and b as in the standard form, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
    d. Find the areas of the ellipses from part a. Are the areas more or less than the area of the circle? Why or why not?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.11.

Image Attributions

1. [Figure 1]
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   Source: https://commons.wikimedia.org/wiki/File:Conic_sections_with_plane.svg;GeoGebra

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6.5.3: Degenerate Conics

Degenerate Conics

The general equation of a **conic** is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. This form is so general that it encompasses all regular lines, singular points and degenerate hyperbolas that look like an X. This is because there are a few special cases of how a plane can intersect a two sided cone. How are these degenerate shapes formed?

Graphing Degenerate Conics

A **degenerate conic** is a conic that does not have the usual properties of a conic. Degenerate conic equations simply cannot be written in graphing form. There are three types of degenerate conics:

1. **A singular point**, which is of the form: $\frac{(x-h)^2}{a} + \frac{(y-k)^2}{b} = 0$. You can think of a singular point as a circle or an ellipse with an infinitely small radius.
2. **A line**, which has coefficients $A = B = C = 0$ in the general equation of a conic. The remaining portion of the equation is $Dx + Ey + F = 0$, which is a line.
3. **A degenerate hyperbola**, which is of the form: $\frac{(x-h)^2}{a} - \frac{(y-k)^2}{b} = 0$. The result is two intersecting lines that make an "X" shape. The slopes of the intersecting lines forming the X are $\pm \frac{b}{a}$. This is because b goes with the y portion of the equation and is the rise, while a goes with the x portion of the equation and is the run.

Examples

Example 1

Earlier, you were asked how degenerate conics are formed.

Solution

When you intersect a plane with a two sided cone where the two cones touch, the intersection is a single point. When you intersect a plane with a two sided cone so that the plane touches the edge of one cone, passes through the central point and continues touching the edge of the other conic, this produces a line. When you intersect a plane with a two sided cone so that the plane passes vertically through the central point of the two cones, it produces a degenerate hyperbola.

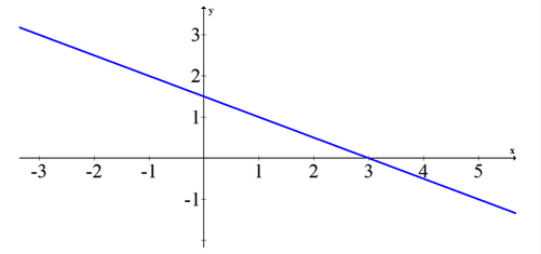
Example 2

Transform the conic equation into standard form and sketch.

$0x^2 + 0xy + 0y^2 + 2x + 4y - 6 = 0$

Solution

This is the line $y = -\frac{1}{2}x + \frac{3}{2}$



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Example 3

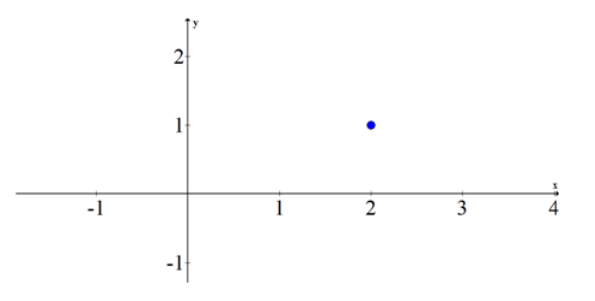
Transform the conic equation into standard form and sketch.

$3x^2 - 12x + 4y^2 - 8y + 16 = 0$

Solution

$$\begin{aligned} 3x^2 - 12x + 4y^2 - 8y + 16 &= 0 \\ 3\left(x^2 - 4x\right) + 4\left(y^2 - 2y\right) &= -16 \\ 3\left(x^2 - 4x + 4\right) + 4\left(y^2 - 2y + 1\right) &= -16 + 12 + 4 \\ 3(x-2)^2 + 4(y-1)^2 &= 0 \\ \frac{(x-2)^2}{4} + \frac{(y-1)^2}{3} &= 0 \end{aligned}$$

The point (2, 1) is the result of this degenerate conic.



Example 4

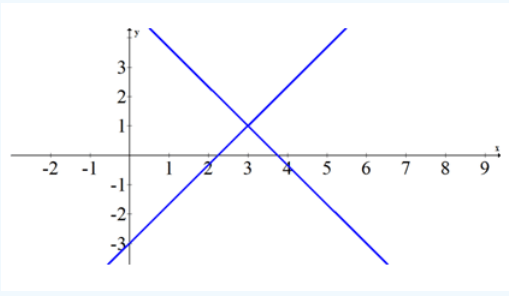
Transform the conic equation into standard form and sketch.

$16x^2 - 96x - 9y^2 + 18y + 135 = 0$

Solution

$$\begin{aligned} 16x^2 - 96x - 9y^2 + 18y + 135 &= 0 \\ 16\left(x^2 - 6x\right) - 9\left(y^2 - 2y\right) &= -135 \\ 16\left(x^2 - 6x + 9\right) - 9\left(y^2 - 2y + 1\right) &= -135 + 144 - 9 \\ 16(x-3)^2 - 9(y-1)^2 &= 0 \\ \frac{(x-3)^2}{9} - \frac{(y-1)^2}{16} &= 0 \end{aligned}$$

This is a degenerate hyperbola.



Example 5

1. Create a conic that describes just the point (4, 7).

Solution

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$(x-4)^2+(y-7)^2=0$

Review

1. What are the three degenerate conics?

Change each equation into graphing form and state what type of conic or degenerate conic it is.

2. $x^2-6x-9y^2-54y-72=0$
3. $4x^2+16x-9y^2+18y-29=0$
4. $9x^2+36x+4y^2-24y+72=0$
5. $9x^2+36x+4y^2-24y+36=0$
6. $0x^2+5x+0y^2-2y+1=0$
7. $x^2+4x-y+8=0$
8. $x^2-2x+y^2-6y+6=0$
9. $x^2-2x-4y^2+24y-35=0$
10. $x^2-2x+4y^2-24y+33=0$

Sketch each conic or degenerate conic.

11. $\frac{(x+2)^2}{4}+\frac{(y-3)^2}{9}=0$
12. $\frac{(x-3)^2}{9}+\frac{(y+3)^2}{16}=1$
13. $\frac{(x+2)^2}{9}-\frac{(y-1)^2}{4}=1$
14. $\frac{(x-3)^2}{9}-\frac{(y+3)^2}{4}=0$
15. $3x+4y=12$

Review (Answers)

To see the Review answers, open this PDF file and look for section 9.6.

Vocabulary

Term	Definition
Conic	Conic sections are those curves that can be created by the intersection of a double cone and a plane. They include circles, ellipses, parabolas, and hyperbolas.
degenerate conic	A degenerate conic is a conic that does not have the usual properties of a conic section. Since some of the coefficients of the general conic equation are zero, the basic shape of the conic is merely a point, a line or a pair of intersecting lines.
degenerate hyperbola	A degenerate hyperbola is an example of a degenerate conic. Its equation takes the form $\frac{(x-h)^2}{a}-\frac{(y-k)^2}{b}=0$. It looks like two intersecting lines that make an “X” shape.

Image Attributions

1. [Figure 1]
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6.5.4: Solving Systems of Lines, Quadratics, and Conics

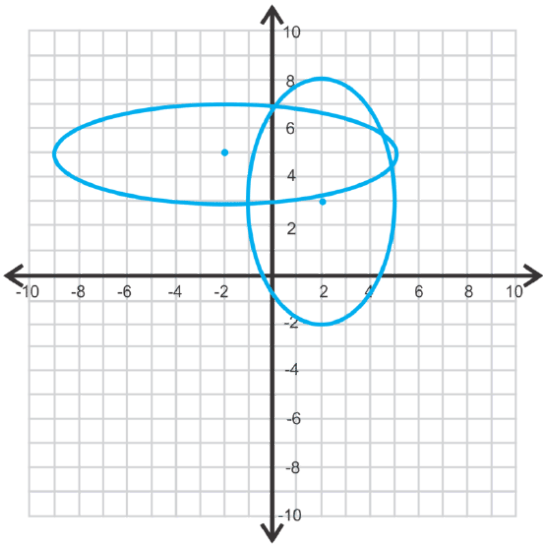
Solving Systems of Lines, Quadratics, and Conics

You are given the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ and the line $y=\frac{3}{2}x+3$. You want to determine at which point(s), if any, the two equations intersect without graphing. Does the line intersect the ellipse? If so, at which point(s) does it do so?

Systems of Lines, Quadratics and Conics

Previously, we solved a system involving two lines or three planes, by using graphing, substitution, and linear combinations. In this concept, we will add circles, parabolas, and ellipses to systems of equations. Because both x and y can be squared in these equations, there will often be more than one answer.

Let's estimate the solutions for the system of equations below.



These two ellipses intersect in four places. They look to be the following points:

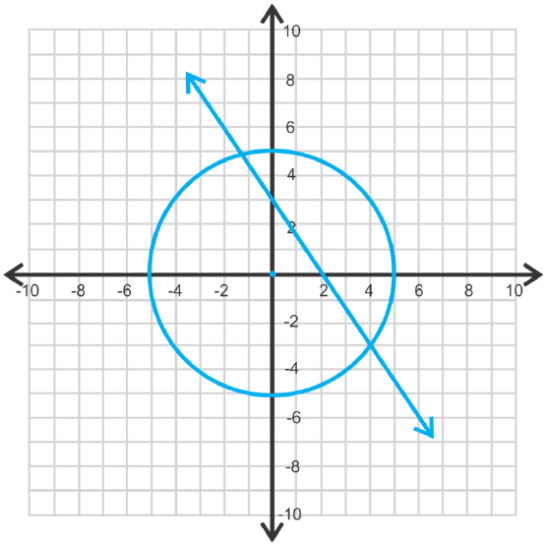
(0, 7), (4.7, 5.5), (4.9, 4.3), and (−1, 2.9)

Keep in mind these are only estimates. In the next problem, we will show how to find the exact answers.

Now, let's solve

$$x^2+y^2=25$$
$$3x+2y=6$$

Let’s solve this system by graphing. The first equation is a circle, centered at the origin, with a radius of 5. The second equation is a line. In slope intercept form, it is $y=-\frac{3}{2}x+3$.



Next, let's estimate where the line and circle intersect. In the second quadrant, it looks like (−1.2, 4.8) and in the fourth quadrant it looks like (4, −3). These are our estimated answers.

To find the exact value of these points of intersection, we need to use substitution. Substitute in the linear equation in for y into the circle equation and solve for x.

$$x^2+\left(-\frac{3}{2}x+3\right)^2=25$$
$$x^2+\frac{9}{4}x^2-9x+9=25$$
$$\frac{13}{4}x^2-9x-16=0$$
$$13x^2-36x-64=0$$

Use the Quadratic Formula:

$$x=\frac{36\pm\sqrt{36^2-4(13)(-64)}}{2(13)}$$
$$=\frac{36\pm\sqrt{4624}}{26}$$
$$=\frac{36\pm 68}{26}$$

The solutions, for x, are $\frac{36+68}{26}=4$ and $\frac{36-68}{26}=-1\frac{3}{13}$. Plug these into either equation to solve for y.

$y=-\frac{3}{2}(4)+3=-3$ and $y=-\frac{3}{2}\left(-\frac{16}{13}\right)+3=4\frac{11}{13}$

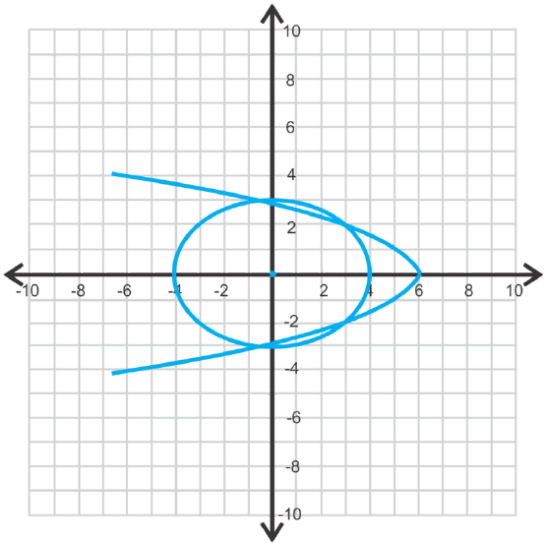
The points are (4, −3) and $\left(-1\frac{3}{13},4\frac{11}{13}\right)$.

The technique used in this problem is how it is recommended you approach each problem. First, graph the system so that you have an idea of how many solutions there are and where they are located. Then, use substitution to solve for the exact answers.

Finally, let's solve

$\frac{x^2}{16}+\frac{y^2}{9}=1$
$y^2=-\frac{4}{3}(x-6)$

Graphing the two equations, we have four points of intersection below. The second equation is solved for y^2, so substitute that into the first equation.



$$\frac{x^2}{16} - \frac{4(x-6)}{3 \cdot 9} = 1$$
$$\frac{x^2}{16} - \frac{4x-24}{27} = 1$$
$$27x^2 - 16(4x-24) = 432$$
$$27x^2 - 64x - 48 = 0$$

Now, use the Quadratic Formula to solve for x.

$$x = \frac{64 \pm \sqrt{(-64)^2 - 4(27)(-48)}}{2(27)}$$
$$= \frac{64 \pm \sqrt{9280}}{54}$$
$$= \frac{32 \pm 4\sqrt{145}}{27}$$

Plugging these into the calculator we get $x = \frac{32+4\sqrt{145}}{27} \approx 2.97$ and $x = \frac{32-4\sqrt{145}}{27} \approx -0.6$. Looking at the graph, we know that there will be two different y-values for each x-value to give four points of intersection. Using the estimations, solve for y. You can choose either equation.

$$y^2 = -\tfrac{4}{3}(2.97-6) \qquad y^2 = -\tfrac{4}{3}(-0.6-6)$$
$$y^2 = 4.04 \quad \text{and} \quad y^2 = 8.8$$
$$y = \pm 2.01 \qquad y = \pm 2.97$$

The points are (2.97, 2.01), (2.97, −2.01), (−0.6, 2.97), and (−0.6, −2.97).

Examples

Example 1

Earlier, you were asked to determine if the line $y = \frac{3}{2}x + 3$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, and if so, at which point(s).

Solution

First, let's get rid of the fractions in the equation of the ellipse to make it easier to work with. To do so, we multiply by the LCD.

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$
$$36\frac{x^2}{4} + 36\frac{y^2}{9} = 36 \cdot 1$$
$$9x^2 + 4y^2 = 36$$

Now we can substitute the equation of the line $y = \frac{3}{2}x + 3$ in for y and solve for x.

$$9x^2 + 4\left(\tfrac{3}{2}x + 3\right)^2 = 36$$
$$9x^2 + 4\left(\tfrac{9}{4}x^2 + 9x + 9\right) = 36$$
$$9x^2 + 9x^2 + 36x + 36 = 36$$
$$18x^2 + 36x = 0$$
$$18x(x + 2) = 0$$

So, $x = 0$ or $x = -2$

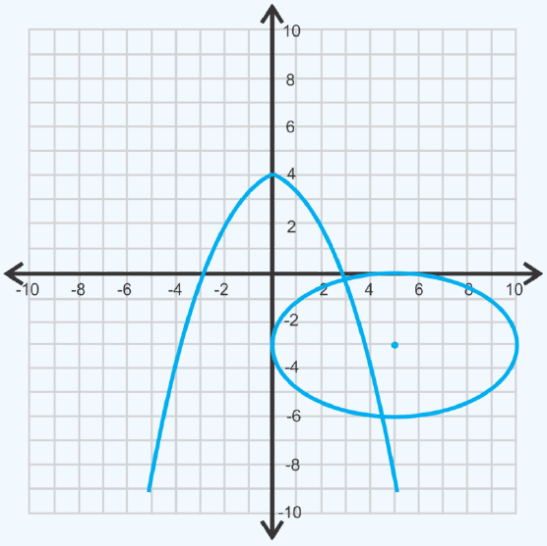
Finally we can substitute these *x* values into the equation of the line to find the corresponding *y* values.

$y = \frac{3}{2}(0) + 3 = 3$
$y = \frac{3}{2}(-2) + 3 = 0$

Therefore, the line intersects the ellipse at points $(0, 3)$ and $(-2, 0)$.

Example 2

Estimate the solutions to the system below.



Solution

(3, −0.1) and (4.5, −6)

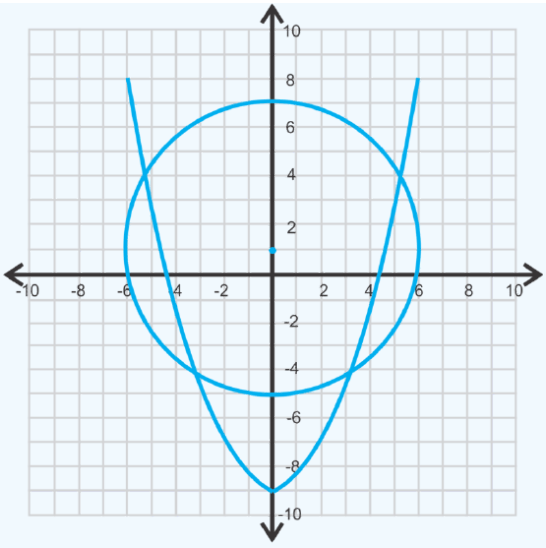
Find the solutions to the systems below.

Example 3

$x^2 + (y - 1)^2 = 36$
$x^2 = 2(y + 9)$

Solution

This is a circle and a parabola that intersects in four places.



Using substitution for x^2, we have:

$$\begin{aligned} 2(y+9)+(y-1)^2 &= 36 \\ 2y+18+y^2-2y+1 &= 36 \\ y^2 &= 17 \\ y &= \pm\sqrt{17} \approx \pm 4.12 \end{aligned}$$

The corresponding x-values are:

$$\begin{aligned} x^2 &= 2(4.12+9) \\ x^2 &= 26.25 \\ x &= \pm 5.12 \end{aligned} \quad \text{and} \quad \begin{aligned} x^2 &= 2(-4.12+9) \\ x^2 &= 9.76 \\ x &= \pm 3.12 \end{aligned}$$

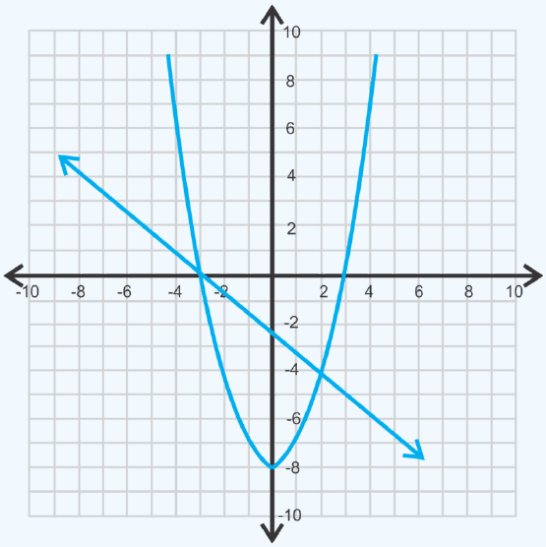
The solutions are: (4.12, 5.12), (4.12, −5.12), (−4.12, 3.12) and (−4.12, −3.12).

Example 4

$$\begin{aligned} x^2 &= y+8 \\ 4x+5y &= 12 \end{aligned}$$

Solution

This is a line and a parabola that intersect in two points.



Solve the first equation for y and substitute into the second.

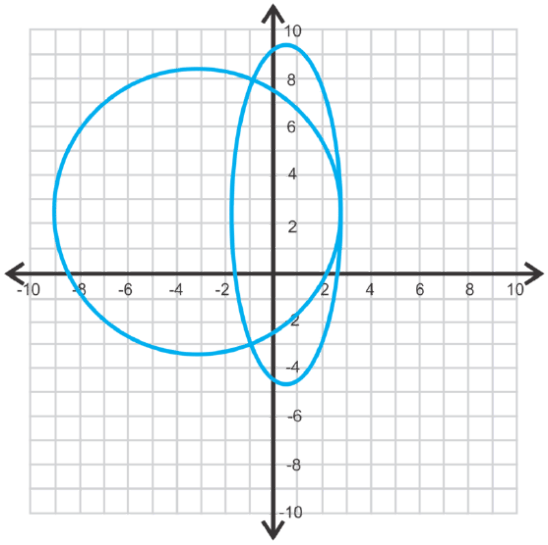
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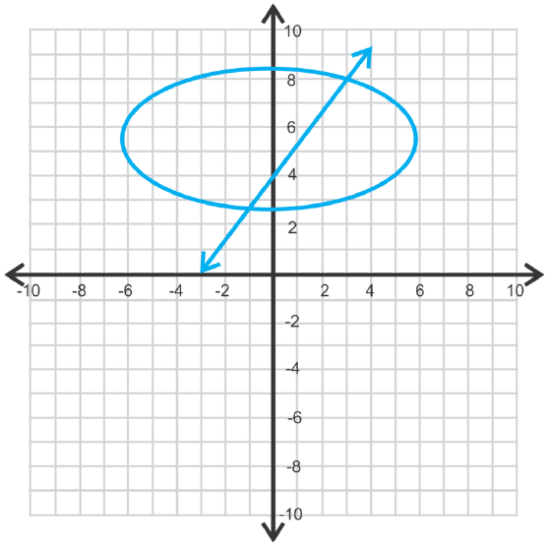
$$4x + 5\left(x^2 - 8\right) = 12$$
$$4x + 5x^2 - 40 = 12$$
$$5x^2 + 4x - 52 = 0$$
$$x = \frac{-4 \pm \sqrt{4^2 - 4(5)(-52)}}{2(5)}$$
$$x = \frac{-4 \pm \sqrt{1056}}{10} \approx 3.65, 2.85$$

Using the first equation, $y = 3.65^2 - 8 = 5.32$ and $y = 2.85^2 - 8 = 0.12$. The points are $(3.65, 5.32)$ and $(2.85, 0.12)$.

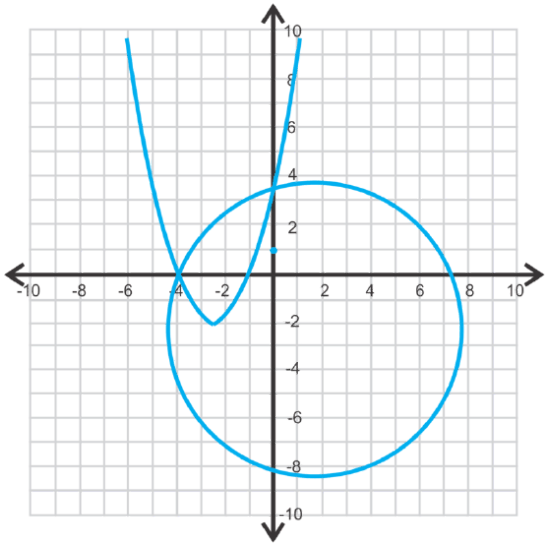
Review

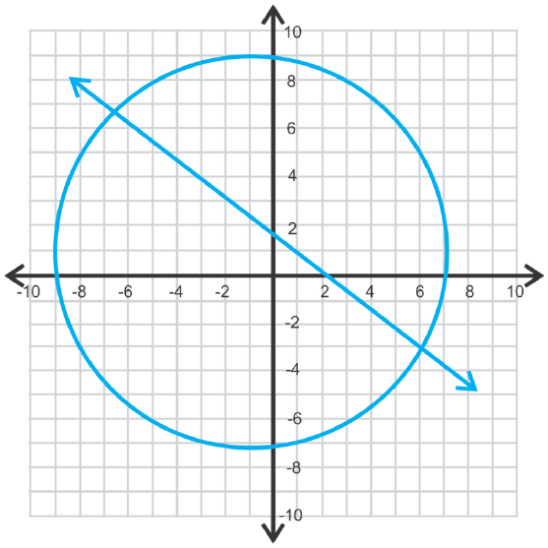
Estimate the solutions for each system of equations below.

1. 

2. 

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3. 

4. 

Solve each system of equations below.

5. $5x^2 + 3y = 17$
   $x - y = 1$
6. $x^2 + y^2 = 7.5$
   $x + 2y = 6$
7. $x^2 = y + 4$
   $\frac{x^2}{4} + (y+2)^2 = 1$
8. $(x-1)^2 + (y-3)^2 = 25$
   $x^2 = -2(y-10)$
9. $x^2 + y^2 = 16$
   $4x - 3y = 18$
10. $(x+4)^2 + (y+1)^2 = 36$
    $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{25} = 1$
11. How many different ways can a circle and a parabola intersect? Draw each possibility.
12. How many different ways can a circle and an ellipse intersect? Draw each possibility.
13. Create a system of two circles with no solution. What would the graph look like?
14. **Challenge** Find the solutions for the system

    $x^2 + y^2 = r^2$
    $y = mx$

Leave your answers in terms of m and r.

15. **Challenge** Determine if the system of three equations below have one common solution.

$$x^2 + 3y^2 = 16$$
$$3x^2 + y^2 = 16$$
$$y = x$$

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 10.12.

Image Attributions

1. [Figure 1]
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CHAPTER OVERVIEW

Unit 7: Sequences, Series, and Mathematical Induction

7.1: Formulas and Notation for Sequences and Series

7.1.1: Recursive Formulas

7.1.2: Explicit Formulas

7.1.3: Sum Notation and Properties of Sigma

7.2: Series Sums

7.2.1: Partial Sums

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7.3.1: Inductive Reasoning from Patterns

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7.7: Geometric Sequences

7.7.1: Finding the nth Term Given the Common Ratio and the First Term

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7.8: Sums of Arithmetic Series

7.8.1: Sums of Finite Arithmetic Series

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SECTION OVERVIEW

7.1: Formulas and Notation for Sequences and Series

Topic hierarchy

7.1.1: Recursive Formulas

7.1.2: Explicit Formulas

7.1.3: Sum Notation and Properties of Sigma

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7.1.1: Recursive Formulas

Describing the Pattern and Writing a Recursive Rule for a Sequence

In 2013, the days a full moon appeared were in the following **sequence** (with Jan. 1 being Day 1). Write a **recursive formula** for the sequence.

9, 38, 67, 96, ...

Source: Moongiant

Recursive Rule

A **recursive rule** for a sequence is a formula which tells us how to progress from one term to the next in a sequence. Generally, the variable n is used to represent the term number. In other words, n takes on the values 1 (first term), 2 (second term), 3 (third term), etc. The variable, a_n represents the nth term and a_{n-1} represents the term preceding a_n.

Example sequence: $4, 7, 11, 16, \ldots, a_{n-1}, a_n$

In the above sequence, $a_1 = 4$, $a_2 = 7$, $a_3 = 11$ and $a_4 = 16$.

Let's describe the pattern and write a recursive rule for the sequence: 9,11,13,15,…

First we need to determine what the pattern is in the sequence. If we subtract each term from the one following it, we see that there is a **common difference** of 29. We can therefore use a_{n-1} and a_n to write a recursive rule as follows: $a_n = a_{n-1} + 29$

Now, let's write a recursive rule for the following sequences.

1. 3,9,27,81,…

   In this sequence, each term is multiplied by 3 to get the next term. We can write a recursive rule: $a_n = 3a_{n-1}$

2. 1,1,2,3,5,8,…

   This is a special sequence called the Fibonacci sequence. In this sequence each term is the sum of the previous two terms. We can write the recursive rule for this sequence as follows: $a_n = a_{n-2} + a_{n-1}$.

Examples

Example 1

Earlier, you were asked to write a recursive formula for the sequence 9, 38, 67, 96, ...

Solution

First we need to determine what pattern the sequence is following. If we subtract each term from the one following it, we find that there is a common difference of 29. We can therefore use a_{n-1} and a_n to write a recursive rule as follows: $a_n = a_{n-1} + 29$

Write the recursive rules for the following sequences.

Example 2

1, 2, 4, 8, …

Solution

In this sequence each term is double the previous term so the recursive rule is: $a_n = 2a_{n-1}$

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Example 3

1, −2, −5, −8, …

Solution

This time three is subtracted each time to get the next term: $a_n = a_{n-1} - 3$

Example 4

1, 2, 4, 7, …

Solution

This one is a little trickier to express. Try looking at each term as shown below:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= a_1 + 1 \\ a_3 &= a_2 + 2 \\ a_4 &= a_3 + 3 \\ &\vdots \\ a_n &= a_{n-1} + (n-1) \end{aligned}$$

Review

Describe the pattern and write a recursive rule for the following sequences.

1. $\frac{1}{4}, -\frac{1}{2}, 1, -2$ …
2. 5, 11, 17, 23, …
3. 33, 28, 23, 18, …
4. 1, 4, 16, 64, …
5. 21, 30, 39, 48, …
6. 100, 75, 50, 25, …
7. 243, 162, 108, 72, …
8. 128, 96, 72, 54, …
9. 1, 5, 10, 16, 23, …
10. 0, 2, 2, 4, 6, …
11. 3, 5, 8, 12, …
12. 0, 2, 6, 12, …
13. 4, 9, 14, 19, …
14. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ …
15. 4, 5, 9, 14, 23, …

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.2.

Vocabulary

Term	Definition
common difference	Every arithmetic sequence has a common or constant difference between consecutive terms. For example: In the sequence 5, 8, 11, 14..., the common difference is "3".
common ratio	Every geometric sequence has a common ratio, or a constant ratio between consecutive terms. For example in the sequence 2, 6, 18, 54..., the common ratio is 3.
index	The index of a term in a sequence is the term's "place" in the sequence.
recursive	The recursive formula for a sequence allows you to find the value of the nth term in the sequence if you know the value of the $(n-1)^{th}$ term in the sequence.
recursive formula	The recursive formula for a sequence allows you to find the value of the nth term in the sequence if you know the value of the $(n-1)^{th}$ term in the sequence.
sequence	A sequence is an ordered list of numbers or objects.

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7.1.2: Explicit Formulas

Explicit Formulas

Rachel and Elaina have started a website where they debate the best color of hair dye. The show is really popular and the visitors to their website are increasing very rapidly. They figure that membership is increasing by about 500 people every three days.

At this rate, how many members will they have on the 48th day? How many days will it be before they reach 25,000 members?

Explicit Formulas

When we represent a **sequence** with a formula that lets us find any term in the sequence without knowing any other terms, we are representing the sequence explicitly.

Given a **recursive** definition of an arithmetic or geometric sequence, you can always find an **explicit formula**, or an equation to represent the nth term of the sequence. Consider for example the sequence of odd numbers we started with: 1, 3, 5, 7, ...

We can find an explicit formula for the nth term of the sequence if we analyze a few terms:

$$a_1 = 1$$
$$a_2 = a_1 + 2 = 1 + 2 = 3$$
$$a_3 = a_2 + 2 = 1 + 2 + 2 = 5$$
$$a_4 = a_3 + 2 = 1 + 2 + 2 + 2 = 7$$
$$a_5 = a_4 + 2 = 1 + 2 + 2 + 2 + 2 = 9$$
$$a_6 = a_5 + 2 = 1 + 2 + 2 + 2 + 2 + 2 = 11$$

Note that every term is made up of a 1, and a set of 2's. How many 2's are in each term?

a_1	= 1
a_2	= 1 + 2 = 3
a_3	= 1 + 2 × 2 = 5
a_4	= 1 + 3 × 2 = 7
a_5	= 1 + 4 × 2 = 9
a_6	= 1 + 5 × 2 = 11

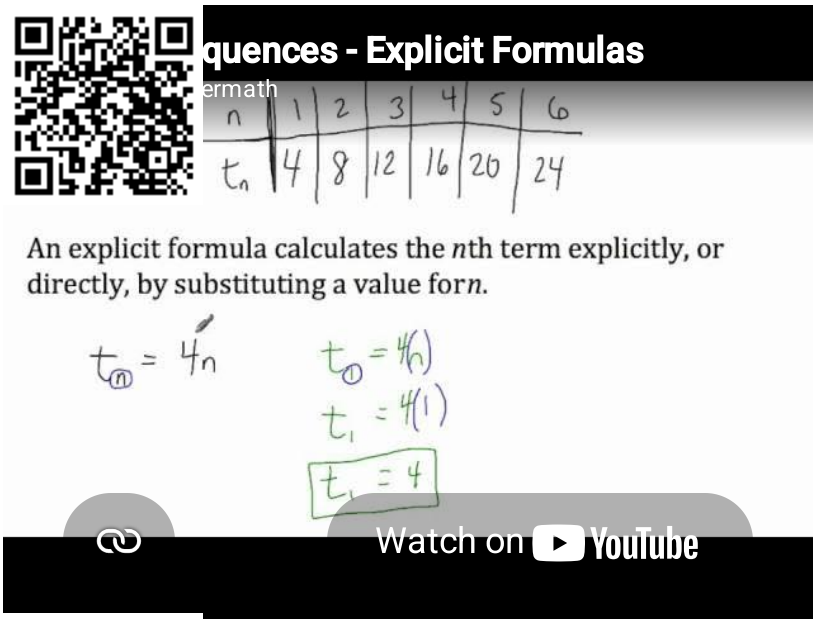
The nth term has (n - 1) 2's. For example, a_{99} = 1 + 98 × 2 = 197 . We can therefore represent the sequence as a_n = 1 + 2(n - 1). We can simplify this expression:

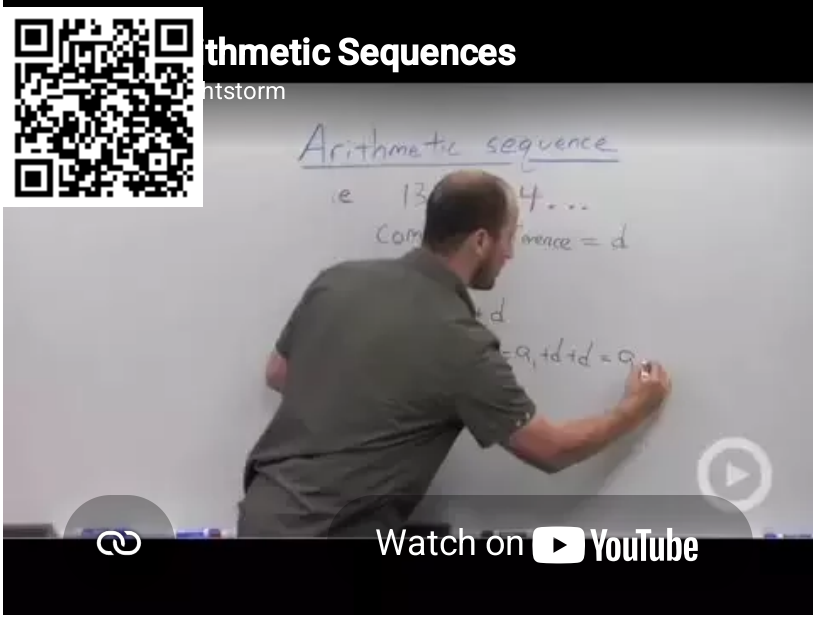
a_n	= 1 + 2(n - 1)
a_n	= 1 + 2n - 2
a_n	= 2n - 1

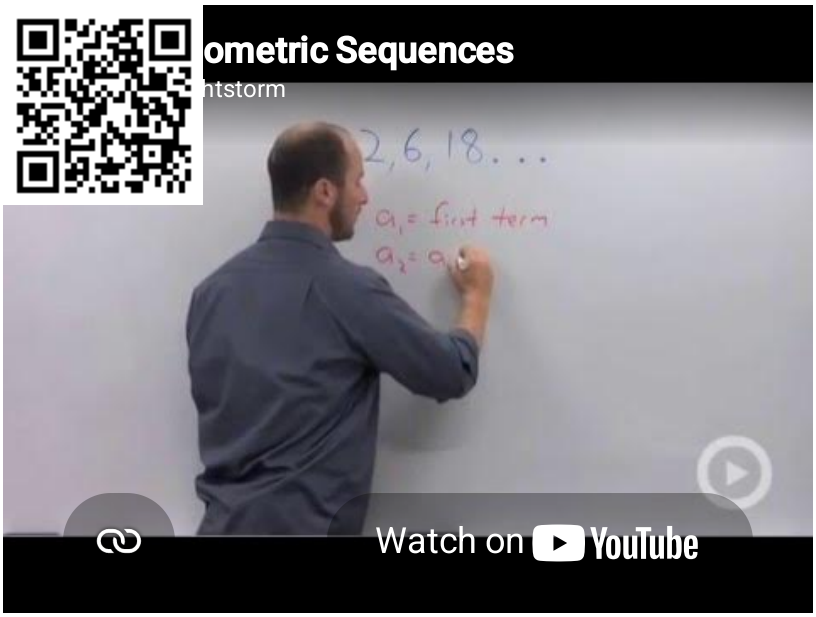
In general, we can represent an **arithmetic sequence** in this way, as long as we know the first term and the **common difference**, d. Notice that in the previous example, the first term was 1, and the common difference, d, was 2. The nth term is therefore the first term, plus d(n - 1):

a_n	= a_1 + d(n - 1)

You can use this general equation to find an explicit formula for any term in an arithmetic sequence.


quences - Explicit Formulas
ermath
n | 1 | 2 | 3 | 4 | 5 | 6
t_n | 4 | 8 | 12 | 16 | 20 | 24
An explicit formula calculates the nth term explicitly, or directly, by substituting a value for n.
$t_n = 4n$
$t_0 = 4(n)$
$t_1 = 4(1)$
$t_1 = 4$
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Arithmetic sequence
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Examples

Example 1

Earlier, you were asked two questions about a website membership.

If memberships are increasing by about 500 people every three days, how many members will they have on the 48th day? How many days will it be before they reach 25,000 members?

Solution

This is actually a fairly simple arithmetic sequence: each day there are 500/3 more members, on average. Use the formula for arithmetic sequences from the Example 2 below.

Example 2

Find an explicit formula for the nth term of the sequence 3, 7, 11, 15... and use the equation to find the 50th term in the sequence.

Solution

$a_n = 4n - 1$, and $a_{50} = 199$

The first term of the sequence is 3, and the common difference is 4.

a_n	$= a_1 + d(n - 1)$
a_n	$= 3 + 4(n - 1)$
a_n	$= 3 + 4n - 4$
a_n	$= 4n - 1$
a_{50}	$= 4(50) - 1 = 200 - 1 = 199$

We can also find an explicit formula for a **geometric sequence**. Consider the following sequence:

		$t_2 = 2t_1 = 2 \times 3 = 6$
$t_1 = 3$	→	$t_3 = 2t_2 = 2 \times 6 = 12$
$t_n = 2 \times t_{n-1}$		$t_4 = 2t_3 = 2 \times 12 = 24$
		$t_5 = 2t_4 = 2 \times 24 = 48$

Notice that every term is the first term, multiplied by a power of 2. This is because 2 is the **common ratio** for the sequence.

t_1	$= 3$
t_2	$= 2 \times 3 = 6$
t_3	$= 2 \times 2 \times 6 = 2^2 \times 6 = 12$
t_4	$= 2 \times 2 \times 2 \times 6 = 2^3 \times 6 = 24$
t_5	$= 2 \times 2 \times 2 \times 2 \times 6 = 2^4 \times 6 = 48$

The power of 2 in the nth term is $(n\text{-}1)$. Therefore the nth term in this sequence can be defined as: $t_n = 3(2^{n-1})$. In general, we can define the nth term of a geometric sequence in terms of its first term and its common ratio, r:

t_n	$= t_1(r^{n-1})$

You can use this general equation to find an explicit formula for any term in a geometric sequence.

Example 3

Find an explicit formula for the nth term of the sequence 5, 15, 45, 135... and use the equation to find the 10th term in the sequence.

Solution

$a_n = 5 \times 3^{n-1}$, and $a_{10} = 98{,}415$

The first term in the sequence is 5, and $r = 3$.

a_n	$= a^1 \times r^{n-1}$
a_n	$= 5 \times 3^{n-1}$
a_{10}	$= 5 \times 3^{10-1}$
a_{10}	$= 5 \times 3^9 = 5 \times 19{,}683 = 98{,}415$

Again, it is always possible to write an explicit formula for terms of an arithmetic or geometric sequence. However, you can also write an explicit formula for other sequences, as long as you can identify a pattern. To do this, you must remember that a sequence is a function, which means there is a relationship between the input and the output. That is, you must identify a pattern between the term and its **index**, or the term's "place" in the sequence.

Example 4

Write an explicit formula for the nth term of the sequence 1, (1/2), (1/3), (1/4)...

Solution

$a_n = (1/n)$

Initially you may see a pattern in the fractions, but you may also wonder about the first term. If you write 1 as (1/1), then it should become clear that the nth term is (1/n).

Example 5

Write an explicit formula for the sequence: 2, 9, 16... and use the formula to find the value of the 20th term.

Solution

For the sequence: 2, 9, 16...

$$a_n = 7n - 5$$
$$\therefore a_{20} = 7(20) - 5$$
$$a_{20} = 135$$

Example 6

Write an explicit formula for the sequence: (1/2), (1/4), (1/8) and use the formula to find the value of the 7_{th} term.

Solution

For the sequence: (1/2), (1/4), (1/8)...

$$a_n = \frac{1}{2^n}$$
$$\therefore a_7 = \frac{1}{2^7}$$
$$a_7 = \frac{1}{128}$$

Example 7

Identify all sequences in the previous two examples that are geometric. What is the common ratio in each sequence?

Solution

The sequence in Example 5 is arithmetic.

The sequence in Example 6 is geometric and has r = 1/2.



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Review

Name the sequence as arithmetic, geometric, or neither.

1. −21, −6, 18, −3, 20, −2
2. $0, \frac{-1}{5}, \frac{-2}{5}, \frac{-3}{5}, \frac{-4}{5}, -1$
3. 1, 3, 9, 27, 81, 243
4. 2, 9, −2, 1, 18, 2

Write the first 5 terms of the arithmetic sequence (explicit).

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5. $a_n = -8 - 9(n-1)$
6. $a_n = 6 - \frac{2}{3}(n-1)$
7. $a_n = 8 + \frac{1}{3}(n-1)$

Solve the following:

8. What are the first five terms of the sequence? $a_n = a_{n-1} - \frac{10}{3}; a_1 = -6$
9. Given the sequence, write a recursive function to generate it: 2, −4, −10, −16, −22, −28
10. Write the equation of a_n without using recursion: $a_n = a_{n-1} - \frac{3}{2}; a_1 = 10$
11. Write as a recursion: $a_n = 6 - \frac{5}{3}(n-1)$
12. Write the equation of a_n without using recursion: $a_n = a_{n-1} + 8; a_1 = 3$
13. What are the first five terms of the sequence? $a_n = a_{n-1} - 1; a_1 = -5$

Write an explicit formula for the n_{th} term of the arithmetic sequence.

14. $-7, \frac{-13}{3}, \frac{-5}{3}, 1, \frac{11}{3}, \frac{19}{3}$
15. 6, −4, −14, −24, −34, −44
16. 9, 16, 23, 30, 37, 44
17. In a particular arithmetic sequence, the second term is 4 and the fifth term is 13. Write an explicit formula for this sequence.

Write the first 5 terms of the geometric sequence.

18. $a_n = 5(-3)^{(n-1)}$
19. $a_n = -6\left(\frac{-10^{(n-1)}}{3}\right)$

Write the explicit formula for the n_{th} term of the geometric sequence.

20. −8, 16, −32, 64, −128, 256
21. $9, \frac{27}{2}, \frac{81}{4}, \frac{243}{8}, \frac{729}{16}, \frac{2187}{32}$

Convert the explicit formula to a **recursive formula**.

22. $a_n = 9\left(\frac{-4}{3}\right)^{(n-1)}$
23. $a_n = -6(-4)^{(n-1)}$
24. $a_n = -5(5)^{(n-1)}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.2.

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Vocabulary

Term	Definition
arithmetic sequence	An arithmetic sequence has a common difference between each two consecutive terms. Arithmetic sequences are also known are arithmetic progressions.
common difference	Every arithmetic sequence has a common or constant difference between consecutive terms. For example: In the sequence 5, 8, 11, 14..., the common difference is "3".
common ratio	Every geometric sequence has a common ratio, or a constant ratio between consecutive terms. For example in the sequence 2, 6, 18, 54..., the common ratio is 3.
Explicit	Explicit formulas define each term in a sequence directly, allowing one to calculate any term in the sequence without knowing the value of the previous terms.
Explicit formula	Explicit formulas define each term in a sequence directly, allowing one to calculate any term in the sequence without knowing the value of the previous terms.
geometric sequence	A geometric sequence is a sequence with a constant ratio between successive terms. Geometric sequences are also known as geometric progressions.
index	The index of a term in a sequence is the term's "place" in the sequence.
Natural Numbers	The natural numbers are the counting numbers and consist of all positive, whole numbers. The natural numbers are the numbers in the list 1, 2, 3... and are often referred to as positive integers.
recursive	The recursive formula for a sequence allows you to find the value of the nth term in the sequence if you know the value of the $(n-1)^{th}$ term in the sequence.
recursive formula	The recursive formula for a sequence allows you to find the value of the nth term in the sequence if you know the value of the $(n-1)^{th}$ term in the sequence.
sequence	A sequence is an ordered list of numbers or objects.

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7.1.3: Sum Notation and Properties of Sigma

Sum Notation and Properties of Sigma

Sayber and Tuscany sell popsicles during the summer for pocket money. One particular weekend, they purchased a package of 30 popsicles from the store.

Usually they just offer the popsicles for free, 1 per customer, and accept tips. This time, Tuscany wonders if they would make more money by charging $0.50 per popsicle. At the same time, Sayber wonders if he might be able to increase his tips by encouraging customers to "outbid" each other. The two children decide to each take 15 popsicles and see who makes the most.

How could you calculate how much money each of them makes, assuming Sayber gets a $0.10 tip from the first customer, and is able to convince each successive customer to double the previous person's tip?

Sum Notation and Properties of Sigma

Consider for example a **sequence** defined by $a_n = 3n$. If we write out the sum of the first 4 terms, we have 3 + 6 + 9 + 12 = 30. But what if we want to write out terms for a larger sum?

Summation notation is a method of writing sums in a succinct form. To write the sum 3 + 6 + 9 + 12 = 30 , we use the Greek letter Sigma, as follows:

$$\sum_{n=1}^{4} 3n$$

The expression $3n$ is called the **summand**, the 1 and the 4 are referred to as the **limits** of the summation, and the n is called the **index** of the sum. Here we have used a “sigma” to write a sum. We can also read a sigma, and determine the sum. For example, we can read the above **sigma notation** as “find the sum of the first four terms of the **series**, where the nth term is $3n$.” We always read the limits from the bottom to the top. The bottom number tells you which term to start with, and the top limit tells you which term is the final term to add. We could then write out the sigma above as:

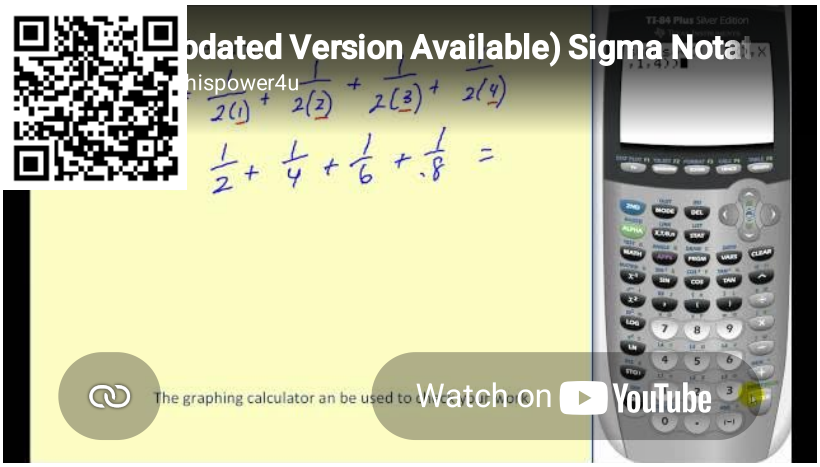
$\sum_{n=1}^{4} 3n$	$= 3(1) + 3(2) + 3(3) + 3(4)$
	= 3 + 6 + 9 + 12 = 30

In general, we can either rewrite a given series in sigma notation, or we can read sigma notation in order to find the value of the sum.

Properties of Sigma

Notice that we can write the sum 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 19 + 20 as 2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10). Therefore $\sum_{n=1}^{10} 2n = 2\sum_{n=1}^{10} n$. In general, we can factor a coefficient out of a sum:

$\sum_{n=1}^{k} ca_n$	$= c\sum_{n=1}^{k} a_n$



Examples

Example 1

Earlier, you were asked to express the incomes of Sayber and Tuscany, who are selling popsicles.

Solution

Tuscany's income can be expressed as: $\sum_{n=1}^{10} .5 \rightarrow .5 \cdot 10 = \$5.00$

Sayber's income can be expressed as:

$\sum_{n=1}^{10} \cdot 10 \cdot 2^n \rightarrow .10\left(2^0\right) + .10\left(2^1\right) + .10\left(2^2\right) \ldots + .10\left(2^{10}\right) \rightarrow \$102.30$

Sayber must be a heck of a salesman to get that last tip!

Example 2

Write the sum using sigma notation: 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 19 + 20.

Solution

$\sum_{n=1}^{10} 2n$

Every term is a multiple of 2. The first term is 2 × 1, the second term is 2 × 2 , and so on. So the summand of the sigma is 2*n*. There are 10 terms in the sum. Therefore the limits of the sum are 1 and 10.

Example 3

Write out the terms of $\sum_{n=2}^{5}(n+7)$ and evaluate the sum.

Solution

Break the sum into two different sums. The sum is 37.

$\sum_{n=2}^{5}(n+7)$	=(2 + 7) + (3 + 7) + (4 + 7) + (5 + 7)
	=2 + 3 + 4 + 5 + 7 + 7 + 7 + 7
	=2 + 3 + 4 + 5 + 7 × 4 = 14 + 28 = 42

Notice we could have written $\sum_{n=2}^{5}(n+7)$ as $\sum_{n=2}^{5} n + \sum_{n=2}^{5} 7$. Also, the second sum does not depend on the index of the sum (i.e. it stays at 7 regardless of the index), only that there are 4 terms to add together. Seeing this can make a sum easier to evaluate.

Example 4

Write out the terms of $\sum_{n=0}^{4} 32\left(\frac{1}{4}\right)^n$ and evaluate the sum.

Solution

In general, we can write a sum as a sum of sums: $\sum_{n=1}^{n}\left(a_n + b_n\right) = \sum_{n=1}^{n}\left(a_n\right) + \sum_{n=1}^{n}\left(b_n\right)$.

The sum is $42\frac{5}{8}$.

$\sum_{i=0}^{4} 32\left(\frac{1}{4}\right)^n$	$= 32\left(\frac{1}{4}\right)^0 + 32\left(\frac{1}{4}\right)^1 + 32\left(\frac{1}{4}\right)^2 + 32\left(\frac{1}{4}\right)^3 + 32\left(\frac{1}{4}\right)^4$
	$= 32 \cdot 1 + 32 \cdot \frac{1}{4} + 32 \cdot \frac{1}{16} + 32 \cdot \frac{1}{64} + 32 \cdot \frac{1}{256}$
	$= 32 + 8 + 2 + \frac{1}{2} + \frac{1}{8} = 42\frac{5}{8}$

Example 5

Write the sum using sigma notation: 2 + 6 + 18 + 54.

Solution

$\therefore \sum_{n=1}^{4} 2\left(3^{n-1}\right)$ or $\sum_{n=0}^{3} 2\left(3^n\right)$

Example 6

Find the series of numbers and the total of those numbers of the **arithmetic series** represented by the following sigma notation: $\sum_{n=8}^{14} 3 + \frac{3}{4}(n-1)$.

Solution

To find the series of numbers, we plug in all the numbers between 8 and 14 for (n):

$3 + \frac{3}{4}((8)-1) = \frac{33}{4}$
$3 + \frac{3}{4}((9)-1) = 9$

We would continue to do this through the number 14. This leaves us with the following series:

$\frac{33}{4} + 9 + \frac{39}{4} + \frac{21}{2} + \frac{45}{4} + 12 + \frac{51}{4}$.

This is fine, if we are just looking for the individual numbers in the sequence, however when asked to evaluate sumations we are being asked to add all the numbers of the series together. It took plenty long enough to find each number, and now we must add them all together. Fortunately there is a formula, not only eliminating our need to find each number, but that lets us also add them all together and arrive at our answer or sum.

The formula is $\frac{k}{2}(a_1 + a_n)$ works like this:

We plug in $n = 8$ to get $a_8 = \frac{33}{4}$

Then we plug in n = 14, to get $a_{14} = \frac{51}{4}$.

Identify the number of terms $(k) = 14 - 8 + 1$ so we use $k = 7$ in the formula below.

Now we can use the formula: $\frac{k}{2}(a_8 + a_{14})$ and we get: $\frac{7}{2}\left(\frac{33}{4} + \frac{51}{4}\right) = \frac{147}{2}$

Example 7

Find the sum of all the numbers in the arithmetic sequence $\sum_{n=8}^{28} 9 - 2(n-1)$.

Solution

Use the formula $\frac{k}{2}(a_1 + a_n)$:

Substitute $n = 8$ to get $a_8 = -5$

Substitute $n = 28$ to get $a_28 = -45$

Identify the number of terms as: $k = 28 - 8 + 1$ so we use $k = 21$ in our formula below:

Now we can use the formula: $\frac{k}{2}(a_8 + a_{14})$ and our answer is −525.

Example 8

Find the series of numbers and the total of those numbers of the **geometric series** represented by the sigma notation $\sum_{n=1}^{8} 7\left(\frac{-1}{2}\right)^{n-1}$.

Solution

To identify and sum the terms in the **geometric series** $\sum_{n=1}^{8} 7\left(\frac{-1}{2}\right)^{n-1}$:

Find the sequence of numbers the same way as we did in Example 6, by plugging in the indicated numbers 1-8 for (n)

$\sum_{n=1}^{8} 7\left(\frac{-1}{2}\right)^{((1)-1)}$ which gives us: 7

We do this for the remaining numbers and our sequence looks like this:

$7 + \frac{-7}{2} + \frac{7}{4} + \frac{-7}{8} + \frac{7}{16} + \frac{-7}{32} + \frac{7}{64} + \frac{-7}{128}$

As in prior examples, we are not just being asked to find the numbers, but add them all together. Again, we do not want to have to take the time to find all the numbers and add them all together. Once again we find ourselves lucky, there is a formula!

The formula $a_1\left(\frac{1-r^k}{1-r}\right)$ works like this:

We plug n=1 into a_1 which gives us 7.

$k = 8$, and $r = \frac{-1}{2}$

Substituting our numbers results: $7\left(\frac{1-\frac{-1}{2}^8}{1-\frac{-1}{2}}\right)$

Which gives us $\frac{595}{128}$.

Example 9

Find the sum of the terms in: $\sum_{n=1}^{11} 9(4)^{n-1}$.

Solution

Let's use the formula for geometric series again:

Identify the terms: $a_1 = 9$, $k = 11$ and $r = 4$

Substituting into our formula, we have: $9\left(\frac{1-4^{11}}{1-4}\right) = 12,582,909$

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Review

Express the sum using sigma notation:

1. $1+3+5+7+9$
2. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{10}$

Find the series of numbers indicated by the sigma notation:

3. $\sum_{n=0}^{14} -2-\frac{10}{3}(n-1)$
4. $\sum_{n=-8}^{14} -7-3(n-1)$

Evaluate:

5. $\sum_{n=-10}^{5} 7-\frac{4}{3}(n-1)$
6. $\sum_{n=-3}^{3} 3-\frac{1}{3}(n-1)$
7. $\sum_{n=-5}^{1} -6+\frac{4}{3}(n-1)$

Find the series of numbers indicated by the sigma notation:

8. $\sum_{n=1}^{2} 3\left(-\frac{1}{2}\right)^{n-1}$
9. $\sum_{n=1}^{5} 5\left(-\frac{4}{3}\right)^{n-1}$

Evaluate:

10. $\sum_{n=1}^{6} 3\left(\frac{1}{2}\right)^{n-1}$
11. $\sum_{n=1}^{7} -5\left(-\frac{1}{2}\right)^{n-1}$
12. $\sum_{n=1}^{11} -7\left(-\frac{4}{3}\right)^{n-1}$
13. $\sum_{n=1}^{6} -7\left(\frac{1}{4}\right)^{n-1}$
14. $\sum_{n=1}^{3} 2\left(-\frac{3}{2}\right)^{n-1}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.3.

Vocabulary

Term	Definition
Σ	Σ (sigma) is the Greek letter meaning "the sum of" when used in mathematics.
arithmetic series	An arithmetic series is the sum of an arithmetic sequence, a sequence with a common difference between each two consecutive terms.
geometric series	A geometric series is a geometric sequence written as an uncalculated sum of terms.
index	The index of a sum is the variable in the sum.
limits	The limits of a sum are written above and below the Σ, and describe the domain to be used in the series calculation.
sequence	A sequence is an ordered list of numbers or objects.
series	A series is the sum of the terms of a sequence.
Sigma	Σ, pronounced syg-mah, is the Greek letter that in math means "the sum of".
sigma notation	Sigma notation is also known as summation notation and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.
summand	A summand is an expression being summed. It directly follows the sigma symbol.
summation	Sigma notation is also known as summation notation and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.

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SECTION OVERVIEW

7.2: Series Sums

Topic hierarchy

7.2.1: Partial Sums

7.2.2: Series Sums and Gauss' Formula

7.2.3: Problem Solving with Series Sums

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7.2.1: Partial Sums

Partial Sums

In the 18th century, mathematician Leonhard Euler solved one of the foremost infinite series problems of his day by examining the series $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$.

Find the first five partial sums of this series and make an observation about the sum of the infinite series.

Source: *Plus Magazine*

Partial Sums

An **infinite series** is a series with an infinite number of terms. In other words, the value of n increases without bound as shown in the series below.

$$\sum_{n=1}^{\infty} 3n+1 = 4+7+10+13+\ldots$$
$$\sum_{n=1}^{\infty} 4(2)^{n-1} = 4+8+16+32+\ldots$$
$$\sum_{n=1}^{\infty} 8\left(\frac{1}{2}\right)^{n-1} = 8+4+2+1+\frac{1}{2}+\ldots$$

These sums continue forever and can increase without bound.

Since we cannot find the sums of these series by adding all the terms, we can analyze their behavior by observing patterns within their **partial sums**. A partial sum is a sum of a finite number of terms in the series. We can look at a series of these sums to observe the behavior of the infinite sum. Each of these partial sums is denoted by S_n where n denotes the index of the last term in the sum. For example, S_6 is the sum of the first 6 terms in an infinite series.

Let's find the first five partial sums of $\sum_{n=1}^{\infty} 2n-1$ and make an observation about the sum of the infinite series.

The first five partial sums are S_1, S_2, S_3, S_4 and S_5. To find each of these sums we will need the first five terms of the sequence: 1, 3, 5, 7, 9. No we can find the partial sums as shown:

$$S_1 = a_1 = 1$$
$$S_2 = a_1 + a_2 = 1+3 = 4$$
$$S_3 = a_1 + a_2 + a_3 = 1+3+5 = 9$$
$$S_4 = a_1 + a_2 + a_3 + a_4 = 1+3+5+7 = 16$$
$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = 1+3+5+7+9 = 25$$

Notice that each sum can also be found by adding the nth term to the previous sum: $S_n = S_{n-1} + a_n$.

For example: $S_5 = S_4 + a_4 = 16 + 9 = 25$

The sequence of the first five partial sums is 1, 4, 9, 16, 25. This pattern will continue and the terms will continue to grow without bound. In other words, the partial sums continue to grow and the infinite sum cannot be determined as it is infinitely large.

Now, let's find the first five partial sums of $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$ and make an observation about the sum of the infinite series.

The first five terms of this sequence are: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$. The partial sums are thus:

$$S_1 = 1$$
$$S_2 = 1.5$$
$$S_3 = 1.75$$
$$S_4 = 1.875$$
$$S_5 = 1.9375$$

Consider what happens with each subsequent term: We start with 1 and add $\frac{1}{2}$ putting us halfway between 1 and 2. Then we add $\frac{1}{4}$, putting us halfway between 1.5 and 2. Each time we add another term, we are cutting the distance between our current sum and 2 in half. If this pattern is continued, we will get ever closer to 2 but never actually reach two. Therefore, the sum is said to "converge to" or "approach" 2.

To support our conjecture further, we can use the calculator to find the 50th partial sum: $S_{50} = 2$. Eventually, if we sum enough terms, the calculator will give us the value to which the sum approaches due to rounding.

Finally, let's find the first five partial sums of $\sum_{n=1}^{\infty} \frac{1}{n}$, the "harmonic series" and make an observation about the sum of the infinite series. (You may need to find addition partial sums to see the behavior of the infinite series.)

Use the calculator to find the following sums:

$S_1 = \text{sum}(\text{seq}(1/x, x, 1, 1)) = 1$
$S_2 = \text{sum}(\text{seq}(1/x, x, 1, 2)) = 1.5$
$S_3 = \text{sum}(\text{seq}(1/x, x, 1, 3)) = 1.833$
$S_4 = \text{sum}(\text{seq}(1/x, x, 1, 4)) = 2.083$
$S_5 = \text{sum}(\text{seq}(1/x, x, 1, 5)) = 2.283$

In this series, the behavior is not quite as clear. Consider some additional partial sums:

$S_{50} = 4.499$
$S_{100} = 5.187$
$S_{500} = 6.793$

In this case, the partial sums don't seem to have a bound. They will continue to grow and therefore there is no finite sum.

Examples

Example 1

Earlier, you were asked to find the first five partial sums of the series $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$, and make an observation about the sum of the infinite series.

Solution

The first five terms of this sequence are: $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}$. The partial sums are thus:

$S_1 = 1$
$S_2 = 1.333$
$S_3 = 1.5$
$S_4 = 1.6$
$S_5 = 1.6666$

If this pattern is continued, we will get ever closer to 2 but never actually reach two. Therefore, the sum is said to "converge to" or "approach" 2.

Find the first five partial sums of the infinite series below and additional partial sums if needed to determine the behavior of the infinite series. Use the calculator to find the partial sums.

Example 2

$\sum_{n=1}^{\infty} 4\left(\frac{3}{2}\right)^{n-1}$

Solution

S_1=4; S_2=10; S_3=19; S_4=32.5; S_5=52.75; The partial sums are growing with increasing speed and thus the infinite series will have no bound.

Example 3

$\sum_{n=1}^{\infty} 500\left(\frac{2}{3}\right)^{n-1}$

Solution

S_1=500; S_2=833.333; S_3=1055.556; S_4=1203.704; S_5=1302.469; Here the sums seems to be growing by smaller amounts each time. Look at the sum additional partial sums to see if there is an apparent upper limit to their growth.

S_{50}=1499.9999…=1500; S_{100}=1500. The sum is clearly approaching 1500 and thus the infinite series has a finite sum.

Example 4

$\sum_{n=1}^{\infty} \frac{5}{6n}$

Solution

S_1=0.833; S_2=1.25; S_3=1.528; S_4=1.736; S_5=1.903; This sequence of partial sums is growing slowly, but will it approach a finite value or continue to grow? Look at additional partial sums: S_{50}=3.749; S_{100}=4.323; S_{500}=5.661. In this case, the sums continue to grow without bound so the infinite series will have no bound.

Review

Find the first five partial sums and additional partial sums as needed to discuss the behavior of each infinite series. Use your calculator to find the partial sums.

1. $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$
2. $\sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1}$
3. $\sum_{n=1}^{\infty} 10(0.9)^{n-1}$
4. $\sum_{n=1}^{\infty} 8(1.03)^{n-1}$
5. $\sum_{n=1}^{\infty} \frac{1}{2}n$
6. $\sum_{n=1}^{\infty} \frac{10}{n}$
7. $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$
8. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
9. $\sum_{n=1}^{\infty} 6(0.1)^{n-1}$
10. $\sum_{n=1}^{\infty} 0.01n + 5$
11. $\sum_{n=1}^{\infty} 2\left(\frac{7}{8}\right)^{n-1}$
12. Which of the series above are arithmetic? Do any of them have a finite sum? Can you explain why?
13. Which of the series above are geometric? Do any of them have a finite sum? Can you explain why?
14. What is the difference between the series that converge and diverge?
15. Make a conjecture about the converging series in this problem set.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.11.

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7.2.2: Series Sums and Gauss' Formula

Series Sums and Gauss's Formula

EyeScreem Ltd is having a special sale on their ice cream cones. In order to get as many people in the store to try out their new flavors, they have decided to run the following promotion:

The first customer to buy an ice cream will pay $6 for a large cone with chocolate fudge.

The second customer will pay only $5.90.

The third customer will only pay $5.80.

Each successive ice cream lover will be charged $0.10 less than the last, until everyone that comes in gets a free cone!

How much money will the store bring in during the sale, assuming at least some customers are given free cones?

Series Sums and Gauss's Formula

Gauss's Formula

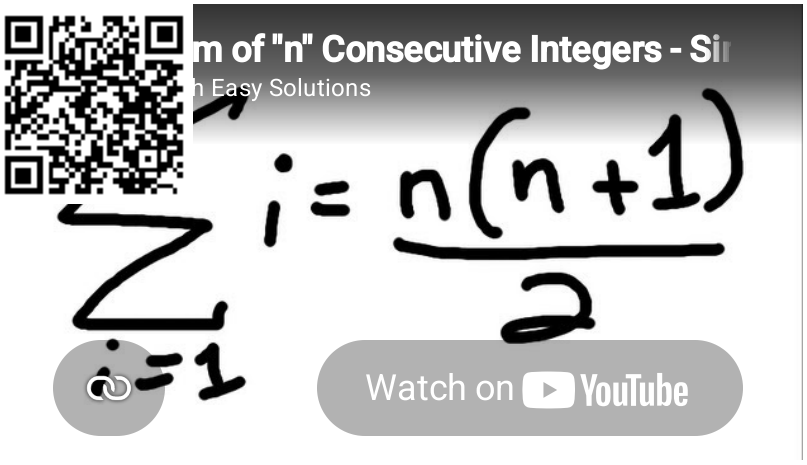
German mathematician C.F. Gauss is often credited with discovering a formula for calculating the sum of a series when he was a young child. The story is likely apocryphal (a legend), but it has been passed down since Gauss lived in the 1700s. According to the story, Gauss's teacher wanted to occupy students by having them add up large sets of numbers. When Gauss was asked to add up the first 100 integers, he found the sum very quickly, by pairing the numbers:

$$\underbrace{1+\underbrace{2+3+4+5+\quad+99}_{2+99=101}+100}_{1+100=101}$$

All of the numbers in the sum could be paired to make groups of 101. There are one hundred numbers being added, so there are such fifty pairs. Therefore the sum is 50(101) = 5050.

The method Gauss used to solve this problem is the basis for a formula that allows us to add together the first *n* positive integers:

$$\sum = \frac{(n)(n+1)}{2}$$



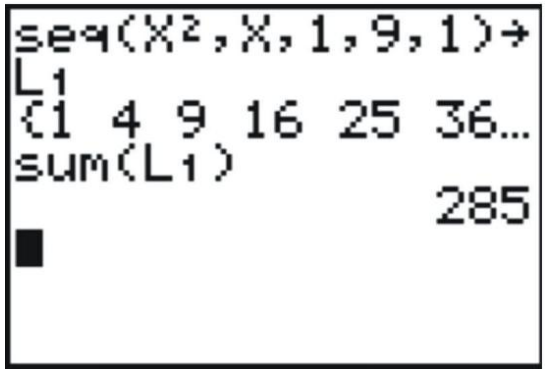
Using a Graphing Calculator

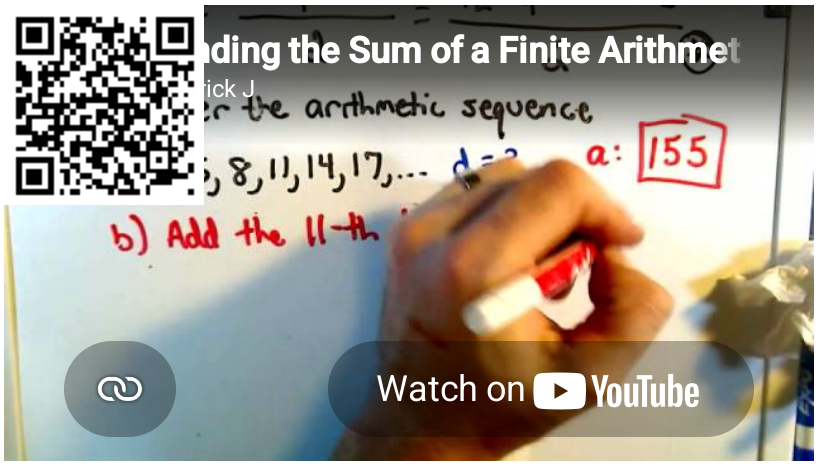
To generate either a sequence or a series, you can use a graphing calculator. The TI-83/84 series gives you several options. You can, for example, work in sequence mode, which allows you to define a sequence and find terms. If you have an explicit formula for a sequence, you can keep your calculator in function mode. For example, consider the sum $\sum_{n=1}^{9} n^2$. It would be time consuming to write out the first 9 squares. The calculator is faster. To generate the 9 terms, press <TI font_2nd> [LIST], then select **OPS**, then option 5, **seq(**. This takes you back to the main screen. You should see **seq(**. After this, enter *x*^2, *x*, 1, 9, 1). (The *x* tells the

calculator that x is the input. The 1 and the 9 tell it the **limits** of the sum. The second 1 tells the calculator to go up in increments of 1.)

Press <TI font_ENTER>, and you should see the list of squares. Scroll to the right to see all of them. The scrolling will end when you reach 81.

If you want to find the sum of the terms, first store the sequence in a list (see screen below), then <TI font_2nd> [LIST], then select **MATH**, then option 5, **sum(.***Then* enter the name of the list and press <TI font_)><TI font_ENTER>. You should get 285.





Examples

Example 1

Earlier, you were asked to find how much money the store would bring in during its ice cream promotion.

Solution

The series sum formula $\sum = \frac{(n)(n+1)}{2}$ is designed for integers, so let's use it to solve for the number of *dimes* brought in (since that is the unit each term reduces by) and then convert to dollars:

$$\sum = \frac{(60)(61)}{2}$$
$$\sum = \frac{3{,}660}{2}$$
$$\sum = 1830.00 \rightarrow \$183.00$$

The store will bring in \$183.00, which will probably *not* cover the costs of the day. However, they will certainly get a lot of people through the door to try out the ice cream!

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Example 2

Expand the **sigma** and find the sum: $\sum_{n=1}^{7}(2n-3)$.

Solution

$\sum_{n=1}^{7}(2n-3)$	=(2×1−3)+(2×2−3)+(2×3−3)+(2×4−3)+(2×5−3)+(2×6−3)+(2×7−3)
	= (−1)+(1)+(3)+(5)+(7)+(9)+(11)
	= 35

Example 3

Expand the sigma and find the sum by adding the terms: $\sum_{n=3}^{6}(n^2-5)$.

Solution

Why is Gauss's formula not recommended for this question?

$\sum_{n=3}^{6}(n^2-5)$	$=(3^2-5)+(4^2-5)+(5^2-5)+(6^2-5)$
	= (4)+(11)+(20)+(31)
	= 66

There are actually a couple of reasons not to use Gauss's formula here, but the biggest is that the formula assumes you are adding all of the integers from 0 to the last number in the series. In the question, you are asked to only sum from the 3rd to 6th term.

Example 4

If the sum of the first *n* integers is 210, what is *n*?

Solution

Using Gauss's formula:

$\sum = \frac{(n)(n+1)}{2}$

$210 = \frac{(n)(n+1)}{2}$

Multiply both sides by 2: $420 = (n)(n+1)$

Distribute: $420 = (n^2+n)$

Complete the square: $420 = (n^2+n+1/4)$

Factor: $420 = (n+1/2)^2$

Square root both sides: $201/2 = n+1/2$

$n = 20$

Example 5

Express the sum using sigma notation: 1 + 3 + 9 + 27 + ...

Solution

$\sum_{n=1}^{\infty}3^{n-1}$ or $\sum_{n=0}^{\infty}3^{n}$

Example 6

Find the sum of $\sum_{n=0}^{12} \frac{-11}{3}(n-1)$.

Solution

Let's look at both ways of solving this one:

1. We could plug in all the numbers between 0 and 12 to get: $\sum_{n=0}^{12} \frac{-11}{3}(n-1)$ and then add them together to get the sum.
2. Using a formula: A slightly modified version of Gauss's formula looks like this: $S_n = \frac{k}{2}(a_0 + a_n)$, where k is the number of terms in the series *plus one*, and a_0 and a_n are the first and last terms in the series

$$\begin{aligned}\left(\frac{13}{2}\right)\left(\left(\frac{-11}{3}\right)(-1)+\left(\frac{-11}{3}\right)(11)\right) &= \left(\frac{13}{2}\right)\left(\frac{11}{3}+\frac{-121}{3}\right)\\ &= \left(\frac{13}{2}\right)\left(\frac{-110}{3}\right)\\ &= \frac{-1430}{6}\\ &= \frac{-715}{3} \to -238\frac{1}{3}\end{aligned}$$

Example 7

Use Gauss's formula to find the sum of the first 200 positive integers.

Solution

Gauss's formula:

$= \frac{(n)(n+1)}{2}$

$n = 200$

$= \frac{(200)(200+1)}{2}$

$\text{Sum} = 20,100$

Review

Calculate the sums of the given series, you may use addition of individual terms, or a series sum formula. You may use a graphing tool for any 3 of them. Try to use each method at least once.

1. $\sum_{n=0}^{16} -10 + 3(n-1)$
2. $64 + 72 + 80 + \ldots + 200$
3. $\frac{-15}{4} - 4 - \frac{17}{4} + \ldots \frac{17}{2}$
4. $\sum_{n=0}^{6} 6 - \frac{1}{2}(n-1)$
5. $\frac{-71}{3} - \frac{67}{3} - 21 + \ldots + \frac{37}{3}$
6. $2 + 4 + 6 + \ldots 26$
7. $-2 - 1 + 0 + \ldots + 12$
8. $-\frac{21}{4} - \frac{13}{2} - \frac{31}{4} + \ldots - \frac{71}{4}$
9. $-\frac{7}{2} - 8 - \frac{25}{2} + \ldots - \frac{43}{2}$
10. $\sum_{n=7}^{20} -1 + (n-1)$
11. $\sum_{n=-6}^{22} 5 + 3(n-1)$
12. $\sum_{n=-2}^{13} -5 - \frac{3}{2}(n-1)$

Consider the sums $\sum_{n=1}^{5}(n+1)$ and $\sum_{n=1}^{5}(n-4)$

13. What is the product of $\left(\sum_{n=1}^{5}(n+1)\right)\left(\sum_{n=1}^{5}(n-4)\right)$
14. What is the sum of $\sum_{n=1}^{5}(n+1)(n-4)$?
15. Look closer at the last two problems, what does this tell you about rules for working with sums?

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Review (Answers)

To see the Review answers, open this PDF file and look for section 7.4.

Vocabulary

Term	Definition
Σ	Σ (sigma) is the Greek letter meaning "the sum of" when used in mathematics.
converge	If a series has a limit, and the limit exists, the series converges.
convergent	If a series has a limit, and the limit exists, the series is convergent.
divergent	If a series does not have a limit, or the limit is infinity, then the series is divergent.
diverges	If a series does not have a limit, or the limit is infinity, then the series diverges.
index	The index of a sum is the variable in the sum.
limits	The limits of a sum are written above and below the Σ, and describe the domain to be used in the series calculation.
series	A series is the sum of the terms of a sequence.
Sigma	Σ, pronounced syg-mah, is the Greek letter that in math means "the sum of".
summand	A summand is an expression being summed. It directly follows the sigma symbol.

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7.2.3: Problem Solving with Series Sums

Problem Solving with Series Sums

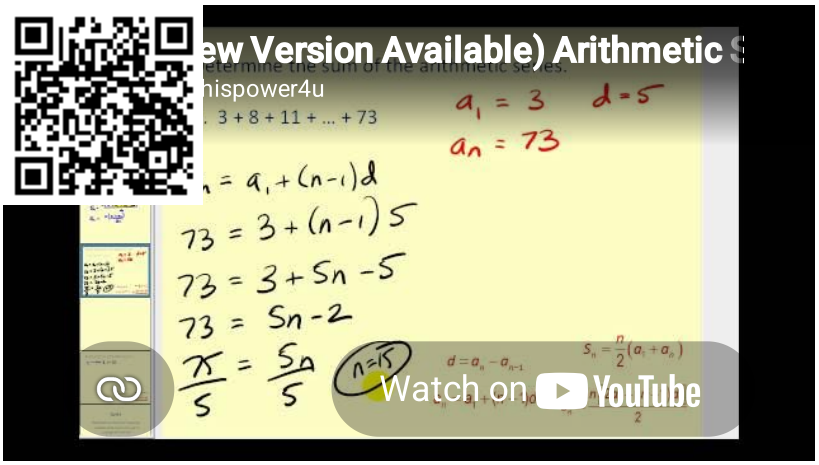
Katrina just started her new job last week. She is making $60 per day, and saving 10%. Her savings account has been slowly building up at $6 per day, and currently has $73 in it.

How much money will she have after working another 15 days? How much after another 27 days?

Problem Solving with Series Sums

In the previous lesson, we discussed Gauss's formula for finding the total of the numbers in a series. In this lesson, we continue to practice that, and also to make use of the modified version of Gauss's formula that is commonly seen in other texts:

The sum of the first *n* terms in an **arithmetic series** is $S_n = \frac{n(a_1+a_n)}{2}$.



Examples

Example 1

Find the sum of the first 50 terms of an arithmetic series if the first term is 5 and the **common difference** is 3.

Solution

$S_{50} = \frac{50(a_1+a_n)}{2}$ ..... Substitute the applicable number of terms in the sequence

$\frac{50(5+(5+49\times3))}{2}$ ..... Substitute the values for the first and 50th terms

$\frac{50(157)}{2} = (25)(127)$ ..... Simplify

3,175

This method is clearly much easier than writing out and adding 50 numbers!

Example 2

Find the sum of the first *40* terms of an arithmetic series in which the first term is *8* and the common difference is 5.

Solution

Use the formula $S_n = \frac{n(a_1+a_n)}{2}$

$S_{40} = \frac{40(8+39\cdot5)}{2}$ ..... Substitute in the number of terms, and the value of the first and last terms

$S_{40} = \frac{40(8+195)}{2}$ ..... Simplify

$S_{40} = \frac{8120}{2}$ ..... Simplify

$S_{40} = 4060$

Example 3

Consider the series: $-13 + -3 + 7 + 17 + 27 + ...$

Solution

a. What is the 25th term?

This is an arithmetic series where each successive term is 10 more than the last.

−13+24(10)=227

b. What is the sum of the first 25 terms?

Use the formula $S_n = \frac{n(a_1+a_n)}{2}$.

This is the sum of the first 25 terms of an arithmetic series with a common difference of 10

$S_{25} = \frac{25(-13+227)}{2}$..... Substitute in the number of terms, and the value of the first and last terms

$S_{25} = \frac{25 \cdot 214}{2}$..... Simplify

$S_{25} = \frac{5350}{2}$..... Simplify

$S_{25} = 2675$

Example 4

A student needed to find the sum of the first 10 terms of the series 4 + 12 + 36 + ... and so he wrote the following:

Solution

$S_{10} = \frac{10(4+4\times3^9)}{2} = \frac{10(78,736)}{2} = 393,680$

Do you agree with the student's work? Explain.

Because the series is geometric, this formula is not appropriate. The work here does not represent the sum of the first 10 terms. Using a graphing calculator, you can find that the sum is 118,096.

Example 5

Given an arithmetic sequence (a_n) determined by $a_1 = 143$ and $d = -3$:

What is the 220th number of the sequence?

Solution

We find the solution by using the following formula: $a_n = a + (n-1)d$.

$a_{220} = a_1 + 219d$
$= 143 + 219(-3)$
$= -514$

Example 6

How do we find the sum of the first 220 numbers of the sequence, given the same information as in Example 5 above?

Solution

We use the following formula: $s_n = \frac{a_1+a_n}{2}(n)$. We set $n = 220$.

To find the sum of the first 220 numbers:

$s_{220} = \frac{a_1+a_{220}}{2}(220)$

$s_{220} = \frac{143+(-514)}{2}(220)$

$s_{220} = \frac{-371}{2} \cdot 220$

The sum of the first 220 numbers of the sequence is −40,810.

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Review

Use the arithmetic series formula to solve the following problems:

1. Given the arithmetic series of numbers: 1, 4, 7, 10, 13... a)Find the 200th number in the sequence b)find the sum of the first 200 numbers.
2. The sum of the first five numbers of an arithmetic sequence is -45. What is the value of the third number? (hint: find a_3 if $s_5 = -45$)
3. You have an arithmetic series of numbers defined by: $a_1 = 45$ and $d = -5$ a)Determine a_{150} b)Identify the sum of: $a_1 + a_2 + ... + a_{150}$
4. The sum of the first three numbers in an arithmetic sequence is 219. The sum of the first nine numbers in the same sequence is 603. What is the 143rd number of the sequence.
5. The first eight numbers of an arithmetic sequence add up to 604. The next eight numbers added up equal 156. Find the first number and the common difference in the sequence.
6. The first number in an arithmetic sequence is 80. Find the common difference if we also know that S_9 is eighteen times a_{11}
7. If a_n is an arithmetic sequence with $a_1 = 1$. Find the second number if we know that the sum of the first five numbers is one-fourth of the sum of the next five numbers.
8. Given $(a_n) = 78, 75, 72, 69...$ Find a_{150} and s_{150}
9. The following conditions exist within a sequence of numbers: $a_{50} = 252$ and $s_{50} = 2800$. What is the first number of the series, and what is the common difference?

10. What are the values of a and d, given that (a_n) is an arithmetic series of numbers, if we know: $a_{15} = 62$ and $s_{20} = 700$?
11. Given the sequence: $a_1 = -16$ and $d = \frac{1}{3}$. Find the values of n, so that $s_n = 50$
12. If $a_1 = 8$ and $d = -3$. What are the values of a_{20} and s_{20}?
13. If $a_{34} = 193$ and $s_{17} = 306$, find a and d.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.5.

Vocabulary

Term	Definition
arithmetic series	An arithmetic series is the sum of an arithmetic sequence, a sequence with a common difference between each two consecutive terms.
common difference	Every arithmetic sequence has a common or constant difference between consecutive terms. For example: In the sequence 5, 8, 11, 14..., the common difference is "3".
infinite series	An infinite series is the sum of the terms in a sequence that has an infinite number of terms.
Mathematical induction	Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
partial sum	A partial sum is the sum of the first "n" terms in an infinite series, where "n" is some positive integer.

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SECTION OVERVIEW

7.3: Mathematical Induction

Topic hierarchy

7.3.1: Inductive Reasoning from Patterns

7.3.1.1: Inductive Proofs

7.3.2: Induction and Factors

7.3.3: Induction and Inequalities

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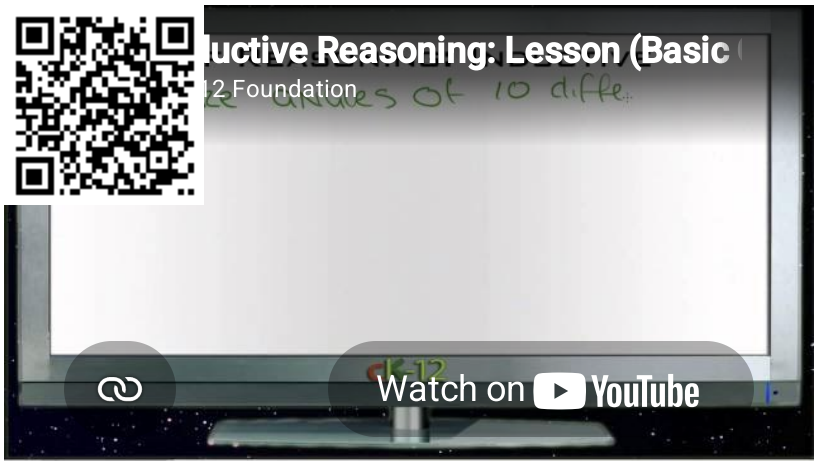
7.3.1: Inductive Reasoning from Patterns

Inductive Reasoning from Patterns

Inductive Reasoning

One type of reasoning is **inductive reasoning**. Inductive reasoning entails making conclusions based upon examples and patterns. Visual patterns and number patterns provide good examples of inductive reasoning. Let's look at some patterns to get a feel for what inductive reasoning is.

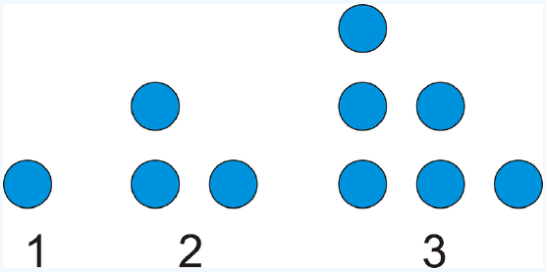
What if you were given a pattern of three numbers or shapes and asked to determine the sixth number or shape that fit that pattern?



Examples

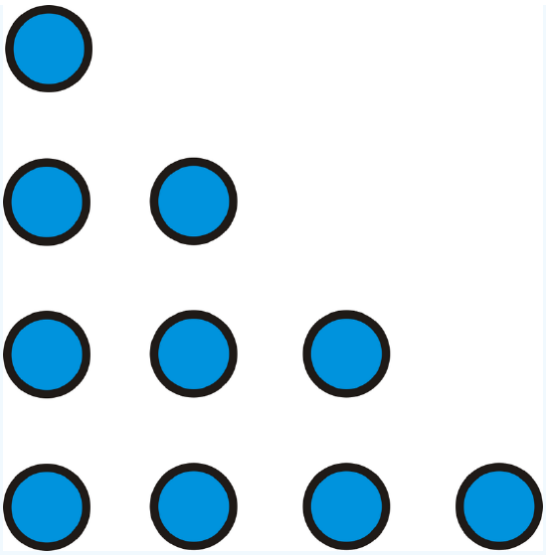
Example 1

A dot pattern is shown below. How many dots would there be in the 4th figure? How many dots would be in the 6th figure?



Solution

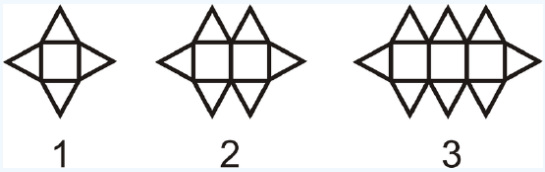
Draw a picture. Counting the dots, there are $4+3+2+1=10$ dots.



For the 6th figure, we can use the same pattern, $6+5+4+3+2+1$. There are 21 dots in the 6th figure.

Example 2

How many *triangles* would be in the 10th figure?



Solution

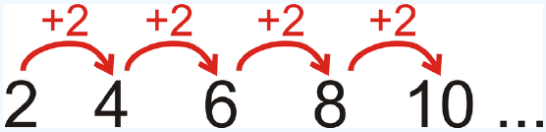
There would be 10 squares in the 10th figure, with a triangle above and below each one. There is also a triangle on each end of the figure. That makes $10+10+2=22$ triangles in all.

Example 3

Look at the pattern 2, 4, 6, 8, 10, … What is the 19th term in the pattern?

Solution

Each term is 2 more than the previous term.



You could count out the pattern until the 19th term, but that could take a while. Notice that the 1st term is 2·1, the 2nd term is 2·2, the 3rd term is 2·3, and so on. So, the 19th term would be 2·19 or 38.

Example 4

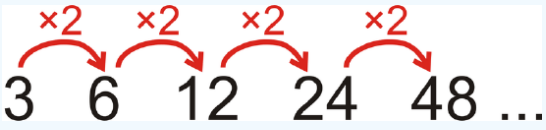
Look at the pattern: 3, 6, 12, 24, 48, …

What is the next term in the pattern? What is the 10th term?

Solution

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Each term is *multiplied* by 2 to get the next term.



Therefore, the next term will be 48·2 or 96.

To find the 10th term, continue to multiply by 2, or $3 \cdot \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{2^9} = 1536$

Example 5

Find the 8th term in the list of numbers: $\frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}$ ...

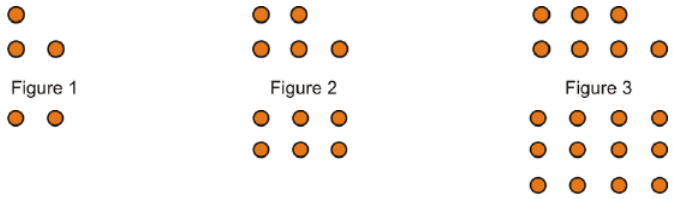
Solution

First, change 2 into a fraction, or $\frac{2}{1}$. So, the pattern is now $\frac{2}{1}, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}$ ... The top is 2, 3, 4, 5, 6. It increases by 1 each time, so the 8th term's numerator is 9. The denominators are the square numbers, so the 8th term's denominator is 8^2 or 64. The 8th term is $\frac{9}{64}$.

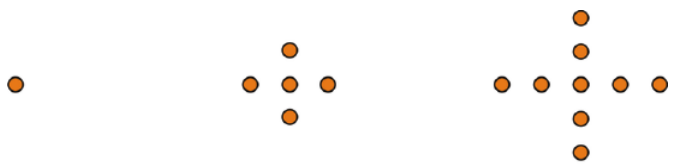
Review

For questions 1-3, determine how many dots there would be in the 4th and the 10th pattern of each figure below.

1. Figure 1 Figure 2 Figure 3



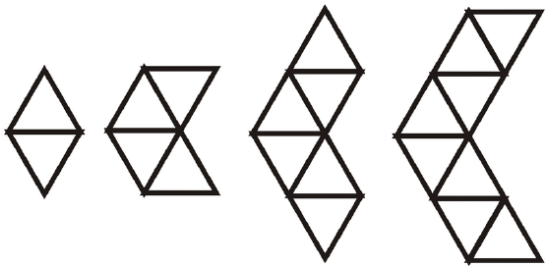
2. Figure 1 Figure 2 Figure 3



3. Figure 1 Figure 2 Figure 3



4. Use the pattern below to answer the questions.
   a. Draw the next figure in the pattern.
   b. How does the number of points in each star relate to the figure number?
5. Use the pattern below to answer the questions. All the triangles are equilateral triangles.



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a. Draw the next figure in the pattern. How many triangles does it have?
b. Determine how many triangles are in the 24th figure.

For questions 6-13, determine: the next three terms in the pattern.

6. $5, 8, 11, 14, 17, \ldots$
7. $6, 1, -4, -9, -14, \ldots$
8. $2, 4, 8, 16, 32, \ldots$
9. $67, 56, 45, 34, 23, \ldots$
10. $9, -4, 6, -8, 3, \ldots$
11. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots$
12. $\frac{2}{3}, \frac{4}{7}, \frac{6}{11}, \frac{8}{15}, \frac{10}{19}, \ldots$
13. $-1, 5, -9, 13, -17, \ldots$

For questions 14-17, determine the next two terms **and** describe the pattern.

14. 3, 6, 11, 18, 27, …
15. 3, 8, 15, 24, 35, …
16. 1, 8, 27, 64, 125, …
17. 1, 1, 2, 3, 5, …

Review (Answers)

To see the Review answers, open this PDF file and look for section 2.1.

Resources



Vocabulary

Term	Definition
Inductive Reasoning	Inductive reasoning is a type of reasoning where one draws conclusions from patterns and previous examples.
Equilateral Triangle	An equilateral triangle is a triangle in which all three sides are the same length.

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7.3.1.1: Inductive Proofs

Inductive Proofs

Proving a theory can be a daunting process. No matter how many times you try something with the same result, how can you be *certain* that it will *always* have the same result, no matter what?

For example, if you were to see someone fill a water balloon with ice water and hold it out the window, you would probably either cringe in anticipation of the shouting below, or eagerly watch, depending on the situation. In either case, your response would be based on the fact that you would be *certain* that a water balloon would pop on someone's head if dropped out the window onto them. Your certainty would be based on your past experience with water balloons and sidewalks, and you'd very likely be correct, but until the balloon actually hits the target, there isn't any way to be absolutely certain it will break.

In math, situations like this occur a lot. Based on repeated experience, you may develop a rule or shortcut to save time or effort when calculating. However, you may be rightly concerned about using such shortcuts on an important exam. After all, how can you be *certain* that the shortcut works in every situation?

Proofs by Induction

In this lesson you will learn about **mathematical induction**, a method of **proof** that will allow you to prove that a particular statement is true for all positive integers.

First let's make a guess at a formula that will give us the sum of all the positive integers from 1 to n for any integer n. If we look closely at Gauss's Formula we used in the last lesson, where the young boy was able to quickly add up all of the numbers between 1 and 100, we can see a general form: there were 100 numbers, hence 50 pairs. So if there were n numbers, there would be $(n/2)$ pairs. The first and last numbers were 1 and 100. They added together to give us 101. This number was the sum of each pair in the overall sum. So in general, we could add together 1 and n to get the sum of each pair. Therefore we might hypothesize that the sum of the first n positive integers is $n((1 + n)/2)$. However, we have not proven that this formula works for all positive integers n. Mathematical induction will allow us to do this.

The overall idea of induction is this: Assume that a statement is true for some arbitrary value of n, and show that if the statement is true for $n = k$, it must also be true for $n = k+1$. This process is used because we can't actually show it is true for every value. For example, you might show that the above equation is true for $n = 100$, and then $n = 101$, and then $n = 102$, but then what about 103? 104? 500? A million?

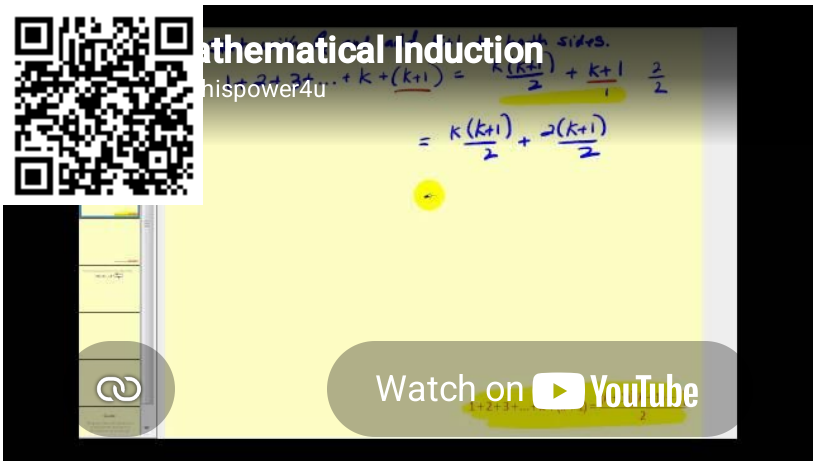
Mathematical induction allows us prove that a statement is true in three steps:

Step 1) The **base case***:* prove that the statement is true for the first value of n. In some cases, this might be $n = 0$. In the case of the integer sum formula above, we would start with $n = 1$. Often with induction you may want to expand the first step by showing that the statement is true for several values of n.

Step 2) The **inductive hypothesis***:* assume that the statement is true for the kth value of n. In the case of the integer sum formula, we would state the following: the sum of the first k positive integers is $k((1 + k)/2)$.

Step 3) The **inductive step***:* use the inductive hypothesis to show that the statement is true for the $k + 1^{th}$ step. In the case of the integer sum formula, we would prove the following: assuming that the sum of the first k positive integers is $k((1 + k)/2)$, the sum of the first k +1 positive integers is $((k + 1) (1 + (k + 1)/2)$.

Carrying out this kind of proof requires that you perform each of these steps., For the third step in particular you must rely on your algebra skills.



Examples

Example 1

Earlier, you were asked how you could be certain that a theory always results in the correct answer.

Solution

Situations like this are custom-made for inductive proofs. If you come up with a shortcut on your math, and want to be absolutely certain it works in every situation, run it through the proof explained in this lesson. If it passes all of the tests, you can be sure it will work with any number you throw at it.

Example 2

Prove: The sum of the first *n* positive integers is $\frac{n(1+n)}{2}$.

Solution

Use the three steps of mathematical induction:

Step 1) The base case:

If *n* = 1, the entire sequence is just 1 and therefore the sum is 1. Also, $\frac{n(1+n)}{2} = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

This establishes the base case.

Step 2) Assume that the sum of the first k positive integers is $\frac{k(1+k)}{2}$.

In other words, assume that $1+2+3+\cdots+k=\frac{k(1+k)}{2}$.

Step 3) We must show that the sum of the first $k+1$ positive integers is $\frac{(k+1)(1+(k+1))}{2}$.

In other words, we must show that $1+2+3+\cdots+k+(k+1)=\frac{(k+1)(1+(k+1))}{2}$

There are two key ideas to keep in mind as you are carrying out this step: (1) remember to use the assumption and (2) remember how sums work.

How does the sum of the first *k* + 1 integers relate to the sum of the first *k* integers? To get the sum of the first *k* + 1 integers we must add up all the integers from 1 to *k* and then add on *k* + 1, since the sum of the first *k* + 1 integers is $1+2+3+\cdots+k+(k+1)$.

Now we must use our assumption. Remember that we are *assuming* that

$1+2+3+\cdots+k=\frac{k(1+k)}{2}$

Substitute $\frac{k(1+k)}{2}$ in for $1+2+3+\cdots+k$ in our expression above for the sum of the first *k* + 1 integers.

Now we have the sum of the first *k* + 1 integers is $\frac{k(1+k)}{2}+(k+1)$.

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Remember that we are trying to show that the sum of the first k + 1 integers is $\frac{(k+1)(1+(k+1))}{2}$. With some algebraic manipulation, we can show that $\frac{k(1+k)}{2}+(k+1)=\frac{(k+1)(1+(k+1))}{2}$.

See below:

$\frac{k(1+k)}{2}+(k+1)$	
$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}$	The common denominator is 2 Add the fractions
$=\frac{k(k+1)+2(k+1)}{2}$	Simplify the numerator
$=\frac{k^2+k+2k+2}{2}$	
$=\frac{k^2+3k+2}{2}$	Factor the numerator
$=\frac{(k+1)(k+2)}{2}$	The term (k+2) is the same as ((k+1)+1)
$=\frac{(k+1)((k+1)+1)}{2}$	

We have shown that our formula for the sum of the first n integers is true for n = 1. We have also shown that whenever it is true for $n = k$ it is also true for $n = k+1$. Since we know it is true for n = 1, it must therefore be true for n = 2. Similarly, since it is true for n = 2, it must therefore be true for n = 3, and it must therefore be true for n = 4,... You should see that we have proven that the sum of the first n positive integers is $\frac{n(1+n)}{2}$ for *all* integer values of n. We can similarly prove a formula for the sum of the first n terms in an **arithmetic series**.

Example 3

Prove that the sum of the first n terms of an arithmetic series is $S_n=\frac{n(a_1+a_n)}{2}$ where a_1 is the first term in the series and a_n is the last term.

Solution

Recall that in an arithmetic sequence or series, there is a common difference, d, between each term, and that the **n^{th} term** is $a_n=a_1+d(n-1)$ We need to keep these ideas in mind in order to complete the proof.

Step 1) Base case: if n=1, then $S_1=a_1$

Using the hypothesized formula, we have

$$S_1=\frac{1(a_1+a_1)}{2}=\frac{2a_1}{2}=a_1$$

Step 2) Assume that $S_k=\frac{k(a_1+a_k)}{2}$

Step 3) Prove that if our formula for S_k is true then $S_{k+1}=\frac{(k+1)(a_1+a_{k+1})}{2}$.

We can think of the sum of the first k + 1 terms as the sum of the first k terms, plus the k + 1 term.

So we have:

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$S_{k+1} = S_k + a_{k+1}$	**Add the \(\ k+1\) term**
$= \frac{k(a_1+a_k)}{2} + a_{k+1}$	Use the formula for S_k from step 2
$= \frac{k(a_1+a_k)}{2} + \frac{2a_{k+1}}{2}$	The common denominator is 2
$= \frac{k(a_1+a_k)+2a_{k+1}}{2}$	Add the fractions
$= \frac{k(a_1+(a_1+(k-1)d))+2(a_1+kd)}{2}$	Use substitution: remember that $a_m = a_1 + (m-1)d$ for any positive integer m. So $a_k = a_1 + (k-1)d$ and $a_{k+1} = a_1 + (k)d$
$= \frac{k(a_1+a_1+kd-d)+2a_1+2kd}{2}$	
$= \frac{ka_1+ka_1+k^2d-kd+2a_1+2kd}{2}$	Distribute and combine like terms
$= \frac{2ka_1+2a_1+k^2d+kd}{2}$	Factor by grouping
$= \frac{2a_1(k+1)+kd(k+1)}{2}$	
$= \frac{(k+1)(2a_1+kd)}{2}$	
$= \frac{(k+1)(a_1+a_1+kd)}{2}$	Again, $a_{k+1} = a_1 + (k)d$
$= \frac{(k+1)(a_1+a_{k+1})}{2}$	

Example 1

Use induction to prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution

Step 1) Base case: $1^2 = 1$

$\frac{1(1+1)(2(1)+1)}{6} = \frac{2(3)}{6} = 1$

Step 2) Inductive hypothesis: $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

Step 3) Inductive step: show that

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

First, note that $\frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$

Now we have:

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$
$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$
$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$
$= \frac{(k+1)[2k^2+k+6k+6]}{6} = \frac{(k+1)[2k^2+7k+6]}{6} = \frac{(k+1)(2k+3)(k+2)}{6}$

Example 5

Use induction to prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Solution

Step 1) Base case: $1 = 1^2$

Step 2) Inductive hypothesis: assume that $1 + 3 + 5 + \ldots + (2k-1) = k^2$

Step 3) Show that $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = (k+1)^2$

We have: $1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + (2k+1)$

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$= k^2 + 2k + 1$
$= (k+1)(k+1) = (k+1)^2$

Example 6

Use induction to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$.

Solution

Step 1) Base case: $1^3 = 1$

$\frac{1^2(1+1)^2}{4} = \frac{2^2}{4} = 1$

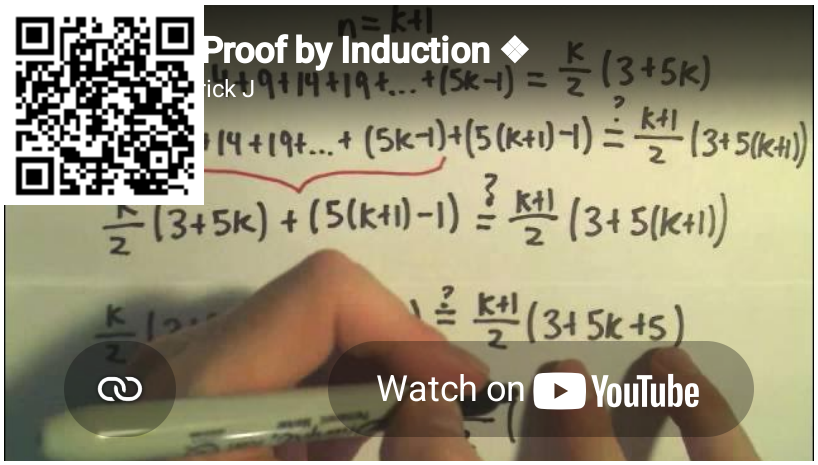
Step 2) Assume that $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$

Step 3) Show that $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

First, note that $\frac{(k+1)^2((k+1)+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$

Now we have:

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$
$= \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$
$= \frac{(k+1)^2[k^2+4(k+1)]}{4} = \frac{(k+1)^2[k^2+4k+4]}{4} = \frac{(k+1)^2(x+2)^2}{4}$



Review

Use induction to prove the following:

1. $-15 - 25 - 35 + \ldots - 10k - 5 = k(-5k - 10)$
2. $1 + 9 + 9^2 + 9^3 + \ldots + 9^k = \frac{9^{k+1}-1}{9-1}$
3. $4 + 8 + 12 + \ldots + 4k = 2k(k+1)$
4. $6 + 8 + 10 + \ldots + 2k + 4 = k(k+5)$
5. $1 + 2 + 3 + \ldots + k = \frac{1}{2}k(k+1)$
6. $1 + 6 + 6^2 + 6^3 + \ldots + 6^k = \frac{6^{k+1}-1}{6-1}$
7. $-1 - 5 - 9 + \ldots - 4k + 3 = k(-2k+1)$
8. $1 + 4 + 4^2 + 4^3 + \ldots + 4^k = \frac{4^{k+1}-1}{4-1}$
9. $3 + 6 + 9 + \ldots + 3k = \frac{3}{2}k(k+1)$
10. $-1 - 3 - 5 + \ldots - 2k + 1 = k(-k)$

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11. $1+3+3^2+3^3+...+3^k=\frac{3^{k+1}-1}{3-1}$
12. $1+7+7^2+7^3+...+7^n=\frac{7^{n+1}-1}{6}$
13. $4+8+12+...+4n=2n(n+1)$
14. $10+18+26+...8n+2=n(4n+6)$
15. $6+12+18+...+6n=3n(n+1)$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.6.

Vocabulary

Term	Definition
arithmetic series	An arithmetic series is the sum of an arithmetic sequence, a sequence with a common difference between each two consecutive terms.
Base Case	In an induction proof, the base case is the anchor step. It is the first domino to fall, creating a cascade and thus proving the statement true for every number greater than the base case.
induction	Induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
inductive hypothesis	In an induction proof, the inductive hypothesis is the step where you assume the statement is true for k.
inductive step	In an induction proof, the inductive step is the proof. It is when you show the statement is true for k+1 using only the inductive hypothesis and algebra.
Mathematical induction	Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
n^{th} term	The nth term in a series commonly refers to the last term in a series, often left unspecified.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.
series sum	The series sum is the total sum of all of the numbers in a series.

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7.3.2: Induction and Factors

Induction and Factors

Proof by **induction** is common in mathematics. In this lesson we will use it to prove different kinds of hypotheses and to practice the use of the process in different situations. Specifically, we will be applying the principles of induction to prove various statements regarding factors and factorability.

Induction and Factors

There are two properties of integers and their factors that will be useful for the proofs in this lesson.

Property 1: If a is a **factor** of b, and a is a factor of c, then a is a factor of the sum

$b + c$.

For example, the set of numbers $a = 3$, $b = 6$, and $c = 9$ satisfies the hypothesis because 3 is a factor of 6 and 3 is a factor of 9. Property 1 states that therefore 3 is a factor of $9 + 6 = 15$, which we know is true because $3 \times 5 = 15$. We can prove that this property is true for all integers if we think about what the term *factor* means. If a is a factor of b, then there exists some **integer** M such that $aM = b$. Similarly if a is a factor of c, then there exists some integer N such that $aN = c$. So we can write the sum $b + c$ as $aM + aN$. We know

$$b + c = aM + aN = a(M + N).$$

Because we can write the sum as a product of a and another number, a is a factor of the sum $b + c$.

Property 2: If a is a factor of b and b is a factor of c, then a is a factor of c.

We can prove this property in a similar manner. If a is a factor of b, then there exists some integer M such that $aM = b$. If b is a factor of c, then there exists some integer N such that $bN = c$. We can write c in the following manner:

$c = bN = (aM)N = a(MN)$

Therefore a is a factor of c because we can write c as the product of a and another integer, MN.

As noted above, these properties of integers are useful for proving statements about integers and factors via induction.

Examples

Example 1

Prove that 3 is a factor of $4^n - 1$ for all positive integers n.

Solution

Proof by induction:

1. The base case: if $n = 1$, then $4^n - 1 = 4 - 1 = 3$. 3 is a factor of itself because $3 \times 1 = 3$. If the base case does not convince you, you can always test out additional values of n. For example, if $n = 2$, $4^2 - 1 = 16 - 1 = 15 = 5 \times 3$.
2. The inductive hypothesis: assume that 3 is a factor of $4^k - 1$.
3. The inductive step: show that 3 is a factor of $4^{k+1} - 1$.

   If 3 is a factor of $4^k - 1$, then there exists some integer M such that $3M = 4^k - 1$. We can write $4^{k+1} - 1$ in a manner that allows us to use the inductive hypothesis:

$4^{k+1} - 1$	
$4(4^k-1) + 3$	Factor out 4, but add 3 in order to keep the value of $4^{k+1}-1$
$4(3M) + 3$	Substitute: $3M = 4^k - 1$

Note that the substitution is the same as using property 2 above: if 3 is a factor of $4^k - 1$, then 3 is a factor of $4(4^k - 1)$. Using the substitution simply makes the fact a bit more obvious. This last step proves that 3 is a factor of $4^{k+1} - 1$, by property 1 above.

The technique of rewriting the $k + 1$ term can also be used to prove statements about polynomials and factors.

Example 2

Prove that $x - y$ is a factor of $x^n - y^n$ for all positive integers n.

Solution

Note: Since we are talking about polynomials that are factorable now, not integers, then we say that if $x - y$ is a factor of $x^n - y^n$, then there exists a polynomial P such that $P(x - y) = x^n - y^n$.

Proof by induction:

1. The base case: If $n = 1$, we have $x^n - y^n = x - y$, and $x - y$ is a factor of itself, as $x - y = 1(x - y)$.

   As we did above, we can also check $n = 2$ in order to convince ourselves. If $n = 2$, we have $x^2 - y^2 = (x - y)(x + y)$, so $x - y$ is clearly a factor.

2. The inductive hypothesis: assume that $x - y$ is a factor of $x^k - y^k$.
3. Show that $x - y$ is a factor of $x^{k+1} - y^{k+1}$.

From the inductive step, we know that there is some polynomial P such that $P(x - y) = x^k - y^k$. We can rewrite $x^{k+1} - y^{k+1}$ in a manner that allows us to use the inductive hypothesis:

$x^{k+1} - y^{k+1}$
$=x^{k+1} - xy^k + xy^k - y^{k+1}$
$=x(x^k - y^k) + y^k(x - y)$
$=x(P(x - y)) + y^k(x - y)$
$=Px(x - y) + y^{k-1}(x - y)$

Again, by property 1 above, this shows that $x - y$ is a factor of $x^{k+1} - y^{k+1}$. Therefore we have shown that $x - y$ is a factor of $x^n - y^n$ for all positive integers n.

Example 3

a. Without adding, determine if 7 a factor of 49 + 70.
b. Consider the sum 23 + 54 = 77. Is 7 a factor of 77? What does this tell you about the first factor property in the lesson?

Solution

a. Use Property 1 from the lesson: *If a is a factor of b, and a is a factor of c, then a is a factor of the sum b + c.*

   7 is a factor of 70, since 7×10=70

   Therefore 7 is a factor of 119, since 49+70=119

b. This is a test of the converse of Property 1, which would be "If a number is a factor of the sum, then it is a factor of the factors of the sum"

   7 is a factor of the sum: 77

   7 is not a factor of 23 or 54

   This tells us that the converse of the property is not necessarily true.

Example 4

Prove that $x^n - 1$ is divisible by $x - 1$ for all positive integers n.

Solution

1. Base case:	If $n = 1$, $x^k - 1 = x - 1 = (x - 1)(1)$
2. Inductive hypothesis:	Assume that $x^k - 1$ is divisible by $x - 1$
3. Inductive step:	Show that $x^{k+1} - 1$ is divisible by $x - 1$.

$x^k - 1$ divisible by $x - 1 \Rightarrow P(x - 1) = (x^k - 1)$ for some polynomial $Px^{k+1} - 1 = x(x^k - 1) + (x - 1) = Px(x - 1) + (x - 1)$, which is divisible by $x - 1$

Example 5

Prove that $n^2 - n$ is even for all positive integers n.

Solution

1. Base case:	If $n = 1$, $1^2 - 1 = 1 - 1 = 0 = 2 \times 0$
2. Inductive hypothesis:	Assume that $k^2 - k$ is even
3. Inductive step:	Show that $(k + 1)^2 - (k + 1)$ is even.

If $k^2 - k$ is even, then $k^2 - k = 2M$ for some integer M $(k + 1)^2 - (k + 1) = k^2 + 2k + 1 - k - 1 = k^2 - k + 2k = 2M + 2k = 2(M + k)$ which is even because $M + K$ is an integer.

Example 6

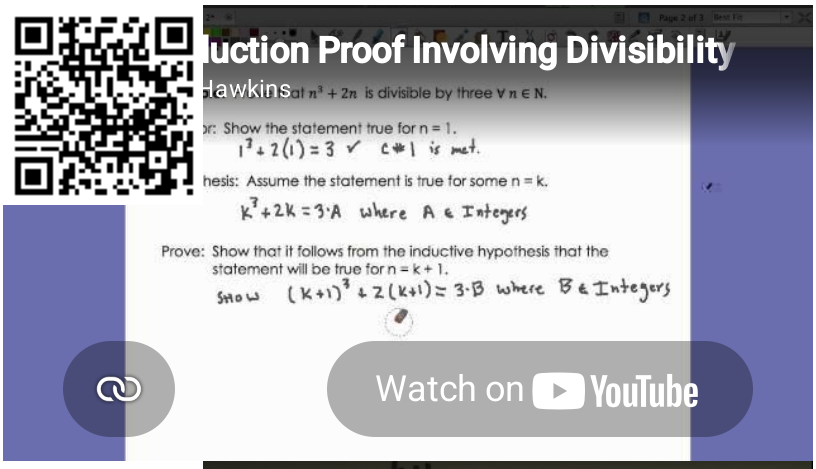
Prove that $5^{2n-1} + 1$ is divisible by 6 for all positive integers n.

Solution

1. Base case:	If $n = 1$, $5^1 + 1 = 5 + 1 = 6 = 6(1)$
2. Inductive hypothesis:	Assume that $5^{2k-1} + 1$ is divisible by 6.
3. Inductive step:	Show that $5^{2(k+1)-1} + 1$ is divisible by 6.

If $5^{2k-1} + 1$ is divisible by 6, then $5^{2k-1} + 1 = 6M$ for some integer M. $5^{2(k+1)-1} + 1 = 5^{2k+1} + 1 = 5^2 (5^{2k-1} + 1) - 24 = 5^2 (6M) - 24$ which is divisible by 6.

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Review

1. Without adding, determine if 7 is a factor of 49 + 70.
2. Consider the sum 23+54=77 Is 7 a factor of 77? What does this tell you about the first factor property in the lesson?
3. Prove that any positive integer n > 1 a) is prime or, b) can be represented as a product of prime factors.
4. Found within set "J" are all positive integers, from the number 1 to 2n. Prove that there are two numbers, one that is a factor of another, from any (n = 1) numbers chosen from set "J"
5. Prove that $\left(x^n + \frac{1}{x^n}\right)$ is also an integer for any positive integer n if the following is an integer: $\left(x + \frac{1}{x}\right)$
6. Prove the formula $n_{k+m} = n_{k-1}u_m + n_k n_{m+1}$ for the sequence of Fibonacci numbers: $n_1 = 1, n_2 = 1, u_{k+1} = n_k + n_{k-1}, k = 2, 3 \ldots$

Prove the following identities.

7. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
8. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$
9. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$
10. $1x \cdot 1! + 2x \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1$
11. $n^2 \frac{(n+1)^2}{4} = 1^3 + 2^3 + 3^3 + \ldots + n^3$
12. $n(n+1)\frac{(2n+1)}{6} = 1^2 + 2^2 + 3^2 + \ldots + n^2$

Prove the following divisibilities.

13. Prove that $n\frac{(n+1)}{2}$ is a factor of $1 + 2 + 3 + \ldots + n$ for all positive integers n
14. Prove that 3 is a factor of $n^3 + 2n$ for all positive integers n.
15. Prove that $n^2 \frac{(n+1)^2}{4}$ is a factor of $1^3 + 2^3 + 3^3 + \ldots + n^3$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.7.

Vocabulary

Term	Definition
factor	Factors are the numbers being multiplied to equal a product. To factor means to rewrite a mathematical expression as a product of factors.
induction	Induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
Integer	The integers consist of all natural numbers, their opposites, and zero. Integers are numbers in the list ..., -3, -2, -1, 0, 1, 2, 3...
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.

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7.3.3: Induction and Inequalities

Induction and Inequalities

This is the third in a series of lessons on mathematical proofs. In this lesson we continue to focus mainly on **proof** by **induction**, this time of inequalities, and other kinds of proofs such as proof by geometry.

Induction and Inequalities

The Transitive Property of Inequality

Below, we will prove several statements about inequalities that rely on the transitive property of inequality:

If $a < b$ and $b < c$, then $a < c$.

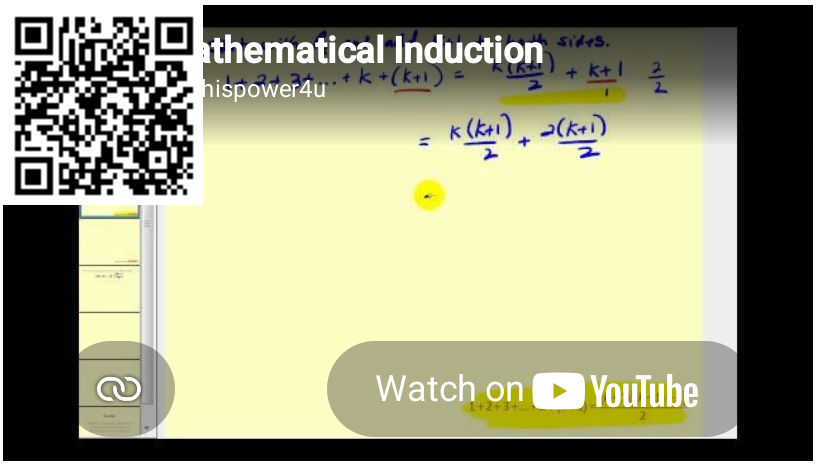
Note that we could also make such a statement by turning around the relationships (i.e., using "greater than" statements) or by making inclusive statements, such as $a \geq b$.

It is also important to note that this property of integers is a **postulate**, or a statement that we assume to be true. This means that we need not prove the transitive property of inequality.

You encountered other useful properties of inequalities in earlier algebra courses:

Addition property: if $a > b$, then $a + c > b + c$.

Multiplication property: if $a > b$, and $c > 0$ then $ac > bc$.



Examples

Example 1

Prove that $n! \geq 2^n$ for $n \geq 4$

Solution

Step 1) The base case is $n = 4$: $4! = 24$, $2^4 = 16$. $24 \geq 16$ so the base case is true.

Step 2) Assume that $k! \geq 2^k$ for some value of k such that $k \geq 4$

Step 3) Show that $(k+1)! \geq 2^{k+1}$

$(k+1)! = k!(k+1)$	Rewrite $(k+1)!$ in terms of $k!$
$\geq 2^k(k+1)$	Use step 2 and the multiplication property.
$\geq 2^k(2)$	$k+1 \geq 5 > 2$, so we can use the multiplication property again.
$= 2^{k+1}$	

Therefore $n! \geq 2^n$ for $n \geq 4$.

Example 2

For what values of x is the inequality $x > x^2$ true?

Solution

The inequality is true if x is a number between -1 and 1 but not 0.

Example 3

Prove that $9^n - 1$ is divisible by 8 for all positive integers n.

Solution

1. Base case:	If $n = 1$, $9^n - 1 = 9-1 = 8 = 8(1)$
2. Inductive hypothesis:	Assume that $9^k - 1$ is divisible by 8.
3. Inductive step:	Show that $9^{k+1} - 1$ is divisible by 8.

$9^k - 1$ divisible by 8 $\Rightarrow 8W = (9^k - 1)$ for some **integer** W
$9^{k+1} - 1 = 9(9^k - 1) + 8 = 9(8W) + 8$, which is divisible by 8

Example 4

Prove that $2^n < n!$ for all positive integers n where $n \geq 4$.

Solution

Use the three steps of proof by induction:

Step 1) Base Case: $2^4 < 4!$

$2 \cdot 2 \cdot 2 \cdot 2 < 1 \cdot 2 \cdot 3 \cdot 4$

$16 < 24$..... This checks out

Step 2) Assumption: $2^k < k!$

Step 3) Induction Step: starting with $2^k < k!$ prove $2^k(k+1) < k!(k+1)$

$2^k(k+1) < (k+1)!$

$2 < k+1$ ..... If $k \geq 4$ then this is true

$2^k \cdot 2 < 2^k(k+1)$ ... Multiply both sides by 2^k

$2^{k+1} < 2^k(k+1)$

$2^{k+1} < (k+1)!$

$\therefore 2^n < n!$ for all positive integers n where $n \geq 4$

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Example 5

Prove that $n^2 < 3^n$ for all integers $n > 2$.

Solution

Use the three steps of proof by induction:

Step 1) Base Case: $(n = 1) 1^2 < 3^1$ or, if you prefer, $(n = 2) 2^2 < 3^2$

Step 2) Assumption: $k^2 < 3^k$

Step 3) Induction Step: starting with $k^2 < 3^k$ prove $(k+1)^2 < 3^{k+1}$

$k^2 \cdot 3 < 3^k \cdot 3$

$2k < k^2$ and $1 < k^2$..... assuming $2 < k$ as specified in the question

$2k + 1 < 2k^2$..... combine the two statements above

$k^2 + 2k + 1 < 3k^2$..... add k^2 to both sides

$(k+1)^2 < 3k^2$

$(k+1)^2 < 3 \cdot 3^k$..... from above

$(k+1)^2 < 3^{k+1}$

$\therefore n^2 < 3^n$ for all integers $n > 2$

Example 6

Prove that $2n + 1 < 2^n$ for all integers $n > 3$

Solution

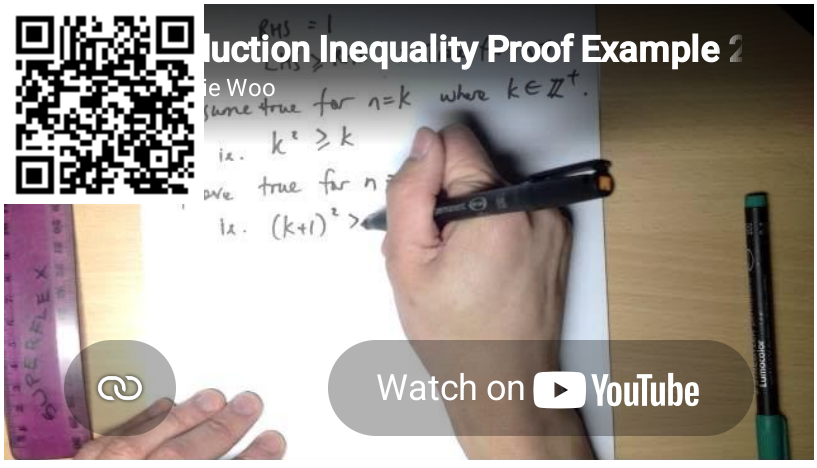
Use the three steps of proof by induction:

Step 1) Base case: If $n = 3, 2(3) + 1 = 7, 2^3 = 8 : 7 < 8$, so the base case is true.

Step 2) Inductive hypothesis: Assume that $2k + 1 < 2^k$ for $k > 3$

Step 3) Inductive step: Show that $2(k+1) + 1 < 2^{k+1}$

$2(k+1) + 1 = 2k + 2 + 1 = (2k+1) + 2 < 2^k + 2 < 2^k + 2^k = 2(2^k) = 2^{k+1}$



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Review

Prove the following inequalities.

1. $5^k < (k+5)!$
2. $1^k < (k+1)!$
3. $4^k < (k+4)!$
4. $2^k < (k+2)!$
5. For what values of x is the inequality $x > x^2$ true?
6. Prove that $3^n > n^2$ for all positive integers n.

Prove the following inequalities.

7. $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n} > \frac{13}{24} (n > 1)$
8. $2^n \geq n^2$ for $n = 4, 5, 6, \ldots$
9. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$
10. Given: $x_1, \ldots, x_n$ are positive numbers, prove the following: $\frac{(x_1 + \ldots + x_n)}{n} \geq (x_1 \cdot \ldots \cdot x_n)^{\frac{1}{n}}$
11. $n! \geq 3^n$ for $n = 7, 8, 9, \ldots$

Complete the following geometric induction proofs.

12. Prove that side length of a quadrilateral is less than the sum of all its other side lengths.
13. Prove that side length of a pentagon is less than the sum of all its other side lengths.
14. Prove that it is possible to color all regions of a plane divided by several lines with two different colors, so that any two neighbor regions contain a different color.

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.8.

Vocabulary

Term	Definition
!	The factorial of a whole number n is the product of the positive integers from 1 to n. The symbol "!" denotes factorial. n!=1·2·3·4...·(n−1)·n.
factorial	The factorial of a whole number n is the product of the positive integers from 1 to n. The symbol "!" denotes factorial. n!=1·2·3·4...·(n−1)·n.
induction	Induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
inequality	An inequality is a mathematical statement that relates expressions that are not necessarily equal by using an inequality symbol. The inequality symbols are <, >, ≤, ≥ and ≠.
Integer	The integers consist of all natural numbers, their opposites, and zero. Integers are numbers in the list ..., -3, -2, -1, 0, 1, 2, 3...
postulate	A postulate is a statement that is accepted as true without proof.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.

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SECTION OVERVIEW

7.4: Sums of Geometric Series

Topic hierarchy

7.4.1: Sums of Finite Geometric Series

7.4.2: Sums of Infinite Geometric Series

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7.4.1: Sums of Finite Geometric Series

Finding the Sum of a Finite Geometric Series

You are saving for summer camp. You deposit $100 on the first of each month into your savings account. The account grows at a rate of 0.5% per month. How much money is in your account on the first day on the 9th month?

Sum of Finite Geometric Series

We have discussed how to use the calculator to find the sum of any series provided we know the nth term rule. For a geometric series, however, there is a specific rule that can be used to find the sum algebraically. Let's look at a finite geometric sequence and derive this rule.

Given $a_n = a_1 r^{n-1}$.

The sum of the first n terms of a geometric sequence is:

$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^{n-2} + a_1 r^{n-1}$.

Now, factor out a_1 to get $a_1 (1 + r^2 + r^3 + ... + r^{n-2} + r^{n-1})$. If we isolate what is in the parenthesis and multiply this sum by $(1 - r)$ as shown below we can simplify the sum:

$$(1-r)S_n = (1-r)(1 + r + r^2 + r^3 + ... + r^{n-2} + r^{n-1})$$
$$= (1 + r + r^2 + r^3 + ... + r^{n-2} + r^{n-1} - r - r^2 - r^3 - r^4 - ... - r^{n-1} - r^n)$$
$$= (1 + r + r^2 + r^3 + ... + r^{n-2} + r^{n-1} - r - r^2 - r^3 - r^4 - ... - r^{n-1} - r^n)$$
$$= (1-r)^n$$

By multiplying the sum by $1 - r$ we were able to cancel out all of the middle terms. However, we have changed the sum by a factor of $1 - r$, so what we really need to do is multiply our sum by $\frac{1-r}{1-r}$, or 1.

$a_1 (1 + r^2 + r^3 + ... + r^{n-2} + r^{n-1}) \frac{1-r}{1-r} = \frac{a_1(1-r^n)}{1-r}$, which is the sum of a finite geometric series.

So, $S_n = \frac{a_1(1-r^n)}{1-r}$.

Let's find the sum of the first ten terms of the geometric sequence $a_n = \frac{1}{32}(-2)^{n-1}$. This could also be written as, "Let's find $\sum_{n=1}^{10} \frac{1}{32}(-2)^{n-1}$."

Using the formula, $a_1 = \frac{1}{32}$, $r = -2$, and $n = 10$.

$S_{10} = \frac{\frac{1}{32}(1-(-2)^{10})}{1-(-2)} = \frac{\frac{1}{32}(1-1024)}{3} = -\frac{341}{32}$

We can also use the calculator as shown below.

$\text{sum}(\text{seq}(1/32(-2)^{x-1}, x, 1, 10)) = -\frac{341}{32}$

Now, let's find the first term and the nth term rule for a geometric series in which the sum of the first 5 terms is 242 and the common ratio is 3.

Plug in what we know to the formula for the sum and solve for the first term:

$$242 = \frac{a_1(1-3^5)}{1-3}$$
$$242 = \frac{a_1(-242)}{-2}$$
$$242 = 121 a_1$$
$$a_1 = 2$$

The first term is 2 and $a_n = 2(3)^{n-1}$.

Finally, let's solve the following problem.

Charlie deposits $1000 on the first of each year into his investment account. The account grows at a rate of 8% per year. How much money is in the account on the first day on the 11th year.

First, consider what is happening here on the first day of each year. On the first day of the first year, $1000 is deposited. On the first day of the second year $1000 is deposited and the previously deposited $1000 earns 8% interest or grows by a factor of 1.08 (108%). On the first day of the third year another $1000 is deposited, the previous year's deposit earns 8% interest and the original deposit earns 8% interest for two years (we multiply by 1.08^2):

Sum Year 1: 1000

Sum Year 2: 1000 + 1000(1.08)

Sum Year 3: 1000 + 1000(1.08) + 1000$(1.08)^2$

Sum Year 4: 1000 + 1000(1.08) + 1000$(1.08)^2$ + 1000$(1.08)^3$

$\vdots$

Sum Year 11: 1000 + 1000(1.08) + 1000$(1.08)^2$ + 1000$(1.08)^3$ + … + 1000$(1.08)^9$ + 1000$(1.08)^{10}$

$*$ There are 11 terms in this series because on the first day of the 11th year we make our final deposit and the original deposit earns interest for 10 years.

This series is geometric. The first term is 1000, the common ratio is 1.08 and $n = 11$. Now we can calculate the sum using the formula and determine the value of the investment account at the start of the 11th year.

$$s_{11} = \frac{1000(1-1.08^{11})}{1-1.08} = 16645.48746 \approx \$16,645.49$$

Examples

Example 1

Earlier, you were asked to find how much money is in your account on the first day of the 9th month.

Solution

There are 9 terms in this series because on the first day of the 9th month you make your final deposit and the original deposit earns interest for 8 months.

This series is geometric. The first term is 100, the common ratio is 1.005 and n=9. Now we can calculate the sum using the formula and determine the value of the investment account at the start of the 9th month.

$$s_9 = \frac{100(1-1.005^9)}{1-1.005} = 918.22$$

Therefore there is $918.22 in the account at the beginning of the ninth month.

Example 2

Evaluate $\sum_{n=3}^{8} 2(-3)^{n-1}$

Solution

Since we are asked to find the sum of the 3rd through 8th terms, we will consider a_3 as the first term. The third term is $a_3 = 2(-3)^2 = 2(9) = 18$. Since we are starting with term three, we will be summing 6 terms, $a_3 + a_4 + a_5 + a_6 + a_7 + a_8$, in total. We can use the rule for the sum of a geometric series now with $a_1 = 18$, $r = -3$ and $n = 6$ to find the sum:

$$\sum_{n=3}^{8} 2(-3)^{n-1} = \frac{18(1-(-3)^6)}{1-(-3)} = -3276$$

Example 3

If the sum of the first seven terms in a geometric series is $\frac{215}{8}$ and $r = -\frac{1}{2}$, find the first term and the nth term rule.

Solution

We can substitute what we know into the formula for the sum of a geometric series and solve for a_1.

$$(l)\ \frac{215}{8} = \frac{a_1\left(1-\left(-\frac{1}{2}\right)^7\right)}{1-\left(-\frac{1}{2}\right)}$$

$$\frac{215}{8} = a_1\left(\frac{43}{64}\right)$$

$$a_1 = \left(\frac{64}{43}\right)\left(\frac{215}{8}\right) = 40$$

The nth term rule is $a_n = 40\left(-\frac{1}{2}\right)^{n-1}$

Example 4

Sam deposits \$50 on the first of each month into an account which earns 0.5% interest each month. To the nearest dollar, how much is in the account right after Sam makes his last deposit on the first day of the fifth year (the 49th month).

Solution

The deposits that Sam make and the interest earned on each deposit generate a geometric series,

$$S_{49} = \underset{\text{last deposit}}{50} + 50(1.005)^1 + 50(1.005)^2 + 50(1.005)^3 + \ldots + 50(1.005)^{47} + \underset{\text{first deposit}}{50(1.005)^{48}},$$

Note that the first deposit earns interest for 48 months and the final deposit does not earn any interest. Now we can find the sum using $a_1 = 50$, $r = 1.005$ and $n = 49$.

$$S_{49} = \frac{50(1-(1.005)^{49})}{(1-1.005)} \approx \$2768$$

Review

Use the formula for the sum of a geometric series to find the sum of the first five terms in each series.

1. $a_n = 36\left(\frac{2}{3}\right)^{n-1}$
2. $a_n = 9(-2)^{n-1}$
3. $a_n = 5(-1)^{n-1}$
4. $a_n = \frac{8}{25}\left(\frac{5}{2}\right)^{n-1}$
5. $a_n = \frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$

Find the indicated sums using the formula and then check your answers with the calculator.

6. $\sum_{n=1}^{4}(-1)\left(\frac{1}{2}\right)^{n-1}$
7. $\sum_{n=2}^{8}(128)\left(\frac{1}{4}\right)^{n-1}$
8. $\sum_{n=2}^{7}\frac{125}{64}\left(\frac{4}{5}\right)^{n-1}$
9. $\sum_{n=5}^{11}\frac{1}{32}(-2)^{n-1}$

Given the sum and the common ratio, find the nth term rule for the series.

10. $\sum_{n=1}^{6} a_n = -63$ and $r = -2$
11. $\sum_{n=1}^{4} a_n = 671$ and $r = \frac{5}{6}$
12. $\sum_{n=1}^{5} a_n = 122$ and $r = -3$
13. $\sum_{n=2}^{7} a_n = -\frac{63}{2}$ and $r = -\frac{1}{2}$

Solve the following word problems using the formula for the sum of a geometric series.

14. Sapna's grandparents deposit \$1200 into a college savings account on her 5th birthday. They continue to make this birthday deposit each year until making the final deposit on her 18th birthday. If the account earns 5% interest annually, how much is there after the final deposit?
15. Jeremy wants to have save \$10,000 in five years. If he makes annual deposits on the first of each year and the account earns 4.5% interest annually, how much should he deposit each year in order to have \$10,000 in the account after the final deposit on the first of the 6th year. Round your answer to the nearest \$100.

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Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.10.

Vocabulary

Term	Definition
induction	Induction is a method of mathematical proof typically used to establish that a given statement is true for all positive integers.
series	A series is the sum of the terms of a sequence.

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7.4.2: Sums of Infinite Geometric Series

Sums of Infinite Geometric Series

Sayber's mom told him to clean his room on Saturday morning.

"But, MOM! It's gonna take forever!" said Sayber.

"Oh, don't be overly dramatic," said mom.

"I am NOT being dramatic!" Sayber said.

"If I start right now, it is going to take me at LEAST an hour to clean this half alone, then it will take another half hour to clean half of the remainder, and 15 mins to clean half of THAT remainder... since I will always have half left, I will never be done!"

Do you agree with Sayber? Will Sayber be stuck with a vacuum in his hand forever? Tune in next week...

Sums of Infinite Geometric Series

Let's return to the situation in the introduction: Poor Sayber is stuck cleaning his room. He cleans half of the room in 60 mins. Then he cleans half of what is left, 30 more minutes, half again for 15 more. If he keeps cleaning half of the remaining area, how will he ever finish the room?

We know that the pieces have to add up to some finite time period (no matter what it feels like, Sayber CAN get the room clean), but how is it possible for the sum of an infinite number of terms to be a finite number?

To find the sum of an infinite number of terms, we should consider some **partial sums**. Three partial sums, relatively early in the series, could be: $S_2 = 90$, $S_3 = 105$, and $S_6 = 118.125$ or $118\frac{1}{8}$

Now let's look at larger values of n:

S_7	$= \frac{60\left(1-\left(\frac{1}{2}\right)^7\right)}{1-\frac{1}{2}} \approx 119.06 \text{ minutes}$
S_8	$= \frac{60\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} \approx 119.5 \text{ minutes}$
S_{10}	$= \frac{60\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} \approx 119.9 \text{ minutes}$

As *n* approaches infinity, the value of S_n seems to approach 120 minutes. In terms of the actual sums, what is happening is this: as *n* increases, the nth term gets smaller and smaller, and so the nth term contributes less and less to the value of S_n. We say that the series **converges,** and we can write this with a limit:

$\lim_{n\to\infty} S_n$	$= \lim_{n\to\infty} \left(\frac{60(1-(\frac{1}{2})^n)}{1-\frac{1}{2}}\right)$
	$= \lim_{n\to\infty} \left(\frac{60(1-(\frac{1}{2})^n)}{\frac{1}{2}}\right)$
	$= \lim_{n\to\infty} \left(120\left(1-\left(\frac{1}{2}\right)^n\right)\right)$

As *n* approaches infinity, the value of $\left(\frac{1}{2}\right)^n$ gets smaller and smaller. That is, the value of this expression approaches 0. Therefore the value of $1-\left(\frac{1}{2}\right)^n$ approaches 1, and $120\left(1-\left(\frac{1}{2}\right)^n\right)$ approaches $120(1) = 120$.

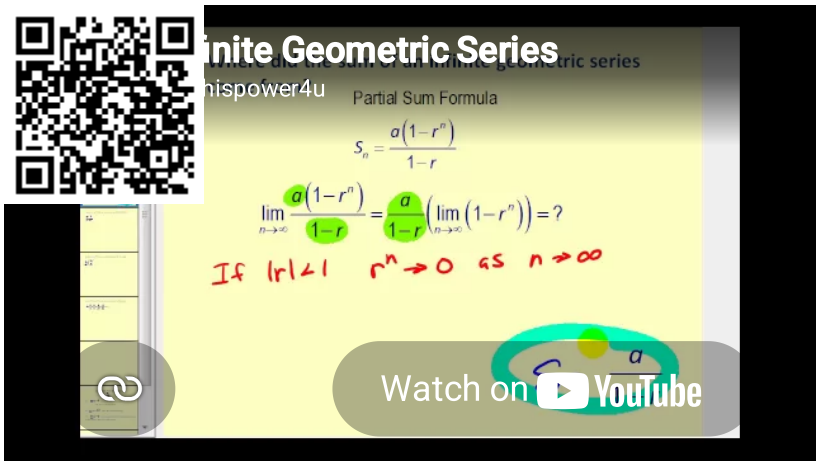
Therefore, no matter how long the process continues, Sayber will not spend more than 2hrs cleaning the room. Of course, it may SEEM like a lot more!

We can do the same analysis for the general case of a geometric series, as long as the terms are getting smaller and smaller. This means that the common ratio must be a number between -1 and 1: |r| < 1.

$\lim_{n\to\infty} S_n$	$= \lim_{n\to\infty} \left(\frac{a_1(1-r^n)}{1-r}\right)$
	$= \frac{a_1}{1-r}$, as $(1-r^n) \to 1$

Therefore, we can find the sum of an infinite geometric series using the formula $S = \frac{a_1}{1-r}$.

When an infinite sum has a finite value, we say the sum **converges.** Otherwise, the sum **diverges.** A sum converges only when the terms get closer to 0 after each step, but that alone is not a sufficient criterion for convergence. For example, the sum $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ does not **converge**.



Examples

Example 1

Find the sum of the **convergent** series: $40 + -20 + 10 + -5 + ...$

Solution

The common ratio is $\frac{-1}{2}$. Therefore the sum converges to:

$$\frac{40}{1-\left(\frac{-1}{2}\right)} = \frac{40}{\frac{3}{2}} = 40\left(\frac{2}{3}\right) = \frac{80}{3}$$

Example 2

Determine if the series converges. If it converges, find the sum.

a. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$

b. $3 + -6 + 12 + -24 + ...$

Solution

a. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...$ converges. The common ratio is (1/3) . Therefore the sum converges to:

$$\frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

b. The series 3 + -6 + 12 + -24 + ... does not converge, as the common ratio is -2.

Remember that the idea of an infinite sum was introduced in the context of a realistic situation, albeit a paradoxical one. We can in fact use infinite geometric series to model other realistic situations. Here we will look at another example: the total vertical distance traveled by a bouncing ball.

Example 3

A ball is dropped from a height of 20 feet. Each time it bounces, it reaches 50% of its previous height. What is the total vertical distance the ball travels?

Solution

We can think of the total distance as the distance the ball travels down + the distance the ball travels back up. The downward bounces form a geometric series:

20 + 10 + 5 +...

The upward bounces form the same series, except the first term is 10.

So the total distance is: $\sum_{n=1}^{\infty} 20\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty} 10\left(\frac{1}{2}\right)^{n-1}$.

Each sum converges, as the common ratio is (1/2). Therefore the total distance is:

$$\frac{20}{1-\frac{1}{2}} + \frac{10}{1-\frac{1}{2}} = \frac{20}{\frac{1}{2}} + \frac{10}{\frac{1}{2}} = 40 + 20 = 60$$

So the ball travels a total vertical distance of 60 feet.

Example 4

Determine if the following series converges or diverges. If it converges, find the sum.

240 + 60 + 15 + ...

Solution

The sum converges. S = 320.

Example 5

In this lesson, we proved the formula for the sum of a geometric series, $S_n = \frac{a_1(1-r^n)}{1-r}$ using induction.

Prove this formula without induction:

Solution

Step 1: Let $S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$

$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$

Step 2: Multiply S_n by r to obtain a second equation

$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^n$

Step 3: Subtract the equations and solve for S_n.

$S_n - rS_n = a_1 - a_1 r^n$
$\Rightarrow S_n(1-r) = a\left(1-r^n\right)$
$\Rightarrow S_n = \frac{a(1-r^n)}{(1-r)}$

Example 6

A ball is dropped from a height of 40 feet, and each time it bounces, it reaches 25% of its previous height.

a. Find the total vertical distance the ball travels, using the method used in the lesson.
b. Find the total vertical distance the ball travels using a single series.

Solution

a. $\sum_{n=1}^{\infty} 40\left(\frac{1}{4}\right)^{n-1} + \sum_{n=1}^{\infty} 20\left(\frac{1}{4}\right)^{n-1} = 66\frac{2}{3}$
b. $\sum_{n=1}^{\infty} 50\left(\frac{1}{4}\right)^{n-1} = 66\frac{2}{3}$

(Hint: write out several terms for each bounce. For example, the first bounce is: 40 feet down + 10 feet up = 50 feet traveled.)

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Example 7

Below are two infinite series that are not geometric. Use a graphing calculator to examine partial sums. Does either series converge?

a. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$
b. $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...$

Solution

a. This series does not converge.
b. This series converges around 1.65. (The actual sum is $\frac{\pi^2}{6}$)

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Review

1. Find the sum of the first 10 terms of $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ using a graphing calculator.
2. Find the sum of the first 20 terms of $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ using a graphing calculator.
3. Conjecture on the possible convergence of the series in questions 1 and 2.

Evaluate the infinite sum of each of the following geometric series:

4. $-2+1-\frac{1}{2}+...$
5. $-6+\frac{24}{5}-\frac{96}{25}+...$
6. $3+\frac{3}{2}-\frac{3}{4}+...$
7. $-6+4-\frac{8}{3}+...$
8. $1+\frac{1}{2}+\frac{1}{4}+...$

Evaluate the infinite sum of each of the following geometric series:

9. $\sum_{n=1}^{\infty} -3\left(\frac{1}{2}\right)^{(n-1)}$
10. $\sum_{n=1}^{\infty} -2\left(\frac{4}{7}\right)^{(n-1)}$
11. $\sum_{n=1}^{\infty} 7\left(\frac{-4}{5}\right)^{(n-1)}$
12. $\sum_{n=1}^{\infty} -9\left(\frac{-1}{5}\right)^{(n-1)}$
13. $\sum_{n=1}^{\infty} 5\left(\frac{-5}{7}\right)^{(n-1)}$
14. $\sum_{n=1}^{\infty} 6\left(\frac{1}{5}\right)^{(n-1)}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 7.10.

Vocabulary

Term	Definition
converge	If a series has a limit, and the limit exists, the series converges.
convergent	If a series has a limit, and the limit exists, the series is convergent.
divergent	If a series does not have a limit, or the limit is infinity, then the series is divergent.
diverges	If a series does not have a limit, or the limit is infinity, then the series diverges.
geometric sequence	A geometric sequence is a sequence with a constant ratio between successive terms. Geometric sequences are also known as geometric progressions.
geometric series	A geometric series is a geometric sequence written as an uncalculated sum of terms.
partial sums	A partial sum is the sum of the first "n" terms in an infinite series, where "n" is some positive integer.

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SECTION OVERVIEW

7.5: Factorials and Combinations

Topic hierarchy

7.5.1: Binomial Theorem and Expansions

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7.5.1: Binomial Theorem and Expansions

Binomial Theorem

The Binomial Theorem tells you how to expand a binomial such as $(2x-3)^5$ without having to compute the repeated distribution. What is the expanded version of $(2x-3)^5$?

Introduction to the Binomial Theorem

The **Binomial Theorem** states:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Writing out a few terms of the summation symbol helps you to understand how this theorem works:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1}b^1 + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n} b^n$$

Going from one term to the next in the expansion, you should notice that the exponents of a decrease while the exponents of b increase. You should also notice that the coefficients of each term are combinations. Recall that $\binom{n}{0}$ is the number of ways to choose objects from a set of n objects.

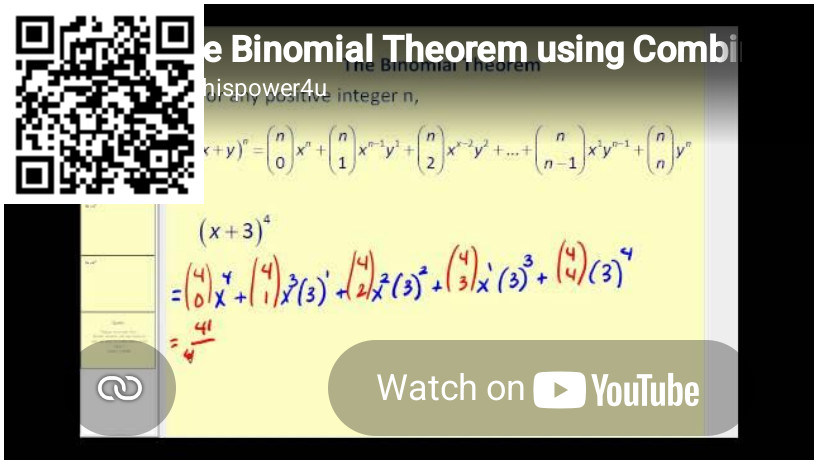
Take the following binomial:

$$(m-n)^6$$

It can be expanded using the Binomial Theorem:

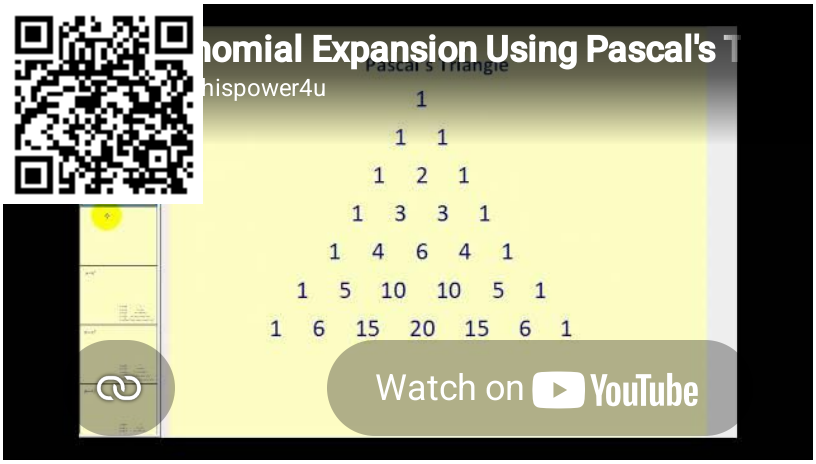
$$\begin{aligned}(m-n)^6 =& \binom{6}{0} m^6 + \binom{6}{1} m^5(-n)^1 + \binom{6}{2} m^4(-n)^2 + \binom{6}{3} m^3(-n)^3 \\ &+ \binom{6}{4} m^2(-n)^4 + \binom{6}{5} m^1(-n)^5 + \binom{6}{6} (-n)^6 \\ =& 1m^6 - 6m^5n + 15m^4n^2 - 20m^3n^3 + 15m^2n^4 - 6m^1n^5 + 1n^6\end{aligned}$$

Be extremely careful when working with binomials of the form $(a-b)^n$. You need to remember to capture the negative with the second term as you write out the expansion: $(a-b)^n = (a+(-b))^n$.



Another way to think about the coefficients in the Binomial Theorem is that they are the numbers from Pascal's Triangle. Look at the expansions of $(a+b)^n$ below and notice how the coefficients of the terms are the numbers in Pascal's Triangle.

$$(a+b)^0 = 1$$
$$(a+b)^1 = 1a + 1b$$
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$\vdots$$



Examples

Example 1

Earlier, you were asked to expand $(2x-3)^5$. The expanded version of $(2x-3)^5$ is:

Solution

$$\begin{aligned}(2x-3)^5 =& \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 \\ &+ \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(-3)^6 \\ =&(2x)^5 + 5(2x)^4(-3)^1 + 10(2x)^3(-3)^2 \\ &+ 10(2x)^2(-3)^3 + 5(2x)^1(-3)^4 + (-3)^5 \\ =&32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

Example 2

What is the coefficient of the term x^7y^9 in the expansion of the binomial $(x+y)^{16}$?

Solution

The Binomial Theorem allows you to calculate just the coefficient you need.

$$\binom{16}{9} = \frac{16!}{9!7!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 11,440$$

Example 3

What is the coefficient of x^6 in the expansion of $(4-3x)^7$?

Solution

For this problem you should calculate the whole term, since the 3 and the 4 in $(3-4x)$ will impact the coefficient of x^6 as well. $\binom{7}{6}4^1(-3x)^6 = 7 \cdot 4 \cdot 729x^6 = 20,412x^6$. The coefficient is 20,412.

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Example 4

Compute the following summation.

$\sum_{i=0}^{4} \binom{4}{i}$

Solution

This is asking for $\binom{4}{0} + \binom{4}{1} + \cdots + \binom{4}{4}$, which are the sum of all the coefficients of $(a+b)^4$.

$1+4+6+4+1=16$

Example 5

Collapse the following polynomial using the Binomial Theorem.

$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

Solution

Since the last term is -1 and the power on the first term is a 5 you can conclude that the second half of the binomial is $(?-1)^5$. The first term is positive and $(2x)^5 = 32x^5$, so the first term in the binomial must be $2x$. The binomial is $(2x-1)^5$.

Review

Expand each of the following binomials using the Binomial Theorem.

1. $(x-y)^4$
2. $(x-3y)^5$
3. $(2x+4y)^7$
4. What is the coefficient of x^4 in $(x-2)^7$?
5. What is the coefficient of x^3y^5 in $(x+y)^8$?
6. What is the coefficient of x^5 in $(2x-5)^6$?
7. What is the coefficient of y^2
8. What is the coefficient of x^2y^6 in $(2x+y)^8$?
9. What is the coefficient of x^3y^4 in $(5x+2y)^7$?

Compute the following summations.

10. $\sum_{i=0}^{9} \binom{9}{i}$
11. $\sum_{i=0}^{12} \binom{12}{i}$
12. $\sum_{i=0}^{8} \binom{8}{i}$

Collapse the following polynomials using the Binomial Theorem.

13. $243x^5 - 405x^4 + 270x^3 - 90x^2 = 15x - 1$
14. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$
15. $128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

Review (Answers)

To see the Review answers, open this PDF file and look for section 12.8.

Vocabulary

Term	Definition
combination	Combinations are distinct arrangements of a specified number of objects without regard to order of selection from a specified set.

Image Attributions

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SECTION OVERVIEW

7.6: Arithmetic Sequences

Topic hierarchy

7.6.1: Finding the nth Term Given the Common Difference and a Term

7.6.2: Finding the nth Term Given Two Terms for an Arithmetic Sequence

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7.6.1: Finding the nth Term Given the Common Difference and a Term

Arithmetic Sequences and Finding the nth Term Given the Common Difference and a Term

Halley's Comet appears in the sky approximately every 76 years. The comet was first spotted in the year 1531. Find the nth term rule and the 10th term for the sequence represented by this situation.

Arithmetic Sequence

In this concept we will begin looking at a specific type of sequence called an **arithmetic sequence**. In an arithmetic sequence the difference between any two consecutive terms is constant. This constant difference is called the **common difference**. We can generalize the equation for an arithmetic sequence below:

$a_n - a_{n-1} = d$, where a_{n-1} and a_n represent two consecutive terms and d represents the common difference.

Since the same value, the common difference, d, is added to get each successive term in an arithmetic sequence we can determine the value of any term from the first term and how many time we need to add d to get to the desired term as illustrated below:

Given the sequence: $22, 19, 16, 13, \ldots$ in which $a_1 = 22$ and $d = -3$

$$
\begin{aligned}
&a_1 = 22 \text{ or } 22 + (1-1)(-3) = 22 + 0 = 22\\
&a_2 = 19 \text{ or } 22 + (2-1)(-3) = 22 + (-3) = 19\\
&a_3 = 16 \text{ or } 22 + (3-1)(-3) = 22 + (-6) = 16\\
&a_4 = 13 \text{ or } 22 + (4-1)(-3) = 22 + (-9) = 13\\
&\quad\vdots\\
&a_n = 22 + (n-1)(-3)\\
&a_n = 22 - 3n + 3\\
&a_n = -3n + 25
\end{aligned}
$$

Now we can generalize this into a rule for the nth term of any arithmetic sequence: $a_n = a_1 + (n-1)d$

Let's find the common difference and n^{th} term rule for the arithmetic sequence: $2, 5, 8, 11 \ldots$

To find the common difference we subtract consecutive terms.

$$
\begin{aligned}
&5 - 2 = 3\\
&8 - 5 = 3, \text{ thus the common difference is } 3\\
&11 - 8 = 3
\end{aligned}
$$

Now we can put our first term and common difference into the nth term rule discovered above and simplify the expression.

$$
\begin{aligned}
a_n &= 2 + (n-1)(3)\\
&= 2 + 3n - 3 \quad \text{, so } a_n = 3n - 1.\\
&= 3n - 1
\end{aligned}
$$

Now, let's find the nth term rule and thus the 100th term for the arithmetic sequence in which $a_1 = -9$ and $d = 2$.

We have what we need to plug into the rule:

$$
\begin{aligned}
a_n &= -9 + (n-1)(2)\\
&= -9 + 2n - 2 \quad \text{, thus the } n^{th} \text{ term rule is } a_n = 2n - 11\\
&= 2n - 11
\end{aligned}
$$

Now to find the 100th term we can use our rule and replace n with 100:

$$a_{100} = 2(100) - 11 = 200 - 11 = 189$$

Finally, let's find the nth term rule and thus the 100th term for the arithmetic sequence in which $a_3 = 8$ and $d = 7$.

This one is a little less straightforward as we will have to first determine the first term from the term we are given. To do this, we will replace a_n with $a_3 = 8$ and use 3 for n in the formula to determine the unknown first term as shown:

$$
\begin{aligned}
a_1 + (3-1)(7) &= 8\\
a_1 + 2(7) &= 8\\
a_1 + 14 &= 8\\
a_1 &= -6
\end{aligned}
$$

Now that we have the first term and the common difference we can follow the same process used in the previous example to complete the problem.

$$\begin{aligned} a_n &= -6 + (n-1)(7) \\ &= -6 + 7n - 7 \quad \text{, thus } a_n = 7n - 13. \\ &= 7n - 13 \end{aligned}$$

Now we can find the 100th term: $a_{100} = 7(100) - 13 = 687$.

Examples

Example 1

Earlier, you were asked to find the nth term rule and the 10th term for the sequence represented by Halley's Comet, which appears in the sky once approximately every 76 years and first appeared in 1531.

Solution

From the information given, we can conclude that $a_1 = 1531$ and $d = 76$.

We now have what we need to plug into the rule:

$$\begin{aligned} a_n &= 1531 + (n-1)(76) \\ &= 1531 + 76n - 76 \quad \text{, thus the } n^{th} \text{ term rule is } a_n = 76n + 1455 \end{aligned}$$

Now to find the 10th term we can use our rule and replace n with 10:

$$a_{10} = 76(10) + 1455 = 760 + 1455 = 2215$$

Example 2

Find the common difference and the nth term rule for the sequence: $5, -3, -11, \ldots$

Solution

The common difference is $-3 - 5 = -8$. Now

$$a_n = 5 + (n-1)(-8) = 5 - 8n + 8 = -8n + 13$$

Example 3

Write the nth term rule and find the 45th term for the arithmetic sequence with $a_{10} = 1$ and $d = -6$.

Solution

To find the first term:

$$\begin{aligned} a_1 + (10-1)(-6) &= 1 \\ a_1 - 54 &= 1 \\ a_1 &= 55 \end{aligned}$$

Find the nth term rule: $a_n = 55 + (n-1)(-6) = 55 - 6n + 6 = -6n + 61$.

Finally, the 45th term: $a_{45} = -6(45) + 61 = -209$.

Example 4

Find the 62nd term for the arithmetic sequence with $a_1 = -7$ and $d = \frac{3}{2}$.

Solution

This time we will not simplify the nth term rule, we will just use the formula to find the 62rd term:

$$a_{62} = -7 + (62-1)\left(\tfrac{3}{2}\right) = -7 + 61\left(\tfrac{3}{2}\right) = -\tfrac{14}{2} + \tfrac{183}{2} = \tfrac{169}{2}.$$

Review

Identify which of the following sequences is arithmetic. If the sequence is arithmetic find the nth term rule.

1. 2, 3, 4, 5, …
2. 6, 2, −1, −3, …
3. 5, 0, −5, −10, …
4. 1, 2, 4, 8, …
5. 0, 3, 6, 9, …
6. 13, 12, 11, 10, …
7. 4, −3, 2, −1, …
8. a, a+2, a+4, a+6, …

Write the nth term rule for each arithmetic sequence with the given term and common difference.

9. $a_1 = 15$ and $d = -8$
10. $a_1 = -10$ and $d = \frac{1}{2}$
11. $a_3 = 24$ and $d = -2$
12. $a_5 = -3$ and $d = 3$
13. $a_{10} = -15$ and $d = -11$
14. $a_7 = 32$ and $d = 7$
15. $a_{n-2} = 3n + 2$, find a_n

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.5.

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7.6.2: Finding the nth Term Given Two Terms for an Arithmetic Sequence

Finding the nth Term Given Two Terms

You are paying off a student loan in monthly installments. After your fifth payment your remaining balance on the loan is $17,500. After your 16th payment, your remaining balance is $12,000. What is the nth term rule for the sequence represented by this situation?

Finding the nth Term

Previously, we were given the common difference directly or two consecutive terms from which we could determine the common difference. In this concept we will find the common difference and write nth term rule given any two terms in the sequence.

Let's find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_7 = 17$ and $a_{20} = 82$.

We will start by using the nth term rule for an arithmetic sequence to create two equations in two variables:

$a_7 = 17$, so $a_1 + (7-1)d = 17$ or more simply: $a_1 + 6d = 17$

$a_{20} = 82$, so $a_1 + (20-1)d = 82$ or more simply: $a_1 + 19d = 82$

Solve the resulting system:

$$\begin{array}{rl} a_1 + 6d = 17 & \\ \underline{-1(a_1 + 19d = 82)} & \Rightarrow \\ & \end{array} \qquad \begin{array}{r} a_1 + 6d = 17 \\ \underline{-a_1 - 19d = -82} \\ -13d = -65 \\ d = 5 \end{array}$$

Replacing d with 5 in one of the equations we get

$$\begin{aligned} a_1 + 6(5) &= 7 \\ a_1 + 30 &= 17 \\ a_1 &= -13 \end{aligned}$$

Using these values we can find the nth term rule:

$$\begin{aligned} a_n &= -13 + (n-1)(5) \\ a_n &= -13 + 5n - 5 \\ a_n &= 5n - 18 \end{aligned}$$

Now, let's find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_{11} = -13$ and $a_{40} = -71$.

Though this is exactly the same type of problem as the previous problem above, we are going to use a different approach. We know that the nth term rule is really just using the first term and adding d to it $n-1$ times to find the nth term. We are going to use that idea to find the common difference. To get from the 11th term to the 40th term, the common difference is added 40−11 or 29 times. The difference in the term values is −71−(−13) or -58. What must be added 29 times to create a difference of -58? We can subtract the terms and divide by the difference in term number to determine the common difference.

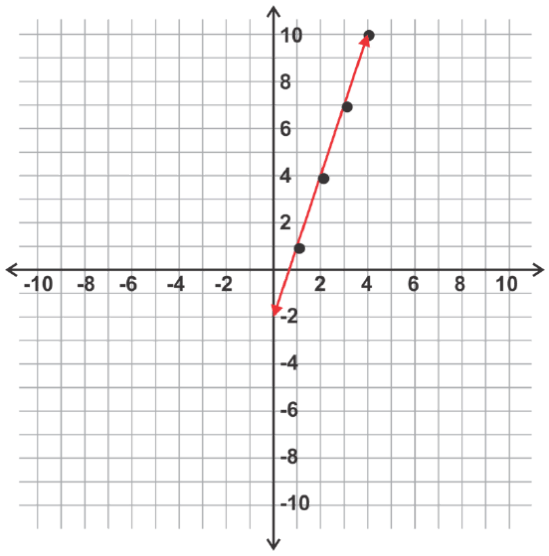
$$\frac{-71-(-13)}{40-11} = \frac{-71+13}{29} = \frac{-58}{29} = -2. \text{ So } d = -2$$

Now we can use the common difference and one of the terms to find the first term as we did previously.

$$\begin{aligned} a_1 + (11-1)(-2) &= -13 \\ a_1 + (-20) &= -13 \\ a_1 &= 7 \end{aligned}$$

Writing the nth term rule we get: $a_n = 7 + (n-1)(-2) = 7 - 2n + 2 = -2n + 9$.

Before we look at the next problem for this concept, we are going to connect the nth term rule for an arithmetic sequence to the equation of a line. Have you noticed that the simplified n^{th} term rule, $a_n = pn + q$, where p and q represent constants, looks a little like $y = mx + b$, the slope-intercept form of the equation of a line? Let's explore why this is the case using the arithmetic sequence 1, 4, 7, 10, … If we create points by letting the x – coordinate be the term number and the y – coordinate be the term, we get the following points and can plot them in the coordinate plane as shown below,



The points are: (1, 1), (2, 4), (3, 7), (4, 10)

Notice, that all of these points lie on the same line. This happens because for each increase of one in the term number (x), the term value (y) increases by 3. This common difference is actually the slope of the line.

We can find the equation of this line using the slope, 3, and the point (1, 1) in the equation $y = mx + b$ as follows:

$$\begin{aligned} 1 &= 3(1) + b \\ 1 &= 2 + b \\ -1 &= b \end{aligned}$$, so the equation of the line is $y = 3x - 1$

The nth term rule for the sequence is thus: $a_n = 3n - 1$.

Finally, let's find the common difference, first term and n^{th} term rule for the arithmetic sequence in which $a_{10} = -50$ and $a_{32} = -182$.

This time we will use the concept that the terms in an arithmetic sequence are actually points on a line to write an equation. In this case our points are (10, −50) and (32, −182). We can find the slope and the equation as shown.

$$m = \frac{-182-(-50)}{32-10} = \frac{-132}{22} = -6$$

Use the point (10, −50) so find the y-intercept:

$$\begin{aligned} -50 &= -6(10) + b \\ -50 &= -60 + b \\ 10 &= b \end{aligned}$$

So $y = -6x + 10$ and $a_n = -6n + 10$.

Examples

Example 1

Earlier, you were asked to find the nth term rule for the sequence represented by the student loan situation.

Solution

In this case our points are (5, 17, 500) and (16, 12, 000). We can find the slope and the equation as shown.

$$m = \frac{12{,}000-17{,}500}{16-5} = \frac{-5500}{11} = -500$$

Use the point (5, 17, 500) to find the y-intercept:

$$\begin{aligned} 17{,}500 &= -500(5) + b \\ 17{,}500 &= -2500 + b \\ 20{,}000 &= b \end{aligned}$$

So $y = -500x + 20,000$ and $a_n = -500n + 20,000$.

Example 2

Find the nth term rule for the arithmetic sequence with $a_6 = -13$ and $a_{15} = -40$.

Solution

From $a_6 = -13$ we get the equation $a_1 + (6-1)d = a_1 + 5d = -13$.

From $a_{15} = -40$ we get the equation $a_1 + (15-1)d = a_1 + 14d = -40$.

Use the two equations to solve for a_1 and d:

$$\begin{array}{r} a_1 + 5d = -13 \\ \underline{-a_1 - 14d = 40} \\ -9d = 17 \\ d = -3 \end{array} \quad \text{Used } d \text{ to find } a_1 \Rightarrow \quad \begin{array}{r} a_1 + 5(-3) = -13 \\ a_1 - 15 = -13 \\ a_1 = 2 \end{array}$$

Find the nth term rule: $a_n = 2 + (n-1)(-3) = 2 - 3n + 3 = -3n + 5$.

Example 3

Find the nth term rule for the arithmetic sequence with $a_6 = 13$ and $a_{22} = 77$.

Solution

The common difference is $\frac{77-13}{22-6} = \frac{64}{16} = 4$. The first term can be found using $a_6 = 13$:

$$\begin{aligned} a_1 + (6-1)(4) &= 13 \\ a_1 + 20 &= 13 \\ a_1 &= -7 \end{aligned}$$

Thus $a_n = -7 + (n-1)(4) = -7 + 4n - 4 = 4n - 11$.

Example 4

Find the nth term rule for the arithmetic sequence with $a_7 = -75$ and $a_{25} = -273$.

Solution

From $a_7 = -75$ we get the point $(7, -75)$. From $a_{25} = -273$ we get the point $(25, -273)$.

The slope between these points is $\frac{-273-(-75)}{25-7} = \frac{-198}{18} = -11$

The y-intercept can be found next using the point $(7, -75)$:

$$\begin{aligned} -75 &= -11(7) + b \\ -75 &= -77 + b \\ 2 &= b \end{aligned}$$

The final equation is $y = -11x + 2$ and the nth term rule is $a_n = -11n + 2$.

Review

Use the two given terms to find an nth term rule for the sequence.

1. $a_7 = -17$ and $a_{25} = -71$
2. $a_{11} = 23$ and $a_{42} = 85$
3. $a_3 = -6$ and $a_{12} = -3$
4. $a_8 = 24$ and $a_2 = 9$
5. $a_6 = -27$ and $a_{10} = -47$
6. $a_4 = 37$ and $a_{12} = 85$

7. $a_{13} = -20$ and $a_{30} = -54$
8. $a_3 = 23$ and $a_9 = 65$
9. $a_{30} = -31$ and $a_{45} = -46$
10. $a_5 = 25$ and $a_{11} = 73$
11. $a_{10} = -2$ and $a_{25} = -14$
12. $a_{16} = 14$ and $a_{28} = 23$
13. $a_2 = 4n + 3$ and $a_{19} = 4n + 65$
14. $a_{10} = -3n + 7$ and $a_{23} = -3n - 16$
15. Which method do you prefer? Why?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.6.

Image Attributions

1. [Figure 1]
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SECTION OVERVIEW

7.7: Geometric Sequences

Topic hierarchy

7.7.1: Finding the nth Term Given the Common Ratio and the First Term

7.7.2: Finding the nth Term Given Two Terms for a Geometric Sequence

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7.7.1: Finding the nth Term Given the Common Ratio and the First Term

Geometric Sequences and Finding the nth Term Given the Common Ratio and the First Term

The following sequence shows the distance (in centimeters) a pendulum travels with each successive swing. Write a general rule for the geometric sequence.

80, 72, 64.8, 58.32, ...

Geometric Sequence

A **geometric sequence** is a sequence in which the ratio between any two consecutive terms, $\frac{a_n}{a_{n-1}}$, is constant. This constant value is called the **common ratio**. Another way to think of this is that each term is multiplied by the same value, the common ratio, to get the next term.

Let's consider the sequence 2, 6, 18 ,54, …

Is this sequence geometric? If so, what is the common difference?

If we look at each pair of successive terms and evaluate the ratios, we get $\frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$ which indicates that the sequence is geometric and that the common ratio is 3.

Now let's see if we can develop a general rule (n^{th} term) for this sequence. Since we know that each term is multiplied by 3 to get the next term, let's rewrite each term as a product and see if there is a pattern.

$$a_1 = 2$$
$$a_2 = a_1(3) = 2(3) = 2(3)^1$$
$$a_3 = a_2(3) = 2(3)(3) = 2(3)^2$$
$$a_4 = a_3(3) = 2(3)(3)(3) = 2(3)^3$$

This illustrates that the general rule is $a_n = a_1(r)^{n-1}$, where r is the common ratio. This even works for the first term since $a_1 = 2(3)^0 = 2(1) = 2$.

Now, let's write a general rule for the geometric sequence 64, 32, 16, 8, …

From the general rule above we can see that we need to know two things: the first term and the common ratio to write the general rule. The first term is 64 and we can find the common ratio by dividing a pair of successive terms, $\frac{32}{64} = \frac{1}{2}$. The nth term rule is thus $a_n = 64\left(\frac{1}{2}\right)^{n-1}$.

Finally, let's find the nth term rule for the sequence 81, 54, 36, 24, … and hence find the 12th term.

The first term here is 81 and the common ratio, r, is $\frac{54}{81} = \frac{2}{3}$. The nth term rule is $a_n = 81\left(\frac{2}{3}\right)^{n-1}$. Now we can find the 12th term $a_{12} = 81\left(\frac{2}{3}\right)^{12-1} = 81\left(\frac{2}{3}\right)^{11} = \frac{2048}{2187}$.

Use the graphing calculator for the last step and **MATH** > **Frac** your answer to get the fraction. We could also use the calculator and the general rule to generate terms seq(81(2/3)^(x−1),x,12,12). Reminder: the seq( ) function can be found in the **LIST** (**2nd STAT**) Menu under **OPS**. Be careful to make sure that the entire exponent is enclosed in parenthesis.

Examples

Example 1

Earlier, you were asked to write a general rule for the sequence 80, 72, 64.8, 58.32, ...

Solution

We need to know two things, the first term and the common ratio, to write the general rule. The first term is 80 and we can find the common ratio by dividing a pair of successive terms, $\frac{72}{80} = \frac{9}{10}$. The nth term rule is thus $a_n = 80\left(\frac{9}{10}\right)^{n-1}$

For Examples 2-4, identify which of the sequences are geometric sequences. If the sequence is geometric, find the common ratio.

Example 2

5, 10, 15, 20, …

Solution

arithmetic

Example 3

1, 2, 4, 8, …

Solution

geometric, $r = 2$

Example 4

243, 49, 7, 1, …

Solution

geometric, $r = \frac{1}{7}$

Example 5

Find the general rule and the 20th term for the sequence 3, 6, 12, 24, …

Solution

The first term is 3 and the common ratio is $r = \frac{6}{3} = 2$ so $a_n = 3(2)^{n-1}$.

The 20th term is $a_{20} = 3(2)^{19} = 1,572,864$.

Example 6

Find the nth term rule and list terms 5 thru 11 using your calculator for the sequence −1024, 768, −432, −324, …

Solution

The first term is -1024 and the common ratio is $r = \frac{768}{-1024} = -\frac{3}{4}$ so $a_n = -1024\left(-\frac{3}{4}\right)^{n-1}$.

Using the calculator sequence function to find the terms and **MATH > Frac**,

$$\text{seq}\ (-1024(-3/4)\text{^}(x-1), x, 5, 11) = \left\{ -324 \quad 243 \quad -\frac{729}{4} \quad \frac{2187}{16} \quad -\frac{6561}{256} \quad \frac{19683}{256} \quad -\frac{59049}{1024} \right\}$$

Example 7

Find the value of a 10 year old car if the purchase price was $22,000 and it depreciates at a rate of 9% per year.

Solution

The first term (value of the car after 0 years) is $22,000. The common ratio is 1−.09 or 0.91.

The value of the car after n years can be determined by $a_n = 22,000(0.91)^n$. For 10 years we get $a_{10} = 22,000(0.91)^{10} = 8567.154599 \approx \$8567$

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Review

Identify which of the following sequences are arithmetic, geometric or neither.

1. $2, 4, 6, 8, \ldots$
2. $\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \ldots$
3. $1, 2, 4, 7, \ldots$
4. $24, -16, \frac{32}{3}, -\frac{64}{9}, \ldots$
5. $10, 5, 0, -5, \ldots$
6. $3, 4, 7, 11, \ldots$

Given the first term and common ratio, write the nth term rule and use the calculator to generate the first five terms in each sequence.

7. $a_1 = 32$ and $r = \frac{3}{2}$
8. $a_1 = -81$ and $r = -\frac{1}{3}$
9. $a_1 = 7$ and $r = 2$
10. $a_1 = \frac{8}{125}$ and $r = -\frac{5}{2}$

Find the nth term rule for each of the following geometric sequences.

11. $162, 108, 72, \ldots$
12. $-625, -375, -225, \ldots$
13. $\frac{9}{4}, -\frac{3}{2}, 1, \ldots$
14. $3, 15, 75, \ldots$
15. $5, 10, 20, \ldots$
16. $\frac{1}{2}, -2, 8, \ldots$

Use a geometric sequence to solve the following word problems.

17. Rebecca inherited some land worth $50,000 that has increased in value by an average of 5% per year for the last 5 years. If this rate of appreciation continues, about how much will the land be worth in another 10 years?
18. A farmer buys a new tractor for $75,000. If the tractor depreciates in value by about 6% per year, how much will it be worth after 15 years?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.8.

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7.7.2: Finding the nth Term Given Two Terms for a Geometric Sequence

Finding the nth Term Given the Common Ratio and any Term or Two Terms

A bacteria sample doubles every hour. After four hours there are 64 bacteria in the sample. What is the nth term rule for the geometric sequence represented by this situation?

Finding the nth Term

We will be using the general rule for the nth term in a geometric sequence and the given term(s) to determine the first term and write a general rule to find any other term.

Let's consider the geometric sequence in which the common ratio is $-\frac{4}{5}$ and $a_5 = 1280$. We'll find the first term in the sequence and write the general rule for the sequence.

We will start by using the term we know, the common ratio and the general rule, $a_n = a_1 r^{n-1}$.

By plugging in the values we know, we can then solve for the first term, a_1.

$$\begin{aligned} a_5 &= a_1 \left(-\frac{4}{5}\right)^4 \\ 1280 &= a_1 \left(-\frac{4}{5}\right)^4 \\ \frac{1280}{\left(-\frac{4}{5}\right)^4} &= a_1 \\ 3125 &= a_1 \end{aligned}$$

Now, the nth term rule is $a_n = 3125\left(-\frac{4}{5}\right)^{n-1}$.

Now, let's find the nth term rule for a sequence in which $a_1 = 16$ and $a_7 = \frac{1}{4}$.

Since $a_7 = \frac{1}{4}$ and we know the first term, we can write the equation $\frac{1}{4} = 16r^6$ and solve for the common ratio:

$$\begin{aligned} \frac{1}{4} &= 16r^6 \\ \frac{1}{64} &= r^6 \\ \sqrt[6]{\frac{1}{64}} &= \sqrt[6]{r^6} \\ \frac{1}{2} &= r \end{aligned}$$

The nth term rule is $a_n = 16\left(\frac{1}{2}\right)^{n-1}$.

Finally, let's find the nth term rule for the geometric sequence in which $a_5 = 8$ and $a_{10} = \frac{1}{4}$.

Using the same method at the previous problem, we can solve for r and a_1. Then, write the general rule.

Equation 1: $a_5 = 8$, so $8 = a_1 r^4$, solving for a_1 we get $a_1 = \frac{8}{r^4}$.

Equation 2: $a_{10} = \frac{1}{4}$, so $\frac{1}{4} = a_1 r^9$, solving for a_1 we get $a_1 = \frac{\frac{1}{4}}{r^9}$.

$$\begin{aligned} \frac{8}{r^4} &= \frac{\frac{1}{4}}{r^9} \\ 8r^9 &= \frac{1}{4}r^4 \\ \frac{8r^9}{8r^4} &= \frac{\frac{1}{4}r^4}{8r^4} \\ r^5 &= \frac{1}{32} \\ \sqrt[5]{r^5} &= \sqrt[5]{\frac{1}{32}} \\ r &= \frac{1}{2} \end{aligned}$$

Thus, $a_1 = \frac{8}{\left(\frac{1}{2}\right)^4} = \frac{8}{\frac{1}{16}} = \frac{8}{1} \cdot \frac{16}{1} = 128$.

The nth term rule is $a_n = \left(\frac{3}{8}\right)(2)^{n-1}$.

Note

In solving the equation above for r we divided both sides by r^4. In general it is not advisable to divide both sides of an equation by the variable because we may lose a possible solution, $r = 0$. However, in this case, $r \neq 0$ since it is the common ratio in a geometric sequence.

Examples

Example 1

Earlier, you were asked to find the nth term rule of the geometric sequence represented by a bacteria sample that doubles every hour.

Solution

We are given that $a_4 = 64$ and because the sample doubles every hour we know that the common ratio is 2. We can therefore plug the known values into the equation $a_n = a_1 r^{n-1}$ to get a_1.

$$\begin{aligned} a_4 &= a_1 r^{n-1} \\ 64 &= a_1 (2)^3 \\ 64 &= 8a_1 \\ 8 &= a_1 \end{aligned}$$

Therefore, there are 8 bacteria in the sample to begin with and the nth term rule is $a_n = 8 \cdot 2^{n-1}$.

Example 2

Find the first term and the nth term rule for the geometric sequence given that $r = -\frac{1}{2}$ and $a_6 = 3$.

Solution

Use the known quantities in the general form for the nth term rule to find a_1.

$$\begin{aligned} 3 &= a_1 \left(-\frac{1}{2}\right)^5 \\ \left(-\frac{32}{1}\right) \cdot 3 &= a_1 \left(-\frac{1}{32}\right) \cdot \left(-\frac{32}{1}\right) \\ a_1 &= -96 \end{aligned}$$

Thus, $a_n = -96\left(-\frac{1}{2}\right)^{n-1}$

Example 3

Find the common ratio and the nth term rule for the geometric sequence given that $a_1 = -\frac{16}{625}$ and $a_6 = -\frac{5}{2}$.

Solution

Again, substitute in the known quantities to solve for r.

$$\begin{aligned} -\frac{5}{2} &= \left(-\frac{16}{625}\right) r^5 \\ -\frac{5}{2}\left(-\frac{625}{16}\right) &= r^5 \\ \frac{3125}{32} &= r^5 \\ \sqrt[5]{\frac{3125}{32}} &= \sqrt[5]{r^5} \\ r &= \frac{5}{2} \end{aligned}$$

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So, $a_n = -\frac{16}{625}\left(\frac{5}{2}\right)^{n-1}$

Example 4

Find the nth term rule for the geometric sequence in which $a_5 = 6$ and $a_{13} = 1536$.

Solution

This time we have two unknowns, the first term and the common ratio. We will need to solve a system of equations using both given terms.

Equation 1: $a_5 = 6$, so $6 = a_1 r^4$, solving for a_1 we get $a_1 = \frac{6}{r^4}$.

Equation 2: $a_{13} = 1536$, so $1536 = a_1 r^{12}$, solving for a_1 we get $a_1 = \frac{1536}{r^{12}}$.

Now that both equations are solved for a_1 we can set them equal to each other and solve for r.

$$\frac{6}{r^4} = \frac{1536}{r^{12}}$$
$$6r^{12} = 1536r^4$$
$$\frac{6r^{12}}{6r^4} = \frac{1536r^4}{6r^4}$$
$$r^8 = 256$$
$$\sqrt[8]{r^8} = \sqrt[8]{256}$$
$$r = 2$$

Now use r to find a_1: $a_1 = \frac{6}{(2^4)} = \frac{6}{16} = \frac{3}{8}$.

The nth term rule is $a_n = \left(\frac{3}{8}\right)(2)^{n-1}$.

Review

Use the given information to find the nth term rule for each geometric sequence.

1. $r = \frac{2}{3}$ and $a_8 = \frac{256}{81}$
2. $r = -\frac{3}{4}$ and $a_5 = \frac{405}{8}$
3. $r = \frac{6}{5}$ and $a_4 = 3$
4. $r = -\frac{1}{2}$ and $a_7 = 5$
5. $r = \frac{6}{7}$ and $a_0 = 1$
6. $a_1 = \frac{11}{8}$ and $a_7 = 88$
7. $a_1 = 24$ and $a_4 = 81$
8. $a_1 = 48$ and $a_4 = \frac{3}{4}$
9. $a_1 = \frac{343}{216}$ and $a_5 = \frac{6}{7}$
10. $a_6 = 486$ and $a_{10} = 39366$
11. $a_5 = 648$ and $a_{10} = \frac{19683}{4}$
12. $a_3 = \frac{2}{3}$ and $a_5 = \frac{3}{2}$
13. $a_5 = \frac{4}{3}$ and $a_{10} = -\frac{128}{3}$

Use a geometric sequence to solve the following word problems.

14. Ricardo's parents want to have $100,000 saved up to pay for college by the time Ricardo graduates from high school (16 years from now). If the investment vehicle they choose to invest in claims to yield 7% growth per year, how much should they invest today? Give your answer to the nearest one thousand dollars.
15. If a piece of machinery depreciates (loses value) at a rate of 6% per year, what was its initial value if it is 10 years old and worth $50,000? Give your answer to the nearest one thousand dollars.

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.9.

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SECTION OVERVIEW

7.8: Sums of Arithmetic Series

Topic hierarchy

7.8.1: Sums of Finite Arithmetic Series

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7.8.1: Sums of Finite Arithmetic Series

Finding the Sum of a Finite Arithmetic Series

A theater's seating is arranged such that each row has two more seats than the one in front of it. The first row has five seats and there are 30 rows of seats in the theater. How many total seats are in the theater?

Sum of a Finite Arithmetic Series

The method of using the calculator to evaluate the sum of a series can be used to find the sum of an arithmetic series as well. However, in this concept we will explore an algebraic method unique to arithmetic series. As we discussed earlier in the unit a series is simply the sum of a sequence so an arithmetic series is a sum of an arithmetic sequence. Let's look at a problem to illustrate this and develop a formula to find the sum of a finite arithmetic series.

Let's find the sum of the arithmetic series: 1+3+5+7+9+11+…+35+37+39.

Now, while we could just add up all of the terms to get the sum, if we had to sum a large number of terms that would be very time consuming. A famous German mathematician, Johann Carl Friedrich Gauss, used the method described here to determine the sum of the first 100 integers in grade school. First, we can write out all the numbers twice, in ascending and descending order, and observe that the sum of each pair of numbers is the same:

$$\begin{array}{cccccccccc} 1 & 3 & 5 & 7 & 9 & 11 & \ldots & 35 & 37 & 39 \\ 39 & 37 & 35 & 33 & 31 & 29 & \ldots & 5 & 3 & 1 \\ & & & & & \vdots & & & & \\ 40 & 40 & 40 & 40 & 40 & 40 & \ldots & 40 & 40 & 40 \end{array}$$

Notice that the sum of the corresponding terms in reverse order is always equal to 40, which is the sum of the first and last terms in the sequence.

What Gauss realized was that this sum can be multiplied by the number of terms and then divided by two (since we are actually summing the series twice here) to get the sum of the terms in the original sequence. For the problem he was given in school, finding the sum of the first 100 integers, he was able to just use the first term, $a_1 = 1$, the last term, $a_n = 100$, and the total number of terms, $n = 100$, in the following formula:

$$\frac{n(a_1+a_n)}{2} = \frac{100(1+100)}{2} = 5050$$

In our problem we know the first and last terms, but how many terms are there? We need to find n to use the formula to find the sum of the series. We can use the first and last terms and the nth term to do this.

$$\begin{aligned} a_n &= a_1 + d(n-1) \\ 39 &= 1 + 2(n-1) \\ 38 &= 2(n-1) \\ 19 &= n-1 \\ 20 &= n \end{aligned}$$

Now the sum is $\frac{20(1+39)}{2} = 400$

Proof of the Arithmetic Sum Formula

The rule for finding the nth term of an arithmetic sequence and properties of summations can be used to prove the formula algebraically. First, we will start with the nth term rule $a_n = a_1 + (n-1)d$. We need to find the sum of numerous n^{th} terms (n of them to be exact) so we will use the index, i, in a summation as shown below:

$\sum_{i=1}^{n} [a_1 + (i-1)d]$ Keep in mind that a_1 and d are constants in this expression.

We can separate this into two separate summations as shown: $\sum_{i=1}^{n} a_1 + \sum_{i=1}^{n} (i-1)d$

Expanding the first summation, $\sum_{i=1}^{n} a_1 = a_1 + a_1 + a_1 + \ldots + a_1$ such that a_1 is added to itself n times. We can simplify this expression to $a_1 n$.

In the second summation, d can be brought out in front of the summation and the difference inside can be split up as we did with the addition to get: $d[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1]$. Using rules you have seen before, $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ and $\sum_{i=1}^{n} 1 = n$. Putting it all together, we can write an expression without any summation symbols and simplify.

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Now, let's find the sum of the first 40 terms in the arithmetic series $35 + 31 + 27 + 23 + \ldots$

For this particular series we know the first term and the common difference, so let's use the rule that doesn't require the nth term: $\frac{1}{2}n[2a_1 + d(n-1)]$, where $n = 40$, $d = -4$ and $a_1 = 35$.

$$\frac{1}{2}(40)[2(35) + (-4)(40-1)] = 20[70 - 156] = -1720$$

We could also find the nth term and use the rule $\frac{1}{2}n(a_1 + a_n)$, where $a_n = a_1 + d(n-1)$.

$a_{40} = 35 + (-4)(40-1) = 35 - 156 = -121$, so the sum is

$\frac{1}{2}(40)(35 - 121) = 20(-86) = -1720$

Next, given that in an arithmetic series $a_{21} = 165$ and $a_{35} = 277$, let's find the sum of terms 21 to 35.

This time we have the "first" and "last" terms of the series, but not the number of terms or the common difference. Since our series starts with the 21st term and ends with the 35th term, there are 15 terms in this series. Now we can use the rule to find the sum as shown.

$$\frac{1}{2}(15)(165 + 277) = 3315$$

Finally, let's find the sum of the arithmetic series $\sum_{i=1}^{8}(12 - 3i)$.

From the summation notation, we know that we need to sum 8 terms. We can use the expression $12 - 3i$ to find the first and last terms as and the use the rule to find the sum.

First term: $12 - 3(1) = 9$

Last term: $12 - 3(8) = -12$

$\sum_{i=1}^{8}(12 - 3i) = \frac{1}{2}(8)(9 - 12) = 4(-3) = -12$

We could use the calculator in this problem as well: *sum*(*seq*(12−3x,x,1,8))=−12

Examples

Example 1

Earlier, you were asked to find the total number of seats in the theater.

Solution

For this particular series we know the first term and the common difference, so let's use the rule that doesn't require the nth term: $\frac{1}{2}n[2a_1 + d(n-1)]$, where $n = 30$, $d = 2$ and $a_1 = 5$.

$$\frac{1}{2}(30)[2(5) + (2)(30-1)] = 15[10 + 58] = 1020$$

Therefore, there are a total of 1020 seats in the theater.

Example 2

Find the sum of the series $87 + 79 + 71 + 63 + \ldots + -105$.

Solution

$d = 8$, so

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$-105 = 87 + (-8)(n-1)$
$-192 = -8n + 8$
$-200 = -8n$
$n = 25$

and then use the rule to find the sum is $\frac{1}{2}(25)(87-105) = -225$

Example 3

Find $\sum_{i=10}^{50}(3i-90)$.

Solution

10th term is $3(10)-90=-60$, 50th term is $3(50)-90=60$ and $n = 50-10+1 = 41$

(add 1 to *include* the 10th term). The sum of the series is $\frac{1}{2}(41)(-60+60) = 0$. Note that the calculator is a great option for this problem: *sum*(*seq*(3x−90,x,10,50))=0

Example 4

Find the sum of the first 30 terms in the series $1+6+11+16+...$

Solution

$d = 5$, use the sum formula, $\frac{1}{2}n\,(2a_1 + d(n-1))$, to get

$\frac{1}{2}(30)[2(1)+5(30-1)] = 15[2+145] = 2205$

Review

Find the sums of the following arithmetic series.

1. $-6+-1+4+...+119$
2. $72+60+48+...+-84$
3. $3+5+7+...+99$
4. $25+21+17+...+-23$
5. Find the sum of the first 25 terms of the series $215+200+185+...$
6. Find the sum of the first 14 terms in the series $3+12+21+...$
7. Find the sum of the first 32 terms in the series $-70+-65+-60+...$
8. Find the sum of the first 200 terms in $-50+-49+-48+...$

Evaluate the following summations.

9. $\sum_{i=4}^{10}(5i-22)$
10. $\sum_{i=2}^{25}(-3i+37)$
11. $\sum_{i=11}^{48}(i-20)$
12. $\sum_{i=5}^{40}(50-2i)$

Find the sum of the series bounded by the terms given. Include these terms in the sum.

13. $a_7 = 39$ and $a_{23} = 103$
14. $a_8 = 1$ and $a_{30} = -43$
15. $a_4 = -15$ and $a_{17} = 24$
16. How many cans are needed to make a triangular arrangement of cans if the bottom row has 35 cans and successive row has one less can than the row below it?
17. Thomas gets a weekly allowance. The first week it is one dollar, the second week it is two dollars, the third week it is three dollars and so on. If Thomas puts all of his allowance in the bank, how much will he have at the end of one year?

Answers for Review Problems

To see the Review answers, open this PDF file and look for section 11.7.

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CHAPTER OVERVIEW

Unit 8: Introduction to Calculus

8.1: Limits in Calculus

8.1.1: Definition of a Limit

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SECTION OVERVIEW

8.1: Limits in Calculus

Topic hierarchy

8.1.1: Definition of a Limit

8.1.2: One-Sided Limits

8.1.3: Infinite Limits

8.1.4: Polynomial Function Limits

8.1.5: Tables to Find Limits

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8.1.1: Definition of a Limit

Limit Notation

When learning about the **end behavior** of a rational function you described the function as either having a **horizontal asymptote** at zero or another number, or going to infinity. **Limit notation** is a way of describing this end behavior mathematically.

You already know that as x gets extremely large then the function $f(x) = \frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ goes to $\frac{8}{3}$ because the greatest powers are equal and $\frac{8}{3}$ is the ratio of the leading coefficients. How is this statement represented using limit notation?

Introduction to Limits

Limit notation is a way of stating an idea that is a little more subtle than simply saying $x = 5$ or $y = 3$.

$\lim_{x \to a} f(x) = b$

"The limit of f of x as x approaches a is b"

The letter a can be any number or infinity. The function $f(x)$ is any function of x. The letter b can be any number. If the function goes to infinity, then instead of writing "= ∞" you should write that the limit does not exist or "*DNE*". This is because infinity is not a number. If a function goes to infinity then it has no limit.

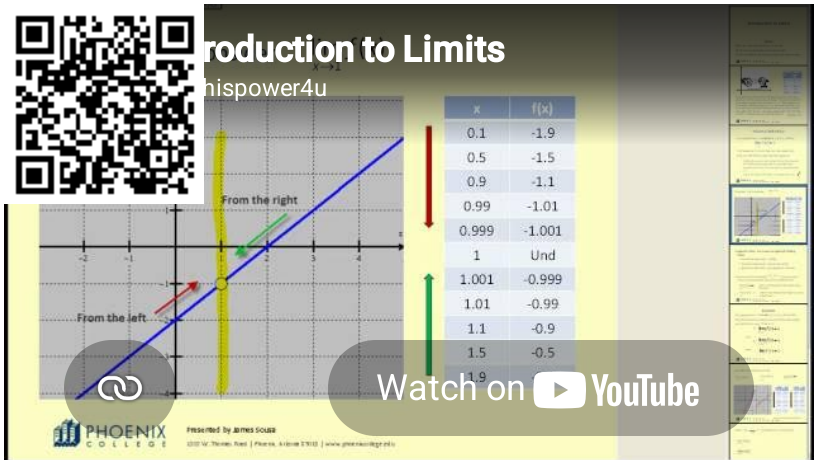
Take the following limit:

The limit of $y = 4x^2$ as x approaches 2 is 16

In limit notation, this would be:

$\lim_{x \to 2} 4x^2 = 16$

While a function may never actually reach a height of b it will get arbitrarily close to b. One way to think about the concept of a limit is to use a physical example. Stand some distance from a wall and then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.



Examples

Example 1

Earlier, you were asked how to write the statement "The limit of $\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ as x approaches infinity is $\frac{8}{3}$" in limit notation.

Solution

This can be written using limit notation as:

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$\lim_{x\to\infty}\left(\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}\right)=\frac{8}{3}$

Example 2

Translate the following mathematical statement into words.

$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i=1$

Solution

The limit of the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ as the number of terms approaches infinity is 1.

Example 3

Use limit notation to represent the following mathematical statement.

$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{2}$

Solution

$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i=\frac{1}{2}$

Example 4

Describe the end behavior of the following rational function at infinity and negative infinity using limits.

$f(x)=\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}$

Solution

Since the function has equal powers of x in the numerator and in the denominator, the end behavior is $-\frac{1}{2}$ as x goes to both positive and negative infinity.

$\lim_{x\to\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=\lim_{x\to-\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right)=-\frac{1}{2}$

Example 5

Translate the following limit expression into words. What do you notice about the limit expression?

$\lim_{h\to0}\left(\frac{f(x+h)-f(x)}{h}\right)=x$

Solution

The limit of the ratio of the difference between f of quantity x plus h and f of x and h as h approaches 0 is x.

You should notice that $h\to0$ does not mean $h=0$ because if it did then you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as h approaches zero. Calculus will enable you to deal with problems that seem to look like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Review

Describe the end behavior of the following rational functions at infinity and negative infinity using limits.

1. $f(x)=\frac{2x^4+4x^2-1}{5x^4+3x+9}$
2. $g(x)=\frac{8x^3+4x^2-1}{2x^3+4x+7}$
3. $f(x)=\frac{x^2+2x^3-3}{5x^3+x+4}$
4. $f(x)=\frac{4x+4x^2-5}{2x^2+3x+3}$
5. $f(x)=\frac{3x^2+4x^3+4}{6x^3+3x^2+6}$

Translate the following statements into limit notation.

6. The limit of $y=2x^2+1$ as x approaches 3 is 19.

7. The limit of $y = e^x$ as x approaches negative infinity is 0.
8. The limit of $y = \frac{1}{x}$ as x approaches infinity is 0.

Use limit notation to represent the following mathematical statements.

9. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}$
10. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.
11. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$
12. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1$

Translate the following mathematical statements into words.

13. $\lim_{x \to 0} \frac{5x^2 - 4}{x+1} = -4$
14. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$
15. If $\lim_{x \to a} f(x) = b$, is it possible that $f(a) = b$ Explain.

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.1.

Vocabulary

Term	Definition
End behavior	End behavior is a description of the trend of a function as input values become very large or very small, represented as the 'ends' of a graphed function.
Horizontal Asymptote	A horizontal asymptote is a horizontal line that indicates where a function flattens out as the independent variable gets very large or very small. A function may touch or pass through a horizontal asymptote.
limit notation	Limit notation is a way of expressing the fact that a function gets arbitrarily close to a value.

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8.1.2: One-Sided Limits

One Sided Limits and Continuity

A **one sided limit** is exactly what you might expect; the limit of a function as it approaches a specific x value from either the right side or the left side. One sided limits help to deal with the issue of a jump discontinuity and the two sides not matching.

Is the following piecewise function **continuous**?

$$f(x) = \begin{cases} -x-2 & x<1 \\ -3 & x=1 \\ x^2-4 & 1<x \end{cases}$$

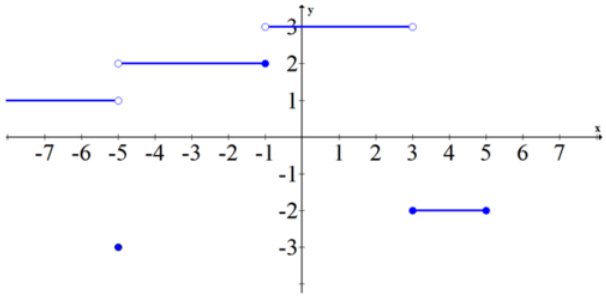
Evaluating One Sided Limits and Continuity

A one sided limit can be evaluated either from the left or from the right. Since left and right are not absolute directions, a more precise way of thinking about direction is “from the negative side” or “from the positive side”. The notation for these one sided limits is:

$\lim_{x\to a^-} f(x), \lim_{x\to a^+} f(x)$

The negative in the superscript of a is not an exponent. Instead it indicates **from the negative side**. Likewise the positive superscript is not an exponent, it just means **from the positive side**. When evaluating one sided limits, it does not matter what the function is doing at the actual point or what the function is doing on the other side of the number. Your job is to determine what the height of the function should be using only evidence on one side.

Take the graph below. What are the one sided limits at -5, -1, 3 and 5?



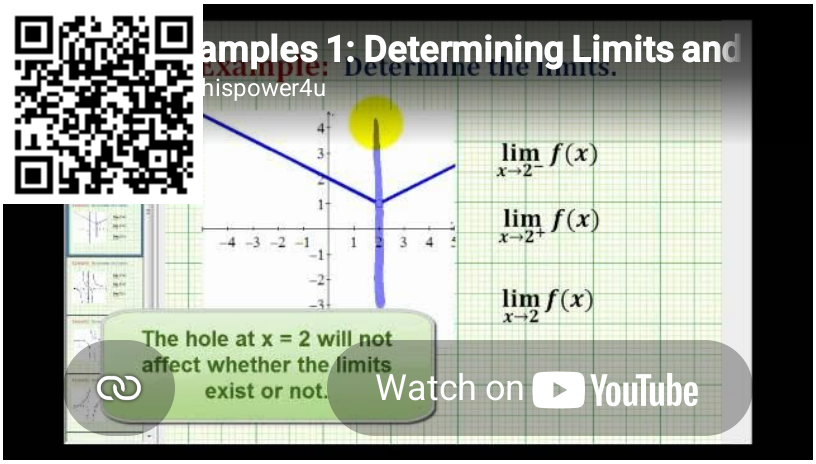
Each point should have two limits, one from the left and one from the right.

$\lim_{x\to -5^-} f(x) = 1$ $\quad$ $\lim_{x\to -5^+} f(x) = 2$

$\lim_{x\to -1^-} f(x) = 2$ $\quad$ $\lim_{x\to -1^+} f(x) = 3$

$\lim_{x\to 3^-} f(x) = 3$ $\quad$ $\lim_{x\to 3^+} f(x) = -2$

$\lim_{x\to 5^-} f(x) = -2$ $\quad$ $\lim_{x\to -5^+} f(x) = DNE$



You have defined continuity in the past as the ability to draw a function completely without lifting your pencil off of the paper. You can now define a more rigorous definition of continuity.

Continuity at a point exists when the left and right sided limits match the function evaluated at that point. In other words, a function is continuous at a if:

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

For an entire function to be continuous, the function must be continuous at every single point in an unbroken domain.

Examples

Example 1

Earlier, you were asked how to confirm the function

$$f(x) = \begin{cases} -x - 2 & x < 1 \\ -3 & x = 1 \\ x^2 - 4 & 1 < x \end{cases}$$

Solution

is continuous. In order to confirm or deny that the function is continuous, graphical tools are not accurate enough. Sometimes **jump discontinuities** can be off by such a small amount that the pixels on the display of your calculator will not display a difference. Your calculator will certainly not display **removable discontinuities**.

You should note that on the graph, everything to the left of 1 is continuous because it is just a line. Next you should note that everything to the right of 1 is also continuous for the same reason. The only point to check is at $x = 1$. To check continuity, explicitly use the definition and evaluate all three parts to see if they are equal.

- $\lim_{x \to a^-} f(x) = -1 - 2 = -3$
- $f(1) = -3$
- $\lim_{x \to 1^+} f(x) = 1^2 - 4 = -3$

Therefore, $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$ and the function is continuous at $x = 1$ and everywhere else.

Example 2

Evaluate the one sided limit at 4 from the negative direction numerically.

$f(x) = \frac{x^2 - 7x + 12}{x - 4}$

Solution

Remember that evaluating numerically means that you should use a table. When creating the table, only use values that are smaller than 4.

x	3.9	3.99	3.999
f(x)	0.9	0.99	0.999

$\lim_{x \to 4^-} \left(\frac{x^2 - 7x + 12}{x - 4} \right) = 1$

Example 3

Evaluate the following limits.

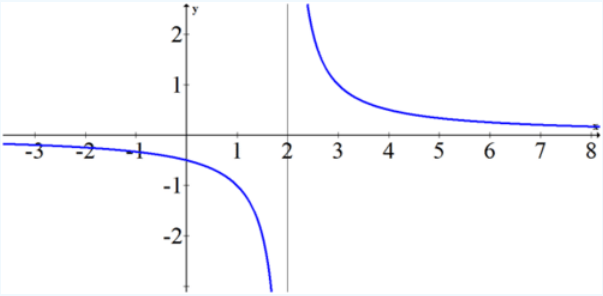
1. $\lim_{x \to 3^-} (4x - 3)$
2. $\lim_{x \to 2^+} \left(\frac{1}{x - 2} \right)$
3. $\lim_{x \to 1^+} \left(\frac{x^2 + 2x - 3}{x - 1} \right)$

Solution

Most of the time one sided limits are the same as the corresponding two sided limit. The exceptions are when there are jump discontinuities, which normally only happen with piecewise functions, and infinite discontinuities, which normally only happen with rational functions.

1. $\lim_{x\to 3^-}(4x-3) = 4\cdot 3 - 3 = 12 - 3 = 9$
2. $\lim_{x\to 2^+}\left(\frac{1}{x-2}\right) = DNE$ or ∞

   The reason why ∞ is preferable in this case is because the two sides of the limit disagree. One side goes to negative infinity and the other side goes to positive infinity (see the graph below). If you just indicate DNE then you are losing some perfectly good information about the nature of the function.



3. $\lim_{x\to 1^+}\left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{(x-1)(x+3)}{(x-1)}\right) = \lim_{x\to 1^+}(x+3) = 1+3 = 4$

Example 4

Evaluate the following limits.

1. $\lim_{x\to 1^-}(2x-1)$
2. $\lim_{x\to -3^+}\left(\frac{2}{x+2}\right)$
3. $\lim_{x\to 2^+}\left(\frac{x^3-8}{x-2}\right)$

Solution

1. $\lim_{x\to 1^-}(2x-1) = 2\cdot 1 - 1 = 2 - 1 = 1$
2. $\lim_{x\to -3^+}\left(\frac{2}{x+2}\right) = \frac{2}{-3+2} = \frac{2}{-1} = -2$

Example 5

Is the following function continuous?

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ 3 & x = -1 \\ -x + 3 & -1 < x \end{cases}$$

Solution

Use the definition of continuity.

- $\lim_{x\to 1^-} f(x) = (-1)^2 - 1 = 1 - 1 = 0$
- $f(-1) = 3$
- $\lim_{x\to -1^+} f(x) = -1 + 3 = 2$

$\lim_{x\to a^-} f(x) \neq f(a) \neq \lim_{x\to a^+} f(x)$ so this function is discontinuous at $x = -1$. It is continuous everywhere else.

Review

Evaluate the following limits.

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1. $\lim_{x \to 6^-} (3x^2 - 4)$
2. $\lim_{x \to 0^-} \frac{3x-1}{x}$
3. $\lim_{x \to 0^+} \frac{3x-1}{x}$
4. $\lim_{x \to 0^+} \frac{x}{|x|}$
5. $\lim_{x \to 0^-} \frac{x}{|x|}$
6. $\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{1+\sqrt{x}}-1}$

Consider

$$f(x) = \begin{cases} 2x^2 - 1 & x < 1 \\ 1 & x = 1 \\ -x + 2 & 1 < x \end{cases}$$

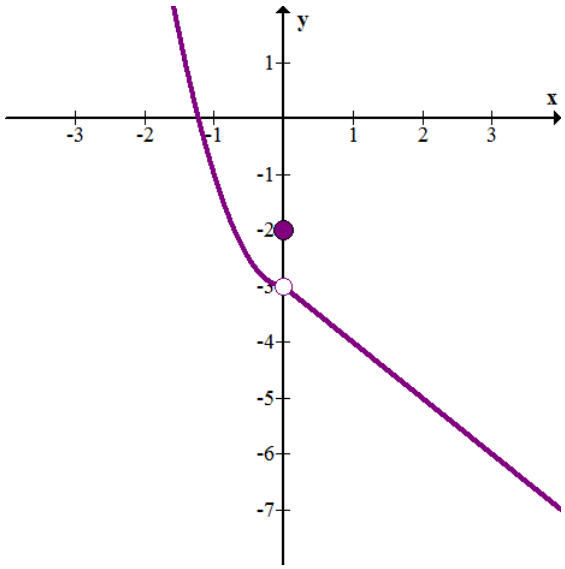
7. What is $\lim_{x \to 1^-} f(x)$?
8. What is $\lim_{x \to 1^+} f(x)$?
9. Is $f(x)$ continuous at $x = 1$?

Consider

$$g(x) = \begin{cases} 4x^2 + 2x - 1 & x < -2 \\ 8 & x = -2 \\ -3x + 5 & -2 < x \end{cases}$$

10. What is $\lim_{x \to -2^-} g(x)$?
11. What is $\lim_{x \to -2^+} g(x)$?
12. Is $g(x)$ continuous at $x = -2$?

Consider $h(x)$ shown in the graph below.



13. What is $\lim_{x \to 0^-} h(x)$?
14. What is $\lim_{x \to 0^+} h(x)$?
15. Is $h(x)$ continuous at $x = 0$?

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.6.

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Vocabulary

Term	Definition
continuity	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
Jump discontinuities	Inverse functions are functions that 'undo' each other. Formally $f(x)$ and $g(x)$ are inverse functions if $f(g(x)) = g(f(x)) = x$.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
Removable discontinuities	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
Removable discontinuity	Removable discontinuities are also known as holes. They occur when factors can be algebraically canceled from rational functions.
two-sided limit	A two-sided limit is the value that a function approaches from both the left side and the right side.

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8.1.3: Infinite Limits

Infinite Limits

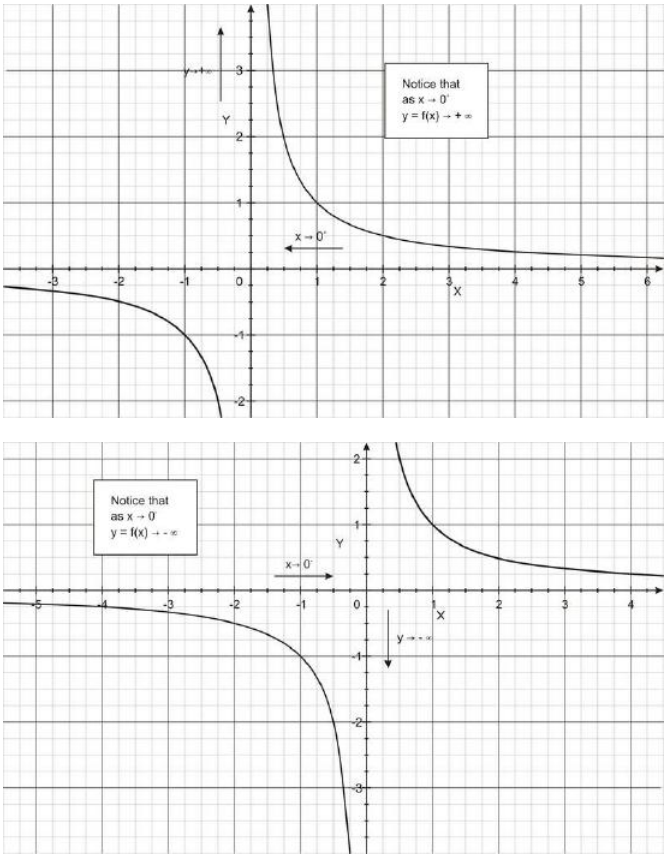
Geeks-R-Us sells titanium mechanical pencils to computer algorithm designers. In an effort to attract more business, they decide to run a rather unusual promotion:

"SALE!! The more you buy, the more you save! Pencils are now $\$\frac{12x}{x-3}$ per dozen!"

If the trillionaire, Spug Dense, comes in and says he wants to buy as many pencils as Geeks-R-Us can turn out, what will the cost of the pencils approach as the order gets bigger and bigger?

Infinite Limits

Sometimes, a function may not be defined at a particular number, but as values are input closer and closer to the undefined number, a **limit** on the output may not exist. For example, for the function $f(x) = 1/x$ (shown in the figures below), as x values are taken closer and closer to 0 from the right, the function increases indefinitely. Also, as x values are taken closer and closer to 0 from the left, the function decreases indefinitely.



We describe these limiting behaviors by writing

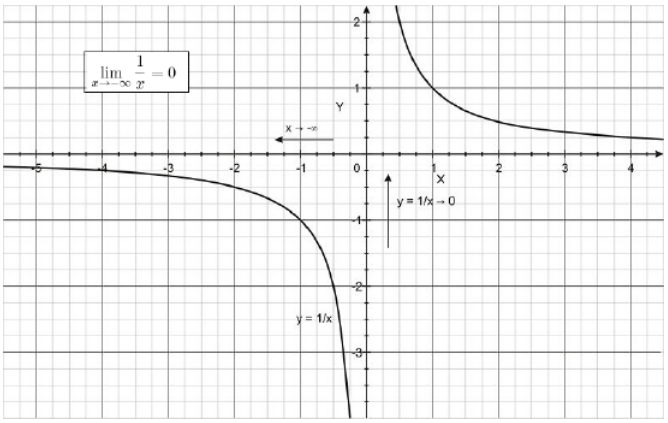
$\lim_{x \to 0^+} \frac{1}{x} = +\infty$
$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

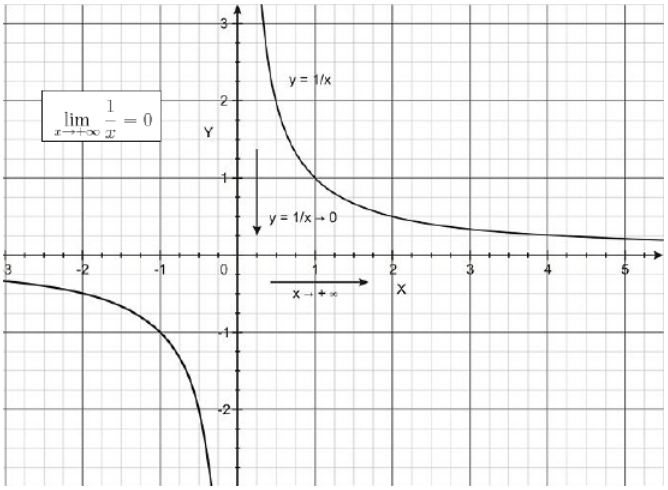
Sometimes we want to know the behavior of $f(x)$ as x increases or decreases without bound. In this case we are interested in the end behavior of the function, a concept you have likely explored before. For example, what is the value of $f(x) = 1/x$ as x increases or decreases without bound? That is,

$\lim_{x \to +\infty} \frac{1}{x} = ?$
$\lim_{x \to -\infty} \frac{1}{x} = ?$

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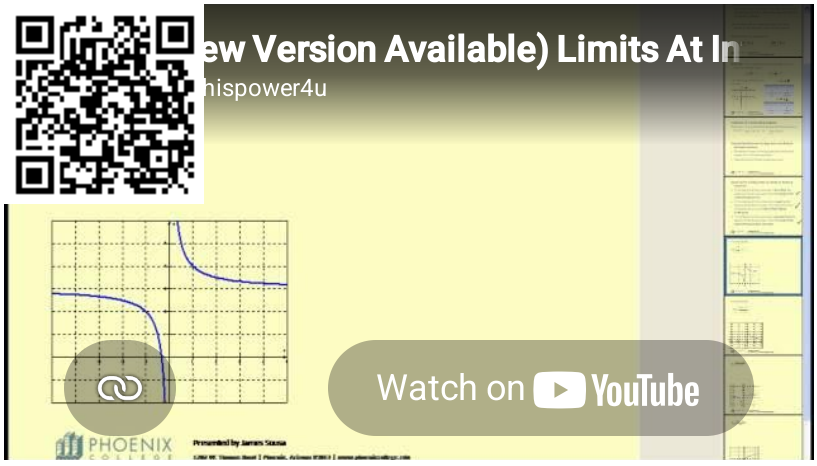
As you can see from the graphs (shown below), as x decreases without bound, the values of $f(x) = 1/x$ are negative and get closer and closer to 0. On the other hand, as x increases without bound, the values of $f(x) = 1/x$ are positive and still get closer and closer to 0.

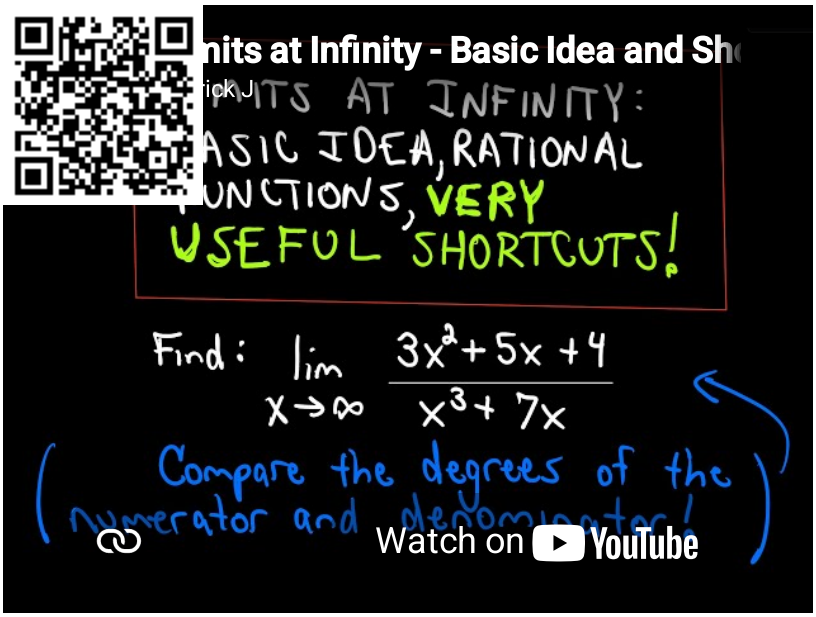




That is,

$\lim_{x \to +\infty} \frac{1}{x} = 0$
$\lim_{x \to -\infty} \frac{1}{x} = 0$





Examples

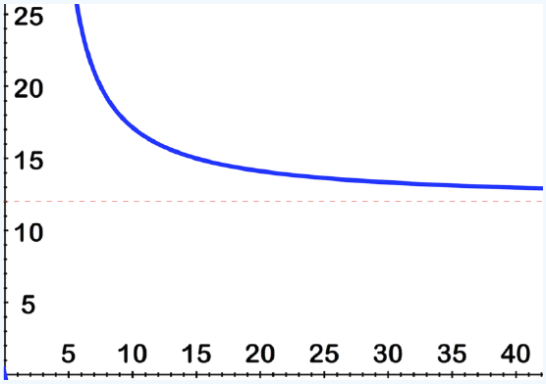
Example 1

Earlier, you were asked a question about buying a lot of pencils.

Solution

As Spug buys more and more pencils, the cost of each dozen will drop quickly at first, and level out after a while, approaching $12 per dozen.

You can see the effect on the graph here:



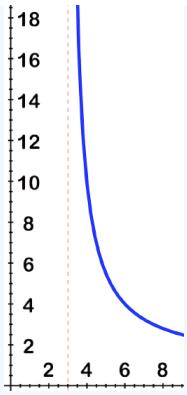
Example 2

Evaluate the limit by making a graph: $\lim_{x \to 3^+} \frac{x+6}{x-3}$

Solution

By looking at the graph:

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We can see that as x gets closer and closer to 3 from the positive side, the output increases right out the top of the image, on its way to ∞

Example 3

Evaluate the limit: $\lim_{x\to\infty} \frac{11x^3-14x^2+8x+16}{9x-3}$.

Solution

To evaluate polynomial function limits, a little bit of intuition helps. Let's think this one through.

First, note that since we are looking at what happens as $x \to \infty$ most of the interesting stuff will happen as x gets *really* big.

On the top part of the fraction, as *x* gets truly massive, the $11x^3$ part will get bigger much faster than either of the other terms. In fact, it increases so much faster than the other terms completely cease to matter at all once *x* gets really monstrous. That means that the important part of the top of the fraction is just the $11x^3$.

On the bottom, a similar situation develops. As *x* gets really, really big, the *-3* matters less and less. So the bottom may as well be just $9x$.

That gives us $\frac{11x^3}{9x}$ which reduces to $\frac{11x^2}{9}$

Now we can more easily see what happens at the "ends." As *x* gets bigger and bigger, the numerator continues to get bigger faster than the denominator, so the overall output also increases.

$\therefore \lim_{x\to+\infty} \frac{11x^3-14x^2+8x+16}{9x-3}$ is $+\infty$

Example 4

Evaluate $\lim_{x\to0} \frac{x+2}{x+3}$

Solution

This one is easier than it looks! As x-->0, leaving just the fraction: ***2/3***

Example 5

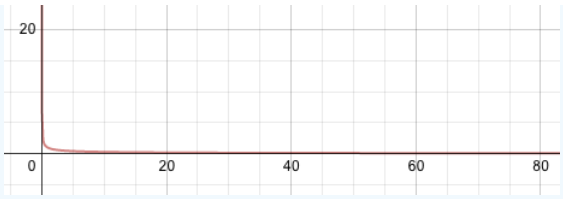
Make a graph to evaluate the limit $\lim_{x\to\infty} \frac{1}{\sqrt{x}}$ and $\lim_{x\to0^+} \frac{1}{\sqrt{x}}$.

Solution

By looking at the image, we see that as *x* gets huge, so does $\sqrt{x}$ which means that 1 is being divided by an ever-larger number, and the result is getting smaller and smaller.

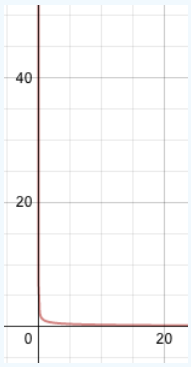
$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

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On the same image, we can see that as x gets closer and closer to zero, so does $\sqrt{x}$ which means that 1 is being divided by an ever smaller number, and the result gets bigger and bigger.

$\lim_{x \to 0^+} \frac{1}{\sqrt{x}}$ is $+\infty$

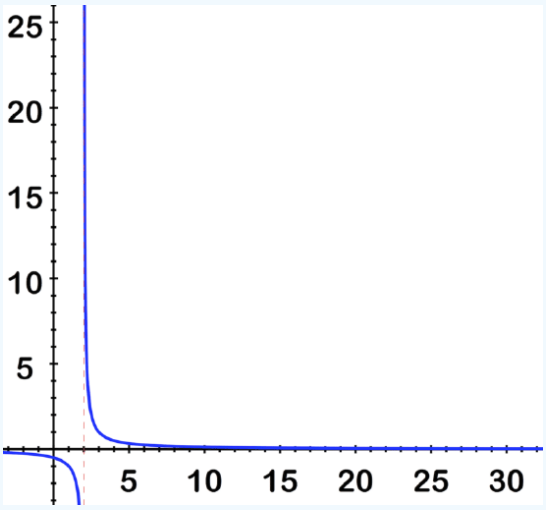


Example 6

Graph and evaluate the limit: $\lim_{x \to 2^+} \frac{1}{x-2}$.

Solution

By looking at the image, we can see that as *x* gets closer and closer to 2 from the positive direction, 1 gets divided by smaller and smaller numbers, so the result gets larger and larger.



$\lim_{x \to 2^+} \frac{1}{x-2}$ is $+\infty$

Review

Evaluate the limits:

1. $\lim_{x \to 3^-} \frac{1}{x-3}$
2. $\lim_{x \to -4^+} \frac{1}{x+4}$

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3. $\lim_{x \to -(\frac{8}{3})^+} \frac{1}{3x+8}$
4. $\lim_{x \to -5^+} \frac{(x^2+11x+30)}{x+5}$
5. $\lim_{x \to -\infty} \frac{(x^2+11x+30)}{x+5}$
6. $\lim_{x \to \infty} \frac{-11x^3+20x^2+15x-17}{-9x^3+5x^2-x-17}$
7. $\lim_{x \to \infty} 13$
8. $\lim_{x \to \infty} \frac{-2x+18}{17x-3}$
9. $\lim_{x \to \infty} 15$
10. $\lim_{x \to \infty} -5x^2 + 5x + 14$
11. $\lim_{x \to \infty} 7x + 12$
12. $\lim_{x \to \infty} -3x + 13$
13. $\lim_{x \to \infty} \frac{13x-8}{19x^3-11x^2+x+4}$
14. $\lim_{x \to \infty} -17x + 14$
15. $\lim_{x \to \infty} -7x^2 - 2x - 13$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.3.

Vocabulary

Term	Definition
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.

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8.1.4: Polynomial Function Limits

Rationalization to Find Limits

Some limits cannot be evaluated directly by substitution and no factors immediately cancel. In these situations there is another algebraic technique to try called **rationalization**. With rationalization, you make the numerator and the denominator of an expression rational by using the properties of conjugate pairs.

How do you evaluate the following **limit** using rationalization?

$\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$

Using Rationalization to Find Limits

Rationalization generally means to multiply a rational function by a clever form of one in order to eliminate radical symbols or imaginary numbers in the denominator. Rationalization is also a technique used to evaluate limits in order to avoid having a zero in the denominator when you substitute.

Tod do this, you will use the properties of **conjugates**.

Conjugates can be used to simplify expressions with a radical in the denominator:

$\frac{5}{1+\sqrt{3}} = \frac{5}{(1+\sqrt{3})} \cdot \frac{(1-\sqrt{3})}{(1-\sqrt{3})} = \frac{5-5\sqrt{3}}{1-3} = \frac{5-5\sqrt{3}}{-2}$

Conjugates can be used to simplify complex numbers with i in the denominator:

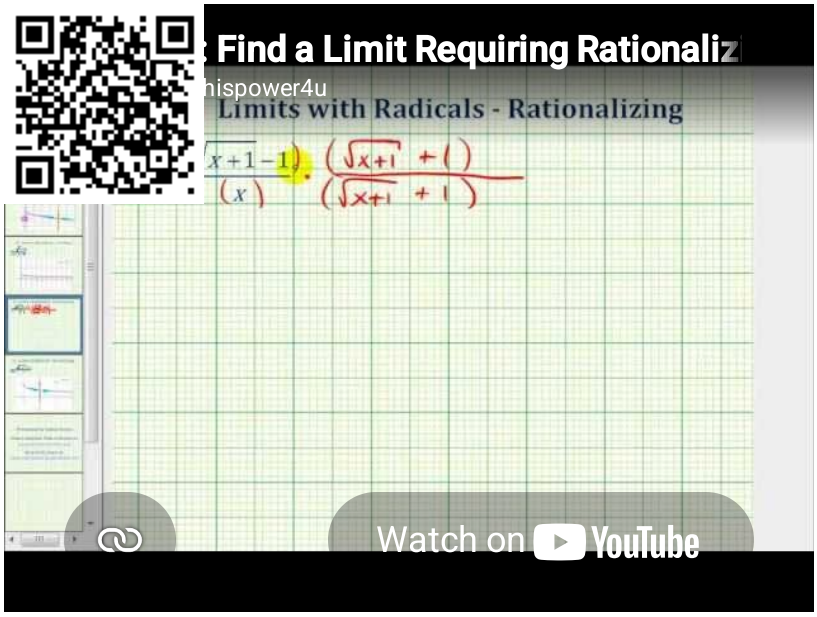
$\frac{4}{2+3i} = \frac{4}{(2+3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{8-12i}{4+9} = \frac{8-12i}{13}$

Here, they can be used to transform an expression in a limit problem that does not immediately factor to one that does immediately factor.

$\lim_{x \to 16} \frac{(\sqrt{x}-4)}{(x-16)} \cdot \frac{(\sqrt{x}+4)}{(\sqrt{x}+4)} = \lim_{x \to 16} \frac{(x-16)}{(x-16)(\sqrt{x}+4)}$

Now you can cancel the common factors in the numerator and denominator and use substitution to finish evaluating the limit.

The rationalizing technique works because when you algebraically manipulate the expression in the limit to an equivalent expression, the resulting limit will be the same. Sometimes you must do a variety of different algebraic manipulations in order avoid a zero in the denominator when using the substitution method.



Examples

Example 1

In order to evaluate the limit of the following rational expression, you need to multiply by a clever form of 1 so that when you substitute there is no longer a zero factor in the denominator.

$\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

Solution

$$\begin{aligned}\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16} &= \lim_{x\to 16} \frac{(\sqrt{x}-4)}{(x-16)} \cdot \frac{(\sqrt{x}+4)}{(\sqrt{x}+4)} \\ &= \lim_{x\to 16} \frac{(x-16)}{(x-16)(\sqrt{x}+4)} \\ &= \lim_{x\to 16} (\sqrt{x}+4) \\ &= 4+4 \\ &= 8\end{aligned}$$

Example 2

Evaluate the following limit: $\lim_{x\to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}}$.

Solution

$$\begin{aligned}\lim_{x\to 3} \frac{(x-3)(x+3)}{(\sqrt{x}-\sqrt{3})} \cdot \frac{(\sqrt{x}+\sqrt{3})}{(\sqrt{x}+\sqrt{3})} &= \lim_{x\to 3} \frac{(x-3)(x+3)(\sqrt{x}+\sqrt{3})}{(x-3)} \\ &= \lim_{x\to 3} (x+3)(\sqrt{x}+\sqrt{3}) \\ &= 6 \cdot 2\sqrt{3} \\ &= 12\sqrt{3}\end{aligned}$$

Example 3

Evaluate the following limit: $\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$.

Solution

$$\begin{aligned}\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7} \frac{(\sqrt{x+2}-3)}{(x-7)} \cdot \frac{(\sqrt{x+2}+3)}{(\sqrt{x+2}+3)} \\ &= \lim_{x\to 7} \frac{(x+2-9)}{(x-7)\cdot(\sqrt{x+2}+3)} \\ &= \lim_{x\to 7} \frac{(x-7)}{(x-7)\cdot(\sqrt{x+2}+3)} \\ &= \lim_{x\to 7} \frac{1}{(\sqrt{x+2}+3)} \\ &= \frac{1}{\sqrt{7+2}+3} \\ &= \frac{1}{6}\end{aligned}$$

Example 4

Evaluate the following limit: $\lim_{x\to 0} \frac{(2+x)^{-1}-2^{-1}}{x}$.

Solution

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$$\begin{aligned}\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x} &= \lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\cdot\frac{(x+2)\cdot 2}{(x+2)\cdot 2}\\ &= \lim_{x\to 0}\frac{2-(x+2)}{2x(x+2)}\\ &= \lim_{x\to 0}\frac{-x}{2x(x+2)}\\ &= \lim_{x\to 0}\frac{-1}{2(x+2)}\\ &= -\frac{1}{2(0+2)}\\ &= -\frac{1}{4}\end{aligned}$$

Example 5

Evaluate the following limit: $\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right)$

Solution

$$\begin{aligned}\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{\sqrt{9-x}}{x\sqrt{9-x}}\right)\\ &= \lim_{x\to 0}\left(\frac{3-\sqrt{9-x}}{x\sqrt{9-x}}\right)\\ &= \lim_{x\to 0}\left(\frac{(3-\sqrt{9-x})}{x\sqrt{9-x}}\cdot\frac{(3+\sqrt{9-x})}{(3+\sqrt{9-x})}\right)\\ &= \lim_{x\to 0}\left(\frac{9-(9-x)}{x\sqrt{9-x}}\right)\\ &= \lim_{x\to 0}\frac{x}{x\sqrt{9-x}}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{9-x}}\\ &= \frac{1}{\sqrt{9-0}}\\ &= \frac{1}{3}\end{aligned}$$

Review

Evaluate the following limits:

1. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$
2. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$
3. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$
4. $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$
5. $\lim_{x\to 4}\frac{\sqrt{3x+4}-x}{4-x}$
6. $\lim_{x\to 0}\frac{2-\sqrt{x+4}}{x}$
7. $\lim_{x\to 0}\frac{\sqrt{x+7}-\sqrt{7}}{x}$
8. $\lim_{x\to 16}\frac{16-x}{4-\sqrt{x}}$
9. $\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$
10. $\lim_{x\to 2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$
11. $\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}$
12. $\lim_{x\to \frac{1}{9}}\frac{9x-1}{3\sqrt{x}-1}$
13. $\lim_{x\to 4}\frac{4x^2-64}{2\sqrt{x}-4}$
14. $\lim_{x\to 9}\frac{9x^2-90x+81}{9-3\sqrt{x}}$
15. When given a limit to evaluate, how do you know when to use the rationalization technique? What will the function look like?

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.5.

Vocabulary

Term	Definition
Conjugates	Conjugates are pairs of binomials that are equal aside from inverse operations between them, e.g. (3+2x) and (3−2x).
Continuous	Continuity for a point exists when the left and right sided limits match the function evaluated at that point. For a function to be continuous, the function must be continuous at every single point in an unbroken domain.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
rationalization	Rationalization generally means to multiply a rational function by a clever form of one in order to eliminate radical symbols or imaginary numbers in the denominator. Rationalization is also a technique used to evaluate limits in order to avoid having a zero in the denominator when you substitute.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.

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8.1.5: Tables to Find Limits

Tables to Find Limits

Calculators such as the TI-84 have a table view that allows you to make extremely educated guesses as to what the **limit** of a function will be at a specific point, even if the function is not actually defined at that point.

How could you use a table to calculate the following limit?

$\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$

Using Tables to Find Limits

If you were given the following information organized in a table, how would you fill in the center column?

3.9	3.99	3.999		4.001	4.01	4.1
12.25	12.01	12.00001		11.99999	11.99	11.75

It would be logical to see the symmetry and notice how the top row approaches the number 4 from the left and the right. It would also be logical to notice how the bottom row approaches the number 12 from the left and the right. This would lead you to the conclusion that the limit of the function represented by this table is 12 as the top row approaches 4. It would not matter if the value at 4 was undefined or defined to be another number like 17, the pattern tells you that the limit at 4 is 12.

Using tables to help evaluate limits **numerically** requires this type of logic. **Numerically** is a term used to describe one of several different representations in mathematics. It refers to tables where the actual numbers are visible.

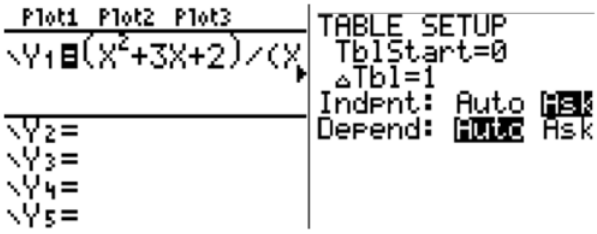
To estimate the limit $\lim_{x \to 2} \frac{x-2}{x^2-x-2}$, complete the table:

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)						

While it is not necessary to use the table feature in the calculator, it is very efficient.

To use a table on your calculator to evaluate a limit:

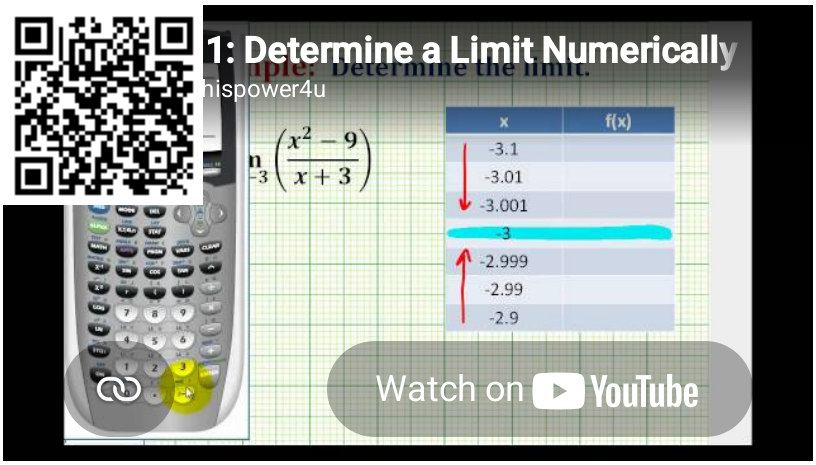
1. Enter the function on the y=screen
2. Go to table set up and highlight “ask” for the independent variable
3. Go to the table and enter values close to the number that x approaches



Another option is to substitute the given x values into the expression $\frac{x-2}{x^2-x-2}$ and record your results. Either way, the completed table is as follows.

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	0.34483	0.33445	0.33344	0.33322	0.33223	0.32258

The evidence suggests that the limit is $\frac{1}{3}$.



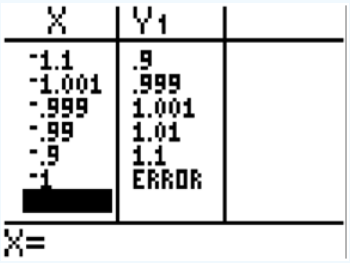
Examples

Example 1

Earlier, you were asked to find the limit of $\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$.

Solution

When you enter values close to -1 in the table you get y values that are increasingly close to the number 1. This implies that the limit as x approaches -1 is 1. Notice that when you evaluate the function at -1, the calculator produces an error. This should lead you to the conclusion that while the function is not defined at $x = -1$, the limit does exist.



Example 2

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)						

Solution

You can trick the calculator into giving a very exact answer by typing in 1.999999999999 because then the calculator rounds instead of producing an error.

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	0.25641	0.25063	0.25006	0.24994	0.24938	0.2439

The evidence suggests that the limit is $\frac{1}{4}$.

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Example 3

Complete the table and use the result to estimate the limit.

$\lim_{x \to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
f(x)						

Solution

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
f(x)	0.29112	0.28892	0.2887	0.28865	0.28843	0.28631

The evidence suggests that the limit is a number between 0.2887 and 0.28865. When you learn to find the limit analytically, you will know that the exact limit is $\frac{1}{2} \cdot 3\frac{1}{2} \approx 0.2886751346$.

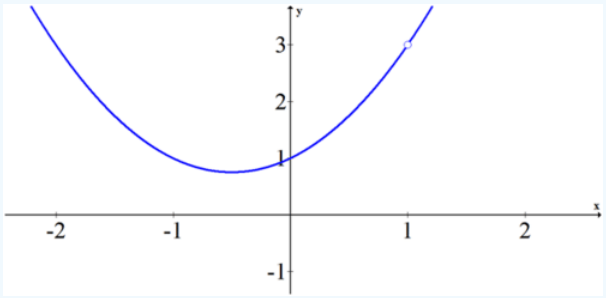
Example 4

Graph the following function and the use a table to verify the limit as x approaches 1.

$f(x) = \frac{x^3-1}{x-1}, x \neq 1$

Solution

$\lim_{x \to 1} f(x) = 3$. This is because when you factor the numerator and cancel common factors, the function becomes a quadratic with a hole at the point $(1, 3)$.



You can verify the limit in the table.

x	f(x)
.75	2.3125
.9	2.71
.99	2.9701
.999	2.997
1	Error
1.001	3.003
1.01	3.0301
1.1	3.31
1.25	3.8125

Example 5

Estimate the limit numerically.

$\lim_{x \to 0} \frac{\left[\frac{4}{x+2}\right]-2}{x}$

Solution

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
f(x)	0.20526	0.02005	0.002	-0.002	-0.02	-0.1952

$\lim_{x \to 0} f(x) = 0$

Review

Estimate the following limits numerically.

1. $\lim_{x \to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x \to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x \to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+2}$
4. $\lim_{x \to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$
5. $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{9}{x^2-9}\right)$
6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
13. $\lim_{x \to 2} \frac{\sqrt{x+3}-2}{x-1}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

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Review (Answers)

To see the Review answers, open this PDF file and look for section 14.3.

Vocabulary

Term	Definition
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
numerically	Numerically is a term used to describe one of several different representations in mathematics. It refers to tables where the actual numbers are visible.

Image Attributions

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SECTION OVERVIEW

8.2: Lines Tangent to a Curve

Topic hierarchy

8.2.1: Tangents to a Curve

8.2.2: Instantaneous Rates of Change

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8.2.1: Tangents to a Curve

Tangents to a Curve

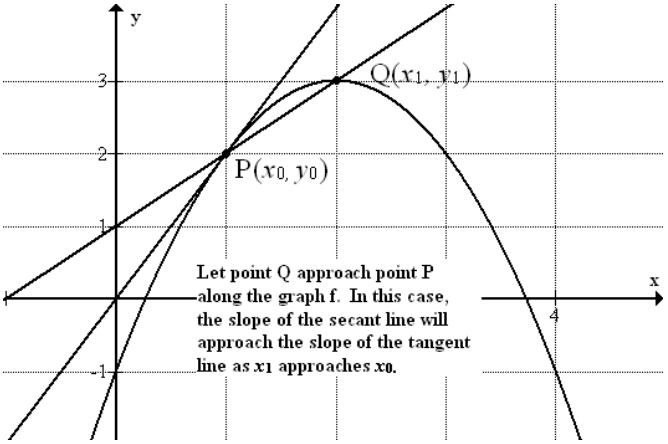
Kevin is learning about the basis of calculus and what calculus is actually used for. Unfortunately, Kevin does not understand why calculus is sometimes necessary to find the equation of a line. In Algebra 1, he learned you can find the equation of a line if you are given two points. You find the **slope** of the line by dividing the up/down difference in the points by the left/right difference, then you use one of the points and the slope to find the y-intercept.

Kevin's teacher, Mr. Banner, offered him extra credit if he could find the slope of a line for the points (4,5) and (4,5) using the method he learned in Algebra 1. Can you see what Mr. Banner did? What is Kevin going to find as he works on those problems?

Tangents to a Curve

Recall from algebra, if points $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two different points on the curve $y = f(x)$, then the **slope** of the **secant line** connecting the two points is given by

$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



Of course, if we let the point x_1 approach x_0 then Q will approach P along the graph f and thus the slope of the **secant** line will gradually approach the slope of the **tangent line** as x_1 approaches x_0. Therefore, (1) becomes

$$m_{sec} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

To simplify our notation, if we let $h = x_1 - x_0$, then $x_1 = x_0 + h$ and $x_1 \to x_0$ becomes equivalent to $h \to 0$. This means that (2) becomes

$$m_{sec} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

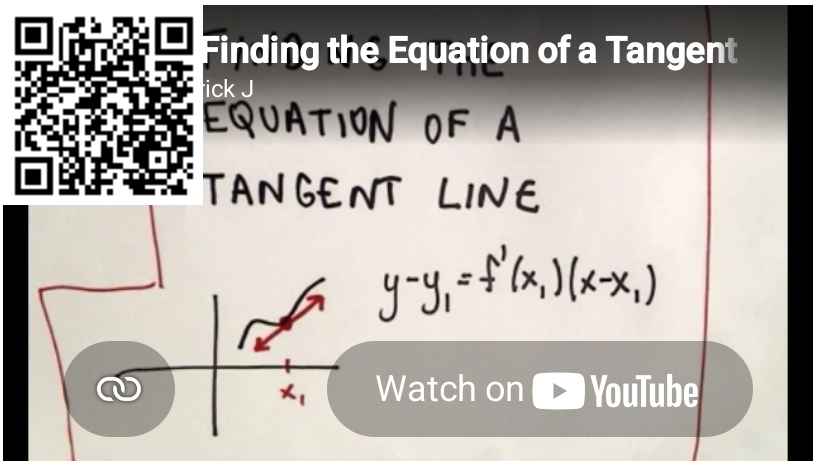
The Slope of a Tangent Line

If the point $P(x_0, y_0)$ is on the curve f, then the tangent line at Phas a slope that is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided that the limit exist.

Recall that the equation of the tangent line through point (x_0, y_0) with slope m is the point-slope form of a line: $y - y_0 = m_{\tan}(x - x_0)$.



Examples

Example 1

Earlier, you were given a problem about Kevin, who is having trouble understanding calculus.

Solution

Mr. Banner asked Kevin to find the equation of a line given the points (4, 5) and (4, 5). The points (4, 5) and (4, 5) are the same, so the $\frac{\text{rise}}{\text{run}}$ would be $\frac{0}{0}$ - Kevin was just introduced to the need for **differential calculus**!

Example 2

Find line tangent to the curve $f(x) = x^3$ that passes through point P (2, 8).

Solution

Since $P(x_0, y_0)$ = (2, 8), using the slope of the tangent equation we have

	mtan	$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$
and we get		
	mtan	$= \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$
		$= \lim_{h \to 0} \frac{(h^3+6h^2+12h+8)-8}{h}$
		$= \lim_{h \to 0} \frac{h^3+6h^2+12h}{h}$
		$= \lim_{h \to 0} (h^2+6h+12)$
		$= 12$

Thus the slope of the tangent line is 12. Using the point-slope formula above, we find that the equation of the tangent line is $y - 8 = 12(x - 2)$ or $y = 12x - 16$.

Example 3

If $f(x) = x^2 - 3$, find $f'(x)$ and use the result to find the slope of the tangent line at $x = 2$ and $x = -1$.

Solution

Since $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ then

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$f'(x)$	$= \lim_{h\to 0} \frac{[(x+h)^2-3]-[x^2-3]}{h}$
	$= \lim_{h\to 0} \frac{x^2+2xh+h^2-3-x^2+3}{h}$
	$= \lim_{h\to 0} \frac{2xh+h^2}{h}$
	$= \lim_{h\to 0}(2x+h)$
	$= 2x$

To find the slope, we simply substitute $x = 2$ into the result $f'(x)$:

	$f'(x)$	$= 2x$
	f'(2)	= 2(2)
		= 4
and		
	f'(x)	= 2x
	f'(-1)	= 2(-1)
		= -2

Thus slope of the tangent line at $x = 2$ and $x = -1$ are 4 and -2 respectively.

Example 4

Find the slope of the tangent line to the curve $y = 1/x$ that passes through the point (1, 1).

Solution

Using the slope of the tangent formula,

		$f'(x)$	$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
and substituting $y = \frac{1}{x}$			
		$f'(x)$	$= \lim_{h\to 0} \frac{(\frac{1}{x+h})-\frac{1}{x}}{h}$
			$= \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h}$
			$= \lim_{h\to 0} \frac{x-x-h}{hx(x+h)}$
			$= \lim_{h\to 0} \frac{-1}{x(x+h)}$
			$= \lim_{h\to 0} \frac{-1}{x(x+h)}$
			$= \frac{-1}{x^2}$
For $x = 1$, the slope is			
		$f'(x)$	$= \frac{-1}{1} = -1$
			$= -1$

Thus the slope of the tangent line at $x = 1$ for the curve $y = 1/x$ is $m = -1$. To find the equation of the tangent line, we simply use the point-slope formula,

		$y - y_0$	$= m(x - x_0)$
Where $(x_0, y_0) = (1, 1)$.			
		$y - 1$	$= -1(x - 1)$
		y	$= -x + 1 + 1$
		y	$= -x + 2$

So the equation of the tangent line is $y = -x + 2$.

Example 5

Given the function $y = \frac{1}{2}x^2$ and the values of $x_0 = 3$ and $x_1 = 4$, find:

a. The **average rate of change** of y with respect to x over the interval $[x_0, x_1]$.
b. The slope of the secant line connecting x_0 and x_1.
c. The **instantaneous rate of change** of y with respect to x at x_0.
d. The slope of the tangent line at x_1.

Solution

a. Identify the two points by substituting 3 and 4 in for x in the function $f(x) = \frac{1}{2}x^2$

Substitute the two points (3, 4.5) and (4, 8) into the average rate of change formula: $m = \frac{y_1 - y_0}{x_1 - x_0}$

Average rate of change $= \frac{7}{2}$

b. The slope of the secant line between x0 and x1 is the slope between (3, 4.5) and (4, 8), which is $\frac{7}{2}$.

c. Instantaneous rate of change is the slope at $x = 3$.

Use the formula: $\frac{f(x+h) - f(x)}{h}$ where $f(x) = \frac{1}{2}x^2$ and $x = 3$

$\frac{f(3+h) - f(3)}{h}$... Substitute 3 for x

$\frac{\frac{1}{2}(3+h)^2 - \frac{1}{2}(3)^2}{h}$... Replace $f(x) \rightarrow \frac{1}{2}x^2$

$\frac{\frac{1}{2}(9) + \frac{1}{2}(6h) + \frac{1}{2}h^2 - \frac{1}{2}9}{h}$... FOIL and Distribute the 1/2

$\frac{6h + h^2}{2h}$... Simplify

$3 + \frac{h}{2}$... Simplify again

3... As $h \rightarrow 0$

$\therefore$ the instantaneous slope at $x = 3$ is 3

d. The slope of the tangent at 4 is the same as the instantaneous rate of change at $x = 4$

This is the same series of steps as with $x = 3$ above

$\therefore$ the slope at $x = 4$ is 4

Example 6

Given the function $f(x) = \frac{1}{x}$ and the values $x_0 = 2$ and $x_1 = 3$, find:

a. The average rate of change of y with respect to x over the interval $[x_0, x_1]$.
b. The slope of the secant line connecting x_0 and x_1.
c. The instantaneous rate of change of y with respect to x at x_0.
d. The slope of the tangent line at x_1.

Solution

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a. Identify the two points by substituting 2 and 3 in for x in the function $f(x) = \frac{1}{x}$ to get $(2, \frac{1}{2}) \mid (3, \frac{1}{3})$

Substitute the two points $(2, \frac{1}{2}) \mid (3, \frac{1}{3})$ into the average rate of change formula: $m = \frac{y_1 - y_0}{x_1 - x_0}$

Average rate of change $= \frac{-1}{6}$

b. The slope of the secant line between x_0 and x_1 is the slope between $(2, \frac{1}{2})$ and $(3, \frac{1}{3})$, which is $\frac{-1}{6}$.

c. Instantaneous rate of change at x_0 is the slope at $x = 2$.

Use the formula: $\frac{f(x+h)-f(x)}{h}$ where $f(x) = \frac{1}{x}$ and $x = 2$

$\frac{f(2+h)-f(2)}{h}$... Substitute 2 for x

$\frac{\frac{1}{2+h}-\frac{1}{2}}{h}$... Replace $f(x) \to \frac{1}{x}$

$\left(\frac{1}{2+h} - \frac{1}{2}\right) \cdot \frac{1}{h}$... We had a fraction divided by a fraction, invert to multiply

$\frac{(2)(1)}{2(2+h)} - \frac{(2+h)(1)}{2(2+h)} \cdot \frac{1}{h}$... Set common denominators

$\frac{(2)-(2+h)}{(2+h)(2)(h)}$... Simplify

$\frac{-h}{4h+2h^2}$... Simplify again

$\frac{-1}{4+2h}$... once more (canceling the h)

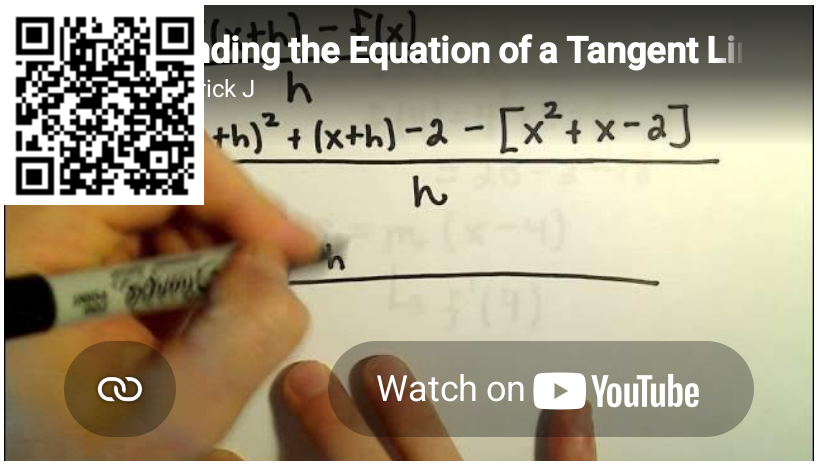
$\frac{-1}{4}$... As $h \to 0$

$\therefore$ the instantaneous slope at $x = 2$ is $\frac{-1}{4}$

d. The slope of the tangent at 3 is the same as the instantaneous rate of change at $x = 3$

This is the same series of steps as with $x = 2$ above

$\therefore$ the slope at $x = 3$ is $\frac{-1}{9}$



Review

1. What is the line connecting two points (x_0, y_0) and (x_1, y_1) on a curve called?
2. As (x_0, y_0) gets immeasurably close to (x_1, y_1) the term describing the line between them becomes: "the _____ line"
3. The expression $f(x_0 + h) - f(x_0)$ is used to describe what distance in the process of finding the slope of a tangent line?
4. When calculating the slope of a tangent, what value is assumed to go to 0 as the two chosen points get closer and closer?
5. What does the concept of limit, discussed in prior lessons, have to do with finding the slope of a line tangent to a curve?

Find the equation of the tangent line:

6. What is the equation of the tangent line at $x = -3$ assuming that $r(-3) = -5$ and $r'(-3) = 1$?
7. What is the equation of the tangent line at $x = 1$ assuming that $r(1) = 3$ and $r'(1) = -5$?
8. What is the equation of the tangent line at $x = 2$ assuming that $g(2) = 1$ and $g'(2) = -3$?

9. What is the equation of the tangent line at $x = 4$ assuming that $u(4) = 4$ and $u'(4) = 3$?
10. What is the equation of the tangent line at $x = -4$ assuming that $t(-4) = 2$ and $t'(-4) = 5$?

Find the equation of the tangent line:

11. Find the equation of the tangent line to the graph of $h(x) = -5x^3 - 3x^2 + x + 3$ at $x = 1$
12. Find the equation of the tangent line to the graph of $t(x) = -2x$ at $x = -2$
13. Find the equation of the tangent line to the graph of $m(x) = 3x^3 + 3x^2 + 4x + 4$ at $x = 1$
14. Find the equation of the tangent line to the graph of $q(x) = -x^3 - 4x^2 + 4x + 3$ at $x = -2$
15. Find the equation of the tangent line to the graph of $t(x) = -4x^2 + 2x - 4$ at $x = -1$
16. Find the equation of the tangent line to the graph of $h(x) = -4x^3 + 2x^2 - 3x + 3$ at $x = -1$
17. Find the equation of the tangent line to the graph of $m(x) = x$ at $x = 0$
18. Find the equation of the tangent line to the graph of $s(x) = -3x^2 - 2x + 3$ at $x = 0$
19. Find the equation of the tangent line to the graph of $c(x) = -3$ at $x = 0$
20. Find the equation of the tangent line to the graph of $b(x) = -5x^4 + 3x^3 - x^2 + 5x - 3$ at $x = -1$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.7.

Vocabulary

Term	Definition
secant	A line that intersects a circle in two points.
tangent	A line that intersects a circle in exactly one point.
Average rate of change	The average rate of change of a function is the change in y coordinates of a function, divided by the change in x coordinates.
Differential calculus	Differential calculus is the branch of calculus based on finding the difference in location between two points that get closer together until the distance between them is infinitely small.
instantaneous rate of change	The instantaneous rate of change of a curve at a given point is the slope of the line tangent to the curve at that point.
secant line	A secant line is a line that joins two points on a curve.
Slope	Slope is a measure of the steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. The slope of a line can be found by calculating “rise over run” or “the change in the y over the change in the x.” The symbol for slope is m
Tangent line	A tangent line is a line that "just touches" a curve at a single point and no others.

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8.2.2: Instantaneous Rates of Change

Instantaneous Rates of Change

You may recall the story about Jim and his track-star girlfriend Becca from the lesson on understanding limits. The two of them were discussing how they might calculate her speed at the instant that Jim snapped a photo of her. The end result of the discussion was Becca pointing out that it is not technically possible to calculate the exact speed of something in a specific instant.

By now, we have explored the related concepts of limits and lines tangent to a curve, so we know it is possible to effectively calculate **instantaneous speed**. What process would be involved with actually calculating Becca's speed at the instant the photo was taken? What are the technical difficulties with identifying instantaneous speed?

Instantaneous Rates of Change

The function f′(x) that we defined in previous lessons is so important that it has its own name: the **derivative**.

The Derivative
The function f' is defined by the formula
$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
where f' is called the derivative of *f* with respect to *x*. The domain of *f* consists of all the values of *x* for which the limit exists.

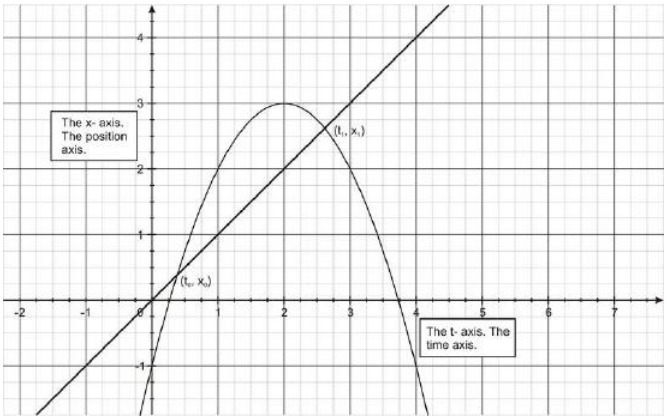
Based on the discussion that we have had in previous section, the derivative f′ represents the slope of the **tangent line** at point *x*. Another way of interpreting it would be that the function $y = f(x)$ has a derivative f′ whose value at *x* is the **instantaneous rate of change** of *y* with respect to point *x*.

One of the two primary concepts of calculus involves calculating the rate of change of one quantity with respect to another. For example, speed is defined as the rate of displacement with respect to time. If a person travels 120 miles in 4 hours, his speed is 120/4 = 30 mi/hr. This speed is called the **average speed** or the **average rate of change** of distance with respect to time. Of course the person who travels 120 miles at a rate of 30 mi/hr for 4 hr probably does not do so continuously. Though he probably slowed down or sped up during the 4-hour period, it does generally suffice to say that he traveled for 4 hours at an average rate of 30 miles per hour. However, if the driver strikes a tree, it would not be his average speed that determines his survival but his speed at the *instant of the collision*. Similarly, when a bullet strikes a target, it is not the average speed that is significant but its **instantaneous speed** at the moment it strikes. So here we have distinct kinds of speeds, average speed and instantaneous speed.

The average speed of an object is defined as the object's displacement Δx divided by the time interval Δt during which the displacement occurs:

$$\text{Average speed } = v = \frac{\Delta x}{\Delta t} = \frac{x_1 - x_0}{t_1 - t_0}$$

Notice that the points (t_0, x_0) and (t_1, x_1) lie on the position versus time curve, as the figure below shows.



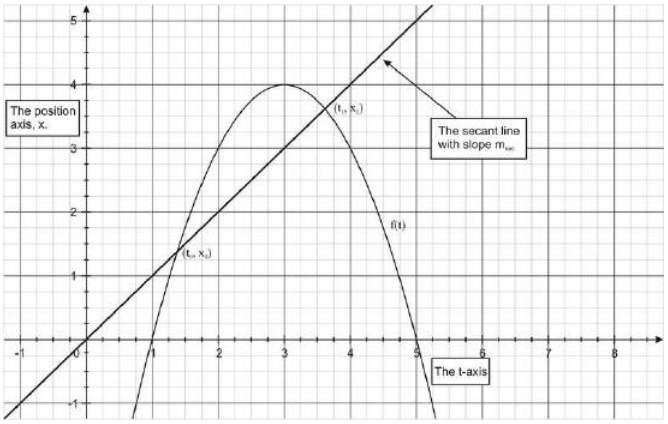
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This expression is also the expression for the slope of a **secant line** connecting the two points. Thus we conclude that the average velocity of an object between time t_0 and t_1 is represented geometrically by the slope of the secant line connecting the two points (t_o, x_o) and (t_1, x_1). If we choose t_1 close to t_o, then the average velocity will closely approximate the instantaneous velocity at time t_o.

Geometrically, the average rate of change is represented by the slope of a secant line (figure *a*, below) and the instantaneous rate of change is represented by the slope of the tangent line (figure *b*, below).

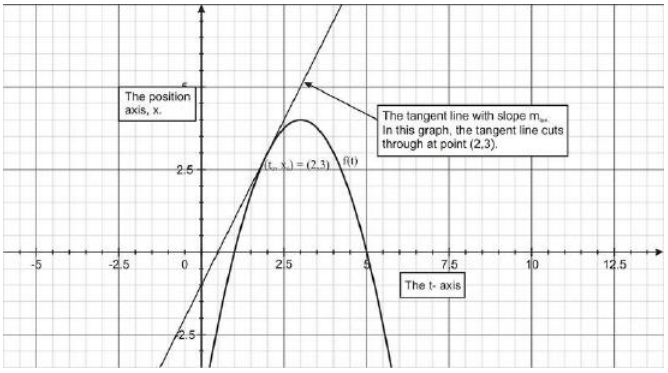
Average Rate of Change (such as *the average velocity*) The average rate of change of $y = f(x)$ over the time interval $[x_0, x_1]$ is the slope m_{sec} of the secant line to the points $(x_o, f(x_0))$ and $(x_1, f(x_0))$ on the graph (figure a):

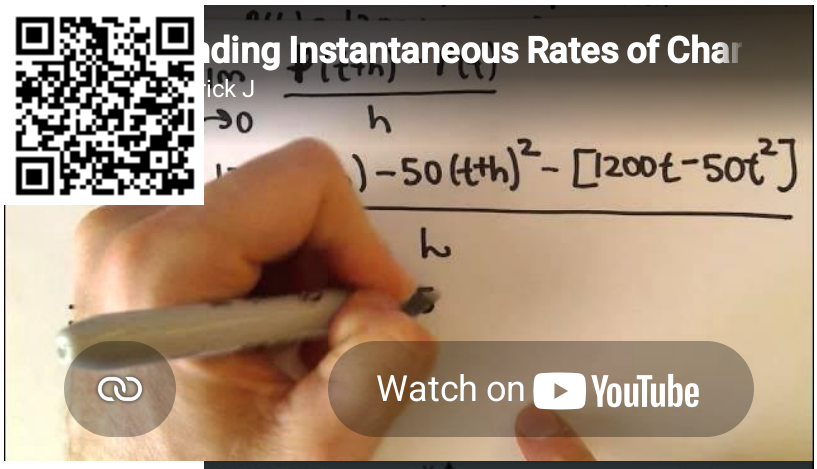
$$m_{sec} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

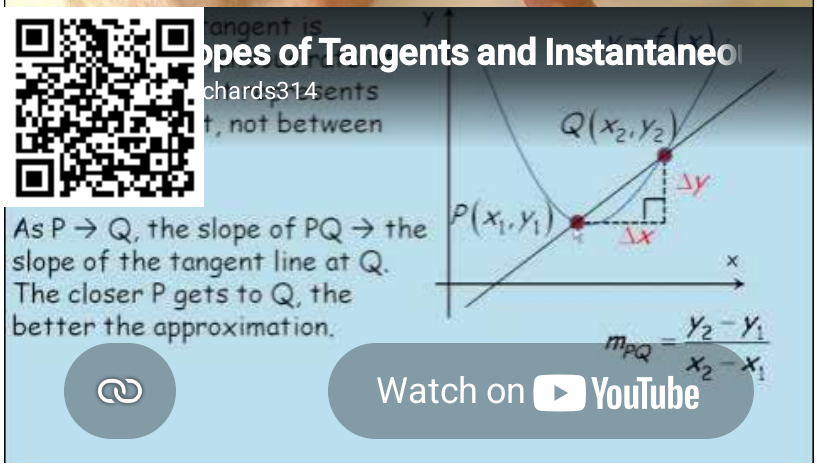


Instantaneous Rate of Change The instantaneous rate of change of $y = f(x)$ at the point x_0 is the slope m_{sec} of the tangent line to the point $x_0$on the graph (figure b):

$$m_{\tan} = f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{t_1 - t_0}$$







Examples

Example 1

Earlier, you were asked a question about the process of calculating Becca's speed in the race.

Solution

Speed is, by definition, a relation between a distance and the time required to cross that distance (recall d=rt from your science class). If the time required is *zero*, then you end up dividing by zero, which is an undefined function.

However, using calculus, you can identify what the **end behavior** of the function would be if you were to get infinitely close, and thereby effectively calculate the speed at a given instant. In fact, that is exactly what you did in the examples above.

Example 2

Calculate: (a) The slope of the line tangent to $y = x^2 + 5$ at the point on the curve $x = 4$ and (b) the equation of that line.

In the previous concept "Tangents to a Curve" we showed how to calculate the **derivative** (the slope of the tangent) of a function of the form $y = x^2 - c$.

a. The slope of the tangent to the curve $y = x^2 + 5$ at $x = 4$ is:
b. Given $m = 8$ from above, we have $y = 8x + b$. All we need is to solve for $b$to have the full equation.

Solution

a. $2(4) = 8$

Recall from algebra the "slope - intercept form" of a straight line: $y = mx + b$, where m is the slope of the line.

b. Substitute 4 (the point we are finding the slope of) in for x in the equation of the curve

$y = x^2 + 5$ to identify a corresponding y

$y = (4)^2 + 5$

$y = 21$

Substitute the values for x, y, and m that we now have into our $y = mx + b$ form of the equation of the tangent line:

$21 = 8(4) + b$

$-11 = b$

Use the calculated values for m and b to complete the equation:

$y = 8x - 11$

Example 3

Suppose that $y = x^2 - 3$.

a. Find the average rate of change of y with respect x over the interval [0, 2] and
b. Find the instantaneous rate of change of y with respect x at the point $x = -1$.

Solution

a. Applying the formula above for secant with $f(x) = x^2 - 3$ and $x_0 = 0$ and $x_1 = 2$, yields

m_{sec}	$= \frac{f(x_1) - f(x_0)}{t_1 - t_0}$
	$= \frac{f(2) - f(0)}{2 - 0}$
	$= \frac{1 - (-3)}{2}$
	$= 2$

This means that the average rate of change of y is 2 units per unit increase in x over the interval [0, 2].

b. Recall, for functions of the form $y = x^2 + c$ that $f'(x) = 2x$ ($f'(x)$ is "f *prime* of x," meaning "the slope (m) of the line tangent to x")

	m_{tan}	= f′(x_0)
		= f′(−1)
		= 2(−1)
		= −2

This means that the instantaneous rate of change is negative. That is, y is decreasing at the point x = -1. It is decreasing at a rate of 2 units per unit increase in x.

Example 4

Find the derivative of $f(x) = \sqrt{x}$ and the equation of the tangent line at x_0 = 1.

Solution

Using the definition of the derivative,

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$f'(x)$	$= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$
	$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$
	$= \lim_{h \to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \right)$
	$= \lim_{h \to 0} \frac{1}{h} \frac{x+h-x}{\sqrt{x+h}+\sqrt{x}}$
	$= \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$
	$= \frac{1}{2\sqrt{x}}$

Thus, the slope of the tangent line at $x_0 = 1$ is

$f'(1)$	$= \frac{1}{2\sqrt{1}} = \frac{1}{2}$

For $x_0 = 1$, we can find y_0 by simply substituting into $f(x)$:

$f(x_0)$	$\equiv y_0$
$f(1)$	$= \sqrt{1} = 1$
y_0	$= 1$

Thus the equation of the tangent line is

$y - y_0$	$= m(x - x_0)$
$y - 1$	$= \frac{1}{2}(x - 1)$
y	$= \frac{1}{2}x + \frac{1}{2}$

Example 5

Find the derivative of $f(x) = \frac{x}{x+1}$.

Solution

Using the derivative formula:

$f'(x)$	$= \lim_{h \to 0} \frac{\left[\frac{x+h}{x+h+1} - \frac{x}{x+1} \right]}{h}$
	$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x+h}{x+h+1} - \frac{x}{x+1} \right]$
	$= \lim_{h \to 0} \frac{1}{h} \left[\frac{(x+h)(x+1)-x(x+h+1)}{(x+h+1)(x+1)} \right]$
	$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2+x+hx+h-x^2-xh-x}{(x+h+1)(x+1)} \right]$
	$= \lim_{h \to 0} \frac{1}{h} \left[\frac{h}{(x+h+1)(x+1)} \right]$
	$= \lim_{h \to 0} \frac{1}{(x+h+1)(x+1)}$
	$= \frac{1}{(x+1)^2}$

Example 6

A rocket is propelled upwards and reaches a height of $h(t) = 4.9t^2$ in t seconds.

a. How high does it reach in 35 seconds?

b. What is the average velocity of the rocket during the first 35 seconds?

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c. What is the instantaneous velocity of the rocket at the end of the 35 seconds?

Solution

a. The height of the rocket at 35 secs is $4.9(35)^2 = 6002.5m$

b. $V_{avg} = \frac{6002.5m}{35s} = 171.5\ \text{m/s}$

c. To find the instantaneous speed, we need to find the derivative of $h(t) = 4.9t^2$

$h'(t) = 9.8t$ Using the instantaneous rate of change formula from above

$9.8 \cdot 35s = 343\ \text{m/sec}$

Example 7

A particle moves in the positive direction along a straight line so that after t nanoseconds, its traversed distance is given by $\chi(t) = 9.9t^3$ nanometers.

a. What is the average velocity of the particle during the first 2 nanoseconds?

b. What is the instantaneous velocity of the particle at t = 2 nanoseconds?

Solution

a. The particle moves $9.9t^3nm$ in t secs

$\therefore 9.9\,(2^3) = \frac{79.2nm}{2s} \rightarrow \frac{39.6nm}{1s}$

b. Using the formula for finding the derivative, we get $\chi'(t) = 29.7t^2$

$\therefore \chi'(2) = 118.8nm/s$

$\therefore$ The instantaneous velocity at $t = 2$ is 118.8nm/sec

Review

Find the average rate of change:

1. $C = f(x)$ and $f(x) = x^2 - 4x + 2$. Find the average rate of change of (C) with respect to (x), when (x) is changed from x = 15, to x = 59.
2. $H = f(x)$ and $f(x) = x^2 - 5x + 201$ Find the average rate of change of (H), with respect to (x) when (x) is changed from x = 10 to x = 11.
3. $N = f(x)$ and $f(x) = 3x^2 - 4x - 1$ Find the average rate of change of (N) with respect to (x) when (x) is changed is changed from x = 20 to x = 64.
4. $H = f(x)$ and f(x)=x^{2}+10 x+201 Find the average rate of change of (H) with respect to (x) when (x) is changed from x + 25 to x = 74.
5. N=f(x) and $f(x) = -5x^2 - 3x - 4$ Find the average rate of change of (N) with respect to (x) when (x) is changed from x = 30 to x = 54.

Find the instantaneous rate of change:

6. If $C = f(x)$ and $f(x) = -4x^2 + 2x + 5$ Find the instantaneous rate of change of (C) with respect to (x) when x = 25.
7. If $N = f(x)$ and $f(x) = 3x^2 - x - 5$ Find the instantaneous rate of change of (N) with respect to x when x = 10.
8. $H = f(x)$ and $f(x) = 4x^2 + 195$ Find the instantaneous rate of change of (H) with respect to x when x = 10.
9. $N = f(x)$ and $f(x) = -x^2 + x - 3$ Find the instantaneous rate of change of (N) with respect to x when x = 150.
10. $C = f(x)$ and $f(x) = -3x^2 + 4x - 4$ Find the instantaneous rate of change of (N) with respect to x when x = 20.

Use the definition of the derivative to find $f(x)$ and then find the equation of the tangent line at $x = x_0$.

11. $f(x) = 6x^2; x_0 = 3$
12. $f(x) = \sqrt{x+2}; x_0 = 8$
13. $f(x) = 3x^3 - 2; x_0 = -1$

14. $f(x) = \frac{1}{x+2}; x_0 = -1$
15. $f(x) = ax^2 - b$, (where a and b are constants); x_0 = b
16. $f(x) = x^{1/3}; x_0 = 1$
17. Suppose that f has the property that $f(x + y) = f(x) + f(y) + 3xy$ and $\lim_{h \to 0} \frac{f(h)}{h} = 4$. Find $f(0)$ and $f'(x)$.
18. Find dy/dx

Solve the rate of change problems.

19. A packing company in the south makes "Mama's Spaghetti Sauce." The cost of producing x jars is J = f(x) dollars. What does f'(100) = 9999 mean in this context?
20. A cherry pie is taken from an oven when its temperature is 202°F and placed on a table in a room where the temperature is 75°F. The temperature of the pie over x minutes is given by T = f(x). What does f'(100) = 102 mean in this context?
21. The number of virus, after (x) hours, in a controlled laboratory experiment is V = f(x). What are the units of f'(x)?
22. The number of people in the US affected by the common cold in the month of November is defined by N = f(x) where x is the day of the month. What is the meaning of f'(x) in this context?
23. The number of households in Florida affected by hurricane season in the month of July is defined by J = f(x) where x is the day of the month. $f(x) = 2x^2 + x + 1$ Find the average rate of change of J with respect to x when days is changed from x = 5 to x = 34.
24. A cake is taken from an oven when its temperature is 196°F and placed on a cooling rack in a room where the temperature is 75°F. The temperature of the cake over (x) minutes is given by H = f(x). $f(x) = 4x^2 + 15x + 196$ Find the instantaneous rate of change of H with respect to x when x = 15.
25. A pan of meatloaf is taken from an oven when its temperature is 205°F and placed on a table in a room where the temperature is 75°F. The temperature of the meatloaf over x minutes is given by H = f(x). $f(x) = 2x^2 + 5x + 205$. Find the average rate of change of H with respect to x when minutes is changed from x = 5 to x = 54.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.8.

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Vocabulary

Term	Definition
Average rate of change	The average rate of change of a function is the change in y coordinates of a function, divided by the change in x coordinates.
Average speed	The average speed of an object is the distance an object travels divided by the travel time
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$ and $\frac{df(x)}{dx}$.
instantaneous rate of change	The instantaneous rate of change of a curve at a given point is the slope of the line tangent to the curve at that point.
Instantaneous speed	The instantaneous speed of an object is the speed of the object at a specific point in time.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
secant line	A secant line is a line that joins two points on a curve.
Slope	Slope is a measure of the steepness of a line. A line can have positive, negative, zero (horizontal), or undefined (vertical) slope. The slope of a line can be found by calculating “rise over run” or “the change in the y over the change in the x.” The symbol for slope is m
Tangent line	A tangent line is a line that "just touches" a curve at a single point and no others.

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SECTION OVERVIEW

8.3: Derivatives

Topic hierarchy

8.3.1: Constant Derivatives and the Power Rule

8.3.2: Derivatives of Sums and Differences

8.3.3: Quotient Rule and Higher Derivatives

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8.3.1: Constant Derivatives and the Power Rule

Constant Derivatives and the Power Rule

The power rule is a fantastic "shortcut" for finding the derivatives of basic polynomials. Between the power rule and the basic definition of the **derivative** of a constant, a great number of polynomial derivatives can be identified with little effort - often in your head!

Constant Derivatives and the Power Rule

In this lesson, we will develop formulas and theorems that will calculate derivatives in more efficient and quick ways. Look for these theorems in boxes throughout the lesson.

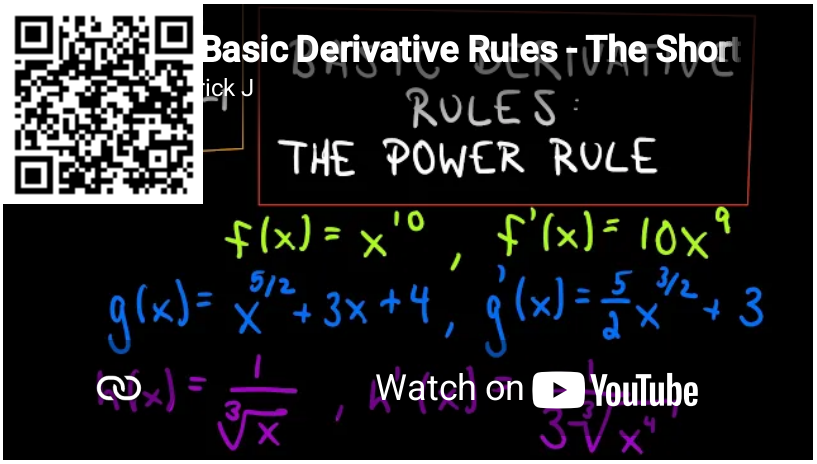
The Derivative of a Constant

Theorem: If f(x)=c where *c* is a constant, then f′(x)=0.
Proof: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = 0$.

Theorem: If c is a constant and f is differentiable at all x, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$.
In simpler notation $(cf)' = c(f)' = cf'$

The Power Rule

Theorem: (The Power Rule) If *n* is a positive integer, then for all real values of *x*
$\frac{d}{dx}[x^n] = nx^{n-1}$.



Examples

Example 1

Find $f'(x)$ for $f(x) = 16$.

Solution

If $f(x) = 16$ for all x, then $f'(x) = 0$ for all x.

We can also write $\frac{d}{dx}16 = 0$.

Example 2

Find the derivative of $f(x) = 4x^3$.

Solution

$\frac{d}{dx}[4x^3]$..... Restate the function

$4\frac{d}{dx}[x^3]$..... Apply the commutative law

$4[3x^2]$..... Apply the power Rule

$12x^2$..... Simplify

Example 3

Find the derivative of $f(x) = \frac{-2}{x^4}$.

Solution

$\frac{d}{dx}\left[\frac{-2}{x^4}\right]$..... Restate

$\frac{d}{dx}[-2x^{-4}]$..... Rules of exponents

$-2\frac{d}{dx}[x^{-4}]$..... By the commutative law

$-2[-4x^{-4-1}]$..... Apply the power rule

$-2[-4x^{-5}]$..... Simplify

$8x^{-5}$..... Simplify again

$\frac{8}{x^5}$..... Use rules of exponents

Example 4

Find the derivative of $f(x) = x$.

Solution

Special application of the power rule:

$\frac{d}{dx}[x] = 1x^{1-1} = x^0 = 1$

Example 5

Find the derivative of $f(x) = \sqrt{x}$.

Solution

Restate the function: $\frac{d}{dx}[\sqrt{x}]$

Using rules of exponents (from algebra): $\frac{d}{dx}[x^{1/2}]$

Apply the power rule: $\frac{1}{2}x^{1/2-1}$

Simplify: $\frac{1}{2}x^{-1/2}$

Rules of exponents: $\frac{1}{2x^{1/2}}$

Simplify: $\frac{1}{2\sqrt{x}}$

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Example 6

Find the derivative of $f(x) = \frac{1}{x^3}$.

Solution

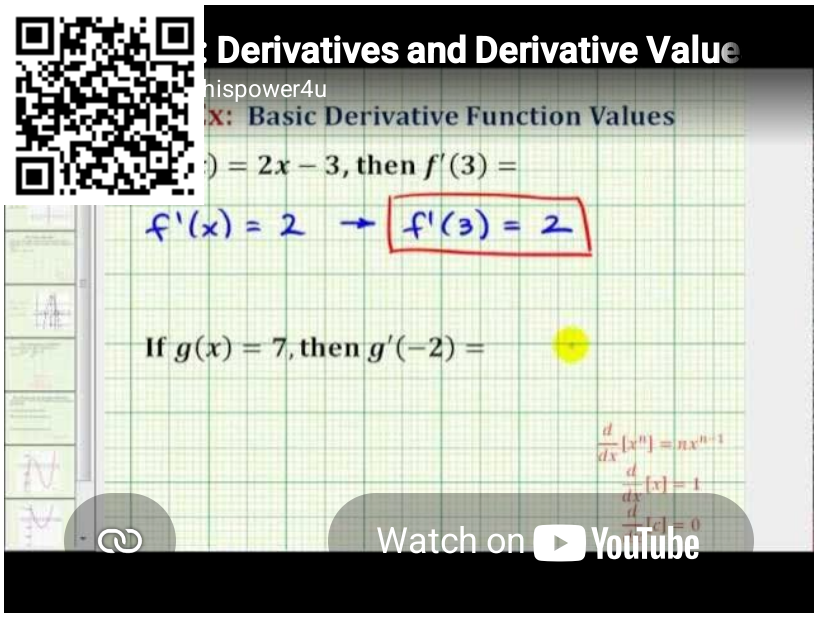
Restate the function: $\frac{d}{dx}\left[\frac{1}{x^3}\right]$

Rules of exponents: $\frac{d}{dx}\left[x^{-3}\right]$

Power rule: $-3x^{-3-1}$

Simplify: $-3x^{-4}$

Rules of exponents: $\frac{-3}{x^4}$



Review

1. State the power rule.

Find the derivative:

2. $y = 5x^7$
3. $y = -3x$
4. $f(x) = \frac{1}{3}x + \frac{4}{3}$
5. $y = x^4 - 2x^3 - 5\sqrt{x} + 10$
6. $y = (5x^2 - 3)^2$
7. Given $y(x) = x^{-4\pi^2}$, find the derivative when $x = 1$.
8. $y(x) = 5$
9. Given $u(x) = x^{-5\pi^3}$, what is $u'(2)$?
10. $y = \frac{1}{5}$ when $x = 4$
11. Given $d(x) = x^{-0.37}$, what is $d'(1)$?
12. $g(x) = x^{-3}$
13. $u(x) = x^{0.096}$
14. $k(x) = x - 0.49$
15. $y = x^{-5\pi^3}$

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.9.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$ and $\frac{df(x)}{dx}$.
proof	A proof is a series of true statements leading to the acceptance of truth of a more complex statement.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.

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8.3.2: Derivatives of Sums and Differences

Derivatives of Sums and Differences

Juan has been out playing with his model rocket all afternoon. Partway through the day, he started taking videos of the flights using his cell phone. Watching the video, he notices that the rockets actually seem to be getting faster after the launch instead of starting off at full speed and slowing down due to gravity.

Juan figures it is reasonable to assume it takes a bit for the engines to get the rocket up to full speed, but the acceleration seems to continue past when he figures that would continue.

After considering for a while, he wonders if the decreased mass of the rocket as it burns fuel might be the cause, assuming he knows the force generated by the engines and the starting and ending weight of the rocket, is there a way he could conjecture whether the increased acceleration might be a result of the decreased mass?

Derivatives of Sums and Differences

Theorem: If f and g are two **differentiable** functions at x then

$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

and

$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

In simpler notation

$(f \pm g)' = f' \pm g'$

The Product Rule

Theorem: (The Product Rule) If f and g are differentiable at x, then

$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$

In a simpler notation

$(f \cdot g)' = f \cdot g' + g \cdot f'$

In words, *The **derivative** of the product of two functions is equal to the first function times the derivative of the second plus the second function times the derivatives of the derivative of the first.*

Keep in mind that $(f \cdot g)' \neq f' + g'$

Examples

Example 1

Earlier, you were asked if Juan could make a conjecture about whether increased acceleration might be a result of decreased mass.

Solution

Yes, he can make the conjecture. Assuming that the force is equal to the change in mass times velocity (momentum) over change in time, then using the power rule and simplifying, he can discover that the acceleration of the rocket is equal to the force minus the velocity multiplied by change in mass over time, all divided by mass, in mathematics this looks like:

$a = \left(\frac{F - v\left(\frac{\delta m}{\delta t}\right)}{m}\right)$

Looking at the upper-right portion of the equation, we can see that as mass decreases, the fraction $\frac{\delta m}{\delta t}$ goes negative. Since $-v$ is multiplied by that fraction, it goes positive, and the overall function increases, meaning the rocket accelerates.

Looks like Juan was right.

Example 2

Find the derivative: $f(x) = 3x^2 + 2x$

Solution

Use the power rule to help:

$\frac{d}{dx}\left[3x^2 + 2x\right]$	$= \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}\left[2x\right]$
	$= 3\frac{d}{dx}\left[x^2\right] + 2\frac{d}{dx}\left[x\right]$
	$= 3[2x] + 2[1]$
	$= 6x + 2$

Example 3

Find the derivative: $f(x) = x^3 - 5x^2$.

Solution

Again, use the power rule to help:

$\frac{d}{dx}\left[x^3 - 5x^2\right]$	$= \frac{d}{dx}\left[x^3\right] - 5\frac{d}{dx}\left[x^2\right]$
	$= 3x^2 - 5[2x]$
	$= 3x^2 - 10x$

Example 4

Find $\frac{dy}{dx}$ for $y = (3x^4 + 2)(7x^3 - 1)$.

Solution

There two methods to solve this problem. One is to multiply to find the product and then use the derivative of the sum rule. The second is to directly use the product rule. Either rule will produce the same answer. We begin with the sum rule.

y	$= (3x^4 + 2)(7x^3 - 1)$
	$= 21x^7 - 3x^4 + 14x^3 - 2$

Taking the derivative of the sum yields

$\frac{dy}{dx}$	$= 147x^6 - 12x^3 + 42x^2 + 0$
	$= 147x^6 - 12x^3 + 42x^2$

Now we use the product rule.

y′	$= (3x^4 + 2) \cdot (7x^3 - 1)' + (3x^4 + 2)' \cdot (7x^3 - 1)$
	$= (3x^4 + 2)(21x^2) + (12x^3)(7x^3 - 1)$
	$= (63x^6 + 42x^2) + (84x^6 - 12x^3)$
	$= 147x^6 - 12x^3 + 42x^2$

Which is the same answer.

Example 5

Given: $t(x) = x - 1$. What is $\frac{dt}{dx}$ when $x = 0$?

Solution

By the difference rule: $(x-1)' = (x)' - (1)' = 0$

$x' = 1$..... By the power rule

$1' = 0$..... The derivative of a constant = 0

So when we evaluate this at x = 0, we get 1, since 1 - 0 = 1

Example 6

What is the derivative of $g(x) = (-x-1)(x+1)$?

Solution

We'll use the difference rule

First, expand $(-x-1)(x+1) \rightarrow -x^2 - 2x - 1$.

By the difference rule: $(-x^2 - 2x - 1)' = (-x^2)' - (2x)' - (1)' = -2x - 2$

Example 7

Given $a(x) = -\pi x^{-0.54} + 6x^4$. What is $\frac{dy}{dx}$?

Solution

We'll use the difference and power rules:

$\frac{d}{dx}(-\pi x^{-0.54} + 6x^4) =$

$\frac{d}{dx}(-\pi x^{-0.54}) + \frac{d}{dx}(6x^4)$..... By the difference rule

$\rightarrow 0.54\pi x^{-1.54} + 24x^3$..... By the power rule

Example 8

What is $\frac{d}{dx}[(-5x)\cos(x)]$?

Solution

We'll use the product rule:

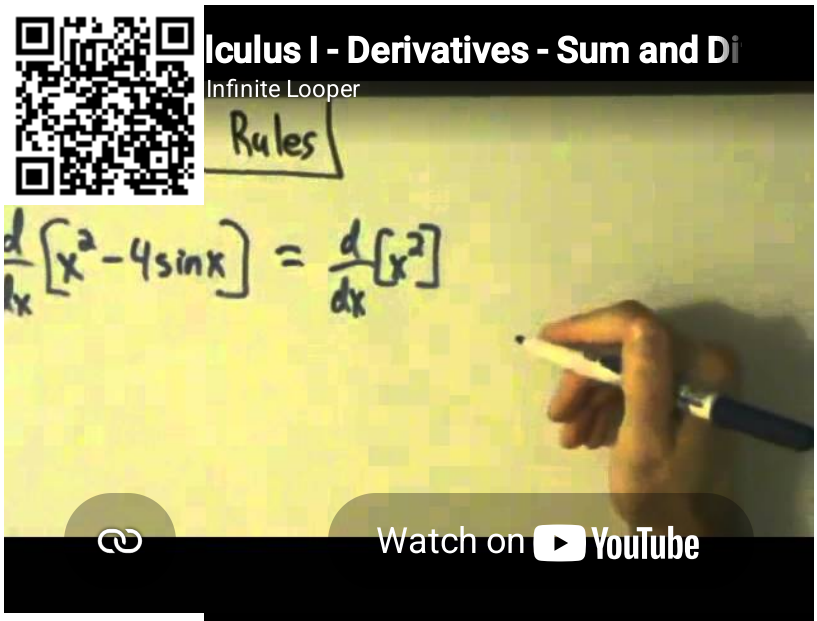
$(pq)' = p'q + pq'$.

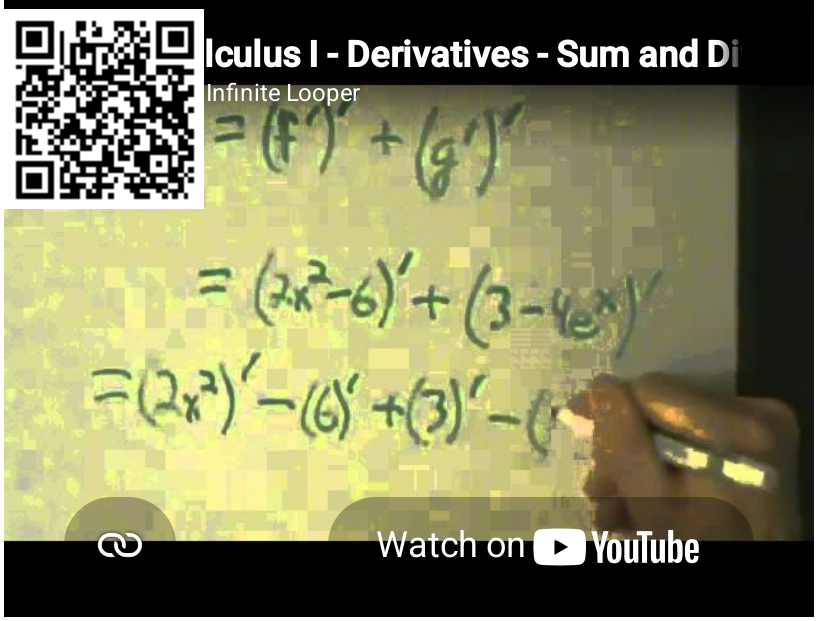
$p(x) = -5x \rightarrow p'(x) = -5$.... By the power rule

$q(x) = \cos(x) \rightarrow q'(x) = -\sin(x)$..... By the power rule and simplifying

So we get $[(-5x)\cos(x)]' = (-5)\cos(x) + (-5x)[-\sin(x)]$

$= -5\cos(x) + (5x)\sin(x)$





Review

Find the derivative using the sum/difference rule.

1. $y = \frac{1}{2}(x^3 - 2x^2 + 1)$
2. $y = \sqrt{2}x^3 - \frac{1}{\sqrt{2}}x^2 + 2x + \sqrt{2}$
3. $y = a^2 - b^2 + x^2 - a - b + x$ (where a, b are constants)
4. $y = x^{-3} + \frac{1}{x^7}$
5. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$
6. $f(x) = (-3x + 4)^2$
7. $f(x) = -0.93x^{10} + (\pi^3 x)^{\frac{-5}{12}}$
8. What is $\frac{d}{dx}(2x + 1)^2$?
9. Given: $a(x) = (-5x + 3)^2$, what is $\frac{dy}{dx}$?
10. If $v(x) = -3x^3 + 5x^2 - 2x - 3$, what is $v'(0)$?

Find the derivative using the product rule.

11. $y = (x^3 - 3x^2 + x) \cdot (2x^3 + 7x^4)$
12. $y = \left(\frac{1}{x} + \frac{1}{x^2}\right)(3x^4 - 7)$
13. What is the derivative of $[(-3x^2 + x + 4)(-3x - 3)]$?
14. $v(x) = (3x - 3) \cdot \cos(x)$
15. Given: $k(-2) = 0, k'(-2) = 18$, find $r(-2)$ when $(kr)'(-2) = 54$.
16. Given $g(x) = (4x^2 - 4x - 5)(3x - 3)$, find $g'(2)$.
17. Find $\frac{d}{dx}[(-4x + 3) \cdot \sin(x)$.
18. Find $\frac{d}{dx}[(x^2 - 3)(-2x^2 + 4x - 1)]$
19. Given $t(1) = 0, t'(1) = 17$, find $a(1)$ when $(ta)'(1) = 272$.
20. Given $d(x) = (2x^2 + 3x - 1)(2x + 1)$, find $d'(-1)$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.10.

Vocabulary

Term	Definition
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x), \frac{dy}{dx}, y', \frac{df}{dx}$ and $\frac{df(x)}{dx}$.
differentiable	A differentiable function is a function that has a derivative that can be calculated.
product rule	In calculus, the product rule states that the derivative of the product of two functions equals the first function times the derivative of the second function, added to the second function times the derivative of the first function.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.

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8.3.3: Quotient Rule and Higher Derivatives

Quotient Rule and Higher Derivatives

You may recall hearing about Becca and her Track and Field competition in a prior lesson. Her boyfriend had taken a picture of her just as she started to pull away from the others on the track. We learned how she might learn to identify her instantaneous speed at just the split second the picture was taken by using calculus to find a derivative.

What if, instead of just finding her *speed* at that split second, she wanted to find her *acceleration*?

Quotient Rule and Higher Derivatives

The Quotient Rule

Theorem: (The Quotient Rule) If f and g are differentiable functions at x and $g(x) \neq 0$, then

$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$

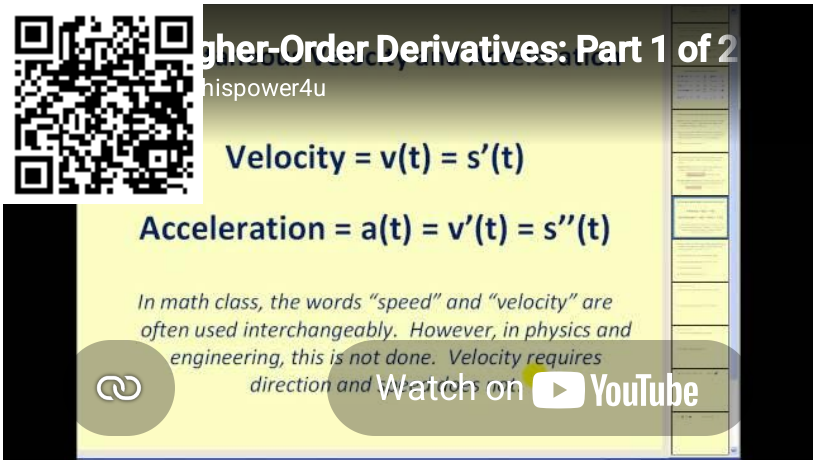
In simpler notation

$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

Keep in mind that the order of operations is important (because of the minus sign in the numerator) and $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$.

Higher Derivatives

If the derivative f′ of the function f is differentiable, then the derivative of f′, denoted by f″ is called the **second derivative** of f. We can continue the process of differentiating derivatives and obtain third, fourth, fifth and higher derivatives of f. They are denoted by f′, f″, f‴, $f^{(4)}$, $f^{(5)}$, . . . ,



Examples

Example 1

Earlier, you were asked how Becca could find her acceleration in addition to her speed.

Solution

Once Becca has calculated her instantaneous speed at a given point on the track by finding the derivative, she could then take the derivative of *that* function to find her **instantaneous acceleration** at the same point in the race.

By finding her instantaneous speed and acceleration at different points in the race, she can learn a lot about what points made a difference in her overall success, and also what points she needs to work on.

Example 2

Find $= \frac{dy}{dx}$ for $y = \frac{x^2-5}{x^3+2}$.

Solution

$\frac{dy}{dx}$	$= \frac{d}{dx}\left[\frac{x^2-5}{x^3+2}\right]$
	$= \frac{(x^3+2)\cdot(x^2-5)' - (x^2-5)\cdot(x^3+2)'}{(x^3+2)^2}$
	$= \frac{(x^3+2)(2x) - (x^2-5)(3x^2)}{(x^3+2)^2}$
	$= \frac{2x^4+4x-3x^4+15x^2}{(x^3+2)^2}$
	$= \frac{-x^4+15x^2+4x}{(x^3+2)^2}$
	$= \frac{x(-x^3+15x+4)}{(x^3+2)^2}$

Example 3

At which point(s) does the graph of $y = \frac{x}{x^2+9}$ have a horizontal tangent line?

Solution

Since the slope of a horizontal line is zero, and since the derivative of a function signifies the slope of the tangent line, then taking the derivative and equating it to zero will enable us to find the points at which the slope of the tangent line is equal to zero, i.e., the locations of the horizontal tangents. Notice that we will need to use the quotient rule here:

y	$= \frac{x}{x^2+9}$	
y′	$= \frac{(x^2+9)\cdot f'(x) - x\cdot g'(x^2+9)}{(x^2+9)^2} = 0$	$= \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = 0$

Multiply both sides by $(x^2+9)^2$,

x^2+9-2x^2	$= 0$
x^2	$= 9$
x	$= \pm 3$

Therefore, at x=−3 and x=3, the tangent line is horizontal.

Example 4

Find the fifth derivative of $f(x) = 2x^4 - 3x^3 + 5x^2 - x - 1$

Solution

To find the fifth derivative, we must first find the first, second, third, and fourth derivatives.

f′(x)	$= 8x^3 - 9x^2 + 5x - x$
f″(x)	$= 24x^2 - 18x + 5$
f‴(x)	$= 48x - 18$
$f^{(4)}(x)$	$= 48$
$f^{(5)}(x)$	$=$

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Example 5

Suppose y'(2) = 0 and (y/q)(2) = 0. Find q(2), assuming y(2) = 0.

Solution

Begin with the quotient rule:

$\left(\frac{y}{q}\right)'(2) = \left(\frac{y'(2)q(2)-y(2)q'(2)}{q(2)^2}\right)$..... Substitute

$(0) = \left(\frac{(0)q(2)-(0)q'(2)}{q(2)^2}\right)$..... Substituting again with given values

$0 = \left(\frac{(0)q(2)}{q(2)^2}\right)$..... Simplify with: $(0)q'(2) = 0$

$0 = \frac{0}{q(2)}$

$q(2) = 0$

Example 6

Find the derivative of $k(x) = \frac{-2x-4}{e^x}$.

Solution

Use the quotient rule: Note: $(-2x-4)' = -2$ and $(e^x)' = e^x$

$\left(\frac{-2x-4}{e^x}\right)' = \frac{(-2)(e^x)-(-2x-4)(e^x)}{e^{2x}}$..... Substitute

$\frac{2x+2}{e^x}$..... Simplify

Example 7

Given $f(x) = (-x^4 - 4x^3 - 5x^2 + 3)$. Find $f''(x)$ when $x = 3$.

Solution

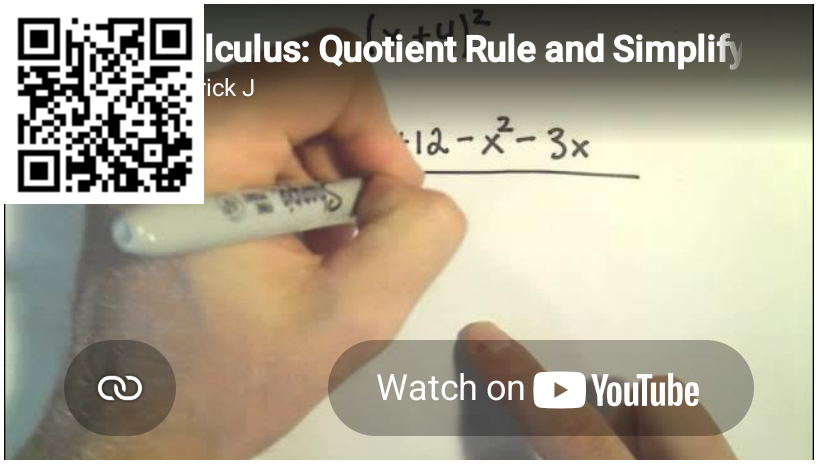
Recall that $f''(x)$ means "The derivative of the derivative of x''

$f'(x) = -4x^3 - 12x^2 - 10x$..... Use the power rule on $f(x)$

$f''(x) = -12x^2 - 24x - 10$..... Use the power rule on $f'(x)$

$f''(3) = -12(3)^2 - 24(3) - 10 \rightarrow -108 - 72 - 10 = -190$..... Substitute 3

$\therefore f''(3) = -190$



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Review

Use the quotient rule to solve:

1. Suppose $u'(0) = 98$ and $\left(\frac{u}{q}\right)'(0) = 7$. Find $q(0)$ assuming $u(0) = 0$.
2. Given: $b(x) = \frac{x^2-5x+4}{-5x+2}$, what is: $b'(2)$?
3. Given: $m(x) = \frac{e^x}{3x+4}$, what is $\frac{dm}{dx}$?
4. What is $\frac{d}{dx} \cdot \frac{\sin(x)}{x-4}$?
5. Find the derivative of $q(x) = \frac{x}{\sin(x)}$.

Solve these higher order derivatives:

6. Given: $v(x) = -4x^3 + 3x^2 + 2x + 3$, what is $v''(x)$?
7. Given: $m(x) = x^2 + 5x$, what is $m''(x)$?
8. Given: $d(x) = 3x^4e^x$, what is $d''(x)$?
9. Given: $t(x) = -2x^5 \sin(x)$, what is $\frac{d^2t}{dx^2}$?
10. What is $\frac{d^2}{dx^2} 3x^5e^x$?

Solve:

11. Find the derivative of $y = \frac{3}{\sqrt{x}+3}$.
12. Find the derivative of $y = \frac{4x+1}{x^2-9}$.
13. Newton's Law of Universal Gravitation states that the gravitational force between two masses (say, the earth and the moon), *m* and *M* is equal two their product divided by the squared of the distance *r* between them. Mathematically, $F = G\frac{mM}{r^2}$ where *G* is the Universal Gravitational Constant (1.602×10^{-11} Nm^2/kg^2). *If the distance* r *between the two masses is changing, find a formula for the instantaneous rate of change of* F *with respect to the separation distance* r.
14. Find $\frac{d}{d\psi}\left[\frac{\psi\psi_0+\psi^3}{3-\psi_0}\right]$, where ψ_0 is a constant.
15. Find $\left.\frac{d^3y}{dx^3}\right|_{x=1}$, where $y = \frac{2}{x^3}$.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.11.

Vocabulary

Term	Definition
differentiable	A differentiable function is a function that has a derivative that can be calculated.
Instantaneous acceleration	The instantaneous acceleration of an object is the change in velocity of the object calculated at a specific point in time.
Instantaneous velocity	The instantaneous velocity of an object is the velocity of the object at a specific point in time.
quotient rule	In calculus, the quotient rule states that if f and g are differentiable functions at x and $g(x) \neq 0$, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$.

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SECTION OVERVIEW

8.4: Integrals

Topic hierarchy

8.4.1: Area Under the Curve

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8.4.1: Area Under the Curve

Area Under a Curve

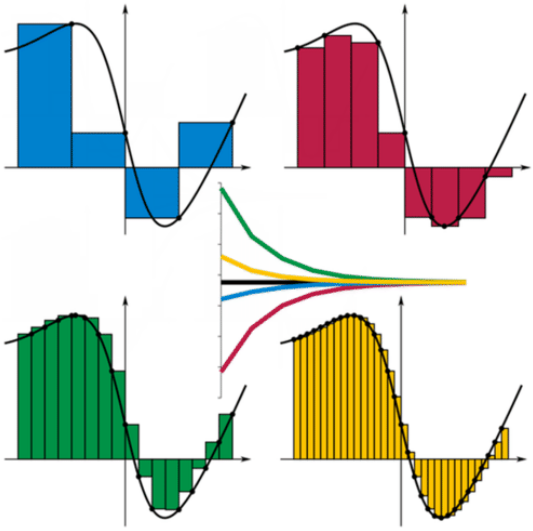
Calculating the area under a straight line can be done with geometry. Calculating the area under a curved line requires calculus. Often the **area under a curve** can be interpreted as the accumulated amount of whatever the function is modeling. Suppose a car's speed in meters per second can be modeled by a quadratic for the first 8 seconds of acceleration:

$s(t) = t^2$

How far has the car traveled in 8 seconds?

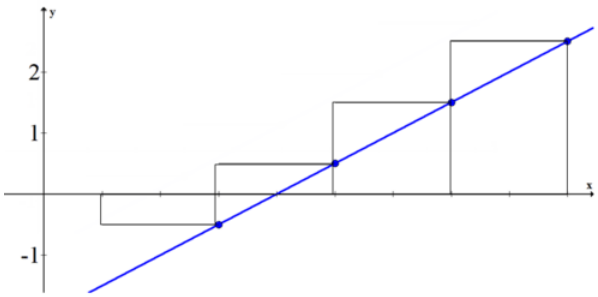
Finding the Area Under a Curve

The area under a curve can be approximated with rectangles equally spaced under a curve as shown below. For consistency, you can choose whether the boxes should hit the curve on the left hand corner, the right hand corner, the maximum value, or the minimum value. The more boxes you use the narrower the boxes will be and thus, the more accurate your approximation of the area will be.



Subintervals are created when an interval is broken into smaller, equally sized intervals. The blue approximation uses right handed boxes for the height of each subinterval. The red approximation assigns the height of the box to be the minimum value of the function in each subinterval. The green approximation assigns the height of the box to be the maximum value of the function in each subinterval. The yellow approximation uses left handed boxes. Rectangles above the x-axis will have positive area and rectangles below the x-axis will have negative area in this context.

Using boxes to estimate the area under a curve is called a **Riemann Sum.** Take the function $f(x) = \frac{1}{2}x - 2$. To calculate the Riemann Sum (area under the curve) between 1 and 9 of the function, first draw the graph and the boxes.



The area of the first box is 2 times the height of the function evaluated at 3:

$2 \cdot \left(\frac{1}{2} \cdot 3 - 2\right) = 3 - 4 = -1$

Because this box is under the x-axis, its area is negative.

The area for each of the rest of the boxes is 2 times the height of the function evaluated at 5, 7 and 9.

$$2 \cdot \left(\frac{1}{2} \cdot 5 - 2\right) = 5 - 4 = 1$$
$$2 \cdot \left(\frac{1}{2} \cdot 7 - 2\right) = 7 - 4 = 3$$
$$2 \cdot \left(\frac{1}{2} \cdot 9 - 2\right) = 9 - 4 = 5$$

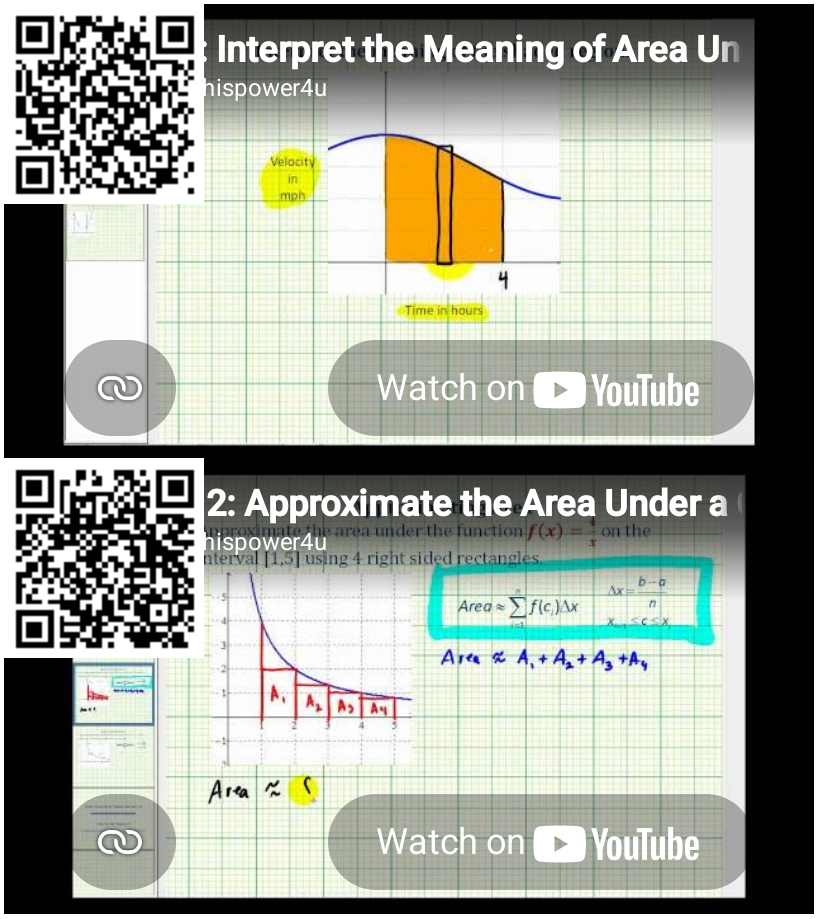
The approximate sum of the total area under the curve is: −1+1+3+5=8 square units.

All four of the area approximations shown earlier get better as the number of boxes increase. In fact, the **limit** of each approximation as the number of subintervals(boxes) increases to infinity is the precise area under the curve.

This is where the calculus idea of an integral comes in. An **integral** is the limit of a sum as the number of summands increases to infinity. A **summand** is one of many pieces being summed together.

$$\int f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} (\text{ Area of box } i)$$

The symbol on the left is the calculus symbol of an integral.



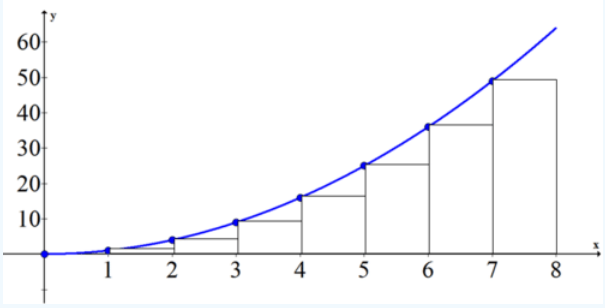
Examples

Example 1

Earlier, you were asked to determine how far a car travels in 8 seconds.

Solution

You can use the area under the curve to find the total distance traveled in the first 8 seconds. Since the quadratic is a curve you must choose the number of subintervals you want to use and whether you want right or left handed boxes for estimating. Suppose you choose 8 left handed boxes of width one.



x	0	1	2	3	4	5	6	7
Area of box to the right	1·0	1·1	1·4	1·9	1·16	1·25	1·36	1·49

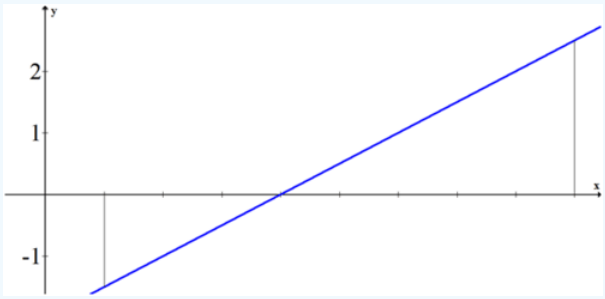
The approximate sum is 1+4+9+16+25+36+49=140. This means that the car traveled approximately 140 meters in the first 8 seconds.

Example 2

Evaluate the exact area under the curve used earlier, $f(x) = \frac{1}{2}x - 2$, using the area formula for a triangle.

Solution

Remember that the area below the x axis is negative while the area above the x axis is positive.



Negative Area: $\frac{1}{2} \cdot 3 \cdot 1.5 = \frac{9}{4}$

Positive Area: $\frac{1}{2} \cdot 5 \cdot 2.5 = \frac{25}{4}$

Area under the curve between 1 and 8: $\frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4$

If you compare this answer to the approximation from earlier, it appears that approximations that are 2 units wide produce an area with significant error.

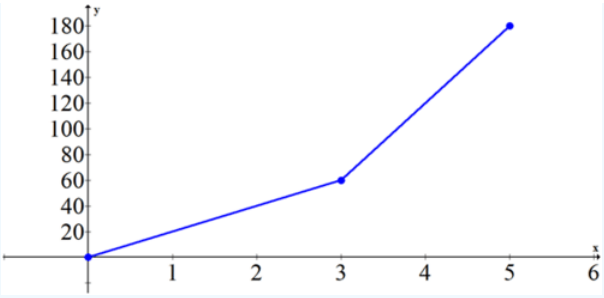
Example 3

Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph. Use an area under the curve argument to connect the two graphs.

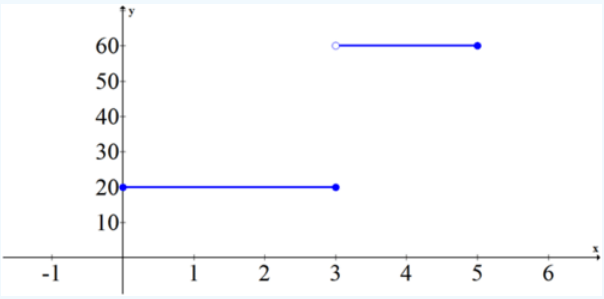
Solution

Distance vs. Time:

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Rate vs. Time:



The slope of the first graph is 20 from 0 to 3 and then 60 from 3 to 5. The second graph is a graph of the slopes from the first graph. If you calculate the area of the second graph at the key points 0, 1, 2, 3, 4 and 5 you will see that they align perfectly with the points on the first graph.

x	***Area under curve from 0 to*** x
0	0
1	20
2	40
3	60
4	120
5	180

Example 4

Approximate the area under the curve using eight subintervals and right endpoints.

$f(x) = 3x^2 - 1, -1 \leq x \leq 7$

Solution

While a graph is helpful to visualize the problem and drawing each box can help give meaning to each summand, it is not always necessary. Since there are going to be 8 subintervals over the total interval of $-1 \leq x \leq 7$, each interval is going to have a width of 1. The height of each interval is going to be at the right hand endpoints of each subinterval (0, 1, 2, 3, 4, 5, 6, 7).

$$\sum \textbf{ height } \cdot \textbf{ width } = \sum_{i=0}^{7} \left(3i^2 - 1\right) \cdot 1 = 412$$

Example 5

Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = x^x, 1 \leq x \leq 3$

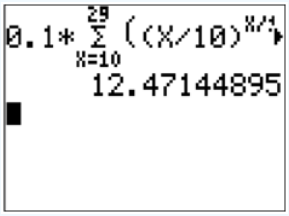
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Solution

When the number of subintervals gets large and the subintervals get extremely narrow it will be impossible to draw an accurate picture. This is why using summation notation and thinking through what the indices and the argument will be is incredibly important. With 20 subintervals between [1,3], each interval will be 0.1 wide. Left endpoints means that the first box has a height of $f(1)$ and the second box has a height of $f(1.1)$.

$$\begin{aligned}\sum \text{height} \cdot \text{width} &= f(1)\cdot 0.1 + f(1.1)\cdot 0.1 + f(1.2)\cdot 0.1 + \cdots + f(2.9)\cdot 0.1 \\ &= 0.1(f(1) + f(1.1) + \cdots f(2.9)) \\ &= 0.1 \cdot \sum_{i=10}^{29} f\left(\frac{i}{10}\right) \\ &= 0.1 \cdot \sum_{i=10}^{29} \left(\frac{i}{10}\right)^{\left(\frac{i}{10}\right)} \\ &\approx 12.47144\end{aligned}$$

Your calculator can compute summations when you go under the math menu.



Review

1. Approximate the area under the curve using eight subintervals and right endpoints.

   $f(x) = x^2 - x + 1, 0 \le x \le 8$

2. Approximate the area under the curve using eight subintervals and left endpoints.

   $f(x) = x^2 - 2x + 1, -4 \le x \le 4$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

   $f(x) = \sqrt{x+3}, 0 \le x \le 4$

4. Approximate the area under the curve using 100 subintervals and left endpoints. Compare to your answer from #3.

   $f(x) = \sqrt{x+3}, 0 \le x \le 4$

5. Approximate the area under the curve using eight subintervals and left endpoints.

   $f(x) = \cos(x), 0 \le x \le 4$

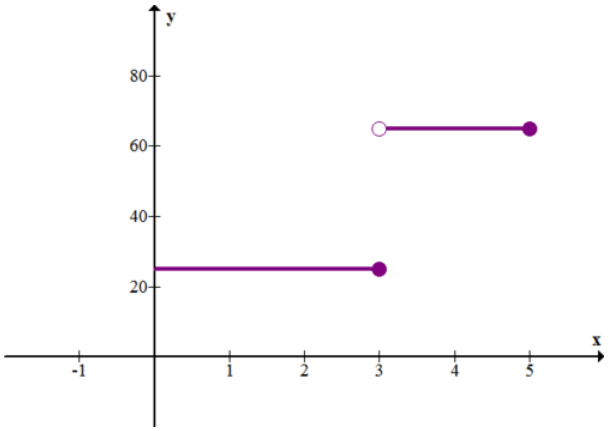
6. Approximate the area under the curve using twenty subintervals and left endpoints.

   $f(x) = \cos(x), 0 \le x \le 4$

7. Approximate the area under the curve using 100 subintervals and left endpoints.

   $f(x) = \cos(x), 0 \le x \le 4$

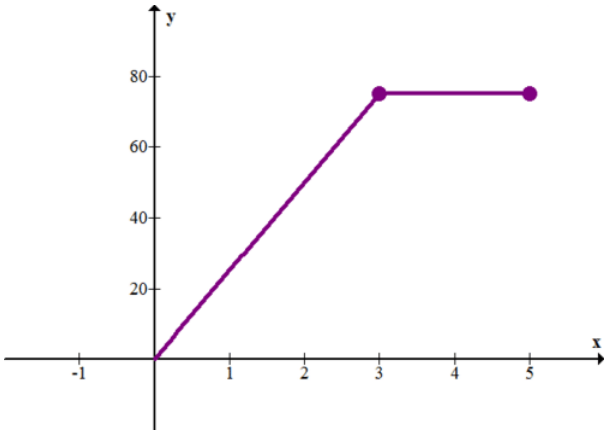
The following graph shows the rate (in miles per hour) vs. time (in hours) for a car.



8. Describe what is happening with the car.

9. How far did the car travel in 5 hours?

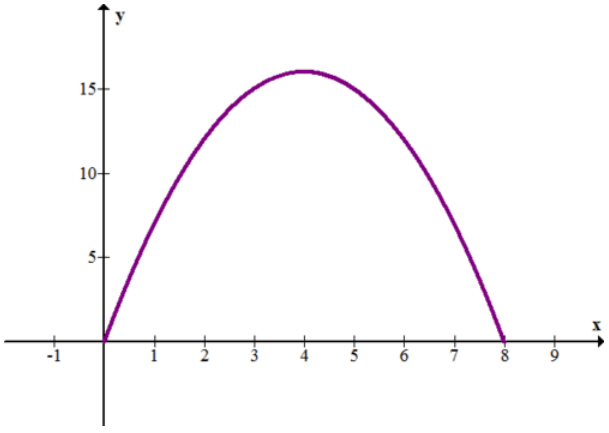
The following graph shows the rate (in feet per second) vs. time (in seconds) for a car.



10. Describe what is happening with the car. In particular, what is happening in the first 3 seconds?

11. How far did the car travel in 5 seconds?

The following graph shows the function $f(x) = -(x-4)^2 + 16$, which represents the rate (in feet per second) vs. time (in seconds) for a runner.



12. Describe what is happening with the runner. In particular, what happens after 4 seconds?

13. Use rectangles to approximate the total distance (in feet) that the runner traveled in the 8 seconds. Try to get as good an approximation as possible.

14. Explain how an integral is like the opposite of a derivative.

15. How do integrals relate to sums?

Review (Answers)

To see the Review answers, open this PDF file and look for section 14.9.

Vocabulary

Term	Definition
Δ	The symbol "Δ", read "delta", is used to denote "change in", as in "the change in velocity over time" $= \frac{\Delta v}{t}$.
definite integral	A definite integral gives the area between the x-axis and a curve over a defined interval.
limit	A limit is the value that the output of a function approaches as the input of the function approaches a given value.
subintervals	Subintervals are created when an interval is broken into smaller equally sized intervals.
summand	A summand is an expression being summed. It directly follows the sigma symbol.

Image Attributions

1. [Figure 1]
   Credit: CK-12 Foundation;KSmrq
   Source: https://commons.wikimedia.org/wiki/File:Riemann_sum_convergence.png
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SECTION OVERVIEW

8.5: Anti-Derivative

Topic hierarchy

8.5.1: Fundamental Theorem of Calculus

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8.5.1: Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Velocity due to gravity can be easily calculated by the formula: $v = gt$, where g is the acceleration due to gravity ($9.8 m/s^2$) and t is time in seconds. In fact, a decent approximation can be calculated in your head easily by rounding 9.8 to 10 so you can just add a decimal place to the time.

Using this function for velocity, how could you find a function that represented the position of the object after a given time? What about a function that represented the instantaneous acceleration of the object at a given time?

Fundamental Theorem of Calculus

Antiderivatives

If you think that evaluating areas under curves is a tedious process you are probably right. Fortunately, there is an easier method. In this section, we shall give a general method of evaluating definite integrals (area under the curve) by using antiderivatives.

Definition: The Antiderivative
If $F'(x) = f(x)$, then $F'(x)$ is said to be the antiderivative of $f(x)$.

There are rules for finding the antiderivatives of simple power functions such as $f(x) = x^2$. As you read through them, try to think about why they make sense, keeping in mind that differentiation reverses integration.

There are rules for finding the antiderivatives of simple power functions such as $f(x) = x^2$. As you read through them, try to think about why they make sense, keeping in mind that differentiation reverses integration.

Rules of Finding the Antiderivatives of Power Functions

- **The Power Rule**
  where C is constant of integration and n is a rational number not equal to -1.
- **A Constant Multiple of a Function Rule**
  where k is a constant.
- **Sum and Difference Rule**
- **The Constant Rule**

where k is a constant. (Notice that this rule comes as a result of the power rule above.)

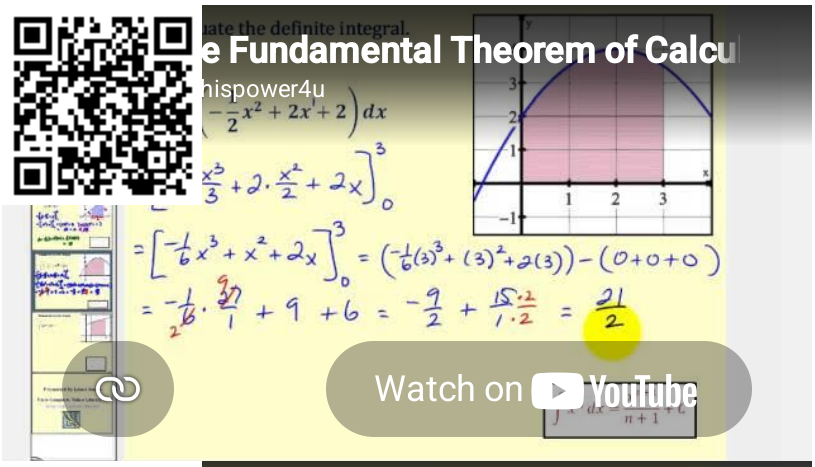
The Fundamental Theorem of Calculus

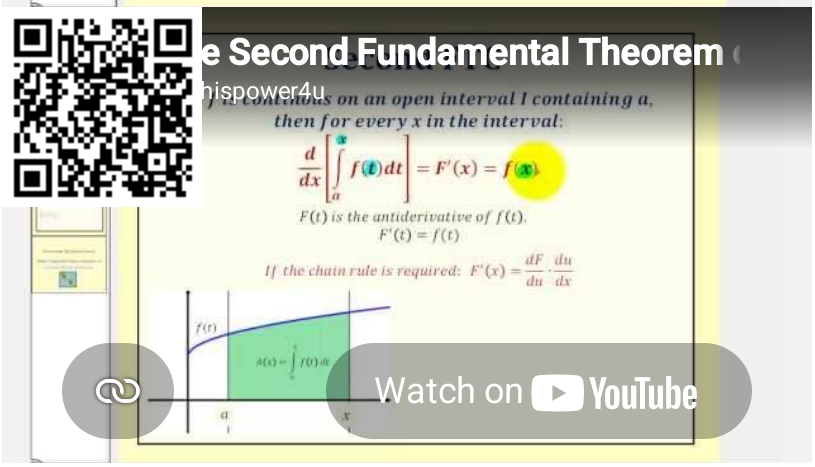
The Fundamental Theorem of Calculus makes the relationship between derivatives and integrals clear. Integration performed on a function can be reversed by differentiation.

The Fundamental Theorem of Calculus
If a function $f(x)$ is defined over the interval $[a, b]$ and if $F(x)$ is the antidervative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b$$
$$= F(b) - F(a)$$

We can use the relationship between differentiation and integration outlined in the Fundamental Theorem of Calculus to compute definite integrals more quickly.





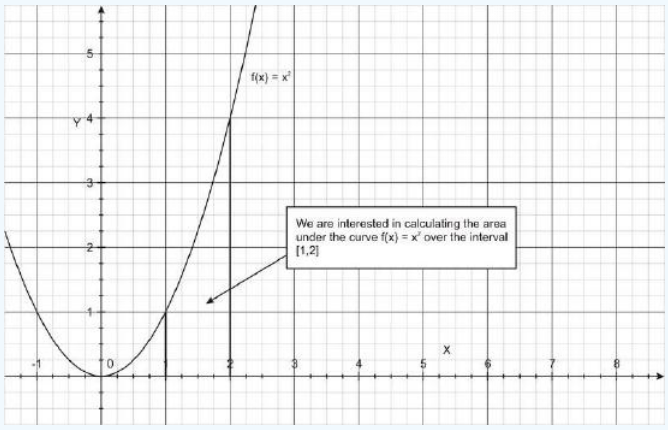
Examples

Example 1

Evaluate $\int_1^2 x^2 dx$.

Solution

This **integral** tells us to evaluate the area under the curve $f(x) = x^2$, which is a parabola over the interval [1, 2], as shown in the figure below.



To compute the integral according to the Fundamental Theorem of Calculus, we need to find the antiderivative of $f(x) = x^2$. It turns out to be $F(x) = (1/3)x^3 + C$, where C is a constant of integration.

How can we get this? Think about the functions that will have derivatives of x^2. Take the **derivative** of $F(x)$ to check that we have found such a function. (For more specific rules, see the box after this example). Substituting into the Fundamental Theorem,

$\int_a^b f(x)dx$	$= F(x)\|_a^b$
$\int_1^2 x^2 dx$	$= \left[\frac{1}{3}x^3 + C\right]_1^2$
	$= \left[\frac{1}{3}(2)^3 + C\right] - \left[\frac{1}{3}(1)^3 + C\right]$
	$= \left[\frac{8}{3} + C\right] - \left[\frac{1}{3} + C\right]$
	$= \frac{7}{3} + C - C$
	$= \frac{7}{3}$

So the area under the curve is (7/3) units^2.

Example 2

Evaluate $\int x^3 dx$

Solution

Since $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$, we have

$\int x^3 dx$	$= \frac{1}{3+1}x^{3+1} + C$
	$= \frac{1}{4}x^4 + C$

To check our answer we can take the derivative of $\frac{1}{4}x^4 + C$ and verify that it is x^3, the original function in our integral.

Example 3

Evaluate $\int 5x^2 dx$

Solution

Using the constant multiple of a power rule, the coefficient 5 can be removed outside the integral:

$$\int 5x^2 dx = 5\int x^2 dx$$

Then we can integrate:

$= 5 \cdot \frac{1}{2+1}x^{2+1} + C$
$= \frac{5}{3}x^3 + C$

Again, if we wanted to check our work we could take the derivative of $\frac{5}{3}x^3 + C$ and verify that we get $5x^2$.

Example 4

Evaluate $\int (3x^3 - 4x^2 + 2)\, dx$.

Solution

Using the sum and difference rule we can separate our integral into three integrals:

$$\int (3x^3 - 4x^2 + 2)\, dx =$$
$$3\left(\int x^3 dx\right) - 4\left(\int x^2 dx\right) + \left(\int 2dx\right)$$
$$\rightarrow 3 \cdot \frac{1}{4}x^4 - 4 \cdot \frac{1}{3}x^3 + 2x + C \rightarrow \frac{3}{4}x^4 - \frac{4}{3}x^3 + 2x + C$$

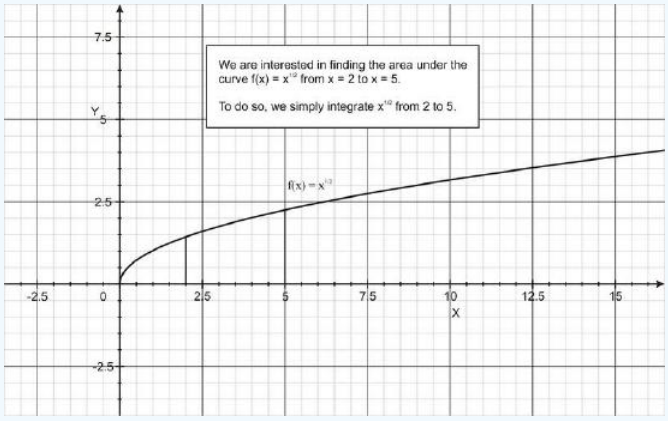
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Example 5

Evaluate $\int_2^5 \sqrt{x}dx$.

Solution

The evaluation of this integral represents calculating the area under the curve $y = \sqrt{x}$ from $x = -2$ to $x = 3$, shown in the figure below.



$\int_2^5 \sqrt{x}dx$	$= \int_2^5 x^{1/2}dx$
	$= \left[\frac{1}{\frac{1}{2}+1}x^{1/2+1}\right]_2^5$
	$= \left[\frac{1}{3/2}x^{3/2}\right]_2^5$
	$= \frac{2}{3}\left[x^{3/2}\right]_2^5$
	$= \frac{2}{3}\left[5^{3/2} - 2^{3/2}\right]$
	$= 5.57$

So the area under the curve is 5.57.

Example 6

Use the Fundamental Theorem of Calculus to solve: $\int_4^6 \frac{dx}{x}$.

Solution

Given what we know, that if $F(x) = \ln x$, then $F'(x) = \frac{1}{x}$

Thus, we apply the Fundamental Theorem of Calculus:

$\int -4^6 \frac{dx}{x} = \ln x \mid -46$

= F(6) - F(4) = [ln(6)] - [ln(4)] = 0.4055

Example 7

Use the Fundamental Theorem of Calculus to solve: $\int_{-2p}^{2p} 3\cos(x)dx$.

Solution

Given what we know, that if F(x) = 3sin(x), then F'(x) = 3cos(x)

So we apply the Fundamental Theorem of Calculus:

$\int_{-2p}^{2p} 3\cos dx = 3\sin(x)|_{-2p}^{2p}$

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= F(8) - F(0) = [3sin(2p)] - [3sin(-2p)] = 1 - 0 = 0

Review

Evaluate the integral:

1. Evaluate the integral $\int_0^3 5x dx$
2. Evaluate the integral $\int_0^1 x^4 dx$
3. Evaluate the integral $\int_1^4 (x-3) dx$

Find the integral:

4. Find the integral of $(x + 1)(2x - 3)$ from -1 to 2.
5. Find the integral of $\sqrt{x}$ from 0 to 9.
6. Find $\int_{-1}^0 -3 dx$
7. Find $\int_{-1}^3 dx$
8. Find $\int_{-p}^{\frac{p}{2}} -4\cos(x) dx$
9. Find $\int_0^2 -dx$
10. Find $\int_2^7 \frac{dx}{x}$
11. Find $\int_{-2}^0 x + 5 dx$
12. Find $\int_{-p}^{\frac{3p}{2}} 6\sin(x) dx$
13. Find $\int_6^7 \frac{dx}{x}$

Challenge yourself:

14. Sketch $y = x^3$ and $y = x$ on the same coordinate system and then find the area of the region enclosed between them (a) in the first quadrant and (b) in the first and third quadrants.
15. Evaluate the integral $\int_{-R}^{R} (\pi R^2 - \pi x^2)\, dx$ where R is a constant.

Review (Answers)

To see the Review answers, open this PDF file and look for section 8.13.

Vocabulary

Term	Definition
antiderivative	An antiderivative is a function that reverses a derivative. Function A is the antiderivative of function B if function B is the derivative of function A.
derivative	The derivative of a function is the slope of the line tangent to the function at a given point on the graph. Notations for derivative include $f'(x)$, $\frac{dy}{dx}$, y', $\frac{df}{dx}$ and $\frac{df(x)}{dx}$.
fundamental theorem of calculus	The fundamental theorem of calculus demonstrates that integration performed on a function can be reversed by differentiation.
integral	An integral is used to calculate the area under a curve or the area between two curves.
theorem	A theorem is a statement that can be proven true using postulates, definitions, and other theorems that have already been proven.

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