UCD GEL 56- INTRODUCTION TO GEOPHYSICS



Magali Billen
University of California, Davis

LibreTexts™

UCD GEL 56- Introduction to Geophysics

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 05/21/2026

TABLE OF CONTENTS

Licensing

Geophysics is everywhere in geology...

- Supporting Topics
- Units, Conversions, and Physical Constants
- 1: Rheology of Rocks
  - 1.0: Introduction to Rheology
  - 1.1: Stress
  - 1.2: Strain and Strain Rate
  - 1.3: Elastic Deformation
  - 1.4: Failure of Rocks
  - 1.5: Viscous Deformation
  - 1.6 Inclined Plane Jupyter Notebook
  - 1.7: Summary
- 2: Diffusion and Darcy's Law
  - 2.0: Introduction to Diffusion and Darcy's Law
  - 2.1: The Diffusion Equation
  - 2.2: Geological Applications of the Diffusion Equation
  - 2.3: Lithospheric Thickness Jupyter Notebook
  - 2.4: Erosion Profile Jupyter Notebook
  - 2.5: Darcy's Law - Flow in a Porous Medium
  - 2.6: Summary
- 3: Planetary Geophysics
  - 3.0: Introduction
  - 3.1: Orbital Mechanics
  - 3.2: Layered Structure of a Planet
  - 3.3: Two Layer Planet Structure Jupyter Notebook
  - 3.4: Isostasy
  - 3.5: Isostasy Jupyter Notebook
  - 3.5: Observing the Gravity Field
  - 3.7: Gravitational Potential, Mass Anomalies and the Geoid
  - 3.8: Summary
- 4: Plate Tectonics
  - 4.0: Introduction to Plate Tectonics
  - 4.1: The Forces Driving Plate Motions
  - 4.2: Magnetic Anomalies on the Seafloor
  - 4.3 Magnetic Anomaly Jupyter Notebook
  - 4.4: Plates Moving on a Plane
  - 4.5: Relative Velocity and Reference Frames
  - 4.6: Relative Plate Velocity Jupyter Notebook
  - 4.7: Plate Motions on a Sphere
  - 4.8: Summary
- 5: Seismology
  - 5.0: Introduction to Seismology

- 5.1: Basics of Wave Propagation
- 5.2: Seismic Refraction (Single Layer)
- 5.3 Seismic Refraction Jupyter Notebook
- 5.4: Seismic Reflection (Single Layer)
- 5.5 Seismic Reflection Jupyter Notebook
- 5.6: Summary

- 6: Earthquakes
  - 6.0: Introduction to Earthquakes
  - 6.1: Earthquake Geometry and Process
  - 6.2: Earthquake Magnitude
  - 6.3: Location and Focal Mechanisms
  - 6.4: Earthquake Distance and Magnitude Jupyter Notebook
  - 6.5: Earthquake Scaling Laws
  - 6.6: Summary

- Index
- Arctan vs Arctan2
- Ipywidgets Interact Example

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

Front Matter

TitlePage

InfoPage

Table of Contents

Supporting Topics

Units, Conversions, and Physical Constants

University of California, Davis

GEL 56: Introduction to Geophysics

Dr. Magali Billen



Rainbow Basin, California (photo courtesy of M. Billen)

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 05/21/2026

TABLE OF CONTENTS

Supporting Topics

Units, Conversions, and Physical Constants

1: Rheology of Rocks

- 1.0: Introduction to Rheology
- 1.1: Stress
- 1.2: Strain and Strain Rate
- 1.3: Elastic Deformation
- 1.4: Failure of Rocks
- 1.5: Viscous Deformation
- 1.6 Inclined Plane Jupyter Notebook
- 1.7: Summary

2: Diffusion and Darcy's Law

- 2.0: Introduction to Diffusion and Darcy's Law
- 2.1: The Diffusion Equation
- 2.2: Geological Applications of the Diffusion Equation
- 2.3: Lithospheric Thickness Jupyter Notebook
- 2.4: Erosion Profile Jupyter Notebook
- 2.5: Darcy's Law - Flow in a Porous Medium
- 2.6: Summary

3: Planetary Geophysics

- 3.0: Introduction
- 3.1: Orbital Mechanics
- 3.2: Layered Structure of a Planet
- 3.3: Two Layer Planet Structure Jupyter Notebook
- 3.4: Isostasy
- 3.5: Isostasy Jupyter Notebook
- 3.5: Observing the Gravity Field
- 3.7: Gravitational Potential, Mass Anomalies and the Geoid
- 3.8: Summary

4: Plate Tectonics

- 4.0: Introduction to Plate Tectonics
- 4.1: The Forces Driving Plate Motions
- 4.2: Magnetic Anomalies on the Seafloor
- 4.3 Magnetic Anomaly Jupyter Notebook
- 4.4: Plates Moving on a Plane
- 4.5: Relative Velocity and Reference Frames
- 4.6: Relative Plate Velocity Jupyter Notebook
- 4.7: Plate Motions on a Sphere
- 4.8: Summary

5: Seismology

- 5.0: Introduction to Seismology
- 5.1: Basics of Wave Propagation
- 5.2: Seismic Refraction (Single Layer)
- 5.3 Seismic Refraction Jupyter Notebook
- 5.4: Seismic Reflection (Single Layer)
- 5.5 Seismic Reflection Jupyter Notebook
- 5.6: Summary

6: Earthquakes

- 6.0: Introduction to Earthquakes
- 6.1: Earthquake Geometry and Process
- 6.2: Earthquake Magnitude
- 6.3: Location and Focal Mechanisms
- 6.4: Earthquake Distance and Magnitude Jupyter Notebook
- 6.5: Earthquake Scaling Laws
- 6.6: Summary

Index

Arctan vs Arctan2

Ipywidgets Interact Example

Supporting Topics

The section numbers below refer to sections in other LibreTexts Books. The links will open in a new window.

Jump to Physics

Math / Calculus

Arctan vs Arctan2

- Arctan vs Arctan2

Converting Radians to Degrees and vice versa

- Units, Conversions, and Physical Constants

Vector Components

- 8.8 Vectors (Overview of Vectors, see the portion on Vector Components)

Physics

Forces

- 2.1 Properties of Force (General Introduction, Newton's Laws, and Free Body Diagrams)
- 2.2 Effects of Force (Newton's Second Law, F = ma)
- 2.3 Types of Force (For more information on gravity, elastic force (springs), normal force, and friction)

Greek Alphabet

- Greek Alphabet (For Greek letter names and symbols)

Unit Abbreviations

- Units, Conversions, and Physical Constants

Jump to Math/Calculus

Jump to the Start of Physics

Units, Conversions, and Physical Constants

Units

Unit Abbreviation	Unit Name	Quantity	Base Units
m	Meter	Length	m
kg	Kilogram	Mass	kg
Pa	Pascal	Pressure	kg / (ms^2)
s	Second	Time	s
N	Newton	Force	kgm / s^2
K	Kelvin	Temperature	K
Hz	Hertz	Frequency	1 / s
rad	Radian	Angle	1
J	Joule	Energy	kgm^2 / s^2
W	Watt	Power	kgm^2 / s^3
AU or au	Astronomical Unit	Length	See Conversions

Common Conversions

1m = 39.37 in = 3.281 ft = 1.094 yd

1 rad = 57.30°

1° = 0.01745 rad = π / 180

1 Pa = 9.869 x 10^{-6} atm

1 kg/m^3 = 1000 g/cm^3

1 AU = 1.495 x 10^8 km = 1.495 x 10^{11} km

1 year (yr) = 3.15 x 10^7 s

1 million years (my or myr) = 3.15 x 10^{13} s

1 day = 86,400 s

1 hour = 3600 s

1 K = -273.15 C = -457.87 F

Physical Constants

Name	Symbol	Value
Acceleration due to Gravity	g	9.81 m/s^2
Universal Gravitational Constant	G	6.674 x 10^{-11} Nm^2/(kg^2)
Speed of Light	c	2.98 x 10^8 m/s
Mass of the Sun	M_{sun}	1.98 x 10^{30} kg
Radius of the Sun	R_{sun}	695700 km

*For Planet and Moon Information see the Other Section

Other

Guide to Greek Letters: Greek Alphabet

Power of 10 Prefixes (ex. kilo-): Metric Prefixes

Planetary Physical and Orbital Constants: Planets and Pluto Physical Characteristics

Selected Moon Planetary and Orbital Constants: Moon Physical Data Note: To get M by itself, just divide the GM number by the Gravitational Constant G

Common Formulas, Trigonometric Identities, Derivatives, and Integrals: Mathematical Formulas

CHAPTER OVERVIEW

1: Rheology of Rocks

1.0: Introduction to Rheology
1.1: Stress
1.2: Strain and Strain Rate
1.3: Elastic Deformation
1.4: Failure of Rocks
1.5: Viscous Deformation
1.6 Inclined Plane Jupyter Notebook
1.7: Summary

This page titled 1: Rheology of Rocks is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.0: Introduction to Rheology

Supporting Topics

Prior to reading the page, please review the following topics if you are unfamiliar with them.

Resources for each topic can be found here: Supporting Topics

Physics

- Force
- Unit Abbreviations
- Greek Alphabet

The movement of tectonic plates is often thought of on a very large scale, but what actually happens to rocks at the plate boundaries as the plates slowly slide past each other? How a rock responds when a force is applied is determined by both the physical properties of the rock and the conditions at which the rock is deforming (e.g., temperature and pressure). The physical properties, such as elasticity and fracture strength also depend on the rock type. **Rheology** describes how rock will deform when subjected to a force and takes into account both the physical properties of the rock and the conditions at which it is deforming. Examples of a rheology include elastic (for materials that deform like a spring) and viscous (for materials that deform like a fluid). Forces on rocks are the results of the weight of overlying rock (also known as the lithostatic stress), **tectonic forces** originating from convection in the Earth's mantle, which moves the tectonic plates, and gravitational forces related to density differences within the crust and lithosphere.

The term rheology was coined by Eugene Bingham in 1929, inspired by the Greek phrase, panta rhei, meaning everything flows. In some disciplines it is used only to describe the flowing behavior of material. In the geosciences it is used more generally to describe all types of deformation experienced by earth materials.

An understanding of rheology is crucial for understanding concepts in geology, geophysics and planetary science as well as petrology, such as what happens during an earthquake, how mountains and faults are formed, how rocks deform in the mantle, the origin of tectonic plates, how magma ascends through the crust and how lava flows.



Figure 1.0.1: Deformed layers on the faulted boundary of the Rainbow Basin Syncline in Barstow, CA. Note people for scale. (courtesy M. I. Billen)

In physics, we learn to consider how forces are applied to particles or blocks and we are interested in the resulting displacement or velocity of the particle or block. In geophysics we turn our attention to how a block (or a rock) deforms internally when forces are applied at its boundaries. In this context, in the place of force we consider the **stress,** which is defined as the force applied over an area. Likewise, in the place of displacement we consider **strain,** which is defined as the change in shape or size of a rock relative to

the original shape or size. Finally, in the place of velocity, we consider the **strain-rate,** which is defined as the change in strain with respect to time.

By the end of this chapter, you should be comfortable with the various ways in which we describe the stresses that act on a rock, some ways a rock can respond to those stresses, and how these stresses are related to geologic structures such as faults or shear zones. First, we will examine stress and strain in a one dimensional (1-D: along a line) context. Second, we will look at various types of rheology (elastic, viscous, and plastic), the timescales over which deformation occurs and the effect this has on some geologic structures. Finally, we will cover how rocks fail (brittle or plastic failure).

This page titled 1.0: Introduction to Rheology is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.1: Stress

Supporting Topics

Prior to reading the page, please review the following topics if you are unfamiliar with them.

Resources for each topic can be found here: Supporting Topics

Calculus

- Vector Components

In introductory physics courses we learn about how objects move when forces act on them. However, we usually consider that the object itself is perfectly rigid and we are not concerned with the internal deformation of the object. However, when we want to consider how an object deforms internally when a force is applied we have to consider the rheology of the material, the orientation of the applied forces, and where these forces are applied. In this section, you will learn how to describe the applied forces acting on an object in terms of the components of stress that exist at a point within the object.

Stress

When a force is applied to a rock it does not act at a single point, but rather over some area. Think about squeezing an eraser between your fingers: the force from your fingers acts over the area of contact between your fingers and the eraser. Stress is the term used to describe the magnitude of the force acting on a surface with some area (Figure 1.1.1). In other words, stress is force divided by area,

$$\sigma = \frac{F}{A} \tag{1.1.1}$$

Since force has units of Newtons (N = $kg\frac{m}{s^2}$) and area has units of m^2, stress has units of $\frac{N}{m^2}$ or in SI (Système International) units $\frac{kg}{ms^2}$. In geosciences, we typically use the unit Pascal (symbol: Pa) to quantify stress and pressure: 1 Pa = 1 $\frac{N}{m^2}$.

Components of Stress

A force can be applied to an area at different angles resulting in two different components of stress: normal stress and shear stress. In the example of squeezing an eraser with your fingers, the force is applied perpendicular to, or normal to, the surface of the eraser. Therefore, this stress is termed a normal stress. **Normal stress** acts at a right angle (perpendicular) to a surface and can either push a rock together (compression) or pull it apart (tension). In geology, compressional stress is positive and tensional stress is negative. In engineering, the opposite sign convention is used. In geological settings compressional stress is typically found in convergent margins where tectonic plates collide. Compressional stress also results from the weight of the rock itself pushing down on the rock below it. Tensional stress is typically found in divergent margins where tectonic plates are being pulled apart.

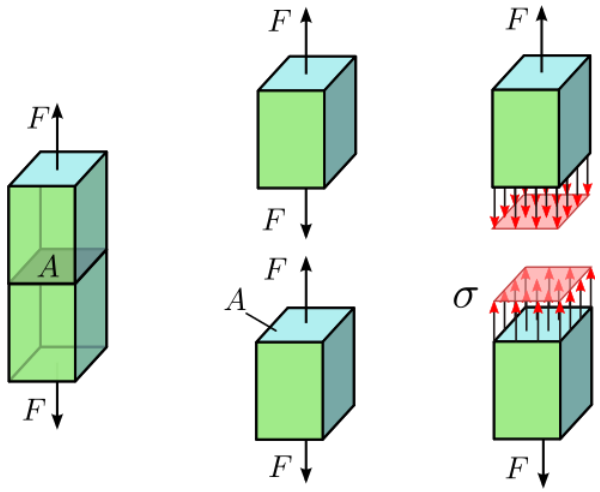


Figure 1.1.1: A force applied perpendicular to a plane of area A results in a normal stress across the plane that is also perpendicular to the plane. (CC BY-SA 3.0; Stolfi, via Wikimedia)

Advanced concept: in the above picture there is a pair of forces with one force acting on each side of the block. Each of these single forces acting on one side of the block or one side of a surface is termed a *traction*. A pair of tractions acting on opposites sides of a plane is a stress. For certain types of analysis (e.g., a force balance on a portion of a tectonic plate) one needs to consider individual tractions. However, for applications in this text we will only be concerned with stresses.

The second type of stress is a **shear stress**, which is a stress that acts tangential (parallel) to a surface. If you slide your hand across the top of your desk, the resistance you feel is a shear stress between your hand and the desk. Similarly, if you slide two blocks past each other, the frictional resistance to sliding generates a shear stress (Figure 1.1.2). Shear stresses are typically found on all faults, resulting from the component of the regional stress that acts parallel to the fault surface.

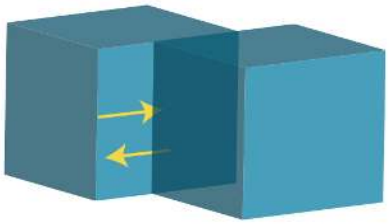


Figure 1.1.2: Forces applied tangential to, or parallel to, a surface result in a shear stress that is also parallel to the surface.

Pressure and Lithostatic Stress

You may have noticed that stress has the same units as pressure, force per unit area. When we normally talk about pressure, we are talking about pressure in a gas (air pressure) or a liquid (the pressure at the bottom of the ocean). **Pressure** is a special state of stress that occurs when the normal stresses acting at a point are the same in all directions. In this special case there are no shear stresses, only normal stresses, and no matter what direction you measure the stress the value is always the same. Over long periods of time the deep interior of the planet behaves as a fluid, so just like in the ocean we can think about the pressure at a point inside the earth.

In geological applications we are also concerned with pressure in the crust or lithosphere, although it is not expected to find stress-states in the crust or lithosphere (or in the mantle) where the state of stress is simply that of constant pressure. Instead we expect the stress-state to be different in different directions even if there are no applied tectonic forces (see next section about elasticity). Instead, we use pressure as a reference state to which tectonic forces are added. This reference stress-state is referred to as a lithostatic stress or lithostatic pressure. The word lithostatic comes from the word Greek word *litho* for rock and the term *static*, meaning not moving.

Lithostatic stress is the stress caused by the weight of the rock above a point. Strictly speaking it is the vertically-oriented normal stress. This is also sometimes referred to as the **overburden**. The normal stresses in the horizontal directions will normally not be equal to the lithostatic stress. However, in geologic applications we often start analysis from a fictional stress-state of lithostatic

pressure. In this case, because we use the word *pressure*, we are saying that the normal stress in all directions is equal to the lithostatic stress.

When one uses the term **lithostatic pressure** (or just *pressure*) there is an implicit assumption that the value of the stress is the same in all directions. Therefore, when considering the state of stress in the crust or lithosphere we often start from a state of lithostatic pressure and then ask what is the magnitude and direction of the stresses acting *in addition* to the lithostatic pressure.

Calculating Lithostatic Stress

Lithostatic stress is the stress due to the weight (force due to gravity) of the overlying rock. Consider a tall rectangular of rock above a point in the crust. This rectangle has a height, h and the area of its base is given by $A = L \times w$. The force due to the weight of the overlying rock is $F = ma = mg$, where m is the mass and g is gravity (9.81 m/s^2). The mass is given by the density, ρ times the volume ($m = \rho Lwh$). The lithostatic stress at the bottom of the rectangle is

$$\sigma_{lith} = \frac{F}{A} = \frac{mg}{A} = \frac{\rho Lwhg}{Lw} = \rho gh \tag{1.1.2}$$

Notice that the dimensions of the area of the rectangle do not appear in this equation, even though the units of stress are N\m^2. This means that so long as the height and density is constant within the rectangle, the lithostatic stress is the same whether it is calculated for a rectangle with an area the size of a football field or a single point, or any other area. For this reason, we often use a unit rectangle with dimensions of 1×1 m^2, when doing calculations of stress.

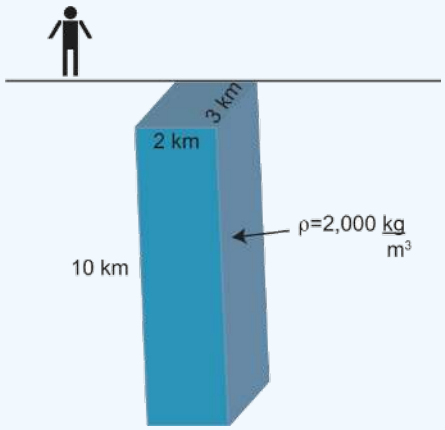


Figure 1.1.6: Schematic for calculating lithostatic stress.

In Figure 1.1.6, the density of the rock is 2000 kg/m^3 and the height of the rock is 10 km (10,000 m). The sides of the base of the rectangle are given, but we do not actually need these numbers. Plugging in these numbers to the equation above (note I'm using 10,000 meters so I'm using SI units for all the numbers),

$$\sigma_{lith} = (2000) * (10,000) * (9.81) = 196,200,000 \text{ N/m}^2$$

You should always check the units to make sure you have the correct units that you expect. Usually it is good practice to change all your numbers to SI units before plugging in.

$$\frac{kg}{m^3} * m * \frac{m}{s^2} = \frac{kg}{ms^2} = \frac{N}{m^2} = Pa$$

So the lithostatic stress is 196,200,000 Pa. This is a very big number, so instead of using just Pascals, we use mega-Pascals (MPa) or giga-Pascals (GPa) to represent stress.

- 1 MPa = 10^6 Pa. (1 million pascals)
- 1 GPa = 10^9 Pa. (1 billion pascals)

For this example, it is most appropriate to use MPa because this gives us a number where we don't need to stop and count zeros

$$\sigma_{lith} = 196,200,000 \ Pa * \frac{1 \ MPa}{10^6 \ Pa}$$
$$= 196 \text{ MPa}$$

> Choosing whether to use MPa or GPa takes some practice, but usually you want your final answer to be between 0.01 and 9999, so that it is an easy number to read.

3D Stress Components

Generally speaking the stress state at a particular location in the crust, lithosphere or mantle, is not simply described by two opposing stresses, but rather a three-dimensional (3-D) stress state. To completely describe such a 3-D stress state requires three pairs of normal stresses oriented in three orthogonal directions (i.e., x, y, z) and 6 sets of shear stress acting on the planes that are perpendicular to the normal stresses. In the figure below these components are labeled with two numbers (indices). The first number represents the orientation of the surface that the stresses are acting on (1 indicates that the normal to the plane points in the x_1 direction. The second number indicates the direction of the force component acting on the plane. For example, note that all three stresses acting on the plane with an x_1 normal (the front of the box) have 1 as the first index. Stress components with two indices that are the same (e.g. σ_{11} are normal stresses because the force is aligned in the same direction as the normal to the plane). Stress components with two different indices are shear stresses. There are six shear stresses because on one plane there can be a component of shear stress: one in the horizontal direction and one component shear stress in the vertical direction (or any two orthogonal directions on the plane). Note that if the rock is not moving (or we are in a reference frame with the moving rock), then some of the shear stress are equal: $\sigma_{12} = \sigma_{21}$ and $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$. Therefore, we usually only need to consider 6 independent components of stress. We will return to these stress components in the next section.

Advanced Concept: Such a 3-D stress state can also be completely represented by a special set of three orthogonal normal stresses called **principal stresses**. The orientation of these three stresses is special because in this particular orientation the shear stress components all have a value of zero (i.e., no shear stress). These principal stress directions are eigenvectors of the stress matrix and the magnitudes of the stress in each direction are the eigenvalues.

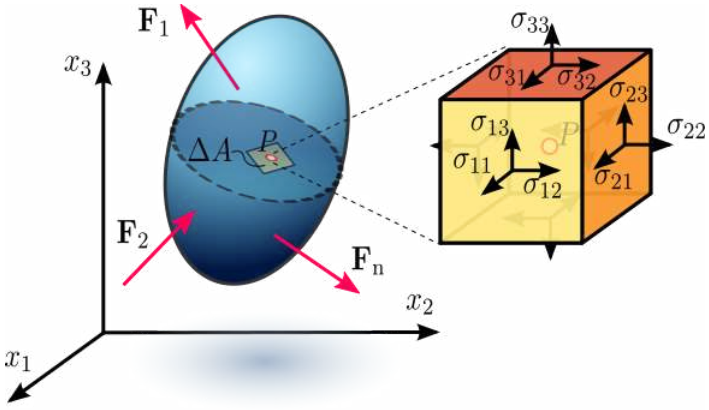


Figure 1.1.1: 3-D Components of stress at a point P. Forces oriented in different directions act on a larger block. The resulting stress at point P is shown as the 9 independent components of stress. Note the matching pair for each component acts on the opposite side of the block. (CC BY-SA 3.0; Sanpaz, via Wikimedia)

This page titled 1.1: Stress is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.2: Strain and Strain Rate

Supporting Topics

Prior to reading the page, please review the following topics if you are unfamiliar with them.

Resources for each topic can be found here: Supporting Topics

Math/Calculus

- Converting Degrees to Radians
- arctan vs arctan2

In introductory physics we learn to relate the forces applied to an object to the resulting displacement or velocity of the object, but we do not consider how an object deforms internally in response to the applied forces. To understand how rocks deform through the slow movement of tectonic plates, or rapid movement occurring during an earthquake, we need to consider the internal deformation of the rocks. Given an applied stress, the amount of deformation that an object experiences depends on the material properties of the object and the orientation of the applied stresses. The final deformation of the object depends on how fast the deformation occurs and for how long. In this section you will learn how to described the internal deformation of a rock in terms of the components of strain, and how to describe the rate of deformation occurring in terms of the strain-rate.

Strain

When a stress is applied to an object (a rock, the crust, the lithosphere), the material deforms or squishes. Think again of squeezing an eraser between your fingers. You can feel the eraser give a little bit. While rocks appear perfectly rigid, the stresses acting in the Earth are large enough that they also cause rocks to deform internally. This internal deformation is called strain. **Strain** is a measure of the change in shape or size of an object relative to its original shape or size.

For example, the change in length of a spring, ΔL relative to the original length of the spring, L describes the strain of the spring, $\epsilon = \Delta L/L$. Both ΔL and L must be measured in the same units (e.g., cm or mm). Notice that the strain, ϵ has units of meters divided by meters, so it ends up without physical units. Strain is also sometimes given the symbol, e instead of the greek symbol, epsilon, ϵ.

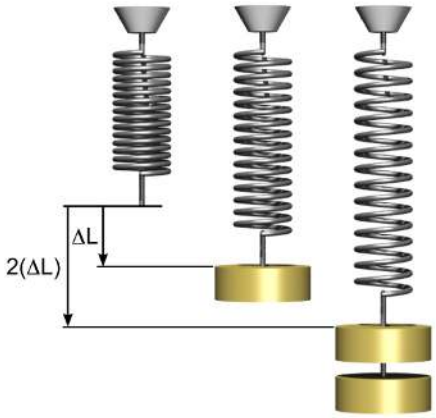


Figure 1.2.1: Linear strain (stretching) of a spring. (CC BY-SA 3.0; Svjo (edited by Taryn Lausch), via Wikimedia)

Normal (Linear) Strain

Normal strain occurs perpendicular to the sides of an object, similar to normal stress. It describes the change in length of the object in one direction relative to the original length of the object in the same direction,

$$\epsilon = \frac{l_f - L}{L} = \frac{\Delta L}{L} \tag{1.2.1}$$

where L is the original length of the material and l_f is the final length. Normal strain is also referred to as **linear strain** or **elongation**. If the final length of the object is smaller than the original object, then the strain is a negative number between 0^- and

-1, and the object has undergone **shortening**. If the final length of the object is greater than the original object, then the strain is a positive number in the range of 0^+ and infinity, and the object has undergone **stretching**.

The linear strain is also often expressed as a **percent strain** given by $100 \times \epsilon$. For example, if the strain is 0.02, then the percent strain is 2%.

Observations of strain in rocks is used to quantify the magnitude and directions of permanent deformation recorded in rocks from the out-crop scale (centimeters) to the plate boundary scale (100s of kilometers). For example, measurements of change in the relative distance between two permanent GPS locations can be used to calculate the strain occurring in the crust in between.

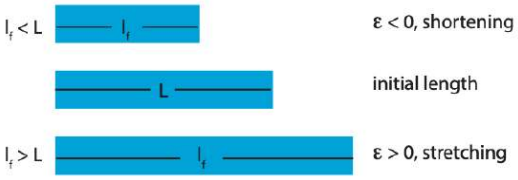


Figure 1.2.2: Normal strain leads to shortening or stretching of an object.

Shear Strain

The second type of strain is shear strain, which results from a shear stress. The **shear strain** is given by the displacement in one direction (x) occurring across a region in the other direction (y). In the figure below, an applied shear stress slides the top of the layer of thickness y, side-ways by an amount Δx. The shear strain is then given by,

$$\epsilon_s = \frac{\Delta x}{y} \tag{1.2.2}$$

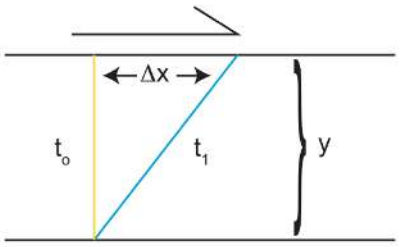


Figure 1.2.3: Shear Strain Variables

The **shear angle**, γ is also used is to describe the shear strain within an outcrop. For example, it may be easier the measure the rotation of a feature on a trilobite fossil than to measure the offset of that feature. The shear strain and the shear angle are related as,

$$\epsilon_s = \tan\gamma = \frac{\Delta_x}{y} \tag{1.2.3}$$

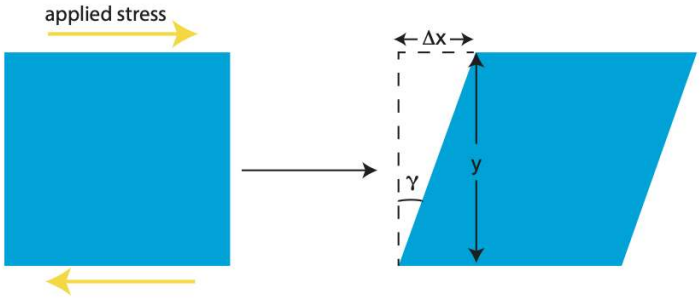


Figure 1.2.4: Shear angle, γ, is also used to describe the shear strain of features.

Convention states that if the angle between the x and y axes decreases, the shear strain is positive, and if the angle increases, the shear strain in negative. Like normal strain, shear strain is a ratio and is thus dimensionless.

Calculating Shear Strain

To calculate the shear strain across a fault or shear zone, we need to first identify a feature that has been offset by the fault or shear zone. Note the brunton for scale. The photo below shows an outcrop in an exhumed section of the mantle within the Josephine Ophiolite in southern Oregon. An ophiolite is a section of oceanic plate preserving all or part of the crust, and sometimes includes the top of the mantle section of the plate. This mantle section is made up of a mix of olivine (weathered, so they appear light brown) and pyroxenes (weathered, so they appear dark reddish-brown or black). The pyroxenes sometime appear in distinct layers that can be traced for tens of meters. The photo shows where one of these pyroxene layers has been offset by a fault or localize shear zone.

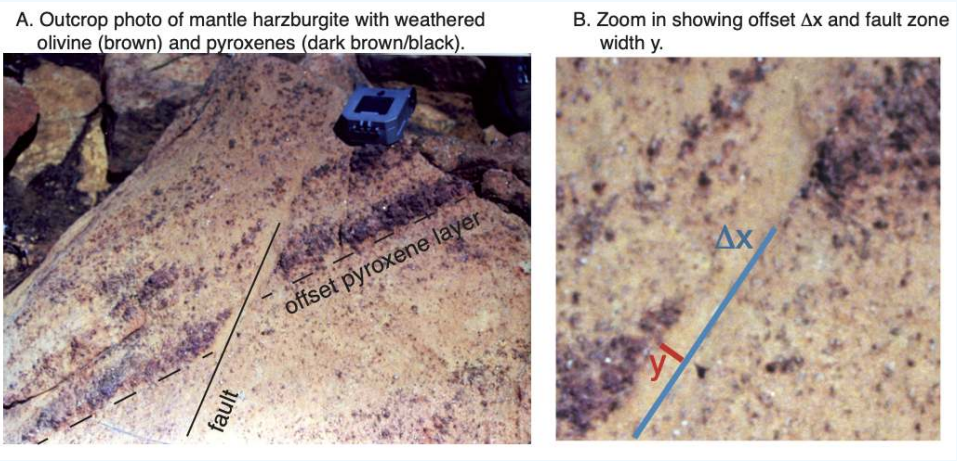


Figure 1.2.5: Small fault or shear zone within an outcrop of mantle harzburgite from the Josephiine Ophiolite in southern Oregon. Photo courtesy of M. I. Billen.

By marking the bottom of the pyroxene layer on either side of the fault we can measure the offset, $\Delta x = 5$ cm. Similarly we can measure the width of the fault zone, $y = 0.5$ cm. The figure below shows these two parameters schematically. (Note there is no sheared feature visible within the fault zone in the photo above. Therefore, the shear strain is

$$\epsilon_s = \frac{5}{0.5} = 10 \tag{1.2.4}$$

This value indicates a localized shear in a narrow zone, with the offset being 10 times greater than the width of the layer.

Similarly, the shear angle is

$$\gamma = tan^{-1}(10) * 180/\pi = 84.3^{\circ} \tag{1.2.5}$$

Note, that the output of the inverse tangent (atan or arctan) is an angle in radians, so it is necessary to convert to degrees.

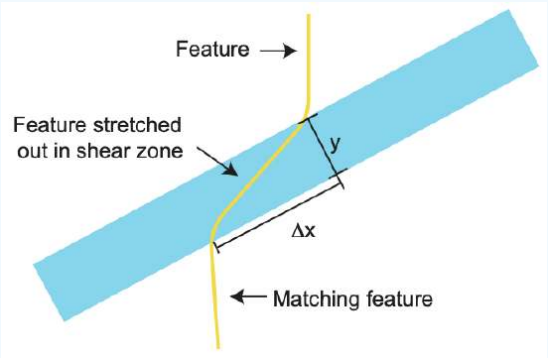


Figure 1.2.6: Schematic showing how to measure shear strain across a fault or shear zone.

Strain Rate

Deformation in the Earth can occur very slowly, such as the motion of plates with speeds of only millimeters or centimeters per year, and it can occur very quickly, such as the motion across a fault with speeds of meters per second. Therefore, strain also accumulates at different rates depending on the process and geologic features involved in the strain accumulation, and can change over time. The figure below shows the net strain accumulating across a shear zone as a function of time. Note the slope of this line changes, being steeper at some times and shallower at other times. The slope of this line, which is the derivative of the strain with respect time, is the strain-rate,

$$\dot{\epsilon} = \frac{d\epsilon}{dt} \tag{1.2.6}$$

For any small interval of time, we can measure the strain-rate as simply the amount of strain accumulated, ϵ, in a time interval, Δt. For example, is the strain between times t_1 and t_2 in the figure is 0.5, and the time interval is t_2 - t_1 = 10 years, then the strain-rate is 0.05 1/yr.

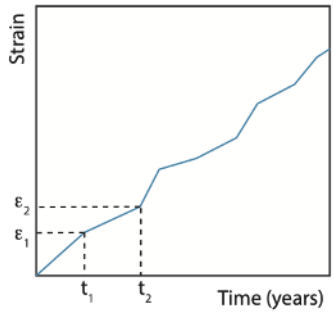


Figure 1.2.7: Schematic plot showing strain accumulation over time. The slope of the line is the strain rate. Strain rate changes over time and can even be zero.

Observing Strain in Rocks

One of the key observations made by geologists is to note the features indicating how rocks are deformed in an outcrop. They look for indications of how the rock has been folded, sheared, or faulted. These observations are used to describe and quantify the internal strain of the rock. It is important to note then that these are measurements of the total strain experienced by the rock since its emplacement. That is, you observed the final result of some unknown history of deformation occurring over some unknown time interval. It is possible to make observations that might indicate the order of different increments of the strain; for example, that a layer was sheared, and then it was faulted. It is also sometimes possible to make measurements that allow you to determine the time over which these increments of strain occurred using radiogenic dating methods. Gathering these different pieces of data from rocks is a painstaking process that eventually allows you to unravel the deformation history recorded in the rocks themselves.



Figure 1.2.8: Sheared and faulted region in a granite outcrop in the Sierra Nevada Mountains, California. Courtesy of M. I. Billen

This page titled 1.2: Strain and Strain Rate is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.3: Elastic Deformation

What Type of Deformation is Recorded in Rocks?

The deformation recorded in the rock record provides a direct measurement of the total strain that the rock has undergone. However, it is important to note two things. First, knowing the total strain does not give us information about the strain path. For example, I can stretch and compress a spring many times and then stop. I can then measure the final strain, but that measurement does not tell me how much I stretched and compressed the rock beforehand. In other words, we know the final strain, but not the history of how that final strain was achieved.

Second, rocks record permanent deformation. Again thinking about the spring. I can stretch and compress the spring many times, but it always returns to its original shape. There is no permanent deformation so there is no record of the deformation. However, if I pull on the spring very hard I can cause it to stay partly stretched even when I let go. This is permanent deformation, which is also referred to as **irrecoverable strain**. In the case of the spring it occurs when the force applied is large enough that it causes the spring to deform plastically (this is a type of failure typical of metals). The same is true for rocks, low stress in a rock causes elastic deformation that is not permanent. This is referred to as **recoverable** strain. This elastic deformation is not recorded in the rock record. The deformation recorded in the rocks occurs when the rocks break (fail) or flows viscously (a type of crystal-scale failure) or plastically (also a crystal-scale failure process).

Elastic deformation is the dominant form of deformation at shallow depths in the crust and lithosphere because both the temperature and pressure are low. However the crust and lithosphere are also brittle and when the stress is big enough, failure by fracture or frictional sliding also occurs. Viscous and plastic deformation typically occur deeper and at higher pressures and temperatures than elastic deformation. In between the shallow elastic region and deeper viscous region there is a region that deforms through both mechanisms (both mechanisms accommodate roughly equal amounts of deformation). This region is referred to as the **brittle-ductile transition** and rocks here include cataclasites (features of viscous flow with pervasive micro-cracks), mylonites and pdeudotachylites (melting).

Even though elastic deformation is not preserved in the rock record, the elastic strength of the rock still supports the stress in the rock up until failure, so we need to be able to quantify the elastic behavior. For example, slow steady motion of tectonic plates causes elastic deformation in the region surrounding a fault. In between earthquakes this elastic deformation builds up until the stress inside the rock exceeds the friction on the fault. So, even though the elastic deformation is released in the surrounding rock, the history of this process is recorded by the earthquake. Similarly, in a laboratory experiment or when computer simulations are used to model the deformation of the crust or lithosphere, we can apply forces that result in a stress in the rock and we can observe or calculate the resulting strain. We will not go into a full description of how this is done, but we will touch on a few key aspects that help a geologist think about the state of stress in the crust or lithosphere.

Elastic Deformation

While the spring is an appropriate model for linear strain, in order to consider the three-dimensional deformation of rocks, we need to consider a block of material or a volume of rock. For the spring the elastic response is determined by a single parameter, the spring constant (or the elasticity of the spring). For the volume of rock we need (at least) two parameters: the Young's modulus and Poisson's ratio.

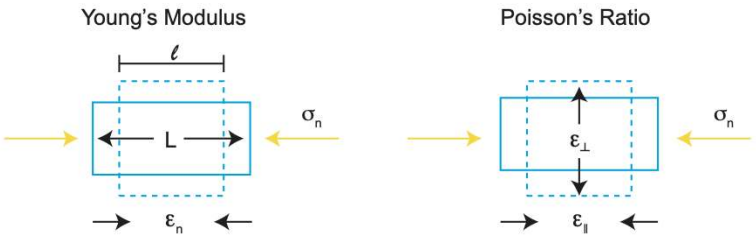


Figure 1.3.1: Young's Modulus quantifies the relationship between stress and strain occurring in the same orientation. Poisson's ratio quantifies the relative size of the strain that occurs in perpendicular directions.

Young's Modulus

Young's modulus, E, quantifies the relationship between an applied linear stress σ_n and the resulting linear strain ϵ_n when no other stresses act on the material:

$$E = \frac{\sigma_n}{\epsilon_n} \tag{1.3.1}$$

Since stress has units of Pascals and strain is dimensionless, Young's modulus has the same units as stress (Pa). The Young's modulus quantifies the ability of an object to resist deformation through the elastic strength arising from the crystal structure of the rock. Typical values for Young's modulus are in the range of 10 MPa to 70 GPa (see Table \PageIndex{1}\)

For Earth materials, Young's modulus is measured in the laboratory by taking small cylinders of rock, compressing them in a vice-like apparatus with a known force, and then measuring the change in length. The stress and strain are determined (knowing the area of the top and bottom of the cylinder and its length) and then Young's modulus is calculated. Note that when this experiment is done there is no stress acting on the curved sides of the cylinder, only on the top and bottom. This state of stress is referred to as a **uniaxial stress-state**. However, note that even though there is no stress on the sides of the cylinder, there is a strain perpendicular to the sides because the compression of the cylinder in one direction causes it to bulge out in the perpendicular directions. The amount that the material bulges out is determined by the second elastic property, the Poisson's ratio.

Table 1.3.1: Typical values of Young's Modulus for Earth Materials.

Rock Type	Young's Modulus	Rock Type	Young's Modulus
Soils	10 - 200 MPa	Granite	10 - 70 GPa
Sand	10 - 70 MPa	Limestone	15 - 55 GPa
Sandstone	1 - 20 GPa	Steel	200 GPa
Shale	1 - 70 GPa	Glass	45 GPa

Poisson's ratio

Poisson's ratio, ν, quantifies the amount that a material deforms in one direction relative to the amount it deforms in the perpendicular direction:

$$\nu = \frac{-\varepsilon_{\perp}}{\varepsilon_{\parallel}} \tag{1.3.2}$$

where $\varepsilon_{\perp}$ is referred to as **lateral strain** and $\varepsilon_{\parallel}$ is referred to as the **longitudinal strain**. (Both lateral and longitudinal strain are normal strains.) Note that there is a minus sign in the equation for Poisson's ratio because either the lateral or longitudinal strain will be negative (depending on whether the applied force causes stretching or shortening). The range of expected values for Poisson's ratio for Earth materials is 0 to 0.5, with typical values of 0.15-0.35. A value of 0.5 corresponds to a perfectly incompressible material (there is not internal compression, so the strain in two perpendicular directions exactly compensates for the applied strain). Water has a Poisson's ratio of 0.5. Poisson's ratio can also be negative (down to -1): these special synthetic materials are called anti-rubber or auxetics, which expand outward as they are stretched. Currently, there are no naturally occurring Earth materials with a negative Poisson's ratio.

Table 1.3.2: Typical values of Poisson's ratio for Earth Materials.

Rock Type	Poisson's ratio	Rock Type	Poisson's Ratio
Sandy soil	0.25 - 0.40	Granite	0.1 - 0.3
Sandstone	0.21 - 0.38	Limestone	0.18 - 0.33
Shale	0.2 - 0.4	Steel	0.3

Elastic Deformation in 3D

The elastic response of an applied stress in 2D can be extended to 3D. Here, for simplicity, we will only consider the normal (linear) components of strain and stress (Similar equations exist for the shear components). Consider the normal strain in one direction resulting from normal stresses applied in three orthogonal directions. As noted above, for an elastic solid, a stress applied in one direction will also cause a strain in the perpendicular directions. Therefore, the linear strain in the x directions depends on the strain caused by the stress in the x direction, but it also depends on the strain caused by the stresses acting in the y and z directions. Therefore, the components of normal strain in an elastic solid, acting in three orthogonal directions are given by:

$$\varepsilon_{xx} = \left(\frac{1}{E}\right)(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) \tag{1.3.3}$$

for the x direction,

$$\varepsilon_{yy} = \left(\frac{1}{E}\right)(\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})) \tag{1.3.4}$$

for the y direction, and

$$\varepsilon_{zz} = \left(\frac{1}{E}\right)(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})) \tag{1.3.5}$$

for the z direction.

Similar to the figure below (and discussed in Section 1.1) in these three equations, the subscript xx indicates a strain or stress in the x direction acting perpendicular to a plane whose normal direction is also in the x direction. You will typically see stress components expressed using either the numbers 1, 2, or 3 or the letters x, y, and z to refer to the orientations of the planes and the forces. It is necessary to know the orientation of the three directions to interpret the orientations of the stresses.

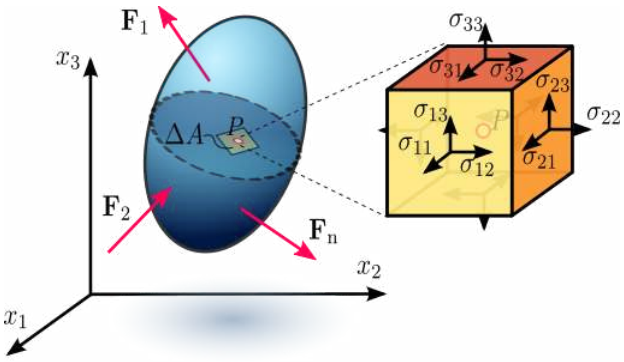


Figure 1.3.2: 3-D Components of stress at a point P. Forces oriented in different directions act on a larger block. The resulting stress at point P is shown as the 9 independent components of stress. Note the matching pair for each component acts on the opposite side of the block. (CC BY-SA 3.0; Sanpaz, via Wikimedia)

Using Stress to Calculate Strain

The picture below shows the crater wall inside Mt. Vesuvius (this picture was taken from the visitors center on the crater wall). The crater wall is 305 meters high. In Section 1.1 you learned how to calculate the lithostatic stress. Most of the lava erupted from Mt Vesuvius is andesitic in composition with density in the range of 2400 - 2800 kg/m^3. Using an intermediate value of 2600 kg/m^3, the lithostatic stress at the base of the wall is 7.8 MPa.

Now, lets ask how much has the rock been shortened at the base of the wall due to the weight of the rock above?

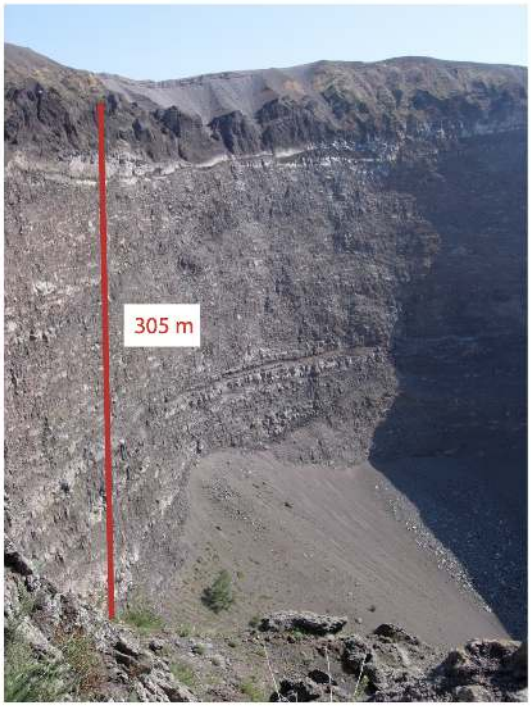


Photo courtesy of M. I. Billen

To answer this question we will use Equation 1.3.5 to calculate the normal strain in the vertical direction, z. For this equation we need to know the three normal components of stress, as well as the Young's modulus (40 GPa) and Poisson's ratio (0.2) from the tables above (use values for granite).

We will use the following values for stress

$\sigma_{zz} = \sigma_{lith}$. This is the lithostatic (vertical) stress.

$\sigma_{xx} = 0$. This is the stress perpendicular to the wall. Since there is air next to the wall this stress is zero.

$\sigma_{yy} = 2\sigma_{zz}$. For this direction we will assume that the horizontal stress equals twice the vertical stress. Why twice? If the wall was not there the state of stress would that of lithostatic pressure (all normal stresses equal. However since no stress is support in the direction of the wall, we assume that this stress must be supported in the orthogonal direction.

With these values, Equation 1.3.5 becomes

$\varepsilon_{zz} = \frac{\sigma_{zz}}{E}(1 - 2\nu)$

Plugging in values we find,

$\varepsilon_{zz} = \frac{7.8e6}{40e9}(1 - 2(0.2)) = 0.000117$

To interpret what this value means, we can use the definition of linear strain and solve for the change in length (l-L) assuming the initial length $L = 305$ m

$l - L = \epsilon_n L = 0.000117 * 305 = 0.035$ m, or about 3.5 cm

Therefore the weight of the rock in the crater wall has elastically shortened (compressed) the rock at the base of the wall by about 3.5 cm. This is elastic strain, so if the rock above were removed, the rock at the base would return to its original size.

This page titled 1.3: Elastic Deformation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.4: Failure of Rocks

At shallow depths, rocks record permanent deformation caused by either fracture of the rock or frictional sliding on existing fractures. This failure often occurs dramatically in the form of earthquakes occurring with displacement across the fault of centimeters to tens of meters. In most cases both fracture and frictional sliding occur through shear deformation (shear strain) along a discrete surface. Both fracture and friction are described by experimentally-determined relationships, which relate the shear resistance to the applied normal stress σ_n and failure properties of the rock. These relationship define the state of stress on the failure plane at the point of failure.

Tectonic Stress and Stress Components on Faults

Tectonic stresses in the crust arise due to density anomalies in the mantle, which drive convection and move the plates. However, if we are concerned with the stresses acting on faults in the crust, we can ignore the details of where these stresses come from and instead treat the tectonic stress as a horizontal normal stresses. In this case, tectonic stresses either act in compression (a positive normal stress) or in extension (a negative normal stress).

To consider the total stress-state in the crust, we start from an assumption that the state of stress is equal to the lithostatic pressure and add the tectonic stress to get the stress component in the horizontal direction:

$$\sigma_H = P_{lith} + \sigma_t \tag{1.4.1}$$

Except in cases of very shallow rock (less than about 500 m), all the components of stress in the rock are positive due to the increase in the lithostatic pressure with depth. This means that the magnitudes of tectonic stresses are smaller than the magnitude of the lithostatic stress at depths greater than about 500 m.

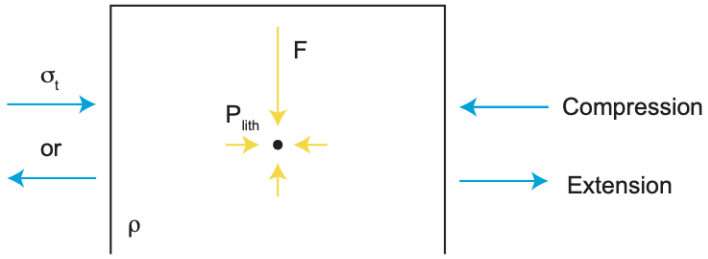


Figure 1.4.3: Compression (positive) and extension (negative) tectonic stresses is added to the background lithostatic pressure to get the total stress-state in the crust.

Andersonian Theory of Faulting

While the magnitude of the horizontal stress is positive, it is the difference in stress magnitude between two orthogonal directions that determine whether a region experiences compression (with reverse faults), extension (with normal faults) or shear (strike-slip faults). Figure 1.4.4 shows the three different kinds of faults arising from three different stress-states. The stress state is described by the maximum, intermediate and minimum magnitude principal stresses, $\sigma_{max} \geq \sigma_{int} \geq \sigma_{min}$. Remember the three principal stress are always orthogonal to each other.These are also referred by as $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (note these only have one number in the subscript, so these refer to the principal stresses). Extension, characterized by normal faulting, occurs where σ_{max} is vertical and extension occurs parallel to the σ_{min} direction. For this case think of squeezing playdoh between your fingers, it will extend outward in the direction of least resistance. Shortening, characterized by reverse faulting, occurs where σ_{max} is horizontal and σ_{min} is vertical. In this case, shortening occurs parallel to the σ_{max} direction. Finally, horizontal shearing characterized by strike-slip faulting, occurs where σ_{max} and σ_{min} are both horizontal. The orientation of the strike-slip fault will be at an angle to both σ_{max} and σ_{min}. The exact angle depends on the value of the friction on the fault, but is usually in the range of 30 - 60°. This first order description of the relationship between stresses and fault orientation is called the **Andersonian Theory of Faulting**.

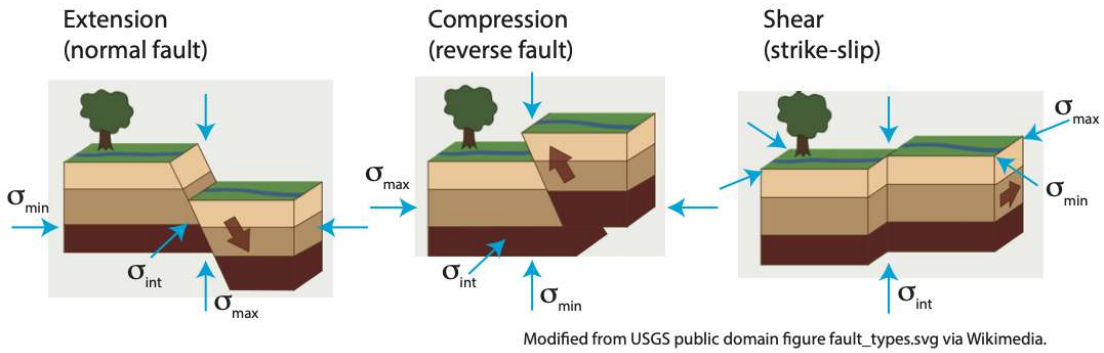


Figure 1.4.4: Orientations of the maximum, minimum and intermediate principal stresses determine the state of stress and the type of faulting.

Stress Components in the Principal Stress Directions

Existing faults and fractures can have any orientation in the crust, and almost all of these orientation will result in both shear stress and normal stress acting on the fault plane. However, there are 3 (and only 3) possible fault/fracture plane orientations that will have a zero shear $\sigma_s = 0$. These three planes are the planes that are perpendicular to the principal stress directions (Figure 1.4.6). This is why the principal directions are normal stresses, and this is why we only need to know these three principal stresses to fully describe the state of stress. Near the surface of the Earth, it is likely that one of the principal stress directions aligns with the vertical, but the two horizontal principal stress can be in any orientation. When looking at the figure below, imagine taking one of the planes and rotating it slightly. The stress component (vector) acting on that plane can now expressed in terms of component parallel to the fault plane and a component perpendicular to the fault plane. All other planes have both shear and normal stresses acting across the plane.



Figure 1.4.5: Example of plane orientations with no shear stress. The normal stresses acting across these planes are principal stresses.

Example: Stress-State on the San Andreas Fault

The San Andreas Fault is a large strike-slip fault the runs most of the length of California. It is the plate boundary between the Pacific Plate to the west and the North American plate to the East. This fault has been active for about 30 million years. During that time the shape of the fault (or its orientation) have changed as the stress state along the plate boundary has changed. Figure 1.4.5 is a schematic map view of a section of the fault showing the orientation of the fault relative to the maximum stress, σ_{H1} and the minimum stress σ_{H2}. Note that in this case, both horizontal components of the lithostatic stress are modified by a tectonic stress: a roughly north-south oriented positive tectonic stress, σ_{t+}, and a roughly east-west oriented negative tectonic stress, σ_{t+}. This state of stress and also be expressed in terms of the stress acting perpendicular and parallel to the fault surface (Figure 1.4.5B). The magnitude of these stresses depend on the exact orientation of the fault relative to the normal stresses. In order for the motion to occur along the fault, the shear stress σ_s must be larger than the frictional resistance to sliding between the two blocks.

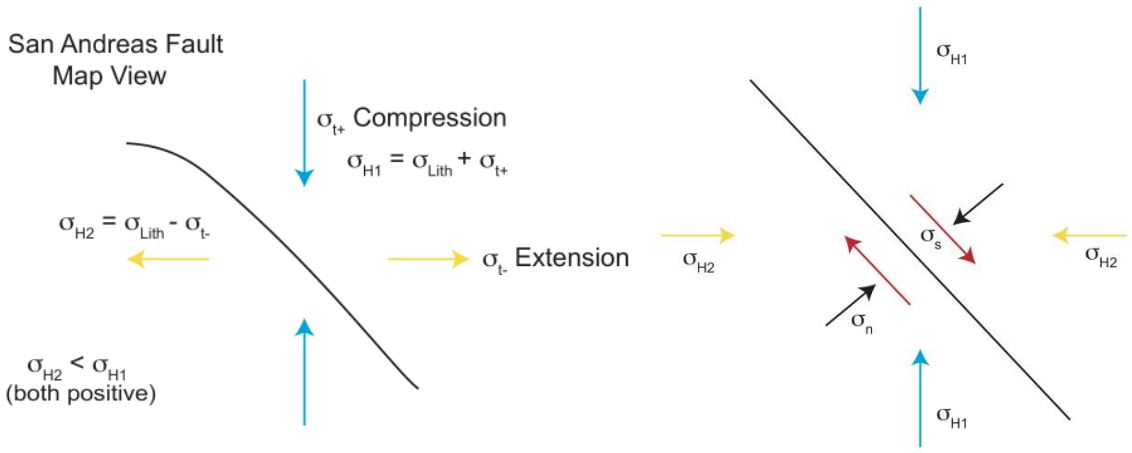


Figure 1.4.6: Left) Schematic showing principal stresses acting on the San Andreas Fault. Right). The principal stresses acting on the fault results in a component of stress normal to the fault and a component of stress acting parallel to the fault.

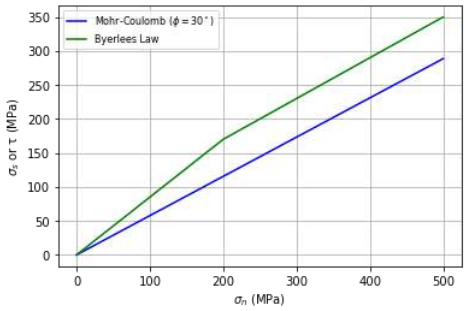
Fracture

Fracture of a rock is described by the Mohr-Coulomb Fracture Criteria. This is an empirical relationship, which states that failure by fracture occurs when the shear stress acting on the rock exceeds the **shear fracture strength** given by

$$\sigma_{s-frac} = tan(\phi)\sigma_n + c \tag{1.4.2}$$

where ϕ is the **angle of internal friction**, and c is the **cohesion**. The term $tan(\phi)$ can be thought of as an internal friction. The angle of internal friction for rock has a typical values 30°, but ranges from $20 - 45^{\circ}$. The cohesion term sets the minimum values of the shear fracture strength, which occurs when the normal stress is zero. Cohesion for rock is typically about 0.1 MPa for silts or clays, but 10 MPa for other rocks. As we will see below, the failure criterion for frictional sliding is defined by a similar equation, with similar magnitudes for the parameters. Therefore, it is not necessary to know if an earthquake occurred on an existing plane or formed a new fracture: the analysis of the stresses or prediction of failure planes will be very similar in both cases.

The plot below shows the Mohr-Coulomb criterion (blue) and Byerlee's law (green; see next topic). These two failure laws predict similar levels of shear stress resistance to failure at a given normal stress acting across the plane. The shear stress resulting from tectonic forces needs to be larger than the shear stress resistance in order for failure to occur.



Frictional sliding

Failure by frictional sliding is described by **Byerlee's law,** an empirically-derived equation based on laboratory experiments originally carried out James Byerlee in the 1960's and 70's**.** The general form of Byerlee's law is given by

$$\tau = \mu_s \sigma_n + C \tag{1.4.3}$$

where tau is the stress stress resisting sliding along the fault, μ_s is the coefficient of friction and C is the cohesion across the fault surface. This equation gives the shear stress needed to cause sliding across the fault. That is, when the tectonically derived shear stress acting on the plane is higher than τ then sliding will occur.

Byerlee showed that the coefficient of friction for most rocks is remarkably uniform, but it does vary systematically depending on the magnitude of the normal stress:

$$\tau = 0.85\sigma_n \qquad \text{when } \sigma_n < 200\,\text{MPa}$$
$$\tau = 50 + 0.6\sigma_n \qquad \text{when } 200 < \sigma_n < 1700\,\text{MPa}$$

where, σ_n and τ are in units of MPa. Note that when $\sigma_n > 200$ MPa, there is a cohesion term of 60 MPa. These equations capture the fact that there is no cohesion at low normal stress, while at higher normal stress there is a threshold shear stress that must be overcome before any sliding occurs. This maybe a familiar phenomena to anyone who has tried to slide furniture across a tile or wood floor. A light chair (low normal stress) moves immediately when we start pushing or pulling it across the floor, whereas as heavy desk or couch seems to resist moving until sufficient force is applied, buy then it readily slides as long as we maintain that same threshold force.

Recall that normal stress in the crust is primarily determined by the lithostatic stress, so the normal stress will increase with depth even if there is no tectonic stress. Tectonic stresses are needed to create a shear stress acting on the fault that exceeds the shear stress resisting sliding. That is, sliding occurs when

$$\tau_{tect} > \tau \tag{1.4.4}$$

While the values for μ_s given by Byerlee's law are appropriate for many rocks, certain clays, talc and serpentinite have coefficient of friction of 0.1-0.2. Clays and talc are often found on exhumed fault zones and in drill cores across active faults. Therefore, the frictional properties on faults themselves may not directly reflect the properties of intact rock surrounding the faults.

Why do the failure criterion for fracture and frictional sliding have the same form and similar values? This is likely because at a microscopic scale fracture occurs through opening of tiny fracture, and frictional sliding on these tiny fracture. The tiny fractures then link up to cause a through-going fracture.

What determines the maximum depth of earthquakes in the crust? Note that shear stress for required to cause fracture or frictional sliding increases with increasing normal stress. Because the normal stress (pressure) increases with depth in the crust, a larger tectonically-derived shear stress is required to cause failure at greater depths. This is why faulting is typically limited to depths of less than about 15 to 30 km except in regions that are very cold (or when other processes can act to reduce the normal stress).

Will the fault fail?

Below is a sketch of a portion of a fault at a depth of 10 km. Both lithostatic stress and tectonic stress are acting on this point to determine the state of stress on the fault. It is been independently determined given the orientation of the fault, the normal stress acting on the fault is σ_n=300 MPa, and the shear stress acting on the fault is $\tau_{tect} = 120$ MPa. Use Byerlee's law to determine if the fault should fail.

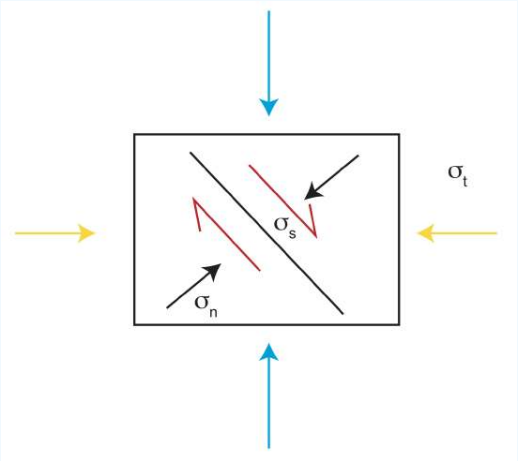


Figure 1.4.2: Tectonic stress and lithostatic stress acting on a rock result in normal stress and shear stress acting on a fault surface.

Because $\sigma_n > 200$ MPa, we use the second part of Byerlee's law:

$$\begin{aligned}\tau &= 50 + 0.6(300) \\ &= 50 + 180 \\ &= 230\,\text{MPa}\end{aligned}$$

Since $\tau_{tect} < \tau$ the fault will not fail: the shear stress resisting sliding (230 MPa) is larger than the shear stress acting along the fault (120 MPa).

This page titled 1.4: Failure of Rocks is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.5: Viscous Deformation

The flow of liquids, like water, honey, oil, and magma, is described as viscous flow or viscous deformation. Similar to elastic deformation we can ask how the deformation is related to the stress during viscous deformation. In this case, the stress is related to the strain-rate $\dot{\epsilon} = \frac{d\epsilon}{dt}$, rather than the strain ϵ through the viscosity η, (the greek symbol eta):

$$\sigma = 2\eta\dot{\epsilon} \tag{1.5.1}$$

This equation states that the stress in a viscous material is proportional to how fast it is deforming (the **strain-rate**). The factor of 2 is there for mathematical convenience in full 3-D descriptions of viscous deformation. The stress has units of Pascals and the strain-rate has units of 1/s, so the viscosity has units of Pa*s. The **viscosity** can be thought of as the resistance to flow. Higher viscosity indicates more resistance to flow. This can be seen more readily, by rewriting the equation above as:

$$\dot{\epsilon} = \frac{\sigma}{2\eta} \tag{1.5.2}$$

For a given applied stress σ a higher viscosity will result in a lower strain-rate $\dot{\epsilon}$. Viscosity is typically dependent on temperature and composition. The viscosity of some familiar liquids are given in Table 1.5.1. Notice that the viscosity ranges over five orders of magnitude (0.001 to 100 Pa s) for these familiar liquids. For example the difference in viscosity of water and oil is a factor of 10, whereas the difference in viscosity between water and honey is 10,000. Materials with lower viscosity deform (flow) faster than materials with a higher viscosity.

Table 1.5.1: Viscosity of some common liquids.

Material	Viscosity (at 20 C) (Pa s)
water	0.0010
milk	0.0020
olive oil	0.010
honey	10.0
toothpaste	70.0
tar	100.00

Viscous Deformation in Solid Rocks or Ice

While we normally think of rocks as being hard, elastic materials that break, the rheology of rocks (and ice) is in fact both elastic and viscous, but these two different responses occur at different time-scales. The short-term deformation of rocks is best described as elastic with brittle failure and applies to processes like earthquakes (timescale of 1-1000 years between events) and propagation of seismic waves (timescale < seconds). This short term response dominates at cold temperatures and low pressures typically found in the crust. However, even at the high pressures and temperatures of the deep mantle the elastic behavior of the rock allows for the transmission of seismic waves due to the very short timescale of this process. The long term deformation of the lithosphere and the mantle is best described as viscous flow. At the higher pressures and temperatures of the deep lithosphere and mantle, deformation of grains by viscous mechanisms occurs more readily than elastic or brittle deformation. However, it is a very slow process with mantle flow occurring at velocities of 1 to 20 cm/yr, at time-scales of a miliion years or more. Viscosity in the mantle is strongly dependent on temperature, as well as stress and rock composition. Typical values of viscosity in the mantle are 10^{18} to 10^{24} Pa s. These are large numbers, but similar to the more common materials discussed above, the variation of rock viscosity in the mantle is about 6 orders of magnitude. Ice is also a viscous and elastic material. The viscous behavior of ice can be see in ice streams that flow downhill carry ice away from large glaciers. The viscosity of ice is typically in the range around 10^{13} at $0^{\circ}C$ to 10^{14} at $-10^{\circ}C$ Pa s.

The viscous deformation of the rock (and ice) occurs by crystal-scale deformation (called creep). In this solid-state creep the grains themselves deform as individual atoms (a lot of them) or planes of atoms (called dislocations) move within the grains. These processes of solid state creep are ultimately controlled by how slowly atoms move through the crystal (a diffusive process), which

is why the time-scale is so long and the viscosity is so high. Its important to remember that at high pressure, the rocks are flowing but they are still solid. High temperatures (around 1200-1400 C) and relatively lower pressure are needed to cause the rock to melt and form magma.

Applications of Viscous Flow in/on the Earth

In this section we will use viscous flow to consider the deformation of solid rock and ice behaving as a viscous fluid (termed solid-state creep) and magma behaving as a viscous fluid. We will consider viscous flow in three settings.

Viscous Drag on a Tectonic Plate

First, we will consider the viscous flow that occurs in the mantle beneath a moving plate (called the asthenosphere). In this case, the tectonics plate (lithosphere) is being moved by two forces. The first force is the slab-pull force caused by sinking tectonic plate (also called a slab) in a subduction zone. The second force moving the plate is the ridge-push force caused by the positive buoyancy of less dense material under a mid-ocean spreading ridge. Figure 1.5.2 shows a profile sketch of a tectonic plate with these forces acting to move the plate across the underlying mantle. The moving plate drags the viscous asthenosphere, and the resistance to this motion is an important force acting on the tectonic plate.

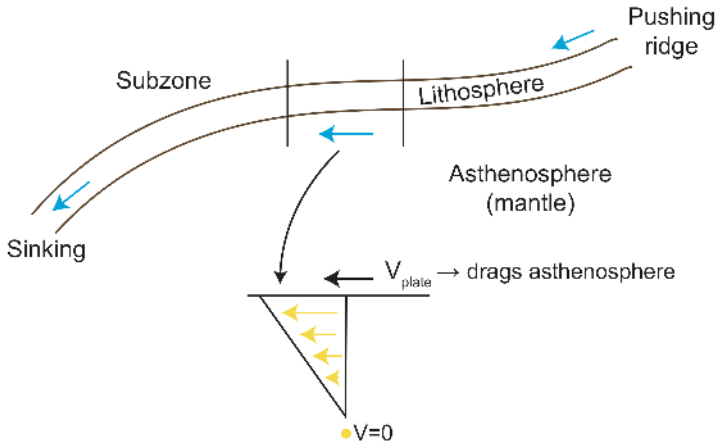


Figure 1.5.2: Example of viscous flow in the mantle caused by motion of a tectonic plate.

Flow Magma Down a Volcano

When a volcano erupts, magma flows from the vent at the top of the volcano down the sides of the volcano. As the magma cools faster at it edges than in its interior, the flowing magma tends to form its own flow channel: cooled magma forms rock levees on the side of the channel. When the eruption rate of magma is high these channels of of magma can carry liquid magma tens of miles from the vent before the magma cools sufficiently to solidify. We will examine the flow-rate of magma in a simplified magma channel in which we ignore how the magma drags against the sides of the channel and just focus on how the middle of a broad magma channel flows. The flow rate will depend on the viscosity of the magma, but the viscosity of the magma depends on its temperature. As the magma cools, it crystallizes and the network of crystals in the magma strongly increase the viscosity of the magma causing to flow slower.

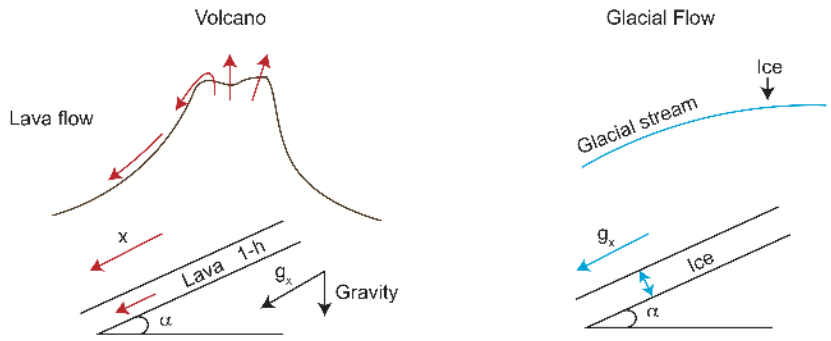


Figure 1.5.3: REMOVE GLACIAL FLOW ADD PICTURE OF MAGMA STREAM FLOWING DOWN SIDE OF HAWAII (USGS PICS SHOULD BE OPEN SOURCE)

Flow of a Glacial Ice Stream

A glacial ice stream is a region of flowing ice that drains ice away form large glaciers in places like Greenland and Antarctica. These rivers of ice move at rates of about 1 to 100 meters per year, depending on the viscosity (temperature) of the ice, as well as the slope of the land glacial ice stream is flow down. Similar to the case of flowing magma, we can gain some intuition about how this flow by simplifying the geometry and considering only the flow at the center of the ice stream, and the effects of resistance at the base of the flowing ice.

ADD SKETCH (MOVED FROM ABOVE AND ADD TOPOGRAPHY) AND PICTURE OF AN ICE STREAM (FROM WIKIPEDIA).

Flow Through a Pipe

To gain a more intuitive understanding of the parameters and conditions that affect viscous flow, let's begin with a simple model of viscous flow through pipe, like a water pipe in your house or a hose in your backyard.

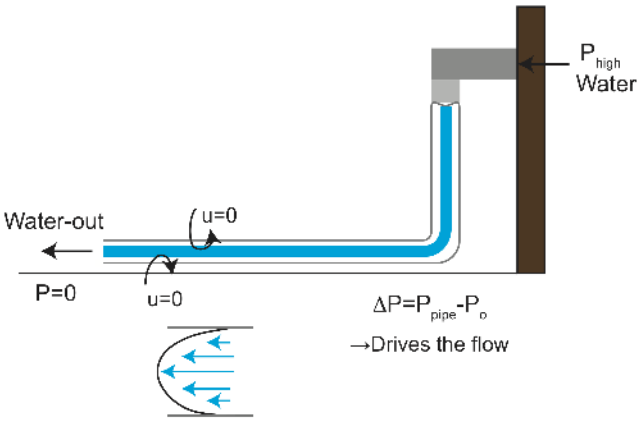


Figure 1.5.1: Viscous Flow

Water flow through the pipe in response to a pressure gradient along the pipe. The pressure at the open end of the pipe P_o=0, while the water is being pushed through the pipe by higher pressure coming from the water being pumped into the street pipes by the utility company. The pressure gradient is given by

$$\Delta P = P_{pipe} - P_o \tag{1.5.3}$$

Fluid always flows from high pressure to low pressure. It is the pressure difference along the pipe that drives the flow.

Now let's consider for a moment what the fluid flow looks like inside the pipe. Note that at the boundary of the pipe, the pipe is not moving, this means that the fluid directly adjacent to the pipe must also have a velocity of zero. However, a tiny distance away from the wall, the fluid is moving. This creates a gradient in the fluid velocity. The magnitude of the fluid velocity in the x direction, u_x, changes in the y direction, so the gradient of the velocity is non-zero

$$\frac{du_x}{dy} \neq 0 \tag{1.5.4}$$

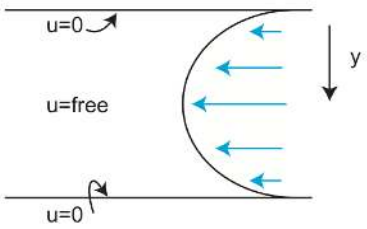


Figure 1.5.5: Inside a pipe the fluid that touches the pipe wall is not moving (u=0). The velocity at the center of the pipe is driven by the pressure gradient in the pipe. Therefore, there is a gradient in the velocity in the pipe.

Let's pause for a moment to consider the units associated with this equation. Velocity has units of meters/second (m/s) while length has units of meters (m)

$$\frac{\frac{m}{s}}{m} = \frac{1}{s} \tag{1.5.5}$$

This has units of strain-rate (1/s)

$$\dot{\epsilon} = \frac{1}{2}\frac{du_x}{dy} \tag{1.5.6}$$

Recall that strain is given by the gradient in displacement

$$\epsilon = \frac{\Delta x}{y} \tag{1.5.7}$$

Likewise, velocity is the rate of displacement, therefore the gradient in the rate of displacement results in a strain rate.

Estimating Strain Rate

It is useful to pause now to get an order of magnitude estimate of strain-rate. To do this, let's look at the plot below comparing strain as a function of time.

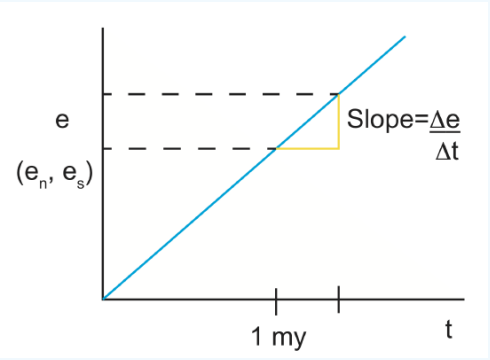


Figure 1.5.12: Strain Rate

The slope is $\frac{\Delta e}{\Delta t}$

Let's assume we have a strain of 0.3 that has accrued over a time period of 1 my:

$$= \frac{0.3}{1my} \cdot \frac{-1my}{1 \times 10^6 yr} \cdot \frac{1yr}{3.15 \times 10^7 s}$$

$$= \frac{0.1}{1 \times 10^{13}} = 1 \times 10^{-14} s^{-1}$$

Strain-rates along plate boundaries and near subducting plates are typically about 1×10^{-15} to $1 \times 10^{-12} s^{-1}$. The rates depend both on the magnitude of the stresses and the magnitude of the viscosity, which is strongly temperature dependent. The

viscosity of the asthenosphere is about 1×10^{18} to 1×10^{21} Pa-s (Pascal seconds), while viscosity inside a sinking tectonic plate may be as high as 1×10^{25} Pa-s. Strain rates are very slow in the solid earth because the viscosity is large. But, other materials can deform much faster, such as an ice flow which deforms at about 1 inch a year. Flowing magma moves even faster, at about 1 inch per minute. Viscosity values for ice range from $10^{11} - 10^{13}$ (Pa)(s) and for lava $10^{2} - 10^{6}$ (Pa)(s).

We often only think about viscous flow in terms of order of magnitude due to the large range in viscosity values. Viscosity depends on a lot of properties of the rock and the physical conditions. Such properties include: composition, grain-size, water in minerals, presence of melt and crystal grains in the melt.

Comparing Elastic and Viscous Deformation

Viscous deformation is different than elastic deformation, in several ways, but one of the primary differences is that for elastic deformation the strain is linear proportional to the stress ($\sigma = E\epsilon$), while for viscous deformation it is the strain-rate that is proportional to stress ($\sigma = \eta\dot{\epsilon}$). To better understand what this difference means, let's first revisit elastic deformation.

The figure below shows the stress resulting from an applied strain in an elastic material. Different material will have different values for the Young's modulus and thus the same amount of strain will result in different amounts of stress in the material.

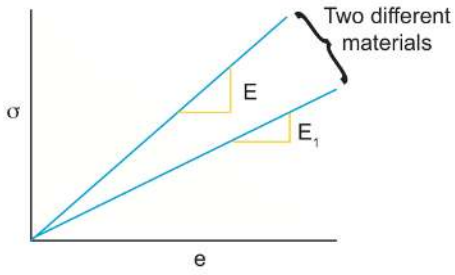


Figure 1.5.13: Different Material Deformation Responses

The next figure shows how the strain in a rock deforming elastically is not permanent, but rather it is recoverable. The stress applied to the rock is ramped up from time 1 to time 2 and then held constant from time 2 to time 3. Then the stress is released from time 3 to time 4. The figure below illustrates that the strain in the rock is proportional to the stress and that once the stress returns to zero, the strain also returns to zero.

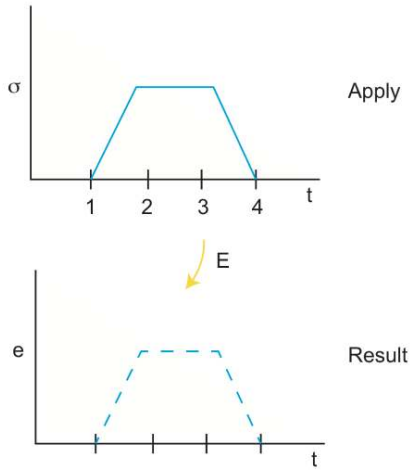


Figure 1.5.14: Applying Elastic Deformation

Returning to viscous deformation, we know that viscosity and stress have a time dependent relationship:

$$\sigma = 2\eta\frac{de}{dt} \tag{1.5.8}$$

When pulling a stick out of a very viscous fluid like honey or tar, the slower you pull the less stress is required, and the faster you pull, the more stress is needed. For a viscous material, consider that the stress is increased instantaneously, the response of the fluid is to start to flow at a proportional strain-rate. As time goes on the amount of strain in the fluid increases linearly. However, when the stress is returned to zero and the fluid stops moving(the strain-rate is zero), therefore the fluid can not flow back to its original position and instead there is permanent (no-recoverable) deformation. The fact the flow is not recovered does not mean it is not reversible. If the stress is reversed exactly the fluid can be returned to its original position.

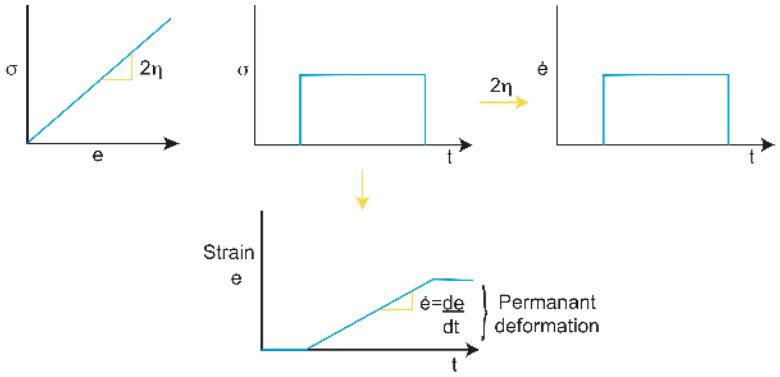


Figure 1.5.15: Viscous Deformation (MIB: first diagram needs to be corrected, should be strain-rate on the bottom axis)

Derivation of Viscous flow Equations

The basic equation describing viscous flow $\sigma = \eta\dot{\epsilon}$ is found by considering conservation of momentum in a fluid. Conservation of momentum is similar to considering a force balance on an element of fluid. Consider an infinitesimal 3D volume of fluid with sides, δ_x, δ_y, δ_z. On each side of the box in the x direction there is a traction acting on the fluid.

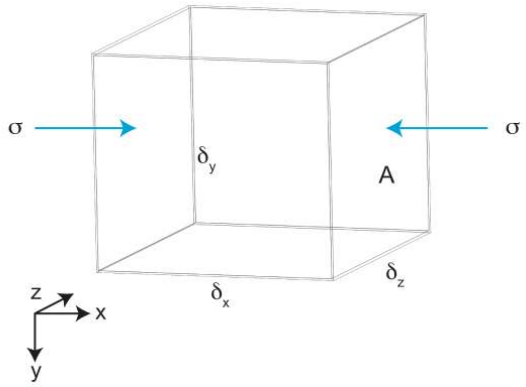


Figure 1.5.4: Force Balance

From the figure, we can see that A=$\delta_z \cdot \delta_y$. Therefore, the force acting on each side is F=σA.

$$F = \sigma\delta_z\delta_y \tag{1.5.9}$$

Dividing by δ_z we get the force per unit length in the z direction.

$$\frac{F}{\delta_z} = \sigma\delta_y \tag{1.5.10}$$

Using the force per unit length allows us to consider just a 2D cross section of the flow.

Next, let's consider all the forces acting on the sides of the element of fluid (in 2D). First, on the top and bottom of the fluid element there are shear stress acting tangential to the top and bottom surface. These shear stresses exist in the fluid because the fluid is being held back at the walls of the pipe, while it is free to move in the interior of the pipe.

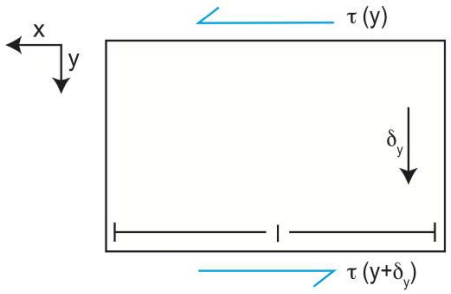


Figure 1.5.6: Shear stresses acting on the top an bottom surface of an element of fluid in the pipe.

Finally, we need to balance all the forces acting on the fluid element due to both the normal stresses acting on the sides of the box and the shear stresses acting on the top and bottom of the box. All the forces are directed in the x direction, so the forces should balance (sum to zero).

The normal stresses on the sides are the pressure in the fluid. Recall that there is a pressure gradient along the pipe in the x direction due to the pressure gradient. Therefore the pressure on the upstream side, P_1 is larger than the pressure on the downstream side, P_o. The force (per unit length) on these sides is by the pressure (stress) times the length δ_y. Similarly the force of the top and bottom are given by the sheear stress times the length, δ_x. The sum of the forces on all four sides is the

$$P_1\delta_y - P_o\delta_y + \tau(y)\delta_x - \tau(y+\delta_y)\delta_x = 0 \tag{1.5.11}$$

Combining terms we get

$$(P_1 - P_o)\delta_y - (\tau(y+\delta_y) - \tau(y))\delta_x = 0 \tag{1.5.12}$$

Next divide through by both δ_x and δ_y

$$\frac{P_1 - P_o}{\delta_x} - \frac{\tau(y+\delta_y) - \tau(y)}{\delta_y} = 0 \tag{1.5.13}$$

Note the first term is the pressure gradient in the pipe in the x direction, and the second term is the gradient in the shear stress across the pipe

$$\frac{dP}{dx} = \frac{d\tau}{dy} \tag{1.5.14}$$

This equation, describes how the pressure gradient drives the fluid flow, which is resisted by shear stress in the fluid. This general relationship is true regardless of the geometry or boundary conditions on the fluid, and it is a useful concept to use in thinking about how a fluid will deform.

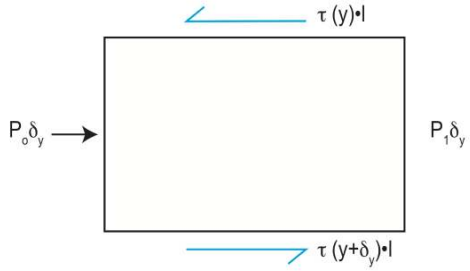


Figure 1.5.7: Conservation of Momentum (MIB: change l to \delta_x)

Finally let's relate shear stress to strain-rate in order to determine the exact form of the fluid flow (velocity) in the pipe. Recall from above that

$$\tau = 2\eta\dot{\epsilon} = \eta\frac{du}{dy} \tag{1.5.15}$$

Substituting in for the shear stress

$$\eta \frac{d^2u}{dy^2} = \frac{dp}{dx} \tag{1.5.16}$$

This is a differential equation, which says that the curvature of the velocity of the fluid in the y direction is equal to the pressure gradient in the x-direction. To determine the shape of the velocity profile, we need to integrate (twice) with respect to y.

$$u(y) = \frac{1}{2\eta}\frac{dp}{dx}y^2 + c_1 y + c_2 \tag{1.5.17}$$

Although we derived this thinking about flow in a pipe, it is the general solution to flow in one direction driven by a pressure gradient. You can see that there are two unknowns in the equation, c_o and c_1, which are the result of integrating twice to get the velocity. To solve for these constants, we must use information we have about how the flow behaves at the boundaries (this information is referred to as the boundary conditions). Different boundary conditions will lead to different shapes for the velocity profiles. Below we discuss two possible solutions referred to as Couette Flow and flow down an inclined plane.

Couette Flow

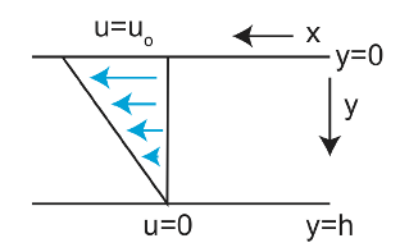


Figure 1.5.8: Couette Flow

Couette flow is defined by a flow in which there is no applied pressure gradient, but one of the boundaries is moving at a fixed velocity.

$\frac{dp}{dx} = 0$ and $u_o \neq 0$

Therefore, in addition to knowing that the term $\frac{dp}{dx} = 0$ we know the velocity at both the top and bottom boundaries:

u(y=h)=0

u(y=0)=u_o

By examining the equation for the u(y) and letting $y = 0$ we see that the constant ($c_2 = u_o$)

Similarly, if we plug in $u = 0$ and $y = h$ we find that

$$u = u_o(1 - \frac{y}{h})$$

This equation gives the shape of the velocity profile, between stationary and moving boundaries, and that profile is linear, as shown in the figure above.

Next, we can find the strain rate in the fluid

$$\dot{\epsilon} = \frac{1}{2}\frac{du}{dy} = -\frac{u_o}{2h}$$

and the shear stress

$$\sigma = 2\eta\dot{\epsilon} = \eta\frac{du}{dy} = -\frac{\eta u_o}{h}$$

The negative sign indicates that the shear strain is directed against the direction of flow, in the -x direction.

Couette flow can be used to approximate the flow in the mantle be dragged by a moving tectonic plate at the surface above. From the figure below, we estimate that the thickness of the shearing layer beneath the plate $h = \approx 100 - 150$ km and the tectonic plate is moving with a velocity of $v_p \approx 5\frac{cm}{yr}$. The solution for the Couette flow can then be used to estimate the strain-rate and the shear stress in the asthenosphere below the plate.

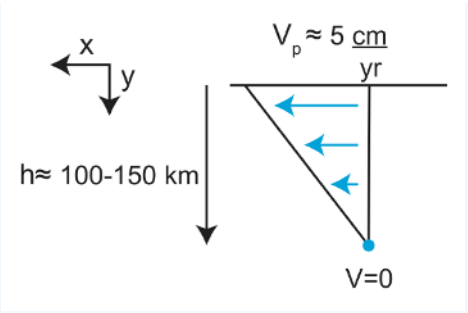


Figure 1.5.8: Flow in the Asthenosphere

Flow Down An Inclined Plane

Now let's consider another geologic application fluid flow down an inclined plane. This geometry can be used to examine flow in a glacial ice stream and flow of magma within channels on the slopes of a volcano.

In this case we consider a layer of flowing fluid with thickness, h, flowing down plane inclined at an angle α. To help solve the problem using the equations we already derived above, we orient the x axis parallel to the inclined plane and the y axis perpendicular with $y = 0$ at the top of the flowing fluid. The boundary conditions are that the shear stress is zero at the top of the fluid (there is nothing but air above the fluid) and the velocity is zero at the bottom of the fluid at $y = h$.

Now, lets think about why the fluid is flowing down the plane. The answer is gravity. While gravity is directed down, there is a component of gravity that is oriented parallel to the inclined surface. This pulls the fluid down the incline due to the weight of the fluid. From the geometry shown below

$$\sin(\alpha) = \frac{g_x}{g}$$

so

$$g_x = g\sin(\alpha).$$

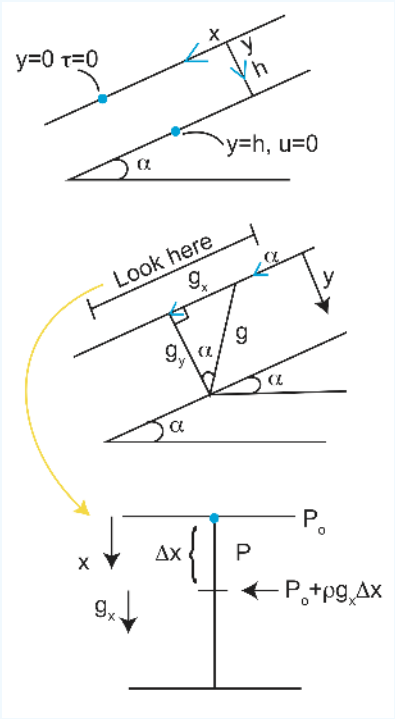


Figure 1.5.9: Flow Down an Inclined Plane (MIB: need to correct middle diagram so that g is directed vertically)

The third panel in the figure below shows how the pressure changes in fluid. At some positions x, the pressure is P_o. At a position Δx down the plane, the pressure is $P_o + pg\sin(\alpha)\Delta x$. Therefore, the pressure difference in the fluid is

$$\Delta P = P_o - (P_o + pg\sin(\alpha)\Delta x$$

or rearranging to get the pressure gradient

$$\frac{\Delta P}{\Delta x} = -\rho g \sin(\alpha)$$

Finally, we get that

$$\frac{dp}{dx} = -\rho g \sin(\alpha)$$

Substitute $\frac{d\tau}{dy} = \frac{dp}{dx}$ into our previous general form of the force balance equation, we can solve to get

$$u = \frac{1}{2\eta}(-\rho g \sin(\alpha))y^2 + c_1 y + c_2.$$

We still need our boundary conditions to solve the equation for the integrations constants.

Using the condition that at y=h, u=0, we find that $c_1 = \frac{\rho g \sin(\alpha)}{2\eta} h$

Using the condition that at y=0, $\tau = \eta \frac{du}{dy} = 0$ we find that

$$u = \frac{\rho g \sin(\alpha)}{2\eta}(h^2 - y^2)$$

Thus, the velocity profile in the inclined plane is a parabola.

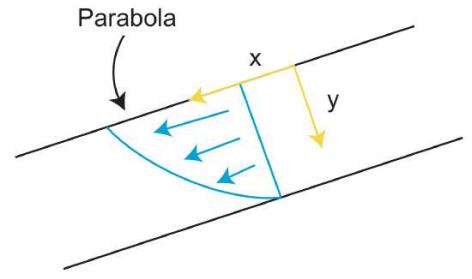


Figure 1.5.10: Parabola

As we can see from the figure, the maximum velocity occurs at the top of the flow, at $y = 0$.

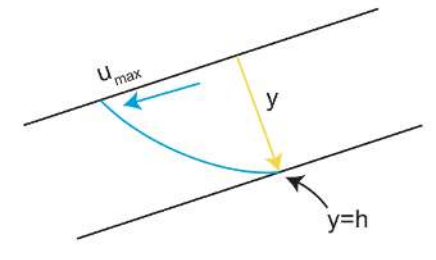


Figure 1.5.11: U_{max}

However, if we use the velocity to determine the strain-rate, we find that

$$\dot{\epsilon} = \frac{1}{2}\frac{du}{dy} = -\frac{\rho g \sin(\alpha)}{2\eta} y$$

Therefore the maximum strain-rate (in terms of magnitude: the sign is not really of importance here) occurs at the bottom of the flow $y = h$.

This illustrates the important point that fast velocity does not necessarily mean high strain-rate. Instead high strain rate occurs where there is a strong spatial gradient in fluid velocity.

This page titled 1.5: Viscous Deformation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

1.6 Inclined Plane Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of how to calculate the velocity and strain rate for flow down an inclined plane. For this example, we'll consider the case of magma flow down an inclined plane. The viscosity of magma primarily depends on temperature, and can range from 10^2 to 10^6 Pa s. For this example we will consider values of 10^2 to 10^4 Pa s. Hill slopes on the side of volcanoes vary depending on the type of volcano and location relative to the (summit versus flanks). Slopes on shield volcanos are typically 5 degrees, while the slopes on stratovolcanos are 5 to 30 degrees. Very close to the vent at the peak of a stratovolcano, the slopes can locally reach up to 70-80 degrees.

Key Questions: Consider these as you work your way through this page.

- What conditions (angle and viscosity) produce the highest flow velocities?
- How do velocity and strain rate change within the flow with depth?
- Where, in the flow profile, do the highest magnitude strain rates occur?
- Where, in the flow profile, do the highest magnitude flow velocities occur?

First, we import the necessary packages and set physical constants. Note: When you first run this box upon loading or refreshing the page it will say "Waiting for kernel...". This is supposed to happen and may take a minute or so.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 g = 9.81 #m/s
5
6 print('Done. Go to next box.')
```

run | run all | restart | restart & run all

Now, let's set some parameters and define functions to calculate the desired quantities, this way it's easier if we want to use the formula multiple times. This function takes 5 input parameters: alpha, rho, eta, y, and h. Alpha is the angle of the inclined plane, rho is the density of the flowing material, eta is the viscosity of the material, and h is the thickness of the flow. y is a range of depths within the flow, starting at 0, and ending at h with an increment defined by step. The density used here is the density of basalt. Then, the function calculates and returns the velocity and strain rate versus depth. The calculations are done using the equations derived in 1.5 Viscous Deformation.

```
h = 50 #m
rho = 3000 # kg / m^3
step = 1
y = np.arange(0,h+step,step) #m

def inclined_plane_velocity(alpha,rho,eta,h,y): #alpha in degrees, rho in kg/m^3, eta in Pa*s, and y an
    v = (rho * g * np.sin(np.radians(alpha)) / (2*eta)) * (h**2 - y**2) # m/s velocity
    return v

def inclined_plane_strain_rate(alpha,rho,eta,y): #rho in kg/m^3, alpha in degrees,  eta in Pa*s, and y
    return -(rho * g * np.sin(np.radians(alpha)) * y / (2*eta)) # (1/s)

print('The parameters and functions are defined, move on to the next cell')
```

run | run all | restart | restart & run all

Throughout this entire exercise, the flow thickness and density will remain constant, and we will consider different angles and viscosities. We need to define the angle and viscosity for this cell, let's vary the angle and hold viscosity constant (eta). This cell uses the functions above to calculate and plot the velocity and strain rate versus depth for each angle in alpha_range. Run the simulation and consider the key questions. You may enter any angle you like between 0 and 80.

```
eta = 1e2 # Pa*s
handle = display(None, display_id=True)

x = np.zeros(len(y))

bbox = dict(boxstyle ="round", fc = '1')
fig, (ax,ax1) = plt.subplots(1,2,figsize=(14,5))
line, = ax.plot(x,y,color='red',label = 'Velocity')
line1, = ax1.plot(x,y,color='blue',label = 'Strain Rate')
ax.set_ylabel('Depth (m)')
ax.set_xlabel('Velocity (km/s)')
ax.set_xlim(0,350)
ax.set_ylim(0,h+step)
ax.invert_yaxis()
ax.invert_xaxis()
ax.set_title('Inclined Plane Flow Velocity versus Depth: Angle Variation')
ax.legend(loc=3)
ax1.set_ylabel('Depth (m)')
ax1.set_xlabel('Strain Rate (1/s)')
ax1.set_title('Inclined Plane Flow Strain Rate versus Depth: Angle Variation')
ax1.set_ylim(0,h+step)
ax1.invert_yaxis()
ax1.set_xlim(-7000,0)

```

run | run all | restart | restart & run all

Now let's hold the angle of the plane constant and vary viscosity. The first lines in the cell take the angle to be 15 degrees, and create a range of viscosities (eta_range). The rest of the cell is set up exactly like the previous one, the only difference is that now

the for-loop is iterating over the viscosities and not angles. Run the simulation and observe the results. Press Enter or Return to change the viscosity in the plot below after you run the cell.

```
alpha = 15 # degrees
eta_range = np.arange(1e2,1.05e4,(1.05e4-1e2)/5) #Pa*s

handle = display(None, display_id=True)

x = np.zeros(len(y))

bbox = dict(boxstyle ="round", fc = '1')
fig, (ax,ax1) = plt.subplots(1,2,figsize=(15,5))
line, = ax.plot(x,y,color='red',label = 'Velocity')
line1, = ax1.plot(x,y,color='blue',label = 'Strain Rate')
ax.set_ylabel('Depth (m)')
ax.set_xlabel('Velocity (km/s)')
ax.set_xlim(0,120)
ax.set_ylim(0,h+step)
ax.invert_yaxis()
ax.invert_xaxis()
ax.set_title('Inclined Plane Flow Velocity versus Depth: Viscosity Variation')
ax.legend(loc=3)
ax1.set_ylabel('Depth (m)')
ax1.set_xlabel('Strain Rate (1/s)')
ax1.set_title('Inclined Plane Flow Strain Rate versus Depth: Viscosity Variation')
ax1.set_ylim(0,h+step)
ax1.invert_yaxis()
```

run | run all | restart | restart & run all

Now let's create a physical model of the flow by rotating the previous velocity graph such that the depth axis is at the angle of the inclined plane. For this example, we will look at three different angles (alpha_range_2) and will hold viscosity constant (eta_2). We will use a rotation matrix to rotate the velocity profile. This involves some basic linear algebra, so you will not be required to come up with something like this on your own for this course, however, pay attention to the output. This cell defines some functions to accomplish this, calls them, and graphs the result. This cell creates normal graphs, not simulations like the above cells.

```
eta_2 = 1e6
alpha_range_2 = np.array([10,40,70]) #degrees
def rotation(alpha,x ,y):
    alpha_rad = np.radians(alpha)
    r = np.matrix([[np.cos(alpha_rad), -np.sin(alpha_rad)], [np.sin(alpha_rad), np.cos(alpha_rad)]])
    Mesh = np.matrix([x,y])
    Rot = r * Mesh
    x_rot = np.array(Rot[0])[0]
    y_rot = np.array(Rot[1])[0]
    return x_rot, y_rot

def physical_model_ip(alpha,rho,eta,h):
    step = 1
    y = np.arange(0,h+step,step)
    Velocities = (rho * g * np.sin(np.radians(alpha)) / (2*eta)) * (h**2 - y**2) # m/s velocity
    vel_rotated, y_rotated = rotation(alpha,Velocities,y)
    q = np.arange(-1,2*np.max(Velocities)) #X-Coordinates for horizontal Lines
    top = np.zeros(len(q)) #Top line
    bottom = np.ones(len(q))*h #Bottom Line
    q_rot_1, top_rot = rotation(alpha,q,top)
    q_rot_2, bot_rot = rotation(alpha,q,bottom)

    # Calculate the aspect ratio of the rotated box so you can rest this to
    # to match an aspect ratio of 1 for the unrotated box.
```

run | run all | restart | restart & run all

Using the figures above, how would you answer the key questions:

- What conditions (angle and viscosity) produce the highest flow velocities?
- How do velocity and strain rate change within the flow with depth?
- Where, in the flow profile, do the highest magnitude strain rates occur?

- Where, in the flow profile, do the highest magnitude flow velocities occur?

1.6 Inclined Plane Jupyter Notebook is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

1.7: Summary

Let's review what we've learned about rheology so far. Rheology describes and defines how a material deforms. To deform a material, stress must be applied, which causes strain. When the stress (σ) placed on a rock is greater than σ_{s-fric} or σ_{s-frac}, the rock will reach its failure point and deform. There are two types of failure a rock can experience, failure by frictional sliding or failure by fracture.

A rock can also deform if it experiences a high degree of stress. The two primary types of deformation are elastic and viscous. Elastic deformation is shallow and has a low magnitude of strain. If the elastic strain is big enough, failure occurs. Viscous deformation occurs deeper and at much higher pressures and temperatures than elastic deformation. Elastic deformation is $\sigma = Ee$, and has a constitutive relation, meaning that it defines rheology. E is Young's modulus which illustrates the relationship between stress and strain in a material. Viscous deformation is $\sigma = 2\mu\dot{\epsilon}$ and is a time derivative ($\frac{d}{dt}$). Two common types of viscous flow are Couette Flow and flow down an inclined plane, as we seen in the asthenosphere. Different materials experience viscous deformation at different rates. There is often a large range in viscosity values for the same material, so it is common to only think about viscous flow in terms of order of magnitude.

When all stress is removed from a material, and there is no net deformation ⇒ this is recoverable deformation.

When stress is removed and the flow stops ⇒ this is non recoverable deformation, but it is still reversible.

Key Terms

Section 1.0 Introduction to Rheology

- rheology
- tectonic forces
- stress
- strain
- strain-rate

Section 1.1 Stress

- normal stress
- shear stress
- principal stresses
- pressure
- lithostatic stress
- overburden
- lithostatic pressure

Section 1.2 Strain and Strain Rate

- strain
- normal or linear strain
- elongation
- shortening
- stretching
- percent strain
- shear strain
- shear angle

Section 1.3 Elastic Deformation

- recoverable
- irrecoverable
- Young's modulus
- Poisson's ratio

- uniaxial
- lateral strain
- longitudinal strain
- Andersonian Theory of Faulting

This page titled 1.7: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

2: Diffusion and Darcy's Law

2.0: Introduction to Diffusion and Darcy's Law
2.1: The Diffusion Equation
2.2: Geological Applications of the Diffusion Equation
2.3: Lithospheric Thickness Jupyter Notebook
2.4: Erosion Profile Jupyter Notebook
2.5: Darcy's Law - Flow in a Porous Medium
2.6: Summary

This page titled 2: Diffusion and Darcy's Law is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

2.0: Introduction to Diffusion and Darcy's Law

Diffusion is a concept many students are first introduced to in physics or chemistry. However, once the partial differential equation aspect of solving the diffusion equation is introduced, diffusion can seem like an intimidating and abstract concept. Simply put, diffusion is one process by which a substance moves from a region of high concentration to a region of low concentration. This is likely already a concept you are familiar with from everyday life, such as when you spray air freshener in one area of a room and then can smell it throughout the entire room, or how helium balloons from a party drop after several days as the helium diffuses out of the balloon.

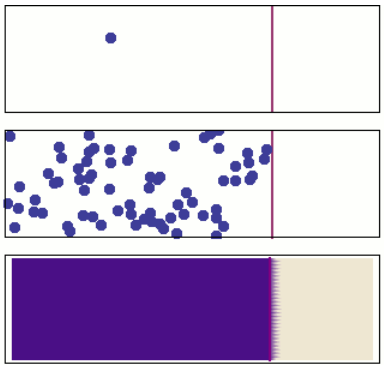


Figure 2.0.1: Basic Gas Diffusion (Public Domain; Sbyrnes321, via Wikimedia)

In this chapter, we will cover the numerous practical uses and applications of the diffusion equation, and break down how exactly to solve the equations, so that they are far more approachable. The concept of diffusion can also be used as a physical model to simulate more complicated process, such as erosion of a fault scarp. Understanding diffusion will give you greater insight into how geologic processes function, such as cooling of a magma dike, the cooling of the lithosphere, diffusion of atoms in/out of grains (e.g., the primary process of rock metamorphism) and erosion of fault scarps. The chapter will also cover Darcy's law, which is based on diffusion and is used to model how a fluid moves through a porous medium.

By the end of this chapter, you should be comfortable with heat flow in one dimension, the origin of the diffusion equation and Darcy's Law, and how to apply these concepts to other processes, such as chemical diffusion and erosion. Additionally, you will gains skills in considering the boundary conditions that are needed to solve the partial differential equation that describes diffusion, specifically for various geologic processes.

Examples of Diffusive Processes in the Geosciences

There are many applications and uses of the diffusion equation in geosciences, from diffusion of an element within a solid at the lattice-scale, to diffusion of heat at a local to regional scale, to diffusion of topography, and diffusion of a chemical species in the crust. Before we dive head first into the diffusion equation, let's look at four examples of how it can be applied. For all of these examples there is a diffusion coefficient (diffusivity) that controls how the parameter of interest (e.g., temperature, concentration) will move through the surroundings.

Heat conduction ($T(\vec{x}, t)$)

A localized temperature perturbation will slowly cool off as it heats up the surrounding region. The temperature profile ($T(x)$) at any time, t, depends on the initial temperature distribution and how quickly heat diffuses into the surrounding material. For this example the thermal diffusivity controls the rate of heat conduction.

- **a. Dike emplacement in the crust**: How long does it take for the heat to diffuse into the surrounding region?

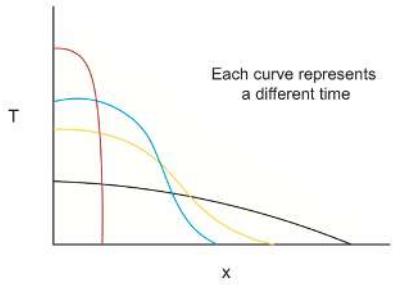


Figure 2.0.1: Temperature profiles showing the diffusion of heat out from an initially warm magma dike intrusion (red curve). The area under the curve is the same at each time (conservation of energy).

- **b. Cooling/aging of the crust/lithosphere**: As the crust and lithosphere cool by thermal conduction the tectonic plate becomes denser and subside. This is the reason that old lithosphere becomes progressively deeper as it moves away from a mid-ocean spreading center. Conduction of heat out the the lithophere is a diffusive process.

Erosion of a fault scarp ($H(\vec{x},t)$)

The sharp topography formed at the surface when a fault ruptures with vertical component of displacement will slowly erode away with time. At first, the cross section will look like a step function. After some amount of time, the sharp corners will fill in or erode away, until slowly the sharp offset becomes a smoothly dipping slope. The shape of this eroding offset with time can be modeled as a diffusive process even though the true physical process is more complex.

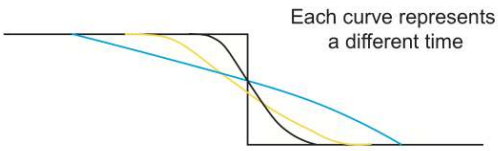


Figure 2.0.2: Sketch of erosion of a fault scarp occurs through a range of physical and chemical weathering and transport process. However, close to the original fault boundary, the net effect of those processes can be modeled as a diffusive process of the height of the fault scarp.

Chemical diffusion ($C(\vec{x},t)$)

Chemical diffusion of elements within a rock lattice structure is also described by a diffusion equation for the concentration of a chemical species. Chemical diffusion is important for understanding radiogenic dating of rocks (when does loss by diffusion stop?), solid-state diffusion (which is the primary process by which rocks metamorphose), diffusion of toxic chemicals from dump sites and diffusion of "chemical plumes" associated with nuclear waste leaks.

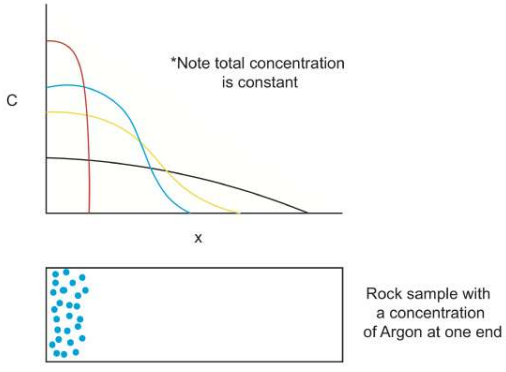


Figure 2.0.3: Sketch of chemical concentration following doping of a mineral sample with argon at one end.

This page titled 2.0: Introduction to Diffusion and Darcy's Law is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

2.1: The Diffusion Equation

We will start the discussion of diffusion by considering the familiar process of thermal conduction.

Thermal Conduction

Diffusion of heat through a crystal lattice occurs through two processes. First, heat can be carried by electrons in the form of kinetic energy in metallic rocks. If a region of the rock is heated at one end, the thermal energy *excites* the electrons, giving them extra energy to move through the rock from regions of high heat to regions of low heat. As the electrons moves, they lose some of this extra energy as heat, thus heating up the region they have moved to. Second, heating a lattice structure adds kinetic energy to the lattice itself, by increasing the vibrational bonds between atoms (think of atoms connected to each other by little springs). The movement of this vibrational energy through the lattice structure is termed a *phonon*. Both modes of passing energy through the lattice structure act to conduct heat from regions of higher temperature to regions of lower temperature, changing the distribution (and perhaps the total amount) of heat within the rock.

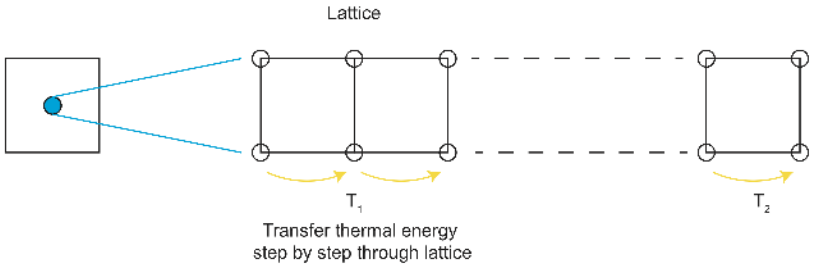


Figure 2.1.4: Heat Diffusion Through a Crystal Lattice

Heat Flow Equation: A 1-D Rod

Start with a cylindrical rod, of length L, and cross-sectional area A, aligned with the z axis. Fix the temperature at $z = 0$ to T_1 and the temperature from $z > 0$ to $z = L$ to T_2, where $T_2 > T_1$. The heat, Q, will move from the high temperature region to the low temperature region, in the -z direction.

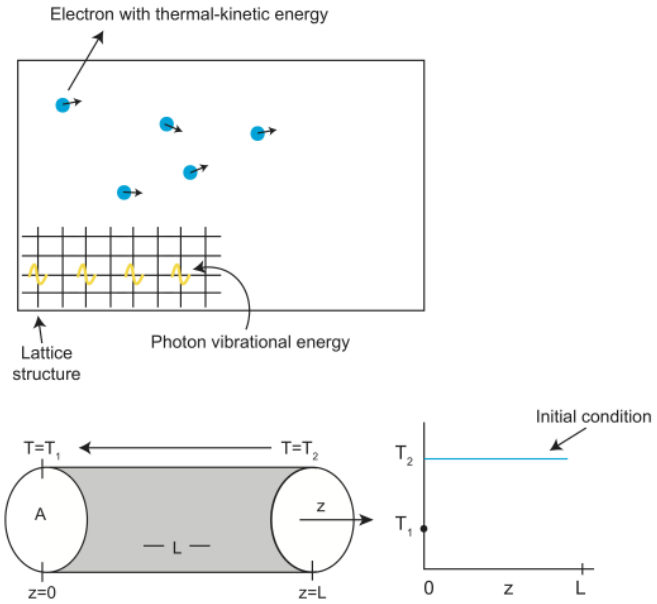


Figure 2.1.5: Top: Modes of transport of heat. Bottom: Heat Flow in a 1D Rod

The change in heat within the rod, ΔQ, is given by:

$$\Delta Q = kA\left(d\frac{T_2 - T_1}{L}\right)\Delta t$$

where k is the **thermal conductivity** with units of $\frac{W}{mK}$(watts per meter Kelvin). Therefore, ΔQ has units of joules (note 1 W=1$\frac{J}{s}$). Rearrange the equation above as,

$$\frac{\Delta Q}{A\Delta t} = k\frac{\Delta T}{L}$$

The left hand side is the change in heat per unit area per unit time, which is a heat flux, given the symbol, q.

$$q = \frac{1}{A}\frac{dQ}{dt}$$

Substituting q into the equation above and then rewriting the other side as a full derivative we arrive at the **Heat Flow Equation:**

$$q = -k\frac{dT}{dz} \tag{2.1.1}$$

Note, that the minus sign shows that the heat moves in the -z direction from high temperature to low temperature.

Diffusion Equation for a 3-D Volume

More generally, we should consider a 3-D volume element (of a rock) and examine how the heat content changes within this volume as a function of time. Consider a cubic volume element with sides of length dx, dy, and dz and volume, $dV = dxdydz$.

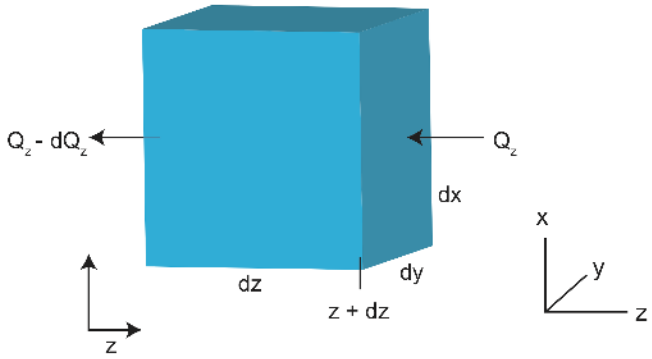


Figure 2.1.6: Heat Flow in 3D

Consider heat flowing along the z axis. The heat that enters the volume at z+dz is Q_z and the heat that leaves the volume at z is $Q_z - dQ_z$. The change in heat within the volume is then:

$$\Delta Q = Q_z - (Q_z - dQz)$$

We an rerewrite as:

$$\Delta Q = Q_z - (Q_z - \left(\frac{dQz}{dz}\right)(-dz))$$

then,

$$\Delta Q = -\left(\frac{dQz}{dz}\right)dz$$

From the heat flow equation (Equation 2.1.1) we also know that

$$q = \frac{1}{A}\frac{dQ}{dt}$$

or

$$dQ = qA\,dt$$

where

$$q = -k\frac{dT}{dz}.$$

Substituting q into the equation for dQ and then subsituting dQ into the equation for Δq we find,

$$\Delta Q = -\frac{d}{dz}(qAdt)dz = -\frac{d}{dz}(-k\frac{dT}{dz})dV\,dt$$

where $dV = Adz = dxdydz$. If the conductivity, k, is independent of position (a constant throughout the rock), then this can be written as:

$$\Delta Q = k(\frac{d^2T}{dz^2})dV\,dt$$

where both negative signs have canceled out so that ΔQ is positive. This equation states that the changes the heat flux in the volume element of the rock are proportional to the **curvature** (second derivative) of the temperature profile. This is interesting, because we usually think of conduction as being faster where the gradient is largest, but its actually faster where the **change in gradient** is the largest.

The total heat within the rock can also be considered from the perspective of the total heat capacity of the rock (how much heat can the rock hold), which depends on the specific heat, c, and the density, ρ, of the rock:

$$\Delta Q = c\rho dV\Delta T$$

Setting the two expressions for ΔQ equal:

$$c\rho dV\Delta T = k(\frac{d^2T}{dz^2})dVdt$$

which can be rearranged as,

$$\frac{dT}{dt} = \frac{k}{c\rho}\frac{d^2T}{dz^2}$$

where dT has been substituted for ΔT. The combination of physical constants, $\frac{k}{c\rho}$ is the **thermal diffusivity** and has units of $(\frac{m^2}{s})$. The thermal diffusivity is given the greek symbol κ (kappa). Substituting in κ we arrive at the **1-D Diffusion Equation**:

$$\frac{dT}{dt} = \kappa\frac{d^2T}{dz^2}$$

Note, this equation relates the change in temperature per unit time to the curvature of the temperature profile in the direction of heat flow.

Full Form of the Diffusion Equation

In general, heat flow can come from any direction, so the temperature will depend on x, y, z, and t. Because $T = T(x, y, z, t)$ and is not just dependent on one variable, it is necessary to rewrite the derivatives in the diffusion equation as partial derivatives:

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

In this case for example, for the partial derivative with respect to time, each of the variables (x, y, z) are treated as constants, and so on for each other variable. Don't let the partial derivatives confuse you, they are just derivatives. In general, you still want to think of this equation as a bunch of small changes in temperature, ΔT, occurring over small intervals of time, Δt, and over small distances (Δz):

$$\frac{\Delta T}{\Delta t} = \kappa\frac{\Delta T}{(\Delta z)^2}$$

This is the "Simplified Diffusion Equation"

Estimating Time/Length-scale for Diffusive Processes

Length-scale of Diffusion

The above simplified version of the diffusion equation can be solved for Δz to give an indication of how far heat can move in a given amount of time:

$$\Delta z \simeq \sqrt{\kappa \Delta t}$$

For the earth, a good average value for the thermal diffusivity is $\kappa = 1x10^{-6} \frac{m^2}{s}$. How far does heat diffuse through a rock in 1 million years?

$$\Delta z = \sqrt{(1x10^{-6})(1x10^{6})(3.15x10^{7})}$$

where there are about 3.15×10^7 sec per year,

$$\Delta z \simeq 5.6\ km \tag{2.1.2}$$

What does this length-scale tell us? It does not tell us the temperature at any particular distance, but instead tells us that a noticeable amount of heat will have travel this distance in the time specified.

Time-scale of Diffusion

Another way of building your intuition for diffusion is to solve the simplified diffusion equation for Δt:

$$\Delta t \simeq \frac{\Delta z^2}{\kappa}$$

We can use this time-scale to consider the question How long does it take for heat magma intrusion to affect the surrounding rock 100 m away?

$$\Delta t = \frac{(100^2)}{(1x10^{-6})(3.15x10^{7})}$$

$$\Delta t = 318\ years. \tag{2.1.3}$$

What does this timescale tell us? It does not tell us the temperature at the distance specified, but instead tells us that a noticeable amount of heat will have reached the distance specified in this time.

It is important to keep in mind that the temperature gradient, which controls the heat flux, changes both spatially and temporally, so heat transport is faster initially, and slower later on. So, the diffusion time-scale and length-scale really just give an order of magnitude estimate of the time-scales and length-scales that will be important in the problem.

Diffusivity and Diffusion in 1, 2, or 3 Dimensions

Note that κ has units of $\frac{m^2}{s}$, which does not depend on temperature or heat flow. Diffusivity, in general, describes the spreading out of some quantity (e.g. temperature, concentration) through an area in some unit of time. Each diffusion processes will have it own diffusivity, which quantifies how the diffuses process occurs for that material. Each diffusivity will have same units, $\frac{m^2}{s}$, no matter which diffusive process is being considered.

If κ is a constant (which it usually is) then diffusion will be isotropic (the same in all directions). However, remember that if there are initial differences in the temperature gradients (due to how the initial temperature disturbance occurred), then the rate of cooling or heating can be different in different directions.

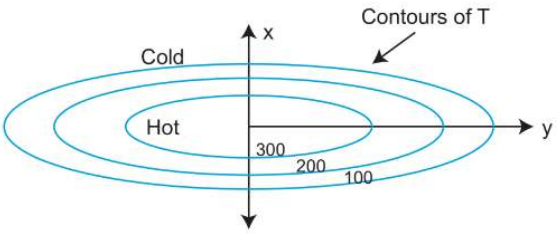


Figure 2.1.1: Heat Diffusion in 2D

The contours on the above figure are analogous to the contours on a topographic map. The contours are spaced closely in the x direction and so the temperature gradient is large there. Thus, the higher the degree of curvature on the graph indicates the direction of maximum cooling or heating.

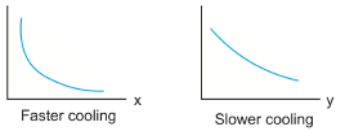


Figure 2.1.2: Cooling Rates

This page titled 2.1: The Diffusion Equation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

2.2: Geological Applications of the Diffusion Equation

In this section we introduce two applications of the diffusion equation to heat transport, followed by applications to erosion of a fault scarp and diffusive transport of a chemical species (atoms or molecules).

For each of the applications we use the 1-D diffusion equation and the solution is found by defining an **initial condition** (the initial temperature, topography, or concentration at every position) and two **boundary conditions** (the value of the temperature, topography, or concentration at two points at either end of the 1-D profile) The boundary conditions can be fixed (constant) in time or change as a function of time. We will go over these conditions in more detail in the next few sections. These initial and boundary conditions, allow us to integrate the differential equations and determine the unknown constants of integration. Because there is one temperature derivative, we need one initial (time) constraint to determine the one constant of integration. Likewise, because there are two derivatives in space, we need two boundary (spatial) conditions to find the two constants of integration.

First, we will go through the a few details related to solving the diffusion equations with these initial and boundary conditions to obtain the temperature profile as a function of time for the specific example of cooling of the oceanic lithosphere. For the other examples, we won't go through all the details, but rather point out the specific differences that arise due to the different boundary conditions.

General Approach to Solving the 1-D Diffusion Equation

We begin with the 1-D diffusion equation for temperature

$$\frac{dT}{dt} = \kappa \frac{d^2T}{dz^2} \tag{2.2.1}$$

where κ is the thermal diffusivity. The solution for this equation is:

$$T(x,t) = (T_s - T_m) erfc(\frac{y}{2\sqrt{\kappa t}}) + T_m \tag{2.2.2}$$

where T_s is the surface temperature, T_m is the mantle temperature, and y is the depth into the lithosphere ($y = 0$ at the surface). This is a commonly used thermal profiles when considering the thermal structure of the oceanic lithosphere through purely conductive processes. It does not take into account reheating due to volcanism, radiogenic heating or mechanical feedback, but it is used to define the initial condition in many calculations or numerical simulations including the thermal structure of the lithosphere. We will use it later on to calculate the ridge push force.

Directly solving the partial differential equations (PDE) is beyond the scope of this text, but understanding that the boundary and initial conditions determine how the solution to a specific diffusion problem will look is valuable for considering each of these physical processes and gaining insight into the geological problem.

$$\frac{dT}{dt} = \kappa \frac{d^2T}{dz^2} \tag{2.2.3}$$

The term on the left side, $\frac{dT}{dt}$ takes the derivative of T in terms of t and holds x constant. Remember from your calculus class, that when we integrate for each derivative there will have a constant of integration. The same is true for solving a PDE. So, from the left side of the equation there will be a constant of integration related to the time derivative. Likewise, on the right-hand side, we have the second derivative of temperature with respect to depth, so there will be two constants of integration. As stated above, the initial condition and the boundary conditions provide the information needed to determine these constants.

- **Initial Condition (IC)**: the starting value for the quantity you are calculating (e.g., temperature) at every point along your profile? This might be a constant or a function of position.
- **Boundary Conditions (BCs)**: the value for the quantity you are calculating e.g., temperature) at each end of your profile (or at infinity depending on the specific problem) for ALL times? These values might be constants or a function of time.

In addition to the IC and BCs, we need to know the values of the physical quantities that show up in our equation. For the diffusion equation, the only constant is the diffusivity, and we need to know this value.

For almost every diffusion equation you come upon, the solution to the equation will include some sort of dependence on the error function, which is defined as,

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta'^2} d\eta' \tag{2.2.4}$$

The parameter η depends on the distance, time, and the diffusion coefficient,

$$\eta = \frac{z}{2\sqrt{\kappa t}}. \tag{2.2.5}$$

Note that if you set your $\eta = 1$, you are back to the simple equation that we used to determine the length-scale and time-scale for diffusion. The value of the error function for any particular value of η can be calculated using a built-in function or looked up in a table. There is also a related function called the **complementary error function**,

$$erfc(\eta) = 1 - erf(\eta) \tag{2.2.6}$$

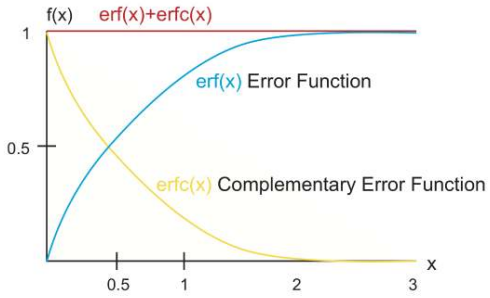


Figure 2.2.3: The error functions, erf(x) and complimentary error function, erfc(x).

Application 1: Cooling of a Lithospheric Plate

Lithosphere forms by cooling of the mantle as the tectonic plate moves away from a spreading ridge. At the ridge, the mantle temperature of T_m almost reaches the surface (z=0). As the material formed at the ridge moves away from the spreading center at a rate equal to the speed of the tectonic plate (u), the mantle cools from above to become the lithosphere. The temperature at the surface of the lithosphere, T_s, is essentially constant because the surface is at the bottom of the ocean ($T = 0^o C$). The age of the plate at a distance x from the spreading center is determined by the speed of the plate: $t = \frac{x}{u}$. A typical value for the thermal diffusion coefficient (thermal diffusivity) is $\kappa_T = 1.0 \times 10^{-6} \frac{m^2}{s}$.

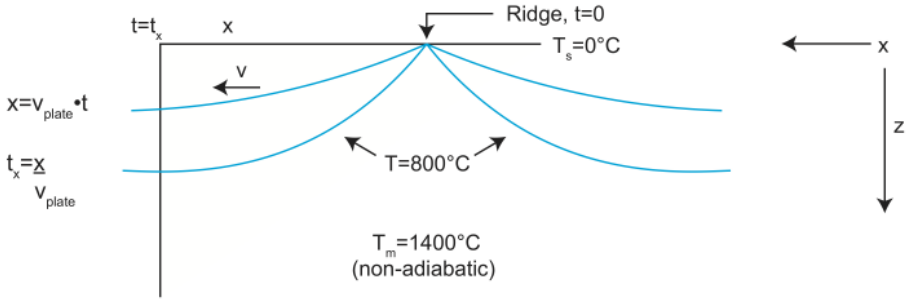


Figure 2.2.4: Oceanic Lithosphere Initial Conditions and Boundary Conditions

What is the initial condition for the temperature?

At time t=0, the temperature is equal to the mantle temperature at all positions in z, $T(z, t = 0) = T_m$. This is even true at the surface. The temperature at the surface changes to zero at the instant of time just after t=0.

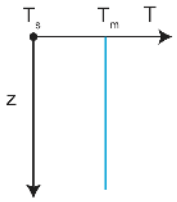


Figure 2.2.2: Initial conditions: $T(z) = T_s$ at $z = 0$ and $T(z) = T_m$ for $z > 0$

What are the boundary conditions for the temperature?

The temperature far away from the surface will stay at the mantle temperature, T_m, and the surface temperature will stay at T_s=0. At z=0, T(0,t)=T_s. As $z \to \infty$, T=T_m. Put in another way:

at t=0 T(z=0)=T_m and, at z=0 T(t>0)=T_s

$t=0_+$ T(z=0)=T_s z=∞ T(t>0)=T_m

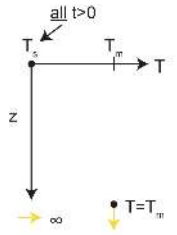


Figure 2.2.2: Boundary conditions: $T = T_s$ at z = 0 for all $t > 0$ and $T = T_m$ at $z = \infty$

What is η in the error function equation?

$$\eta = \frac{x}{2\sqrt{\kappa t}} \tag{2.2.7}$$

Solution

$$T(x,t) = (T_s - T_m)erfc(\frac{x}{2\sqrt{\kappa t}}) + T_m \tag{2.2.8}$$

We can also rewrite this in terms of erf instead of $erfc$ as

$$\begin{aligned} erfc(\eta) &= (1 - erf(\eta)) \\ &= (T_s - T_m)(1 - erf(\eta)) + T_m \\ &= (T_s - T_m) - (T_s - T_m)erf(\eta) + T_m \\ &= T_s - (T_s - T_m)erf(\eta) \\ &= T_s + (T_m - T_s)erf(\eta) \end{aligned}$$

Application 2: Fault Scarp Erosion

The change in shape of a fault scarp profile as a function of time due to erosional processes is often modeled using the diffusion equation. In this case the topography (mass) **diffuses** (erodes) from the uplifted block to the down-dropped block. For the case of an initial vertical offset, as the diffusive erosion progresses the sharp discontinuity in surface topography will become smoother with the slope across the fault with decreasing time.

For this example, we will consider a 1-D topography profile across a fault with total vertical offset $\Delta H(x = 0) = 2a$, and $H_{x_{max}} = a$ and $H_{x_{min}} = -a$. The fault could occur in a region with an overall background slope given by b (units of $\frac{m}{km}$; b=0 is a flat surface). The diffusion coefficient for topography will depend on the type of rock forming the fault and surface processes. Typical diffusion coefficient values are: $\kappa_H = 0.5 - 1.0 \frac{m^2}{kyr}$, which is $\kappa_H = 1.5 - 3.0$ x10^{-11} $\frac{m^2}{s}$.

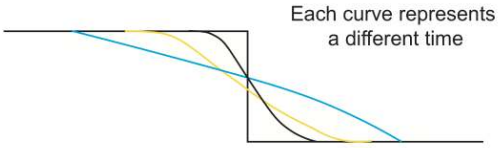


Figure 2.2.2: Sketch of erosion of a fault scarp occurs through a range of physical and chemical weathering and transport process. However, close to the original fault boundary, the net effect of those processes can be modeled as a diffusive process of the height of the fault scarp.

What is the initial condition for topography?

At time t=0, the fault has just formed so the topography on either side of the fault has a specified value. For a flat background surface:

At t=0 and $x \leq 0, H(x) = a$ or $H(x) = a + bx$

At t=0 and $x > 0, H(x) = -a$ or $H(x) = -a - bx$

What are the boundary conditions for the topography?

For the fault scarp diffusion problem, erosion will only modify the topography in the region near the fault. Far away from the fault, the total offset will not change from the initial value:

As $x \to -\infty$, for x≤ 0, $H(x,t) \to a + bx$

As $x \to \infty$, for x> 0, $H(x,t) \to -a - bx$

What is η in the error function equation?

$$\eta = \frac{x}{2\sqrt{\kappa_H t}} \tag{2.2.9}$$

For the case of erosion of a fault scarp with initial height, H, the diffusion equation can be written as:

$$\frac{\partial H}{\partial t} = \kappa_H (\frac{\partial^2 H}{\partial x^2}) \tag{2.2.10}$$

where κ_H is the diffusivity for fault scarp topography.

Solution

Using the initial and boundary conditions the diffusion equation can be solved for the change in shape of the topography as a function of time and position:

$$H(x,t) = erf(\frac{x}{2\sqrt{\kappa_H t}})a + bx \tag{2.2.11}$$

where a is half the total fault offset and b is the background slope of topography.

Application 3: Diffusion of Chemical Species (a Solute in Water)

Contamination of groundwater by toxic spills can be modeled as a diffusive process, where the concentration of a toxic substance diffuses out away from the source due to a chemical gradient. Basically, the toxin wants to move form a region of high concentration to a region of low concentration in order to minimize the total energy of the system (thermodynamics concept).

For this example imagine there is a gasoline station with a leaking subsurface gasoline storage compartment. The rate of leakage is constant, so at the position of the leak there is a constant input value of toxin per volume, or constant concentration, C_s (s-spill). Far away from the leak, the concentration is zero (or in general some background value, C_o, depending on the solute of interest). For a 1-D profile, with the leakage located at x=0, the diffusion equation can be used to determine how the concentration of the toxin changes with time and distance away from the leakage. The diffusion coefficient for toxins will depend on the type of toxin, the medium it is diffusing through (groundwater in this example) and temperature. Typical concentration diffusion coefficients (κ_C), in groundwater at $T = 25^oC$, are: MTBE (Methyl tert-butyl ether, in gasoline) 9.4x10^{-10} $\frac{m^2}{s}$, Benzene (in gasoline) 1.1x10^{-9} $\frac{m^2}{s}$, and Methanol 1.65x10^{-9} $\frac{m^2}{s}$.

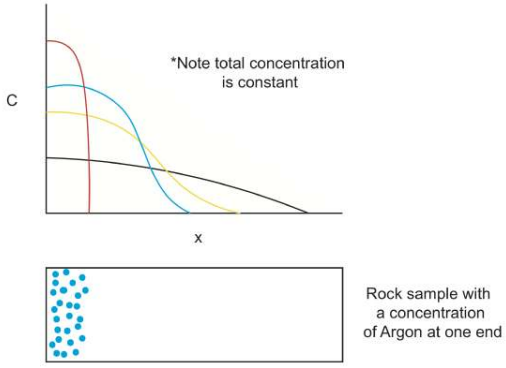


Figure 2.2.3: Sketch of chemical concentration following doping of a mineral sample with argon at one end.

What is the initial condition for the solute concentration?

At time t=0, the leakage starts, so the concentration at x=0 is $C=C_s$. Elsewhere the concentration is equal to the background value. But also note that at all times $C=C_s$ since the spill continues at a constant rate.

What are the boundary conditions for the solute concentration?

Far away from the leakage, the concentration will be unaffected by the leakage: As $x \rightarrow \pm\infty$, $C=C_o$ (this is in fact two boundary conditions at plus and minus infinity). However, in addition, at the location of the spill the concentration stays constant because the leakage continues at a constant rate: $C=C_s$ at x=0 for all times.

What is η in the error function equation?

$$\eta = \frac{x}{2\sqrt{\kappa_C t}} \tag{2.2.12}$$

For the case of diffusion of some chemical element denoted by C, the diffusion equation can be written as:

$$\frac{\partial C}{\partial t} = \kappa_C(\frac{\partial^2 C}{\partial x^2}) \tag{2.2.13}$$

where κ_C is the **diffusivity** of the chemical element and C is the concentration of the chemical element.

Solution

Using the initial and boundary conditions the diffusion equation can be solved for the change in concentration of the toxin in the groundwater as a function of time and position

$$C(x,t) = (C_s - C_o)erfc(\frac{x}{2\sqrt{\kappa_C t}}) + C_o \tag{2.2.14}$$

This page titled 2.2: Geological Applications of the Diffusion Equation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

2.3: Lithospheric Thickness Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of how to calculate and plot the thickness of the lithosphere over time and distance from a spreading center.

Key Questions: Consider these as you work your way through this page.

- Describe how does the thickness of the lithosphere change as it moves away from the ridge in time and distance. Follow one isotherm and observe how it changes. Is the rate of thickening constant? Where/when is it faster or slower?
- Why does the thickness change with distance from the ridge?
- Why is the lithosphere thicker at a distance 1000 km for u = 2 cm/yr compared to u = 5 cm/yr?

First, import the necessary packages:

```
1 import matplotlib.pyplot as plt
2 import numpy as np
3 from scipy.special import erfcinv
4
5 print('The packages have been imported, move on the the next cell.')
```

run | run all | restart | restart & run all

Next, we define parameters and set up the necessary function. kappa is the thermal diffusion coefficient, age is the age of the lithosphere (it is created at the ridge), and Ts is the surface temperature. Tm is the temperature of the mantle, which is taken to be 1400C (ignoring the adiabatic depth dependence of temperature in the mantle). yr2my and sec2yr are conversion factors, and T is the range of isotherms that will be plotted. u_range is the range of spreading rates that we will consider. Then a diffusion profile function is created. This function does some unit conversions and solves for depth given the diffusion coefficient, age, spreading

rate and the surface, mantle, and isotherm temperatures. The equation used in the function is the half-space cooling equation derived in 2.2: Geological Applications of the Diffusion Equation, but solved for depth.

```
kappa = 1e-6 # m^2 / s
age = np.arange(0.1,101,1) # my
Ts = 0 # C
Tm = 1400 # C
yr2my = 1e6   # yr/my
sec2yr = 3600*24*365   # s/yr
T = np.arange(600,1300,200) # C
u_range = np.arange(1,6) #cm/yr

def diff_profile_z(Ts,Tm,kappa,age,T,u):
    u = u / 3.15e7 / 100 #Convert to m/s
    agesec = age*yr2my*sec2yr #Converts age to seconds
    zm = 2*np.sqrt(kappa*agesec)*erfcinv((T-Tm)/(Ts-Tm)) # Calculates Halfspace cooling equation, but s
    zkm = zm/1000 #Convert to km
    return zkm

print('The parameters and function have been defined. Move to the next cell.')
```

run | run all | restart | restart & run all

The following cell plots the thickness of the lithosphere vs time and distance away from the ridge. The for loop changes the spreading rate and updates the graph accordingly in each iteration. u_current is the current spreading rate in the for loop from the array of spreading rates defined above. The first set of code in the for loop calculates the distance traveled given u_current. The ax1.plot commands are dummy lines needed to provide a second x-axis for age to the plot. For each spreading rate in the for loop, the diffusion function is called to calculate the depth for each isotherm (this is done in the lines starting with line.set_ydata-line3.set_ydata). The horizontal distance traveled is also calculated here for each velocity, x_new.

```
handle = display(None, display_id=True)

y= np.zeros(len(age))

bbox = dict(boxstyle ="round", fc = '1')
fig, ax = plt.subplots(figsize = (15,5))
ax1 = ax.twiny()
line, = ax.plot(age,y,color='purple')
line1, = ax.plot(age,y,color='blue')
line2, = ax.plot(age,y,color='green')
line3, = ax.plot(age,y,color='red')
line4, = ax1.plot(age,-np.ones(len(age)))
ax.set_ylim(130,0)
ax.set_xlim(0,5000)
ax.set_xlabel('Distance Traveled Away From Ridge (km)')
ax1.set_xlabel('Age (myr)')
ax.set_ylabel('Depth (km)')
ax.annotate('Ridge $ T_s = 0^\circ C $', (0,0),(500,10),arrowprops = {'arrowstyle':'->','color':'black'
ax.set_title('Lithospheric Thickness versus Distance from the Ridge (Calculated to an age of 100 Myr)')
for u_current in u_range:
    print('Now Plotting:', u_current, 'cm/yr')
    agesec = age*yr2my*sec2yr #Convert age from myr to s
    x_new = agesec * (u_current/ 3.15e7 / 100) #m
    age_new = np.arange(0.5000)*1000 / (u_current/ 3.15e7 / 100) / yr2mv / sec2vr #Mvr
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

- Describe how does the thickness of the lithosphere change as it moves away from the ridge in time and distance. Follow one isotherm and observe how it changes. Is the rate of thickening constant? Where/when is it faster or slower?

- Why does the thickness change with distance from the ridge?
- Why is the lithosphere thicker at a distance 1000 km for u = 2 cm/yr compared to u = 5 cm/yr?

2.3: Lithospheric Thickness Jupyter Notebook is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

2.4: Erosion Profile Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of how to calculate and plot the fault scarp erosion application of diffusion. A fault scarp is an offset in topography created when there is near vertical relative motion on a fault at the surface. Land on one side of the fault ends up higher in elevation, and the other side is lower, creating a step-like appearance. The change in height of the scarp due to erosion over time can be approximated with a diffusion profile, as demonstrated in 2.2: Geological Applications of the Diffusion Equation.

Key Questions: Consider these as you work your way through this page.

- How does the geomorphology (shape) of the fault scarp change over time?
- Why is the fault scarp changing in this way? Be able to explain how this process works.
- What role does the diffusion coefficient play in this process?

First, we import the necessary packages:

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.special import erf

print('Packages are imported, move on to the next cell.')
```

run | run all | restart | restart & run all

Next lets set our parameters, and define a function to calculate the diffusion equation. a is 1/2 of the scarp height at time = 0. b is the overall background slope. x is the horizontal distance from the fault that will be considered. kappa is the diffusion coefficient, and t is a range of times that we will consider. t0 is the initial time. The next few lines define a function that will calculate the

height of the scarp given a, b, kappa, x, and t using the equation derived in 2.2: Geological Applications of the Diffusion Equation. H is the height of the scarp at time = t.

```
a = 0.25 # m
b = 1/1000 # m / km
x = np.arange(-50,51,.5) # m
t = np.logspace(-1,3.5,7) # kyr
kappa = 0.5 # m^2 / kyr
t0 = 0.0001 # kyr, this cannot be 0 because then we would get a 0 in the denominator of the diffusion e

def erosion(a,b,kappa,x,t):
    H = ((a*(erf(x/(2*np.sqrt(kappa * t))))) + (b*x)) # m
    return H

print('The parameters and function are defined, move on to the next cell.')
```

run | run all | restart | restart & run all

The first thing this cell does is call the above function to determine the initial height of the scarp at the initial time, this is assigned to the variable H0. The rest of this cell sets up a graph to show how the height of the scarp changes over time. H0 is plotted each time as a dashed line for reference. H_current is the height of the scarp at time t_current, using the function above. This line is updated each time through the for loop to create the simulation. Once you run the cell, press enter or return to move through the simulation.

```
H0 = erosion(a,b,kappa,x,t0) #m
# Create a named display
handle = display(None, display_id=True)

y= np.zeros(len(x))

bbox = dict(boxstyle ="round", fc = '1')
fig, ax = plt.subplots(figsize = (7,5))
line, = ax.plot(x,y,color='blue',label = 'Current')
ax.set_ylim(-.4,.4)
ax.plot(x,H0,color = 'black',linestyle = 'dashed',label = 'Original')
ax.set_xlabel('Distance from Fault (m)')
ax.set_ylabel('Height (m)')
ax.grid('True')

ax.set_title('Fault Scarp Erosion - Time Variation')
ax.legend()
for t_current in t:
    print('Now Plotting:', round(t_current,1), 'kyr')
    H_current = erosion(a,b,kappa,x,t_current) #m
    line.set_ydata(H_current)
    annotation = ax.annotate('Time = ' + str(round(t_current,1)) + ' kyr',(-40,0.13),bbox=bbox,color =
    handle.update(fig)
    if t_current == t[0]:
```

run | run all | restart | restart & run all

The next cell defines new parameters in order to consider the role of the diffusion coefficient in this process. Instead of initial time, we need an initial diffusion coefficient, kappa_0. t1 is the single instance in time that we will be considering for this simulation. The initial height, H0 must be recalculated using kappa_0 and t1. The rest of the code sets up the graph and runs the simulation. Enter typical diffusion coefficient values between: $0.5 - 1.0 \frac{m^2}{kyr}$.

```
kappa_0 = 0.5 # m^2 / kyr
t1 = 251 #kyr
H0 = erosion(a,b,kappa_0,x,t1) # m
# Create a named display
handle = display(None, display_id=True)

y = np.zeros(len(x))

bbox = dict(boxstyle ="round", fc = '1')
fig, ax = plt.subplots(figsize = (7,5))
line, = ax.plot(x,y,color='blue',label = 'Current')
ax.set_ylim(-.4,.4)
ax.plot(x,H0,color = 'black',linestyle = 'dashed',label = 'Reference: $ \kappa $ = 0.5 $ m^2/kyr $')
ax.set_xlabel('Distance from Fault (m)')
ax.set_ylabel('Height (m)')
ax.grid('True')

ax.set_title('Fault Scarp Erosion - Diffusion Coefficient Variation')
ax.legend()
print('Enter a Diffusion Coefficient between 0.5 and 1 (m^2/kyr)')
kappa_current = float(input())
while kappa_current >= 0.5 and kappa_current <= 1.0:
    print('Now Plotting:', kappa_current, 'm^2/kyr')
```

run | run all | restart | restart & run all

Using the figures above, how would you answer the key questions:

- How does the geomorphology (shape) of the fault scarp change over time?
- Why is the fault scarp changing in this way? Be able to explain how this process works.
- What role does the diffusion coefficient play in this process?

2.4: Erosion Profile Jupyter Notebook is shared under a CC BY-SA license and was authored, remixed, and/or curated by LibreTexts.

2.5: Darcy's Law - Flow in a Porous Medium

Darcy's law is crucial to understanding many branches of geology, especially hydrogeology. Before we look at the law and what it can tell us, let's look at how it was developed. Darcy's law is named after Henry Darcy, a 19th century French engineer who developed an underground pressurized pipe system to deliver water around the city of Dijon. The system, which also provided water to the famous Dijon fountains, revolutionized city water and sewage systems. The system required no pumps and was driven purely by gravity. During the process of developing the new system, Darcy conducted a series of experiments where he tried to move water solely using gravity.

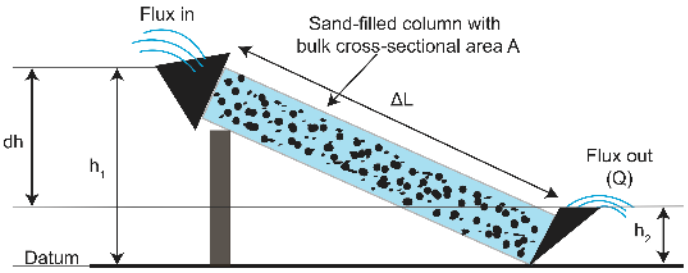


Figure 2.5.1: Model of Darcy's Experiment (MIB: need to correct drawing, ΔL should be L)

From the experiments, collected data on the length of the flume (L), the cross sectional area of the flume (A), the height difference ($h_1 - h_2$), and the flux of water coming out Q both with and without granular (filtering) material in the flume. He then plotted the flux measurement normalized by the area versus the ratio of the height difference to the length of the flume ($\frac{h_1-h_2}{l}$). What he found is that there is a linear relationship, and the slope depends on the granular material he used in the flume:

$$\frac{Q}{A} \propto \frac{h_1 - h_2}{L} \tag{2.5.1}$$

which can be rewritten as a derivative,

$$Q = -KA\frac{dh}{dx} \tag{2.5.2}$$

where K is the hydraulic conductivity and $\frac{dh}{dx}$ is the hydraulic gradient. Q, the total discharge rate, has units of $\frac{m^3}{s}$, the volume of water per time. The negative sign is due to the fact that the fluid flows down (negative) the hydraulic gradient from higher values to lower values.

The **hydraulic conductivity** is a physical parameter that accounts for how easily the fluid can move through the pore space for the material. In Darcy's original experiments, he used sands with different grain sizes, however, the hydraulic conductivity can be determined for solid rocks as well, as most rocks have some pore space.

Following on Darcy's original experiments, further work was done to understand the physical properties of the fluid and rock that determine the effective hydraulic conductivity. From these experiments it was found that,

$$K = k\frac{\gamma}{\eta} \tag{2.5.3}$$

where k is the intrinsic permeability of the porous media (the solid), η is the dynamic viscosity of the fluid, and $\gamma = \rho g$ is the specific weight of the fluid, which depends on the fluid density, ρ and gravity, g.

The highlighted column in the figure is K, the hydraulic conductivity in $\frac{m}{s}$; these are the units that we will be using. The hydraulic conductivity is also sometimes reported in terms of the **intrinsic permeability**, k, with units of darcys or cm^2. The full range of values for hydraulic conductivity is $1\text{-}10^{-13}$ $\frac{m}{s}$. Experiments like Darcy's are used to measure K in real materials.

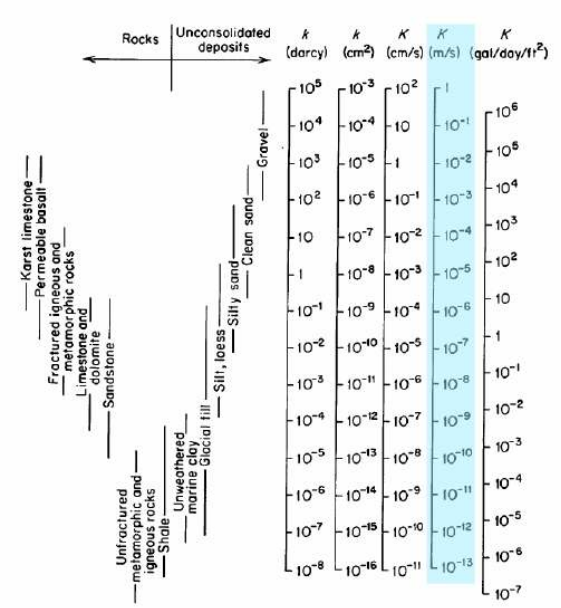


Figure 2.5.3: Hydraulic Conductivity of Various Geologic Materials (CC BY-SA 4.0; Freeze, R. A., & Cherry, J. A, edited by Taryn Lausch, via Wikimedia)

The discharge rate Q is a volume flux (m^3/s), but we usually think of fluid flow in terms of a speed or velocity. The discharge questions can be rearranged by dividing through by the area to define the Darcy Flux

$$q = \frac{Q}{A} = -K\frac{dh}{dx}\ [m/s] \tag{2.5.4}$$

Or in terms of the intrinsic permeability and pressure gradient,

$$q = -\frac{k}{\eta}\frac{dp}{dx} \tag{2.5.5}$$

where η is the fluid viscosity (see below for the derivation of the second equation).

It is important to note that Darcy flux does **not** equal the fluid velocity, even though it is in units of velocity. The fluid velocity is found through the average linear velocity (v_a), which is average of the velocity of all possible fluid paths through the porous media: but finding this value is outside the scope of this class. Instead, we will note that the fluid velocity is

$$v = \frac{q}{\varphi} \tag{2.5.6}$$

where φ is the porosity and v is the fluid velocity. The **porosity**, φ is calculated as ratio of the volume of void space to the the total volume of a material $\varphi = \frac{V_{void}}{V_{tot}}$, and is usually expressed as a fraction between 0 and 1 or as a percent. Note that if $\varphi \leq 1$ then the fluid velocity is greater the darcy flux.

Another term that arises when discussing flow through rocks and unconsolidated materials is permeability. While porosity is the actual fraction of pores (voids) in the rock, these voids can have different shapes and different connectivity, which affects how easily a fluid can move through the pore space. The **permeability** is a measure of the ease with which liquids and gases can pass through a rock. The more consolidated the material, the lower its permeability. Thus "loose" materials like gravel have high permeability. Some rocks can also have anisotropic permeability, meaning that fluids can flow easily in one direction, but not in another. An example of a rock with anisotropic permeability is a shale. Fluids can move easily within a layer of the shale, but can not move across layers.

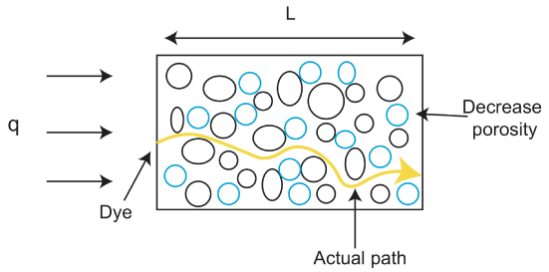


Figure 2.5.4: From the figure we can see that $v = \frac{L}{t}$ in the absence of any granular material. Adding in the pebbles decreases the pore space, and thus decreases the porosity, and each particle of fluid must follow a longer path to traverse the box. With the new longer path created by adding the smaller pebbles, to achieve the same flux, the fluid velocity increases due to the larger L component.

Fluid Velocity

Let's do a basic example.

We are given that $q = 2\frac{m}{min}$ and φ is 25% and want to find the fluid velocity.

We first convert 25% to 0.25.

Then, $v = \frac{2}{0.25}\frac{m}{min}$

and

$v = 8\frac{m}{min}$ is the fluid velocity.

Hydraulic Head

Returning to Darcy's experiments, what is physical process is actually driving the flow of fluid through the flume?

The term $\frac{dh}{dx}$ is referred to as the **hydraulic head**, which is the liquid pressure due to the weight of a fluid relative to some reference location: a pressure gradient. However, $\frac{dh}{dx}$ does not have units of a pressure per unit length, so why is it called the hydraulic head?

Let's consider what the pressure is in the flume in Darcy's experiments. In order to do the experiments, Darcy's had a tank of water at the top end of the flume providing a constant level of water at the top of the flume, equivalent to the height of the flume, h_1. Similarly, there was a tank at the bottom with a constant level of water equivalent to h_2. Therefore the fluid pressure in each of the tanks due to the weight of the water is given by $P = \rho g h$. The pressure gradient from between the top and the bottom of the plume is then

$$= \rho g \frac{dh}{dx} \tag{2.5.7}$$

or

$$\frac{dh}{dx} = \frac{1}{\rho g}\frac{dP}{dx} \tag{2.5.8}$$

Substituting into Darcy's Law,

$$q = -K\frac{dh}{dx} \tag{2.5.9}$$

$$= \frac{K}{\rho g}\frac{dp}{dx} \tag{2.5.10}$$

Recall, from above that the hydraulic conductivity can be written in terms of the intrinsic permeability as $K = k\frac{\gamma}{\eta}$ which is equivalent to $k\frac{\rho g}{\eta}$ or rearranging, $\frac{K}{\rho g} = \frac{k}{\eta}$. Therefore, we can substitute to give two equivalent expressions for Darcy's Law, with one interms of the gradient in height,

$$q = -K\frac{dh}{dx} \tag{2.5.11}$$

and the second in terms of the gradient in pressure,

$$q = \frac{-k}{\eta}\frac{dp}{dx} \tag{2.5.12}$$

Thus, $\frac{dh}{dx}$ is called the hydraulic head because the physical origin of the term in Darcy's Law is the pressure gradient in the fluid, which drives the flow.

Water leaving an Aquifer

Let's now do an example calculating the fluid velocity of water entering an aquifer. In the sketch below the aquifer height follows the topography on the hill adjacent to the valley with the stream. Water flows from high pressure (head height) toward valley and leaves the aquifer at the stream. We can use Darcy's Law to estimate the rate of water flow form the aquifer.

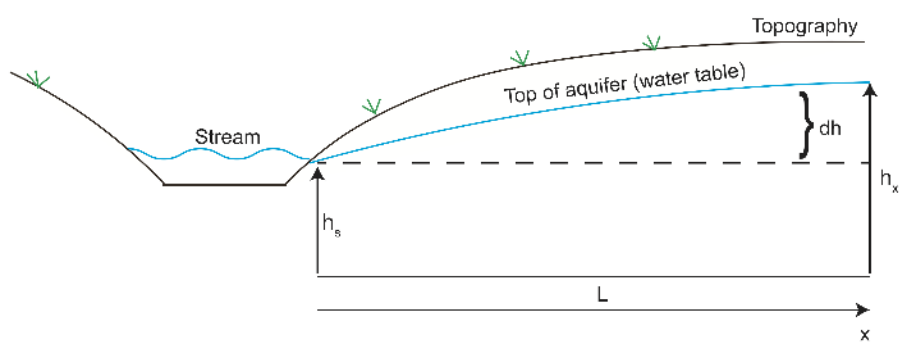


Figure 2.5.6: Water Entering an Aquifer

Using geophysical imaging, the height of the aquifer is found to be 10 m (dh) above the stream height about 100 meters (L) away. Assume the hydraulic conductivity is $K = 10^{-6}$ m/s and the porosity is $\varphi = 30\%$.

The darcy flux is

$$q = K\frac{dh}{L} \tag{2.5.13}$$

$$q = 10^{-6}(\frac{10}{100}) \tag{2.5.14}$$

$$q = 10^{-7}\,\frac{m}{s} \tag{2.5.15}$$

The fluid velocity is

$$v = \frac{q}{\varphi} \tag{2.5.16}$$

$$v = \frac{10^{-7}}{0.3}\,\mathrm{m/s} \tag{2.5.17}$$

$$v = 3 \times 10^{-8}\,\mathrm{m/s}(3.15 \times 10^{7})\mathrm{s/yr} \tag{2.5.18}$$

$$v \approx 0.9\,m/yr \tag{2.5.19}$$



Figure 2.5.7: Laminar to Turbulent Flow of a Plume from a Candle Flame (CC BY-SA 3.0; Gary Settles, via Wikimedia)

Laminar versus Turbulent Flow

One of the implicit assumptions in Darcy's law is that the flow is laminar. Laminar flow is characterized as consisting of thin layers or laminae of fluid all moving parallel to one another. In contrast turbulent flow is characterized by swirls, vortices and eddies, in which adjacent layers of fluid diverge and take different paths down stream.

One way that we can quantify whether a fluid is laminar or turbulent is by calculating the Reynold's number for the flow, defined as

$$Re = \frac{\rho u_o L}{\eta} \text{ Reynold's number} \tag{2.5.20}$$

The Reynold's number is found by considering the two terms in the momentum balance for the flow. The first term is the inertial force per unit volume, which drives the flow, $\rho u \frac{du}{dx}$. The second term is the viscous force per unit volume, which resists flow, $\eta \frac{d^2u}{dx^2}$. These two terms must balance, so we can set them equal,

$$\rho u \frac{du}{dx} = \eta \frac{d^2u}{dx^2} \tag{2.5.21}$$

Next, we rewrite the terms, in a simplified way to capture only the relevant dimensions of the variables. To do, we go through what is called a dimensional analysis, in which we replace each variable, with a reference values, times a non-dimensional form of the variable.

For example for the term x is rewritten as just Lx', where L is the reference length (the fluid depth) and x' is the non-dimensional distance. Similarly, we let $u = u_o u'$. Substituting these into the equation, we get

$$\rho u_o \frac{u_o}{L} \frac{du'}{dx'} = \eta \frac{u_o}{L^2} \frac{d^2u'}{dx'^2} \tag{2.5.22}$$

Next we combine the constant values out in front

$$\frac{\rho u_o L}{\eta} \frac{du}{dx} = \frac{d^2u}{dx^2} \tag{2.5.23}$$

Note that the combination of variables in front is the Reynold's number

$$Re = \frac{\rho u_o L}{\eta} \tag{2.5.24}$$

where the variables are u_o flow velocity, L fluid depth, and the variables we have seen earlier, ρ fluid density, and η fluid viscosity. By going through this analysis, we find that any flow that is characterized by the balance of these two terms can by compared by simply knowing the value of the Reynolds' number. For example, if one flow has a viscosity that is 10 times the viscosity in another flow, the two flows will behave the same is the density (or velocity or length) is also increased by a factor of 10 for the higher viscosity case; that is, the Reynolds number for the two flows is the same.

To have Darcy flow, the Reynold's number must be less than about 1-10 (laminar). For $Re > 1 - 10$ the flow is not strictly laminar, but it is not yet turbulent (non-linear, not-laminar). To have turbulent flow, the Reynold's number must be greater than about 2000.

Understanding that the constants in the numerator and denominator affect the characteristics of the flow in the opposite way, also help to think about how different fluids will behave under similar external conditions. For example, a denser fluid that also has a fast velocity will have a higher Re and exhibit turbulent behavior. In contrast, a fluid such as ice that has a high viscosity and moves very slowly will have a low Re and will likely have laminar flow. The high viscosity of the ice acts to stabilize the flow. As another example, consider two fluids could both be moving at the same velocity behave differently if the other values in the Reynold's number $\frac{\rho L}{\mu}$ are different. Therefore, the flow type depends not just on the flow velocity, but on the combination of variables in the Reynold's number.

Application of Darcy's Flow

Darcy's law is the main equation that governs flow in aquifers and wells. Without it, we would not be able to drill wells or understand water movement in aquifers, especially in California's central valley, where both these ideas are crucial to both agricultural productivity and groundwater subsidence.



Figure 2.5.8: Typical Aquifer Cross-Section (Public Domain; Hans Hillewaert, via Wikimedia)

This page titled 2.5: Darcy's Law - Flow in a Porous Medium is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

2.6: Summary

Let's review what we've learned about diffusion and Darcy's law so far.

The Diffusion Equation

Diffusion is how a substance moves from a region of high concentration to a region of low concentration. It can be described using the diffusion equation

$$\frac{dT}{dt} = \kappa \frac{d^2T}{dz^2} \tag{2.6.1}$$

To solve this PDE so that we can apply it to real problems in geology, we take several steps.

1. We first determine what is the initial condition for the variable we are given?
2. What are the boundary conditions for the variable?
3. What is η in the error function equation?
4. Use the initial conditions and boundary conditions to determine the constants of integration in the general solution

Geological Applications of the Diffusion Equation and Darcy's Law

We applied these PDE solving skills to several problems, the Cooling of a Lithospheric Plate, Fault Scarp Erosion, and Diffusion of a Chemical Species.

We also looked at Darcy's law and how it applies to flow in closed spaces such as pipes and groundwater. Darcy's law can be written as

$$q = -K\frac{dh}{dx} \quad or \quad q = \frac{-k}{\mu}\frac{dp}{dx} \tag{2.6.2}$$

where q is Darcy flux, k is the permeability in m^2, K is the hydraulic conductivity, μ is the fluid viscosity in Pa·s, and $\frac{dh}{dx}$ is the hydraulic gradient. It describes how a fluid flows in a porous medium.

On a smaller scale, fluid flow can either be laminar or turbulent. This depends on the fluid's Reynold's number (Re).

$$Re = \frac{\rho u_o L}{\mu} \quad Reynold's\ number \tag{2.6.3}$$

where u_o is flow velocity and L is fluid depth.

To have Darcy flow, the Re<1-10 (laminar) or Re>1-10 (non-linear/not-laminar). An Re>1-10 may not be laminar flow, but it is not turbulent either. To have turbulent flow, the Re>2000.

This page titled 2.6: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

3: Planetary Geophysics

3.0: Introduction
3.1: Orbital Mechanics
3.2: Layered Structure of a Planet
3.3: Two Layer Planet Structure Jupyter Notebook
3.4: Isostasy
3.5: Isostasy Jupyter Notebook
3.5: Observing the Gravity Field
3.7: Gravitational Potential, Mass Anomalies and the Geoid
3.8: Summary

This page titled 3: Planetary Geophysics is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.0: Introduction

Planetary geophysics is the study of the internal structure of the objects in our solar system and other solar systems, including planets, moons, and asteroids. It does not include the structure of the star (sun), but these studies do sometimes get into questions of planetary formation which occurs at the same time star/sun forms. Several remote observations of planets and moons can be used to determine their internal structure.

In this chapter we will first focus on two calculations used to determine the basic structure of a planet (mass and core/mantle layering from the moment of inertia) using observations of the that can be made remotely (orbital properties, spin and precession). Since orbiters are needed for the some of these observations, we will also consider the question of how long it takes to send a space craft from one planet to another (the Hohman Transfer Orbit).

Next, we will explore the gravitational field of the Earth, how it is related to plate tectonics and mantle convection and how we measure the variations in gravity at the surface of the Earth. The image below shows the "anomalies" in the gravitational field. If the Earth was only made up of concentric layers, there would not be any anomalies. Therefore, these anomalies give us information about lateral variations in density and the viscosity inside the mantle. We will finish up by considering the concept of isostasy and how it relates to the gravitational field.

Figure 3.0.1: The Geoid (Public Domain; NASA, via Wikimedia)

Characterizing Planetary Structure

Before we can consider whether another planet (or moon) might have plate tectonics or mantle convection, it is necessary to gather basic information about the structure of the planet. This information can be organized in to different levels of knowledge. First order information is the most basic and most fundamental. Second order information adds details or refinement to the first order information, and so on.

In terms of planetary structure, we can divide up the information into three levels (see figure below):

- **1st Order**: Mass and radius: from the radius we can determined the volume, and then together with mass determined the bulk/average density $\rho = \frac{m}{v}$
- **2nd Order**: Layers (stratification): what is the radius of the mantle R_m and core R_c? This requires information about the moment of inertia and possible densities of the mantle and core $\rho_m, \rho_c \rightarrow \rho_{bulk}$
- **3rd Order**: Internal (non-layered) density anomalies. If there is mantle convection, we expect regions of dense (ΔP, cold, and/or composition) material and buoyant (-ΔP, hot, and/or composition) material. These density variations can be detected as perturbations (anomalies) in the gravity field.

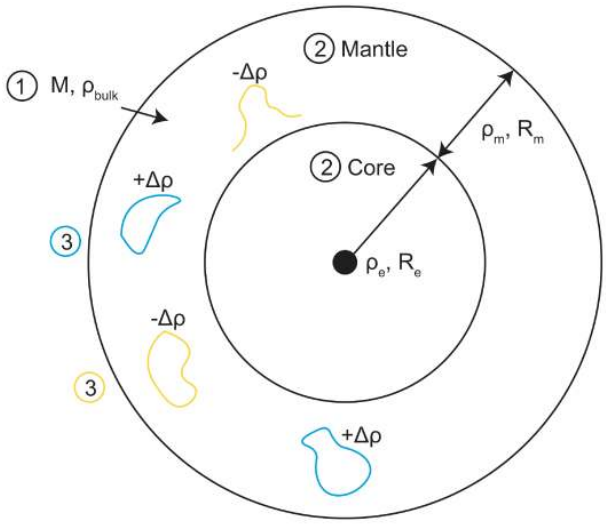


Figure 3.0.1: Planetary Structure

This page titled 3.0: Introduction is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

https://geo.libretexts.org/@go/page/4703

3.1: Orbital Mechanics

Orbital mechanics is a branch of planetary physics that uses observations and theories to examine the Earth's elliptical orbit, its tilt, and how it spins. Observations of the orbital behavior of planets, moons or satellites (orbiters) can provide information about the planet being orbited through an understanding of how these orbital properties are related to gravitational forces.

Humans have been studying orbital mechanics since 1543, when Copernicus discovered that planets, including the Earth, orbit the sun, and that planets with a larger orbital radius around their star have a longer period and thus a slower velocity. While these may seem straightforward to us today, at the time these were radical ideas. Johannes Kepler elaborated on Copernicus' ideas in the early 1600's, stating that orbits follow elliptical paths, and that orbits sweep out equal area in equal time (Figure 3.1.1). The same (blue) area is swept out in a fixed time period. The green arrow is velocity. The purple arrow directed towards the Sun is the acceleration. The other two purple arrows are acceleration components parallel (tangent to the orbit) and perpendicular to the velocity.

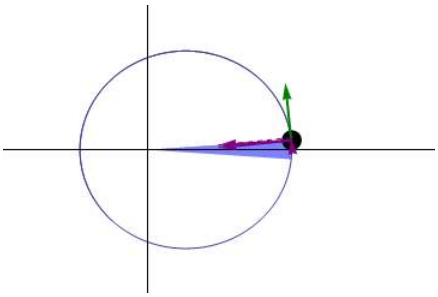


Figure 3.1.1: Kepler's Second Law for a Planetary Orbit (CC BY-SA 3.0; Gonfer, via Wikimedia)

Following on this observations Kepler also observed the orbital periods and orbital radius for several planets. For each planet he considered various relationships between these two parameters to determine how they were related. He determined that there is a constant relationship for all the planets orbiting the sun

$$T^2 \propto R^3 \tag{3.1.1}$$

That is, for each planet orbiting another (much larger) object (the Sun), the square of the orbital period is proportional to the cube of the orbital radius. In addition, he found that the constant of proportionality was the same for all the planets orbiting the sun. This relationship is true for any set of smaller objects (planets) orbiting a (much) larger object, which is why this is now known as Kepler's Third Law:

$$\frac{T^2}{R^3} = Constant \tag{3.1.2}$$

Below we will see that this constant is related to Newton's Law of Universal Gravitation, and therefore can also give us information about the mass of the object being orbited. But first, let's see how one can use Kepler's third law to for two applications.

Orbital Period or Radius of a Satellite or other Object

Consider two planets (1 and 2) orbiting the sun. If the proportionality above it true for each planet, then we can set the fractions equal to each other, and rearrange to find,

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3} \tag{3.1.3}$$

Why would we do this? Well, suppose we want to launch a satellite into outer space that will orbit the Earth at a specified orbital radius, R_s. For example, the best height for taking Google Earth imagery is about 6 times the Earth's radius, R_e. We can use Kepler's Third Law to determine the orbital period, T_s of the satellite.

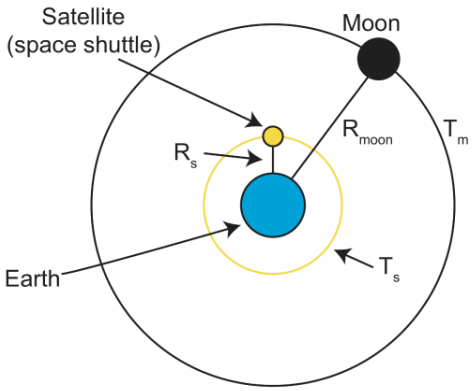


Figure 3.1.2: Application of Kepler's Third Law to determine the orbital radius of a satellite given a desired orbital period. Note that orbital radius is measured from the center of the Earth.

We are know the orbital period of the moon is $T_m = 27.3217$ days and the orbital radius of the moon is $R_m = 60 \times R_e$ where R_e is the radius of the Earth. Substituting

$$\left(\frac{T_s}{T_m}\right)^2 = \left(\frac{R_s}{R_m}\right)^3$$

$$T_s^2 = T_m^2 \left(\frac{R_s}{R_m}\right)^3$$

$$T_s = T_m \left(\frac{R_s}{R_m}\right)^{\frac{3}{2}}$$

$$= 27.3217 \left(\frac{6R_e}{60R_e}\right)^{\frac{3}{2}}$$

$$= 27.3217 \left(\frac{1}{10}\right)^{\frac{3}{2}}$$

$$= 27.3217\,(0.0317)$$

$$= 0.86\ days$$

So, the orbital period is about 1 day (with more precise numbers, you will find it is exactly one day... a **geosynchonous orbit**). You could also start with T_s and determine the orbital radius. A note about units: you should use what units make sense as long as they are consistent, ie., they are the same for both of the orbital periods and both orbital radii, so they cancel out.

Many geological and geophysical observations are made with orbiting satellites, including missions that measure Earth's gravity field, topography, changes in topography related to earthquakes and volcanoes (and other things), and the magnetic field. Other satellites monitor ice mass, vegetation, and all sorts of chemical signatures in the atmosphere. Planetary scientists also send orbiters to other planets to make similar measurements (okay not vegetation).

Kepler's Third Law can also be used to study distant solar systems. Using a telescope, one can detect other planets around stars by observing a drop in the brightness of the star as the planet transits between the star and the telescope. By observing the time between transits, we know the orbital period. Kepler's Third law can be used to determine the orbital radius of the planet if the mass of the orbiting star is known ($R^3 = T^2 - M_{star}/M_{sun}$, the radius is in AU and the period is in earth years).

The Fastest Path from one Planet to Another

Kepler's 3rd law can also be used to determine the fast path (orbit) from one planet to another. This fastest path is called a Hohmann transfer orbit, named for the german scientist Walter Hohmann who first published the orbit in 1952 (see more in this article). In the above discussion of Kepler's Law we referred to R as the orbital radius. In fact, because almost no planet, satellite, or moon is actually on a perfectly circular orbit R is the semi-major axis of the elliptical path of the orbiting object. For the Hohmann Transfer orbit, we need to be more explicit about treating the orbits as elliptical.

Take for example Mars orbiting the Sun. The Sun is not located at the center of the ellipse, but slightly to one side (at one of the two foci of the ellipse). Mars is closest to the Sun at Perihelion and farthest away at Aphelion.

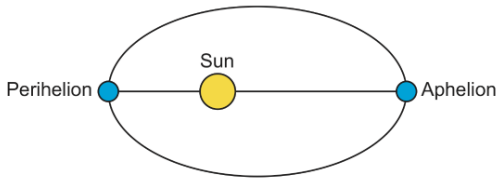


Figure 3.1.4: Perihelion and Aphelion. (Mars is closest to the sun in its orbit at perihelion and farthest at aphelion).

Now, let's consider the fastest path from Earth to Mars using Kepler's Third Law.

Hohmann Transfer Orbits

A transfer orbit is an intermediate elliptical orbit that is used to move a satellite or other object from one circular, or largely circular, orbit to another. Using Figure 3.1.3, we will calculate how long it would take to reach Mars in the most efficient orbit. Note from the figure, that the when Earth is at Perihelion and Mars is a Aphelion, the path connecting the two planets is an ellipse. This path is the Hohmann Transfer Orbit and is the shortest (in time) path between the two planets. Note that when the satellite leaves the Earth, Mars will not yet be at Perihelion, so the timing is important. The other important thing to note, is that it is not very often that the orbits line up exactly such that a Hohmann transfer orbit is possible. However, knowing that it is the fastest path places clear limits on missions to Mars (and similarly missions to other planets) including sending manned missions.

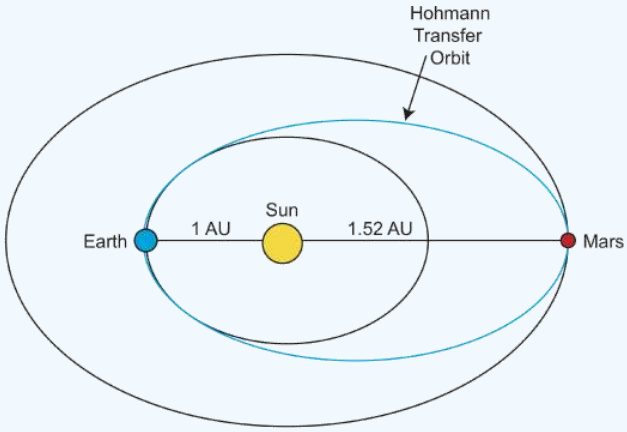


Figure 3.1.3: Hohmann Transfer Orbit

Solution

We know that the path is an elliptical orbit around the sun, and it grazes the orbit of Mars at aphelion.

For Hohmann Transfer orbit, the semi-major axis of the elliptical orbit is R_n and is the average of the Earth's distance from the sun (at Perihelion), R_e and the distance of Mars from the sun (at Aphelion), R_m,

$$\begin{aligned} R_n &= \frac{1}{2}(R_e + R_m) \\ &= \frac{1}{2}(1 + 1.524) \\ &= 1.262\ AU \end{aligned}$$

Next, noting that both the Earth and the object traveling on the Hohmann Transfer Orbit are both orbiting the sun, we use this Kepler's Law to determine the period of the object on the Hohmann Transfer orbit,

$$\left(\frac{T_n}{T_e}\right)^2 = \left(\frac{R_n}{R_e}\right)^3$$

$R_e = 1$ AU and $T_e = 1$ earth-year

$$(T_n)^2 = (R_n)^3$$
$$(T_n)^2 = (1.262)^3$$
$$(T_n)^2 = 2.0099$$
$$T_n = 1.412\ years$$

This is the full orbit time, but a a transfer takes only a half orbit (1.412/2 = 0.7088 year).

Therefore the shortest orbital path to Mars from Earth takes about 8 months.

Both the examples above illustrate the way that Kepler's Third Law can be used determine orbital information about planets, moons or satellites. The next step is to connect Kepler's 3rd law to the object being orbited. That it, we want to know the constant of proportionality between the T^2 and R^3. What is the physical meaning of this constant and what does it depend on?

Determining the Mass of a Planet

In the late 1600s, Newton laid the groundwork for this idea with his three laws of motion and the law of universal gravitation. As you were likely told in elementary school, legend states that while attempting to escape an outbreak of the bubonic plague, Newton retreated to the countryside, sat in an orchard, and was hit on the head with an apple. This lead him to develop his ideas on gravity, and equate that when an apple falls or planets orbit, the same physics apply.

Newton's Law of Gravity

Newton, building on other people's observations, showed that the force between two objects is proportional to the product of their masses and decreases with the square of the distance:

$$F = \frac{GMm}{r^2}$$

where $G = 6.67 \times 10^{-11}$ m^{3}kg s^2 is the gravitational constant.

This gravitational force acts along a line extending from the center of one mass to the center of the second mass.

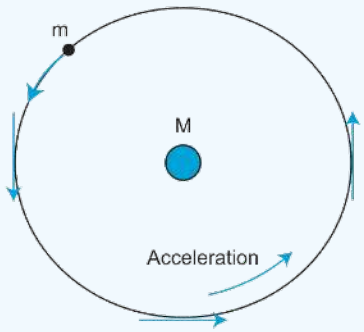


Figure 3.1.5: Centripetal Force

For an object orbiting another object, Newton also observed that the orbiting object must be experiencing an acceleration because the velocity of the object is constantly changing (change direction, not speed, but this is still an acceleration). Newton's second Law states that without such an acceleration the object would simple continue in a straight line. Since the object is experiencing an acceleration, then there must also be a force on the object. This is force is called the Centripetal force and is proportional to the velocity of the orbiting object, but decreases proportional to the distance.

$$F = \frac{mv^2}{r}$$

The last step is to recognize that the acceleration of the orbiting object is due to gravity. Therefore we can set these two forces equal,

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

Or, solving for the velocity of the orbiting object,

$$v^2 = \frac{GM}{r}$$

Next, the velocity of the orbiting object can be related to its radius and period, by recognizing that the distance = velocity x time, where the distance is the length of the circular path and time is the period of the orbit, so

$$v = \frac{d}{t} = \frac{2\pi r}{T}$$

where 2πr is the circumference and T is the orbital period.

Substituting for v, we find

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \tag{3.1.4}$$

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{r} \tag{3.1.5}$$

And finally we solve for T^2,

$$T^2 = \frac{4\pi^2}{GM} r^3$$

We can rearrange this equation to find the constant of proportionality constant for Kepler's Third law,

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM} \tag{3.1.6}$$

The constant of proportionality depends on the mass, M of the object being orbited and the gravitational constant, G.

Using 3.1.6, we can determine the constant of proportionality for objects orbiting our sun as a check of Kepler's third Law.

Planets Orbiting Sun

The mass of the sun is

$M = M_{sun} = 1.9891 \times 10^{30}$ kg

We rewrite Kepler's 3rd Law as

$$T^2 = KR^3 \tag{3.1.7}$$

where K is a constant of proportionality. Now, we calculate K,

$$K = \frac{4\pi^2}{GM}$$

$$= 2.97 \times 10^{-19} \frac{s^2}{m^3}$$

For any object orbiting the sun, $T^2/R^3 = 2.97 \times 10^{-19}$

Also note, that if R is in AU (astonomical units, 1 AU=1.49x10^{11} m) and T is in earth-years, then

$$K = 0.9916 \approx 1$$

Now knowing this proportionality constant is related to the mass of the object being orbited, gives us the means to determine the mass this object by observing the orbiting objects. Solving Equation 3.1.6 for mass, we find

$$M = \frac{4\pi^2}{G} \frac{R^3}{T^2} \tag{3.1.8}$$

By observing the orbital period and orbital radius of small objects orbiting larger objects, we can determine the mass of the larger objects.

This is the how planetary scientists determined the mass of Earth, the mass of other planets in our solar system that have moons, the mass of the moon using an orbiter, and the mass of other stars when orbiting planets can be observed.

Knowing the mass of a planet is the most fundamental geophysical observation of that planet, and with other observations it can be used to determine the whether another planet has a core, and relative size of the core and mantle. With this information, model of the planets can be made to determine if they might be convecting like Earth, and if they might have plate tectonics.

Mass of the Earth

What is the mass of the Earth, M_e?

Solution

This is a direct application of Equation 3.1.8.

$$M_e = \frac{4\pi^2}{G}\left(\frac{R_{moon}^3}{T_{moon}^2}\right)$$

with $R_{moon} = 384 \times 10^6\ m$ and $T_{moon} = 27.3\, days = 2358720\, sec$.

$$M_e = 6.02 \times 10^{24} kg \tag{3.1.9}$$

This is quite close to the accepted value for the mass of the Earth, which is $5.98 \times 10^{24} kg$.

See the NASA Planetary Fact Sheet, for fundamental planetary data for all the planets, and some moons in our solar system.

This page titled 3.1: Orbital Mechanics is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.2: Layered Structure of a Planet

Once the mass of a planet or moon is known, the next thing a planetary scientist want to know is what is the layered structure inside the planetary object. Does the object have a dense iron core? Or, is the object homogeneous? How thick is observed crustal (or ice) layer that is observed at the surface. This information is more difficult to determine, but with just one more piece of information and some educated assumptions about the plausible composition and density), it is possible to determine the basic layered structure. The additional piece of necessary information is the moment of inertia. Below, we will describe one way that the moment of inertia can be determined by using an orbiter to detect measure the shape of the gravitational field around the planetary object. First, let's review moment of inertia.

Moment of Inertia

A moment of inertia quantifies the distribution of mass of an object relative to an axis. Consider the irregular shaped object in the figure below, with an axis P. The mass of the object can be considered as many individual point masses, m_i, each located a distance r_i from the axis. Note this distance, r_i is measured along a line that is perpendicular to the axis located at the height of the point mass. The moment of inertia of axis P is given as the sum over all the point masses,

$$I_P = \sum_{i=1}^{N} m_i {r_i}^2 \tag{3.2.1}$$

This sum gives a scalar value. Also, note that if a different axis is used, the moment of inertia will be different.

For most commonly-used shapes this sum can be calculated as an explicit integral and there are analytic expressions that depend on the total mass of the object. For example the moment of inertia of a solid disk about an axis through the center is $0.5MR^2$, whereas the moment of inertia for a sphere is $0.4MR^2$. More examples of moment of inertia for different shapes can be found here.

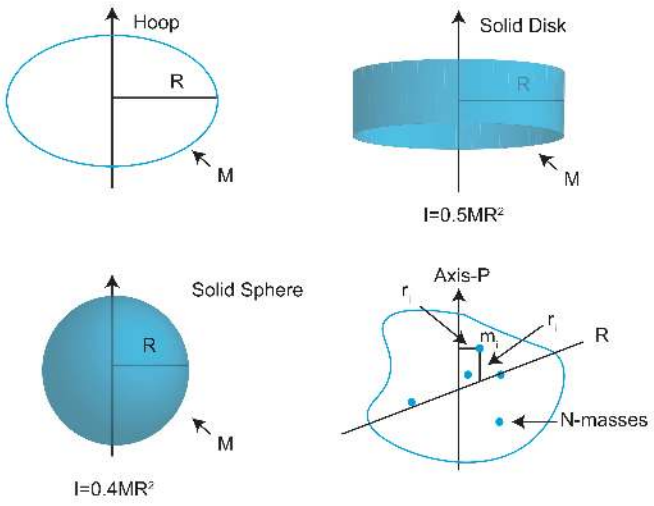


Figure 3.2.2: Moments of Inertia

Of particular interest for planetary structure is the how the moment of inertia changes as one goes from a solid sphere with a uniform density to a sphere with higher density material at is core. For a sphere with a higher density core and lower density outer layer, the moment of inertia will decrease relative to the moment of inertia of a homogeneous sphere.

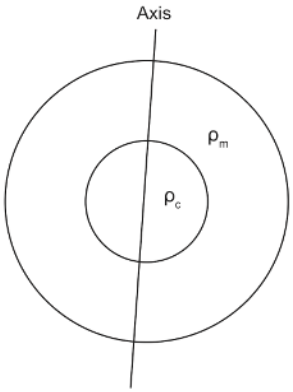


Figure 3.2.3: A sphere with a higher density core will have a lower moment of inertia than a homogeneous sphere with the same total mass.

When $\rho_c > \rho_m$, more mass is closer to the axis compared to a uniform sphere, this means that more mass is at smaller distances from an axis through the center of the sphere, and therefore the sums of $m_i r_i^2$ will be smaller, so the moment of inertia will be smaller for the layered sphere

$$I_{layered} < 0.4MR^2$$

Caution: this is for a layered sphere with higher density material at the core relative to the outer layer. If the denser layer was on the outside the moment of inertia would be bigger. However, for planets, the denser material sinks to the core due to gravitational segregation, so the layered structure always has a higher density for the deeper layer.

In addition, its important to notice that the moment of inertia depends on the mass, but also on the radius. Therefore, both a higher density or larger radius of the inner layer could lead a larger moment of inertia, so there are multiple models with the same total mass that could give the same moment of inertia.

Example: Trade-off between density and Radius

The bulk density of the earth is about $\rho_{bulk} = 5550$ kg/m^3 and the average moment of inertia is $0.33MR_e^2$.

These two models for the mantle density, ρ_m, the core density ρ_c and layer thickness both give the same total mass and moment of inertia:

- Model 1: $\rho_m = 4000$ kg/m^3, $\rho_c = 10800$ kg/m^3, with the ratio of radii as $\frac{r_c}{r_m} = 0.6$
- Model 2: $\rho_m = 2850$ kg/m^3, $\rho_c = 8120$ kg/m^3, with the ratio of radii as $\frac{r_c}{r_m} = 0.8$

Moment of Inertia of Ellipsoidal Object, like Earth

It is important to remember that Earth, and planets in general, are not spheres, they are ellipsoids. This means that there shape is described by two moments of inertia. The moment of inertia around the axis aligned with the spinning axis is usually given then symbol C, while the moment of inertia around an axis through the equator is given the symbol A. The axis through the equator can have any orientation since the equator is an actual circle. (If the shape were also elliptical, then the axis perpendicular to A would be denoted B).

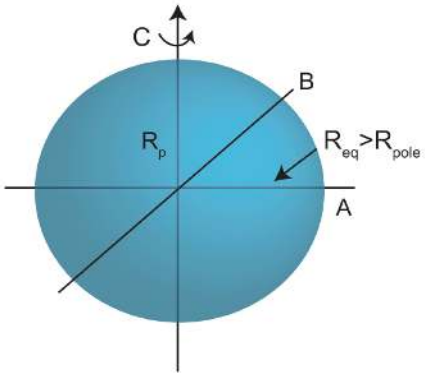


Figure 3.2.4: Ellipsoidal Body

If we knew C and A, we could use m, R, and ρ_{bulk} to estimate the internal layered structure of the planet. Even with this information there will be trade-offs between the assumed values of the densities and thickness of the layers. Therefore, other observation of the surface of the planet, including spectral information, can provide constraints on the composition (and density) of the surface layer. For more precise models, other data is needed that constrains the bulk composition of the planet and then petrologic calculations are used to determine the plausible compositions. For the Earth, this extra information comes from the composition of meteorites, which were the building blocks of the planet, as well as xenoliths (mantle rocks brought to the surface through volcanic activity).

Advanced Topic: Determining Moment of Inertia for Planets

If we know the moment of inertia of a planet and its mass, it is possible to determine simple models of the layered density structure of the planet. The layered density structure is defined by the thickness of the layers and the density of each layer. Usually, it is possible to determine reasonable two-layered structures (i.e., radius of the core and thickness of the mantle) if we have additional information about the composition and/or density at the surface of the planet.

? Procedure for Determining Moment of Inertia

The main assumption in the following procedure is that the planet behaves like a fluid (flows viscously) over long periods of time (e.g., a million years). This is called the **hydrostatic assumption**. This is a reasonable assumption for rocky planets, which have mantle compositions similar to the Earth.

- The shape of spinning planets is that of an oblate spheroid that is slightly broader at the equator and shorter at the poles. The oblate shape is quantified by the the **flattening**, f. The flattening of a rotating, fluid planet, with a polar radius, c, and equatorial radius, a, is given by:

$$f = \frac{a - c}{a} \tag{3.2.2}$$

The equatorial radius is larger because the spinning motion of the planet (centripetal force) causes mass to move to the equator and away from the poles. While this is a simple equation, it is actually not easy to get accurate measurements of the equatorial and polar radii of a planetary object. However, the flattening can be determined by measuring the gravitational potential field, which will mimic the oblate shape of the planet, through the orbit of satellites or an orbiter. The orbit will be perturbed by the non-spherical potential field, and these perturbations can be related to the actual shape of the gravitational potential.

The shape of the gravitational potential field is usually expressed in terms of functions called spherical harmonics. Think of these as sines and cosines, but instead of depending on just one angle, they depend on both longitude and latitude. It is possible to use an infinite sum of sines and cosines, of varying frequency and amplitude, to represent any other function that depends on just one variable. Likewise, it is possible to represent any 3D field in spherical coordinates using an infinite sum of spherical harmonics. It turns out that if you do this, the amplitude of the third term in the sum is directly related to the flattened (oblate) shape of an object. This term, called the **ellipticity coefficient**, is usually given the symbol, J_2, and depends on two of the moments of inertia of the object,

$$J_2 = \frac{C - A}{Ma^2} \tag{3.2.3}$$

Therefore, by measuring the orbits of satellite, we can get a first-order estimate of the flattening by just keeping two terms from the spherical harmonic representation of the gravity field,

$$f_{hyd} \approx \frac{3}{2} J_2 + \frac{1}{2} \frac{a^3 \omega^2}{2GM} \tag{3.2.4}$$

However, remember that we are trying to determine the internal layering structure of a planet, so we are interested in the moment of inertia $\frac{C}{Ma^2}$, but the ellipticity depends on two moments, C and A. Therefore, planetary scientists came up with another, approximate, way to express the flattening that only depends on $\frac{C}{Ma^2}$. You know it is not easy to come up with the approximation by the fact that this equation has a name, the Darwin-Radau Approximation (not the Darwin of survival of the fittest).

- The Darwin-Radau Approximation provides a second approximate expression for the hydrostatic flattening,

$$f_{hyd} \approx \frac{\frac{5}{2} \frac{a^3 \omega^2}{GM}}{1 + \frac{25}{4} \left(1 - \frac{3}{2} \frac{C}{Ma^2}\right)^2} \tag{3.2.5}$$

In this equation, everything can be measured/observed, including f_{hyd} from Equation 3.2.4, which means that the equation can be used to solve for the moment of inertia factor $\frac{C}{Ma^2}$,

$$\frac{C}{Ma^2} \approx \frac{2}{3} \left(1 - \sqrt{\frac{2}{5} \frac{\frac{a^3 \omega^2}{GM}}{f_{hyd}} - \frac{4}{25}}\right) \tag{3.2.6}$$

Using this equation, with observations of the planetary mass, the equatorial radius and f_{hyd} from the gravitational potential field, the moment of inertia factor of the planet can be determined. With this information, simple models of the the two layered structure can be made that also match the total mass of the planet. It is usually necessary to also have some educated guesses as to the expected average density of the mantle and the core.

This page titled 3.2: Layered Structure of a Planet is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.3: Two Layer Planet Structure Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive model to explore the layered structure of far-away planetary interiors from the limited information available to us, assuming there are two distinct layers.

Key Questions: Consider these as you work your way through this page.

Initial Reference Model

- How are the initial density and core size related? For example, does a smaller starting density produce a large or small core radius when compared to a larger starting density?

Density Variation Example

- Find a density value that creates a non-physical (NaN: not-a-number) density for the core of your chosen planet? Why does this happen?
- Find a density value where the model shows a core that is larger than the planet itself? Why does this happen?

Radius Variation Example

- Find a radius value that creates a non-physical (NaN) density for the core of your chosen planet? Why does this happen?
- Find a radius value where the model shows a core is larger than the planet itself? Why does this happen?

Mass Variation Example

- Is there an upper bound on what the mass can be in the model? That is, is there a high value that breaks the model like the previous examples; ignore whether it is realistic or not.
- Find a mass value that creates a non-physical (NaN) density for the core of your chosen planet? Why does this happen?
- Find a mass value where the model shows a core that is larger than the planet itself? Why does this happen?

First let's call the necessary functions, and define a dictionary called planets holding the relevant parameters of different planets.

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Circle
from ipywidgets import interact, interactive, fixed, interact_manual
import ipywidgets as widgets

planets = {'Jupiter':{'MOI':0.254,'Mass':1898.19e24,'Radius':69911000,'Surface Density':0.16,'Compositi
           'Neptune':{'MOI':0.26,'Mass':102.413e24,'Radius':24622000,'Surface Density':0.45,'Compositio
           'Earth':{'Mass':5.9723e24,'Radius':6371000,'MOI':0.3308,'type':'Olivine and Peridotite Upper
           'Compositional Density':4000}}

print('The packages have been imported and the dictionary defined; move to the next cell')
```

```
```

run | run all | restart | restart & run all

Reference Planet

Now to start your own model. Select a planet from the prompt once you run this cell.

```
style = {'description_width': 'initial'}
layout = {'width': '400px'}
planet = widgets.Dropdown(
    options=['Jupiter', 'Earth', 'Neptune'],
    value='Earth',
    description='Select a planet for your model:',
    disabled=False,
    style = style,
    layout = layout
)


display(planet)
```

run | run all | restart | restart & run all

Now, choose which density type you would like to use. The surface density can be measured, and the compositional density is the density of the assumed material that the mantle of the planet is made out of.

```
planet_choice = planet.value
print('The Surface Density of', planet_choice, 'is', planets[planet_choice]['Surface Density'], 'kg/m^3
density = widgets.Dropdown(
    options=['Surface', 'Compositional', 'Other'],
    value='Surface',
    description='Select a density for your model:',
    disabled=False,
    style = style,
    layout = layout
)

display(density)
```

run | run all | restart | restart & run all

This cell will gather all the required parameters for your choices, and prompt you for your density value if you selected 'other' in the previous cell.

```
density_choice = density.value

if density_choice.lower() == 'other':
    print('Please enter your desired density in kg/m^3, for now stay between the Surface and Compositio
    rho = float(input())
    while rho <= planets[planet_choice]['Surface Density'] or rho >= planets[planet_choice]['Compositio
        print('Please enter a density between the Surface and Compositonal Densities')
        rho = float(input())

print('Here are your selected parameters:')
print('Planet:',planet_choice)
if density_choice.lower() == 'surface':
    print('Surface Density of',planet_choice,'is',planets[planet_choice]['Surface Density'],'kg/m^3')
    rho = planets[planet_choice]['Surface Density']
    if planet_choice == 'Earth':
        print('The Surface Density of Earth is taken to be the Crustal Density')
elif density_choice.lower() == 'other':
    print('Your Entered Density of',planet_choice,'is',rho,'kg/m^3')
else:
    print('Compositional Density of',planet_choice,'is',planets[planet_choice]['Compositional Density']
    rho = planets[planet_choice]['Compositional Density']

M = planets[planet_choice]['Mass']
R = planets[planet_choice]['Radius']
```

run | run all | restart | restart & run all

This cell defines the parameters beta, a, and b as defined in 3.2: Layered Structure of a Planet. beta depends on the planet's mass and radius. a and b depend on beta, alpha (which is 5/2 times the Moment of Inertia), and the density rho. a is the fraction between the core and total radius, and b is the proportionality constant to calculate the core density from the original density. Therefore, once you calculate a, you can get the core radius by multiplying a by the total radius (however, for simplicity this model only plots a and not a * R). The density of the core is found by multiplying b by the original density, which is done in the plotting function below.

```
def beta(M,R):
    denominator = (4 * np.pi * R**3)
    return ((3*M)/denominator)

def a_and_b(beta,rho,alpha):
    w = beta - rho
    q = alpha * beta
    a = np.sqrt((q-rho)/w)
    b = (w / ((a**3) * rho)) + 1
    return a,b

def plotting(M,R,rho,I):
    beta1 = beta(M,R)
    a,b = a_and_b(beta1,rho,I*(5/2))
    fig, ax = plt.subplots(figsize = (7,7))
    ax.add_patch(Circle((0,0),R/R,alpha = 0.1,ec='black',fc='blue'))
    ax.add_patch(Circle((0,0),R/R,fill=False,ec='black'))
    ax.add_patch(Circle((0,0),a,fill=False,ec='black'))
    ax.add_patch(Circle((0,0),a,fc='red',color = 'red',alpha=0.2,ec='black'))
    ax.annotate('$p_1$ = '+'{:.2f}'.format(rho)+' $ kg/m^3 $',(0.05,0.97-((1-a)/2)))
    ax.annotate('$p_{core}$ = '+'{:.0f}'.format(b*rho)+' $ kg/m^3 $',(0.05,0.05))
    ax.set_ylabel('Fraction of total radius: '+ str(R/1000) +' km')
    ax.set_xlabel('Fraction of total radius: '+ str(R/1000) +' km')
    plt.show()
```

run | run all | restart | restart & run all

Density Variation Example

Take note of the appearance of the above model. This is generally what the model looks like when it is physically possible for the input parameters to yield a solution. That's why you were limited in your choice for the range for densities if you chose the other

option above. In the next series of examples, we are going to consider how the initial mass, radius, and density affect the model, and if certain values of these parameters will create an impossible planet. First, let's change the initial density of the planet. See how changing the density changes the model, and why certain density choices are impossible. You can tell if a choice is impossible if the graph does not look nice and neat like the previous one; you may also encounter non-physical values in the form of nan values (nan/NaN stands for not-a-number) or the radius of the core could end up being larger than the radius of the planet itself.

```
print('Enter your new surface density value, it must be greater than 0 and less than 5600 in kg/m^3:')
rho = float(input())
while rho <= 0 or rho > 5600:
    print('Invalid Density: it must be greater than 0 or less than or equal to 5600 kg/m^3')
    rho = float(input())

handle = display(None, display_id=True)
fig, ax = plt.subplots(figsize = (7,7))
ax.add_patch(Circle((0,0),R/R,alpha = 0.1,ec='black',fc='blue'))
ax.add_patch(Circle((0,0),R/R,fill=False,ec='black'))
while rho != 0:

    print('Here is your new model of',planet_choice,'using a density of',rho,'kg/m^3')
    beta1 = beta(M,R)
    a,b = a_and_b(beta1,rho,I*(5/2))



    patch1 = ax.add_patch(Circle((0,0),a,fill=False,ec='black'))
    patch2 = ax.add_patch(Circle((0,0),a,fc='red',color = 'red',alpha=0.2,ec='black'))
    annot = ax.annotate('$p_1$ = '+'{:.2f}'.format(rho)+' $ kg/m^3 $',(0.05,0.97-((1-a)/2)))
    annot2 = ax.annotate('$p_{core}$ = '+'{:.0f}'.format(b*rho)+' $ kg/m^3 $',(0.05,0.05))
    ax.set_ylabel('Fraction of total radius: '+ str(R/1000) +' km')
    ax.set_xlabel('Fraction of total radius: '+ str(R/1000) +' km')
```

run | run all | restart | restart & run all

Radius Variation Example

Now let's set the density to a constant value, we will use the compositional density of the planet you selected. This time, let's change the radius and see what happens. Like before, take note of when and why the model gives a nan or a core radius larger than the planet itself. Note that the given range of radii are given to accommodate all three planets, it may be easier to find the nan and large cores near the radius of the initial planet.

```
rho = planets[planet_choice]['Compositional Density'] # Going to hold rho constant at the compositional
print('The original radius of your planet is',planets[planet_choice]['Radius']/1000,'km')
print('Enter your new radius value in km, it must be greater than 0 or less than 100,000 km:')
R = float(input())
while R <= 0 or R>100000:
    print('Invalid Radius: it must be greater than 0 or less than 100,000 km')
    R = float(input())
R *= 1000 #convert to m
handle = display(None, display_id=True)
fig, ax = plt.subplots(figsize = (7,7))
ax.add_patch(Circle((0,0),R/R,alpha = 0.1,ec='black',fc='blue'))
ax.add_patch(Circle((0,0),R/R,fill=False,ec='black'))
while R != 0:

    print('Here is your new model of',planet_choice,'using a radius of',R,'m')
    beta1 = beta(M,R)
    a,b = a_and_b(beta1,rho,I*(5/2))



    patch1 = ax.add_patch(Circle((0,0),a,fill=False,ec='black'))
    patch2 = ax.add_patch(Circle((0,0),a,fc='red',color = 'red',alpha=0.2,ec='black'))
    annot = ax.annotate('$p_1$ = '+'{:.2f}'.format(rho)+' $ kg/m^3 $',(0.05,0.97-((1-a)/2)))
    annot2 = ax.annotate('$p_{core}$ = '+'{:.0f}'.format(b*rho)+' $ kg/m^3 $',(0.05,0.05))
```

run | run all | restart | restart & run all

Mass Variation Example

Finally, lets set the radius to its original length and change the mass of your planet. Like before, take note of when and why the model gives a nan or a core radius larger than the planet itself.

```
R = planets[planet_choice]['Radius'] # Going to hold radius constant for this example
print('The original mass of your planet is',planets[planet_choice]['Mass'])
print('Enter your new mass value, it must be greater than 0 and in kg:')
print('The number you enter will be multiplied by 10^24')
M = float(input())
while M <= 0:
    print('Invalid Mass: it must be greater than 0')
    M = float(input())
M *= 10**24
handle = display(None, display_id=True)
fig, ax = plt.subplots(figsize = (7,7))
ax.add_patch(Circle((0,0),R/R,alpha = 0.1,ec='black',fc='blue'))
ax.add_patch(Circle((0,0),R/R,fill=False,ec='black'))
while M != 0:

    print('Here is your new model of',planet_choice,'using a mass of','{:.2e}'.format(M),'kg')
    beta1 = beta(M,R)
    a,b = a_and_b(beta1,rho,I*(5/2))



    patch1 = ax.add_patch(Circle((0,0),a,fill=False,ec='black'))
    patch2 = ax.add_patch(Circle((0,0),a,fc='red',color = 'red',alpha=0.2,ec='black'))
    annot = ax.annotate('$\rho_1$ = '+'{:.2f}'.format(rho)+' $ kg/m^3 $',(0.05,0.97-((1-a)/2)))
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

Initial Reference Model

- How are the initial density and core size related? For example, does a smaller starting density produce a large or small core radius when compared to a larger starting density?

Density Variation Example

- Find a density value that creates a nan density for the core of your chosen planet? Why does this happen?
- Find a density value where the model shows a core is larger than the planet itself? Why does this happen?

Radius Variation Example

- Find a radius value that creates a nan density for the core of your chosen planet? Why does this happen?
- Find a radius value where the model shows a core is larger than the planet itself? Why does this happen?

Mass Variation Example

- Is there an upper bound on what the mass can be in the model? ie is there a high value that breaks the model like the previous examples; ignore whether it is realistic or not.
- Find a mass value that creates a nan density for the core of your chosen planet? Why does this happen?
- Find a mass value where the model shows a core is larger than the planet itself? Why does this happen?

3.3: Two Layer Planet Structure Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

3.4: Isostasy

Very broad or long wave-length topography is more likely to be compensated than smaller scale topography. This is because large topography is more massive and causes the lithosphere to bend into the deeper viscous mantle. Smaller topographic features can be supported by the strength of the lithosphere, and therefore no compensating mass anomaly forms beneath the topography.

Compensated topography is also referred as isostatically compensated topography, where the word isostatic refers to equal pressure (or stress) in the viscous lithosphere or mantle beneath adjacent columns of static (not moving up or down, or moving very, very slowly) mass. In Figure 3.4.2, the pressures, P_1 and P_2 are equal at a depth, D_c, called the depth of compensation. Beneath D_c there are no differences in the density structure of the two columns.

$$P = \sum_{i=1}^{n} \rho_i h_i g \tag{3.4.1}$$

The concept of isostasy is used to understand why the average height of one region of a plate is different than another region of plate.

Start with familiar concept: ice floating in water.

- Thicker ice floats higher than thinner ice because its mass is less.
- However, there is also much more ice beneath the water line than above the water line because,
- the density difference between ice and water is smaller than the density difference between ice and air.

Consider a continent ocean system:

- Two adjacent columns of mass: continental and oceanic
- The pressure in the viscous (flowing) layer at the bottom is due to the weight ($\sum \rho g h$ of the column above.
- In isostatic equilibrium, the pressure at the bottom of each column (below where there are any differences in the density) is the **same** in both columns ($P_1 = P_2$
- Write out the sum for the pressure in each column (this reduces to the sum of the density times the thickness in each layer)
- Set the pressures equal.
- This equation can be used to determine the height difference between the two columns or the difference in the thickness of the crust extending into the viscous lithosphere.
- However, a second equation is needed, otherwise there are two unknowns.
- The second equation comes from equating the heights. Then one of the unknowns can be solved for and this equation is substituted into the pressure balance equation.

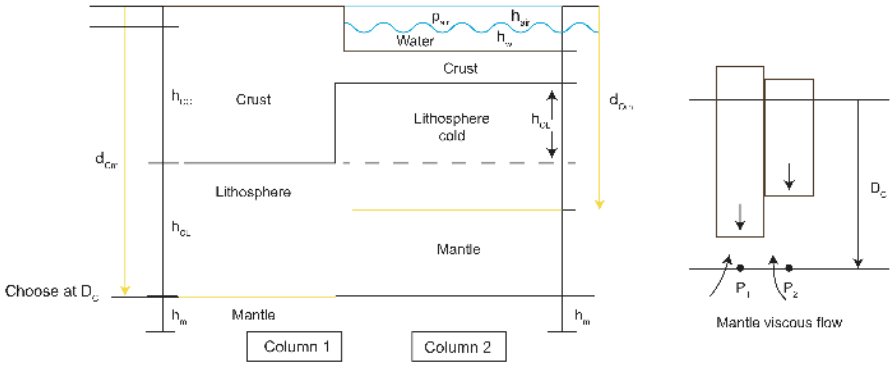


Figure 3.4.2: Adjacent continent and oceanic crust and lithosphere in isostatic equilibrium.

Isostasy Problem Recipe:

The approach to any isostasy problem is the same

- Draw a picture
- Determine D_c as the depth where there are no more differences in density between the two columns
- Write down the equation $P_1 = P_2$

- Simplify: cancel g's and combine like terms
- Write down $\sum H_{1i} = \sum H_{2i}$ and use this to get rid of extra unknowns (solve for the unknown you don't want to know)
- Solve for desired unknown
-

Example relative Height of Continent and Oceans

Why is the average elevation of continents higher than the average height of oceans?

Assume that we know all the densities in the problem, and we know the crustal thicknesses ($h_{cc} and h_{oc}$ and the thickness of the water layer h_w.

From the drawing we see that we can take the depth of compensation at the base of the continent because below this depth there are no differences in density between the to two columes

First, we write out the sums of pressure for both columns and set them equal:

$$\rho_{cc} h_{cc} g = \rho_{air} h_{air} g + \rho_w h_w g + \rho_{oc} h_{oc} g + \rho_L h_L g \tag{3.4.2}$$

(here I have used the subscript L instead of oL for the thickness and density of the lithosphere mantle in the oceanic column.

Next we can cancel all g's, and Equation 3.4.2 now reads:

$$\rho_{cc} h_{cc} = \rho_{air} h_{air} + \rho_w h_w + \rho_{oc} h_{oc} + \rho_L h_L \tag{3.4.3}$$

We can now see that we have 2 unknowns, h_{air} and h_L and thus we will need a second equation. The total thickness of the crust is equal to the sum of the thicknesses in the oceanic column:

$$h_{cc} = h_{air} + h_w + h_{oc} + h_L \tag{3.4.4}$$

We solve this equation to for h_L because we don't know it and do not want to know it (we want to know h_{air}):

$$h_L = h_{cc} - h_{air} - h_w - h_{oc} \tag{3.4.5}$$

Now substitute the equation above for h_L into the pressure balance equation. This removes h_L from the equation allowing you to solve for h_{air}:

$$\rho_{cc} h_{cc} = \rho_{air} h_{air} + \rho_w h_w + \rho_{oc} h_{oc} + \rho_L h_{cc} - \rho_L h_{air} - \rho_L h_w - \rho_L h_{oc} \tag{3.4.6}$$

Note that now you have three negative terms that all depend on ρ_L but have different thicknesses. The next step is to combine these terms with the positive terms that have the same thickness.

Combine the "like terms", that is combine the terms that have the same thickness (keeping h_a on the left-hand side)

$$0 = +h_{cc}(\rho_L - \rho_{cc}) - h_a(\rho_L - \rho_a) - h_w(\rho_L - \rho_w) - h_{oc}(\rho_L - \rho_{oc}) \tag{3.4.7}$$

Note that, for the last three terms we have taken a negative sign out front so the density difference is a positive number.

Next, re-arrange by adding the negative h_a term to the other side

$$h_a(\rho_L - \rho_a) = (\rho_L - \rho_{cc}) h_{cc} - h_w(\rho_L - \rho_w) - h_{oc}(\rho_L - \rho_{oc}) \tag{3.4.8}$$

Finally divide $(\rho_L - \rho_a)$ to get h_a alone:

$$h_a = h_{cc}\left(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}\right) - h_w\left(\frac{\rho_L - \rho_w}{\rho_L - \rho_a}\right) - h_{oc}\left(\frac{\rho_L - \rho_{oc}}{\rho_L - \rho_a}\right) \tag{3.4.9}$$

Note that all terms on the right are fractional heights weighted by the density difference across each layer relative to the density different between the air and the continental crust. This is the outcome of every isostatic balance problem an illustrates how the pressure balance is achieved for each layer.

Example Time Changes

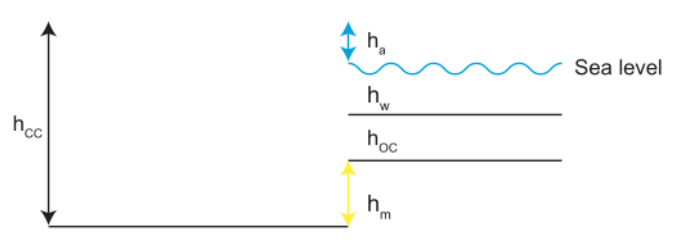


Figure 3.4.3: Time Changes

At some time T_1 the height of mountain range above sea level is h_{a1} and the thickness of the crust is h_{cc}. After erosion at time T_2 the thickness of the crust is smaller by Δh. Because there is no less mass weighting the continent down it will rise buoyantly (like removing weight from a floating ship). What is the height of the mountain range at time T_2 (assuming isostatic balance).

First, let's consider what stays the same: the thickness of the water h_w and the oceanic crust h_{oc}

Second, what changes? Both the height of the continent above sea-level h_a and the thickness of the mantle root h_m change.

Start by writing the isostatic balance for time T_1 using the solution given above in the previous example:

$$h_{a1} = h_{cc}\left(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}\right) - h_w\left(\frac{\rho_L - \rho_w}{\rho_L - \rho_a}\right) - h_{oc}\left(\frac{\rho_L - \rho_{oc}}{\rho_L - \rho_a}\right) \tag{3.4.10}$$

Now, let $A_1 = -h_w(\frac{\rho_L - \rho_w}{\rho_L - \rho_a})$ and $A_2 = -h_{oc}(\frac{\rho_L - \rho_{oc}}{\rho_L - \rho_a})$. These terms do not change between time 1 and time 2 because h_w and h_{oc} do not change.

Next, write the equation for the height of the continent at time T_2:

$$h_{a2} = (h_{cc} - \Delta h)(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}) + A_1 + A_2 \tag{3.4.11}$$

Expand the first term

$$h_{a2} = \left(h_{cc}(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}) + A_1 + A_2\right) - \Delta h(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}) \tag{3.4.12}$$

Now note that the first three terms are equal to where h_{a1}.

$$h_{a2} = h_{a1} - \Delta h(\frac{\rho_L - \rho_{cc}}{\rho_L - \rho_a}) \tag{3.4.13}$$

So the final height of the continent after erosion is equal to original height, minus a density weighted fraction of the thickness that was eroded. The density fraction is less than 1, so change in height is less than Δh. The reason is that the continent moves up a little bit in response to the removal of mass by erosion.

Example Depth of an Ocean Basin

Mid ocean ridge are usually found at depths of about 2500 m (2.5 km) below sea level. As the oceanic plate moves away from the mid ocean ridge, the lithosphere cools becoming thicker. Because colder rock has a higher density than warmer rock this thickening lithosphere increases the weight of the tectonic plate, and so it sinks into mantle. By the time the lithosphere reaches 80 My years old, the ocean depth is more like 4500 to 5000 m.

Isostatic balance can be used to determine the amount the ocean depth increases relative to its original depth, $\Delta d(t)$. The total depth is given by:

$$d(t) = d_o + \Delta d(t)? \qquad (3.4.14)$$

We know, from the solving the diffusion equation, that the lithosphere cools and thickens. The thickness of the lithosphere, $w(t)$, corresponds to following an isotherm (e.g., 1000 C) and assuming that the mass associated with the warmer material is small and can be ignored.

Notice that we can ignore the thickness of water above the ridge, because this water also exists everywhere else. Therefore, when balancing the pressure in two columns, one at the ridge and one some distance away, we only need to consider the layers that exist beneath the ridge height. At the ridge this is $h_c + h_a$, where h_a is the asthenosphere. At some distance away (corresponding to time of cooling or plate age of, t), there are three layers:

- the deepening ocean relative to the ridge given by $\Delta d(t)$
- the oceanic crust, h_c
- the lithosphere thickness (not including the crust), $w(t)$

Also note that, h_c exists in both columns, so $h_a = \Delta d(t) + w(t)$. To complete the problem, write down the pressure balance equation and solve for $\Delta d(t)$.

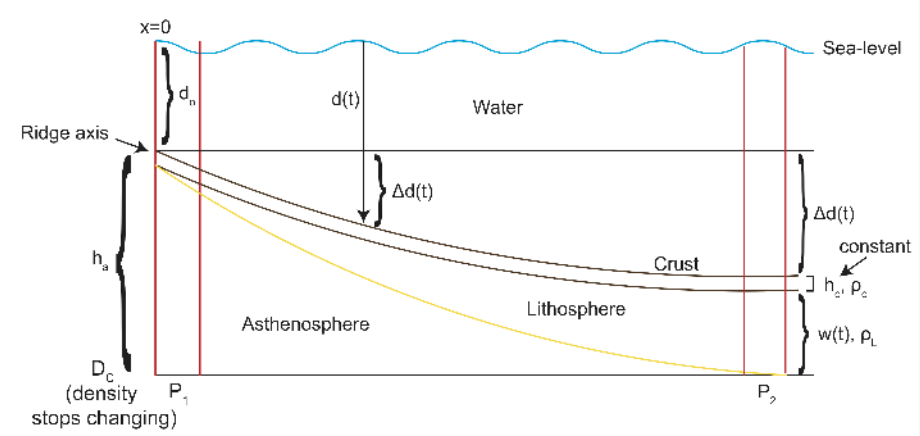


Figure 3.4.4: Depth of an Ocean Basin

This page titled 3.4: Isostasy is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.5: Isostasy Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

Interactive examples of the principle of isostasy starting with an iceberg, and then moving to more complicated examples of the crust and the mantle, and a crust with multiple thicknesses.

Key Questions: Consider these as you work your way through this page.

Iceberg in Water Example

- For a given thickness, does the iceberg extend father down into the water, or extend higher above the water?
- How does the height of the iceberg relative to its depth change between thicknesses of 10 m, 60 m, and 100 m? (Does the height/depth ratio increase, decrease, or remain the same?) What combination of parameters in the equation determines this ratio?

Crust in Mantle "Ocean" Example

- Run the simulation for a few different crustal thicknesses (1, 20, and 40 km): does the height/depth ratio increase, decrease, or remain the same?
- Compare how the height/depth ratio depends on layer thickness for this example with the iceberg example. Why are they the same or different?

Continent Isostasy Example

- As the oceanic thickness increases, does the elevation of the continent increase, decrease, or remain the same? What combination of parameters controls this relationship?
- Run the simulation at a few different oceanic crustal thicknesses (1, 5, and 20 km), does the height/depth ratio of the continental crust increase, decrease, or remain the same? In other words, how does the thickness of the oceanic crust affect the height/depth ratio of the continental crust?
- What happens when you use an oceanic crustal thickness above 22 km? Why does this happen?

First, let's import the required packages and define constants for this problem. rho_w is the density of seawater at the surface and rho_i is the density of ice.

```
import numpy as np
import matplotlib.pyplot as plt

rho_w = 1028 #kg/m^3
rho_i = 917 #kg/m^3

print('Done, Move on to the next cell.')
```

```
```

run | run all | restart | restart & run all

Iceberg in Water

We can use Archimedes' Principle to calculate the height and depth of the iceberg by setting the weight of the iceberg equal to the weight of the displaced water. Let's define that function here first, then run a simulation to see what happens as the thickness of the iceberg changes. The T in the function stands for the thickness of the iceberg. x is the horizontal distance range that will be considered, for simplicity the width of the iceberg will remain constant at 20 m. After the function, there is a plotting routine that will update the plot every time you enter a new thickness. After you run the cell, enter a thickness between 1 - 100 m and consider the key questions.

```
def iceberg_isos(rho_w,rho_i,T):
    depth = (rho_i / rho_w) * T
    height = T - depth
    return -depth, height

handle = display(None, display_id=True)
x = np.arange(-25,25) # Horizontal distance in the Graph

bbox = dict(boxstyle ="round", fc = '1')
fig, ax = plt.subplots(figsize = (9,7))
ax.hlines(0,-25,-10,color='blue')
ax.hlines(0,10,25,color='blue')
ax.fill_betweenx((0,-100),-25,-10,color = 'blue')
ax.fill_betweenx((0,-100),10,25,color = 'blue')
ax.set_xlim(-25,25)
ax.set_title('Iceberg Isostasy')
ax.set_ylabel('Elevation (m)')
ax.set_xlabel('Distance from the Center of the Iceberg (m)')
ax.set_ylim(-100,15)
ax.annotate('Air',(18,3))
print('Enter a Iceberg Thickness between 1 and 100 m: ')
T = float(input())
while T< 1 or T > 100:
    print('Invalid Thickness, please enter a number between 1 and 100 km')
```

run | run all | restart | restart & run all

Crust in a Mantle "Ocean"

Now, let's consider an example more closely related to geophysics. Here, we replace the water with mantle rock, treating the mantle like an ocean. We then replace the ice with crustal rock. (Think of the first crust to crystallize in the magma ocean that covered the early Earth). To be more realistic with the appropriate length scales, we also move from meters to kilometers in this example. In this scenario, the math is exactly the same as above, so we will reuse our iceberg_isos function from above, replacing rho_i with rho_crust, and rho_w with rho_mantle.

```
rho_crust = 2850 # kg/m^3
rho_mantle = 4000 # kg/m^3

handle = display(None, display_id=True)
x = np.arange(-25,25) # Horizontal distance in the Graph


bbox = dict(boxstyle ="round", fc = '1')
```

```
fig, ax = plt.subplots(figsize = (9,7))
ax.hlines(0,-25,-10,color='red')
ax.hlines(0,10,25,color='red')
ax.fill_betweenx((0,-50),-25,-10,color = 'red')
ax.fill_betweenx((0,-50),10,25,color = 'red')
ax.set_xlim(-25,25)
ax.set_title('Crust-Mantle Ocean Isostasy')
ax.set_ylabel('Elevation (km)')
ax.set_xlabel('Distance from the Center of the Crustal Block (km)')
ax.set_ylim(-40,20)
ax.annotate('Air',(18,3))
print('Enter a Crustal Thickness between 1 and 40 km: ')
T = float(input())
while T< 1 or T > 40:
    print('Invalid Thickness, please enter a number between 1 and 40 km')
```

run | run all | restart | restart & run all

Continent Isostasy

Now let's build on your understanding from the two above examples to move to a complex example that resembles a more realistic situation. This time we will consider a setup that has air, ocean, lithospheric mantle, and oceanic crust in the outer columns. We will fix the continental crust thickness and the thickness of the ocean while varying the thickness of the oceanic crust. The first portion of this cell defines the known densities and thicknesses. The next part defines a new isostasy function that will take in all of the known thicknesses and densities, and calculates the two unknown thicknesses: air and mantle. The equations within this function can be found in section 3.6 Isostasy. The rest of the cell is simply a plotting routine. Go ahead and run the cell, and then answer the key questions relating to this example.

Note: in the present-day earth the thickness of oceanic crust formed at a mid-ocean ridge is remarkably constant, and is about 7.5 km, with a range of 1-10 km. However, this thickness is determined by melting processes and so it could be different in the past or future when the mantle was/is hotter or colder. It could also be different on different planet with a different bulk composition.

The average elevation of the continents above sea level is about 800m, or 0.8km. You should find a number close to that if you use the average oceanic thickness of 7.5km.

```
rho_air = 1 # kg/m^3
rho_mantle = 3300 # kg/m^3
rho_crust = 2900 # kg/m^3
rho_oc = 3000 # kg/m^3
h_c = 35 # km
h_w = 4 # km
rho_water = 1030 # kg/m^3

def isostasy(rho_a,rho_m,rho_c,rho_oc,h_c,h_oc,rho_w,h_w):
    h_a = ((h_c*(rho_m - rho_c)) - (h_w*(rho_m - rho_w)) - (h_oc*(rho_m - rho_oc))) / (rho_m - rho_a)
    h_m = h_c - h_oc - h_a - h_w
    return h_a, h_m

print('Please enter the thickness of the oceanic crust between 1 and 40 km')
h_oc = float(input())
while h_oc < 1 or h_oc > 40:
    print('Invalid thickness, enter a number between 1 and 40 km')
    h_oc = float(input())


handle = display(None, display_id=True)
fig, ax = plt.subplots(figsize = (9,7))
ax.set_xlim(-25,25)
ax.set_ylim(-50,20)
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

Iceberg in Water Example

- For a given thickness, does the iceberg extend father down into the water, or extend higher above the water?

- How does the height of the iceberg relative to its depth change between thicknesses of 10 m, 60 m, and 100 m? (Does the height/depth ratio increase, decrease, or remain the same?) What combination of parameters in the equation determines this ratio?

Crust in Mantle "Ocean" Example

- Run the simulation for a few different crustal thicknesses (1, 20, and 40 km): does the height/depth ratio increase, decrease, or remain the same?
- Compare how the height/depth ratio depends on layer thickness for this example with the iceberg example. Why are they the same or different?

Continent Isostasy Example

- As the oceanic thickness increases, does the elevation of the continent increase, decrease, or remain the same? What combination of parameters controls this relationship?
- Run the simulation at a few different oceanic crustal thicknesses (1, 5, and 20 km), does the height/depth ratio of the continental crust increase, decrease, or remain the same? In other words, how does the thickness of the oceanic crust affect the height/depth ratio of the continental crust?
- What happens when you use an oceanic crustal thickness above 22 km? Why does this happen?

3.5: Isostasy Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

3.5: Observing the Gravity Field

Observing (Measuring) the Gravity Field on Earth

The gravity field of the Earth is determined in two ways:

1. Measuring the orbits of satellites and using these to determine the gravity field. Early on this was done with single satellites and this provided the long wavelength observation of the gravity field (able to capture features down to about 100 km by the year 2000). Now these observations are done with twin satellites that orbit in tandem known as the GRACE mission (ADD LINK). Because satellites are not at the Earth's surface but rather are thousands of kilometers above the Earth's surface and gravity decreases with distances, these measurements always provide a smoother picture of gravity than what actually exists at the surface.
2. Because 70% of the earth's surface is covered in water, a higher resolution observations of the gravity field can be measured by measuring the height of the ocean surface. This is done using a technique called satellite radar altimetery [ADD LINK] in which the distance between a satellite and the sea surface is measured. Repeated measurements removes the signal due to waves and currents (this signal is used by oceanographers to map where the currents are flowing).

Satellite altimetry provides a very high resolution image of the gravity field. This measurement is so high resolution that it can be converted into a map of seafloor topography and used to locate plate boundaries, fracture zones and seamounts both big and small. It is not high resolution enough to use for navigation of a real boat. However, if you use google maps and turn topography on, and got hunting around off the coast, the background topography images for the globe is the topography derived from satellite altimetry. If you zoom in really close near a port, you will see that there are regions with much finer resolution, this is because there is shipboard mapping of the seafloor that is "stitched" onto the background image.

Geoid Height from Satellite Altimetry

Some things to note about and what influences it:

- There is a layer of water → ocean
- U is parallel to surface, g is perpendicular
- Add a mass anomaly
- U moves out
- g is deflected → it now has a horizontal component
- Water flows in created a bump
- Now the sea surface follows U_{obs}

Satellite altimetry is measuring the geoid. Gravity (g) is then calculated from the geoid by taking the spatial derivative.

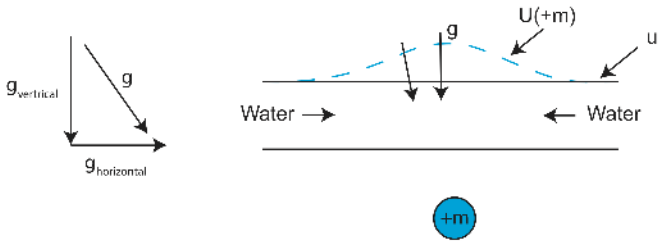


Figure 3.5.1: Sea-level

Because the sea surface follows the geoid, we can visualize the geoid height as that height of the ocean surface above or below a reference surface. Note that the fact that water surface follows the geoid means means that sea-level is not at the same height (relative to the center of the Earth) everywhere.

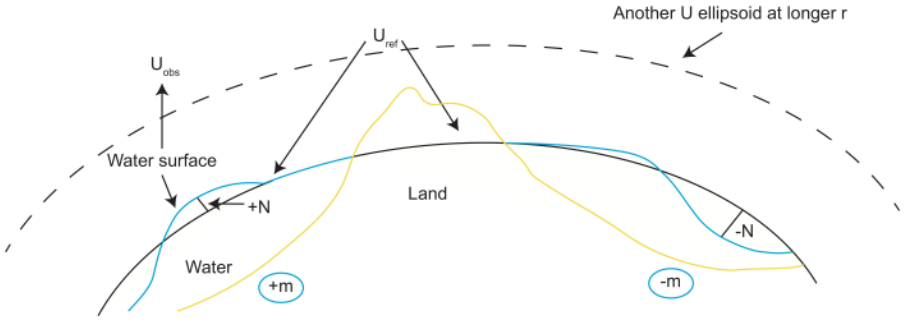


Figure 3.5.2: U_{obs} and U_{ref}

- Land and water
- U_{ref} → cuts through these
- U_{obs} → undulations due to mass
- U_{obs}-U_{ref} → geoid height → N

+N over +m and -N over -m. |N| → ±100 meters

Compensated and Uncompensated Topography [need to move to its own section]

Topography at the Earth's surface is not always visible through the gravity field. For example, there is not an offset in gravity associated with going from the oceans to the continents despite the fact that the continents sit several kilometers higher. For some topography the positive mass anomaly of the topography is balanced (compensated) by a negative mass anomaly due to thicker crust. This is the case the continents. The continents are higher because the continental crust is thicker and lower density than the crust of the oceanic regions. Only uncompensated topography will have a signal in the gravity field. Some mass anomalies can also be "partially" compensated, so that the gravity signal is less than expected based on the topography.

The figure below illustrates the concepts of uncompensated and compensated topography. On the left imagine a brick on a stiff board that doesn't bend. This brick is an uncompensated mass on top of the board. It will have a signal in the gravity field. On the right imagine putting that brick on a small boat filled with air. The boat is pushed into the water by the weight of the brick until the weight is balanced (compensated). In this case there is not gravity signal because the anomaly from the brick is cancelled out by the anomaly of the submerged portion of the boat.

We will return to this concept in the discussion of isostasy [ADD LINK].

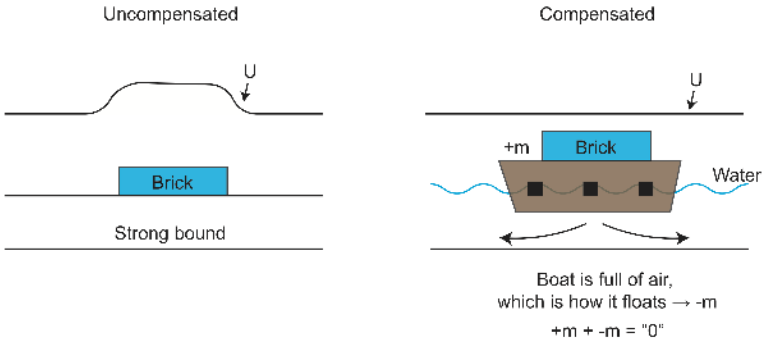


Figure 3.5.6: Uncompensated and Compensated Topography

Smaller features

- seamounts
- fracture zones
- trenches
- These are all compensated and supported by the strength of the lithosphere
- ⇒gravity anomaly

Bigger features

- continents
- mountain ranges
- compensated or "floating"
- Uncompensated and supported by viscous stresses, often mantle has flowed away⇒isotasy
- ⇒no gravity anomaly

Hudson Bay

Let's look at an example of Hudson Bay during the last ice age. During the ice age, the bay had a kilometers thick ice cap sitting on top. This is a compensated example because the lithosphere deforms. Flashing forward to the bay today, the ice has melted and the lithosphere has slowly rebounded from its deformed state. It is now in an uncompensated state. An important thing to note is that the bay may never fully return to its undeformed state, as the bay is filling in with sediments.

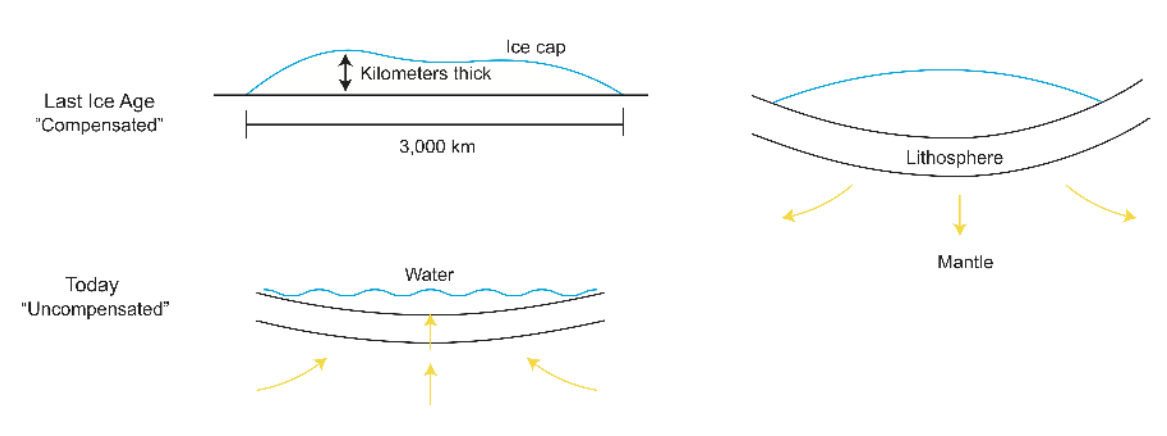


Figure 3.5.7: Hudson Bay Deformation

Subduction Zones

Sinking tectonic plates are large mass anomalies in the Earth. However, the gravity signal above them can be much less than predicted for the slab by itself because of flow-induced mass anomalies that compensate for the mass anomaly of the slab.

First, consider the topography above a subduction zone, which can include large mountain ranges created by compression. This mountain ranges are mostly compensated by thicker continental crust. (The picture should show a much thicker crust under the Andes; and the hieght of the mountain is definitely not drawn to scale).

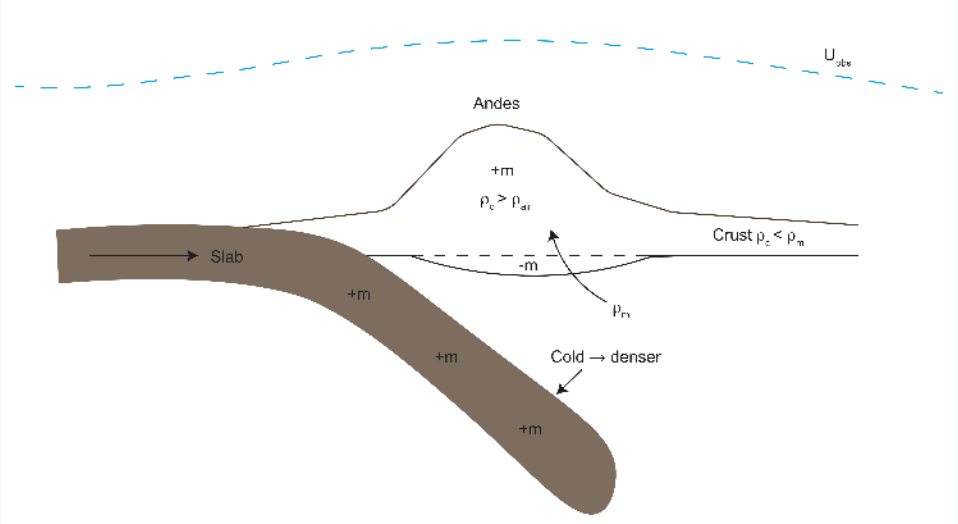


Figure 3.5.8: Andes Subduction Zone

Second, the sinking of the slab will pull down on the surface creating a region of lower mass that compensates for the mass anomaly of the slab. In the Earth, the fact that the viscosity increases by about 30 x at around 660 km, means that the surface is not dragged down as much as it would be if the mantle had a uniform viscosity. So there is less compensation of the mass anomaly of the slab.

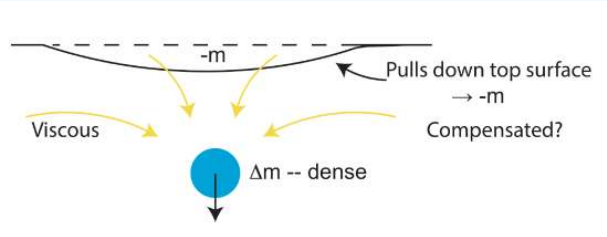


Figure 3.5.9: Density Anomaly

In summary, deformation (viscous flow and/or bending) in response to one mass anomaly can create mass anomalies of opposite sign. These are compensating anomalies. The gravitational field from all the mass anomalies sum up to give the total gravitational field. Gravity anomalies over compensated masses are much smaller (or zero) than the gravity anomaly of the initial mass anomaly.

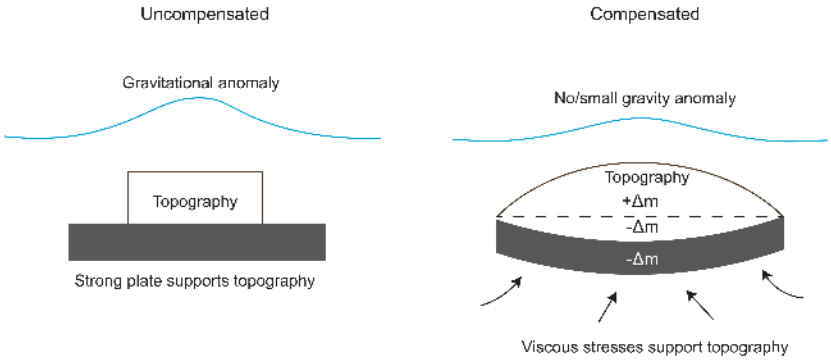


Figure 3.5.1: Uncompensated and Compensated Topography

This page titled 3.5: Observing the Gravity Field is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.7: Gravitational Potential, Mass Anomalies and the Geoid

Determining the layered structure of a planet requires knowing the gravitational field around a planet and in particular the elliptical shape of the gravity field. Recall from the previous section, that the oblate (ellipsoidal) shape of the planet is a direct result of the viscous flow of a spinning planet. However, the gravity field can also provide information about density anomalies inside the planet and the viscosity layering in the convecting (mantle) part of the planet. Therefore, here we spend more time describing the relationship between the gravity (acceleration) we are familiar with from introductory physics, and the gravitational potential field.

The gravitational potential, U is defined as,

$$U = -\frac{Gm}{r} \tag{3.7.1}$$

and **gravity**, g the acceleration caused be the gravitational potential is the gradient of the potential field

$$g = -\bigtriangledown U = -\left(\frac{dU}{dx} + \frac{dU}{dy} + \frac{dU}{dz}\right) \tag{3.7.2}$$

While we don't usually think of gravity as a field quantity, we are probably familiar with the concept of a field from magnetism. Consider a bar magnet, with positive (north) end and negative (south) end. We usually illustrate the magnetic field as lines that come out one end of the magnet and go back in at the other side of the magnet. This magnetic field is create by two magnetic poles (positive and negative). The strength of the magnetic field is strongest near the poles and decays (and switches sign) along these field lines.

Similarly, we can think of gravity as originating at a single "pole", a point mass, with gravity field lines that point radially outward. The strength of gravitational field also decays along these field line. The gravitational potential around a point mass is represented by spherical surfaces centered on the point mass. The magnitude of the gravitational potential, $U(r)$ is a constant on each of these surfaces: therefore, these surfaces are referred to as **equipotential surfaces.** Spherical surfaces at greater distances have lower values because (U decreases proportional to $1/r$).

Gravity, g, or the gravitational acceleration at any point, is the gradient of this three-dimensional potential field, and has both a magnitude and a direction. For a point mass or a perfect sphere, g, points radially outward (it is perpendicular to the spherical equipotential surface) and its magnitude is the same everywhere on one of the equipotential surfaces (for a perfect sphere, but not other shapes, as we will see). When we say that gravity is 9.81 m/s^2, we are actually stating the average value of g on the equipotential surface that is close to concentric with the surface (sea-level) of the Earth. If we move to an equipotential surface further from the Earth, the gradient will be lower and therefore the value of g decreases. Note that, for a point mass or a perfect sphere

$$g = -\bigtriangledown U = \frac{dU}{dr} = \frac{d}{dr}\left(-\frac{Gm}{r}\right) = \frac{Gm}{r^2} \tag{3.7.3}$$

which gives the more familiar equation for the gravitational acceleration from a point mass, $g = \frac{Gm}{r^2}$.

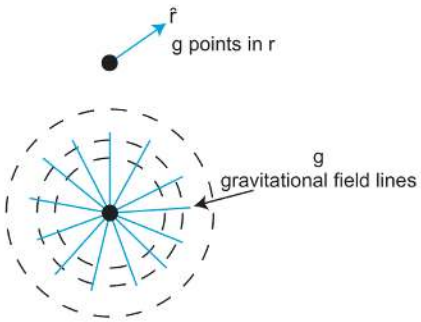


Figure 3.7.1: Equipotential surfaces around a sphere are also spherical. The gravity field lines are perpendicular to the equipotential surfaces, so the point radially outwards.

Reference Equipotential Gravity Surface, U_{ref}

Note, that the because the Earth's shape is ellipsoidal (as discussed in the previous section), the gravitational potential surfaces will also be ellipsoidal: there first-order shape follows the oblate shape of the planet. This is what allowed us to use measurements of the gravitational potential to determine the J_2 term needed to find the moment of inertia factor.

It is useful to introduce the idea of a reference gravitational potential surface, U_{ref} because then we can measure perturbations to a planet's gravitational field and related these to mass anomalies inside.

$$U_{ref} = -\frac{GM}{r} + J_2 \left(\frac{GMa^2}{2r^3} \right) [3\sin^2\phi - 1] - \frac{1}{2}\omega^2 r^2 \cos^2\phi \tag{3.7.4}$$

where the first term is due to the underlying spherical shape, the second term accounts for the equatorial bulge (of the viscous "hyrostatic" solid mantle), and the third term is due to the actual spin of the planet. This reference equipotential surface is either predicted (given the known structure of the planet) or determined by fitting the observed field, which is how we get the observed value of J_2. Note that this is the same equipotential surface that is used to determine the hydrostatic flattening term, f_{hyd}. The term hydrostatic refers to the assumption of a viscously deformable planet.

Mass Anomalies On and In a Planet

Planets like Earth, which has mantle convection, have regions where the rock is either more dense or less dense than the average layered structure. For example, where cold tectonic plates sink back into the mantle, this cold rock is denser than the surrounding. Where hot plumes rise from the core mantle boundary, this rock is less dense (more buoyant) than the surrounding rock. These regions of higher or lower density are termed **mass anomalies**.

How does the shape of the gravitational potential and the direction of the gravity field lines change if the planet is not a homogeneous or concentrically layered planet?

To answer this question, there are two key points to remember:

1. The gravity of the planet can be represented a the sum of the gravity for point mass distributed throughout the planet. This means that if I add mass locally, I can imagine adding a spherical gravitational potential field from that mass to the overall gravitational potential for the whole planet.
2. For a given equipotential surface the magnitude of gravity on the surface is a constant and the magnitude is directly proportional to the mass.

Consider the sketch below. This fictitious planet has a giant (not to scale) topographic mass anomaly on one side. The gravitational field from this mass is added to the gravitational field for the rest of the planet. Therefore, near this mass the equipotential surfaces are shifted outward away from the extra mass.

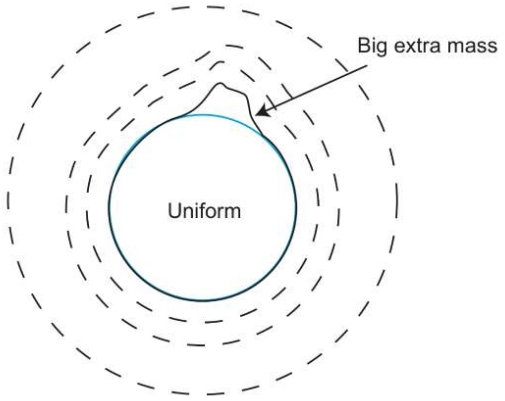


Figure 3.7.2: Non-Uniform Mass: such as Olympus Mons on Mars.

Zooming in on the regions of the planet with the extra topographic mass, we can also consider how the change in the shape of the equipotential surface, from that of a perfect sphere, changes the orientation of the gravity field lines. Remember that the gravity

always points perpendicular to the gradient of the potential field. This direction is always mostly radial on the planet, but locally, the direction can be tilted away from the radial direction due to the extra (or missing) mass.

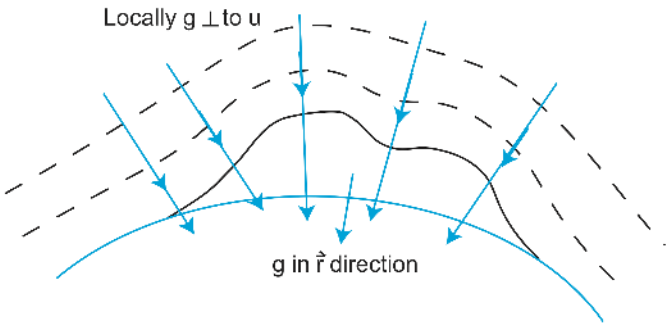


Figure 3.7.3: Local g

While it may be easier to visualize the mass anomaly from a huge mountain on the surface of a planet, it turns out the biggest undulations of the gravity field come from mass anomalies inside the Earth. This is because most of the mass anomalies due to topography are balanced (compensated) by a mass anomaly of lower density crust immediately underneath (see section on isostasy).

The sketch (meant to be purely schematic) below shows two mass anomalies in an otherwise homogeneous planet. The equipotential surface moves outward above the region with more mass (higher density) and it more inward in the region with less mass (lower density).

Now consider how gravity g changes due to the mass anomalies. How does the gradient in the potential surface change near these anomalies? Imagine that the blue (solid) line is the surface of the planet. As you walk north along this line starting at the equator you measure a constant value of gravity g_o until you reach the region affect by the extra mass. As you keep walking north, the magnitude of the gravity increases to a peak and then decreases again on the other side. The value of g changes because you are crossing over equipotential surfaces that are at smaller distances (r) from the extra mass and therefore have larger magnitudes of both U and the gradient of the U.

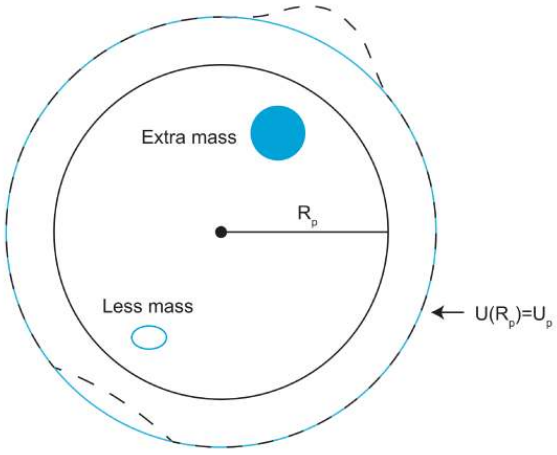


Figure 3.7.4: Mass Anomalies in the planet perturb the gravitational potential field.

Height of the Geoid or "The Geoid"

Because the undulations in the gravitational equipotential surface are related to mass anomalies in the earth, it is useful to find a way of just representing the undulations rather than always considering the full potential field. The sketch below has to surfaces. The blue dashed line represents the reference equipotential surface discussed above, which accounts all of the signal related the layered structure of the earth. The solid black line represents the observed shape of the equipotential surface that is closest to sea-level: this surface is called the geoid. The height of the geoid above or below the reference equipotential surface,

$$N = U_{obs} - U_{ref} \tag{3.7.5}$$

The height of the geoid is measured in meters: it is the radial distance between the two surfaces at every point. Note that N is often referred to as ***the geoid height*** and also (mistakenly) sometimes referred to as "the geoid" (a confusing shorthand jargon that one gets use to). The geoid height is positive above extra mass and negative above regions with less mass. However, interpreting these undulations is not always straightforward because there can be mass anomalies at multiple depths and sinking or rising mass anomalies can create other mass anomalies at the core-mantle boundary and the Earth's surface due to viscous flow that couples to these boundaries.

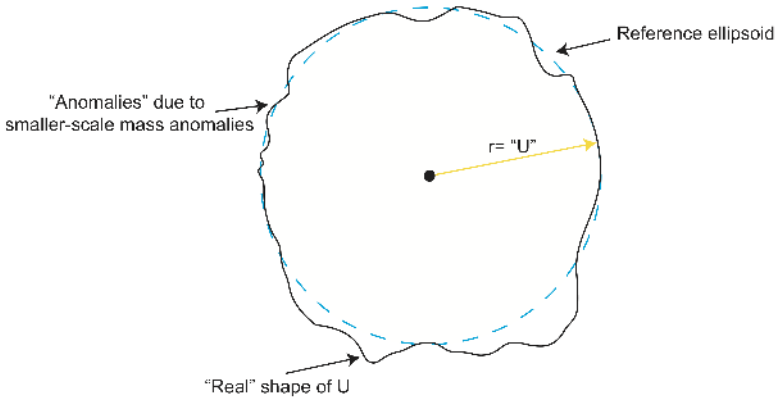


Figure 3.7.7: Geoid height is the distance between the observed and reference equipotential surfaces.

Flow Induced Mass Anomalies

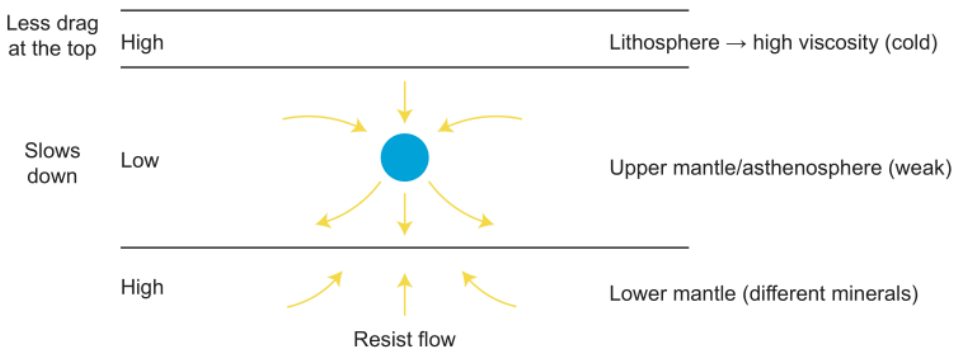
[THIS SECTION NEEDS A NEW FIGURE THAT SHOWS HOW THE SURFACES ABOVE AND BELOW AND DRAGGED/PUSHED DOWNWARD AND CREATE MASS ANOMALIES]

In earth, there are viscous layers as seen below.

- dragging down of the top layer creates a region of less mass because air or water fill in where the lithosphere pulled down.
- below the sinking mass the viscous flow pushes down on the core mantle boundary also creating a region of less mass because less dense mantle material fills in where the denser core material pushed downward.
- the amount the top surface or core-mantle-boundary are deflected depends on viscosity structure of the mantle. In the Earth, the lower mantle has a higher viscosity than the upper mantle. This means that the stress pushing down on the core-mantle boundary will be bigger than the stress pulling down at the surface. So, the mass anomaly at the core-mantle boundary will have a bigger volume than the one at the surface.

The total change to the gravity signal at the surface is due to the original sinking mass anomaly and the flow-induced mass anomalies. The amount each mass anomaly contributes depends on its distance from the surface. In some cases the flow induced mass anomaly at the surface can lead to negative geoid height above a higher density sinking mass because the missing mass at the surface is closer to the surface so it has a bigger effect on the gravity at the surface.

Modeling of the viscous flow and comparison to the observations is one of the primary ways we know the layered viscosity structure inside the mantle.



Spherical Harmonic Representation of the Geoid

The observed gravitational potential field and the geoid height are usually represented in terms of spherical harmonics. Spherical harmonics are like sines and cosines of varying wavelengths, but they depend on (θ, ϕ) lat/lon. For the purposes of this explanation, let's consider the geoid height, which is just the distance above or below the reference. The figure below shows what the spherical harmonic functions look like on a sphere. Let's consider light regions as being positive geoid height and dark regions as being negative geoid height. Each of the spheres represents one function in the series of spherical harmonics that can be added together to represent the observed geoid height. The first sphere (all white) gives that average value on the sphere. When adding each of the higher order functions, the magnitude of a constant factor in front of each term determines how much of that term is added to the sum. The more functions we include in the summation (higher order) the more detailed (short wavelength) we include in our approximation. Expressing an observation in terms of a sum of spherical harmonic functions is also referred to as a spherical harmonic expansion.

As and example, the expansion of the gravitational potential in terms of spherical harmonics is given by,

$$U = \frac{Gm}{r}\left(1 - \sum_{\ell=1}^{\infty} J_2 \ell \left[\frac{a}{r}\right]^{2\ell} P_{2\ell}(\cos\theta)\right) \tag{3.7.6}$$

where $\frac{Gm}{r}$ is the point mass. At $r > a$ (a is the equatorial radius), the value of the fraction decreases. The ellipticity, J_2 is the value of $J_{2\ell}$ when $\ell = 1$ and is the largest term in the summation.



Figure 3.7.6: Spherical Harmonics (CC BY-SA 3.0; Inigo.quilez and Cyp, via Wikimedia)

This page titled 3.7: Gravitational Potential, Mass Anomalies and the Geoid is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

3.8: Summary

Let's review what we've learned about the many aspects of gravity so far.

Orbital Mechanics

We fist covered orbital mechanics, which uses observations and theories to examine the Earth's elliptical orbit, its tilt, and how it spins. One of the most important part of orbital mechanics is Kepler's Third Law, in which the relationship $T^2 \propto r^3$ between the planet's orbital period T and its radius r can be represented in this equation:

$$\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3} \tag{3.8.1}$$

This relationship is very useful for determining the orbital periods or radii of a variety of bodies including suns, moons orbiting planets, and satellites orbiting planets or the sun. Now relating gravity to mass, we found that

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM} \tag{3.8.2}$$

By observing the orbital radius and period of the orbiting object, we can then calculate the mass of the object being orbited.

Moments of Inertia

We next discovered that Earth is not a perfect sphere and is in fact an ellipsoid. Understanding this can help us calculate the planet's moment of inertia and thus learn more about its internal structure. We can usually determine reasonable 2-layered planetary structures with additional information about the composition of density at the surface.

In determining the moment of inertia, several steps must be taken. First, we must assume that the planet behaves like a fluid over long periods of time, ie the hydrostatic assumption.

We can use the Darwin-Radau Approximation:

$$f_{hyd} \approx \frac{\frac{5}{2}\frac{a^3\omega^2}{GM}}{1 + \frac{25}{4}(1 - \frac{3}{2}\frac{C}{Ma^2})^2} \tag{3.8.3}$$

Where we can measure/observe everything except the moment of inertia $\frac{C}{Ma^2}$

To remedy this, the Darwin Radau equation should be solved for the moment of inertia:

$$\frac{C}{Ma^2} \approx \frac{2}{3}(1 - \sqrt{\frac{2}{5}\frac{\frac{a^3\omega^2}{GM}}{f_{hyd}} - \frac{4}{25}}) \tag{3.8.4}$$

Geoid

We next dived into gravitational potential, U, which has the relationship to variables we discussed previously

$$U = \frac{Gm}{r}. \tag{3.8.5}$$

Our ultimate goal is to accurately represent the gravitational potential on the surface of a planet, and to do this we must use spherical harmonics. Summing these spherical harmonies together allows us to represent any surface on a sphere

$$U \approx -\frac{Gm}{r} + \frac{Gma^2}{2r^3}J_2[3\sin^2\phi - 1] - \frac{1}{2}\omega^2 r^2 \cos^2\phi \tag{3.8.6}$$

Isostasy and Satellite-derived Gravity and Geoid

One of the final topics we covered in this chapter is isostasy, which is topography supported by viscous stresses ie pressure, and can be compensated or uncompensated. Smaller feature such as seamounts or trenches are compensated and supported by the strength of the lithosphere thus they have a gravity anomaly. Larger features such as continents and mountain ranges are supported by viscous stresses, ie isostasy, and have no gravity anomaly.

The ultimate goal is to obtain isostatic balance and use known ρ_i and h_i's to find an unknown ρ_i or h_i. Each real world example of isostasy such a continent-ocean or depth of an ocean basin can be written in its simplest form as

$$P = \sum_{i=1}^{n} \rho_i h_i g \; with \; n \; layers \tag{3.8.7}$$

This page titled 3.8: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

4: Plate Tectonics

4.0: Introduction to Plate Tectonics
4.1: The Forces Driving Plate Motions
4.2: Magnetic Anomalies on the Seafloor
4.3 Magnetic Anomaly Jupyter Notebook
4.4: Plates Moving on a Plane
4.5: Relative Velocity and Reference Frames
4.6: Relative Plate Velocity Jupyter Notebook
4.7: Plate Motions on a Sphere
4.8: Summary

This page titled 4: Plate Tectonics is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.0: Introduction to Plate Tectonics

The surface of the Earth is divided into rigid blocks, called **tectonic plates** that are bounded by narrow regions of high deformation called plate boundaries. Tectonic plates are comprised of both crust (oceanic or continental) and mantle rock and owe their rigidity to the stiffness of mantle rock at low temperatures. Tectonic plates come in various shapes and sizes and are continuously changing shape, either through the addition of new crust and lithosphere at mid-ocean spreading centers, or through the loss of material at subduction zones. Tectonic plate shape can also change due to breaking of the plate or accretion of a piece of an adjacent plate.

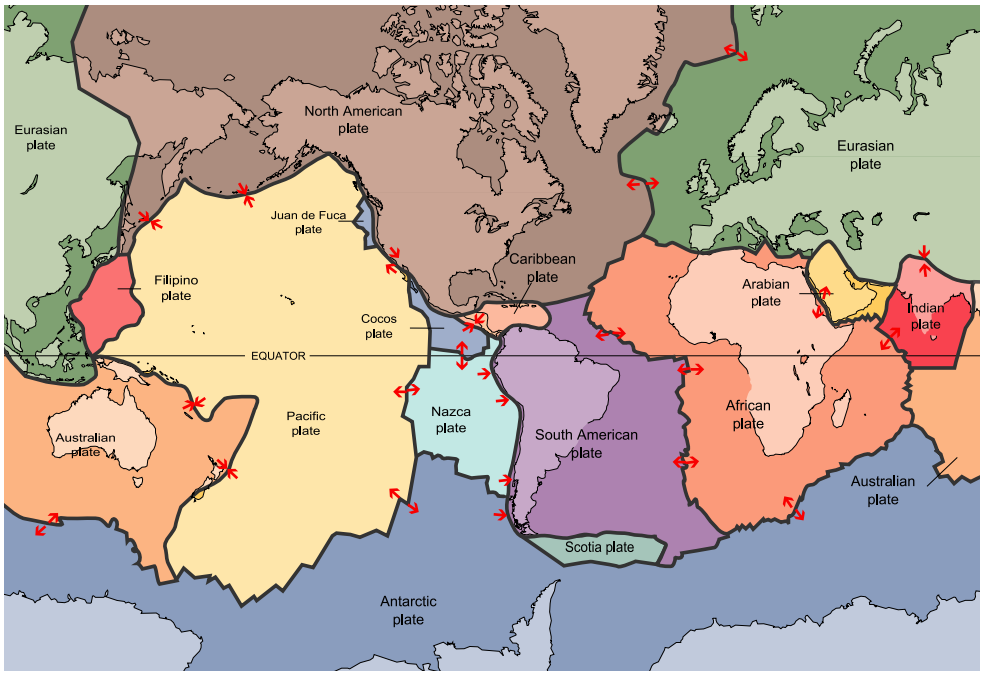


Figure 4.0.1: Plates of the World (Public Domain by USGS-Scott Nash; via Wikimedia)

Tectonic plates move at rates of a few to tens of centimeters per year. However, just as the shaped of the plates are continuously changing, so to are the motion of the plates - both in terms of speed and direction. The complex shapes of the tectonic plates, the geometric constraints of plates moving on the confined surface of a sphere and changes in the forces acting on the plates all lead to changes in plate motion over time. The motion of tectonic plates are primarily driven by gravitational buoyancy forces associated with sinking of cold material into the mantle at subduction zones and rising of hot material at mid-ocean spreading centers. This motion is resisted by viscous forces on the base of the plates, by the strength of the plate itself, which resists bending into the mantle at subduction zones, and by the frictional and viscous forces acting between adjacent plates.

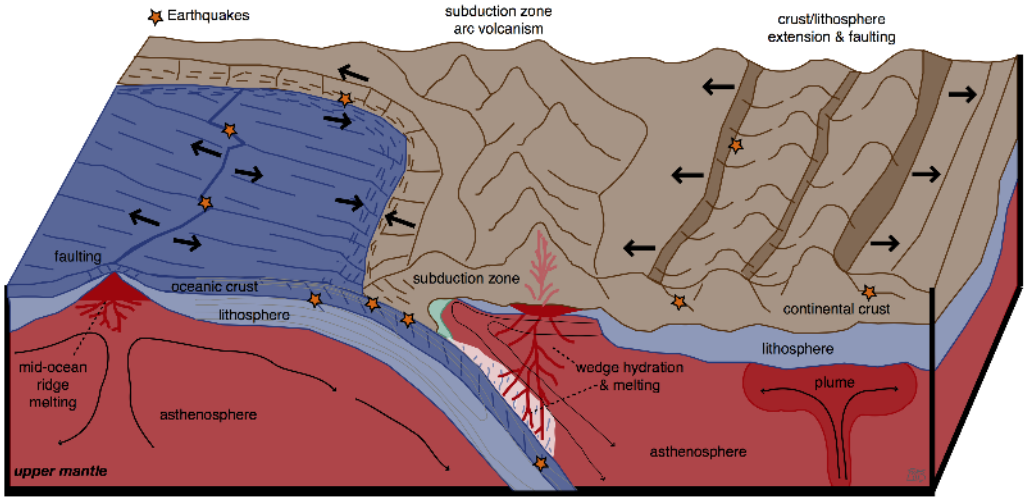


Figure 4.0.2: Plate Tectonics

Detailed observation of both the present and past motions of tectonic plates are essential for addressing many questions in the geosciences, such as understanding the forces driving plate tectonics, the origins of intra-plate deformation, how deformation at plate boundaries manifests as earthquakes, and the physical structure of the deep mantle. Therefore, it is important for geophysicists to be able to both use observations to determine plate motions, and conversely to use to plate motions to make predictions about how those motions are connected to other observations.

The goals of this section is to get you comfortable thinking about plate motions on the surface of a sphere, and to learn some of the fundamental quantitative tools for determining plate motions (i.e., speed and direction) or to use plate motions to calculate related observations (e.g., the relative motion between two points). First, we will consider plate motions on a 2-D plane, how they are related to plate boundaries, and what it means to choose a reference frame. Second, we will review some of the ways in which we can observe plate motions, such as using magnetic anomalies on the seafloor or GPS measurements. Third, we will move to defining plate motions on a sphere using Euler poles and how to determine an Euler pole from observations. Finally, we will learn how to use a model of present-day plate motion to answer questions related to relative plate motions (e.g., How long until San Diego is at the same latitude as Sacramento?) and changes in plate boundaries (e.g., How long until the Juan de Fuca ridge reaches the Cascadia subduction zone?).

This page titled 4.0: Introduction to Plate Tectonics is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.1: The Forces Driving Plate Motions

The motion of tectonic plates is driven by convection in the mantle. In simple terms, convection is the idea that dense, cold things sink, and buoyant, warm things rise. In the earth the cold sinking things are slabs (subducting plates) and the warm things are plumes, or just rising material from deeper in the mantle.

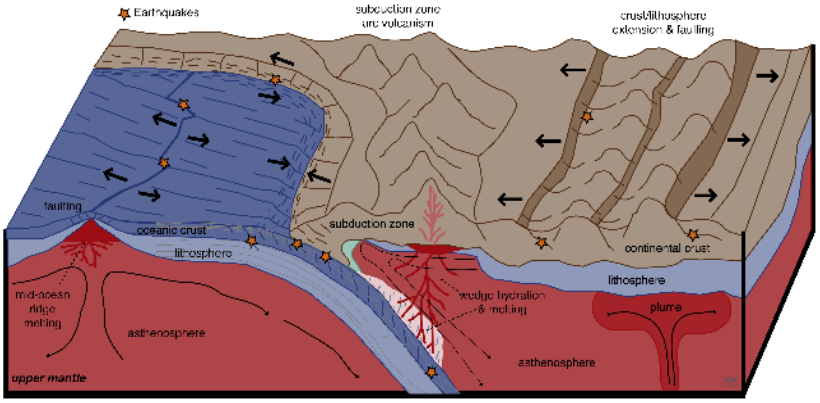


Figure courtesy of Magali Billen, Dept. of Earth and Planetary Sciences, UC Davis)

There are three main forces that determine the rate at which tectonic plates move as part of the mantle convection system:

- slab pull: the force due to the weight of the cold, dense sinking tectonic plate
- ridge push: the force due to the buoyancy of the hot mantle rising to the surface beneath the ridge.
- viscous drag: the force opposing motion of the plate and slab past the viscous mantle underneath or on the side.

This force balance is given by:

$$F_{ridge-push} + F_{slab-pull} - F_{viscous-drag} = 0 \tag{4.1.1}$$

Because the viscous drag depends explicitly on the plate velocity, it is possible to predict the speed plate motion when the forces are balanced (under certain simplifying assumptions).

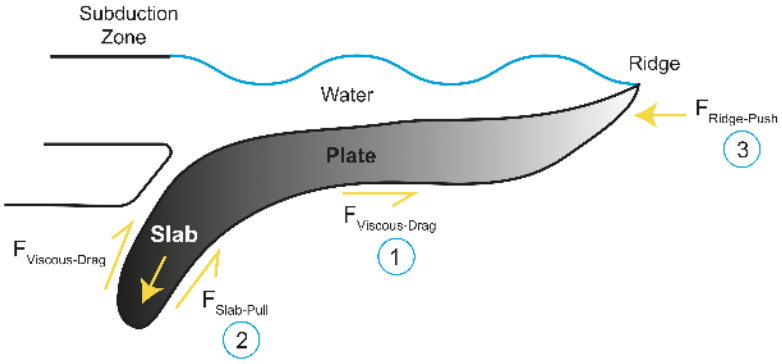


Figure 4.1.2: Forces of Plate Tectonics

Recall from Chapter 1 that stress is related to force as $\sigma = \frac{F}{A}$ where $A = LW$. Rearranging, $F = \sigma A$. When considering the force balance on the a plate it is useful to using a simplified geometry as shown below and consider the force per unit length (or ridge or subduction zone)

$$F = \sigma A / W = \sigma L \tag{4.1.2}$$

As seen in the figure below, W is the length of the plate along the ridge and subduction zone. By considering the force per unit length (in the W direction), we can then consider the forces working along a profiles from the ridge to the subduction zone and ignore any variation in the ridge-parallel direction.

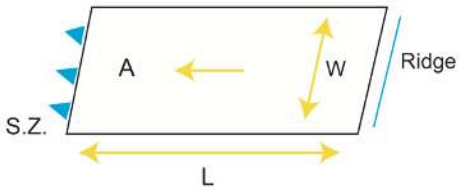


Figure 4.1.3: Simplified plate geometry between the ridge and subduction zone.

Viscous Drag

One of the mechanisms for plate motion that is a result of convection is **viscous drag**. This concept relates to ideas we discussed earlier, including the Couette Flow and viscous stress. The drag on the base of the oceanic lithosphere can both drive or resist plate motion, depending on the relative motion between the plate and the underlying mantle. For this discussion, we will consider that the plate is being pulled or pushed by slab-pull and ridge push and drags the asthenosphere with it. Therefore viscsous drag oppose the motion of the plate (or slab) and acts to slow it down.

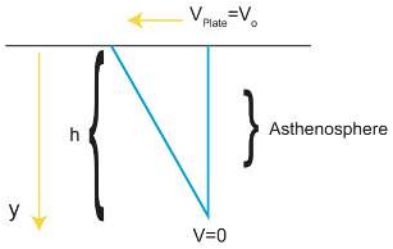


Figure 4.1.4: Plate Velocity

From the figure above, we can derive an equation to describe plate motion due to viscous drag.

$$V = V_o(1 - \frac{y}{h}) \tag{4.1.3}$$

$$\dot{\varepsilon} = \frac{1}{2}\frac{dV}{dy} = -\frac{V_o}{2h} \tag{4.1.4}$$

$$\begin{aligned} \sigma &= 2\eta\dot{\varepsilon} \\ &= 2\eta(\frac{-V_o}{2h}) \\ &= \frac{-V_o\eta}{h} \end{aligned}$$

The negative sign indicates an opposition to plate motion, i.e., the viscous drag. We now have an equation that describes the magnitude of drag:

$$|\sigma| = \frac{V_o\eta}{h} \tag{4.1.5}$$

placing this into our previous force equation (Equation 4.1.2):

$$F = \sigma L = \frac{V_o\eta}{h}L \tag{4.1.6}$$

Let's do an example calculating viscous drag under a plate.

Example Viscous Drag Under the Plate

Assume that:

- the plate drags the mantle underneath in a layer that is 100 km thick (h~100 km).
- the mantle viscosity is η 1×10^{18} Pa(s).
- the plate velocity is V_o~5 cm/yr. (Note that this is equivalent to 0.05 m/yr = 50 mm/yr = 50 km/my
- the plate length from the ridge to the subduction zone is about 5000 km

From the strain-rate, we can get the stress as shown above, and then multiply by the length of the plate

Using these numbers, we can calculate the force on the descending plate, $F = 1.59 \times 10^{11}$ N/m.

The above example is for the viscous drag just under the plate, but now we will also consider a case where there are stresses acting on both sides of the subducting slab. As the slab sinks into the mantle, the viscous mantle on both sides resists the sinking motion of the slab. For this simple example,

$$\sigma = \frac{V_o \eta_{av}}{h} \tag{4.1.7}$$

$$F = \sigma L_{slab} \tag{4.1.8}$$

$$F_{vis.} = 2F = 2\sigma L_{slab} = \frac{2V_o \eta_{av}}{h} L_{slab} \tag{4.1.9}$$

Example Viscous Drag on Two Sides of a Plate

As we did in the previous example, consider viscous drag on a slab, but now on both sides of the plate.

We are given that

- The slab sinks at the same speed that the plate is moving: V_o=V_{plate}~5 $\frac{cm}{yr}$,
- Again, the shearing layer on both sides is 100 km wide (h~100 km),
- The viscosity of the mantle increases with depth, for this example we'll assume its constant $\eta_{av} = 10^{21}$ Pa s ,
- The slab length (from the base of the lithosphere to the upper/lower mantle boundary is L_{slab} = 560 m.

In this case, $F = 78 \times 10^{13}$ N/m. If we change η to 10^{21}, then equals $F = 1.78x10^{10}$ N/m.

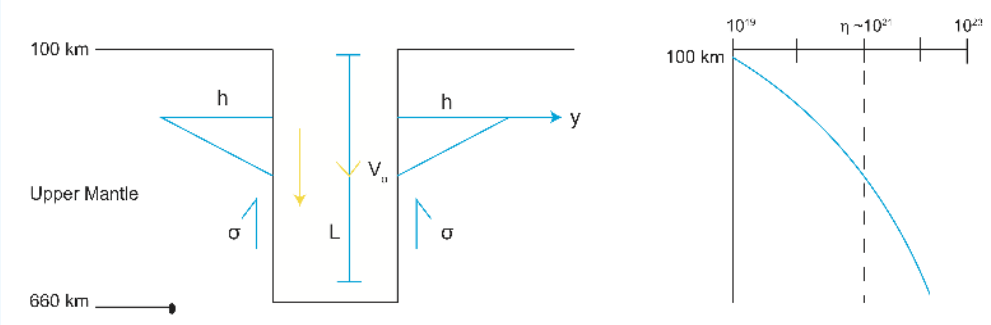


Figure 4.1.5: Force on a Descending Plate

On the sides of the slab, the above figures demonstrate the forces that are present. We can also see that the slab is sinking vertically, which is not often the case in the real world. From the right figure, $\eta \approx 10^{21}$. This is actually too high because of non linear viscosity, but we are ignoring other things so it works for this example.

Notice that the stress is proportional to the velocity $\sigma \alpha V$ so as the velocity increases so does the stress opposing the sinking.

However, the biggest changes come from the viscosity because it can changes by x10-100 in the upper mantle, thus this change has the biggest effect on the viscous drag.

The viscous drag is also proportional to the length of the slab, so the resistance also increases as the slab gets longer (but so does the slab pull force).

Slab-Pull Force

The main force on a subducting plate is the slab-pull force. This is the force due to the density contrast between a slab and its surroundings, ie., the mass anomaly of the slab in the mantle. We can approximate this force by thinking of the sinking slab a a vertical rectangle (like the viscous drag example) with a width, w, hanging down a distance, d from the Earth's surface with a temperature ΔT different than its surroundings.

The density as a function of temperature is given by

$$\rho(T) = \rho_m(1 + \alpha(T_m - T)) \tag{4.1.10}$$

where ρ_m is the reference mantle density at $T = T_m$ and cold material has a higher density, and α is the thermal expansion coefficient and is ~2×10^{-5} 1/K in the sinking slab.

For the slab, we can rewrite this as:

$$\rho(T) = \rho_m(1 + \alpha\Delta T) \tag{4.1.11}$$

where $\Delta T = T_m - T_{slab}$

The density anomaly is then

$$\Delta\rho = \rho_m\alpha\Delta T \tag{4.1.12}$$

This is the extra density of the slab relative to the mantle. To calculate the force per unit area, we need to determine the mass anomaly per unit area, $\Delta M = \Delta\rho A_{slab}$, where A_{slab} is the cross sectional area of the slab along the profile (width across times length into the mantle).

The temperature in the slab is related to the temperature of the subducting plate. Recall from the chapter on thermal diffusion that the temperature of a tectonic plate is given by the half-space cooling model. Once the plate starts sinking, the cold slab warms up, and the surrounding mantle cools down. However, the total temperature anomaly (temperature difference times volumes) stays the same. (Remember that there is conservation of energy, so the total energy of the cold slab is conserved). So, if we integrate the temperature profile solution from the half-space cooling model over the thickness of this plate, this will also be the thermal anomaly of the the slab. Finally, because the density anomaly is just

$$\Delta\rho = \rho_m\alpha\Delta T \tag{4.1.13}$$

, we can get the total density anomaly by just multiplying by $\rho_m\alpha$.

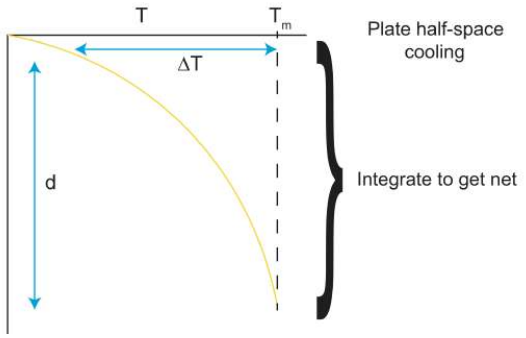


Figure 4.1.6: Temperature in the subducting plate before subduction.

So, the trick to figuring out the slab pull force is the first determine the density anomaly of the subducting plate just before it enters the trench. Then imagine taking a length of plate equal to the length of the slab and rotating it en masse in to the mantle. The density anomaly of the sinking slab will be the same.

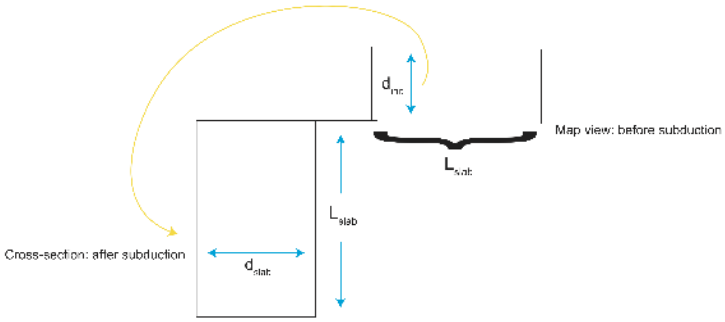


Figure 4.1.7: Temperature is Not Lost

To get the density anomaly, we integrate the temperature equation for the plate from the half-space cooling model.

$$T(z,t) = (T_s - T_m)erfc(\frac{z}{2\sqrt{\kappa t}}) + T_m \tag{4.1.14}$$

Here z is the depth through the subducting plate, *before it sinks*, or the width through the slab *while it is sinking*.

$$\Delta T = T_m - T(z,t) \tag{4.1.15}$$

$$\Delta T = -(T_s - T_m)erfc(\frac{z}{2\sqrt{\kappa t}}) \tag{4.1.16}$$

$$\Delta T = (T_s - T_m)erfc(\frac{z}{2\sqrt{\kappa t}}) \tag{4.1.17}$$

Then, multiplying the $\rho_m \alpha$,

$$\Delta \rho = \rho_m \alpha \Delta T \tag{4.1.18}$$

$$\Delta \rho = \rho_m \alpha \Delta T(z) \tag{4.1.19}$$

Next we integrate over the width (depth) to the get the mass per unit length (length parallel to the trench) in a *single profile*, (we will need to sum this over the length of the slab):

$$\Delta m = \int_0^{z \to \infty} \Delta \rho(z) dz \tag{4.1.20}$$

$$\Delta m = \rho_m \alpha (T_m - T_s) \int_0^{\infty} erfc(\frac{z}{2\sqrt{\kappa t}}) dz \tag{4.1.21}$$

Now we use variable substitution so we can solve the integral

Let $q = \frac{z}{2\sqrt{\kappa t}}$ where $z \to \infty$ and $q \to \infty$.

$$\frac{dq}{dz} = \frac{1}{2\sqrt{\kappa t}} \tag{4.1.22}$$

$$dz = 2\sqrt{\kappa t} dq \tag{4.1.23}$$

Now substitute into the integral,

$$\Delta m = \rho_m \alpha (T_m - T_s) 2\sqrt{\kappa t} \int_0^{\infty} erfc(q) dq \tag{4.1.24}$$

The complementary error function integral solution can be found in any integral table (or search online) and equals $\sqrt{\frac{1}{\pi}}$

$$\Delta m = \rho_m \alpha (T_m - T_s) 2\sqrt{\frac{\kappa t}{\pi}} \tag{4.1.25}$$

The above equation gives the mass anomaly (relative to the surrounding mantle) per unit length.

Finally, we can determine the total mass anomaly by multiplying by the length of the slab:

$$\Delta M = \Delta m L_{slab} \tag{4.1.26}$$

The slab pull force is then given by,

$$F_{slab-pull} = \Delta m L_{slab} g \tag{4.1.27}$$

and subsituting for Δm,

$$F_{sp} = 2\rho_m \alpha (T_m - T_s) g L_{slab} \sqrt{\frac{\kappa t}{\pi}} \tag{4.1.28}$$

Example Slab-Pull Force

We are given that

- ρ_o=3300 kg/m^3,
- $\alpha = 2e-5$ 1/K, g=9.81 m/s^2,
- T_m-T_s=1400 K,
- $\kappa = 1e-6$ m^2/s,
- t=age of slab *at time of subduction*,
- L_{slab}=560 km.

We know that $F_{sp} = \Delta m L_{slab} g$

Thus, the force per unit length of the slab-pull is F_{sp}=2.88x10^{13} N/m

You might notice that this force is 100 - 1000x more than the viscous drag force.

Ridge-Push Force

Let's cover a final force a subducting plate would experience, the ridge-push force. This force results from the elevation of oceanic ridges above the seafloor. This difference in height leads to pressure that 'pushes' the plate away from the ridge. Ridge push is the simplest force in some ways, as all its components can be easily examined.

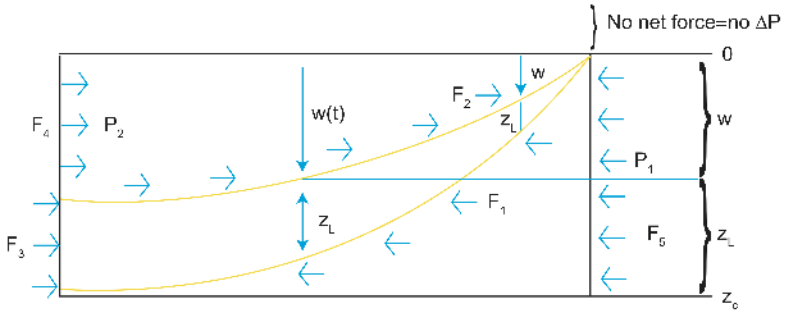


Figure 4.1.9: Ridge Push Forces

Simply put, to determine ridge-push forces, we need to do two things. First, look at the isostatic balance and the depth of oceans w(t). Second, we need to look at the force balance on the plate.

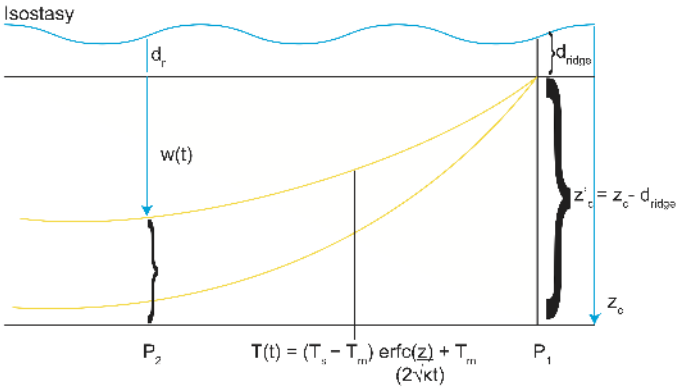


Figure 4.1.10: Ridge Push

$$T(t) = (T_s - T_m) erfc(\frac{z}{2\sqrt{\kappa t}}) + T_m \tag{4.1.29}$$

$$\rho(T) = \rho_m(1 + \alpha \Delta T) \tag{4.1.30}$$

$$\Delta T = T_m - T(t) \tag{4.1.31}$$

T(t) is $<T_m+\Delta T$ because of more density.

Balance pressures:

$P_1=P_2$

$$drp_w g + \int_{dr}^{dr+w} \rho(T) g dz + \int_{dr+w}^{z_c} \rho(T) g dz = (dr+w)\rho_w g + \int_{dr+w}^{z_c} \rho(T) g dz \tag{4.1.32}$$

$$\int_{dr}^{dr+w} \rho_m dz + \int_{dr+w}^{z_c} \rho_m dz = w p_w + \int_{dr+w}^{z_c} \rho(z) dz \tag{4.1.33}$$

$$w\rho_m - w p_w = \int_{dr+w}^{z_c} \rho(z) - \rho_m dz \tag{4.1.34}$$

$$\rho(z) = \rho(T) \rightarrow T(z) \tag{4.1.35}$$

Like we did above, we will use the complementary error function and find its solution in a integral table.

$$w(\rho_m - \rho_w) = \rho_m \alpha (T_m - T_s) \int_0^\infty erfc(\frac{z'}{2\sqrt{\kappa t}}) dz' \tag{4.1.36}$$

The complementary error function integral solution is $2\sqrt{\frac{\kappa t}{\pi}}$

$$w = \frac{2\rho_m \alpha (T_m - T_o)}{\rho_m - \rho_w} \sqrt{\frac{\kappa t}{\pi}} \tag{4.1.37}$$

An important thing to note is that the depth of the ocean basins deepen at $\sim\sqrt{t}$ (the square root of age)

⇒balanced pressures (isostasy)

Ridge-push is very similar to the balanced pressures idea, but uses balanced forces instead.

⇒ΔP⇒drives flow across A. No net force⇒no ΔP.

$$F \backsim \Delta P A \tag{4.1.38}$$

This is the force per unit width.

Example Ridge-Push

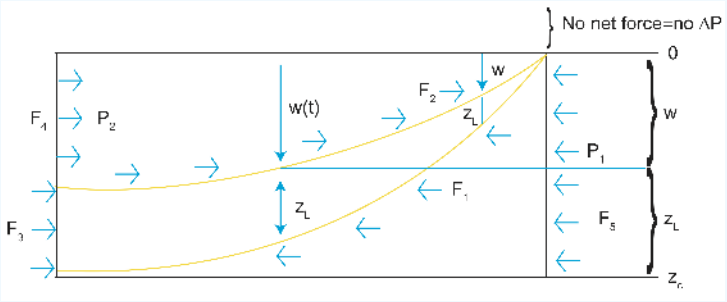


Figure 4.1.9: Ridge Push Forces

The forces in this example should balance. After we write the integrals for the various forces, we should have an expression where $F_{RP}=F_1-F_2-F_3$.

$$F_1 = \int_0^{z_c} \rho_m g z dz = F_s \tag{4.1.39}$$

Here, $\rho_m g z$ represents the pressure, dz is XA⇒Area⇒wXL$\frac{area}{w}$=L

We can do a simple integral in depth, this is the same pressure from the mantle below a plate.

Rewriting F_1 $0\rightarrow z_c$, $0\rightarrow$ w + w-zL

$$F_1 = \int_0^w \rho_m g z dz + \int_w^{w+z_c} \rho_m g z dz + \int_0^{2L} \rho_m g \bar{z} d\bar{z} \tag{4.1.40}$$

Where we let $\bar{z} = z - w$

$$F_2 = \int_o^w \rho_w g z = F_4 \tag{4.1.41}$$

$\rho_w g z$ is P_2. Here, we do not need to integrate over shape of w.

$$F_3 = \int_0^{z_L} P_L d\bar{z} \tag{4.1.42}$$

$$P_L = \rho_w g w + \int_0^{\bar{z}} \rho_L(T) g d\bar{z} \tag{4.1.43}$$

Here, $\rho_w g w$ is the pressure due to the water and ρ_L is the temperature-dependent density of the lithosphere.

Remember our force balance equation:

$$F_{RP} = F_1 - F_2 - F_3 \tag{4.1.44}$$

Plugging everything in and skipping some of the basic algebra we get:

$$F_{RP} = g[\frac{1}{2}(\rho_m - \rho_w)w^2 + \kappa t \rho_m \alpha (T_m + T_s)] \tag{4.1.45}$$

We get the term w^2 because $P_L \sim w$ and we integrated P_L.

From isostasy, we can plug in

$$w(t) = \frac{2\rho_m \alpha (T_m - T_s)}{(\rho_m - \rho_w)} \sqrt{\frac{\kappa t}{\pi}} \tag{4.1.46}$$

$$F_{RP} = g\rho_m \alpha (T_m - T_s)\kappa t (1 + \frac{2p_m \alpha (T_m - T_s)}{\pi(\rho_m - \rho_w)}) \tag{4.1.47}$$

Notes on Ridge-Push

A few things to note: isostasy → controlling w(t)

The pressure across the lithosphere → ridge-push force

$F_{RP} \alpha t$ age of the oldest part of plate

Now that we have our general solution for the ridge-push force, if we are given actual values, we can solve it. If g=9.81, k=1e^{-6} $\frac{m^2}{s}$, ρ_m=3300 $\frac{ky}{m^3}$, T_m-T_s=1400 K, α=2x10^{-5} $\frac{1}{K}$, ρ_m=1000 $\frac{kg}{m^3}$, we can finally find the force

$$F_{RP} = 3.24x10^{12} \frac{N}{m} \tag{4.1.48}$$

This page titled 4.1: The Forces Driving Plate Motions is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.2: Magnetic Anomalies on the Seafloor

Information about the motion of tectonics plates comes from both direct measurement of the plates location during the present day and information about the age and geometry of plate boundaries preserved in the rocks themselves. For tectonic plates with continents, it is possible to measure the present-day motion of the plates using GPS (Global Positioning System). To measure the motion accurately enough, special GPS measuring stations are established and continuously record the location of the station. By then calculating the change in location over a time interval, we can determine the velocity of that point on the plate. By repeating this at multiple locations, the overall motion of the plate can be determined. However, for tectonic plates beneath the oceans, or for past plate motions we must rely on information recorded by the rocks themselves.

While there are multiple ways to determine the age of rocks, such as radiometric dating and fossil dating, for large-scale plate tectonic studies the most useful way of determining the age of plates is using magnetic stratigraphy. When sea floor is created at spreading centers magma is emplaced at shallow depth or erupted at the surface to form the crust of the growing plate. Certain minerals in the magma (e.g., magnetite) are sensitive to the Earth's magnetic field. As the magma cools, magnetic domains in these minerals will align with the Earth's magnetic field locking in the orientation (dip relative to horizontal) and polarity (field lines pointing out or field lines pointing in) of the magnetic field at that location. This would not be all that useful except that the Earth's magnetic field reverses direction in an aperiodic (non-repeating) pattern. Also, the Earth's magnetic field is dominated by a dipole field similar to what one gets from a simple bar magnet with a "north" end (positive end: magnetic field lines leave the magnet) and a "south" end (negative end: magnetic field lines enter the magnet). Therefore, at certain times the positive (north) pole of the magnetic field is close to the north pole of the Earth, while at other times the positive pole of the magnetic field is close to the south pole. Because this pattern of reversals is non-repeating, it acts like a bar code or finger print with a distinct pattern associated with different time intervals in the geologic past. At present, the negative magnetic pole located near the geographic north pole: this is termed a "normal" orientation. Times when the positive magnetic pole is located near the geographic north pole are termed "reversed".

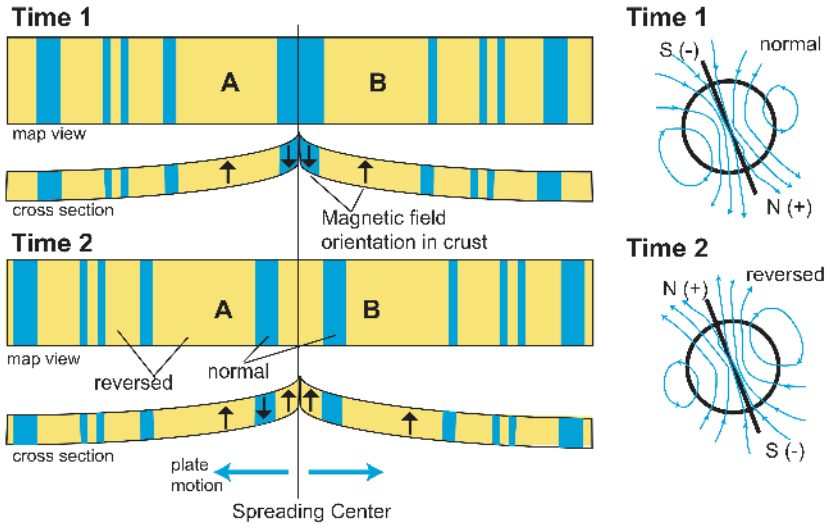


Figure 4.2.1: Magnetic Anomalies

Using lava eruptions on land, and dating these using radiometric dating methods, scientist have determined the pattern of reversals including the start and end times of each reversal going back about 250 million years. Using this "bar code" (called the Geomagnetic reversal time scale) one can determine the age of oceanic crust by measuring the present-day magnetic field, removing the contribution from the current magnetic field, and then analyzing the magnetic "anomalies" that remain. One must have some sense of whether the plate was in the northern or southern hemisphere at the time it formed. This is needed so we can determine whether a positive magnetic orientation is indicative of a "normal" orientation of the magnetic field or a reversal. The geographic orientation of the ridge can also cause the measured anomalies to appear asymmetric or skewed: this effect can be explored by calculating what anomalies would be expected for different orientations using calculation of the dipole field for the

earth. Also, because the Earth's magnetic field is parallel to the Earth's surface at the magnetic equator, there is no information about the orientation of the magnetic field, and the crust in these locations can not be dated using magnetic stratigraphy.

Geomagnetic Timescale for the Neogene and Quaternary

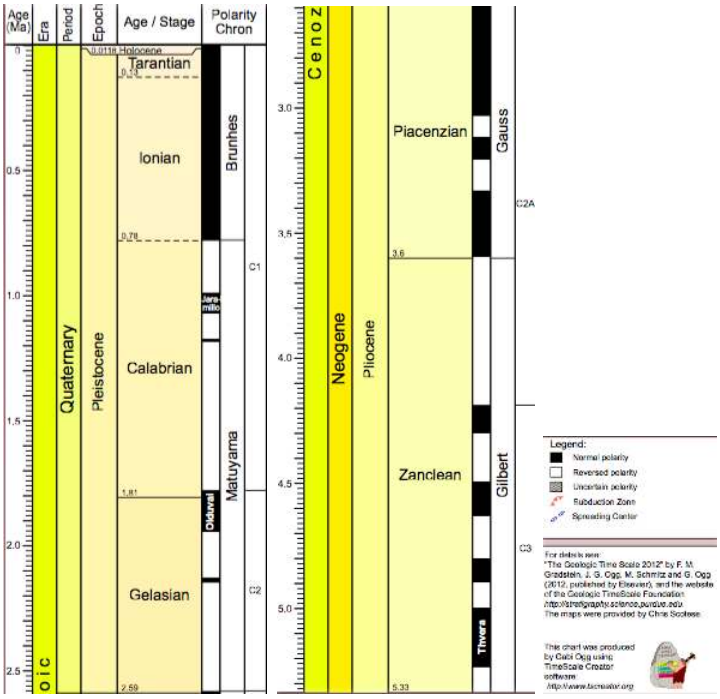


Figure 4.2.2: Neogene and Quaternary Timescale

Geomagnetic Time-scale for 0 to 145 my

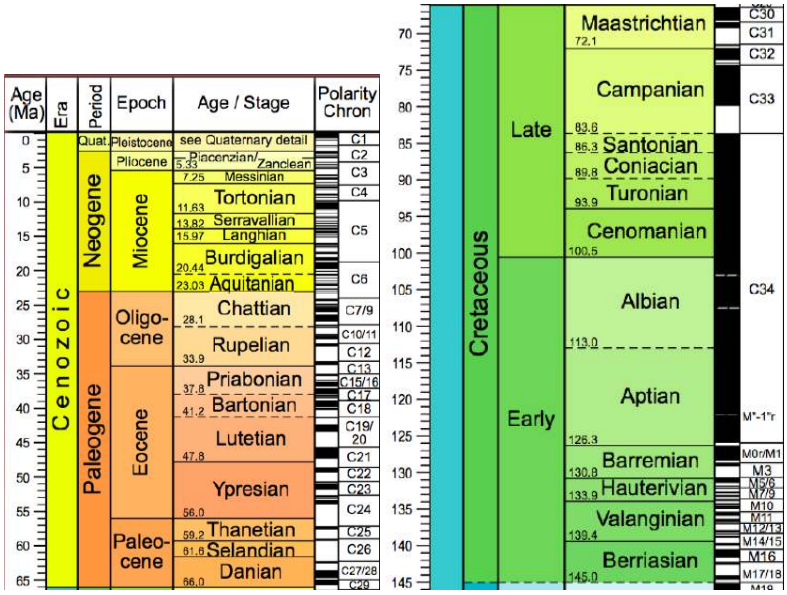


Figure 4.2.3: 0-145 My Geomagnetic Timescale

Geomagnetic Time-scale for 145 to 250 my

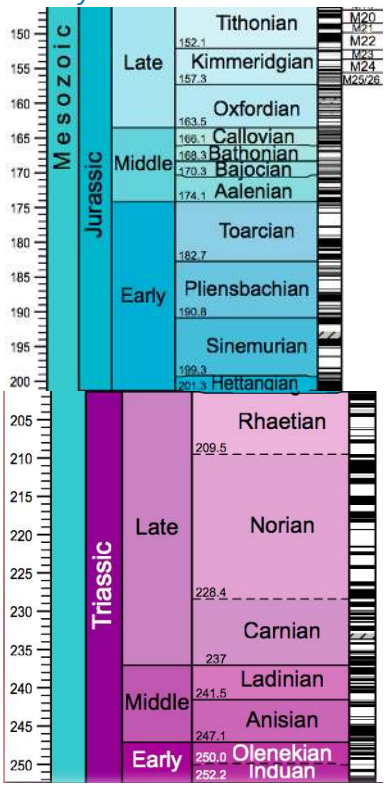


Figure 4.2.4: 145-250 My Geomagnetic Timescale

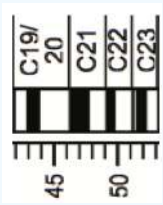
Take a little time to check out the patterns in the geomagnetic timescale shown above. What do you see? First note that when we just focus on the last 5 my, there are some very short reversals of the time-scale. Some are so short that they could be missed completely when looking at seafloor anomalies, especially at very slow spreading ridges in which time is represented by smaller widths of seafloor parallel to the spreading ridge. Second, notice the non-repeating nature of the pattern. There really is no pattern to speak of - this is exactly what makes it so useful for determining the age of seafloor. Third, notice that most of the periods of normal or reversed polarity at less than a million years to a few million years. There is one big exception to this and this is the very long period of normal polarity in Cretaceous, which extends from 126.3 to 83.6 my, a duration of 42.7 my. This long period of normal polarity is referred to as the Cretaceous quiet zone - quiet referring to the lack of magnetic field reversals.

Spreading Rate from Magnetic Anomalies

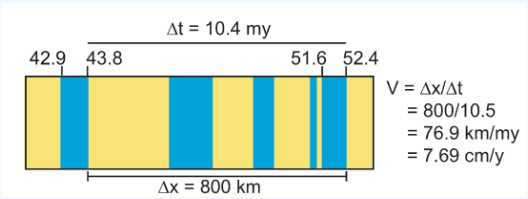
Determining the spreading rate (rate of crust accreted to the plate) from the magnetic anomalies is done in several steps. We assume that the magnetic anomalies have already been analyzed to identify the normal and reverse polarity anomalies taking into account the location (northern/southern hemisphere) and orientation (north-south versus east-west) of the ridge (north/south). You then have a "bar code" of normal and reverse polarity intervals of varying lengths. First, just look at the pattern (see example below) -- what do you see? Are there lots of reversals, or just a few. Are the reversals all similar length or different lengths? Is the pattern symmetric with respect to any point on the profile? This last question is key because a symmetric pattern indicates that there is an active or extinct spreading center in the profile, and therefore, you should only be considering the anomalies on one side of the profile in trying to match the pattern of reversals. Next, try to identify some specific pattern... short-short-long-long-short ... and find a similar pattern in the reference geomagnetic timescale. Here's a test section. Try finding where this fits in the time-scale above (hint its in the Cenozoic):



Note, you are looking at pattern, not the specific width of the reversals as these will depend on the actual spreading rate that formed the crust. A fast spreading rate will form wider bands because more crust is formed during each time interval. A slow spreading rate will form narrower bands. Initially, you should assume that the spreading rate was constant for the whole time interval. Once you think you have identify a section of the reference time-scale that matches your observation, look to the adjacent anomalies and see whether they also match with what comes next. If it matches - great, you can start marking down which normal and reverse isochrons match your profile. If they don't match, repeat the procedure until you find a consistent match of normal and reversed periods for the whole profile. Here's the solution to the above test section:



Once the anomalies are matched, the spreading rate is calculated by noting the start and end time of an anomaly at each end of the profile. Then calculate the time duration between the start or end of the first anomaly and the second anomaly Δt and the distance Δx between these two points on the profile. The spreading rate (velocity) is $v_s = \frac{\Delta x}{\Delta t}$.



More Practice in Determining Spreading History

If *we have time we can add this section with updated magnetic plots*

This practice is very similar to what you will do in class.

For each profile:

- Determine if there is a pattern to the symmetry in the profile or not
- Look for distinct patterns that you can search for in the magnetic time-scale plots
- When you think you have a match, check that the adjacent patterns also fit
- Once you have a match, determine the age for the ends of the profile and symmetry position
- Determine the spreading history (the scale is in km)
- Describe in words a scenario for the history

FIGURE a

FIGURE f

FIGURE geomagnetic reversal time scale

FIGURE a answer

FIGURE f answer

10 my$\Rightarrow \frac{100km}{10my} = 100\frac{km}{my}$

$\Rightarrow$Dead ridge:was spreading at $\sim\frac{100mm}{10yr}$ for 10 my then died at 155 my.

This page titled 4.2: Magnetic Anomalies on the Seafloor is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.3 Magnetic Anomaly Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An example about the distribution and patterns of seafloor magnetic anomalies, using real data.

Key Questions: Consider these as you work your way through this page.

- What is unique about the pattern of the seafloor stripes when there is a ridge in the data?
- When the spreading rate is increased, do the magnetic anomalies get closer to each other or farther apart? Stated differently, when the spreading rate is increased is the distance between the youngest and oldest age larger or smaller?
- When the spreading rate is decreased, why does it become difficult to distinguish individual anomalies?

First, let's import the necessary libraries, and load the data file. This cell is going to output a lot of information from the wget command, this is not an error and you may ignore this. This command searches the given url to load the file into the code cell, then np.loadtxt loads the data into a numpy array.

```
import numpy as np
import matplotlib.pyplot as plt
from pylab import rcParams
!wget "https://geo.libretexts.org/@api/deki/files/9811/polarity.txt?origin=mt-web"
data = np.loadtxt('polarity.txt?origin=mt-web',usecols=[0,1])
print('The libraries are imported and the data is loaded, move on to the next cell')
```

run | run all | restart | restart & run all

Now that the data is loaded it must be organized. This dataset only listed the times that bound polarity changes, so to plot, we must assign a value of 0 for reverse polarity, or 1 for normal polarity. The last piece of the cell defines some constants for conversion.

```
my_array = data.flatten()
seafloor_data = np.empty((len(my_array),2))
```

```
seafloor_data[:,0] = my_array
for i in range(len(my_array)):
    if i % 2 == 0:
        seafloor_data[i,1] = 0 # Reverse Polarity
    else:
        seafloor_data[i,1] = 1 # Normal Polarity

yr2my = 1e6   # yr/my
sec2yr = 3600*24*365   # s/yr

print('The data is organized and conversion constants defined')
```

run | run all | restart | restart & run all

This cell will guide you to create a magnetic polarity profile. The minimum time refers to the youngest age that you wish to view, and maximum time refers to the oldest age you wish to view. The rest of the questions are self-explanatory, so go ahead and create your model and use it to answer the key questions. Note: If you select the dead ridge option, the changes in the graph may not be that obvious so be sure to look at the distance axis and the information printed below the graph.

```
bbox = dict(boxstyle ="round", fc = '1')
print('First Let\'s examine an age range to look at the data. It is recommended not to exceed a 25 Ma a
print('Please Enter Your Minimum Age as a multiple of 5 between 0 (active ridge) and 110 Ma:')

start_time = float(input())
while start_time < 0 or start_time > 110 or start_time % 5 != 0:
    print('Invalid minimum age, Please Enter a time between 0 and 110 Ma (as a multiple of 5)')
    start_time = float(input())
print('Now enter a maximum age, between',start_time,'Ma and 115 Ma (as a multiple of 5)')
end_time = float(input())
while end_time <= start_time or end_time > 115 or end_time % 5 != 0:
    print('Invalid maximum age, Please Enter a time between', start_time,' and 115 Ma (as a multiple of
    end_time = float(input())
if start_time == 0:
    ridge = 'yes'
else:
    print('Is the start time you Selected the Location of a dead ridge? Yes or No')
    print('Note: If the location is not actually a dead ridge in the data, it will still be plotted sym

    ridge = input()
    while ridge.lower() != 'yes' and ridge.lower() != 'no':
        print('Invalid responce: please answer yes or no')
        ridge = input()
ridge = ridge.lower()
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

- What is unique about the pattern of the seafloor stripes when there is a ridge in the data?
- When the spreading rate is increased, do the magnetic anomalies get closer to each other or farther apart? Stated differently, when the spreading rate is increased is the distance between the youngest and oldest age larger or smaller?
- When the spreading rate is decreased, why does it become difficult to distinguish individual anomalies?

The data for this example was retrieved from http://deeptow.whoi.edu/gpts.html and are from Cande and Kent, 1995.

4.3 Magnetic Anomaly Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

4.4: Plates Moving on a Plane

Relating Plate Motion to Plate Boundaries

Let's start by considering the motion of two rectangular plates: plate A and plate B. Let's hold plate B fixed in space so it is not moving. If plate A moves away from plate B space is created between the two plates. On the surface of the earth, this space is filled in through melting of the mantle beneath the plate boundary and emplacement or eruption of the magma to form oceanic crust. This plate boundary is referred to as divergent boundary or a spreading center. If plate A moves toward plate B, it either needs to compress or slide above or beneath plate B. On the surface of the Earth, this motion is accommodated either through shortening of plates A and B in a collision (case 1) or plate A needs to sink into the mantle below plate B (or vice versa), a process known as **subduction** (case 2). This type of plate boundary is referred to as a convergent boundary or as collision zone (case 1) or subduction zone (case 2). If plate A slides past plate B parallel to the plate boundary, then no plate is created or destroyed at the plate boundary. This plate boundary is referred to as transform boundary and exhibits strike-slip motion (meaning motion parallel to the strike of the boundary.

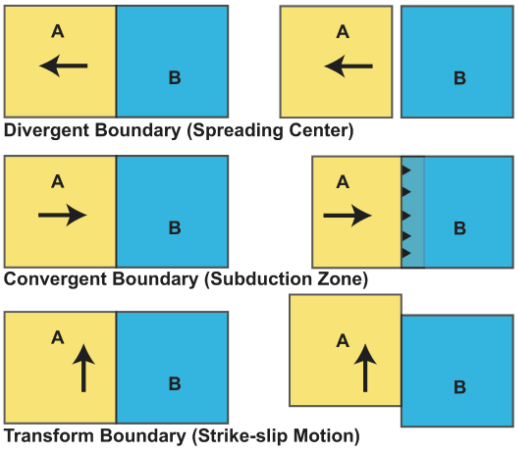


Figure 4.4.1: Non-oblique Plate Boundaries

Now consider a slightly more interesting geometry in which plate A has a triangle shape. Depending on the direction that plate A moves relative to plate B and relative to the plate boundaries, the motion across the plate boundaries changes, and thus the type designation for each plate boundary changes. For example, if plate A moves parallel to the side 2 and toward plate B, then the boundary at side 2 is a transform and the boundary at side 3 is a convergent boundary. However, with a small change in the direction of plate A to the north, side 2 becomes a spreading center. In this case because the motion of plate A is not perpendicular to or parallel to the plate boundary, we say that the motion is oblique. There is oblique divergence at side 2 and oblique convergence at side 1. Such oblique motions are often manifest in different geometry of faults than at non-oblique boundaries. These faults work together to partition the deformation into parallel and perpendicular components.

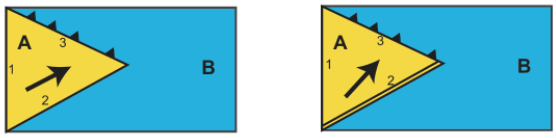


Figure 4.4.2: Oblique Plate Boundaries

From these two simple examples, we can see that plate boundary types, in a plate tectonic sense, are defined by the relative velocity of the plates across the boundary. As the forces driving plate motion change, the relative velocity can change. If the change in velocity is large enough it can lead to a change in the type of plate boundary. As an example, this type of plate boundary change is occurring across the Macquarie Ridge south of New Zealand. This boundary was a spreading center where the Pacific plate was moving away from the Indian Plate. However, in recent times, the plate motions have change such that the two plates are converging across the same plate boundary turning it into an incipient subduction zone.

Let's do an example together demonstrating how plates move.

Plate Motion on a Plane

From the figure below, we can see that A is fixed, and the trench is fixed to A. The ridge moves with B and leaves a trail behind. Let's say we started out with the below configuration at 10 MA. What does it look like now?

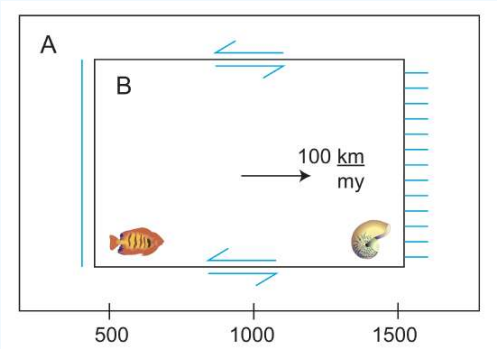


Figure 4.4.3: Plate Configuration at 10 MA

Solution

10 my⇒10 my $\frac{100km}{my}$=1000 km

Which plate moves 1000 km? Plate B. But, plate B is also growing at $\frac{1}{2}V$, as is plate A. So how do we account for that to determine the total plate motion?

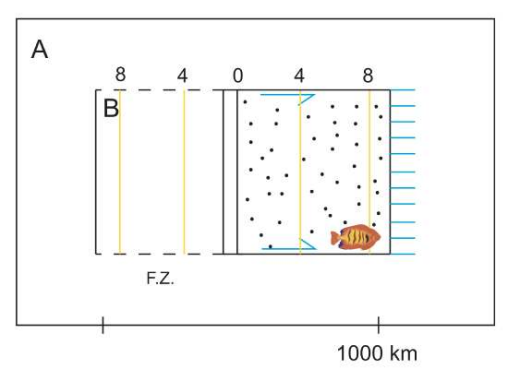


Figure 4.4.4: New Plate Configuration

1. Fish moves 1000 km
2. Ridge moves half as far
3. Stipple in plate B
4. What are the trails of the Transform Fault?⇒F.Z.
5. Ridge is at 0. Old ridge is at 10 MA, ridge was moving ⇒ at rate of 50 $\frac{km}{my}$

At 8 Ma, 2my (50$\frac{km}{my}$)=100 km

At 4 Ma, 4 my (50$\frac{km}{my}$)=200 km

This page titled 4.4: Plates Moving on a Plane is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.5: Relative Velocity and Reference Frames

Reference Frames, Relative and Absolute Plate Velocity

Relative velocity is the velocity with respect to some frame of reference. Considering the relative motion of plates, the reference frame can be something external (like the location of rising plumes from the core mantle boundary) or it can be one of the plates, or even a plate boundary. The **reference frame** is set by the object that is being held fixed. In other words it is the object that the velocities are measure with respect to.

If you have two plates, A and B, the velocity of plate B relative to plate A is denoted by ${}_AV_B$, where the first subscript is the *fixed* plate (A) and the second subscript is the moving plate (B). The relative motion is calculated as:

$$ {}_AV_B = V_B - V_A \tag{4.5.1}$$

Note that you subtract the velocity of the fixed plate from the moving plate. Similarly, we can reverse which plate is being held fixed and hold plate B fixed. Then:

$$ {}_BV_A = V_A - V_B \tag{4.5.2}$$

gives the velocity of plate A relative to plate B.

As an example, take $V_A = 60$ mm/yr and $V_B = 20$ mm/yr, then

$$ {}_AV_B = 20 - 60 = -40 \tag{4.5.3}$$

$$ {}_BV_A = 60 - 20 = 40 \tag{4.5.4}$$

Note that ${}_AV_B = -40$ mm/yr and ${}_BV_A = 40$ mm/yr, so

$$ {}_AV_B = -{}_BV_A \tag{4.5.5}$$

The magnitude of the motion between the two plates is the same (40 mm/y in the example): this is the **relative velocity**, which does depend on which plate I choose as the reference frame. The sign, however does depend on which plate is being used as the reference frame, and indicates direction of relative motion.

Finally, if an external reference frame is being used, then we may refer to the motion of a plate as an **absolute plate velocity**. This indicates that the plate velocity is not the relative motion between two plates, but rather it is the motion of each plate relative to the external frame of reference (e.g., the lighthouse in the example below). The relative motion of the two plates, can be found by differencing the absolute plate velocities (see example below).

Example Relative Velocity

To better understand relative velocity, consider a simple example of people on a ship.

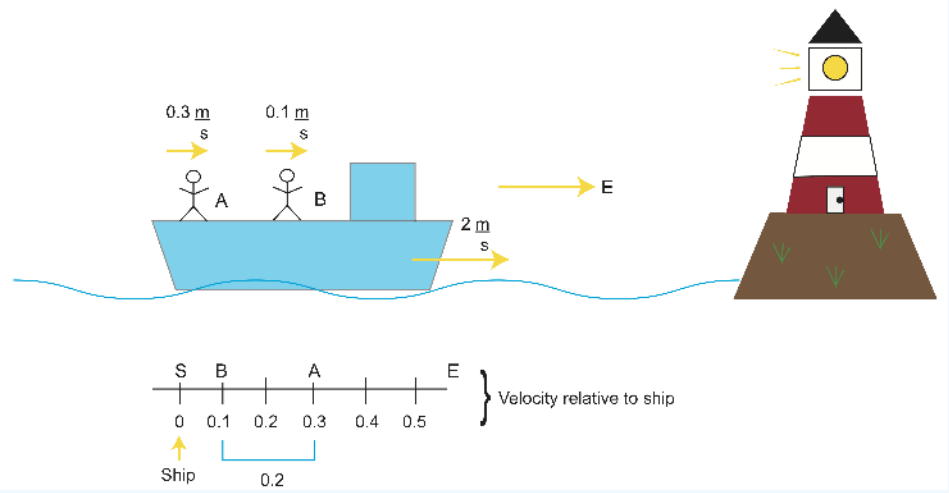


Figure 4.5.1: Relative Velocity

The velocity line indicates velocity relative to the ship.

We want to know 'how fast is person A approaching person B'? Another way to phrase this is "the motion of person B relative to person A". Translating these words to an algebraic expression we get:

$$\begin{aligned} {}_AV_B &= V_A - V_B \\ &= 0.2\frac{m}{s} \\ &= 0.3 - 0.1 \end{aligned}$$

Now we want to know, "how fast are the people moving towards the lighthouse"?

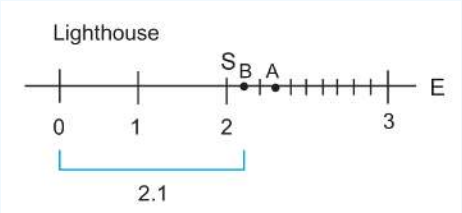


Figure 4.5.2: Lighthouse Velocity Line

$$ {}_LV_B = {}_L\ V_S + {}_S\ V_B \tag{4.5.6}$$

The two "S" cancel, and the "L" indicates movement relative to the lighthouse.

$$\begin{aligned} {}_LV_B &= 2.0 + .01 \\ &= 2.1 \end{aligned}$$

2.1 is the "absolute velocity"

Now what if person A turns around and starts walking the other way? We want to know 'what is $_AV_B$'?

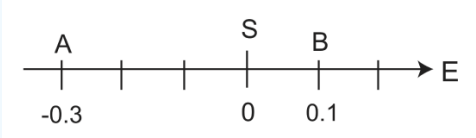


Figure 4.5.3: $_AV_B$ Velocity Line

How do we make V_A=0? ⇒ Add to 0.3. We now know that V_A=-0.3.

$$ {}_AV_B = {}_S\ V_B - {}_S\ V_A \tag{4.5.7}$$

$$ {}_SV_A = -{}_AV_S \tag{4.5.8}$$

$$ {}_AV_B = {}_A\ V_S + {}_S\ V_B \tag{4.5.9}$$

Once again the "S" cancel.

$$ {}_AV_B = 0.3 + 0.1 \tag{4.5.10}$$

$$ {}_AV_B = 0.4 \tag{4.5.11}$$

One final question, what is the motion of person A relative to the lighthouse?

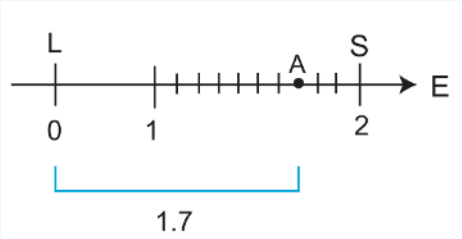


Figure 4.5.4: $_LV_A$ Velocity Line

$$ {}_{L}V_{A} = {}_{L}V_{S} + {}_{S}V_{B} \tag{4.5.12} $$

Where ${}_{L}V_{A}$ indicates the absolute velocity or velocity relative to L, and ${}_{s}V_{B}$ is the relative velocity.

$$ {}_{L}V_{B} = 2.0 + (-0.3) \tag{4.5.13} $$

$$ {}_{L}V_{B} = 1.7 \tag{4.5.14} $$

Relative Velocity Between Plates

So now let's try this with plates. Consider again our two plate system with plate A diverging away from plate B perpendicular to the boundary between the two plates. We assume plate B is not moving (i.e. plate B is fixed). The velocity of plate A relative to plate B is denoted ${}_{B}V_{A}$. If instead we wanted to hold plate A fixed and consider plate B moving away from plate, then this would be denoted ${}_{A}V_{B}$. As a way to remember this nomenclature, note: ${}_{fixed}V_{moving}$.

Calculating Relative Plate Velocity

With the age of the plate adjacent to the spreading ridge determined from magnetic anomalies, we can calculate the relative velocity from the distance Δx between the spreading ridge and the edge (start or end) of a dated magnetic anomaly with age t_a as

$$ {}_{A}V_{B} = \frac{\Delta x}{t_a} \tag{4.5.15} $$

Let's say the age is 5 my and the distance is 100 km. Then, in this case,

$$ {}_{A}V_{B} = \frac{-100\,\text{km}}{5\,\text{my}} = -20\,\text{km/my} = -20\,\text{mm/y} = -2\,\text{cm/y} \tag{4.5.16} $$

We consider velocity in the negative x direction to be negative, and velocity in the positive direction to be positive. Therefore, the motion of plate B (in the positive direction) relative to plate A is ${}_{B}V_{A} = +2$ cm/yr Notice that when we choose to hold a plate fixed, the spreading center necessarily moves as both plates grow symmetrically during the spreading process. The spreading center moves away from the plate being held fixed.

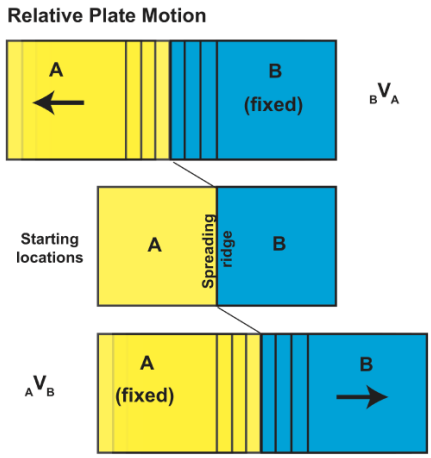


Figure 4.5.5: Relative Plate motion from a Spreading Ridge

When the diverging motion of the plates is perpendicular to the spreading center, then the relative velocity of the two plates is the same as the spreading rate. However, the motion of the plate need not be perpendicular to the plate boundary. In this case, the magnetic anomalies still form parallel to the spreading center, but calculating the relative motion must take into account the distance between anomalies in the direction of spreading. The direction (orientation) of spreading relative to the spreading center can be determined by the orientation of transform faults between ridge segments. The transform faults are active faults between the ridge segments and are parallel to the relative plate motion. Beyond the ridge segments, the previous location of the transform fault is recorded as a topographic feature called a fracture zone. There is no relative motion across the fracture zone, it resides wholly

within the plate on that side of the spreading center. When a transform fault or fracture zone indicates oblique spreading, the relative velocity can be determined by first calculating the apparent velocity perpendicular to the ridge, V and then correcting for the actual orientation of the plate motion given by the angle between the perpendicular direction and the actual plate motion direction α and is given by

$$ {}_AV_B = \frac{V}{\cos\alpha} \tag{4.5.17}$$

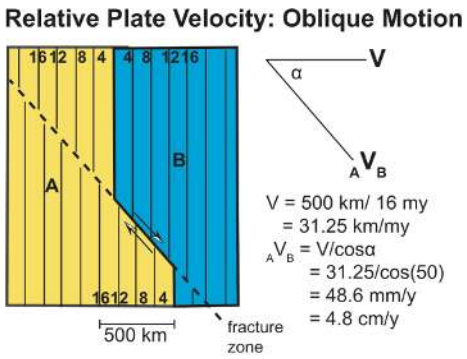


Figure 4.5.6: Relative Plate Motion: Oblique Motion

Reference Frames

In the above example we arbitrarily hold one of the plates fixed and calculate the relative velocity between the two plates. However, one could imagine holding the spreading center fixed or some other external reference frame. In considering the motion of tectonic plates there is not an obvious choice for the reference frame, because the plate are moving above a mantle that is dynamically mixing and changing in time. At one time it was thought that the plumes feeding hot-spot volcanoes might serve as a fixed reference frame if these plumes originate at the core-mantle boundary and then rise through the mantle without being entrained by the flow in the mantle. Locally and for short periods of time this assumption appears true, but globally the plumes move relative to each other and simulations show that their origination point can move through time. Therefore, different research groups make different choices about which reference frame to use depending on the question they are asking... e.g., is it about recent plate motions, or does the physical interpretation depend on the absolute motion of the plate. It is therefore also useful to be comfortable with changing reference frame as needed.

First, it is important to note that that relative plate velocity does not depend on choice of reference frame. We saw this above when we changed from a fixed-plate-A reference frame to a fixed-plate-B reference frame. Consider again the two plate A and B diverging at a spreading ridge with ${}_AV_B = +20$ cm/y. Now, consider that Plate A is in fact subducting under Plate C with ${}_CV_A = -10$ cm/y (note plate C is now fixed). Then, what is the relative velocity between Plate B and Plate C?

The Velocity Line

One way to consider this question is using a velocity line. First choose which plate to hold fixed. Let's hold plate A fixed. Then ${}_AV_C = +10$ cm/y and ${}_AV_B = +20$ cm/y. To determine the relative velocity between Plate B and Plate C, we need to hold one of these plates fixed. Let's hold plate C fixed. On the velocity line, we do this by subtracting the velocity of plate C from all the points, so it now lies at the origin and is fixed. The relative velocity of plate C with respect to plate B can then be read from the velocity line, ${}_CV_B = +10$ cm/y. Therefore, regardless of the which plate is held fixed (the reference frame), the relative velocity between to plate is always just the difference in the velocity of the two plates (expressed in the same reference frame). As one last example, consider that instead of holding any of the plates fixed, we wanted to use a "hot-spot" reference frame. Plate A is moving -15 cm/y with respect to the the hot-spot reference frame. The velocity of each of the plate can be adjusted by subtracting 5 cm/y. In the hot-spot reference frame all the plate are moving with some velocity and their absolute velcoity have all shifted compared to the fixed plate reference frames. However, the relative velocities remain the same.

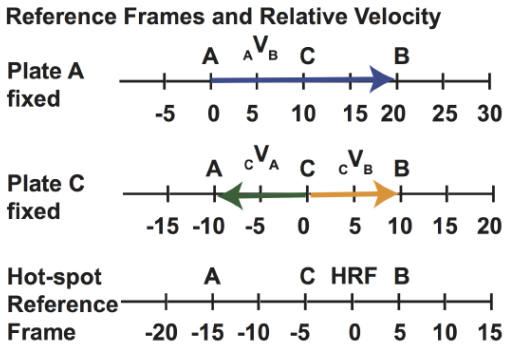


Figure 4.5.7: Reference Frames and Relative Velocity

The Velocity Plane

Unlike the example above in which all the plates are moving a long a single direction (one-dimensional, 1D motion), plates actually move in two-dimensions (either on a plate or the surface of a sphere). Therefore, to consider relative plate motions and reference frames, we need to deal with the plate velocity as a vector that has both a magnitude and a direction. As our reference coordinate system, we use the east-west direction east is positive) and the north-south direction (north is positive). The velocity of a plate is then given by its speed V, and its direction, θ, an angle measured clockwise from north: these are polar coordinate (r, θ). The Cartesian components of motion of the plate in the north and south directions are then given by:

$$V^E = V \sin(\theta) \tag{4.5.18}$$

$$V^N = V \cos(\theta) \tag{4.5.19}$$

The polar coordinates can also be determined from the Cartesian components as:

$$V = \sqrt{V^{E2} + V^{N2}} \tag{4.5.20}$$

$$\theta = \tan^{-1}(V^E / V^N) \tag{4.5.21}$$

Finding Relative Plate Motion in 2D

The relative motion of plates in 2D can then be found by adding the components in Cartesian coordinates, and the converting the result to polar coordinates if desired. For example, if the motion of plate A with respect to plate B is given by ${}_BV_A = 5$ cm/y, $\theta_{BA} = 30^\circ$ and the plate motion of plate C with respect to plate A is given by ${}_AV_C = 10$ cm/y, $\theta_{AC} = 50^\circ$, then we first convert both velocities to Cartersian coordinates.

Plate B-A:

$${}_BV_A^E = {}_B\,V_A \sin(\theta_{BA} = 5 \sin(30) = 2.5\,\text{cm/y} \tag{4.5.22}$$

$${}_BV_A^N = {}_B\,V_A \sin(\theta_{BA} = 5 \cos(30) = 4.33\,\text{cm/y} \tag{4.5.23}$$

Plate A-C:

$${}_AV_C^E = {}_A\,V_C \sin(\theta_{AC} = 10 \sin(50) = 7.66\,\text{cm/y} \tag{4.5.24}$$

$${}_AV_C^N = {}_A\,V_C \sin(\theta_{AC} = 10 \cos(50) = 6.42\,\text{cm/y} \tag{4.5.25}$$

Next, we add the north components together and the east components together :

$${}_BV_C^N = {}_B\,V_A^N + {}_A\,V_C^N = 4.33 + 6.42 = 10.75\,\text{cm/y} \tag{4.5.26}$$

$${}_BV_C^E = {}_B\,V_A^E + {}_A\,V_C^E = 2.5 + 7.66 = 10.16\,\text{cm/y} \tag{4.5.27}$$

and then convert back to polar coordinates:

$${}_BV_C = \sqrt{{}_BV_C^{E2} + {}_BV_C^{N2}} = \sqrt{10.75^2 + 10.16^2} = 14.79\,\text{cm/y} \tag{4.5.28}$$

$$\theta_{BC} = \tan^{-1}({}_BV_C^E / {}_BV_C^N) = \tan^{-1}(10.16/10.75) = 43.38^\circ \quad (4.5.29)$$

This process of adding plate velocity vectors can also be illustrated on a velocity plane. Similar to the velocity line the axis represent components of velocity in North and East directions. The velocity vectors are added by drawing the velocity vectors in the appropriate direction and magnitude. In the case describe above, we first draw the velocity of Plate A relative to Plate B. Therefore, Plate B is fixed and is located at the origin. Next, we draw the velocity of Plate C relative to A. In this case, the origin of the vector is the located at the point marked A (the point having the velocity components of Plate A relative to Plate B. The velocity of Plate C is then determined by drawing the vector that closes the velocity triangle, going from point B (fixed) to plate C. Also similar to the a change in reference frame on the velocity line, a change in reference frame in 2-D simply translate the velocity vectors within the velocity plate. Therefore, again, the relative velocities stay the same, by the absolute velocity magnitude depend on which reference frame is used.

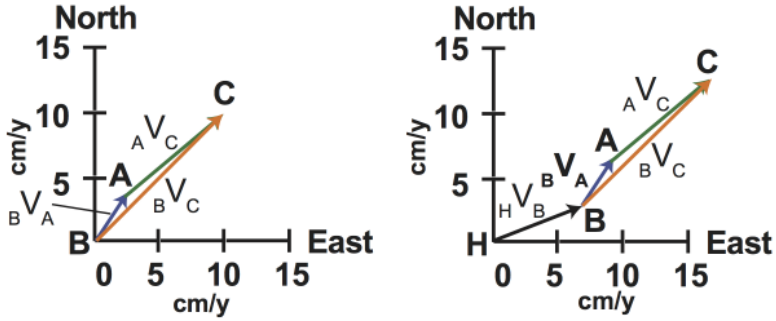


Figure 4.5.8: Relative Velocities

Let's now reverse the procedure. We are going to try determining the velocity diagram from plate boundaries.

Determining the Velocity Diagram From Plate Boundaries.

We are going to look at the motion of plate C with respect to plate A.

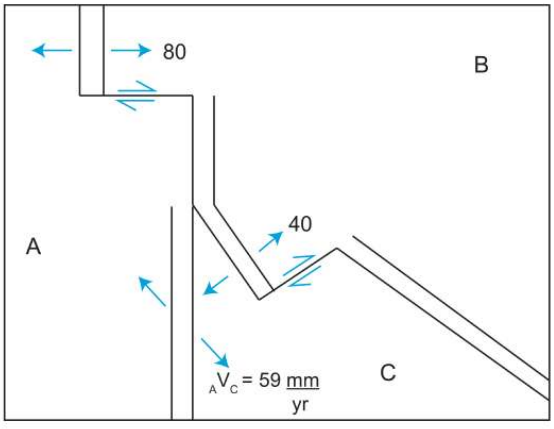


Figure 4.5.9: Plate Boundary Velocities

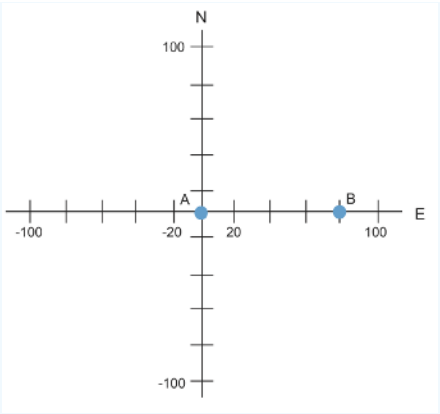


Figure 4.5.10: Initial Plotted Velocities

1.Hold A fixed (Because we have AV_B)→plot at origin. We could also hold B fixed because we have ${}_BV_A$ and ${}_BV_C$.

2. B moves east at 80$\frac{mm}{yr}$ with respect to A.

${}_AV_B$=80$\frac{mm}{yr}$

3. C moves southwest (45º) from B at 40$\frac{mm}{yr}$

E=40cos45º=28.3

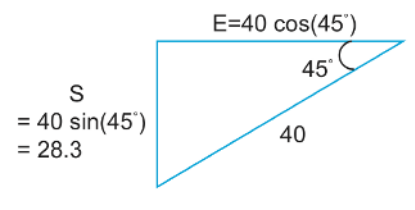


Figure 4.5.11: Calculating Velocities

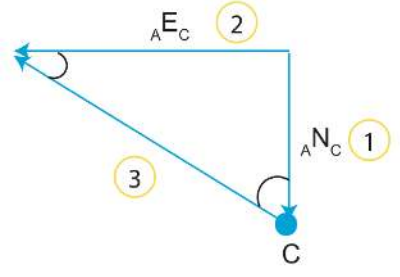


Figure 4.5.12: Calculating Velocities

$$ {}_AN_C = -28.3 \tag{4.5.30}$$

$$ {}_AE_C = {}_A\ E_B + {}_B\ E_C \tag{4.5.31}$$

$$ {}_AE_C = 80 + (-28.3) \tag{4.5.32}$$

$$ {}_AE_C = 51.7 \tag{4.5.33}$$

$$ {}_AV_C = \sqrt{(51.7)^2 + (28.3)^2} \tag{4.5.34}$$

$$ {}_AV_C = 58.9 \tag{4.5.35}$$

$$ {}_AV_C \approx 59 \tag{4.5.36}$$

Now that we have our velocities, we can place them on the velocity graph.

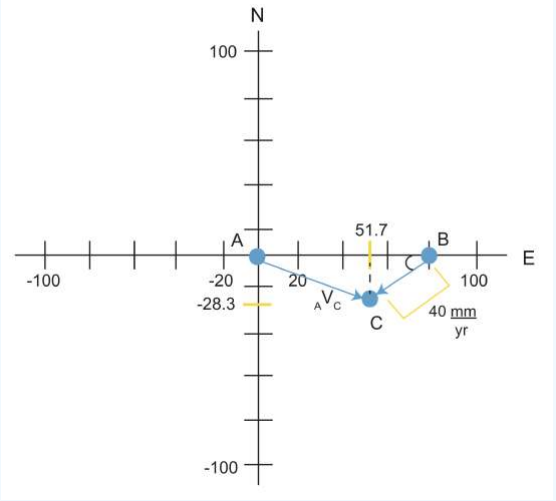


Figure 4.5.13: Final Plotted Velocities

This page titled 4.5: Relative Velocity and Reference Frames is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.6: Relative Plate Velocity Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An example about relative plate velocities.

Key Questions: Consider these as you work your way through this page.

Holding Plate A fixed:

- What is the plate boundary type between Plates A and B?
- What is the relative velocity of Plate B with respect to Plate A?

Holding Plate B fixed:

- Has the plate boundary type changed?
- What is the relative velocity of Plate A with respect to Plate B?

First, let's import the necessary libraries.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 print('The libraries are imported')
```

run | run all | restart | restart & run all

Now let's plot our plates. The first portion of this code will ask you for the plate velocities and their directions. Directions are given in azimuth in degrees measured clockwise from north (e.g., south is 180). Then it fills in the two plates and plots your chosen

velocity as vectors. At the bottom of the graph is the list of velocity components, you will need these for the key questions. Run this cell, and answer the key questions.

```
print('Please enter the absolute plate velocity for plate A between 0 and 10 cm/yr:')
va = float(input())
while va <= 0 or va > 10:
    print('Invalid Velocity, please enter a velocity between 0 and 10 cm/yr:')
    va = float(input())

print('Please enter the azimuth of the plate velocity between 0 and 360 degrees:')
aza = float(input())
while aza < 0 or aza > 360:
    print('Invalid Azimuth: please enter a value between 0 and 360 degrees')
    aza = float(input())

print('Please enter the absolute plate velocity for plate B between 0 and 10 cm/yr:')
vb = float(input())
while vb <= 0 or vb > 10:
    print('Invalid Velocity, please enter a velocity between 0 and 10 cm/yr:')
    vb = float(input())

print('Please enter the azimuth of the plate velocity between 0 and 360 degrees:')
azb = float(input())
while azb < 0 or azb > 360:
    print('Invalid Azimuth: please enter a value between 0 and 360 degrees')
    azb = float(input())

```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

Holding Plate A fixed:

- What is the plate boundary type between Plates A and B?
- What is the relative velocity of Plate B with respect to Plate A?

Holding Plate B fixed:

- Has the plate boundary type changed?
- What is the relative velocity of Plate A with respect to Plate B?

4.6: Relative Plate Velocity Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

4.7: Plate Motions on a Sphere

Now that we understand plate motion on a plane, let's move to a sphere. In order to move to a sphere, we need to master five ideas/concepts.

1. Spherical Coordinates
2. Euler Pole
3. Angular Distance
4. Relative motion from Euler Poles
   - Reference Frame
5. Velocity and direction of motion at a point

Spherical Coordinates

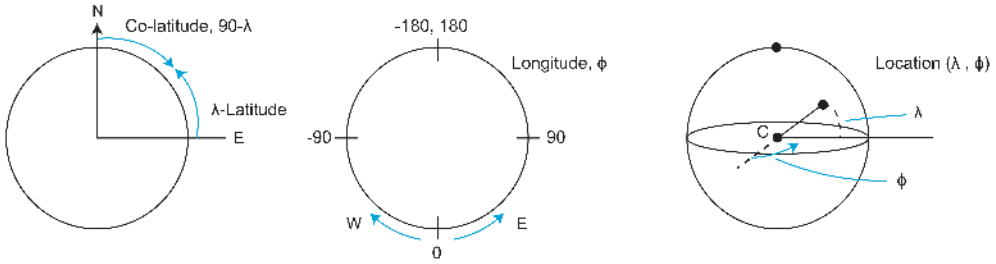


Figure 4.7.1: Latitude and Longitude

Now move point C to the top.

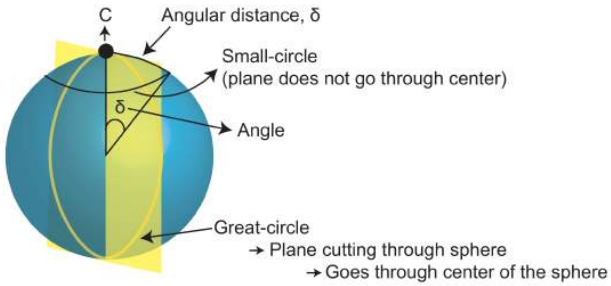


Figure 4.7.2: Point C

In all the equations below, spherical coordinates are given as λ (latitude) and ϕ (longitude). Cartesian coordinates are given with subscripts x, y, and z, where x and y are in the equatorial plane and z points towards the north pole. The subscript p indicates a Euler Pole. A Euler Pole is typically defined by its location (λ_p, ϕ_p) and the magnitude of the rotation, ω, in degrees/my. Latitude locations that are *south* of the equator should be entered as negative. Longitude locations that are *west* of the prime meridian should be entered as negative. Remember to convert angles to radians before using them in a cosine, sine, and tangent functions. Similarly, you need to convert the output of an inverse cosine, inverse sine, or inverse tangent from radians to degrees.

- Converting Cartesian to Spherical Coordinates

$$A_x = \cos(\lambda)\cos(\phi)$$
$$A_y = \cos(\lambda)\sin(\phi)$$
$$A_z = \sin(\lambda)$$

- Converting Spherical to Cartesian Coordinates

$$\lambda = \sin^{-1}(A_z) \tag{4.7.1}$$

$$\phi = \tan^{-1}(\frac{A_y}{A_x}) \tag{4.7.2}$$

- Convert Angular Distance Between Locations
    1. Convert the coordinates of each location from Spherical to Cartesian (see above). The Cartesian coordinates define a vector $A = [A_x, A_y, A_z]$ from the center of the Earth to the location on the Earth's surface.
    2. Use the dot product (or projection) of the two vectors to determine the angular distance, δ between the two points on the surface of the Earth:

$$\delta = \cos^{-1}(A \cdot B) \tag{4.7.3}$$

$$\delta = \cos^{-1}(A_x B_x + A_y B_y + A_z B_z) \tag{4.7.4}$$

- Convert a Rotation Vector (Euler Pole) from Spherical to Cartesian

$$\omega_x = \omega \cos(\lambda_p) cos(\phi_p) \tag{4.7.5}$$

$$\omega_y = \omega \cos(\lambda_p) sin(\phi_p) \tag{4.7.6}$$

$$\omega_z = \omega \sin(\lambda_p) \tag{4.7.7}$$

- Convert a Rotation Vector (Euler Pole) from Cartesian to Spherical

$$\phi_p = \tan^{-1}(\frac{\omega_y}{\omega_x}) \tag{4.7.8}$$

$$\lambda_p = \tan^{-1}(\frac{\omega_z}{\sqrt{\omega_x^2 + \omega_y^2}}) \tag{4.7.9}$$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} \tag{4.7.10}$$

- Calculate Relative Velocity and Azimuth of Plate Motion at a Location Using the Euler Pole
    - For the relative motion of the two plates (λ_p, ϕ_p), determine the velocity ($\frac{mm}{yr}$) and azimuth (degrees from north) of the relative motion at the location given by λ_x, ϕ_x. Note that this calculation is defined for any point, but only makes sense if the point is on one of the two plates. The equations use a triangle defined on the surface of the sphere, with sides a, b, and c and angles A, B, and C, where C is the angle opposite to side c. The lengths of the sides are measures in degrees.
        1. Calculate the angular length of side a of the triangle.

$$a = \cos^{-1}(\sin(\lambda_x) sin(\lambda_p) + \cos(\lambda_x) \cos(\lambda_p) \cos(\phi_p - \phi_x)) \tag{4.7.11}$$

        2. Calculate the angle C

$$C = \sin^{-1}(\frac{\cos(\lambda_p) \sin(\phi_p - \phi_x)}{\sin(a)}) \tag{4.7.12}$$

        3. Calculate the magnitude of the relative velocity.

$$v = \omega R \sin(a) \tag{4.7.13}$$

        4. Calculate the azimuth (in degrees).

$$\beta = 90 + C(\frac{180}{\pi}) \tag{4.7.14}$$

Euler Pole

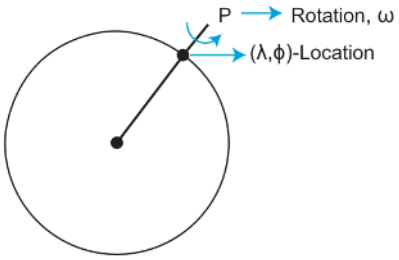


Figure 4.7.3: Euler Pole

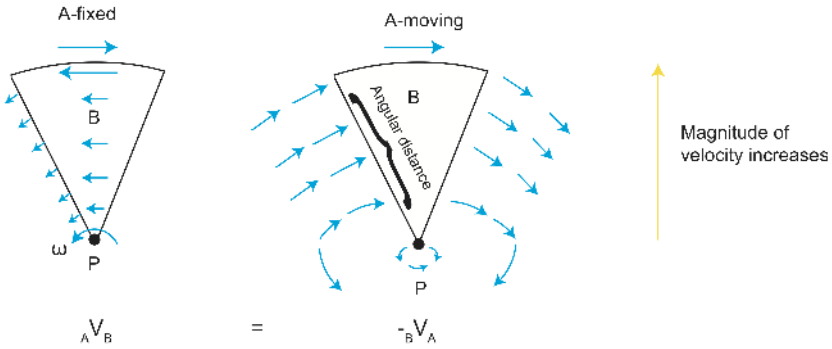


Figure 4.7.4: A-fixed vs A-moving

From the above figures, we can see that there is a magnitude of velocity that increases with distance from pole to 90°.

$$v = f(\omega, distance) = \omega R sin(\theta) \tag{4.7.15}$$

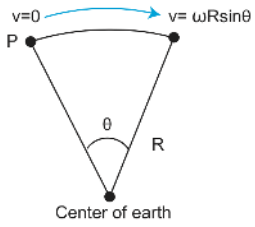


Figure 4.7.5: $v = \omega R sin(\theta)$

Angular Distance

How do we determine angular distance? ⇒It is easier in Cartesian coordinates

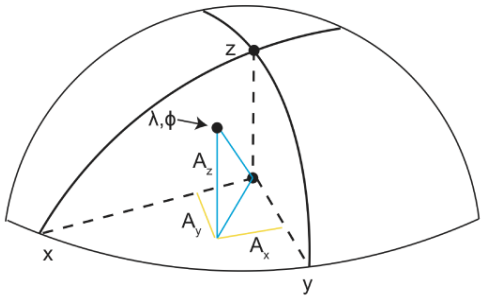


Figure 4.7.6: Cartesian Coordinates

In Cartesian coordinates, the orientation on x, y, and z are very important. As we noted above, x and y are in the equatorial plane and z points towards the north pole.

If given λ_1, ϕ_1, and λ_2, ϕ_2

1. Convert $\lambda_1, \phi_1 \rightarrow A_x, A_y, A_z$
2. Convert $\lambda_2, \phi_2 \rightarrow B_x, B_y, B_z$
3. Use dot product $\delta = cos^{-1}(A \cdot B)$

We can use a similar approach to express a Euler Pole in Cartesian components

$$\vec{\omega} = (\lambda_p, \phi_p, \omega) \; Spherical \; coordinates \tag{4.7.16}$$

This is the same as Cartesian → Spherical that we did above, but it now defines the magnitude.

$$\omega_{Cart.} = (\omega_x, \omega_y, \omega_z) \tag{4.7.17}$$

For plate reconstruction we need to be able to combine poles for different plates.

Example Plate Reconstruction

Given Euler Pole for ${}_{PAC}\omega_{NAM}$ and ${}_{PAC}\omega_{SAM}$, what is ${}_{SAM}\omega_{NAM}$? To solve this problem, what we need to do is convert the Euler Pole into something more manageable. We need components that we know how to add together. Thus, we are going to convert the Euler Pole into addable vectors, aka Cartesian components.

$${}_{SAM}\vec{\omega}_{NAM} = {}_{SAM}\omega_{PAC} + {}_{PAC}\omega_{NAM} \tag{4.7.18}$$

Convert to Cartesian

SAM-NAM SAM-PAC PAC-NAM

$\omega_x = \omega_x + \omega_x$

$\omega_y = \omega_y + \omega_y$

$\omega_z = \omega_z + \omega_z$

Convert back to Spherical (see equations above)

Determine relative velocity (magnitude and direction) at a location (λ, ϕ).

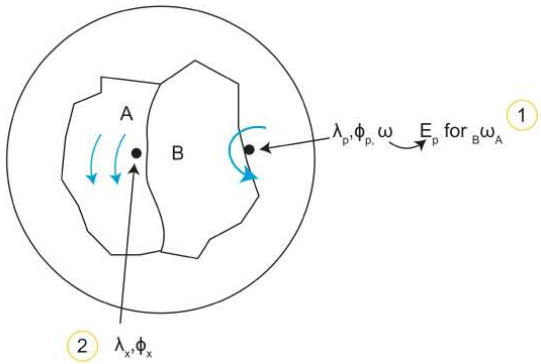


Figure 4.7.7: Determining Relative Velocity at Points

Location 1: $\lambda_p, \phi_p, \omega \rightarrow$ E$_p$ for ${}_B\omega_A$.

What is V, β (azimuth) at location 2? (Repeat for locations on whole plate)

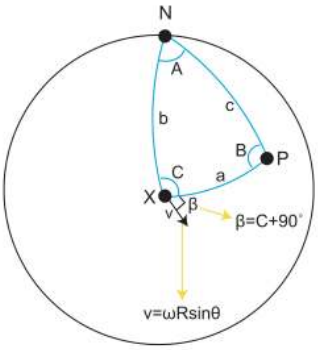


Figure 4.7.8: Great Circles

The sides a, b, c are the great circle paths, and the angles A, B, C are in the circles.

$\beta = C + 90$

$$v = \omega R sin\theta = \omega R sin(a) \tag{4.7.19}$$

$\theta = a$, P-X angular distance

So now we need C and a. We can use known trig relations for triangles on a sphere, for example:

$$cos(a) = cos(b)cos(c) + sin(b)sin(c)cos(A) \tag{4.7.20}$$

$$\frac{sin(a)}{sin(A)} = \frac{sin(c)}{sin(C)} \tag{4.7.21}$$

Write b, c in terms of λ_p, λ_x and ϕ_p, ϕ_x (λ_x is measured from the equator)

b=90-λ_x

c=90-λ_p

A=ϕ_p, ϕ_x (does not depend on λ_p)

Substitute, solve for a and c.

$$a = cos^{-1}[sin\lambda_x sin\lambda_p + cos\lambda_x cos\lambda_p cos(\phi_p - \phi_x)] \tag{4.7.22}$$

$$C = sin^{-1}[\frac{cos\lambda_p sin(\phi_p - \phi_x)}{sin(a)}] \tag{4.7.23}$$

$$\beta = 90 + C \tag{4.7.24}$$

$$V = \omega R sin(a) \tag{4.7.25}$$

Determining Relative Velocity at a Location

Now let's apply some real numbers determine relative velocity at a location on a plate.

Point 28°S, 71° W on Peru-Chile Trench

Use Nazca-South America Motion

We are given that λ_x=-28, ϕ_x=-71, λ_p=56, ϕ_p=94, and ω=7.6x10^{-7} degrees per year.

Plug into: a=86.26, C=-12.65°⇒β=77.35°, and finally v=8.43$\frac{cm}{yr}$

The plate motion can be seen in the below figure.

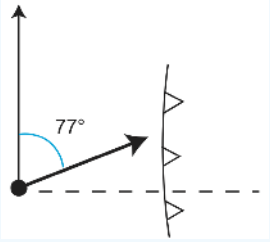


Figure 4.7.9: Motion of a Point on Peru-Chile Tench

This page titled 4.7: Plate Motions on a Sphere is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

4.8: Summary

Plate tectonics is a large subject with a great deal of sub topics, which can often make it confusing. It is also one of the most important subjects in modern day geology. Let's review what we've leaned in this chapter to make sure that you completely understand and feel confident with the subject.

Forces Driving Plate Motion

We first discussed how exactly plate tectonics happen, ie what are the forces behind plate movement? The simple answer is convection. Convection can be broken up into 3 sub forces, viscous drag, slab push, and ridge pull. These forces should all balance to 0.

$$F_{ridge-push} + F_{slab-pull} - F_{viscous-drag} = 0 \tag{4.8.1}$$

Viscous drag opposes plate motion and the magnitude of drag under a plate can be represented by

$$|\sigma| = \frac{V_o \eta}{h} \tag{4.8.2}$$

For drag on both sides of a plate we have the equation:

$$F_{vis.} = \frac{2V_o \eta_{av}}{h} L_{slab} \tag{4.8.3}$$

Another main force a subducting plate will encounter is the slab-pull force. This is the force due to the density contrast between a slab and its surroundings, ie the mass anomaly of the slab in the mantle. This force is represented as:

$$F_{sp} = \Delta m L_{slab} g \tag{4.8.4}$$

A final force a subducting plate would experience is the ridge push force. Ridge push results from the elevation of oceanic ridges relative to the seafloor. To determine ridge-push forces we first look at the isostatic balance and the depth of oceans w(t), and second look at the force balance on the plate. Balancing all forces gives us

$$F_{RP} = F_1 - F_2 - F_3 \tag{4.8.5}$$

which will eventually give us

$$F_{RP} = g\kappa t \rho_m \propto (T_m - T_s)(1 + \frac{2p_m \propto (T_m - T_s)}{\pi(\rho_m - \rho_w)}) \tag{4.8.6}$$

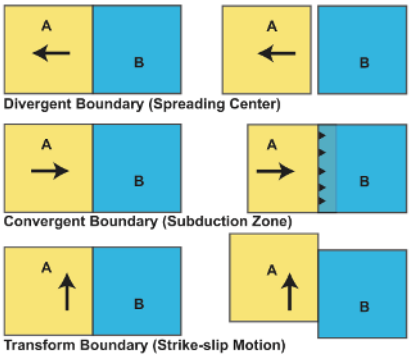
Magnetic Anomalies on the Seafloor

The most useful way of determining the age of plates is using magnetic stratigraphy. When sea floor is created at spreading centers magma is emplaced at shallow depth or erupted at the surface to form the crust of the growing plate. Certain minerals in the magma (e.g., magnetite) are sensitive to the Earth's magnetic field. The Earth's magnetic field reverses direction in an a-periodic (non-repeating) pattern, and the oriented minerals act like a bar code or finger print with a distinct pattern associated with different time intervals in the geologic past. Using the Geomagnetic reversal time scale one can determine the age of oceanic crust by measuring the present-day magnetic field, removing the contribution from the current magnetic field, and then analyzing the magnetic "anomalies" that remain.

Determining the spreading rate from the magnetic anomalies is done in several steps. First, just look at the pattern. Are the reversals all similar length or different lengths? Is the pattern symmetric with respect to any point on the profile? This last question is key because a symmetric pattern indicates that there is an active or extinct spreading center in the profile, and therefore, you should only be considering the anomalies on one side of the profile in trying to match the pattern of reversals. Next, try to identify some specific pattern... short-short-long-long-short ... and find a similar pattern in the reference geomagnetic timescale. Once the anomalies are matched, the spreading rate is calculated by noting the start and end time of an anomaly at each end of the profile. Then calculate the time duration between the start or end of the first anomaly and the second anomaly Δt and the distance Δx between these two points on the profile. The spreading rate (velocity) is $v_s = \frac{\Delta x}{\Delta t}$.

Plates Moving on a Plane

In this section, we learned about how the different types of plate boundaries relate to plate movement.



We can see that plate boundary types, in a plate tectonic sense, are defined by the relative velocity of the plates across the boundary. As the forces driving plate motion change, the relative velocity can change. If the change in velocity is large enough it can lead to a change in the type of plate boundary.

Relative Velocity and Reference Frames

Relative velocity is the velocity with respect to some frame of reference. To better understand this, we did several examples with moving and non moving components. Common notation for velocity of an object relative to another object is ${}_{fixed}V_{moving}$, or ${}_{\mathrm{x}}\mathrm{V}_{\mathrm{y}}$. The question in this case would be what is the motion of object y relative to object x? This can easily be applied to plates, where one is fixed and one is moving. When motion is not directly perpendicular to the plate boundary, it is oblique. The relative velocity in oblique spreading can be determined by first calculating the apparent velocity perpendicular to the ridge, V and then correcting for the actual orientation of the plate motion given by the angle between the perpendicular direction and the actual plate motion direction α and is given by

$$ {}_{A}V_{B} = \frac{V}{\cos\alpha} \tag{4.8.7} $$

Relative velocity can also be indicated with a velocity line, where regardless of the which plate is held fixed (the reference frame), the relative velocity between to plate is always just the difference in the velocity of the two plates (expressed in the same reference frame). However, plates actually move in two-dimensions (either on a plate or the surface of a sphere). Therefore, to consider relative plate motions and reference frames, we need to deal with the plate velocity as a vector that has both a magnitude and a direction.

$$ V^{E} = V\sin(\theta) \tag{4.8.8} $$

$$ V^{N} = V\cos(\theta) \tag{4.8.9} $$

$$ V = \sqrt{V^{E2} + V^{N2}} \tag{4.8.10} $$

$$ \theta = \tan^{-1}(V^{E}/V^{N}) \tag{4.8.11} $$

The relative motion of plates in 2D can then be found by adding the components in Cartesian coordinates, and the converting the result to polar coordinates if desired. This process of adding plate velocity vectors can also be illustrated on a velocity plane. Similar to the velocity line the axis represent components of velocity in North and East directions. The velocity vectors are added by drawing the velocity vectors in the appropriate direction and magnitude.

Plate Motions on a Sphere

We then moved on to understanding plate motions of a sphere instead of a plane. Spherical coordinates, as the name suggests, are ideal for dealing with spherical objects. We learned a number of ways to convert between different coordinates including:

- Converting Cartesian to Spherical Coordinates

$$A_x = cos(\lambda)cos(\phi) \tag{4.8.12}$$

$$A_y = cos(\lambda)cos(\phi) \tag{4.8.13}$$

$$A_z = sin(\lambda) \tag{4.8.14}$$

- Converting Spherical to Cartesian Coordinates

$$\lambda = sin^{-1}(A_z) \tag{4.8.15}$$

$$\phi = tan^{-1}(\frac{A_y}{A_x}) \tag{4.8.16}$$

We also learned about the significance of the Euler Pole. A Euler Pole is typically defined by its location (λ_p, ϕ_p) and the magnitude of the rotation, ω, in degrees/my.

To determine angular distance: If given λ_1, ϕ_1, and $\lambda_2, \phi_2\backslash$

1. Convert $\lambda_1, \phi_1 \rightarrow A_x, A_y, A_z$
2. Convert $\lambda_2, \phi_2 \rightarrow B_x, B_y, B_z$
3. Use dot product $\delta = cos^{-1}(A \cdot B)$

In Cartesian coordinates, the orientation on x, y, and z are very important. X and y are in the equatorial plane and z points towards the north pole.

Other necessary conversions and variables can be found using known trig relations for triangles on a sphere.

This page titled 4.8: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

5: Seismology

5.0: Introduction to Seismology
5.1: Basics of Wave Propagation
5.2: Seismic Refraction (Single Layer)
5.3 Seismic Refraction Jupyter Notebook
5.4: Seismic Reflection (Single Layer)
5.5 Seismic Reflection Jupyter Notebook
5.6: Summary

This page titled 5: Seismology is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

5.0: Introduction to Seismology

In this chapter, we will begin a discussion on seismology, which is the branch of geology that studies earthquakes and how seismic waves move through the earth. We will study earthquakes in much greater detail in the next chapter, so for now we will focus on building a foundation to understand how waves move in the earth. Wave movement is the basis for seismic surveys, which can tell us a great deal about the earth's subsurface structures. We can learn about the large layers in the earth, such as the mantle and core, or much more locally, like the layers of rock at a building site.

Figure 5.0.1: 2004 Indonesia Tsunami (Public Domain; Vasily V. Titov with NOAA Center for Tsunami Research via Wikimedia).

In our discussion of seismology, we will cover some of the basic types of seismic surveys, like reflection and refraction for a single layer. We will also discuss how we actually get a picture of the subsurface and its varied layers and structures. By the end of this chapter, you should feel comfortable with the basics of wave propagation (like reflection vs refraction and Snell's Law) and single layer seismic surveys (both refraction and reflection).

This page titled 5.0: Introduction to Seismology is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

5.1: Basics of Wave Propagation

To understand some of the more complex aspects of seismology, we must first start at the beginning and get a handle on the basics of wave propagation. In this section, we will examine three primary concepts:

1. The Basics of Waves
2. Types of Seismic Waves
3. Optics: Reflection, Transmission (Refraction), and Snell's Law

The Basics of Waves

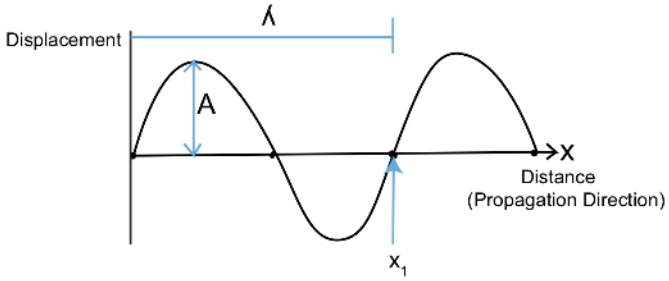


Figure 5.1.1: Amplitude and Wavelength

In the figure above, λ is the wavelength in meters and A is the amplitude in $\mu m - cm$. If you were to stand at x_1 and watch the wave go by, you would see Figure 5.1.1:

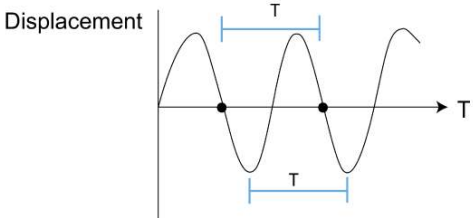


Figure 5.1.2: Period

where T is the period in s and f is the frequency in Hz. Frequency and period have the relation $f = \frac{1}{T}$, and thus $Hz = \frac{1}{s}$.

In terms of the relation between sound velocity and seismic velocity:

$$v = f\lambda \left[\frac{m}{s}\right] \tag{5.1.1}$$

Light has a similar relationship between frequency and wavelength, $c = f\lambda$, where different values for $f\lambda$ give different colors of light.

Types of Seismic Waves

One category of seismic waves are body waves. Body waves are waves you have likely heard of before, P waves and S waves. P waves act like an accordion, and move parallel to the propagation direction.

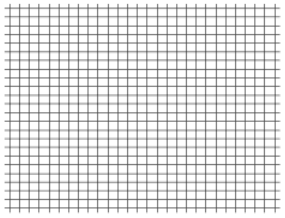


Figure 5.1.3: Plane P-wave (CC BY-SA 3.0; Christophe Dang Ngoc Chan (cdang) via Wikimedia)

S waves can have two components of motion, vertical and horizontal. Most S waves usually have both components, S-vert and S-horiz, which can be polarized.

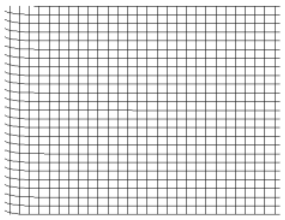


Figure 5.1.4: Plane S-wave (CC BY-SA 3.0; Christophe Dang Ngoc Chan (cdang) via Wikimedia)

The other category of seismic waves are surface waves, which decay with depth. Thus, as the name suggests, they are the strongest at the surface. There are two main surface waves, Love waves and Rayleigh waves. Love waves have particle motion like that of the S-H component.

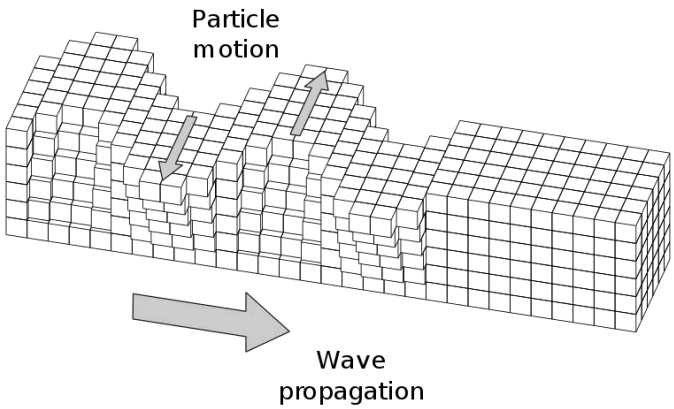


Figure 5.1.5: Love wave (CC BY 4.0; Nicoguaro via Wikimedia)

Rayleigh waves have the components of a P wave and a S-V wave. The particle motion is retrograde.

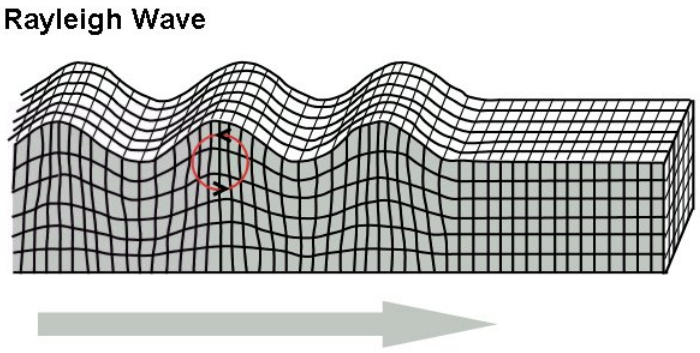


Figure 5.1.6: Rayleigh wave (Public Domain; Starmenclock via Wikimedia).

How fast do these waves actually move? Wave speed depends on two things, the density of the material the waves are traveling through and the elastic constant or 'stiffness'. Density is key to wave speed because the wave has to move mass. Imagine moving two materials in a wave-like pattern, a string and a metal cable. It takes more energy to move the more massive metal cable than it does the string because the cable is much denser. To understand the elastic constant, imagine moving a yarn and a stiff rope. In the yarn, it is hard to get the wave to move because the material lacks stiffness. With these two properties, we can write a relationship for wave velocity:

$$v \propto \frac{\text{Elastic constant}}{\text{Density}} \tag{5.1.2}$$

As v↑, $\rho \downarrow$, the elastic constant ↑.

Additional useful relationships when dealing with stress-strain due to seismic waves are

$$\sigma = E\epsilon \tag{5.1.3}$$

where E is **Young's modulus** that we discussed in Chapter 1.

$$F = -k\Delta x \tag{5.1.4}$$

Where k is the spring constant.

$$\nu = \frac{\epsilon_1}{\epsilon_3} \tag{5.1.5}$$

In addition to stress-strain deformation, we also have **shear deformation**.

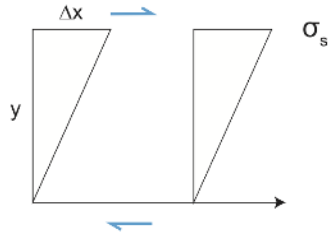


Figure 5.1.7: Shear Deformation Variables

$$\epsilon_s = \frac{1}{2}\frac{\Delta x}{y} \tag{5.1.6}$$

$$\sigma_s = 2G\epsilon_s \tag{5.1.7}$$

where G is the shear modulus.

$$G = \frac{\sigma_s}{2\epsilon_s}\left[\frac{N}{m^2}\right] \tag{5.1.8}$$

For reference, sometimes the shear modulus is represented by μ.

We can also have **volumetric strain**.

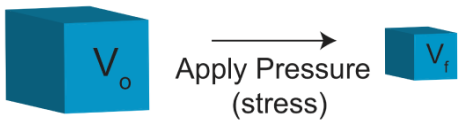


Figure 5.1.8: Volumetric Strain

$$\frac{\Delta V}{V} = \frac{V_f - V_o}{V_o} \tag{5.1.9}$$

$$P = \kappa\frac{\Delta V}{V} \tag{5.1.10}$$

Where κ is the bulk Modulus

$$\kappa = \frac{P}{(\frac{\Delta V}{V})}[\frac{N}{m^2}] \tag{5.1.11}$$

To summarize and review some of the constants that we have learned so far:

- E-Young's Modulus (elongation dilation)
- ν-Poisson's Ratio (compensation deformation)
- G or μ - shear muodulus
- κ- bulk modulus

In seismic calculations, you can use one set (E and ν) or the other (G and κ).

Now let's learn how to actually calculate seismic velocities. For P-waves:

$$v_p = \sqrt{\frac{\kappa + \frac{4}{3}G}{\rho}} \; or \; \sqrt{\frac{E}{\rho}\frac{(1-\nu)}{(1-2\nu)(1+\nu)}} \tag{5.1.12}$$

For S-waves, we can calculate velocity as:

$$v_s = \sqrt{\frac{G}{\rho}} \; or \; \sqrt{\frac{E}{\rho}\frac{1}{(2(1+\nu))}} \tag{5.1.13}$$

To find the ratio of P and S-wave velocities:

$$\frac{v_p}{v_s} = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}} \tag{5.1.14}$$

Thus, we can see that this only depends on ν. This ratio is a good indicator of gas/liquids in exploration geology

- $\frac{v_p}{v_s} < 2$ indicates gas+sand
- $\frac{v_p}{v_s} > 2$ indicates sand only

In general, we find that $v_p \sim 1.7 v_s$ (~50% faster).

Unfortunately, for surface waves, we cannot easily write down equations for v_R and v_L because of depth dependence. Thus, to determine their velocity, we must solve the wave equation with boundary conditions.

Optics: Reflection, Transmission (Refraction), and Snell's Law

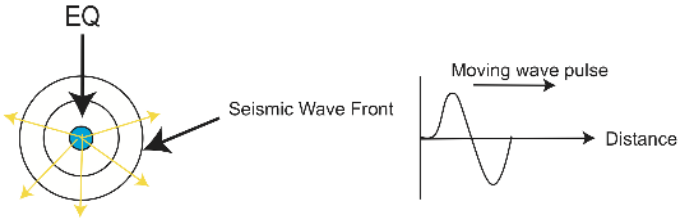


Figure 5.1.9: Seismic Wave Front Model

We can use the ray path to think about what happens to a wave when it encounters a boundary.

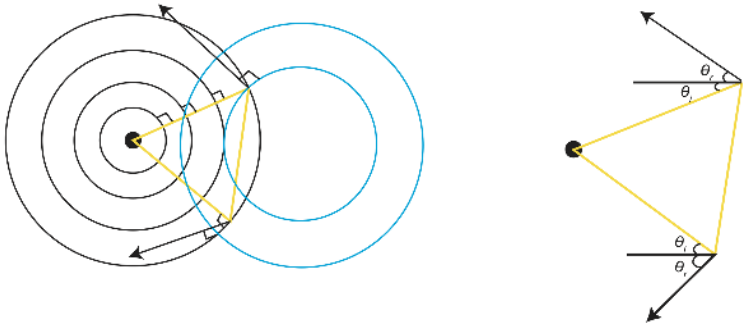


Figure 5.1.10: Wave Fronts

The most important information you can take from the above figure is that $\theta_i = \theta_r$, i.e. the angle of incidence=the angle of reflection. We've now considered reflection of waves, but what about refraction?

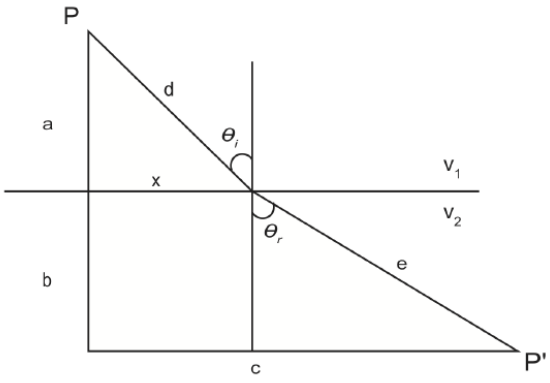


Figure 5.1.11: Refraction

$$T_{p-p'} = \frac{d}{v_1} + \frac{e}{v_1} = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(c-x)^2}}{v_2} \tag{5.1.15}$$

Fermat's Principle of least time states that the ray will follow the path that takes the shortest time. We can use this principle to derive one of the most important laws in seismology, Snell's Law.

$$\frac{dT}{dx} = 0$$

$$= \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{(c-x)}{v_2\sqrt{b^2+(c-x)^2}}$$

where $\sin\theta_i = \frac{x}{\sqrt{a^2+x^2}}$ and $\sin\theta_r = \frac{(c-x)}{\sqrt{b^2+(c-x)^2}}$

Now we can rewrite the equation as:

$$\frac{dT}{dx} = \frac{\sin\theta_i}{v_1} - \frac{\sin\theta_r}{v_2} \tag{5.1.16}$$

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_r}{v_2} \tag{5.1.17}$$

The above equation is Snell's Law. To better understand Fermat's Principle and why a ray takes the path of shortest time, we can also say that $\frac{\sin\theta_i}{v_1}$=p=slowness. This slowness is constant along a ray path.

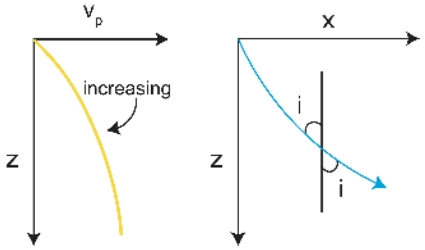


Figure 5.1.12: Curving Ray Path

$$\frac{sin(i)}{v} = constant = p \tag{5.1.18}$$

The ray path curves to adjust for changing velocity between different mediums. Thus, the ray takes the shortest time path. Let's now take a better look at Snell's Law, what it tells us, and what makes it so useful.

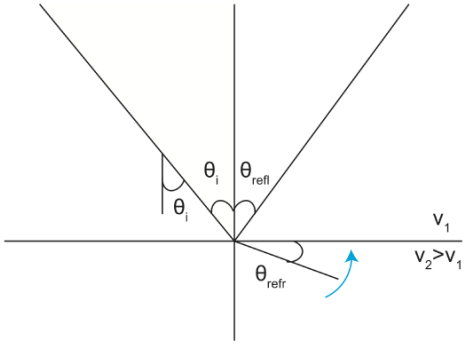


Figure 5.1.13: Reflection vs. Refraction

The figure shows the difference between reflection and refraction when a ray encounters a boundary.

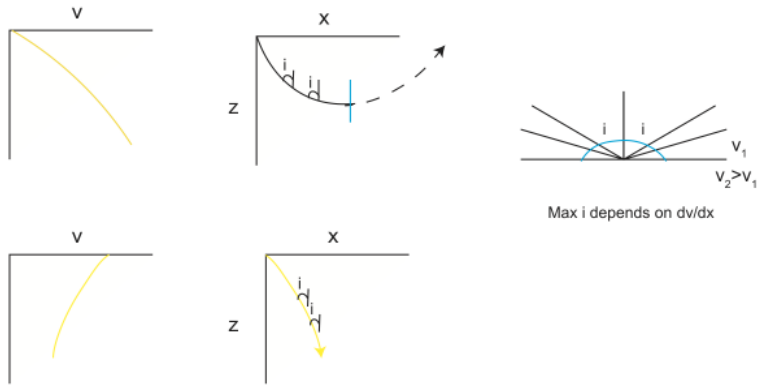
$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_{refr}}{v_2} \tag{5.1.19}$$

If we keep increasing v_2, θ_r increases up to 90°. At 90°, we have critical refraction, where $sin(90^o) = 1$. Plugging this into Snell's Law we get:

$$\frac{\sin\theta_i}{v_1} = \frac{1}{v_2} \tag{5.1.20}$$

$$\theta_{ic} = sin^{-1}(\frac{v_1}{v_2}) \tag{5.1.21}$$

If $\theta_{ir} > \theta_{ic}$, we get total reflection. What does Snell's Law actually tell us? (slowness)



v increases with depth, i increases.

v decreases with depth, i decreases.

Figure 5.1.14: Changes in i with respect to v

This page titled 5.1: Basics of Wave Propagation is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

5.2: Seismic Refraction (Single Layer)

One of the most important uses of seismic waves is for seismic surveys. Seismic surveys can be used to determine shallow (~100s of meters) crustal structure. The surveys shed light on the layers of rock, including each ones thickness and seismic velocity (and from seismic velocity we can estimate rock type). We can also determine structures such as faults and basins, history of deposition, and look for oil and gas. Primarily, we use P-waves for seismic surveys as they are faster and easier to see on seismograms. There are two types of seismic surveys

1. Refraction ie 'critical refraction' which is useful for determining velocities of layers
2. Reflection, which is useful for determining layers and structure

In seismic surveys we can use geometry and travel time of waves to determine structure and velocity. Today we are going to focus on refraction surveys. First, we are going to quickly review what we learned in the last section: slowness/Snell's Law and Reflection and Transmission

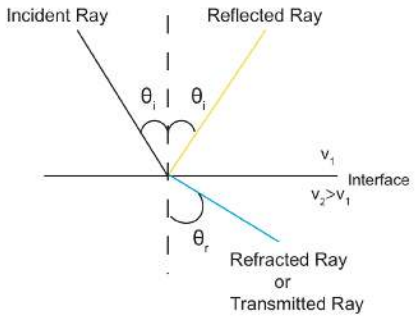


Figure 5.2.1: Snell's Law

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_r}{v_2} \tag{5.2.1}$$

$v_2 > v_1$ is the most common interface in the earth, and $\theta_r > \theta_i$ is due to $v_2 > v_1$.

Now that we've reviewed, let's first look at two sample ray paths.

The direct wave travels horizontally along the surface. The Rs represent receiving stations. What is the arrival time at each station?

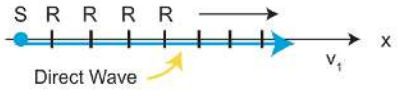


Figure 5.2.2: Direct Wave

$$distance = velocity \cdot time \tag{5.2.2}$$

$$x = v_1 \cdot t \tag{5.2.3}$$

Solve for t.

$$t = \frac{x}{v_1} \tag{5.2.4}$$

The above equation is the direct ray travel/arrival time. The equation is a line whose slope is $\frac{1}{v_1}$ and intercept=0.

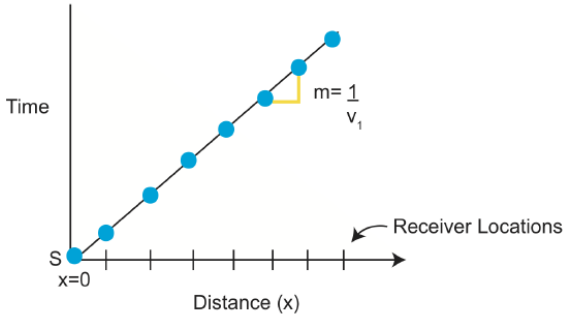


Figure 5.2.3: Plot of Direct Ray Travel/Arrival Time

Using the vertical incidence of a reflected ray, we can derive an equation to calculate the time the ray took to reach the receiver:

$$d = v \cdot t \tag{5.2.5}$$

$$2h = v_1 \cdot t \tag{5.2.6}$$

$$t = \frac{2h}{v_1} \tag{5.2.7}$$



Figure 5.2.4: Vertical Incidence of a Reflected Ray

Now, let's look at what happens as θ_i increases.

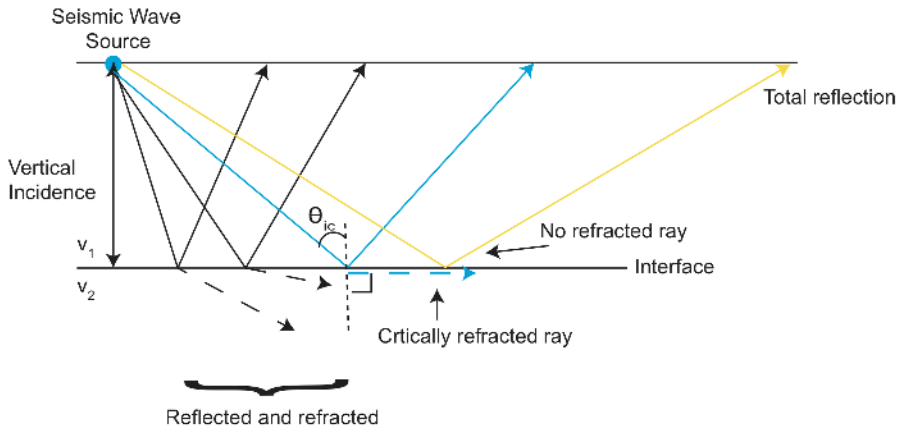


Figure 5.2.5: Changes as θ_i Increases

We can see from the figure that after the hammer blow/weight drop/small explosion (the seismic wave source), the waves go out in all directions. If there is a large angle reflection of a wave, all energy is reflected and none is refracted. This is called total reflection. When θ_r increases until 90^o, the wave is critically refracted. The critically refracted ray is crucial to surveys.

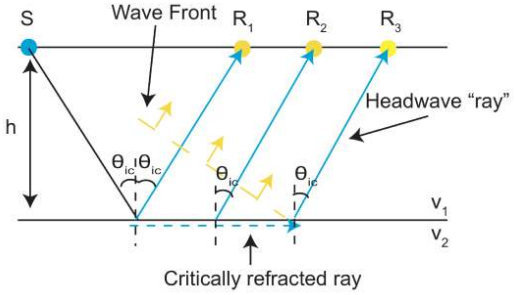


Figure 5.2.6: Critically Refracted Ray

The critically refracted ray generates a 'head wave'. Receivers (R_1, R_2, R_3) on the surface detect head wave at t_1, t_2, t_3, which correspond with increasing distance from the source. We can use geometry and arrival times to determine v_1, v_2, and h. One of the cons of momentum and fitting boundary conditions is that it sends energy back into the top layer.

Now we're going to look at refraction geometry

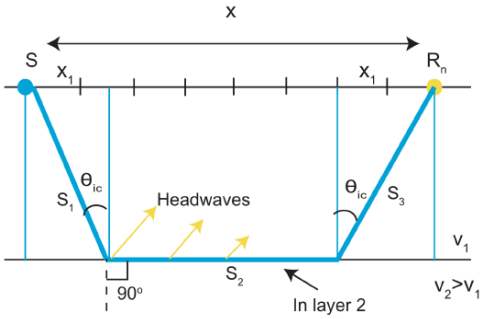


Figure 5.2.7: Refraction Geometry

We know that

$$\frac{\sin\theta_{ic}}{v_1} = \frac{sin90^o}{v_2} \tag{5.2.8}$$

$$\sin\theta_{ic} = \frac{v_1}{v_2} \tag{5.2.9}$$

To get refraction, we must have $v_2 > v_1$. Looking at the above figure, how do we figure out what are the distances s_1, s_2, and s_3? What are velocities v_1, v_2, and v_3? To get the travel time for each segment,

$$[t = t_1 + t_2 + t_3 \tag{5.2.10}$$

$$= \frac{s_1}{v_1} + \frac{s_2}{v_2} + \frac{s_3}{v_3} \tag{5.2.11}$$

We know that $s_1 = s_3$ so:

$$t = \frac{2s_1}{v_1} + s_2 \tag{5.2.12}$$

Now we need to know 'what is s_1'?

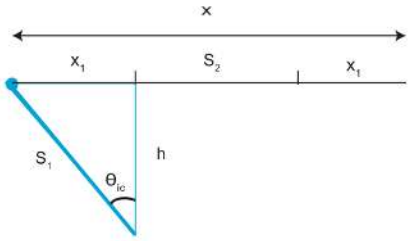


Figure 5.2.8: Refraction Geometry Variables

From the triangle we can derive that:

$$s_1 = \frac{h}{\cos\theta_{ic}} \tag{5.2.13}$$

$$x_1 = h \cdot \tan\theta_{ic} \tag{5.2.14}$$

$$s_2 = x - 2x_1 \tag{5.2.15}$$

Substituting:

$$s_2 = x - 2h \cdot \tan\theta_{ic} \tag{5.2.16}$$

$$t = \frac{2h}{v_1 \cos\theta_{ic}} + \frac{(x - 2h \cdot \tan\theta_{ic})}{v_2} \tag{5.2.17}$$

We can use

$$\tan\theta_{ic} = \frac{\sin\theta_{ic}}{\cos\theta_{ic}} \tag{5.2.18}$$

We know that $\sin\theta_{ic} = \frac{v_1}{v_2}$ from $\frac{\sin\theta_{ic}}{v_1} = \frac{sin(90)}{v_2}$ and that $sin^2\theta_{ic} + cos^2\theta_{ic} = 1$

$$t = \frac{x}{v_2} + \frac{2h(v_2^2 - v_1^2)^{\frac{1}{2}}}{v_1 v_2} \tag{5.2.19}$$

Or we can write it as

$$t = \frac{x}{v_1} + \frac{2h}{v_1}(1 - \frac{v_1^2}{v_2^2})^{\frac{1}{2}} \tag{5.2.20}$$

So what kind of equation is this exactly? Its a line with slope $m = \frac{1}{v_2}$ and $t(o) = \frac{2h(v_2^2 - v_1^2)^{\frac{1}{2}}}{v_1 v_2}$

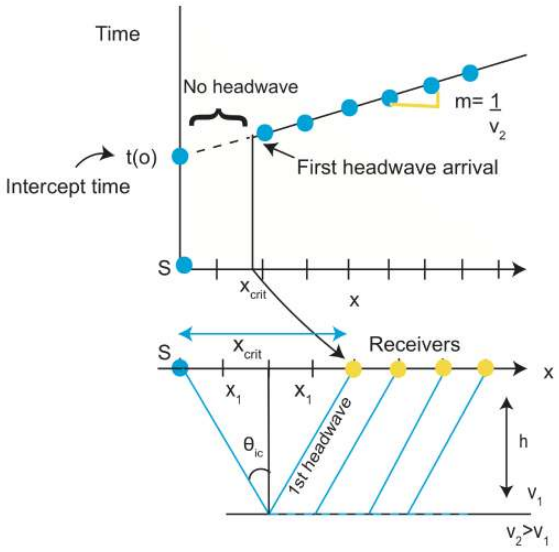


Figure 5.2.9: $X_{critical}$

From Figure 5.2.9, we can also derive that

$$x_{crit} = 2x_1 \tag{5.2.21}$$

$$= 2h \cdot \tan\theta_{ic} \tag{5.2.22}$$

Another important value that can be graphed is the crossover distance, or x_{co}.

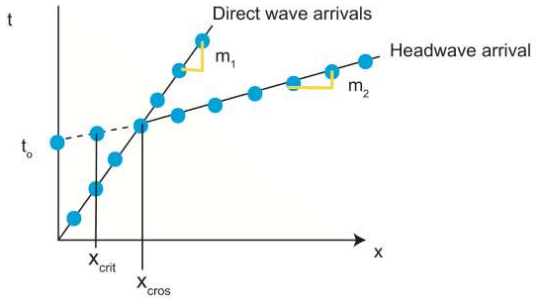


Figure 5.2.10: $X_{crossover}$

The x_{co} is when the t_{dir}=$t_{head(rfr)}$.

$$x_{co} = 2h(\frac{v_2 + v_1}{v_2 - v_1})^{\frac{1}{2}} \tag{5.2.23}$$

To summarize what we learned in this section:

How do you determine subsurface structure?

- Identify direct wave arrivals
  - Measure slope → m_1
  - Calculate v_1(layer 1) → $v_1 = \frac{1}{m_1}$
- Identify headwave arrivals
  - Measure slope, m, intercept, t_o
  - Calculate v_2(layer 2) → $v_2 = \frac{1}{m_2}$
- Use t_o, v_1, and v_2 to calculate h
  - $t_o = \frac{2h(v_2^2 - v_1^2)^{\frac{1}{2}}}{v_1 v_2}$
  - $h = \frac{t_o v_1 v_2}{2(v_2^2 - v_1^2)^{\frac{1}{2}}}$

This page titled 5.2: Seismic Refraction (Single Layer) is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

5.3 Seismic Refraction Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of single-layer seismic refraction.

Key Questions: Consider these as you work your way through this page.

- Holding the layer 1 thickness and velocity fixed, how does the angle of the incident wave (and refracted wave) change as you increase the layer 2 velocity?
- How does the layer thickness affect the geometry of the ray path?
- What happens if the velocity of the second layer is less that the first layer? Why does this happen?

First, we import the necessary libraries.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 print('The Libraries are imported.')
```

run | run all | restart | restart & run all

Now let's create our single-layer refraction model. This cell will prompt you to select 2 p-wave velocities and a layer thickness. It will then calculate the critical angle and plot the model. The model is created by plotting lines that reflect the desired geometry. Go ahead and run the cell, then consider the key questions.

```
1 print('Typical p-wave velocities are between 1000 and 4000 m/s')
2 print('Please enter the p-wave velocity of layer 1')
3 v1 = int(input())
4 print("Please enter the p-wave velocity of layer 2")
```

```
v2 = int(input())
print('Please enter the thickness of layer 1 between 0 and 500 m')
h = int(input())
while h <= 0 or h>500:
    print('Invalid thickness, enter a number between 0 and 500 m')
    h = float(input())

h2 = 10 # m

theta_ic = np.arcsin(v1 / v2)
theta = np.pi/2 - theta_ic
handle = display(None, display_id=True)
bbox = dict(boxstyle ="round", fc = '1')
while h > 0 and h <= 500:
    fig, ax = plt.subplots(figsize=(10,5))

    ax.set_xlabel('Distance from Source (m)')
    ax.set_ylabel('Depth (m)')
    ax.set_title('Seismic Refraction')
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

- Holding the layer 1 thickness and velocity fixed, how does the angle of the incident wave (and refracted wave) change as you increase the layer 2 velocity?
- How does the layer thickness affect the geometry of the ray path?
- What happens if the velocity of the second layer is less that the first layer? Why does this happen?

5.3 Seismic Refraction Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

5.4: Seismic Reflection (Single Layer)

Reflection surveys follow the same basic principles as refraction surveys, except of course they use reflected waves instead of refracted waves. We can use reflected waves to get h and v_1, but we cannot get v_2 unless we have a reflection from a deeper layer.

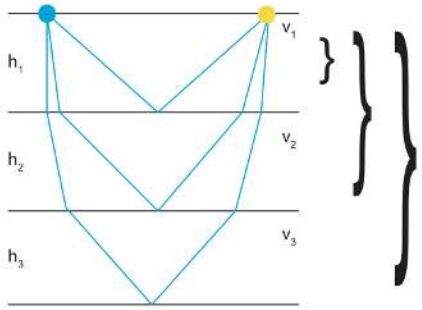


Figure 5.4.1: Multiple Layer Reflection

If there are multiple layers, we can determine each layers data such as height and velocity by working down one layer at a time. However, today we will only focus on a single layer subsurface. We can use the layer geometry to get a travel-time equation.

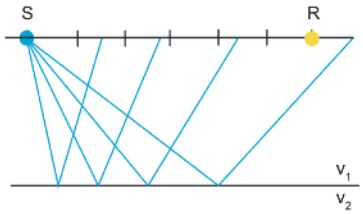


Figure 5.4.2: Basic Single Layer Reflection

In the case of reflection, we can have $v_2 > v_1$ or $v_2 < v_1$, we will get a reflected wave either way.

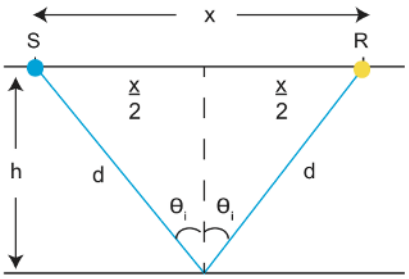


Figure 5.4.3: Reflection Variables

Starting with our distance equation,

$$d = v \cdot t \tag{5.4.1}$$

$$t = \frac{d}{v} \tag{5.4.2}$$

$$d^2 = (\frac{x}{2})^2 + h^2 \tag{5.4.3}$$

The total distance a reflected wave will travel is 2d, because the wave must go down and then back up to the receiver.

$$2d = 2((\frac{x}{2})^2 + h^2)^{\frac{1}{2}} \tag{5.4.4}$$

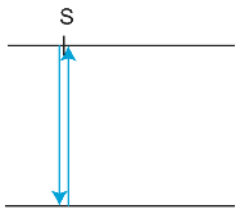
$$2d = (x^2 + 4h^2)^{\frac{1}{2}} \tag{5.4.5}$$

$$t = \frac{2d}{v_1} \tag{5.4.6}$$

$$t = \frac{\sqrt{x^2 + 4h^2}}{v_1} \tag{5.4.7}$$

What is the intercept time?

$t(x=0)=t_o=\frac{2h}{v_1}$



For a vertically reflected wave, if you know v_1, you can get h.

$$h = \frac{v_1 t_o}{2} \tag{5.4.8}$$

We've just said that calculating h from a vertically reflected wave is easy if you know v_1. So how do we get v_1? We are going to assume that we can't measure the direct wave (this is not usually true for layer 1, but definitely true for deeper layers-there is no direct wave here).

From the last section, we know that the graph of a refracted wave is a line. So what is the shape of t(x) for a reflected wave?

Figure 5.4.4: Vertically Reflected Wave

$$t^2 = \frac{x^2}{v_1^2} + \frac{4h^2}{v_1^2} \tag{5.4.9}$$

We know that $\frac{4h^2}{v_1^2}$ can also be written as t_o^2.

$$t_o^2 = t^2 - \frac{x^2}{v_1^2} \tag{5.4.10}$$

$$1 = \frac{t^2}{t_o^2} - \frac{x^2}{v_1^2 t_o^2} \tag{5.4.11}$$

The above equation is the equation for a hyperbola where $a=t_o$ and $b=v_1 t_o$. The intercept is at t_o. The slope at large $x=\frac{a}{b} = \frac{t_o}{v_1 t_o} = \frac{1}{v_1}$.

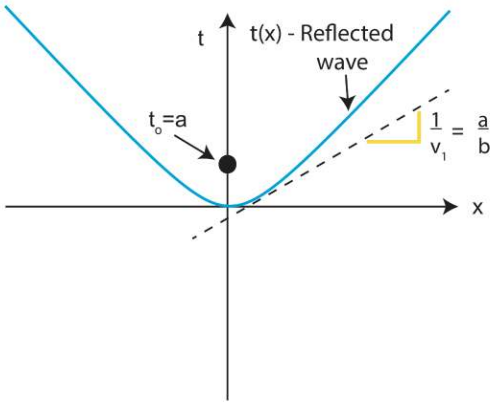


Figure 5.4.5: Reflected Wave Hyperbola

We can derive that the equation for t_o is:

$$t_o = \frac{2h}{v_1} \tag{5.4.12}$$

The figure below shows how the hyperbola changes with layer thickness. The hyperbola will increase with increasing layer thickness.

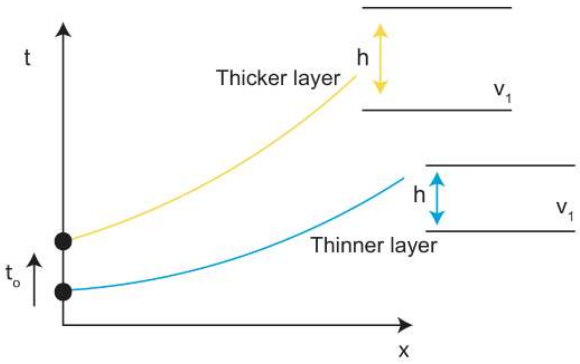


Figure 5.4.6: Effects of Varying Layer Thicknesses

What if two layers have the same thickness but different velocities?

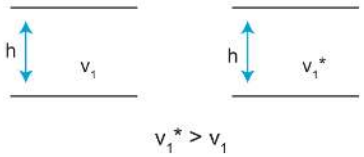


Figure 5.4.7: $v_1^*>v_1$

In the above figure, $v_1^*>v_1$. Using our equation $t_o = \frac{2h}{v_1}$, $t_o\downarrow$ for $v_1\uparrow$. Since the slope of the hyperbola $\frac{a}{b}$ is $\frac{1}{v_1}$, we can see that as v_1 increases, we will get a lower slope (a broader shape).

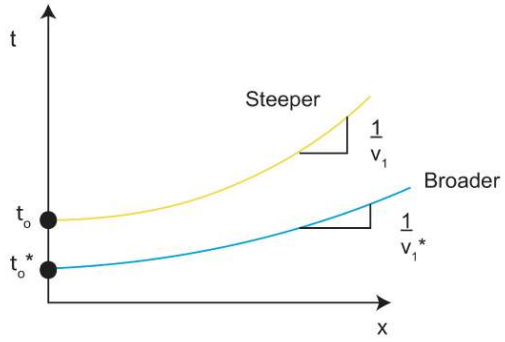


Figure 5.4.8: Effects of Varying Velocities

In the case of multiple layers, the travel time equation is also a hyperbola. We just used a single layer example to understand the basic concepts. Now that we have a basic understanding of the different terms and how the data can be graphed, we are going to analyze it using a method called Normal Move-out (NMO).

$$T_{NMO} = t(x) - t_o \tag{5.4.13}$$

The equation for T_{NMO} gives us the "shape of the hyperbola". If we could correct for NMO, all reflections would look like they come from a vertical reflection. As you 'move-out' along the hyperbola, the path the ray takes gets longer and longer. Thus by correcting for NMO, we are looking at data close to the first few recievers, ie those that have taken shorter paths and have not 'moved-out'.

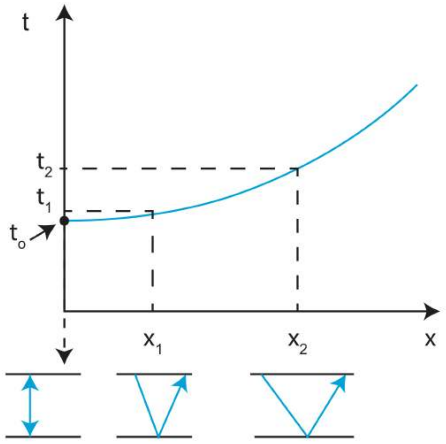


Figure 5.4.9: Normal Move Out

At small x, we can approximate

$$t(x) \approx t_o + \frac{x}{2v_1^2 t_o} \tag{5.4.14}$$

and that

$$t(x)^2 = t_o^2 + \frac{x^2}{v_1^2} \tag{5.4.15}$$

Using binomial expansion, we get that

$$t(x) = t_o(1 + (\frac{x^2}{v_1 t_o})^2)^{\frac{1}{2}} \tag{5.4.16}$$

then

$$T_{NMO} = t(x) - t_o \tag{5.4.17}$$

Substituting for t(x):

$$T_{NMO} \approx \frac{x^2}{2v_1^2 t_o} \tag{5.4.18}$$

Solve for v_1

$$v_1 \approx \frac{x}{\sqrt{2t_o T_{NMO}}} \tag{5.4.19}$$

When you are actually analyzing a travel time graph, there are several steps you can take to get information on the subsurface.

1. Measure (or read off) t_o, t(x), and x
2. If x has a small value, calculate T_{NMO}=t(x)-t_o
3. Calculate $v_1 \approx \frac{x}{\sqrt{2t_o T_{NMO}}}$
4. Calculate $h = \frac{v_1 t_o}{2}$

For real processing of reflection data

- We guess v_1 and h
- Calculate predicted NMO
- Correct arrivals
- Are they flat? If not, try new values (loop through many values and find peak sum)

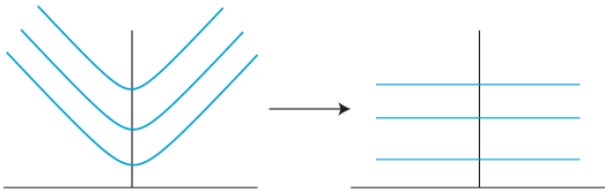


Figure 5.4.10: Correcting Reflection Data

This page titled 5.4: Seismic Reflection (Single Layer) is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

5.5 Seismic Reflection Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of single-layer seismic reflection.

Key Questions: Consider these as you work your way through this page.

- Do waves with larger or smaller incidence angles travel farther?
- Compare the two segments of the various ray path to each other: how are these similar or different?
- Does the bounce point spacing depend on layer thickness? Why or why not?

First, let's import the necessary libraries.

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 print('The libraries are imported.')
```

run | run all | restart | restart & run all

Now let's create our 1-layer model. This cell will prompt you to choose a layer thickness and the spacing between receivers, and will then plot your model. Mathematically, this model is created by selecting the receiver locations and calculating the required incidence angles to achieve those locations given the layer thickness.

```
1 print('Please enter the thickness of the layer between 0 and 500 m')
2 h = float(input())
3 while h <= 0 or h>500:
4     print('Invalid thickness, enter a number between 0 and 500 m')
5     h = float(input())
```

```
6
7  print('Please enter your desired reciever spacing')
8  print('There will be 5 recievers spaced evenly using this spacing')
9  spacing = float(input())
10 v = 1000 #m/s
11 recievers = np.arange(spacing,5*spacing+spacing,spacing)
12 half_x = recievers/2
13 handle = display(None, display_id=True)
14 lines = []
15 bbox = dict(boxstyle ="round", fc = '1')
16
17 while h > 0 and h <= 500:
18     theta_i = np.arctan(half_x/h)
19     theta = np.pi/2 - theta_i
20
21     print('Plotting for thickness',h,'m')
22
23     fig, ax = plt.subplots(figsize=(10,5))
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

- Do waves with larger or smaller incidence angles travel farther?
- Compare the two segments of the various ray path to each other: how are these similar or different?
- Does the bounce point spacing depend on layer thickness? Why or why not?

5.5 Seismic Reflection Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

5.6: Summary

Let's review what we learned about seismology in this chapter:

Basics of Wave Propagation

The basic ways we measure waves are through λ (the wavelength in meters) and f (the frequency in Hz). They have the relation to velocty:

$$v = f\lambda\ [\frac{m}{s}] \tag{5.6.1}$$

One category of seismic waves are body waves. Body waves are P waves and S waves. P waves act like an accordion, and move parallel to the propagation direction. S waves can have two components of motion, vertical and horizontal. Most S waves usually have both components, S-vert and S-horiz, which can be polarized. The other category of seismic waves are surface waves, which decay with depth. There are two main surface waves, Love waves and Rayleigh waves. Love waves have particle motion like that of the S-H component. Rayleigh waves have the components of a P wave and a S-V wave. The particle motion is retrograde.

We also reviewed some constants we learned earlier that will be helpful in determining P and S wave velocity.

E-Young's Modulus (elongation dilation)

μ-Poisson's Ratio (compensation deformation)

G-shear strain

κ-volume change

In seismic calculations, you can use one set (E and μ) or the other (G and κ).

For P-waves:

$$v_p = \sqrt{\frac{\kappa + \frac{4}{3}G}{\rho}}\ or\ \sqrt{\frac{E}{\rho}\frac{(1-\nu)}{(1-2\nu)(1+\nu)}} \tag{5.6.2}$$

For S-waves, we can calculate velocity as:

$$v_s = \sqrt{\frac{G}{\rho}}\ or\ \sqrt{\frac{E}{\rho}\frac{(1-2\nu)}{(2(1+\nu))}} \tag{5.6.3}$$

Using Fermat's Principle and the fact that $\theta_i = \theta_r$, we derived Snell's Law:

$$\frac{sin\theta_i}{v_1} = \frac{sin\theta_r}{v_2} \tag{5.6.4}$$

In special cases such as critical refraction,

$$\theta_{ic} = sin^{-1}(\frac{v_1}{v_2}) \tag{5.6.5}$$

Seismic Refraction (Single Layer)

One of the most important uses of seismic waves is for seismic surveys. There are two types of seismic surveys:

1. Refraction ie 'critical refraction' which is useful for determining velocities of layers
2. Reflection, which is useful for determining layers and structure

$$t = \frac{x}{v_1} \tag{5.6.6}$$

The above equation is the direct ray travel/arrival time. The equation is a line whose slope is $\frac{1}{v_1}$ and intercept=0. Using the vertical incidence of a reflected ray, we can derive an equation to calculate the time the ray took to reach the receiver:

$$t = \frac{2h}{v_1} \tag{5.6.7}$$

There are important values that can be seen when the refracted waves are graphed such as the cross over distance, x_{co} is when the $t_{dir}=t_{head(rfr)}$.

$$x_{co} = 2h(\frac{v_2 + v_1}{v_2 - v_1})^{\frac{1}{2}} \tag{5.6.8}$$

In this section, we were essentially learning how to interpret/extract information from a seismic refraction survey. This allows us to determine the subsurface structure.

How do you determine subsurface structure?

- Identify direct wave arrivals
  - Measure slope $\rightarrow m_1$
  - Calculate v_1(layer 1) $\rightarrow v_1 = \frac{1}{m_1}$
- Identify headwave arrivals
  - Measure slope, m, intercept, t_o
  - Calculate v_2(layer 2) $\rightarrow v_2 = \frac{1}{m_2}$
- Use t_o, v_1, and v_2 to calculate h
  - $t_o = \frac{2h(v_2^2 - v_1^2)^{\frac{1}{2}}}{v_1 v_2}$
  - $h = \frac{t_o v_1 v_2}{2(v_2^2 - v_1^2)^{\frac{1}{2}}}$

Seismic Reflection (Single Layer)

Reflection surveys follow the same basic principles as refraction surveys, using reflected waves instead of refracted waves. We can use reflected waves to get h and v_1, but we cannot get v_2 unless we have a reflection from a deeper layer. The time the wave traveled down and back up to the receiver (2d) can be represented as:

$$t = \frac{\sqrt{x^2 + 4h^2}}{v_1} \tag{5.6.9}$$

The intercept time is:

$t(x=0)=t_o=\frac{2h}{v_1}$

For a vertically reflected wave, if you know v_1, you can get h.

$$h = \frac{v_1 t_o}{2} \tag{5.6.10}$$

Reflected waves graph as hyperbolas instead of lines, where $a=t_o$ and $b=v_1 t_o$. The slope at large $x=\frac{a}{b} = \frac{t_o}{v_1 t_o} = \frac{1}{v_1}$.

$$1 = \frac{t^2}{t_o^2} - \frac{x^2}{v_1^2 t_o^2} \tag{5.6.11}$$

We analyze data using a method called Normal Move-out (NMO).

$$T_{NMO} = t(x) - t_o \tag{5.6.12}$$

The equation for T_{NMO} gives us the "shape of the hyperbola". By correcting for NMO, we are looking at data close to the first few recievers, ie those that have taken shorter paths and have not 'moved-out'.

$$v_1 \approx \frac{x}{\sqrt{2 t_o T_{NMO}}} \tag{5.6.13}$$

When you are actually analyzing a travel time graph, there are several steps you can take to get information on the subsurface.

1. Measure (or read off) t_o, t(x), and x
2. If x has a small value, calculate $T_{NMO}=t(x)-t_o$
3. Calculate $v_1 \approx \frac{x}{\sqrt{2 t_o T_{NMO}}}$
4. Calculate $h = \frac{v_1 t_o}{2}$

This page titled 5.6: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

6: Earthquakes

6.0: Introduction to Earthquakes
6.1: Earthquake Geometry and Process
6.2: Earthquake Magnitude
6.3: Location and Focal Mechanisms
6.4: Earthquake Distance and Magnitude Jupyter Notebook
6.5: Earthquake Scaling Laws
6.6: Summary

This page titled 6: Earthquakes is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.0: Introduction to Earthquakes

If you have lived in California for some time, you are no stranger to earthquakes. They seemingly occur out of nowhere and can inspire both awe and fear. Every time one occurs, the public is always briefly interested in learning a bit more about the mysterious quakes. Instead of letting them be a cryptic event, in this chapter we will break down how and where they happen. Having knowledge of earthquakes is useful not only in the moment while they are actually happening, but also in helping you plan your future life. Factors such as where you want to live and if you have earthquake preparedness supplies are important considerations.

Figure 6.0.1: Destruction in the M_w 9.5 1960 Valdivia Earthquake (Public Domain; Pierre St. Amand via Wikimedia)

In our discussion of earthquakes, we will cover the types of faults, how we locate where earthquakes have occurred, and what a 'beach ball' is. We will also cover the different ways we can measure earthquakes and the various waves they create. By the end of this chapter, you should have a good understanding of how we measure earthquakes, where they occur and how we model them, and the frequency of large vs small earthquakes.

This page titled 6.0: Introduction to Earthquakes is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.1: Earthquake Geometry and Process

Generally speaking, it is the movement of faults that leads to earthquakes. A fault is a plane with localized displacement separating two blocks of rock. As you might already know from your intro to geology class, there are three basic types of faults; strike slip, normal, and reverse. The figures below show the three types of faults.

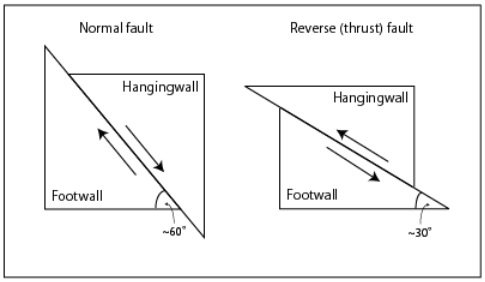


Figure 6.1.1: Normal and Reverse Faults (CC BY-SA 3.0; Cferrero at en.Wikipedia, transferred by Adrignola, via Wikimedia)

A thrust fault is a low angle reverse fault.

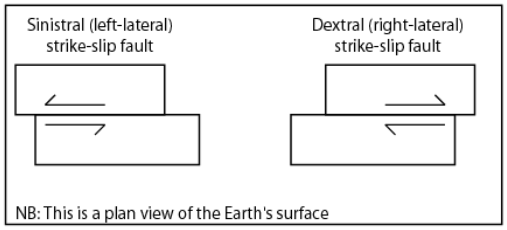


Figure 6.1.2: Strike Slip Faults (CC BY-SA 3.0; Cferrero at en.Wikipedia, transferred to Commons, via Wikimedia)

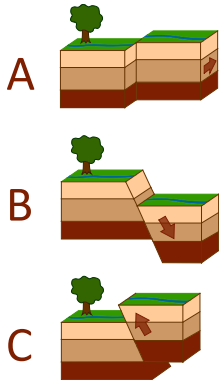


Figure 6.1.3: 3D Fault View: A. Strike-slip, B. Normal, C. Reverse (Public Domain; Karta24, via Wikimedia)

On a fault plane, there are several main locations that help us qualify earthquakes.

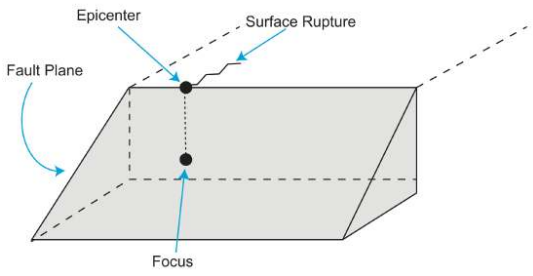


Figure 6.1.4: Fault Plane

The focus is the location where the earthquake initiates. The epicenter is the projection of the focus to the surface. On the surface, the rupture above the focus is visible.

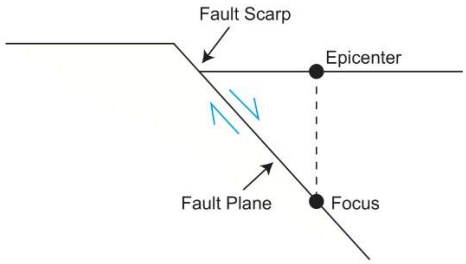


Figure 6.1.5: Fault Scarp

Looking at the area from a different view, we can see that the fault scarp is separated from the epicenter. If we look at the area from yet another different angle (map view), we can develop a model of stick-slip faulting.

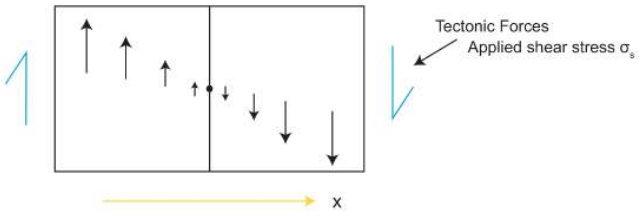


Figure 6.1.6: Model of Stick-slip Faulting

Tectonic forces are moving in opposite directions and cause applied shear stress, σ_s.

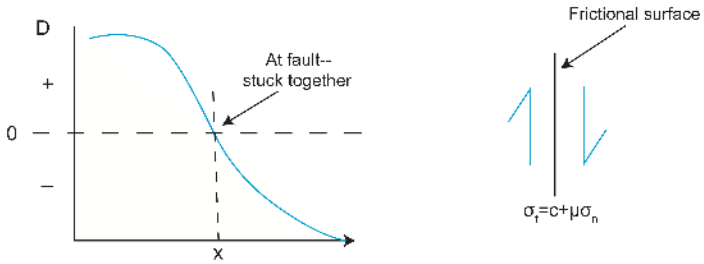


Figure 6.1.7: Frictional Strength

The equation for frictional strength is

$$\sigma_f = c + \mu\sigma_n \tag{6.1.1}$$

If $\sigma_s > \sigma_f$, then failure (an earthquake) occurs.

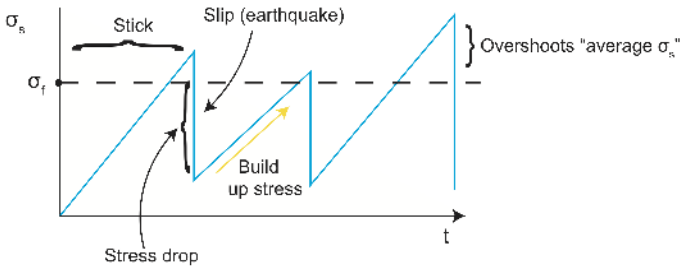


Figure 6.1.8: Stick-slip Graph

The top part of the graph, above the σ_f line, shows where the force overshoots 'average σ_s' because some places will be stronger/weaker and σ_s is heterogeneous. There needs to be a 'large enough' area above σ_f to trigger a rupture.

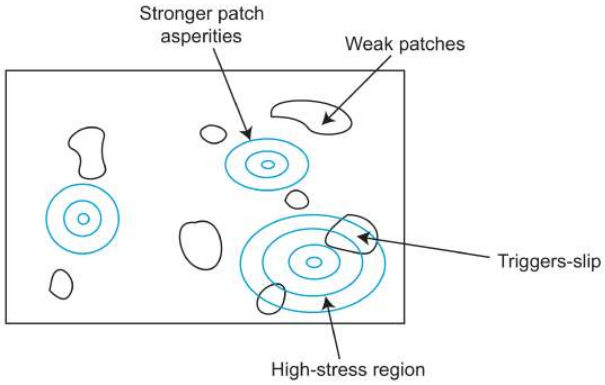


Figure 6.1.9: High Stress Regions

The above figure shows that if there is a large enough region with high enough stress, this will trigger an earthquake. Now that we know a bit about where earthquakes occur, let's go through how they actually happen.

In a fault rupture, the earthquake initiates at the focus.

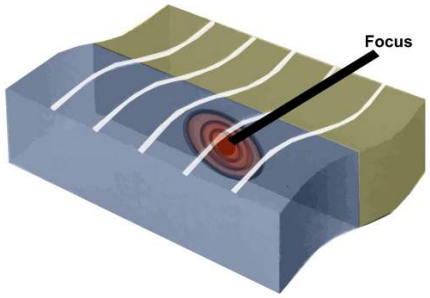


Figure 6.1.10*a*: Rupture at t=0

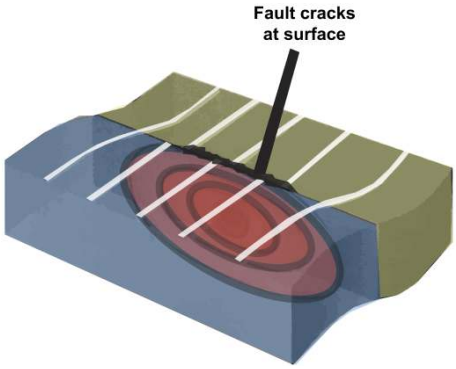


Figure 6.1.10*b*: Rupture at t=5

At 5 seconds, the rupture continues to expand as a crack along the fault plane. When the rupture front reaches the surface, displacements occur along the surface trace, and rocks at the surface begin to rebound from their deformed state.

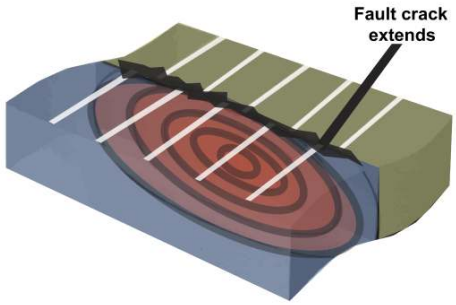


Figure 6.1.10*c*: Rupture at t=10

At 10 seconds, the rupture front progresses down the fault plane, reducing the stress and allowing the rocks on either side to rebound. Seismic waves continue to be emitted in all directions as the fault propagates.

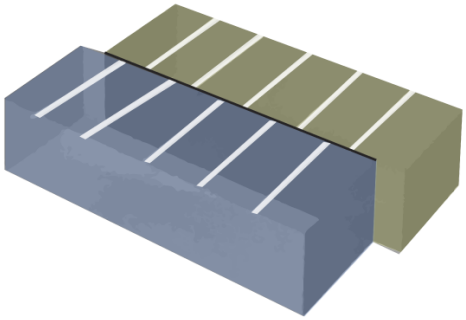


Figure 6.1.10*d*: Rupture at t=20

At 20 seconds, the rupture has progressed along the entire length of the fault. The fault has reached its maximum displacement, and the earthquake stops.

In general, backtracking seismic energy allows us to see its source on the fault. We can also get total slip this way.

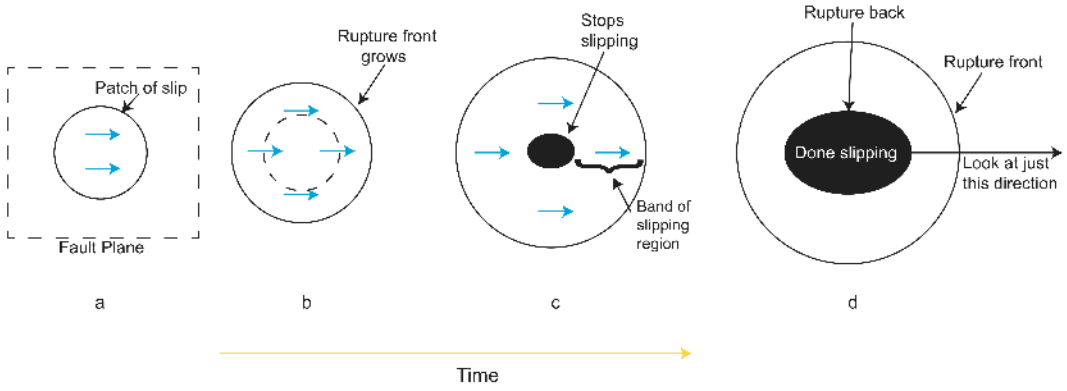


Figure 6.1.11: Rupture with Time

Only a finite width behind the rupture front is slipping. So slipping stops at the focus while regions further away are still slipping.

This slip movement can also be visualized graphically.

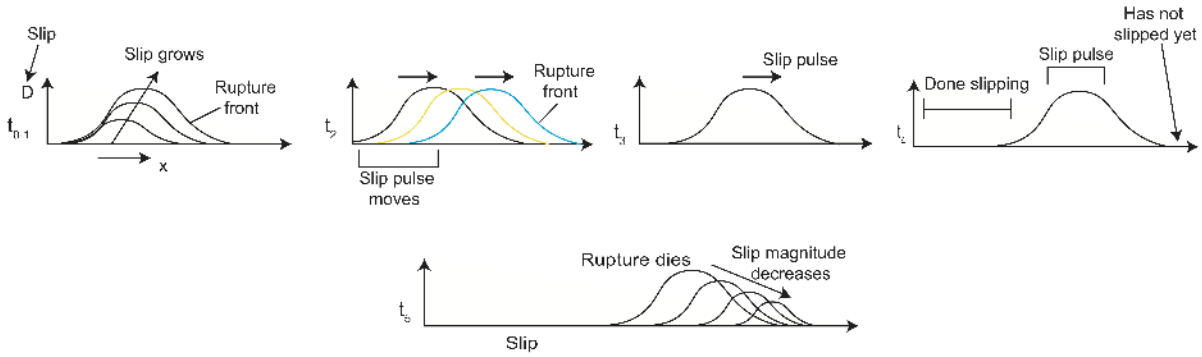


Figure 6.1.12: Graph of Rupture with Time

Another useful tool for tracking earthquakes are seismograms.

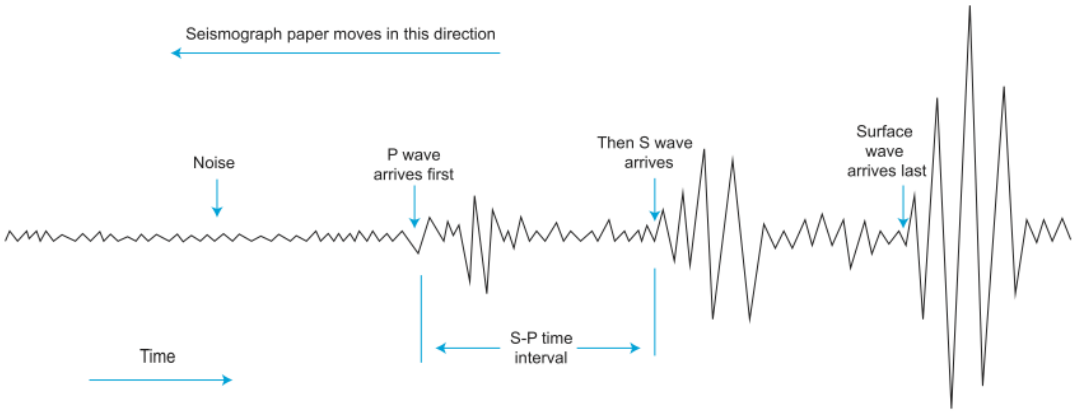


Figure 6.1.13: Interpreting a Seismogram

We can get lots of information from seismograms. They are used for determining:

1. Time of an earthquake and distance to epicenter
2. Earthquake location
3. Magnitude (energy release) of the quake
4. Type of fault (normal, reverse, strike slip)
5. Distribution of slip on the fault surface
6. Structure of the interior of the Earth

We can infer the distribution of slip on a fault surface from seismograms.

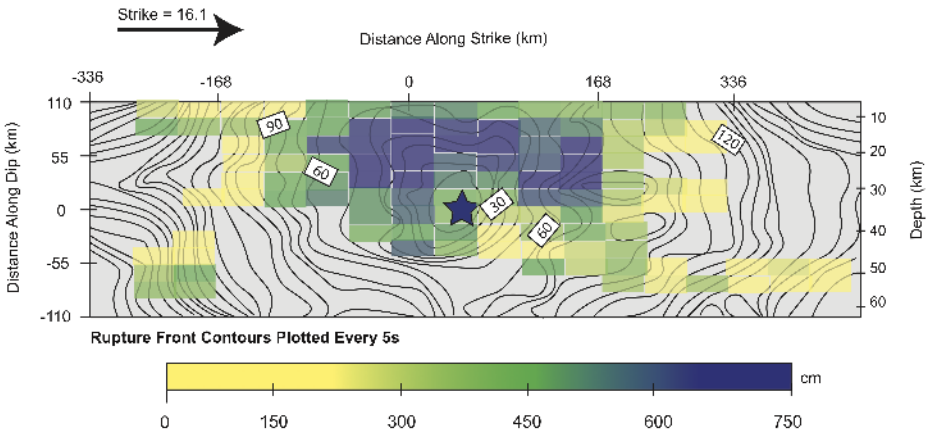


Figure 6.1.14: Distribution of Slip

In the above figure, contours indicate location of the rupture front. The star indicates where the earthquake started.

This page titled 6.1: Earthquake Geometry and Process is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.2: Earthquake Magnitude

In order to measure the size and intensity of earthquakes, we have devised several scales to measure them. The **Richter Magnitude** scale is one such scale that you have likely heard of.

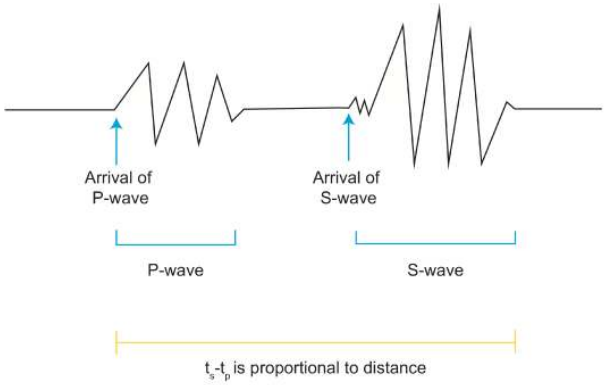


Figure 6.2.1: Seismogram

One issue with measuring earthquakes is that as the waves propagate, the energy is spread out over more area.

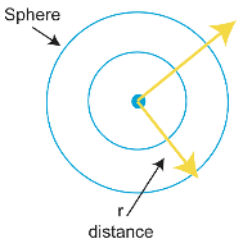


Figure 6.2.2: Distance from Source

As $\frac{E}{Area} \downarrow$, the amplitude decreases with distance. So how do we get around this problem? We discovered that amplitude $\propto$ the initial energy of the earthquake. Thus, we can measure earthquake size/magnitude. Charles Richter took this principle and devised a magnitude scale to relate amplitude of seismic waves to 'size' of the event. This scale takes into account distance from the source and instrument response. The Richter scale is based off of using a particular seismometer, the Wood-Anderson seismometer. The scale was developed to work best only in Southern California, and can not accurately measure earthquakes above a 6.8. To get an idea of how the Richter scale measures earthquakes, look at the figure below.

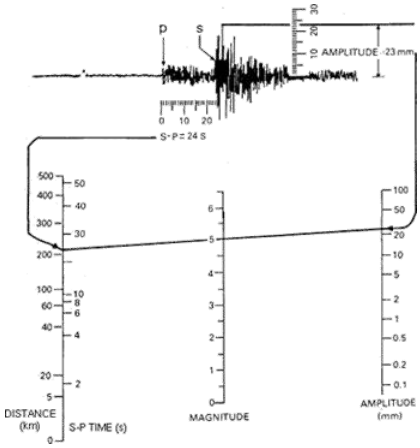


Figure 6.2.3: Richter Scale. (Image is from Richter's (1958) book Elementary Seismology, via USGS)

First, we measure the amplitude. Then, we also know that the difference in arrival times between the P and S waves is proportional to the distance, ie t_s-t_p $\propto$ distance. So we next measure the t_s-t_p $\propto$ distance. Then, draw a line between the amplitude and the

distance. The line will cross a point on the Richter magnitude scale and give the approximate earthquake size. The actual equation for determining Richter magnitude is:

$$M_L = log_{10}A - log_{10}A_0(\delta) = log_{10}[\frac{A}{A_0(\delta)}] \tag{6.2.1}$$

where A is the maximum excursion of the Wood–Anderson seismograph and A_0 depends on the distance of the station from the epicenter (δ).

Logarithm Review

Logarithms are useful for looking at really large data sets (like earthquakes). In case its been a while since you last dealt with logarithms, here is a quick review.

$$\log_{10} y = x$$

The 10 is the base of the logarithm, and 'base 10' is the most common base for a logarithm. The above equation can also be written as

$$10^x = y$$

where x becomes the power of 10. Another example of rewriting logarithms:

$$\begin{aligned} \log_{10}(100) &= y \\ 10^y &= 100 \\ y &= 2 \end{aligned}$$

We could also go the other direction and solve with a calculator:

$$\begin{aligned} 10^y &= 980 \\ \log_{10}(980) &= y \\ y &= 2.99 \end{aligned}$$

A basic log vs log plot looks like this:

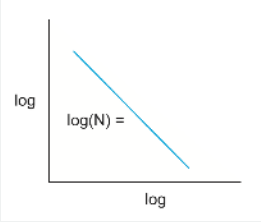


Figure 6.2.4: Log Plot

Using a log scale when plotting is very useful when modeling frequency-magnitude relationships as we will see in section 6.4.

The Richter scale both logarithmic and 'empirical'. Empirical means that it is chosen to fit a range of observation of both very small and very large earthquakes. The fact that the scale is logarithmic indicates that there is an order of magnitude difference between each number. A 0 magnitude would have a relative size (energy) of 10^0, a 5 would have 10^5 energy, and an 8 would have 10^8 energy. Thus you can see that an increase by even one order of magnitude, like from a 7 to 8, is a 32-fold increase in energy.

Another problem we have in measuring earthquakes is how to compare the physical size rupture area from the aftershocks and the surface rupture. The magnitude stopped increasing because big earthquakes put more energy into the surface waves and therefore the energy in S waves does not continue to increase as the size of the earthquake grows. We need to measure all the energy in order to get an accurate measure of the size of the earthquake. To solve this problem, we need to integrate the full seismogram (sum up all the energy) and then match the Richter magnitude scale for moderate sized earthquakes. We can also relate this magnitude to the physical size of the earthquake, in terms of the rupture area, and something else, called **moment**. Moment is literally a torque.

$$\tau = F \cdot l \tag{6.2.2}$$

An earthquake can be thought of as two pairs of opposing torques.

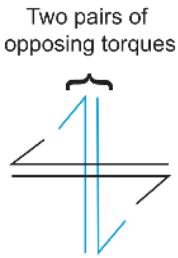


Figure 6.2.5: Torque Model of an Earthquake

For earthquakes, the physical moment can be written as

$$m_o = \mu A \cdot D \tag{6.2.3}$$

where μ is the rock rigidity (usually ~ 30 GPa), A is the fault area that slipped, and D is the average slip. The famed seismologist Hiroo Kanamori took these principals to create a new magnitude scale, the Moment magnitude scale.

$$M_w = \frac{2}{3}(log(M_o) - 10.7) \tag{6.2.4}$$

M_o integrates seismogram data and relates to physical size of the earthquake. Instead of solving for M_w, we can also solve for M_o.

$$M_o = 10^{\frac{3}{2}M_w + 10.7} \tag{6.2.5}$$

The M_w is what is most commonly reported however. The largest recorded earthquake in history (since 1900) is a M_w 9.5, which occurred in Chile in 1960. The earthquake lasted nearly 10 minutes.

Example Expected Magnitude

Here is an example to get you thinking about the concepts we have just discussed.

If I have a fault of A=100 km^2 and D=5 m, where μ=30 GPa, what is the expected magnitude?

Or given M_w and A from aftershocks, determine the amount of slip.

Try to answer these questions now and we will go over them in class.

This page titled 6.2: Earthquake Magnitude is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.3: Location and Focal Mechanisms

When an earthquake occurs, how can we determine the location of the epicenter? The answer is that we can use the seismic waves themselves to tell us where they are coming from. The drawing below is a map showing the epicenter of an earthquake with several seismograph stations. Each seismograph station is at a different distance from the epicenter and records the seismic waves as they arrive at the station. The seismographs show that the time difference for the arrival of the P and S wave at each station is proportional to the distance of the station from epicenter.

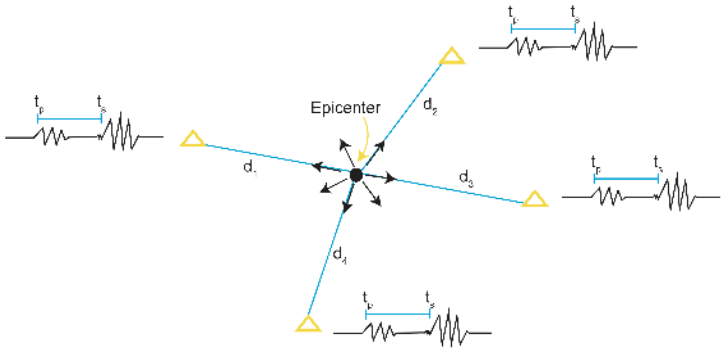


Figure 6.3.1: Station Distance from Epicenter

To find the epicenter of an earthquake there are four basic steps.

1. Identify seismic wave phase arrival times at stations
2. Calculate distance between individual stations and the epicenter
3. Triangulate-use determined distances for multiple stations to locate epicenter
4. Calculate origin time

Step 1 involves using the difference in arrival times of P and S waves that we talked about earlier. P and S waves travel the same distance, but have different arrival times. Thus, the distance in arrival time=difference in travel time.

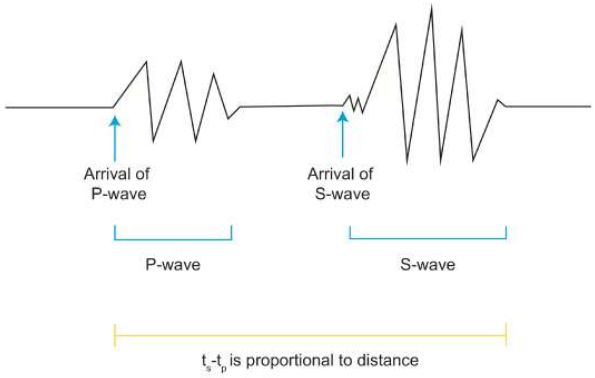


Figure 6.3.2: Seismogram

We can write a basic equation for distance traveled by starting with the equation: velocity= $\frac{distance}{time}$. Rearrange and we get

$$d = tv \tag{6.3.1}$$

We know from experiments V_p, the P wave velocity, and V_s, the S wave velocity. We can also record on the seismograph t_p and t_s, the P and S wave travel times. We are thus trying to find x, the distance the waves traveled. For P waves:

$$x = v_p t_p \tag{6.3.2}$$

$$t_p = \frac{x}{v_p} \tag{6.3.3}$$

For S waves:

$$x = v_s t_s \tag{6.3.4}$$

$$t_s = \frac{x}{v_s} \tag{6.3.5}$$

Using the difference in arrival times:

$$\Delta t = t_p - t_s \tag{6.3.6}$$

$$= \frac{x}{v_p} - \frac{x}{v_s} \tag{6.3.7}$$

$$= x\left(\frac{1}{v_p} - \frac{1}{v_s}\right) \tag{6.3.8}$$

$$= x\left(\frac{v_s}{v_s v_p} - \frac{v_p}{v_p v_s}\right) \tag{6.3.9}$$

$$= x\left(\frac{v_s - v_p}{v_p v_s}\right) \tag{6.3.10}$$

And finally

$$x = \Delta t\left[\frac{v_p v_s}{v_s - v_p}\right] \tag{6.3.11}$$

gives you the distance traveled.

Now that we know have the equation to find wave distance traveled, let's do a real example.

Example 6.3.1: Earthquake Location

Given that $\Delta t = -10\,s$, $v_p = 5\frac{km}{s}$, and $v_s = 3\frac{km}{s}$, how far did the earthquake tarvel?

Solution

This is a simple application of Equation 6.3.11.

$$x = \Delta t[\frac{v_p v_s}{v_s - v_p}]$$

$$= -10[\frac{5 \cdot 3}{3 - 5}]$$

$$= -10[\frac{15}{-2}]$$

$$= 75\ km$$

Now we have the distance, but the earthquake could lie anywhere on a circle centered at that station with a radius=distance.

In order to find the exact location of the earthquake, we must repeat the distance calculation for at least two more stations. This will give us at least three circles with radius=distance, which we can cross reference as in the figure below.

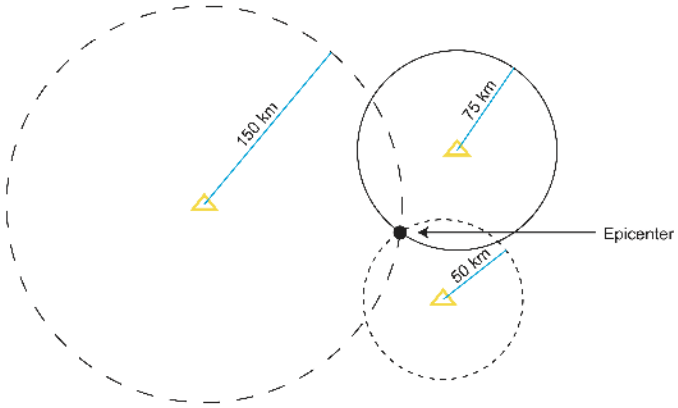


Figure 6.3.3: Triangulating the Epicenter

To locate the focus/hypocenter, you would need at least four stations to draw spheres in 3D in order to get depth.

Origin Time

We have now learned how to find: the arrival time, distance, velocity, and travel time. Now using those variables we can find the origin time.

$$origin\ time = arrival\ time - travel\ time \tag{6.3.12}$$

Example 6.3.2: Origin Time

Using the values we were given/calculated in the above example (v_p=5$\frac{km}{s}$, v_s=3$\frac{km}{s}$, and x=75 km), we can calculate the travel time:

$$\frac{75\ km}{5\ \frac{km}{s}} \tag{6.3.13}$$

$$15\ s \tag{6.3.14}$$

Note that the travel time can be calculated with any station and both P and S wave arrivals.

If we know that the arrival time was at 11:30:45 (11:30 and 45 sec), the origin time=11:30:45-00:00:15. Thus the origin time is 11:30:30.

Focal Mechanisms

Focal mechanisms give information about the type of deformation in the source region that generates the seismic waves. In other words, focal mechanisms show a simple seismic model of faulting. They are also a 'moment tensor' or 'beach ball diagram'. A moment tensor is like a stress tensor, it gives you the state of stress. Focal mechanisms are a double representation of slip, this refers to the coupling of forces, like in Figure 6.3.4.

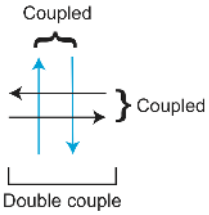


Figure 6.3.4: Coupling of Forces

Slip on a fault also causes volumetric changes. Look at the figure below to better understand this.

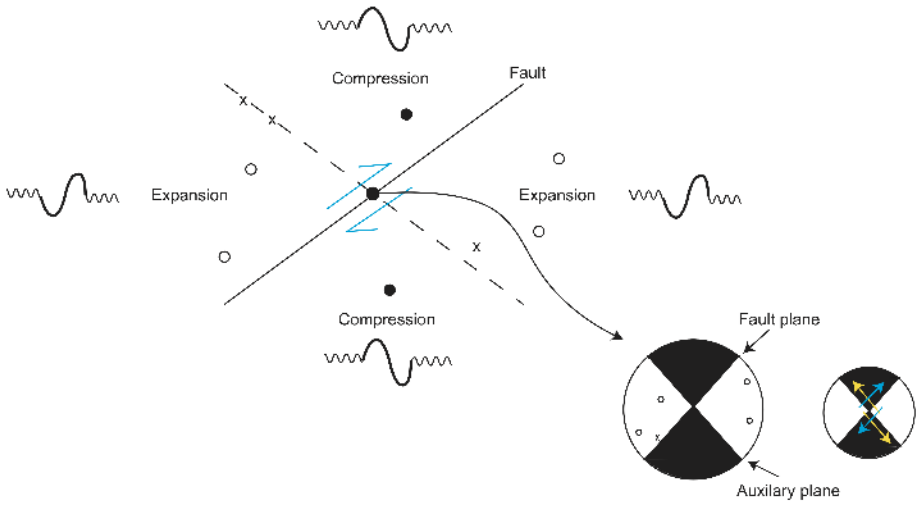


Figure 6.3.5: Expansion and Compression Due to Slip

The first motion of the fault that causes the volumetric changes is the P waves that are recorded at stations. The various motions of the fault are represented on what is known as a beach ball. The regions of compression indicate a first motion of up and are represented by a closed dot ●. The regions of expansion indicate a first motion of down and are represented by an open dot ○. If there is no detected signal, this is represented by an ×. All together, the motions of the fault look like the figure below.

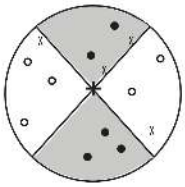


Figure 6.3.6: Volumetric Changes on a Fault

To combine all of the data into a cohesive picture of fault motion, we will use a beach ball. A basic beach ball is seen below.

Focal mechanism 02.jpg

Figure 6.3.7: Constructing a Beach Ball (CC BY-SA 3.0; Mikenorton, via Wikimedia)

Visualizing fault planes on a sphere is often difficult, so let's look at it from some different angles.

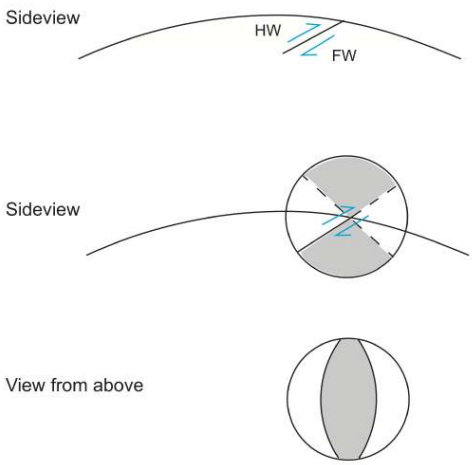


Figure 6.3.8: Thrust Fault

When drawing a beach ball, we must ensure that the auxiliary plane is perpendicular to the fault plane. Remember that the fault plane is where displacement occurs. These two planes are also called nodal planes.

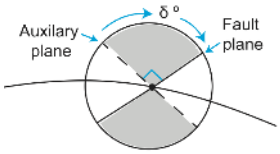


Figure 6.3.9: Nodal Planes

There are three principal axes on a beach ball, the N axis, P axis, and T axis. These axes are all orthogonal to each other.

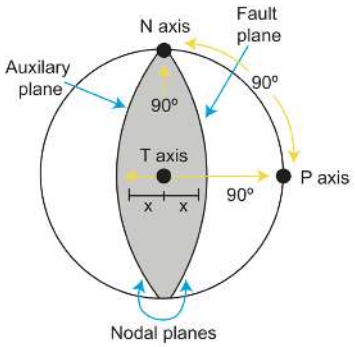


Figure 6.3.10: 3 Principal Axes

The N axis, also called the b axis, is the null axis. It is the intersection of the two nodal planes. The P axis is the pressure axis. It bisects the angle between the nodal planes, and has the largest amplitude down first motion. The T axis is the tension axis. It also bisects the angle between the nodal planes, and has the largest amplitude up first motion.

To get a better understanding of a how different fault types are modeled on a beach ball:

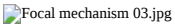


Figure 6.3.11: Fault Types Modeled as Beach Balls (CC BY-SA 3.0; Mikenorton, via Wikimedia)

How do you determine which nodal plane is the fault plane?

1. Look at the geology
2. Look at the aftershock pattern

FIGURE Magali will make figure for use

Summary

That was a lot of new information to cover in one section. Here are some quick glance takeaways of focal mechanisms can tell you:

- Approximate strike of fault
- What type of faulting occurred
- Overall sense of motion in a region
- Regions of compression and tension

This page titled 6.3: Location and Focal Mechanisms is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.4: Earthquake Distance and Magnitude Jupyter Notebook

Instructions

- To use this page, read the text between each of the python code windows, then press RUN to execute the code in the box.
- It is necessary to run each of the boxes in order.
- Don't click restart in the cells unless you come back to the top of the page and start over.
- If you modify the code, this modification will not remain after you leave or refresh this page.
- You must run simulations/cells that require user input to completion. Make sure you see the Message informing you a cell is complete before continuing to the next cell. Here's why you must do this:
  - These cells may not be re-run until the simulation is completed, otherwise no output will be generated.
  - All following cells will not run until the user input cell is completed.
- For Dropdown Menus: You do not need to rerun the cells with the menu to change your selection. However, you must rerun all of the following cells to implement the change.

Be patient, some times it can take 1-2 minutes for the juypter kernel to start.

An interactive example of calculating the distance from an earthquake from seismic wave arrivals and magnitude from the empirical formula for Richter Magnitude.

Key Questions: Consider these as you work your way through this page.

For Earthquake Distance:

- Compare the distance predicted for the same ts-tp time for different rock types: which rock type gives the largest distance? Explain why.

For Richter Magnitude:

- As you increase the seismogram amplitude, does the magnitude increase, decrease, or stay the same?
- Can you generate an unrealistic magnitude ($M \geq 10$)? If so, why? If not, which parameter value prevents this?

First, we import the necessary libraries and create a dictionary containing p and s-wave velocities in different materials.

```
1 import numpy as np
2 from ipywidgets import interact, interactive, fixed, interact_manual
3 import ipywidgets as widgets
4
5 rock_types = {'shale':{'P':3000,'S':2600},'limestone':{'P':4250,'S':3100},'sandstone':{'P':2700,'S':240
6
7 print('The libraries are imported and the dictionary is defined.')
```

run | run all | restart | restart & run all

Earthquake Distance

Now, let's learn about how we can determine how far away we are from an earthquake from a seismogram. The p and s-wave arrival times can be picked from a seismogram, and their difference between them is related to how far away the earthquake was. In this cell, you will be prompted to enter the difference in arrival times of the waves. That information will be used to calculate the distance between you and the earthquake assuming the waves are traveling through a layer of granite. The function distance_from_eq accomplishes this, and the formula is derived in 6.3: Location and Focal Mechanisms.

```
print('Please enter the amount of time between the P-wave and S-wave arrivals between 1 and 60 s')
arrival_diff = float(input())
while arrival_diff < 1 or arrival_diff > 60:
    print('Invalid time difference: please enter a number between 1 and 120s')
    arrival_diff = float(input())

def distance_from_eq(vp,vs,delta_t):
    x = (-delta_t * vp * vs) / (vs - vp)
    return x

vp = 4450 #m/s, granite ranges from about 3000 to 5900 m/s
vs = 3500 #m/s, granite ranges from about 3400 to 3600 m/s

x = distance_from_eq(vp,vs,arrival_diff)
print('The p-wave velocity in granite is',vp,'m/s')
print('The s-wave velocity in granite is',vs,'m/s')
print('The distance from the earthquake is ' + '{:.2f}'.format(x/1000)+' km')
print('The calculation is complete.')
```

run | run all | restart | restart & run all

Now let's explore how different materials affect the results. Changing the material will change the p and s-wave velocities. This cell will prompt you to select a new arrival time difference and allow you to pick a material. Note: To change the material later on, you do not need to rerun this cell, simply change the dropdown menu selection. However, you will need to run all cells after this one.

```
print('Please enter the amount of time between the P-wave and S-wave arrivals between 1 and 60 s') #Ask
arrival_diff = float(input())
while arrival_diff < 1 or arrival_diff > 60:
    print('Invalid time difference: please enter a number between 1 and 60s')
    arrival_diff = float(input())

style = {'description_width': 'initial'}
layout = {'width': '400px'}
rock_dropdown = widgets.Dropdown(
    options=['shale', 'limestone', 'sandstone','alluvium'],
    value='shale',
    description='Select a material for your model',
    disabled=False,
    style = style,
    layout = layout
)


display(rock_dropdown)
```

run | run all | restart | restart & run all

Now run the next cell to do the calculation, do this for all the materials and consider the key questions.

```
rock = rock_dropdown.value
```

```
print('Now plotting for',rock)



vp = rock_types[rock]['P']
vs = rock_types[rock]['S']


x = distance_from_eq(vp,vs,arrival_diff)
print('The p-wave velocity in',rock,'is',vp,'m/s.')
print('The s-wave velocity in',rock,'is',vs,'m/s.')
print()
print('The distance from the earthquake is ' + '{:.2f}'.format(x/1000)+' km.')
print()


print('Go up to the above cell, select a new rock type, and re-run this cell')
print('Note: You do not need to re-run the cell with the dropdown menu')


print('The calculation is complete.')
```

run | run all | restart | restart & run all

Richter Magnitude

Now let's take it a step further and calculate the Richter Magnitude. The Richter Magnitude depends on the amplitude on a Wood-Anderson Seismometer and the distance from the earthquake. This cell will prompt you to select an arrival time difference and a material type for this model.

```
print('Please enter the amount of time between the P-wave and S-wave arrivals between 1 and 60 s') #Ask
arrival_diff = float(input())
while arrival_diff < 1 or arrival_diff > 60:
    print('Invalid time difference: please enter a number between 1 and 60s')
    arrival_diff = float(input())

display(rock_dropdown)
```

run | run all | restart | restart & run all

This cell will first calculate the distance from the earthquake just like before. However, to calculate magnitude we will also need the seismometer amplitude so you will be prompted to enter that here. To calculate the magnitude, the function richter_magnitude is defined. This formula is shown partially in 6.2: Earthquake Magnitude, however that version of the formula leaves $A_0(\delta)$, where delta is the distance, as an unknown function. In this example, an empirical formula is used for $A_0(\delta)$. Go ahead and run the cell, and see how the magnitude varies with different amplitudes.

```
rock = rock_dropdown.value
vp = rock_types[rock]['P']
vs = rock_types[rock]['S']
x = distance_from_eq(vp,vs,arrival_diff)
print('The p-wave velocity in',rock,'is',vp,'m/s.')
```

```
print('The s-wave velocity in',rock,'is',vs,'m/s.')
print('The distance from the earthquake is ' + '{:.2f}'.format(x/1000)+' km')
print()

print('Enter an amplitude of shaking from a Wood Anderson Seismogram in mm:')
print('The normal range of amplitudes is between 0.1 and 500 mm')
A = float(input())


def richter_magnitude(A,x):
    if x <= 200:
        M = (np.log10(A)  + 1.6*np.log10(x/1000) - 0.15)
    else:
        M = (np.log10(A)  + 3*np.log10(x/1000) - 3.38)
    return M

while A > 0:
    print('Now calculating magnitude for an amplitude of',A,'mm')
```

run | run all | restart | restart & run all

Key Questions: Consider these as you work your way through this page.

For Earthquake Distance:

- Compare the distance predicted for the same ts-tp time for different rock types: which rock type gives the largest distance? Explain why.

For Richter Magnitude:

- As you increase the seismogram amplitude, does the magnitude increase, decrease, or stay the same?
- Can you generate an unrealistic magnitude ($M \geq 10$)? If so, why? If not, which parameter value prevents this?

6.4: Earthquake Distance and Magnitude Jupyter Notebook is shared under a not declared license and was authored, remixed, and/or curated by LibreTexts.

6.5: Earthquake Scaling Laws

Big earthquakes, those 7 and above, are not very common. Smaller magnitude earthquakes occur much more frequently. There is a quantitative way of looking at the distributions of the occurrence of different magnitude earthquakes, the Gutenberg-Richter frequency-magnitude scaling law. Frequency-magnitude describes the number of occurrences as a function of magnitude.

$$log_{10}(N_c) = -bM + a \tag{6.5.1}$$

where M=magnitude, N_c=cumulative number of earthquakes that occurred with M ≥ 0 (i.e. all earthquakes), a=total seismicity, and b='b value' which is the scaling value relating the number of large events to small events. a and b are constants calculated for a specific region and time.

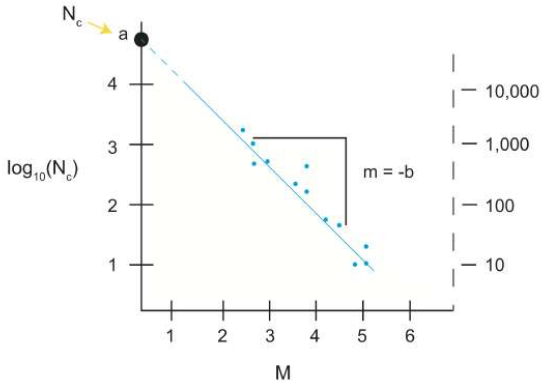


Figure 6.5.1: Gutenburg-Richter Scaling Law

The scaling law is plotted on the above graph. The equation is a line (y=mx+b), where a gives the y-intercept and -b provides a negative slope.

Take a look at the three figures below:

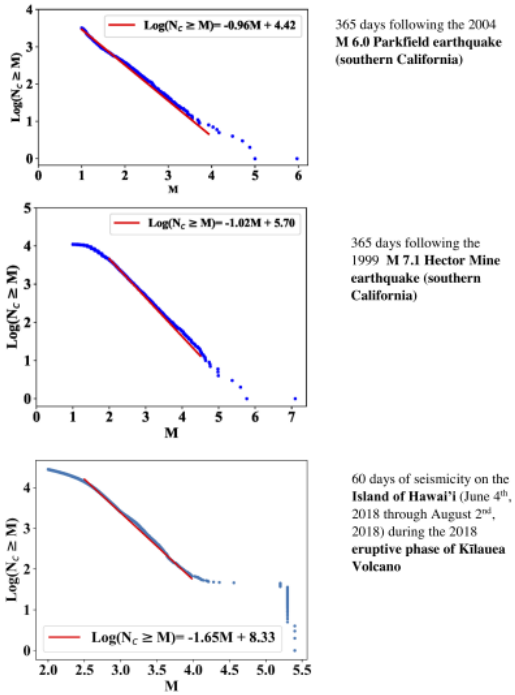


Figure 6.5.2: Gutenburg-Richter Diagrams (Rebecca Fildes, UC Davis. Earthquake data obtained from the United States Geological Survey Earthquake Catalog: USGS Data accessed on 6-13-2018 (Parkfield), 5-30-2018 (Hector Mine) and 8-22-2018 (Kilauea) and plotted using Python)

We can see that linearity and fit of the Gutenberg-Richter law occurs in the mid-magnitude range. There are two reasons for this. One is roll-off. This is when small events are either not recorded or not detected among larger events (typically ~M 2). The other reason is a higher magnitude drop off. It is not physically possible to sustain this consistently for large magnitude events (typically ~M 4-5). In the Parkfield figure, there is a lower roll off. This is due to the dense network of seismometers around the area, which were installed as part of an experiment.

The most talked about part of the scaling relationship is the b value. The b value is typically $0.8 \leq b \leq 1.2$ with b $\simeq$ 1. Since the relationship is logarithmic, when b=1, m=-1. Each M you go up by, you go down an order of magnitude of occurrence.

10,000 M2 events

1,000 M3

100 M4

10 M5

1 M6

b $\neq$1 always. As b>1, you expect more smaller events relative to larger events (than for b=1).

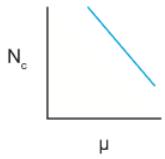


Figure 6.5.3: b>1 (MIB: TYPO IN THIS FIGURE: μ should be M (magnitude))

As b<1, you expect more larger events relative to smaller events (than for b=1).

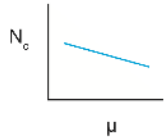


Figure 6.5.4: b<1 (MIB: TYPO IN THIS FIGURE: μ should be M (magnitude))

a values are harder to compare than b values because they involve the total level of seismicity for a specific region and time.

This page titled 6.5: Earthquake Scaling Laws is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

6.6: Summary

Let's review what we learned about earthquakes in this chapter:

Earthquake Geometry and Process

Generally speaking, it is the movement of faults that leads to earthquakes. There are three basic types of faults; strike slip, normal, and reverse. The focus is the location where the earthquake initiates. The epicenter is the projection of the focus to the surface.

Tectonic forces are moving in opposite directions and cause applied shear stress, σ_s.

The equation for frictional strength is

$$\sigma_f = c + \mu\sigma_n \tag{6.6.1}$$

If $\sigma_s > \sigma_f$, then failure (an earthquake) occurs.

We also learned how earthquakes actually happen. In a fault rupture, the earthquake initiates at the focus.

- At 5 seconds, the rupture continues to expand as a crack along the fault plane. When the rupture front reaches the surface, displacements occur along the surface trace, and rocks at the surface begin to rebound from their deformed state.
- At 10 seconds, the rupture front progresses down the fault plane, reducing the stress and allowing the rocks on either side to rebound. Seismic waves continue to be emitted in all directions as the fault propagates.
- At 20 seconds, the rupture has progressed along the entire length of the fault. The fault has reached its maximum displacement, and the earthquake stops.

Seismograms record earthquake waves, and are used for determining:

1. Time of an earthquake and distance to epicenter
2. Earthquake location
3. Magnitude (energy release) of the quake
4. Type of fault (normal, reverse, strike slip)
5. Distribution of slip on the fault surface
6. Structure of the interior of the Earth

Earthquake Magnitude

In order to measure the size and intensity of earthquakes, we have devised several scales to measure them. The Richter Magnitude scale is one such scale. One issue with measuring earthquakes is that as the waves propagate, the energy is spread out over more area. We discovered that amplitude $\propto$ the initial energy of the earthquake. The actual equation for determining Richter magnitude is:

$$M_L = \log_{10} A - \log_{10} A_0(\delta) = \log_{10}[\frac{A}{A_0(\delta)}] \tag{6.6.2}$$

where A is the maximum excursion of the Wood–Anderson seismograph and A_0 depends on the distance of the station from the epicenter (δ). The Richter scale is both logarithmic and 'empirical'. Empirical means that it is chosen to fit a range of observation of both very small and very large earthquakes. The fact that the scale is logarithmic indicates that there is an order of magnitude difference between each number.

Another problem we have in measuring earthquakes is how to compare the physical size rupture area from the aftershocks and the surface rupture. To solve this problem, we need to integrate and scale a typical amplitude vs time diagram, and convert it to area, or moment. Moment is literally torque. The famed seismologist Hiroo Kanamori used these ideas to create a new magnitude scale, the Moment magnitude scale.

$$M_w = \frac{2}{3}(log(M_o) - 10.7) \tag{6.6.3}$$

M_o integrates seismogram data and relates to physical size of the earthquake. Instead of solving for M_w, we can also solve for M_o.

$$M_o = 10^{\frac{3}{2}M_w + 10.7} \tag{6.6.4}$$

Location and Focal Mechanisms

To find the simplified location of an earthquake there are four basic steps.

1. Identify seismic wave phase arrival times at stations
2. Calculate distance between individual stations and the epicenter
3. Triangulate-use determined distances for multiple stations to locate epicenter
4. Calculate origin time

The equation to find wave distance traveled:

$$x = \Delta t[\frac{v_p v_s}{v_s - v_p}] \tag{6.6.5}$$

Using this we were able to find distance, but the earthquake could lie anywhere on a circle centered at that station with a radius=distance.

In order to find the exact location of the earthquake, we must repeat the distance calculation for at least two more stations. This will give us at least three circles with radius=distance, which we can cross reference as in the figure below. To locate the focus/hypocenter, you would need at least four stations to draw spheres in 3D in order to get depth.

To find information about the type of deformation in the source region that generates the seismic waves, we used focal mechanisms. Focal mechanisms show a simple seismic model of faulting. They are also a 'moment tensor' or 'beach ball diagram'. The various motions of the fault are represented on what is known as a beach ball. The regions of compression indicate a first motion of up and are represented by a closed dot. The regions of expansion indicate a first motion of down and are represented by an open dot. If there is no detected signal, this is represented by an x. The auxiliary plane and the fault plane are also called nodal planes.

There are three principal axes on a beach ball, the N axis, P axis, and T axis which are all orthogonal to each other. The N axis is the null axis. It is the intersection of the two nodal planes. The P axis is the pressure axis. It bisects the angle between the nodal planes, and has the largest amplitude down first motion. The T axis is the tension axis. It also bisects the angle between the nodal planes, and has the largest amplitude up first motion.

Here are some quick glance takeaways of focal mechanisms can tell you:

- Approximate strike of fault
- What type of faulting occurred
- Overall sense of motion in a region
- Regions of compression and tension

Earthquake Scaling Laws

There is a quantitative way of looking at the distributions of the occurrence of different magnitude earthquakes, the Gutenberg-Richter frequency-magnitude scaling law. Frequency-magnitude describes the number of occurrences as a function of magnitude.

$$log_{10}(N_c) = -bM + a \tag{6.6.6}$$

where

- M is the magnitude,
- N_c is the cumulative number of earthquakes that occurred with magnitude $\geq$ M,
- a is the total seismicity, and
- b is the 'b value' which is the scaling value relating the number of large events to small events.

a and b are constants calculated for a specific region and time. The equation is a line (y=mx+b). Linearity and fit of the Gutenberg-Richter law occurs in the mid-magnitude range due to roll-off and a higher magnitude drop off.

The most talked about part of the scaling relationship is the b value. The b value is typically $0.8 \leq b \leq 1.2$ with $b \simeq 1$. Each |(M\) you go up by, you go down an order of magnitude of occurrence.

10,000	1,000	100	10	1
M2 events	M3	M4	M5	M6

This page titled 6.6: Summary is shared under a CC BY-SA license and was authored, remixed, and/or curated by Magali Billen.

CHAPTER OVERVIEW

Back Matter

Index

Arctan vs Arctan2

Ipywidgets Interact Example

Index

A

Andersonian theory of faulting
- 1.3: Elastic Deformation

B

Byerlee's law
- 1.3: Elastic Deformation
- 1.4: Failure of Rocks

C

compensated topography
- 3.4: Isostasy

Couette Flow
- 1.5: Viscous Deformation

D

Darcy's Law
- 2.0: Introduction to Diffusion and Darcy's Law
- 2.5: Darcy's Law - Flow in a Porous Medium

Darwin Radau equation
- 3.2: Layered Structure of a Planet

diffusion
- 2.0: Introduction to Diffusion and Darcy's Law

diffusion equation
- 2.1: The Diffusion Equation
- 2.2: Geological Applications of the Diffusion Equation
- 2.6: Summary

Diffusivity
- 2.2: Geological Applications of the Diffusion Equation

E

elastic deformation
- 1.3: Elastic Deformation

ellipticity coefficient
- 3.2: Layered Structure of a Planet

Euler Pole
- 4.7: Plate Motions on a Sphere

G

geoid
- 3.7: Gravitational Potential, Mass Anomalies and the Geoid

Geomagnetic Timescales
- 4.2: Magnetic Anomalies on the Seafloor

geosynchonous orbits
- 3.1: Orbital Mechanics

H

headwave
- 5.2: Seismic Refraction (Single Layer)

Hohmann Transfer Orbit
- 3.1: Orbital Mechanics

hydraulic conductivity
- 2.5: Darcy's Law - Flow in a Porous Medium

hydraulic head
- 2.5: Darcy's Law - Flow in a Porous Medium

hydrostatic assumption
- 3.2: Layered Structure of a Planet

hydrostatic flattening
- 3.2: Layered Structure of a Planet

I

intrinsic permeability
- 2.5: Darcy's Law - Flow in a Porous Medium

isostasy
- 3.4: Isostasy
- 3.5: Isostasy Jupyter Notebook

L

lateral strain
- 1.3: Elastic Deformation

longitudinal strain
- 1.3: Elastic Deformation

Love waves
- 5.1: Basics of Wave Propagation

M

magnetic stratigraphy
- 4.2: Magnetic Anomalies on the Seafloor

Moments of Inertia
- 3.2: Layered Structure of a Planet

N

normal stress
- 1.1: Stress

O

Origin Time
- 6.3: Location and Focal Mechanisms

P

permeability
- 2.5: Darcy's Law - Flow in a Porous Medium

phonon
- 2.1: The Diffusion Equation

planetary structure
- 3.2: Layered Structure of a Planet

plate tectonics
- 4: Plate Tectonics
- 4.0: Introduction to Plate Tectonics

Poisson's ratio
- 1.3: Elastic Deformation

porosity
- 2.5: Darcy's Law - Flow in a Porous Medium

principal stresses
- 1.1: Stress

R

Rayleigh wave
- 5.1: Basics of Wave Propagation

Relative Plate Velocity
- 4.5: Relative Velocity and Reference Frames

Relative Velocity
- 4.5: Relative Velocity and Reference Frames

rheology
- 1.0: Introduction to Rheology

richter scale
- 6.2: Earthquake Magnitude

S

seismic reflection
- 5.4: Seismic Reflection (Single Layer)

seismic refraction
- 5.2: Seismic Refraction (Single Layer)

seismic survey
- 5.2: Seismic Refraction (Single Layer)

Seismogram
- 6.3: Location and Focal Mechanisms

seismology
- 5: Seismology

shear angle
- 1.2: Strain and Strain Rate

shear stress
- 1.4: Failure of Rocks

strain
- 1.2: Strain and Strain Rate

stress
- 1.1: Stress

subduction
- 4.4: Plates Moving on a Plane

surface waves
- 5.1: Basics of Wave Propagation

T

Thermal Conduction
- 2.1: The Diffusion Equation

traction
- 1.1: Stress

U

uncompensated topography
- 3.4: Isostasy

V

viscosity
- 1.5: Viscous Deformation

Viscous Deformation
- 1.3: Elastic Deformation
- 1.5: Viscous Deformation

W

wave propagation
- 5.1: Basics of Wave Propagation

Y

Young's modulus
- 5.1: Basics of Wave Propagation

Arctan vs Arctan2

Normally when you calculate the inverse tangent, you are doing so for a question related to a triangle in which the two sides, x and y are lengths (positive numbers), and the resulting angle is between 0 and 90° ($\pi/2$).

However, it turns out that there are two different definitions of the inverse tangent, and which one you use matters if x and y can both be positive or negative. This is exactly the case when using the Cartesian components of location on a sphere to determine the longitude and latitude of that point.

The version of tangent that you find on your calculator is a two-quadrant inverse tangent (atan) and it expects a single input value (y/x). This value can be positive or negative and the output will have the same sign and have values between 0 and $\pi/2$ (90 °). In the plot below, atan only gives back results in the right half of the circle.

Now let's think about a point (x,y,z) on the equator of the earth, where z=0. Note, that the ratio of y/x=(−y)/(−x), so that atan will return the same value for the angle (the longitude), but these two points do not lie in the same location on the sphere.

This is why, there is also a four-quadrant inverse tangent, atan2(y,x). Note this version of the inverse tangent requires two input values. This allows the function to determine in which of the four quadrants (+x,+y), (+x,-y), (-x,+y) or (-x,-y) the result should lie. The output of the function range from 0 to ±180° ($\pm\pi/2$), with positive values (east longitudes) for positive y values, and negative values (west longitudes) for negative y values.

Below, I include the example for x = -1 and y = -1 (blue circle)

* atan returns 45°, because -1/-1 = 1, which is the same as 1/1 (red circle)

* atan2 returns 135°, which is the correct answer accounting for the signs of x and y.

```
import matplotlib.pyplot as plt
import numpy as np

r2d = 180/np.pi
fig = plt.figure(figsize=(7,7))
ax = fig.add_subplot(1,1,1)
x = np.arange(-1,1.,0.01)
y = np.arange(1,-1.,-0.01)

ycirc = np.sqrt(1 - x**2)
x1 = -1
y1 = -1
print('atan2 for x = -1, y = -1',np.arctan2(y1,x1)*r2d)
print('atan,for x = -1, y = -1',np.arctan(y1/x1)*r2d)

ax.plot(np.zeros(y.shape),y)
ax.plot(x, np.zeros(x.shape))
ax.plot(x, ycirc,'black')
ax.plot(x, -ycirc,'black')
ax.plot([0, -1], [0,-1],color='blue')
ax.plot([0, 1], [0,1],color='red')
ax.plot(1,1,marker='o',color='red')
ax.plot(-1,-1,marker='o',color='blue')

```

run | run all | restart | restart & run all

For more information on the math behind arctan, click here, or for arctan2, click here.

Numpy Documentation: Arctan and Arctan2

Another Explanation:

The normal inverse tangent function has a range between -$\pi/2$ and $\pi/2$, or -90 to 90 degrees. In mathematical notation -$\pi/2$ $\leq \arctan(z) \leq$ $\pi/2$ or in degrees, -90° $\leq \arctan(z) \leq$ 90°, where z = y/x = opposite/adjacent. This is because the function cannot distinguish between all four quadrants.

Recall that $\tan(\theta) = \sin(\theta)/\cos(\theta)$, and in right triangles $\tan(\theta) = y/x$

If $\tan(\theta)$ is positive, that could mean that either the angle is in Quadrant I (where both $\sin(\theta)$ and $\cos(\theta)$ are positive) or Quadrant III (where both $\sin(x)$ and $\cos(\theta)$ are negative).

Similarly if $\tan(\theta)$ is negative, that could mean that either the angle is in Quadrant II or Quadrant IV, where $\sin(\theta)$ and $\cos(\theta)$ are opposite signs.

Arctan takes only one input value, and therefore cannot determine which of the two quadrants the angle lies in each case. By default, it provides the angle either in Quadrant I or Quadrant IV depending on the sign of $\tan(\theta)$. It is perhaps easier to think of this in terms of triangles, and the one value taken by arctan is really the quotient of the opposite and adjacent sides of the triangle or y/x, so $\arctan(y/x)$. In order to determine the correct quadrant, the sign of y and x must be known, however, the function only receives the sign of x/y, and not x and y separately and cannot distinguish quadrants.

This can occasionally be an issue, especially if we are interested in an angle that can range from -180 to 180 degrees, such as longitude. Arctan2 is the 4-quadrant inverse tangent. It can get around the previous issue by taking both x and y in as separate arguments. Whereas the syntax for arctan is $\arctan(y/x)$, the syntax for arctan2 is $\arctan 2(y, x)$. Knowing the signs of x and y separately can determine if the angle lies in any of the four quadrants. The range of arctan2 is $-\pi \leq \arctan 2(y, x) \leq \pi$ or in degrees, $-180^\circ \leq \arctan 2(y, x) \leq 180^\circ$.

In conclusion, if you are calculating something that ranges between -90 and 90 degrees like latitude, use arctan. If calculating an angle that can be between -180 and 180 degrees, use arctan2.

This explanation for arctan vs arctan2 was adapted and expanded from stackexchange

Ipywidgets Interact Example

Students, if you happen to come across this page, you may ignore it.

Example of How we would like to use interact from Ipywidgets. Working properly, a slider would appear and could be used to change the graph.

```
import numpy as np
import matplotlib.pyplot as plt
from ipywidgets import interact, interactive, fixed, interact_manual
import ipywidgets as widgets

g = 9.81 #m/s

def inclined_plane_velocity_and_strain_rate(alpha,rho,eta,h): #alpha in degrees, rho in kg/m^3, eta in
    step = 1
    y = y = np.arange(0,h+step,step)

    Velocities = (rho * g * np.sin(np.radians(alpha)) / (2*eta)) * (h**2 - y**2) # m/s velocity
    Strain_Rate = -(rho * g * np.sin(np.radians(alpha)) * y / (2*eta)) # (1/s) strain rate


    fig, (ax,ax1) = plt.subplots(1,2,figsize = (15,5))
    ax.plot(Velocities,y,color = 'red')
    ax.set_ylabel('Depth (m)')
    ax.set_xlabel('Velocity (m/s)')
    ax.set_ylim(0,h+step)
    ax.invert_yaxis()
    ax.invert_xaxis()
    ax.set_xlim(350000,0)
    ax.set_title('Inclined Plane Flow Velocity vs Depth')
```

run | run all | restart | restart & run all

Here is an alternative method if IntSlider() can be displayed but interact doesn't work:

```
import ipywidgets as widgets
w = widgets.IntSlider()
display(w)
```

run | run all | restart | restart & run all

Error displaying widget

```
print(w.value)
```

run | run all | restart | restart & run all

```
        hello world
```


https://geo.libretexts.org/@go/page/11311

Index

A

Andersonian theory of faulting
- 1.3: Elastic Deformation

B

Byerlee's law
- 1.3: Elastic Deformation
- 1.4: Failure of Rocks

C

compensated topography
- 3.4: Isostasy

Couette Flow
- 1.5: Viscous Deformation

D

Darcy's Law
- 2.0: Introduction to Diffusion and Darcy's Law
- 2.5: Darcy's Law - Flow in a Porous Medium

Darwin Radau equation
- 3.2: Layered Structure of a Planet

diffusion
- 2.0: Introduction to Diffusion and Darcy's Law

diffusion equation
- 2.1: The Diffusion Equation
- 2.2: Geological Applications of the Diffusion Equation
- 2.6: Summary

Diffusivity
- 2.2: Geological Applications of the Diffusion Equation

E

elastic deformation
- 1.3: Elastic Deformation

ellipticity coefficient
- 3.2: Layered Structure of a Planet

Euler Pole
- 4.7: Plate Motions on a Sphere

G

geoid
- 3.7: Gravitational Potential, Mass Anomalies and the Geoid

Geomagnetic Timescales
- 4.2: Magnetic Anomalies on the Seafloor

geosynchonous orbits
- 3.1: Orbital Mechanics

H

headwave
- 5.2: Seismic Refraction (Single Layer)

Hohmann Transfer Orbit
- 3.1: Orbital Mechanics

hydraulic conductivity
- 2.5: Darcy's Law - Flow in a Porous Medium

hydraulic head
- 2.5: Darcy's Law - Flow in a Porous Medium

hydrostatic assumption
- 3.2: Layered Structure of a Planet

hydrostatic flattening
- 3.2: Layered Structure of a Planet

I

intrinsic permeability
- 2.5: Darcy's Law - Flow in a Porous Medium

isostasy
- 3.4: Isostasy
- 3.5: Isostasy Jupyter Notebook

L

lateral strain
- 1.3: Elastic Deformation

longitudinal strain
- 1.3: Elastic Deformation

Love waves
- 5.1: Basics of Wave Propagation

M

magnetic stratigraphy
- 4.2: Magnetic Anomalies on the Seafloor

Moments of Inertia
- 3.2: Layered Structure of a Planet

N

normal stress
- 1.1: Stress

O

Origin Time
- 6.3: Location and Focal Mechanisms

P

permeability
- 2.5: Darcy's Law - Flow in a Porous Medium

phonon
- 2.1: The Diffusion Equation

planetary structure
- 3.2: Layered Structure of a Planet

plate tectonics
- 4: Plate Tectonics
- 4.0: Introduction to Plate Tectonics

Poisson's ratio
- 1.3: Elastic Deformation

porosity
- 2.5: Darcy's Law - Flow in a Porous Medium

principal stresses
- 1.1: Stress

R

Rayleigh wave
- 5.1: Basics of Wave Propagation

Relative Plate Velocity
- 4.5: Relative Velocity and Reference Frames

Relative Velocity
- 4.5: Relative Velocity and Reference Frames

rheology
- 1.0: Introduction to Rheology

richter scale
- 6.2: Earthquake Magnitude

S

seismic reflection
- 5.4: Seismic Reflection (Single Layer)

seismic refraction
- 5.2: Seismic Refraction (Single Layer)

seismic survey
- 5.2: Seismic Refraction (Single Layer)

Seismogram
- 6.3: Location and Focal Mechanisms

seismology
- 5: Seismology

shear angle
- 1.2: Strain and Strain Rate

shear stress
- 1.4: Failure of Rocks

strain
- 1.2: Strain and Strain Rate

stress
- 1.1: Stress

subduction
- 4.4: Plates Moving on a Plane

surface waves
- 5.1: Basics of Wave Propagation

T

Thermal Conduction
- 2.1: The Diffusion Equation

traction
- 1.1: Stress

U

uncompensated topography
- 3.4: Isostasy

V

viscosity
- 1.5: Viscous Deformation

Viscous Deformation
- 1.3: Elastic Deformation
- 1.5: Viscous Deformation

W

wave propagation
- 5.1: Basics of Wave Propagation

Y

Young's modulus
- 5.1: Basics of Wave Propagation

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: GEL 56: Introduction to Geophysics

Webpages: 74

All licenses found:

- CC BY-SA 4.0: 79.7% (59 pages)
- Undeclared: 20.3% (15 pages)

By Page

- GEL 56: Introduction to Geophysics - *CC BY-SA 4.0*
  - Front Matter - *CC BY-SA 4.0*
    - TitlePage - *CC BY-SA 4.0*
    - InfoPage - *CC BY-SA 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - Geophysics is everywhere in geology... - *CC BY-SA 4.0*
    - Front Matter - *CC BY-SA 4.0*
      - TitlePage - *CC BY-SA 4.0*
      - InfoPage - *CC BY-SA 4.0*
      - Table of Contents - *Undeclared*
      - Supporting Topics - *Undeclared*
      - Units, Conversions, and Physical Constants - *Undeclared*
    - 1: Rheology of Rocks - *CC BY-SA 4.0*
      - 1.0: Introduction to Rheology - *CC BY-SA 4.0*
      - 1.1: Stress - *CC BY-SA 4.0*
      - 1.2: Strain and Strain Rate - *CC BY-SA 4.0*
      - 1.3: Elastic Deformation - *CC BY-SA 4.0*
      - 1.4: Failure of Rocks - *CC BY-SA 4.0*
      - 1.5: Viscous Deformation - *CC BY-SA 4.0*
      - 1.6 Inclined Plane Jupyter Notebook - *CC BY-SA 4.0*
      - 1.7: Summary - *CC BY-SA 4.0*
    - 2: Diffusion and Darcy's Law - *CC BY-SA 4.0*
      - 2.0: Introduction to Diffusion and Darcy's Law - *CC BY-SA 4.0*
      - 2.1: The Diffusion Equation - *CC BY-SA 4.0*
      - 2.2: Geological Applications of the Diffusion Equation - *CC BY-SA 4.0*
      - 2.3: Lithospheric Thickness Jupyter Notebook - *CC BY-SA 4.0*
      - 2.4: Erosion Profile Jupyter Notebook - *CC BY-SA 4.0*
      - 2.5: Darcy's Law - Flow in a Porous Medium - *CC BY-SA 4.0*
      - 2.6: Summary - *CC BY-SA 4.0*
    - 3: Planetary Geophysics - *CC BY-SA 4.0*
      - 3.0: Introduction - *CC BY-SA 4.0*
      - 3.1: Orbital Mechanics - *CC BY-SA 4.0*
      - 3.2: Layered Structure of a Planet - *CC BY-SA 4.0*
      - 3.3: Two Layer Planet Structure Jupyter Notebook - *Undeclared*
      - 3.4: Isostasy - *CC BY-SA 4.0*
      - 3.5: Isostasy Jupyter Notebook - *Undeclared*
      - 3.5: Observing the Gravity Field - *CC BY-SA 4.0*
      - 3.7: Gravitational Potential, Mass Anomalies and the Geoid - *CC BY-SA 4.0*
      - 3.8: Summary - *CC BY-SA 4.0*
    - 4: Plate Tectonics - *CC BY-SA 4.0*
      - 4.0: Introduction to Plate Tectonics - *CC BY-SA 4.0*
      - 4.1: The Forces Driving Plate Motions - *CC BY-SA 4.0*
      - 4.2: Magnetic Anomalies on the Seafloor - *CC BY-SA 4.0*
      - 4.3 Magnetic Anomaly Jupyter Notebook - *Undeclared*
      - 4.4: Plates Moving on a Plane - *CC BY-SA 4.0*
      - 4.5: Relative Velocity and Reference Frames - *CC BY-SA 4.0*
      - 4.6: Relative Plate Velocity Jupyter Notebook - *Undeclared*
      - 4.7: Plate Motions on a Sphere - *CC BY-SA 4.0*
      - 4.8: Summary - *CC BY-SA 4.0*
    - 5: Seismology - *CC BY-SA 4.0*
      - 5.0: Introduction to Seismology - *CC BY-SA 4.0*
      - 5.1: Basics of Wave Propagation - *CC BY-SA 4.0*
      - 5.2: Seismic Refraction (Single Layer) - *CC BY-SA 4.0*
      - 5.3 Seismic Refraction Jupyter Notebook - *Undeclared*

- 5.4: Seismic Reflection (Single Layer) - *CC BY-SA 4.0*
- 5.5 Seismic Reflection Jupyter Notebook - *Undeclared*
- 5.6: Summary - *CC BY-SA 4.0*

- 6: Earthquakes - *CC BY-SA 4.0*
  - 6.0: Introduction to Earthquakes - *CC BY-SA 4.0*
  - 6.1: Earthquake Geometry and Process - *CC BY-SA 4.0*
  - 6.2: Earthquake Magnitude - *CC BY-SA 4.0*
  - 6.3: Location and Focal Mechanisms - *CC BY-SA 4.0*
  - 6.4: Earthquake Distance and Magnitude Jupyter Notebook - *Undeclared*
  - 6.5: Earthquake Scaling Laws - *CC BY-SA 4.0*
  - 6.6: Summary - *CC BY-SA 4.0*
- Back Matter - *CC BY-SA 4.0*
  - Index - *CC BY-SA 4.0*
  - Arctan vs Arctan2 - *Undeclared*
  - Ipywidgets Interact Example - *CC BY-SA 4.0*

- Back Matter - *CC BY-SA 4.0*
  - Index - *CC BY-SA 4.0*
  - Glossary - *Undeclared*
  - Detailed Licensing - *Undeclared*