INTRODUCTION TO LINEAR TIME-INVARIANT DYNAMIC SYSTEMS FOR STUDENTS OF ENGINEERING



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Book: Introduction to Linear, Time-Invariant, Dynamic Systems for Students of Engineering (Hallauer)

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About this Book

The book's general organization is:

- Chapters 1-10 deal primarily with the ODEs and behaviors of first-order and second-order dynamic systems;
- Chapters 11 and 12 discuss the ODEs and behaviors of mechanical systems having two degrees of freedom, *i.e.*, fourth-order systems;
- Chapters 13 and 14 introduce classical feedback control;
- Chapter 15 presents the basic features of proportional, integral, and derivative types of classical control;
- Chapters 16 and 17 discuss methods for analyzing the stability of classical control systems.

The general minimum prerequisite for understanding this book is the intellectual maturity of a junior-level (third-year) college student in an accredited four-year engineering curriculum. A mathematical second-order system is represented in this book primarily by a single second-order ODE, not in the *state-space* form by a pair of coupled first-order ODEs. Similarly, a two-degrees-of-freedom (fourth-order) system is represented by two coupled second-order ODEs, not in the state-space form by four coupled first-order ODEs. The book does not use *bond graph* modeling, the general and powerful, but complicated, modern tool for analysis of complex, multidisciplinary dynamic systems. The homework problems at the ends of chapters are very important to the learning objectives, so the author attempted to compose problems of practical interest and to make the problem statements as clear, correct, and unambiguous as possible. A major focus of the book is computer calculation of system characteristics and responses and graphical display of results, with use of basic (not advanced) MATLAB commands and programs. The book includes many examples and homework problems relevant to aerospace engineering, among which are rolling dynamics of flight vehicles, spacecraft actuators, aerospace motion sensors, and aeroelasticity. There are also several examples and homework problems illustrating and validating theory by using measured data to identify first- and second-order system dynamic characteristics based on mathematical models (*e.g.*, time constants and natural frequencies), and system basic properties (*e.g.*, mass, stiffness, and damping). Applications of real and simulated experimental data appear in many homework problems. The book contains somewhat more material than can be covered during a single standard college semester, so an instructor who wishes to use this as a one-semester course textbook should not attempt to cover the entire book, but instead should cover only those parts that are most relevant to the course objectives.

The book is intended to enable students to:

- Solve first-, second-, and higher-order, linear, time-invariant (LTI) ordinary differential equations (ODEs) with initial conditions and excitation, using both time-domain and Laplace-transform methods;
- Solve for the frequency response of an LTI system to periodic sinusoidal excitation and plot this response in standard form;
- Explain the role of the *time constant* in the response of a first-order LTI system, and the roles of *natural frequency*, *damping ratio*, and *resonance* in the response of a second-order LTI system;
- Derive and analyze mathematical models (ODEs) of low-order mechanical systems, both translational and rotational, that are composed of inertial elements, spring elements, and damping devices;
- Derive and analyze mathematical models (ODEs) of low-order electrical circuits composed of resistors, capacitors, inductors, and operational amplifiers;
- Derive (from ODEs) and manipulate Laplace transfer functions and block diagrams representing output-to-input relationships of discrete elements and of systems;
- Define and evaluate *stability* for an LTI system;
- Explain *proportional*, *integral*, and *derivative* types of feedback control for single-input, single-output (SISO), LTI systems;
- Sketch the locus of characteristic values, as a control parameter varies, for a feedback-controlled SISO, LTI system;
- Use MATLAB as a tool to study the time and frequency responses of LTI systems.

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1: FIRST AND SECOND ORDER SYSTEMS; ANALYSIS; AND MATLAB GRAPHING

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1.1: Introduction

The subject of this book is the *dynamic behavior of physical systems*, with some emphasis on simple mechanical and electrical systems representative of or analogous to those often encountered in aerospace and mechanical engineering. A *system*, as defined in this book, is a combination of two or more simple physical *elements* or *components*, these being connected together in such a way that they all influence the dynamic behavior of the entire system. An *element* or *component*, as defined in this book, is usually a discrete object, such as a mechanical spring or an electrical resistor. This object usually produces a discrete effect, such as a motion-induced force or a voltage drop. *Dynamic behavior* is the variation in time of some physical response quantity of the system, for example, the position of a mass, or the voltage at some location in an electrical circuit.

The general subject of this book is relevant to courses that are offered in most engineering colleges for students who major in aerospace engineering, mechanical engineering, engineering mechanics, ocean engineering or naval architecture, electrical engineering, and chemical engineering. Many of the specific topics addressed within chapters and in homework problems following chapters are relevant especially to the study and practice of aerospace engineering.

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1.2: LTI Systems and ODEs

We consider physical systems that can be modeled with reasonable engineering fidelity as *linear, time-invariant* (LTI) systems. Such a system is represented mathematically by an *ordinary differential equation* (ODE), or by a set of coupled ODEs, for which the single independent variable is time, denoted as t. These ODEs are linear, and they have constant coefficients, so we describe them as linear, time-invariant (LTI), the same as the systems they represent[1]. For example, suppose we denote a dependent variable as $x(t)$, here a general symbol representing some physical dynamic *response* quantity for which we want to solve. Then an LTI ODE that models an LTI physical system might have the form

$$\frac{dx}{dt} - a\,x = b\,u(t) \tag{1.2.1}$$

in which a and b are constant multiplying coefficients, and known function $u(t)$ is the *excitation* and is independent of the response. In the study of systems, an independent excitation $u(t)$ is often called an *input*, and a dependent response $x(t)$ is often called an *output*.

Hereafter, we will usually employ the common shorthand dot notation for denoting derivatives with respect to time: $\frac{dx}{dt} = \dot{x}$, $\frac{d^2x}{dt^2} = \ddot{x}$ *etc.*, so that Equation 1.2.1 can be written more simply as

$$\dot{x} - a\,x = b\,u(t).$$

The *linearity* of Equation 1.2.1 is manifested by the linear appearance of $x(t)$ and all of its derivatives in the ODE. The following are some similar ODEs that are not linear (they are *nonlinear*) for obvious reasons: $\dot{x} - a\ x^2 = b\ u(t)$; $sin(\dot{x}) - a\ x = b\ u(t)$; $\sqrt{(}\dot{x}) - a\ tan(x) = b\ u(t)$. Linear ODEs are almost always easier to solve (at least in closed form, *i.e.*, as equations involving standard functions) than nonlinear ODEs. Moreover, the important *principle of superposition* applies to linear ODEs, but not to nonlinear ODEs. An example of the application of this principle is: let the response to input $u_1(t)$ be $x_1(t)$, and let the response to another input $u_2(t)$ be $x_1(t)$; if a third input is the sum of multiplied terms $u_3(t) = c_1\ u_1(t) + c_2\ u_2(t)$, in which c_1 and c_2 are constants, then the response to $u_3(t)$ is $x_3(t) = c_1\ x_1(t) + c_2\ x_2(t)$. This result is easy to derive just by multiplying two ODEs such as Equation 1.2.1 by the constants, then adding the multiplied ODEs. The principle of superposition allows us to solve accurately for the responses of linear systems to any physically realistic inputs. (See Section 8.10 for a derivation of system response to an arbitrary physically realistic input by direct application of superposition.)

The *time invariance* of Equation 1.2.1 is manifested by the constant coefficients of $x(t)$ and all of its derivatives in the ODE. ODEs with time-invariant coefficients model the behavior of systems assumed to have physical properties that either remain constant in time or vary so slowly and/or slightly that the variation is negligible for engineering purposes. But many practically important systems have time-varying physical properties. For example, a vehicle such as a space shuttle between liftoff and achievement of orbital position has rapidly varying (decreasing) mass as propellant is burned and external fuel tanks and boosters are released. The following is a linear equation somewhat similar to Equation 1.2.1, but with an obviously time-varying coefficient: $\dot{x} - 3\ x\ (1 - e^(-2\ t))\ x = b\ u(t)$. The study of systems with time-varying physical properties is generally more complicated, not fundamental, so only time-invariant systems and ODEs are considered in this book.

The form of Equation 1.2.1, $\dot{x} - a\ x = b\ u(t)$, is widely regarded as **the** *standard form* for a first order LTI ODE, and we will use it as such in this book. Beginning in the next section, we will study idealized physical systems whose dynamic behaviors are described by equations that are directly analogous to Equation 1.2.1. We will express the mathematical constants a and b in terms of specific physical constants. Also, the roles of input $u(t)$ and output $x(t)$ in Equation 1.2.1 will be assumed by some specific physical quantities, such as force, velocity, voltage, *etc.*, and we will denote them with relevant symbols [often different than $u(t)$ and $x(t)$] when appropriate.

Although only first order ODEs are discussed in this section, we certainly will encounter and study systems and ODEs of second and higher orders.

[1]LTI ODEs are also sometimes described as linear, constant-coefficient, or LCC.

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1.3: The Mass-Damper System I - example of 1st order, linear, time-invariant (LTI) system and ordinary differential equation (ODE)

Consider a rigid body of mass m that is constrained to sliding translation $x(t)$ in only one direction, Figure 1.3.1. The mass is subjected to an externally applied, arbitrary force $f_x(t)$, and it can slide on a thin, viscous liquid layer such as water or oil. The viscous force acting on the mass due to sliding on the liquid layer is opposite to the direction of velocity, $v(t) \equiv \dot{x}(t)$, and we assume that the magnitude of viscous force is proportional to velocity with constant of proportionality c, called the **viscous damping constant**. Mass m and viscous damping constant c are positive physical quantities. **All** of the forces acting on the mass are as shown on the *free-body diagram* (FBD) of Figure 1.3.1.

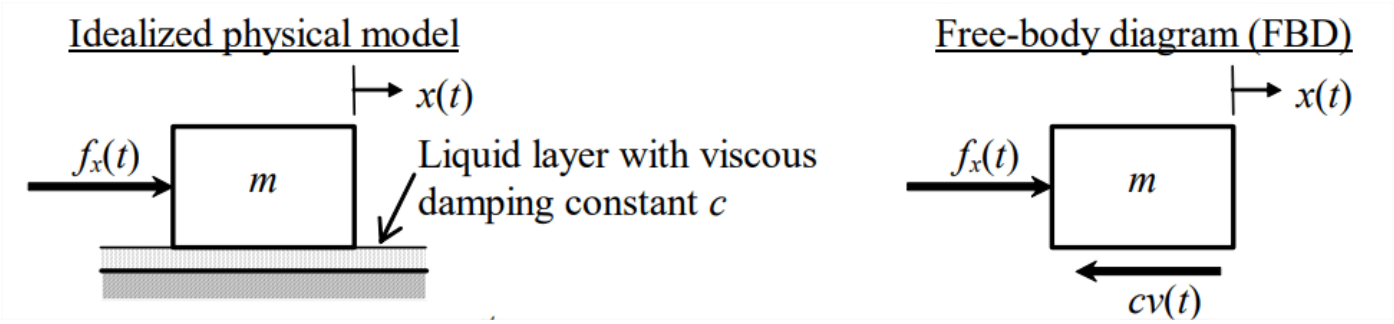


Figure 1.3.1: 1st order mass-damper mechanical system.

Next, we use (from your engineering dynamics course) the free-body diagram (FBD) of Figure 1.3.1 and Newton's 2nd law of motion (after English physicist and mathematician Isaac Newton, 1642-1727) for translation in a single direction, to write the equation of motion for the mass:

$$\sum (\text{ Forces })_x = \text{ mass } \times (\text{ acceleration })_x$$

where

$$(acceleration)_x = \frac{dv}{dt} = \dot{v}$$

$$f_x(t) - cv = m\dot{v}.$$

As is customary in writing ODEs, we collect all terms involving the dependent variable and its derivatives on the left-hand side, and put all independent input functions on the right-hand side:

$$m\dot{v} + cv = f_x(t) \tag{1.3.1}$$

ODE Equation 1.3.1 is clearly linear in the single dependent variable, velocity $v(t)$, and time-invariant, assuming that m and c are constants. The highest derivative of $v(t)$ in the ODE is the first derivative, so this is called a 1st order ODE, and the mass-damper system is called a 1st order system. If $f_x(t)$ is defined explicitly, and if we also know some *initial condition* (IC) of the velocity, $v_0 \equiv v(t_0)$ at time $t = t_0$, then we can, at least in principle, solve ODE Equation 1.3.1 for velocity $v(t)$ at all times $t > t_0$. (In this book, we will usually define the initial time as t_0= 0 second.)

Equation 1.3.1 expressed in the form of the standard 1st order LTI ODE Equation 1.2.1 becomes $\dot{v} - av = bf_x(t)$, where $a = -c/m$ and $b = 1/m$. Since m and c are positive physical constants, a is clearly negative. This negative polarity is characteristic of most physical systems that we will study; we shall see that it has an important influence on the general nature of the transient response of systems.

Note that after solving for velocity $v(t)$, we can solve by direct integration another ODE for position $x(t)$, provided that we know the initial position $x_0 \equiv x(t_0)$ at time $t = t_0$. One systematic method for finding $x(t)$ is based upon the derivative definition:

$$\text{ODE:} \quad \dot{x}(t) \equiv \frac{dx(t)}{dt} = v(t)$$

The following shows careful **definite** integration of both sides of the ODE, using τ as the "dummy" variable of integration to distinguish it from the upper limit, time t:

$$\begin{aligned} \int_{t=1}^{t=t} \frac{dx(\tau)}{d\tau} d\tau = \int_{t=t_0}^{\tau=t} v(\tau) d\tau \quad &\Rightarrow \quad x(t) - x\left(t_0\right) = \int_{\tau=t_0}^{\tau=t} v(\tau) d\tau \\ \Rightarrow \quad x(t) &= x_0 + \int_{\tau=t_0}^{t=t} v(\tau) d\tau \end{aligned} \tag{1.3.2}$$

Another popular method of solution is to find the antiderivative (indefinite integral) of the ODE and add a constant of integration C, which then must be determined in terms of the initial condition:

$$x(t) = \int v(t)dt + C \Rightarrow x\left(t_0\right) = \left[\int v(t)dt\right]_{t=t_0} + C \Rightarrow C = x_0 - \left[\int v(t)dt\right]_{t-t_0} \tag{1.3.3}$$

$$\Rightarrow x(t) = x_0 + \int v(t)dt - \left[\int v(t)dt\right]_{t-t_0} \tag{1.3.4}$$

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1.4: A Short Discussion of Engineering Models

The mass-damper of Figure 1.3.1 can be used to represent **approximately** (*i.e.*, to *model*) some actual physical systems. One such system is a surface ship moving over the water under its own propulsion or being pushed/pulled by a tugboat. Another is an automobile hydroplaning on a wet road. You can probably think of other similar real systems. However, it is important for us, as engineers, to recognize that the mass-damper system is **not** the actual system, but only an **approximate** *idealized physical model* of the actual system. We are able to derive from this idealized physical model the solvable *mathematical model*, which consists of ODE Equation 1.3.3 and known values for $f_x(t)$ and v_0. The actual physical system, on the other hand, might be so complicated that it cannot be characterized mathematically with absolute precision. For example, the ideal viscous damping model used in the derivation of ODE Equation 1.3.3 is almost certainly not an exact representation of the liquid drag forces acting on either a surface ship or a hydroplaning car.

The same general observation applies for almost any idealized physical model and associated mathematical model developed for engineering purposes: the physical model is, at best, a reasonably accurate approximation of the actual physical system. The fidelity of a model usually depends on a number of factors, including system complexity, uncertainties, the costs of modeling and mathematical/computational solutions, time constraints, modeling skills of the engineer, *etc*.

But a reasonably accurate approximate model often suffices for engineering purposes. Engineering systems are usually designed conservatively, with redundancies and factors of safety to compensate for severe overloads, unexpected material flaws, operator error, and the many other unpredictable influences that can arise in the functioning of a system. As engineers, we almost never require 100 % accuracy; we are usually satisfied if our mathematical/computational predictions of system behavior are qualitatively correct and are quantitatively within around $\pm$10 % (in a general sense) of the actual behavior.

The main point of this discussion is to emphasize that any idealized physical model used for engineering analysis and design is only an approximation of an actual physical system. Moreover, the primary subjects of this book are the fundamental dynamic characteristics of idealized physical models, because a great deal of practical experience has shown that these are also the characteristics of many real engineering systems. Therefore, this book does not consider in depth the development of idealized physical models to represent actual systems; rather, we shall focus on deriving mathematical models (mostly ODEs) that describe idealized physical models, and on solving the mathematical equations and exploring the characteristics of the solutions.

The process of developing idealized physical models to represent real systems involves both science (theory and experimental data) and “art” (experience and intuition); you probably will encounter this process in laboratory and design courses, and later in your professional practice of engineering.

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1.5: The Mass-Damper System II - Solving the 1st order LTI ODE for time response, given a pulse excitation and an IC

An input of limited duration, typically called a *pulse*, is a very common type of excitation imposed onto systems. For example, when a hammer strikes a nail, the force imposed on the nail by the hammer is a pulse. A real pulse such as hammer impact force is often modeled as a *half-sine pulse*. Let the force acting on the mass in Figure 1.3.1 be the half-sine pulse described by the following figure and Equation ???:

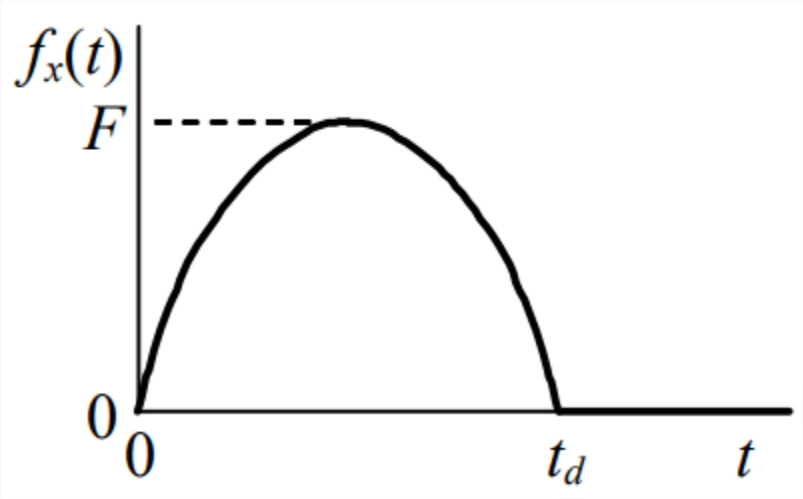


Figure 1.5.1: Half-sine pulse.

\[f_{x}(t)=\left\{\begin{array}{l}
F \sin \left(\pi \frac{t}{t_{d}}\right), 0 \leq t \leq t_{d} \\
0, t_{d} \end{array}\right.\label{eqn:1.5} \]

In Equation ??? for $f_x(t)$, t_d denotes the pulse duration. The notation will be more manageable in this problem if we express the time-varying sinusoid in the form $\sin \omega t$, where ω denotes the *circular frequency* of oscillation, in radians per second. In this case, clearly the circular frequency is expressed in terms of the pulse duration as $\omega = \pi/t_d$. Let us specify that the initial velocity of the mass at time t = 0 is some known value v_0. The mathematical statement of the problem for finding the velocity time-history is:

$$\begin{array}{rl} \underline{\text{ODE :}} & m\dot{v} + cv = f_x(t) \\ \underline{\text{IC :}} & v(0) = v_0 \\ \underline{\text{Find :}} & v(t) \quad \text{for all} \quad t > 0 \end{array}$$

To solve this problem in closed form, we will use a method with which you should be familiar from your previous study of ODEs. First, we find the *homogeneous* (also called *complementary*) solution $v_h(t)$, which is the solution of the homogeneous ODE, the version of Equation 1.3.3 with zero right-hand side:

$$m\dot{v}_h + cv_h = 0 \tag{1.5.1}$$

A homogeneous LTI ODE **always** has solutions in time-linear powers of e = 2.71828... (the base of natural logarithms), with some initially unknown constant coefficients:

$$v_h(t) = Ce^{\lambda t},$$

in which constants C and λ are unknown at this stage. To find λ, we substitute the solution into the homogeneous ODE 1.5.1:

$$mC\lambda e^{\lambda t} + cCe^{\lambda t} = (m\lambda + c)Ce^{\lambda t} = 0$$

If $Ce^{\lambda t}$ = 0, we get the useless trivial solution $v_h(t)$ = 0, so a useful solution requires that t $m\lambda + c$ = 0, which is known as the *characteristic equation* of the ODE. Solution of this equation gives the so-called *characteristic value*, $\lambda = -c/m$, leading to:

$$v_h(t) = Ce^{-(c/m)t} \tag{1.5.2}$$

Note that we still have not solved for constant C. We can find C only **after** we have determined a *particular* solution, also known as the *non-homogeneous* solution because it is a solution that satisfies the complete ODE Equation 1.3.3 for the given right-hand side $f_x(t)$. For this problem, we will need two particular solutions, because $f_x(t)$ is defined differently over two different intervals of time, Equation ???. First, we find a particular solution $v_p(t)$ valid over the pulse duration, $0 \le t \le t_d$, for which the ODE is:

$$m\dot{v}_p + cv_p = F\sin\omega t, \text{ where } \omega = \pi/t_d \tag{1.5.3}$$

To find a particular $v_p(t)$ that satisfies ODE Equation 1.5.3, we apply the method of undetermined coefficients, which entails making an educated guess of the functional character of the solution, using multiplicative coefficients that will be determined by substituting the candidate solution back into ODE Equation 1.5.3. The right-hand-side sine function of ODE Equation 1.5.3 has a finite set of derivatives: the derivative of a sine is a cosine, the derivative of a cosine is a sine, *etc*. Therefore, we assume a form of solution consisting of a linear sum of the function and all of its derivatives:

$$v_p(t) = P_1\sin\omega t + P_2\cos\omega t,$$

with coefficients P_1 and P_2 undetermined at this stage. Substitute this candidate solution back into ODE Equation 1.5.3:

$$m\omega\left(P_1\cos\omega t - P_2\sin\omega t\right) + c\left(P_1\sin\omega t + P_2\cos\omega t\right) = F\sin\omega t$$

Collect terms that multiply $\sin\omega t$ and $\cos\omega t$ on both sides of the equation:

$$\left(-m\omega P_2 + cP_1\right)\sin\omega t + \left(m\omega P_1 + cP_2\right)\cos\omega t = (F)\sin\omega t + (0)\cos\omega t$$

Functions $\sin\omega t$ and $\cos\omega t$ are *linearly independent* of each other, which requires that the left-hand-side and right-hand-side terms multiplying $\sin\omega t$ must equal each other, and the same for the terms multiplying cosωt , leading to two algebraic equations for the coefficients P_1 and P_2:

$$cP_1 - m\omega P_2 = F \quad \text{and} \quad m\omega P_1 + cP_2 = 0$$

The second equation gives $P_2 = -(m\omega/c)P_1$, and substituting this into the first equation to eliminate P_2 leads to:

$$P_1 = \frac{cF}{(m\omega)^2 + c^2} \quad \text{and} \quad P_2 = \frac{-m\omega}{c}P_1 \tag{1.5.4}$$

Rather than write out messy algebraic formulas for all the coefficients in this problem, it is convenient to express all others in terms of P_1 and P_2, as is in Equation 1.5.4.

To obtain the **complete** solution for the pulse duration, $0 \le t \le t_d$, we now combine the homogeneous and particular solutions:

$$v(t) = v_h(t) + v_p(t) = Ce^{-(c/m)t} + P_1\sin\omega t + P_2\cos\omega t, \text{ for } 0 \le t \le t_d \tag{1.5.5}$$

Coefficient C in Equation 1.5.5 is still not known; but now, finally, we can apply the initial condition (IC) to determine C:

$$v(0) = v_0 = C(1) + P_1(0) + P_2(1) \Rightarrow C = v_0 - P_2 = v_0 + (m\omega/c)P_1 \tag{1.5.6}$$

Equations 1.5.4 through 1.5.6 describe the velocity response during the pulse duration, $0 \le t \le t_d$, so we still need to find the post-pulse response, for $t_d < t$. To do so, we should recognize two facts: (1) $f_x(t) = 0$ for $t_d < t$; and (2) velocity $v(t)$ cannot suddenly change at $t = t_d$ (because acceleration cannot be infinite), rather, velocity must equal Equation 1.5.5 evaluated at $t = t_d$. Fact 1 means that the ODE for $t_d < t$ is homogeneous; hence, the particular solution is zero, and we have only a homogeneous solution, but now with a different coefficient, D, than before:

\[v(t)=D e^{-(c / m) t}, \text { for } t_{d}

To find D, we use Fact 2, which essentially is the IC for $t_d < t$, and Equation 1.5.5:

$$De^{-(c/m)t_d} = v\left(t_d\right) = Ce^{-(c/m)t_d} + P_1\sin\omega t_d + P_2\cos\omega t_d \tag{1.5.7}$$

$$\Rightarrow \quad D = v\left(t_d\right)e^{(c/m)t_d}$$

\[\Rightarrow \quad v(t)=v\left(t_{d}\right) e^{-\frac{c}{m}\left(t-t_{d}\right)}, \text { for } t_{d}

Equation ???, with Equation 1.5.7 for $v(t_d)$, combined with Equation 1.5.4 and Equation 1.5.6 for coefficients P_1, P_2 and C, represents the response for $t_d < t$. Because mass m and viscous damping constant c are positive physical quantities, Equation ??? is a pure exponential decay, which approaches zero as $t \to \inf$.

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1.6: The Mass-Damper System III - Numerical and Graphical Evaluation of Time Response using MATLAB

For the mass-damper response solution developed in the previous section, consider the following numerical case, with all quantities expressed in SI units: m = 5 kg, c = 2 N-s/m, F = 18 N, t_d = 7.5 s, v_0 = 3.3 m/s. A MATLAB script M-file, named MATLABdemo11.m, to calculate and graph the response from 0 to 25 seconds is given below. The MATLAB commands are supplemented with explanatory comments, so you should be able to follow and understand the M-file without much difficulty. Writing comments in this manner is good practice for your own programs; comments added while you are writing a computer program are especially helpful if you need to revise or refer back to the program long after you have forgotten the details.

```
%MATLABdemo11.m

%Mass-damper system response to IC + half-sine pulse forcing

m=5;c=2; %system mass & viscous damping coefficient, SI units

F=18;td=7.5; %half-sine pulse, amplitude (N), pulse duration (sec)

vo=3.3; %initial velocity (m/sec)

w=pi/td; %circular frequency of half-sine pulse (rad/sec)

t1=0:0.05:td; %time instants for forced response

f1=F*sin(w*t1); %force pulse

P1=c*F/((w*m)^2+c^2); P2=P1*(-w*m/c);C=vo-P2; %constants

v1=C*exp(-c/m*t1)+P1*sin(w*t1)+P2*cos(w*t1);%time series of forced velocity

nt1end=length(t1);v2o=v1(nt1end);%initial velocity for post-pulse response

t2=td:0.1:25; %time instants for post-pulse unforced response

v2=v2o*exp(-c/m*(t2-td));%time series of post-pulse unforced velocity

f2=zeros(1,length(t2)); %null force after pulse

plot(t1,f1/10,'k',t1,v1,'k',t2,f2,'k',t2,v2,'k'),grid,xlabel('Time t (sec)')

ylabel('Force f_x(t) (daN), and Velocity v(t) (m/sec)')

title('Response of mass-damper system to IC + half-sine pulse forcing')
```

To execute in MATLAB an M-file that is stored on a folder (directory) of your computer's hard disk, you must have added that folder to the so-called MATLABpath. In Versions 6 and higher of MATLAB, you can add the folder to the MATLABpath by specifying the folder as the "Current Directory" in the formatting toolbar above the MATLAB command window. The command line below executes the script M-file.

MATLAB command:

```
>> MATLABdemo11
```

The resulting MATLAB (Versions 6 and higher) graph is Figure 1.6.1. The graph was edited in the MATLAB figure window, mainly to reduce the size and to add curve labels and arrows.

Note that the unit of force "daN" on Figure 1.6.1 is a deka-newton, which means 10 newtons. All of the mechanical units used in

this book are described in Chapter 3.

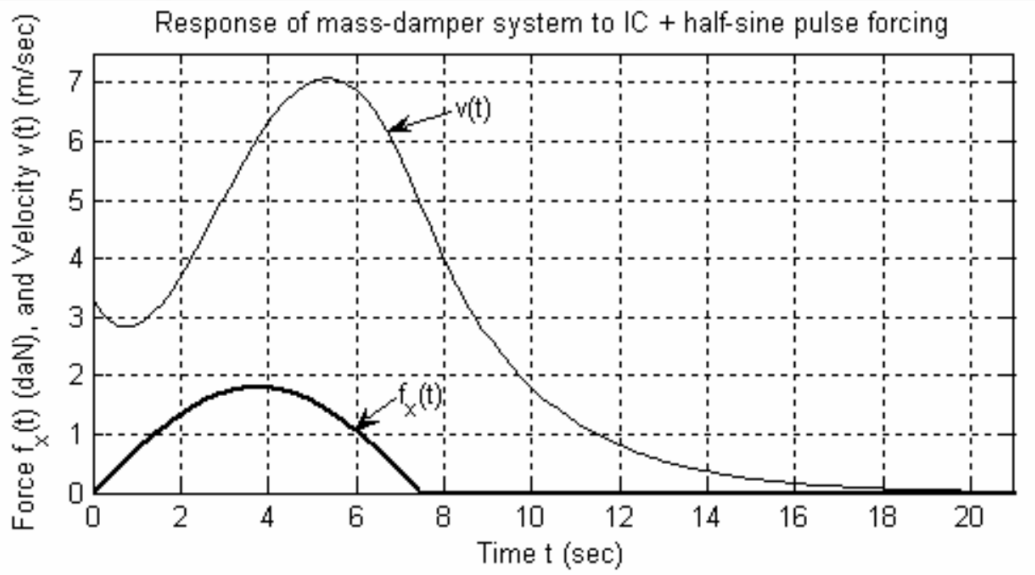


Figure 1.6.1: Excitation and response of a mass-damper system

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1.7: Good Engineering Graphical Practice

- Always label both axes, and always include the **units** of physical quantities.
- Always write an explanatory one-line title. Such a short title cannot explain everything about the graph, but any title you use will almost certainly help the reader to understand the graph.
- It is usually good practice to add grids to a graph. Grids help the reader to perceive values correctly. For example, the grids on Figure 1.6.1 show clearly that the peak velocity response is just above 7 m/s at a little after 5 s.
- The commands in the script file that specify the densities of computed points are, first, `t1=0:0.05:td` while the pulse acts and, second, `t2=td:0.1:25` following the pulse. For example, the first line directs MATLAB to compute the response at 0.05 s intervals; then `plot(…,t1,v1,'k',…)` directs MATLAB to plot small points for those instants, and to **connect each two adjacent points with a straight line**. If you use high point densities, then the overall plotted line will capture all important features of the response (extreme values, for example), and will appear to be a smooth curve. Indeed, the overall line should look like a smooth curve, because it represents a continuous physical response; neither the response itself nor its low-order derivatives should be discontinuous. On the other hand, if you use low point densities, then you might miss important features of the response, and the overall line will be unnaturally piecewise linear, with kinks (slope discontinuities). To visualize an extreme example of low point density, suppose that for the mass-damper system you were to compute and plot the velocity response at 4 s intervals; the velocity graph in Figure 1.6.1 would consist only of straight lines connecting the computed points at 0, 4, 8, 12, 16, and 20 seconds, which would badly represent the actual response. So always specify high point density on graphs of continuous physical response. You might not know initially what point density you should use, especially if you are analyzing an unfamiliar system. But try some plausible point density. If your plotted response curve appears unnaturally kinky, then increase the point density appropriately and run the M-file again. It will cost you nothing more than the little time required to edit and re-run the M-file (or any other graphing computer program).

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1.8: Plausibility Checks of System Response Equations and Calculations

We all make mistakes in the process of analyzing engineering problems. Most common are mistakes in arithmetic, algebra, calculus, theory, and calculation algorithms (usually computer programming). Also, sometimes we simply use incorrect data. It seems that there are countless ways to make mistakes. Therefore, it is important **always** to check your mathematical, numerical, and computational operations and results in every way possible. An important type of check for any problem with physical results is the *plausibility check*, known more colloquially as *reality check* and *sanity test*. Essentially you examine the results to determine if they are physically plausible (believable, credible, reasonable). Do the results make sense physically? A classic example of implausible result that often appears on exam papers in structures courses is the structural deformation on the order of 10^3 or 10^6 inches, when it ought to be on the order of 10^{-3} inches.

To illustrate a plausibility check, let us examine Figure 1.6.1 for the velocity response of the mass-damper system to an initial condition and half-sine pulse excitation. First, the specified initial velocity is v_0 = 3.3 m/s, and the response curve at time t = 0 correctly reflects that initial condition. Next, for about the first half-second of response, the velocity decreases due to viscous drag force cv. But then, as the applied force $f_x(t)$ increases, the velocity dips to a local minimum and subsequently increases. Applied force $f_x(t)$ peaks at $t = \frac{1}{2}t_d$ = 3.75 s, and the graph shows that the slope of the velocity curve, acceleration $\dot{v}$, is maximum at about the same time. The velocity itself peaks at a bit past 5 s. Because velocity is the integral of acceleration (area under the acceleration curve), this lag of the velocity peak behind the force pulse peak is quite plausible. After the velocity peaks, it decreases monotonically toward zero as the applied force decreases to zero at $t = t_d$ = 7.50 s and remains at zero thereafter. So the entire response, as depicted graphically, is physically plausible.

Plausibility of a calculated response does not guarantee that it is correct, although it is reassuring. On the other hand, **implausibility** of a calculated response almost certainly guarantees that it is **wrong**. The primary justification for conducting a plausibility check is to catch incorrect engineering predictions before they cause trouble (which can vary from point loss on homework or exams, to a major disaster, the unpredicted failure of an engineering system).

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1.9: The Mass-Damper-Spring System - A 2nd Order LTI System and ODE

Consider a rigid body of mass m that is constrained to sliding translation $x(t)$ in only one direction, Figure 1.9.1. The mass is subjected to an externally applied, arbitrary force $f_x(t)$, and it slides on a thin, viscous, liquid layer that has linear viscous damping constant c. Additionally, the mass is restrained by a linear spring. The force exerted by the spring on the mass is proportional to translation $x(t)$ relative to the undeformed state of the spring, the constant of proportionality being k. Parameters m, c, and k are positive physical quantities. **All** of the horizontal forces acting on the mass are shown on the FBD of Figure 1.9.1.

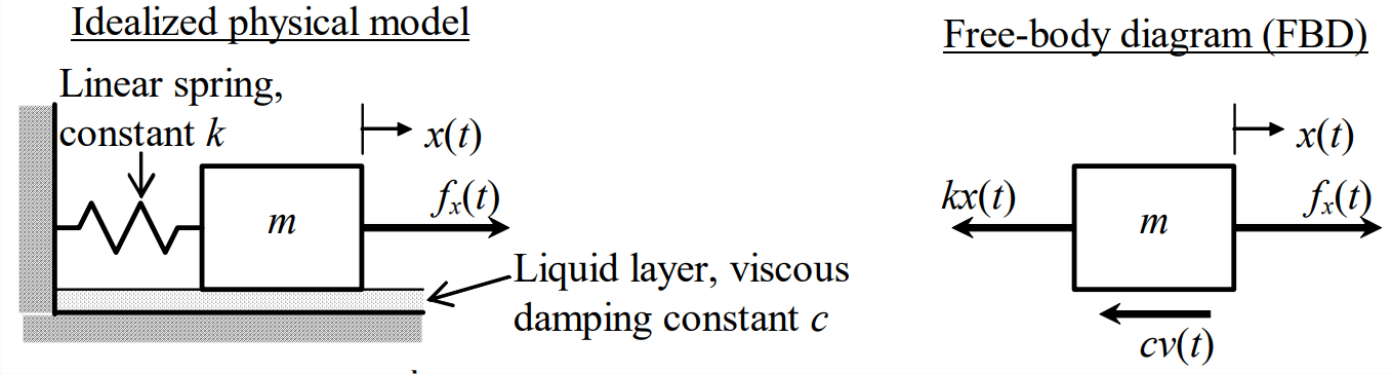


Figure 1.9.1: 2nd order mass-damper-spring mechanical system.

From the FBD of Figure 1.9.1 and Newton's 2nd law for translation in a single direction, we write the equation of motion for the mass:

$$\sum (\text{ Forces })_x = \text{ mass } \times (\text{ acceleration })_x$$

where $(acceleration)_x = \dot{v} = \ddot{x}$;

$$f_x(t) - cv - kx = m\dot{v}.$$

Re-arrange this equation, and add the relationship between $x(t)$ and $v(t)$, $\dot{x} = v$:

$$m\dot{v} + cv + kx = f_x(t) \tag{1.9.1}$$

$$\dot{x} - v = 0 \tag{1.9.2}$$

Equations 1.9.1 and 1.9.2 are a pair of 1st order ODEs in the dependent variables $v(t)$ and $x(t)$. The two ODEs are said to be coupled, because each equation contains both dependent variables and neither equation can be solved independently of the other. Such a pair of coupled 1st order ODEs is called a 2nd order set of ODEs.

Solving 1st order ODE Equation 1.3.3 in the single dependent variable $v(t)$ for all times $t > t_0$ requires knowledge of a single IC, which we previously expressed as $v_0 = v(t_0)$. Similarly, solving the coupled pair of 1st order ODEs, Equations 1.9.1 and 1.9.2, in dependent variables $v(t)$ and $x(t)$ for all times $t > t_0$, requires a known IC for each of the dependent variables:

$$v_0 \equiv v\left(t_0\right) = \dot{x}\left(t_0\right) \text{ and } x_0 = x\left(t_0\right) \tag{1.9.3}$$

In this book, the mathematical problem is expressed in a form different from Equations 1.9.1 and 1.9.2: we eliminate v from Equation 1.9.1 by substituting for it from Equation 1.9.2 with $v = \dot{x}$ and the associated derivative $\dot{v} = \ddot{x}$, which gives[1]

$$m\ddot{x} + c\dot{x} + kx = f_x(t) \tag{1.9.4}$$

ODE Equation 1.9.4 is clearly linear in the single dependent variable, position $x(t)$, and time-invariant, assuming that m, c, and k are constants. The highest derivative of $x(t)$ in the ODE is the second derivative, so this is a 2nd order ODE, and the mass-

damper-spring mechanical system is called a 2nd order system. If $f_x(t)$ is defined explicitly, and if we also know ICs Equation 1.9.3 for both the velocity $\dot{x}(t_0)$ and the position $x(t_0)$, then we can, at least in principle, solve ODE Equation 1.9.4 for position $x(t)$ at all times $t > t_0$. We shall study the response of 2nd order systems in considerable detail, beginning in Chapter 7, for which the following section is a preview.

[1]An alternative derivation of ODE Equation 1.9.4 is presented in Appendix B, Section 19.2. The rate of change of system energy is equated with the power supplied to the system.

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1.10: The Mass-Spring System - Solving a 2nd order LTI ODE for Time Response

Suppose that we have a system of the type depicted on Figure 1.9.1 for which the damping force $c\dot{x}$, in Equation 1.9.6, is negligibly small in comparison with inertial force $m\ddot{x}$ and structural force kx. Figure $1.10.1$ is a photograph of a real system[1] with so little damping that, under some circumstances, we may neglect the damping force. The mass carriage of this system rides back and forth on low-friction linear ball bearings, which are enclosed underneath the carriage and not visible in the photograph. The entire length of this system, from the left (fixed) end of the spring, to the rightmost edge of the mass carriage is $8\frac{1}{2}$ inches (21.6 cm), and each of the three light-colored metal slabs attached to the carriage has mass of $\frac{1}{2}$ kilogram.

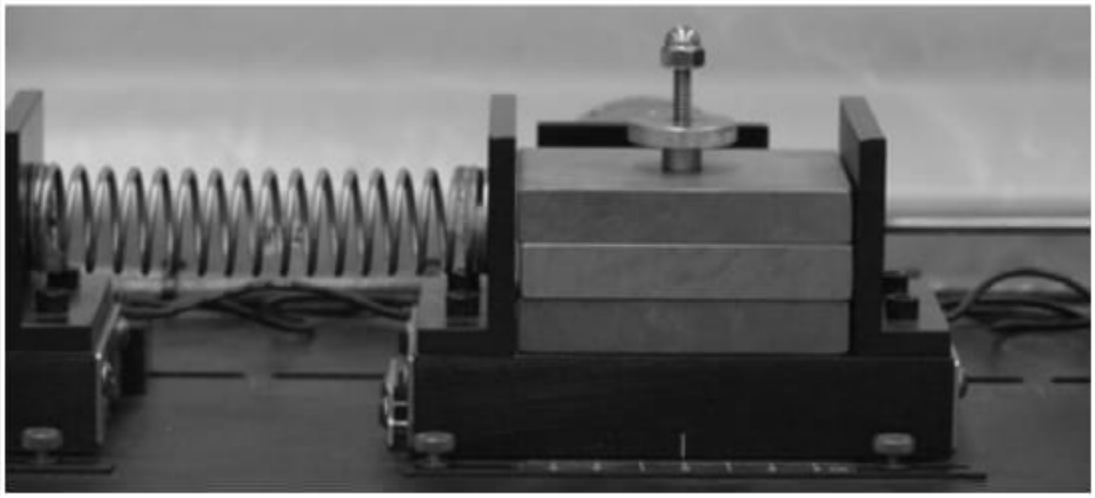


Figure $1.10.1$: Laboratory mass-spring system

If we may neglect the damping force in a system such as that of Figure $1.10.1$, then the term $c\dot{x}$ drops out of Equation 1.9.6, and we are left with the simpler 2nd order ODE,

$$m\ddot{x} + kx = f_x(t) \tag{1.10.1}$$

If we know ICs (Equation 1.9.5) and excitation force $f_x(t)$, then we can solve Equation $1.10.1$ for $x(t)$. For future reference, note that mass quantity m and spring stiffness constant k are intrinsically positive values. Also, observe that $f_x(t)$ may be applied to the system of Figure $1.10.1$ through the link visible at the right-hand side of the mass carriage.

It will be instructive to determine a time response for this 2nd order mass-spring ($m-k$) system, by applying the standard ODE solution procedure described in Section 1.5. We shall find the complete algebraic solution as the sum of homogeneous and particular solutions, $x(t) = x_h(t) + x_p(t)$. Suppose that at time t = 0 the spring is undeformed and the mass is at rest, so that ICs Equation 1.9.5 are

$$\dot{x}(0) = 0 \text{ and } x(0) = 0 \tag{1.10.2}$$

Suppose also that the excitation is a force pulse described by the equation $f_x(t) = F_m(t/t_m)e^{(1-t/t_m)}$; by applying calculus to this function, you can easily prove that it rises from zero at t = 0 to the maximum value F_m at time $t = t_m$, and, thereafter, it gradually drops back to zero (see Figure $1.10.2$). In dynamics, a linear function of time such as t/t_m is often called a “ramp” function; since our excitation consists of a declining exponential function multiplied by a ramp function, we call this excitation a “ramped exponential” force pulse.

The homogeneous ODE associated with Equation $1.10.1$ is $m\ddot{x}_h + kx_h = 0$. We cast this equation into a familiar form by dividing through by mass m and defining the positive quantity $\omega_n^2 = k/m$, giving the ODE

$$\ddot{x}_h + \omega_n^2 x_h = 0 \tag{1.10.3}$$

The positive root $\omega_n = \sqrt{k/m}$ is called the *natural frequency* of this $m-k$ system, and ω_n has physical significance that will be discussed below with the final solution. You might remember from your previous study of ODEs, or you can easily verify by substitution into Equation 1.10.3, that the solution can be expressed as

$$x_h(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t \tag{1.10.4}$$

Constants C_1 and C_2 are unknown at this point in the solution process.

With excitation by the ramped exponential pulse defined above, any particular solution $x_p(t)$ must satisfy the non-homogeneous ODE

$$m\ddot{x}_p + kx_p = F_m\,(t/t_m)\,e^{(1-t/t_m)} \tag{1.10.5}$$

To find a particular $x_p(t)$ that satisfies ODE Equation 1.10.5, we apply the method of undetermined coefficients, just as in Section 1.5. It is easy to show by successive differentiations that all derivatives with respect to time of the function $(t/t_m)e^{(1-t/t_m)}$ produce **only** constant multiples of $(t/t_m)e^{(1-t/t_m)}$ itself, and of $e^{(1-t/t_m)}$. Therefore, we seek a particular solution that consists of a linear sum of these two functions:

$$x_p(t) = P_1\,(t/t_m)\,e^{(1-t/t_m)} + P_2 e^{(1-t/t_m)} \tag{1.10.6}$$

We determine constants P_1 and P_2 by substituting Equation 1.10.6 into Equation 1.10.5, equating the coefficients of $(t/t_m)e^{(1-t/t_m)}$ and $e^{(1-t/t_m)}$ on the two sides of the resulting algebraic equation, and then solving for P_1 and P_2 [homework Problem 1.10(a)]. This process produces the results

$$P_1 = \frac{F_m}{m/t_m^2 + k} \quad \text{and} \quad P_2 = \frac{2m/t_m^2}{m/t_m^2 + k} P_1 \tag{1.10.7}$$

The complete (but not yet final) solution is

$$x(t) = x_h(t) + x_p(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t + P_1\,(t/t_m)\,e^{(1-t/t_m)} + P_2 e^{(1-t/t_m)} \tag{1.10.8}$$

We now can determine the remaining unknown constants by substituting Equation 1.10.8 into ICs Equation 1.10.2, $\dot{x}(0) = 0$ and $x(0) = 0$, and solving algebraically for constants C_1 and C_2 [homework Problem 1.10(b)]. The results are

$$C_1 = \frac{P_2 - P_1}{\omega_n t_m} e^1, \text{ with } \omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad C_2 = -P_2 e^1 \tag{1.10.9}$$

The complete and final solution is Equation 1.10.8, with constants C_1, C_2, P_1, and P_2 defined in terms of the basic parameters m, k, F_m, and t_m by Equations 1.10.7 and 1.10.9. Let us calculate and plot a time response for the system of Figure 1.10.1. The total mass of the carriage plus the three $\frac{1}{2}$-kilogram attached metal slabs is m = 2.20 kg, and the stiffness of the spring is k = 770 N/m. Suppose that the ramped exponential force pulse has maximum value F_m = 6 N at the instant t_m = 1/12 s. The MATLAB command lines on the next page calculate and print out the frequency quantity $f_n = \omega_n/2\pi$, and they calculate and plot the time response over the interval $0 \le t \le 2$ s. Note that it was not necessary to assemble these command lines into a formal M-file; rather, the lines were composed in a text-editing program, then the entire group of lines was copied together and pasted onto the prompt "`>>` " on the MATLAB command window. Also, since solution Equation 1.10.8 includes the product of two functions of time in the term $(t/t_m)e^{(1-t/t_m)}$, it is necessary to implement this product using MATLAB array multiplication (`.*`) in the operation `prod=tt.*efun`.

```
>> %Mass-spring response: rest ICs; ramped exponential pulse input m=2.20;k=770; %system
mass and stiffness, SI units wn=sqrt(k/m);fn=wn/2/pi; disp('Natural frequency (cycles/sec) fn
='),disp(fn) Fm=6.0;tm=1/12; %excitation parameters t=0:.005:2; %time sequence for excitation/
response tt=t/tm;efun=exp(1-tt);prod=tt.*efun;f=Fm*prod; %excitation a1=m/tm^2;a2=a1+k;P1=Fm/
a2;P2=2*a1*P1/a2;   %P   constants   C1=(P2-P1)*exp(1)/wn/tm;C2=-P2*exp(1);   %C   constants
wt=wn*t;x=C1*sin(wt)+C2*cos(wt)+P1*prod+P2*efun;          %response          plot(t,100*f/
k,'k.',t,100*x,'k'),grid,xlabel('Time t [sec]'),... ylabel('Translations reference f_x(t)/k and
actual x(t) [cm]'),... title('Response of mass-spring system to ramped exponential pulse')

Natural frequency (cycles/sec) fn = 2.9775
```

Figure 1.10.2 is the MATLAB graph produced by the sequence of commands above. The graph was edited in the MATLAB figure

window, mainly to reduce the size and to add the legend.

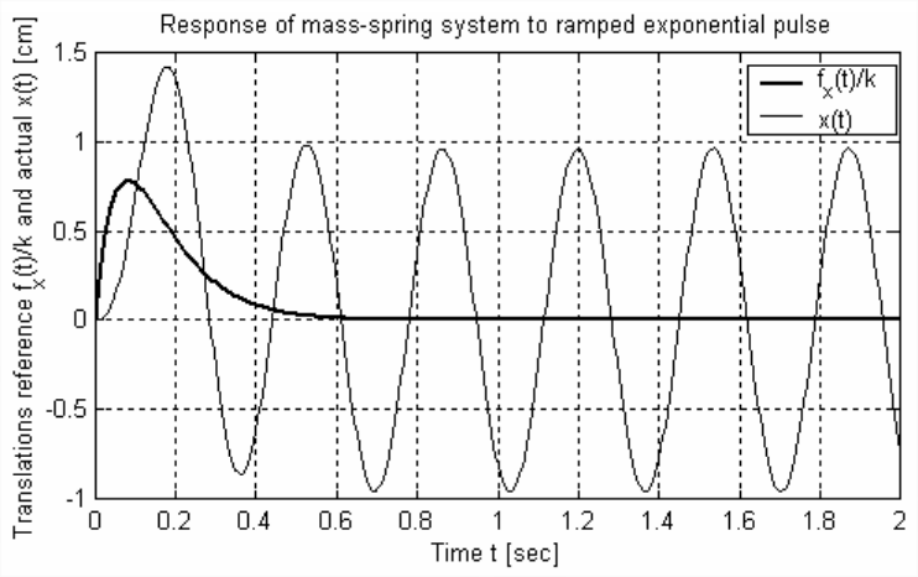


Figure 1.10.2: Excitation and response of a mass-spring system.

The reference quantity $f_x(t)/k$ plotted on Figure 1.10.2 represents the excitation force, but divided by k, so that it has the units of translation. You can see from the basic ODE Equation 1.10.1, $m\ddot{x} + kx = f_x(t)$, that the plotted quantity physically represents the spring deformation that dynamic excitation $f_x(t)$ would produce if mass m were zero or negligible. Because this reference quantity is calculated by neglect of the entire dynamic inertial force, $m\ddot{x}$, it is often called the *pseudo-static* response.

Observe on Figure 1.10.2 that the actual dynamic translation $x(t)$ of the mass greatly overshoots the pseudo-static translation initially (with peak response of about +1.4 cm at about 0.18 s), and that the actual response persists as significant *vibration* after the pulse excitation has died away. These are important dynamic characteristics of many systems that we encounter in engineering practice. In reality, all mechanical systems that consist of purely passive masses and passive structural members have some degree of positive damping; but, if the damping forces are low in comparison with inertial and structural forces, then these real systems will respond to excitation similarly, at least in a qualitative sense, to the "actual" response of Figure 1.10.2. For example, the real system of Figure 1.10.1 is positively but lightly damped; in response to the given force pulse, this system would experience just about the same initial response and overshoot as shown on Figure 1.10.2, but the vibration would die away gradually, and motion would cease completely after three or four seconds.

The terms in Equation 1.10.8 that primarily represent vibration are the sinusoidal functions $\sin \omega_n t$ and $\cos \omega_n t$, with the angle (in radians) argument of these periodic functions being $\theta(t) = \omega_n t$. The *period* of such a function is the time required for the function to complete a full cycle and begin repeating itself. In this case, let us denote that period as T_n and observe that its natural physical units are seconds/cycle. The associated *frequency* f_n, in the units cycles/second,[2] is obviously just the inverse of the period, $f_n = 1/T_n$. Thus, the angle (in radians) argument of the periodic sinusoidal functions can be written in the following equivalent forms: $\theta(t) = \omega_n t = 2\pi(t/T_n = 2\pi f_n t$. The important relationship between *circular* natural frequency ω_n, in radians/second, and *cyclic* natural frequency f_n, in cycles/second, is $f_n = \omega_n/2\pi$. For the values of mass m and stiffness k used in the MATLAB calculations above, the printed-out value of f_n is 2.98 cycles per second, and this is the frequency that is most obvious in Figure 1.10.2; for example, as closely as we can detect by eye, almost exactly three complete cycles of vibration occur between t = 0.7 s and t = 1.7 s.

[1]This is part of a Model 210a Rectilinear Plant designed and fabricated by Educational Control Products of Bell Canyon, California, USA.

[2]More commonly in modern terminology, a cycle/second is designated as a *hertz*, with the abbreviation Hz.

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1.11: Homework problems for Chapter 1

1. Dynaman.
    1. Super hero Dynaman is cruising along at 126 km/hr when the Dynamobile suddenly encounters wet pavement and begins hydroplaning. Naturally, Dynamobile is headed straight for a thick, solid steel barrier and will crash if it cannot stop. Also naturally, Dynamobile is equipped not only with conventional brakes (useless against hydroplaning) but also with jet reverse thrusters that can provide emergency braking in the form of a half-sine pulse. Dynaman hits the panic button, which activates all sensors and the onboard computer. The sensors instantly detect initial velocity v_0 = 35.0 m/s, Dynamobile total mass m = 1 500 kg, and hydroplaning viscous damping constant c = 7.70 N-s/m. The computer calculates that a possible disaster-avoidance action is to deploy the maximum available braking thrust amplitude of F = –7 200 N (–7.2 kN) in a half-sine pulse extending over 10.0 s [$f_x(t) = F\sin(0.100\pi t$ for $0 \le t \le 10.0$ s]. Your task is to demonstrate the effectiveness of this braking thrust by making a MATLAB graph that shows the velocity of Dynamobile over 15 seconds (the 10-s braking period plus another 5 seconds of coasting). Be sure to use good engineering graphical practice: provide grids, a title, appropriate labels, and high point density. Submit your MATLAB script as well as your graph.
    2. Consider the same scenario as in part (1.1.1), but with the following different data: v_0 = 40.0 m/s, m = 1 700 kg, c = 130 N-s/m, with braking force amplitude F = –30 000 N (–30.0 kN) in the pulse $f_x(t) = F\sin(0.250\pi t)$ for $0 \le t \le 4.000$ s. Make a MATLAB graph that shows the velocity of Dynamobile over 8 seconds (the 4-s braking period plus another 4 seconds of coasting).
    3. Consider the same scenario as in part (1.1.1), but with the following different data: v_0 = 60.0 m/s, m = 1 700 kg, c = 1 100 N-s/m, with braking force amplitude F = –120 000 N (–120 kN) in the pulse $f_x(t) = F\sin(\pi t)$ for $0 \le t \le 1.00$ s. Make a MATLAB graph that shows the velocity of Dynamobile over 3 seconds (the 1-s braking period plus another 2 seconds of coasting).
2. Consider again the hydroplaning Dynamobile of Problem 1.1 with total mass m, hydroplaning viscous damping constant c, and initial velocity $v(0) \equiv v_0$.
    1. Integrate $\dot{x} = v(t)$)(given in Equation 1.5.13 to derive an algebraic equation for position $x(t)$ while the braking pulse is active. In order to keep the equation algebraically simple (relatively, anyway), leave it in terms of constants C, P_1 and P_2 (**do not** write those constants in terms of m, c, v_0, F, and t_d). Make sure that your $x(t)$ equation gives the initial position as zero, $x(0) = 0$, by either (i) carefully evaluating the **definite** integral of $v(t)$ between the limits 0 and t [see Equation 1.3.5], or (ii) taking the antiderivative of the $v(t)$ equation and adding an appropriate constant of integration [see Equation 1.3.7]. Using the numerical data of Problem 1.1 [part (1) or (2) or (3), whichever you solved before], calculate the distance traveled by Dynamobile at the end of the braking (pulse) period. (Comment: in reality, to prevent collision with the barrier, would not you need to know Dynamobile's initial distance from the barrier as well as its initial velocity?)
    2. Differentiate $v(t)$ given in Equation 1.5.13 to derive an algebraic equation for **acceleration** $a(t) = \dot{v}$ while the braking pulse is active. In order to keep the equation algebraically simple, leave it in terms of constants C, P_1, P_2. Using the numerical data of Problem 1.1 [part 1.1.1 or 1.1.2 or 1.1.3, whichever you solved before], make a MATLAB graph that shows the acceleration in G's of Dynamobile over the braking (pulse) period. To calculate acceleration in G's, divide acceleration in m/s^2 by the nominal SI acceleration of gravity, g = 9.807 m/s^2. [Comment: we presume that Dynaman would not survive the collision if there were no braking, but, in fact, would he survive the deceleration (perhaps many G's) required to prevent the collision?]
3. Consider again the hydroplaning Dynamobile of Problems 1.1 and 1.2. Suppose that all parameters are known except F, the pulse magnitude. Now we want to find the **optimum** pulse magnitude, $F = F_{opt}$, defined here as the value that will bring Dynamobile to a dead stop at the end of the pulse duration, $v(t_d) = 0$.
    1. Use Equations 1.5.12-1.5.14 for $v(t)$ to derive an algebraic equation for F_{opt} in terms of the parameters m, c, v_0, and t_d. Next, evaluate your equation numerically for m = 1 700 kg, c = 130 N/m/s, v_0 = 40.0 m/s, and t_d = 4.00 s. (answer: F_{opt} = –22.87 kN)
    2. Use the algebraic equation for position $x(t)$ from Problem 1.2.1, and the value from Problem 1.3.1 to calculate how far Dynamobile would travel after activation of thrust braking before coming to a dead stop. (answer: $x(t_d)$ = 75.18 m)
4. Consider a mass-damper system with a cosine forcing function, as described by the 1st order, LTI ODE $m\dot{v} + cv = F\cos\omega t$, in which velocity $v(t)$ is the dependent variable, and the known constant parameters are mass m, viscous damping constant c, force amplitude F, and circular frequency of forcing ω. Use the method of undetermined coefficients to derive an algebraic equation (in terms of the given constants) for the **particular** solution $v_p(t)$ of this ODE and forcing function.
5. The "standard" 1st order ODE is $\dot{x} - ax = bu(t)$.
    1. Suppose that $u(t) = U\sin\omega t$, $t > 0$, where U is a constant amplitude, and that $x(0) = 0$. Derive the following solution of

the standard 1st order ODE:

$$x(t) = \frac{-abU}{\omega^2 + a^2}\left(-\frac{\omega}{a}e^{at} + \sin\omega t + \frac{\omega}{a}\cos\omega t\right), t \geq 0$$

2. Suppose that $u(t) = U\cos\omega t$, $t > 0$, where U is a constant amplitude, and that $x(0) = 0$. Derive the following solution of the standard 1st order ODE:

$$x(t) = \frac{-abU}{\omega^2 + a^2}\left(-e^{at} - \frac{\omega}{a}\sin\omega t + \cos\omega t\right), t \geq 0$$

6. Symbolic software such as Mathematica[1] and MATLAB (the Symbolic Math Toolbox) can solve some ODEs, including those discussed in Chapter 1. For an introduction to this type of solution, solve Problem 1.5.1 in MATLAB. Begin with the following MATLAB commands:

```
>> syms a b U w t
```

```
>> syms a b U w t
```

The MATLAB response will probably be a long, algebraically undisciplined equation for $x(t)$. In order to express the equation in a more economical form, type the command

```
>> x=simple(x)
```

The simplified equation will still be in rather awkward computer notation, so, to express the equation in a more conventional form, type the command:

```
>> pretty(x)
```

A few words of warning about symbolic software are appropriate. For more complex ODEs, symbolic software might give algebraic solutions that are correct, but are expressed in an unfamiliar form, or in an algebraic form that must be simplified by hand in order to be useful. For this reason and others, engineers often regard symbolic software as useful mainly for providing **checks** on mathematical results that have been derived the old-fashioned way, by hand.

7. Imagine a spherical cannon ball having the diameter d and mass m of a regulation baseball, but with a very smooth surface (not somewhat rough, like a baseball's stitched cowhide cover). Suppose that we launch this ball vertically upward from sea level against Earth's gravity, with known launch (muzzle) vertical locity v_0 being sufficiently low that the acceleration of gravity g remains essentially constant over the entire ascent and descent of the ball. Denote the varying altitude (vertical position) of ball as $y(t)$, positive upwards, so the ball's velocity is $v(t) = \dot{y}(t)$.

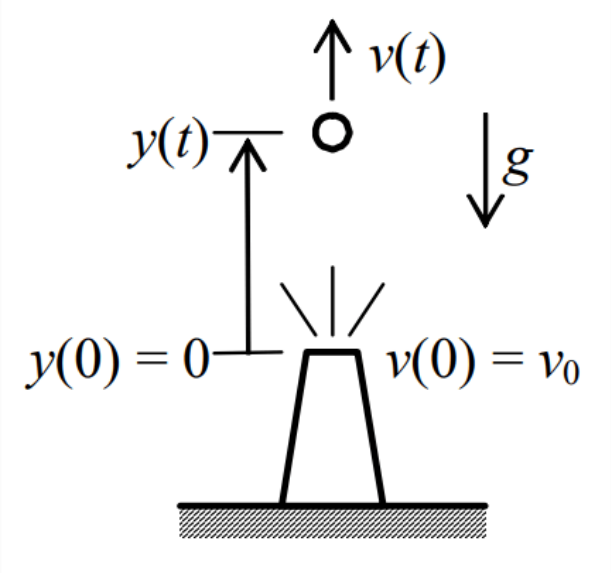


Figure 1.11.1

1. First, let us idealize the aerodynamic drag force as being linearly viscous with damping constant c, $D_1 = cv$ (subscript 1

denotes this mathematical model of drag as proportional to the 1st power of velocity, v^1). Draw and label an appropriate free-body diagram (FBD), showing the direction of drag force, by convention in analysis of aerodynamics, as being opposite to the direction of velocity $v(t)$. Using your FBD, apply Newton's 2nd law to derive the ODE of motion, $m\dot{v} + cv = mg$.[2]

2. Solve the ODE of part (a) to determine an equation for velocity $v(t)$, $t \geq 0$, given the known IC, $v(0) = v_0$. First, observe that the homogeneous ODE is the same as Equation 1.5.3, so that Equation 1.5.6 is the homogeneous solution. Next, find the simplest possible particular solution, v_p, a constant. Next, enforce the initial condition to determine the unknown constant of the homogeneous solution. From your $v(t)$ equation, show that $v_p = v_{t1}$, the constant *terminal velocity* of the ball's descent for drag D_1. Also, derive an equation for the instant of time t_z1 when velocity is zero, which, of course, corresponds to the peak altitude, y_{max_1} of the ball. (*Partial answers*: $V_{t1} = -mg/c$; $t_{z1} = (m/c)\ln(1 - v_0/v_{t1})$, in which ln denotes the natural logarithm, i.e., the logarithm to base e.)
3. Integrate $v(t)$ of part 1.7.2 to derive an equation for the ball's altitude, $y(t)$, $t \geq 0$; define the launch point to be the reference position, *i.e.*, set the IC as $y(0) \equiv 0$. (*Answer*: $y(t) = (m/c)(v_0 - v_{t1})(1 - e^{-(c/m)t}) + v_{t1}t$.)
4. Next, let us consider the mathematical model for the aerodynamic drag force that is generally considered most appropriate for a smooth ball: $D_2 = \pm\bar{q}SC_D$, with the plus sign for ascent ($v \geq 0$), and the minus sign for descent ($v < 0$). The terms in this equation are: $\bar{q} = \frac{1}{2}\rho v^2$, the dynamic pressure, with sea-level air density ρ = 0.002 377 slug/ft^3 = 0.002 377 lb-s^2/ft^4 ; $S = \pi d^2/4$, the ball's projected area, with baseball diameter about d = 2.90 inches; and C_D, the dimensionless drag coefficient, which is about C_D = 0.50 for the smooth ball, provided its airspeed is less than about 140 mph (miles/hour). Write two 1st order ODEs for $v(t)$ with drag model D_2, one ODE applying to ascent, the other to descent. Explain what it is about these ODEs that makes them nonlinear. Do **not** attempt to solve these ODEs **in general**, but use the appropriately simplified form of the descent ODE to find an algebraic equation for v_{t2}, the constant *terminal velocity* of the ball's descent for drag D_2; and then calculate the value of v_{t2} in mph. Use the average weight of baseballs, $mg = 5\frac{1}{8}$ ounces, with g = 32.17 ft/s^2 . Note: 60 mph = 88 ft/s, 1 foot = 12 inches, and 1 lb = 16 ounces. (*Partial answer*: v_{t2} = –74.91 mph. It might seem counterintuitive that the terminal velocity of a stitched baseball is –95 mph, a much greater speed than that of an otherwise equivalent smooth ball; see Adair, 1994, pages 5-12.[3])
5. Let us attempt to establish some sort of "equivalent" linear mathematical model relative to the nonlinear model of part 1.7.4. The only comparable results that we have determined for this purpose are the equations for terminal velocity. Therefore, calculate an "equivalent" linear viscous damping constant c by equating $v_{t1} = v_{t2}$. Suppose that v_0= 110 mph, the average speed at which a baseball rebounds from a bat on the ball's way to a 400-foot major league home run. Calculate the total ascent time t_{z1} (in seconds) and the peak altitude $y(t_{z1}) \equiv y_{max\_1}$ (in feet), quantities that are defined in parts 1.7.2 and 1.7.3. It happens that both nonlinear ODEs of part 1.7.4 can be solved in terms of standard mathematical functions.[4] In particular, solution of the nonlinear ODE for ascent leads to the equation for total ascent time, $t_{z2} = [(-v_{t2})/g]\tan^{-1}[v_0/(-v_{t2})]$, and the equation for peak altitude, $y(t_{z2}) \equiv y_{\max_2} = (v_{t2}^2/g)\ln\sqrt{1 + (v_0/v_{t2})^2}$. Calculate t_{z2} and y_{max_2}, and compare them, respectively, with t_{z1} and y_{max_1}; note, however, that we cannot infer from this limited comparison the general quality of the "equivalent" linear model relative to the nonlinear model. (*Partial answers*: c = 2.955 × 10^{-3} lb-s/ft; y_{max_1} = 210.7 ft; t_{z2} = 3.299 sec)

8. The ODE solution procedure illustrated in Section 1-5 for a 1st order ODE can be used to solve any LTI ODE or system of LTI ODEs. Consider, for example, the 2nd order ODE Equation 1.9.6 for a mass-damper-spring system, $m\ddot{x} + c\dot{x} + kx = f_x(t)$. The physical parameters m, c, and k are known constants.
   1. Seek a homogeneous solution in the form $x_h(t) = Ce^{\lambda t}$ as follows. First, show that the characteristic equation is a quadratic polynomial in the unknown λ. Next, solve the polynomial equation and show in detail that there are two characteristic values (roots), $\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$. Assume that $\left(\frac{c}{2m}\right)^2 > \frac{k}{m}$. Therefore, the general homogeneous solution must have the form $x_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$, with two initially unknown constants.
   2. Suppose that the excitation force has the sinusoidal form $f_x(t) = F\sin\omega t$, in which force amplitude F and circular frequency ω (in radians/second) are considered to be known. Seek a particular solution by using the method of undetermined coefficients. First, express the solution as $x_p(t) = P_1\sin\omega t + P_2\cos\omega t$. Now, substitute this into the non-homogeneous ODE to find algebraic equations for and in terms of the constants that m, c, k, F, and ω. (Partial answer: $P_2 = \frac{-c\omega}{(k - m\omega^2)^2 + (c\omega)^2}F$)
   3. Express the complete solution as $x(t) = x_h(t) + x_p(t)$. Suppose that the mass m is initially at rest with ICs: initial position $x(0) = 0$ and initial velocity $\dot{x}(0) = 0$. Use these ICs to write two linear equations with unknowns C_1 and C_2. Solve for C_1 and C_2; show that they can be written as $C_1 = \frac{P_2\lambda_2 - P_1\omega}{\lambda_1 - \lambda_2}$ and $C_2 = \frac{P_1\omega - P_2\lambda_1}{\lambda_1 - \lambda_2}$.

      Note that a reasonably efficient algebraic equation for the final result $x(t)$ is derived by defining constants C_1 and C_2 in terms of the previously defined λ_i and P_i, rather than writing them in terms of all the basic parameters, m, c, k, F, and ω.

9. Solve Problem 1.8 in MATLAB. Begin with the following MATLAB commands:

```
>> syms x t m c k F w

>> x=dsolve('D2x+c/m*Dx+k/m*x=F/m*sin(w*t)','x(0)=0','Dx(0)=0')
```

The MATLAB response will probably be an equation for $x(t)$ of truly breathtaking length. You can try simplification operations such as `x=simple(x)` and `pretty(x)` and the `subexpr` command, but they will not necessarily lead to a more useful equation for $x(t)$. At some point, you might decide that, rather than continuing to flail away at the symbolic software, your time is used more efficiently if you just solve by hand and define intermediate symbols in terms of the basic parameters, as is done in Problem 1.8.

10. This problem relates to the details of the mass-spring-system ODE solution presented in Section 1-10.
    1. Substitute the assumed particular solution Equation 1.10.6 into non-homogeneous ODE Equation 1.10.5, then carry out the process in all algebraic detail to verify Equations 1.10.7 for P_1 and P_2.
    2. Substitute ICs Equation 1.10.2, $\dot{x}(0) = 0$ and $x(0) = 0$ into complete solution Equation 1.10.9, then carry out the process in all algebraic detail to verify Equations 1.10.10 for C_1 and C_2.
11. Consider a mass-spring system with an "exponential step" forcing function, as described by the 2nd order LTI ODE $m\ddot{x} + kx = f_x(t) = F\left(1 - e^{-t/t_c}\right)$, in which position $x(t)$ is the dependent variable, and the known constant parameters are mass m, spring stiffness constant k, force amplitude F, and *time constant* t_c. The time constant is the time required for the "exponential unit-step" function, $1 - e^{-t/t_c}$, to rise from 0 at t = 0 to the value $1 - e^{-1}$ = 0.6321, on its way to the asymptotic value 1 as $t \to \inf$.
    1. Use the method of undetermined coefficients to derive an algebraic equation (in terms of the given constants) for the **particular** solution $x_p(t)$ of this ODE and forcing function.
    2. Let the ICs for this m-k system be $\dot{x}(0) = 0$ and $x(0) = 0$. Use the result of part (a) and whatever else is required to derive in all detail the following complete algebraic solution for $x(t)$, $t \geq 0$, in which $\omega_n = \sqrt{k/m}$:
    $$x(t) = \frac{F}{k}\left\{1 - \frac{1}{1 + (1/\omega_n t_c)^2}\left[e^{-t/t_c} + (1/\omega_n t_c)\sin\omega_n t + (1/\omega_n t_c)^2 \cos\omega_n t\right]\right\}$$
    3. Consider the specific numerical case of mass m = 8.03 kg, spring stiffness constant k = 317 N/m, and force amplitude F = 4.50 N. Calculate the circular natural frequency ω_n and the cyclic natural frequency f_n. Let excitation time constant $t_c = 1/\omega_n$, a much shorter interval than the m-k system's natural period $T_n = 2\pi/\omega_n$. Compose and run a MATLAB program that does the following: calculate the actual dynamic response $x(t)$ and the pseudo-static response $f_x(t)/k$ over the time interval $0 \leq t \leq 2.5$ s; plot both $x(t)$ and $f_x(t)/k$ on the same graph in units of either centimeters or millimeters. Explain in a sentence or two what feature of your plot of $x(t)$ demonstrates and conforms with your calculation of f_n.
12. Consider a mass-spring system with a sinusoidal forcing function, as described by the 2nd order LTI ODE $m\ddot{x} + kx = f_x(t) = F\sin\omega t$, in which position $x(t)$ is the dependent variable, and the known constant parameters are mass m, spring stiffness constant k, force amplitude F, and excitation frequency ω.
    1. Use the method of undetermined coefficients to derive an algebraic equation (in terms of the given constants) for the **particular** solution $x_p(t)$ of this ODE and forcing function, a solution that is valid provided $\omega^2 \neq k/m$.
    2. Let the ICs for this m-k system be $\dot{x}(0) = 0$ and $x(0) = 0$. Use the result of part 1.12.1 and whatever else is required to derive in all detail the following complete algebraic solution for $x(t)$, $t \geq 0$, in which $\omega_n = \sqrt{k/m}$:
    $$x(t) = \frac{F}{k}\frac{1}{1 - (\omega/\omega_n)^2}\left[\sin\omega t - (\omega/\omega_n)\sin\omega_n t\right], \text{ valid for } \omega \neq \omega_n$$
    3. Consider the specific numerical case of mass m = 0.230 lb-s^2 /inch (which weighs about 88.8 lb), spring stiffness constant k = 227 lb/inch, and force amplitude F = 45.0 lb. Calculate the circular natural frequency ωn and the cyclic natural frequency f_n. For excitation frequency $\omega = 1.2\omega_n$, compose and run a MATLAB program that does the following: calculate the actual dynamic response $x(t)$ and the pseudo-static response $f_x(t)/k$ over the time interval $0 \geq t \geq 2$ s; plot both $x(t)$ and $f_x(t)/k$ on the same graph. Your $x(t)$ plot should exhibit the phenomenon of *beating*, which is analyzed more completely in Section 10-6.
    4. The $x(t)$ equation derived in part 1.12.2 is not valid if excitation frequency ω equals natural frequency ω_n, but note that in this case the $x(t)$ equation has the indeterminate form 0/0, so that we can apply l'Hospital's rule (which is described in any calculus textbook) to find the limit-case response solution. Define $\beta = \omega/\omega_n$, so that $\omega t = \beta\omega_n t$. Now, hold ω_n constant and take the limit as $\beta \to 1$ of the $x(t)$ equation in order to determine the correct response $x(t)$ equation for the case $\omega = \omega_n$.

[1]Mathematica ® is a registered trademark of Wolfram Research, Inc.

[2]In Appendix B, Section B-2, this ODE is derived by an alternative method using energy and power.

[3]Literature references such as this are described in detail in the References section following Chapter 17.

[4]For examples of these solutions, see: an ascent solution in Greenwood, 1965, Problem 3-4 on pages 128 and 503; and a descent solution in Sommerfeld, 1964, pages 21-22.

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2: COMPLEX NUMBERS AND ARITHMETIC, LAPLACE TRANSFORMS, AND PARTIAL-FRACTION EXPANSION

2.1: Review of Complex Numbers and Arithmetic

2.2: Introduction to Application of Laplace Transforms

2.3: Partial-Fraction Expansion

2.4: Additional Useful Functions and Laplace Transforms - Step, Sine, Cosine, and Definite Integral

2.5: Chapter 2 Homework

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2.1: Review of Complex Numbers and Arithmetic

We will find many uses in system dynamics for analysis with complex numbers and variables. It is instructive first to imagine how complex numbers might have been developed historically. It seems likely that they appeared first in solutions for roots of polynomial equations. To see how, let us examine the 2nd degree polynomial, the standard quadratic equation, and its solutions. The quadratic equation is

$$az^2 + bz + c = 0 \tag{2.1.1}$$

where z is the unknown and a, b, and c are known constants. The standard solution for roots z is

$$z = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \tag{2.1.2}$$

Provided that $b^2 - 4ac \geq 0$, the solution via Equation 2.1.2 can be calculated easily. But suppose the constants have values such that $b^2 - 4ac < 0$, so that Equation 2.1.2 indicates the square root of a negative number. That must have seemed strange to early mathematicians. Nevertheless, Equation 2.1.2 is the correct solution **regardless** of the sign of $b^2 - 4ac$, which you can verify simply by substituting Equation 2.1.2 back into Equation 2.1.1. So it was necessary to recognize the existence of the square root of a negative number and to invent notation and terminology for describing such a number.

For $b^2 - 4ac < 0$, we can write

$$\sqrt{b^2 - 4ac} = \sqrt{(-1)\,(4ac - b^2)} = \sqrt{-1} \times \sqrt{4ac - b^2} \equiv j\sqrt{4ac - b^2}$$

in which we define the *unit imaginary number* as $j \equiv \sqrt{-1}$. Perhaps early mathematicians used the adjective “imaginary” because such a number seemed unreal to them. With this definition, we can write Equation 2.1.2 in the form

$$z = -\frac{b}{2a} \pm j\frac{1}{2a}\sqrt{4ac - b^2}$$

This is called a *complex* number, which is the sum of a normal, or *real* number and an imaginary number (j times a real number being a general imaginary number).

If x and y are reals, then any complex number can be written in the *rectangular* form:

$$z = x + jy, \quad \text{with real part} \quad x \equiv \mathrm{Re}(z), \quad \text{and imaginary part} \quad y \equiv \mathrm{Im}(z) \tag{2.1.3}$$

This form is called “rectangular” because it expresses the complex number in terms of its rectangular, or Cartesian components, x and y. Equation 2.1.3 is similar mathematically to a two-dimensional vector **v** expressed in terms of Cartesian components and unit vectors $\mathbf{1_x}$ and $\mathbf{1_y}$: $\mathbf{v} = v_x\mathbf{1_x} + v_y\mathbf{1_y}$. Therefore, we can represent a complex number $z = x + jy$ graphically in the Cartesian form similar to that used for two-dimensional vectors, as shown in Figure 2.1.1.

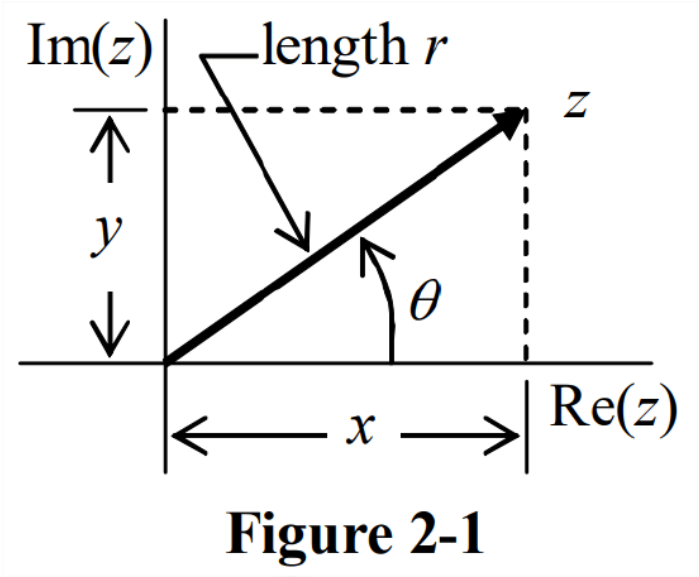


Figure 2-1

Figure 2.1.1

From Figure 2.1.1, we can infer another useful way to represent a complex number. By defining the *magnitude* (*length*, *absolute value*, *modulus*) of z as the positive number

$$r = \sqrt{x^2 + y^2} \tag{2.1.4}$$

we can write

$$z = x + jy = r\left(\frac{x}{r} + j\frac{y}{r}\right) = r(\cos\theta + j\sin\theta) \tag{2.1.5}$$

In Equation 2.1.5, we employ the standard trigonometric functions of angle θ originating on the positive real axis and defined positive in the counterclockwise (CCW) sense, Figure 2.1.1. Angle θ is often called the *argument* of z. The general definition is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{2.1.6}$$

It is necessary to recognize Equation 2.1.6 as the *four-quadrant inverse tangent* (or *arctangent*). For example, if y is positive and x is negative, then Equation 2.1.6 should give angle θ in the second quadrant, $90^\circ < \theta < 180^\circ$; or, if both y and x are negative, then Equation 2.1.6 should give angle θ in the 3rd quadrant, $-180^\circ < \theta < -90^\circ$ (which also can be written as $180^\circ < \theta < 270^\circ$). Unfortunately, the *tan*$^{-1}$ or *arctan* function on hand calculators does **not** calculate the four-quadrant inverse tangent, but only considers the sign of the quotient y/x, and produces a θ value in the 1st or 4th quadrants, $-90^\circ < \theta < +90^\circ$. So, if you are evaluating Equation 2.1.6 on your hand calculator, **you** need to account for the signs of both y and x, and you need to modify appropriately the answer given by your calculator. If you are evaluating Equation 2.1.6 in MATLAB, then you should use MATLAB's four-quadrant inverse tangent function, `atan2(y,x)`.

Equation 2.1.5 is just an intermediate result. To achieve the desired final form, we rewrite Equation 2.1.5 using the well-known Taylor power series expansions for sine, cosine and exponential functions:

$$z = r(\cos\theta + j\sin\theta) = r\left[\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\right]$$

$$z = r(\cos\theta + j\sin\theta) = r\left(1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \ldots\right) \equiv re^{j\theta}$$

$$z = re^{j\theta} \tag{2.1.7}$$

The important result

$$\cos\theta + j\sin\theta = e^{j\theta} \tag{2.1.8}$$

is known as *Euler's equation* (after Leonhard Euler, 1707-1783, Swiss mathematician and physicist). See homework Problem 2.1 for other equations closely related to Equation 2.1.8.

A complex number in the form $z = re^{j\theta}$ is called the *polar* form because it employs polar coordinates r and θ. We can visualize the complex exponential term $e^{j\theta}$ itself as a clock hand in the complex plane, Figure 2.1.1; the clock hand has unit length, and it is rotated CCW by angle θ from the 3 o'clock position. Similarly, the full complex number $z = re^{j\theta}$ represents a clock hand of length r that is rotated CCW by angle θ from the 3 o'clock position, so the notation $z = r\angle\theta$ is often used, where the symbol $\angle$ stands for "angle."

We will occasionally need to convert a complex number from rectangular form into polar form using Equations 2.1.4 and 2.1.6. The MATLAB functions `abs` and `angle` are especially useful for this task, as is illustrated in Complex-arithmetic Example 2 below.

We also will need to use complex arithmetic, so let's review (from your mathematics courses) some of the basics, supposing that we have two complex numbers z and w expressed in both rectangular and polar forms:

$z = x + jy = re^{j\theta}$ and $w = u + jv = qe^{j\phi}$, where x, y, r, θ, u, v, q, and ϕ all are real.

Addition or subtraction, rectangular: $z \pm w = (x \pm u) + j(y \pm v)$.

Addition or subtraction, polar: $re^{j\theta} \pm qe^{j\phi} \equiv Z \equiv Re^{j\Theta}$.

$$Z = r\cos\theta + q\cos\phi \pm j(r\sin\theta + q\sin\phi)$$

$$R = \sqrt{[\mathrm{Re}(Z)]^2 + [\mathrm{Im}(Z)]^2} = \sqrt{r^2 + q^2 + 2rq\cos(\theta - \phi)}$$

$$\Theta = \tan^{-1}[\mathrm{Im}(Z)/\,\mathrm{Re}(Z)] = \tan^{-1}[\pm(r\sin\theta + q\sin\phi)/(r\cos\theta + q\cos\phi)]$$

Multiplication, rectangular: $z \times w = xu + j(xv + yu) + j^2(yv) = (xu - yv) + j(xv + yu)$.

Multiplication, polar: $z \times w = rqe^{j(\theta+\phi)}$.

The *complex conjugate* of is defined as $z = x + jy = re^{j\theta}$ is defined as $\bar{z} = x - jy = re^{-j\theta}$, so that the product of a complex number and its conjugate is real, and equal to the square of the magnitude: $z\bar{z} = x^2 + j(-xy + yx) - j^2y^2 = x^2 + y^2 = r^2$, or more easily in polar form, $z\bar{z} = re^{j\theta} \times re^{-j\theta} = r^2e^{j(\theta-\theta)} = r^2$. Suppose that we have the quotient of two complex numbers z by w, both in rectangular form, and we wish to cast the result into standard rectangular form; it is common practice first to multiply both denominator and numerator by the conjugate of the denominator, thus forming a real denominator:

Division, rectangular: $\frac{z}{w} = \frac{x+jy}{u+jv} = \frac{(x+jy)(u-jv)}{(u+jv)(u-jv)} = \frac{xu+yv}{u^2+v^2} + j\frac{-xv+yu}{u^2+v^2}$

Division, polar, $\frac{z}{w} = \frac{r}{q}e^{j(\theta-\phi)}$

Example 2.1.1 Complex-arithmetic

Cast the following complex fraction into standard rectangular form, then convert it into polar form:

$$z = \frac{3 - j4}{1 + j2} = \frac{(3 - j4)(1 - j2)}{(1 + j2)(1 - j2)} = \frac{3 - 8 + j(-6 - 4)}{5} = -1 - j2$$

Solution

$$r = \sqrt{(-1)^2 + (-2)^2} = 2.236$$

$$\theta = \tan^{-1}\left(\frac{-2}{-1}\right) = -116.6^\circ \times \frac{\pi\mathrm{rad}}{180^\circ} = -2.034\mathrm{rad}$$

$$\Rightarrow \quad z = \frac{3 - j4}{1 + j2} = -1 - j2 = 2.236e^{j(-116.6^\circ)} \equiv 2.236\angle(-2.034\mathrm{rad})$$

Note that when you take $\tan^{-1}(2)$ on your hand calculator, it gives 63.43°. Because you are really trying to find $\tan^{-1}(-2/-1)$, you must recognize that the correct angle is in the 3rd quadrant and, therefore, add +180° or –180° to the calculator's solution in order to obtain the correct result, +243.4° or –116.6°.

Example 2.1.2 Complex-arithmetic

Cast the following complex fraction into standard rectangular form, then convert it into polar form:

$$z = \frac{(3-j2)(4+j)}{2-j3} - 7 + j4 = \frac{(14-j5)(2+j3)}{(2-j3)(2+j3)} - 7 + j4 = \frac{43+j32}{13} - 7 + j4$$

Solution

$$z = \left(\frac{43}{13} - 7\right) + j\left(\frac{32}{13} + 4\right) = -3.692 + j6.462$$

$$r = \sqrt{(-3.692)^2 + (6.462)^2} = 7.442$$

$$\theta = \tan^{-1}\left(\frac{6.462}{-3.692}\right) = -60.255^\circ + 180^\circ = 119.7^\circ$$

$$\Rightarrow \quad z = \frac{(3-j2)(4+j)}{2-j3} - 7 + j4 = -3.692 + j6.462 = 7.442e^{j119.7^\circ}$$

A solution of Complex-arithmetic Example 2 in terms of MATLAB operations follows:

```
>> z=(3-2j)*(4+j)/(2-3j)-7+4j

z = -3.6923 + 6.4615i

>> r=abs(z),theta=angle(z)

r = 7.4421

theta = 2.0899

>> thetadeg=theta*180/pi

thetadeg = 119.7449
```

Note that MATLAB prefers, and gives results in terms of the unit imaginary number denoted as i, but it recognizes our notation of j. Note also that the radian is MATLAB's default unit for angles: if you want a MATLAB angle result in degrees, you must convert from radians; also, if you wish to calculate a trigonometric function of an angle, you must express that angle in radians.

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2.2: Introduction to Application of Laplace Transforms

The *Laplace transform* (after French mathematician and celestial mechanician Pierre Simon Laplace, 1749-1827) is a mathematical tool primarily for solving ODEs, but with other important applications in system dynamics that we will study later. In Laplace transformation, we deal with a *complex variable* denoted as s, which is usually expressed in terms of its real and imaginary parts as

$$s = \sigma + j\omega \tag{2.2.1}$$

in which σ and ω both are real. We define a complex function of s, $F(s)$. The type of function that we will encounter often takes the form of a ratio of two polynomials:

$$F(s) \equiv \frac{\mathrm{Num}(s)}{\mathrm{Den}(s)} = \frac{b_1 s^m + b_2 s^{m-1} + ... + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + ... + a_{n+1}} = \frac{b_1 (s - z_1)(s - z_2) \cdots (s - z_m)}{a_1 (s - p_1)(s - p_2) \cdots (s - p_n)} \tag{2.2.2}$$

In the first polynomial form of Equation 2.2.2, a_1, ..., a_{n+1} and b_1, ..., b_{m+1} are real constants (with the symbols and numbering system keyed to MATLAB notation); numerator $\mathrm{Num}(s)$ is an mth degree polynomial in s, and denominator $\mathrm{Den}(s)$ is an nth degree polynomial in s, with $m \leq n$ in general. In the second polynomial form, with factored $\mathrm{Num}(s)$ and $\mathrm{Den}(s)$, complex constants z_1, z_2 , ..., z_m are called zeros of $F(s)$ because $F(s)$ is zero if s equals any one of them, and complex constants p_1, p_2, ..., p_n are called poles of $F(s)$ because $F(s)$ is infinite if s equals any one of them. If $m < n$, $F(s)$ in Equation 2.2.2 also goes to zero as $s \to \mathrm{inf}$.

Solving a simple ODE problem with Laplace transforms is a gentle introduction to the subject. Consider the 1st order LTI ODE written in standard form: $\dot{x} - ax = bu(t)$, Equation 1.2.1. Let us solve this ODE with a known IC, $x(0) = x_0$, and with a specific exponential input function, $u(t) = Ue^{-wt}$, U being a dimensional magnitude; in any physically realistic problem, constant w would be a real number, but for generality here we allow it to be a complex number. So the complete problem statement is:

$$\dot{x} - ax = bUe^{-wt}, x(0) = x_0, \text{ solve for } x(t), t > 0 \tag{2.2.3}$$

To begin the solution, we multiply the ODE by a complex exponential, e^{-st}, then take the definite integral over time from t = 0 to $t = \mathrm{inf}$ of the entire multiplied ODE:

$$\int_{t=0}^{t=\infty} e^{-st}\left(\dot{x} - ax = bUe^{-wt}\right) dt \Rightarrow \int_{t=0}^{t=\infty} e^{-st}\dot{x}dt - a\int_{t=0}^{t=\infty} e^{-st}x dt = bU\int_{t=0}^{t=\infty} e^{-st}e^{-wt}dt \tag{2.2.4}$$

In Equation 2.2.4, s is a complex variable that must have values for which the integrals exist. Based upon Equation 2.2.4, we define the *forward Laplace transform* of dependent variable $x(t)$[1]:

$$L[x(t)] \equiv X(s) \equiv \int_{t=0}^{t=\infty} e^{-st}x(t)dt \tag{2.2.5}$$

In Equation 2.2.5, the function x of time t is *transformed* by the definite integration into a function X of Laplace variable s. Also, the ODE of Equation 2.2.3 is transformed into Equation 2.2.4, which, as we will find, becomes an easily solvable **algebraic** equation in the unknown $X(s)$. After we solve that algebraic equation for $X(s)$, then we will reverse the process to find the original unknown $x(t)$ by applying the inverse Laplace transform, denoted as $L^{-1}[X(s)] \equiv x(t)$. For any Laplace-transformable function $f(t)$, and its transform $F(s)$, the companion equations $L[f(t)] = F(s)$ and $L^{-1}[F(s)] = f(t)$ are called a *Laplace transform pair*.

$$\int_{t=0}^{t=\infty} e^{-st}e^{-wt}dt = \int_{t=0}^{t=\infty} e^{-(s+w)t}dt = \frac{1}{-(s+w)}\int_{t=0}^{t=\infty} d\left[e^{-(s+w)t}\right] = \frac{1}{-(s+w)}\left[e^{-(s+w)t}\right]_{t=0}^{t=\infty}$$
$$= \frac{1}{-(s+w)}\left[e^{-(s+w)\infty} - 1\right] = \frac{1}{s+w}$$

An important step in this derivation involves the complex exponential, with use of Equations 2.1.5 and 2.1.12: $e^z = e^{(x+jy)} = e^x(\cos y + j\sin y)$. Since $\sin y$ and $\cos y$ vary periodically as y varies, the only way for e^z to $\to 0 + 0j$ for all values of y is for $x \to -\infty$. Thus, in the last step of the integration, we assume that $\mathrm{Re}(s + w) > 0$, or $\mathrm{Re}(s) > -\mathrm{Re}(w)$, so that $e^{-(s+w)\infty} = 0 + j0$. The integration above establishes the following Laplace transform pair, which is one of the most important of all pairs for applications:

$$L\left[e^{-wt}\right] = \frac{1}{s+w} \quad \text{and} \quad L^{-1}\left[\frac{1}{s+w}\right] = e^{-wt} \tag{2.2.6}$$

For your convenience, transform pair Equation 2.2.6 and all other fundamental Laplace transform pairs used in this book are tabulated in Appendix A at the end of the book. Appendix A also includes some of the longer but less instructive transform derivations, should you wish to study them.

Next, let us evaluate the first term on the left-hand side of Equation 2.2.4 by the standard method of *integration by parts* in the form $\int_{t=0}^{t=\infty} udv = [uv]_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty} vdu$:

$$\int_{t=0}^{t=\infty} e^{-st} \cdot \dot{x} dt = \left[e^{-st}x(t)\right]_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty} x(t) \times (-s)e^{-st}dt = \left[e^{-sx\infty}x(\infty) - e^{-0}x(0)\right] + sX(s) \tag{2.2.7}$$

$$= -x(0) + sX(s) \tag{2.2.8}$$

To obtain this result, we assume that $e^{-(s\times\infty)}x(\infty) = 0$, which generally means $\mathrm{Re}(s) > 0$. This result is the important general Laplace transform of the first derivative of any transformable function $f(t)$:

$$L[\dot{f}(t)] = sF(s) - f(0) \tag{2.2.9}$$

Equation 2.2.9 is the basis for derivation of the Laplace transform of a derivative of any order n, which we will use later (Ogata[2], 1998, pp. 25-26):

$$L\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - \overset{(n-1)}{f}(0) \tag{2.2.10}$$

For example, the Laplace transform of the 2nd derivative is (homework Problem 2.7):

$$L[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0) \tag{2.2.11}$$

Proceeding with the solution of ODE + IC Equation 2.2.3, we now substitute Equations 2.2.5, 2.2.6, and 2.2.8 back into Equation 2.2.4 to obtain an algebraic equation for transform $X(s)$, which is easily solved:

$$sX(s) - x(0) - aX(s) = \frac{bU}{s+w} \quad \Rightarrow \quad (s-a)X(s) = x_0 + \frac{bU}{s+w}$$

$$\Rightarrow \quad X(s) = \frac{1}{(s-a)}x_0 + \frac{bU}{(s-a)(s+w)} \tag{2.2.12}$$

To solve for the ultimate unknown $x(t)$, we now need to find the inverse transform $X(s)$ of in (2-18). Using Equation 2.2.6 easily gives us the inverse of the first term on the righthand side:

$$L^{-1}\left[\left(\frac{1}{s-a}\right)x_0\right] = x_0L^{-1}\left[\frac{1}{s-a}\right] = x_0e^{at} \tag{2.2.13}$$

Inverting the second term on the right-hand side of Equation 2.2.12 is a greater challenge, which requires us to use **partial-fraction expansion** in order to expand that term into two simpler terms, each of which has the easily invertible form of a constant divided by a single factor of the form (s - p). We first write the troublesome term as two simpler fractions with unknown constant numerators, C_1 and C_2, which are called **residues of the partial-fraction expansion**. (A more detailed justification for this form is derived in the next section.):

$$\frac{1}{(s-a)(s+w)} = \frac{C_1}{s-a} + \frac{C_2}{s+w} \Rightarrow \frac{C_1}{s-a} + \frac{C_2}{s+w} = \frac{1}{(s-a)(s+w)} \tag{2.2.14}$$

There are two common methods for finding C_1 and C_2, which we shall call the “laborsaving” method and the “brute-force” method.

Consider first the labor-saving method. Let us go in great detail through the steps required to find C_1, for example. First, multiply both sides of Equation 2.2.14 by , the denominator of the term:

$$(s-a)\left(\frac{C_1}{s-a} + \frac{C_2}{s+w}\right) = C_1 + (s-a)\left(\frac{C_2}{s+w}\right) = (s-a)\left(\frac{1}{(s-a)(s+w)}\right)$$

Isolate C_1 on the left-hand side:

$$C_1 = -(s-a)\left(\frac{C_2}{s+w}\right) + (s-a)\left(\frac{1}{(s-a)(s+w)}\right)$$

Now, in order to eliminate the C_2 term on the right-hand side, set $s = a$. Note that this does not eliminate the second right-hand-side term because $(s-1)$ is in both the denominator and the numerator.

$$\begin{aligned} C_1 &= \left[-(s-a)\left(\frac{C_2}{s+w}\right) + (s-a)\left(\frac{1}{(s-a)(s+w)}\right)\right]_{s=a} \\ &= \left[(s-a)\left(\frac{1}{(s-a)(s+w)}\right)\right]_{s=a} \end{aligned}$$

Thus, we obtain the required equation for C_1:

$$C_1 = \left[(s-a)\left(\frac{1}{(s-a)(s+w)}\right)\right]_{s=a} = \frac{1}{a+w}$$

Now that we see the logic of the labor-saving method, we can dispense with most of the intermediate steps and quickly write the corresponding equation for C_2 directly from Equation 2.2.14:

$$C_2 = \left[(s+w)\left(\frac{1}{(s-a)(s+w)}\right)\right]_{s=-w} = \frac{1}{-w-a} = -\frac{1}{a+w}$$

Next, the brute-force method begins with use of the traditional algebraic method for combining the two fractions of Equation 2.2.14 into a single fraction:

$$\frac{1}{(s-a)(s+w)} = \frac{C_1}{s-a} + \frac{C_2}{s+w} = \frac{C_1(s+w)+C_2(s-a)}{(s-a)(s+w)} = \frac{s\,(C_1+C_2)+(C_1w-C_2a)}{(s-a)(s+w)}$$

Equate coefficients of like powers of s in the numerators:

$$s^1 : 0 = C_1 + C_2 \Rightarrow C_2 = -C_1$$

$$s^0 : 1 = C_1w - C_2a = C_1(w+a) \Rightarrow C_1 = \frac{1}{a+w} = -C_2$$

In this simple problem, the brute-force method is not much more demanding algebraically than the labor-saving method. However, for a slightly more complex original fraction, say, one with three denominator polynomial factors instead of just two, the brute-force method can require orders of magnitude more algebra than the labor-saving method. The point of this discussion: exert the mental energy to understand the laborsaving method, and always use it rather than the brute-force method [see homework Problem 2.9.2].

Using the partial-fraction expansion developed above, we now express the complete second term on the right-hand side of Equation 2.2.12 in the easily invertible form

$$F(s) = \frac{bU}{(s-a)(s+w)} = \frac{bU}{a+w}\left(\frac{1}{s-a} - \frac{1}{s+w}\right)$$

Using the fundamental inverse transform from 2.2.6 gives

$$f(t) = \frac{bU}{a+w}\left(e^{at} - e^{-wt}\right) \tag{2.2.15}$$

Finally, combining Equations 2.2.13 and 2.2.15 gives the inverse of Equation 2.2.12 and the final desired solution of ODE + IC Equation 2.2.3:

$$x(t) = x_0e^{at} + \frac{bU}{a+w}\left(e^{at} - e^{-wt}\right), \text{ for } t \geq 0 \tag{2.2.16}$$

The introduction to Laplace transformation in this section is not mathematically rigorous. The focus in introductory system dynamics is more with applying Laplace transforms than with the detailed theory. It is relevant, however, to comment on the

existence of Laplace transforms: a transform generally exists (*i.e.*, the defining integral Equation 2.2.5 can be evaluated, in principle) for any function $f(t)$ for which the product $e^{-at}|f(t)| \to 0$ as $t \to \infty$, where a is some finite, positive, real constant (Hildebrand, 1962, Section 2.2). Practically speaking, this means that **any** physically realizable function, the type of function that we encounter in engineering, has a Laplace transform.

[1]In Section 8.4 we develop a more general definition, Eq. (8-12), in order to accommodate the ideal unit-impulse function, which is important in linear-system theory and applications.

[2]Literature sources such as this are described in detail in the References section following Chapter 17.

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2.3: Partial-Fraction Expansion

Let us examine in more detail the justification for the form of the partial-fraction expansion presented in Equation 2.2.15. As an example, consider the following sum of three fractional terms:

$$F_3(s) = \sum_{k=1}^{3} \frac{C_k}{s - p_k} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \frac{C_3}{s - p_3} \tag{2.3.1}$$

$$= \frac{C_1 (s - p_2)(s - p_3) + C_2 (s - p_1)(s - p_3) + C_3 (s - p_1)(s - p_2)}{(s - p_1)(s - p_2)(s - p_3)} \tag{2.3.2}$$

In the combination of the three fractional terms into a single fraction, the denominator is a cubic polynomial ($n = 3$), and the degree of the numerator polynomial is, at most, $m = 2$. Depending upon the values of constants C_k, the degree of the numerator polynomial can be from 0 to 2, so that $0 \leq m < n$.

Next, let us generalize the observation of the previous paragraph by considering a general sum of fractional terms in the form

$$F_n(s) = \sum_{k=1}^{n} \frac{C_k}{s - p_k} \tag{2.3.3}$$

If we combine all n of these terms into a single ratio using the traditional algebraic method illustrated in Equation 2.3.2, then the result will have the following general form:

$$F_n(s) = \frac{c_1 s^m + c_2 s^{m-1} + \ldots + c_{m+1}}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

The denominator is a polynomial of degree n, and the numerator is a polynomial of, **at most**, degree $m = (n - 1)$. We therefore can conclude the following: a ratio of polynomials, in which the numerator has a lower degree than that of the denominator, can usually be expanded into the simple partial-fraction form Equation 2.3.3. In other words, **provided that** $0 \leq m < n$, we can **usually** find finite residues C_k in the partial-fraction expansion:

$$F_n(s) = \frac{b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n+1}} = \frac{b_1 (s - z_1)(s - z_2) \cdots (s - z_m)}{a_1 (s - p_1)(s - p_2) \cdots (s - p_n)} = \sum_{k=1}^{n} \frac{C_k}{s - p_k} \tag{2.3.4}$$

with the residues given by the labor-saving method as[1]

$$C_k = \left[(s - p_k) F_n(s) \right]_{s=p_k}, k = 1, 2, \ldots, n \tag{2.3.5}$$

Consider, for example, part of the partial-fraction expansion of a quadratic divided by a cubic:

$$F_3(s) = \frac{b_1 s^2 + b_2 s + b_3}{(s - p_1)(s - p_2)(s - p_3)} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \frac{C_3}{s - p_3}$$

Using Equation 2.3.5, to determine, for example, residue C_1 gives:

$$C_1 = \left[(s - p_1) F_3(s) \right]_{s=p_1} = \frac{b_1 p_1^2 + b_2 p_1 + b_3}{(p_1 - p_2)(p_1 - p_3)}$$

This equation for C_1 reveals an exception to the rule: this equation clearly would not be valid if the denominator polynomial were to have repeated roots, $p_1 = p_2$ or $p_1 = p_3$; in that case, a form different than Equations 2.3.4 and 2.3.5 would be appropriate (Ogata, 1998, pp. 33-34). That is a special case which we shall address in this book only as the need arises.

Finally, observe that it is easy to check the validity/correctness of a partial-fraction expansion after we have solved for residues C_k. Simply combine the individual fractions into a single ratio, as is illustrated in Equation 2.3.2; the resulting ratio should equal the original ratio of polynomials that we expanded into partial fractions.

[1]An interesting alternative form of Equation 2.3.5 is developed in Appendix A, Section A-2.

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2.4: Additional Useful Functions and Laplace Transforms - Step, Sine, Cosine, and Definite Integral

We shall describe and transform several different useful mathematical functions. A common feature of most of these functions is that they are defined to have non-zero values only for positive time, *i.e.*, they are zero before t = 0. The fundamental function of this type is the basic **Heaviside unit-step function** (after English electrical engineer, physicist, and applied mathematician Oliver Heaviside, 1850-1925) shown in Figure 2.4.1:

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{2.4.1}$$

$H(t)$ is dimensionless, and it is undefined mathematically at the discontinuity at $t = 0$. If we wish to write an equation for a physical input quantity that is applied quickly and remains constant thereafter, we can use $H(t)$ to represent it approximately; for example, a "step" force $f_x(t)$ can be described with the equation

$$f_x(t) = F \times H(t) \equiv FH(t)$$

where F is the dimensional force magnitude. For many physical input quantities, a step function is an idealized approximation; such a quantity increases quickly but continuously, not as a discontinuous pure step, from zero to a constant value. [The notation $H(t)$ used here for the unit-step function is common but not standard; in fact, there is no standard symbol in engineering literature.]

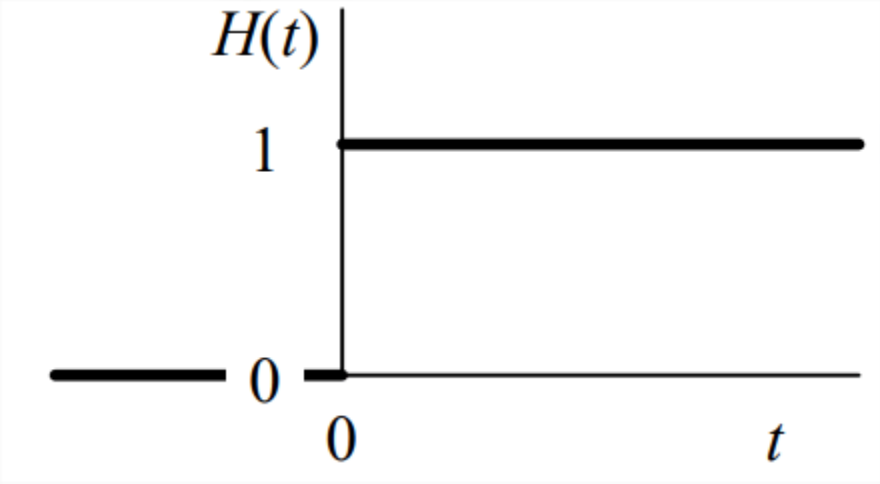


Figure 2.4.1

A more general unit-step function describes a step that occurs at some time different than the instant defined to be t = 0. This function shown in Figure 2.4.2 is:

\[H\left(t-t_{s}\right)=\left\{\begin{array}{ll}
0 & \text { for } t 1 & \text { for } t>t_{s}
\end{array}\right.\label{eqn:2.27} \]

If the time of discontinuity is $t_s = 0$, then this more general function becomes the basic function of Equation 2.4.1. The Laplace transform of $H\left(t - t_s\right)$ is

$$L\left[H\left(t-t_s\right)\right] = \int_{t=0}^{t=\infty} e^{-st} H\left(t-t_s\right) dt = \int_{t=t_s}^{t=\infty} e^{-st} dt = \frac{1}{-s} \int_{t-t_s}^{t=\infty} d\left(e^{-st}\right) = -\frac{1}{s}\left(e^{-s\times\infty} - e^{-st_s}\right)$$

$$L\left[H\left(t-t_s\right)\right] = \frac{e^{-st_s}}{s} \tag{2.4.2}$$

Note, in particular, the version of Equation 2.4.2 for the basic unit-step function with t_s = 0:

$$L[H(t)] = \frac{1}{s} \tag{2.4.3}$$

We will frequently use sine and cosine functions of time with circular frequency ω, the functions being non-zero only for positive time, $t \geq 0$. Using precise notation, such a sine function, for example, should be denoted $H(t) \times \sin \omega t$; however, with few exceptions, we will consider mostly $t \geq 0$, so we can almost always omit the "$H(t) \times$" part of the equation, with the implicit understanding that the analysis applies only for $t \geq 0$.

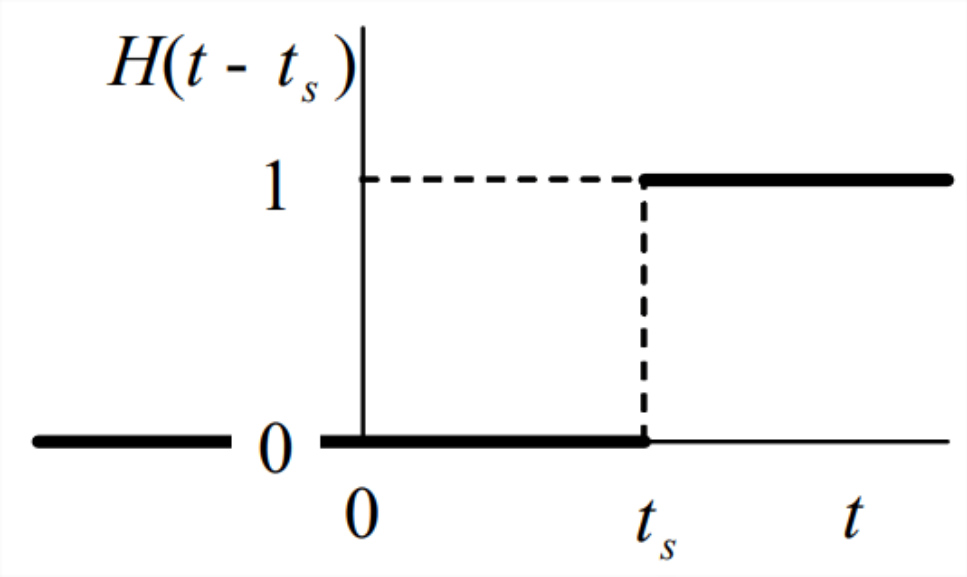


Figure 2.4.2

Next we derive the Laplace transform of the sine function by expressing the sine in terms of complex exponential functions (homework Problem 2.1) and using the basic forward transform of Equation 2.4.1:

$$L[\sin \omega t] = L\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{1}{2j}\left(\frac{s + j\omega - (s - j\omega)}{(s - j\omega)(s + j\omega)}\right)$$

$$L[\sin \omega t] = \frac{1}{2j}\left(\frac{2j\omega}{s^2 + \omega^2}\right) = \frac{\omega}{s^2 + \omega^2} \tag{2.4.4}$$

Using a similar process, you can derive (homework Problem 2.10) the following Laplace transform of the cosine function:

$$L[\cos \omega t] = \frac{s}{s^2 + \omega^2} \tag{2.4.5}$$

Another useful Laplace transform is that of a definite integral. Suppose that a physically realistic function $f(t)$ has Laplace transform $F(s) = L[f(t)]$, and that we need the transform of the definite integral $\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau$. Note the lower limit of $\tau = -\infty$; we will usually consider $f(t)$ only for $t \geq 0$, but occasionally the integral of $f(t)$ over previous time, t < 0, is also needed. The general transform, which is derived in Appendix A, Section A-3, is:

$$L\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right] = \frac{1}{s}F(s) + \frac{1}{s}\int_{\tau=-\infty}^{\tau=0} f(\tau)d\tau \tag{2.4.6}$$

For most applications, we will have $f(t)$ = 0 for t < 0, for which the simpler transform is:

$$L\left[\int_{\tau=0}^{\tau=t\geq 0} f(\tau)d\tau\right] = \frac{1}{s}F(s) \tag{2.4.7}$$

If we regard the integral of $f(t)$ as being the first "negative" derivative (antiderivative), then we see that transform Equation 2.4.6 is logically consistent with transform Equation 2.2.9 for a "positive" derivative, with respect to both power of s and the initial value term.

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2.5: Chapter 2 Homework

1. Refer to Euler's equation, Equation 2.1.12, which is $e^{j\theta} = \cos\theta + j\sin\theta$. The cosine is an even (symmetric) function, $\cos(-\theta) = \cos\theta$, and the sine is an odd (antisymmetric) function, $\sin(-\theta) = -\sin\theta$, so the conjugate to Euler's equation is
$$e^{-j\theta} = e^{j(-\theta)} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$$
    1. Add the conjugate to Euler's equation to prove $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$. Subtract the conjugate from Euler's equation to prove $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$.
    2. Use the results of part 2.1.1 to prove the trigonometric identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$. Use these results to prove $\cos^2\theta - \sin^2\theta = \cos 2\theta$.
    3. Use the results of part 2.1.1 to prove the identity $\sin\theta \times \cos\theta = \frac{1}{2}\sin 2\theta$.
2. **Given** the following complex numbers:

    First, express the complex number in rectangular form, $z = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers. Next, express the complex number in polar form $re^{j\theta}$, that is, calculate by hand and calculator the radius r and the phase angle θ , where θ is in degrees, –180° $\leq \theta <$ +180°. You may check your hand calculations by using the `abs` and `angle` commands appropriately in MATLAB.

    1. $\frac{1}{-0.4-j0.3}$
    2. $5j \times (2-j)$
    3. $\frac{0.7-j2.4}{0.5+j1.2}$
    4. $\frac{-10+j5}{0.6+j0.8}$
3. **Given**: two complex numbers expressed in polar form $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, with r_1, r_2, θ_1,and θ_2 all real numbers.
    1. Use complex arithmetic with complex numbers in polar form to show that the conjugate of a product equals the product of conjugates, which is expressed in equation form as \(\bar{\left(z_{1} \times z_{2}\right)}=\bar{z}_{1} \times \bar{z}_{2}
    2. Use complex arithmetic with complex numbers in polar form to show that the conjugate of a quotient equals the quotient of conjugates, which is expressed in equation form as $(z_1/z_2) = \bar{z}_1/\bar{z}_2$
4. Express a complex number in rectangular form, $z = x + jy$, and prove the following identity: $z + \bar{z} = 2\,\mathrm{Re}(z) = 2x$.
5. **Given**: complex numbers $A = A_r + jA_i$ and $B = B_r + jB_i$ (A_r, A_i, B_r, B_i all real) and complex variable $z = x + jy$ (with real, non-zero values of x and y, which are independent of each other), in the equation $C = Az + B\bar{z}$.

    Prove: use complex arithmetic with complex values in rectangular form to show that C can be **real**, $C = C_r + j0$, only if $B = \bar{A}$. (Note that you can also prove this by applying the results of homework Problems 2.3.1 and 2.4.)

6. The following complex function will arise when we study the frequency response of a 2nd order system: $Z(\beta) = \frac{1}{(1-\beta^2)+j2\zeta\beta}$, in which β is a positive, real, **dimensionless** forcing frequency (not dimensional frequency ω), and ζ is a positive, real, dimensionless ratio that represents viscous damping.
    1. Convert the $\mathrm{Z}(\beta)$ equation above into polar form $\mathrm{Z}(\beta) = r(\beta)e^{j\theta(\beta)}$. In particular, carry out the detailed algebraic steps to show that the magnitude and phase angle are $r(\beta) = \frac{1}{\sqrt{(1-\beta^2)^2+(2\zeta\beta)^2}}$ and $\theta(\beta) = \tan^{-1}\left(\frac{-2\zeta\beta}{1-\beta^2}\right)$.
    2. Calculate values of the magnitude and phase angle for several values of forcing frequency β and for two values of damping ratio ζ (specifics below). Make these calculations with a hand calculator, not with MATLAB or some other computer software (although you may check your hand calculations with computer software). These calculations will give you a small preview of frequency response, and (especially) they should help you to understand the care required when you evaluate the four-quadrant arctangent. Specifically, for **each** of the damping ratios ζ = 0 and ζ = 0.1 (or another value assigned by your instructor), calculate the quantities in the empty cells of the table below (to at least three significant figures, preferably four, but no more); also, calculate phase angles using the standard engineering convention, –180° $\leq \theta <$ +180°, **not** 0° $\leq \theta <$ +360°.

        Hints for the ζ = 0 case: (1) for the $r(\beta)$ calculations, recognize that the equation requires $r(\beta) > 0$, even if $\beta > 1$; (2) for the $\theta(\beta)$ calculations, imagine that ζ has an **extremely small**, positive value, say $\zeta = +1 \times 10^{-10}$.

β	$r(\beta)$	$\theta(\beta)$ degrees	$\theta(\beta)$ radians
0			
0.5			
1			
2			
5			

Figure 2.5.1

7. Prove Equation 2.2.11, $L[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$: first define $g(t) = \dot{f}(t)$; now use the fundamental equation (2-15b) twice, first as $G(s) = L[\dot{f}] = sF(s) - f(0)$, and then as $L[\dot{g}] = sG(s) - g(0)$.
8. Consider the Laplace transform shown below, both in the first form, which is derived directly from an ODE, and in a partial-fraction expansion,

$$F(s) = \left(\frac{1}{s-a}\right)\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{C_1}{s-a} + \frac{C_2}{s-j\omega} + \frac{C_3}{s+j\omega}$$

in which a and ω are real constants. Find residues C_1, C_2, and C_3 in terms of a and show that C_1 is real. Use the labor-saving method directly, **without** first completing the products in the form derived from an ODE.
9. Find the inverse Laplace transform $f(t)$, $t \leq 0$, for the given function $F(s)$ below. First use partial-fraction expansion to express $F(s)$ as a sum of simple quotients, and then apply the appropriate inverse transformation equations. Also, find the values of any zeros and poles of $F(s)$.
    1. $F(s) = \frac{s+3}{(s+1)(s+5)}$ [*Answer*: $f(t) = \frac{1}{2}\left(e^{-t} + e^{-5t}\right), t \geq 0$]
    2. $F(s) = \frac{2(s+1)}{s(s+3)(s+4)}$ [*Answer*: $f(t) = \frac{1}{6} + \frac{4}{3}e^{-3t} - \frac{3}{2}e^{-4t}, t \geq 0$]
10. Derive the Laplace transform $L[\cos\omega t] = \frac{s}{s^2+\omega^2}$. Follow the same procedure, with appropriate changes in details, as the derivation of $L[\sin\omega t]$, Equation 2.4.7. Show all steps, just as in the derivation of $L[\sin\omega t]$.
11. Solution Equation 2.2.26 of ODE problem Equation 2.2.3 was derived by Laplace transformation, a technique that is probably new to you, so you might be a little skeptical. To convince yourself that Equation 2.2.26 is indeed the correct solution of Equation 2.2.3, substitute Equation 2.2.26 back into the ODE and IC of Equation 2.2.3, and show that the original equations are satisfied. This method can always be used to check the correctness of an ODE solution.
12. Consider the standard 1st order ODE $\dot{x} - ax = bu(t)$ with IC $x(0) \equiv x_0$. Use Laplace transformation to solve for $x(t)$ with the $u(t)$ functions given below. It will ease your algebraic burden greatly if you will make liberal use of the identities involving complex numbers proved in homework Problems 2.3 and 2.4. Also, use where appropriate the transforms of $\sin\omega t$ from Equation 2.4.7 and $\cos\omega t$ from Equation 2.4.8 or homework Problem 2.10.
    1. Let $u(t) = U\sin\omega t$, $t > 0$, where U is a constant amplitude.

    [*Answer*: $x(t) = x_0e^{at} + \frac{bU\omega}{\omega^2+a^2}\left(e^{at} - \frac{a}{\omega}\sin\omega t - \cos\omega t\right), t \geq 0$]

    2. Let $u(t) = U\cos\omega t$, $t > 0$, where U is a constant amplitude.

    [*Answer*: $x(t) = x_0e^{at} + \frac{bU}{\omega^2+a^2}\left(ae^{at} - a\cos\omega t + \omega\sin\omega t\right), t \geq 0$]

13. Suppose that t is a real number representing time, and that ω is imaginary, $\omega = j\mu$, where μ is real. Use the results of homework Problem 2.1 and, from calculus, the definitions of the hyperbolic cosine, $\cosh\mu t$, and the hyperbolic sine, $\sinh\mu t$, to prove the following: $\cos\omega t = \cosh\mu t$, and $\sin\omega t/\omega = \sinh\mu t/\mu$.
14. For a mass-spring system with negligible damping, the appropriate ODE is Equation 1.10.1, $m\ddot{x} + kx = f_x(t)$. Consider the case of this mass-spring system having imposed initial conditions at time $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$, and then after $t = 0$ being subjected to constant force F, which we can express mathematically as $f_x(t) = FH(t)$, or equivalently, $f(t) = F$ for $t > 0$. In this application, unit-step function $H(t)$ is used to remind you that the Laplace transform is $L[f_x(t)] = F/s$. (In fact, force F could have been applied even before $t = 0$, but that would be irrelevant because the initial conditions establish the state of the system at $t = 0$, regardless of whatever force acts before $t = 0$.) Your task is to solve the problem by using Laplace transformation.
    1. First take the Laplace transform of the ODE, accounting for the ICs, and solve for the transform of the output to find

$$L[x(t)] \equiv X(s) = x_0 \frac{s}{s^2+\omega_n^2} + \frac{v_0}{\omega_n}\frac{\omega_n}{s^2+\omega_n^2} + \frac{F}{m}\frac{1}{s\,(s^2+\omega_n^2)}$$

In this equation, we define the *natural frequency of vibration*, $\omega_n = \sqrt{k/m}$, from Equation 1.10.3. (See Section 1-10 for an explanation of the physical significance of ω_n.) Show all of your work leading to your result, as if the correct answer above were not given.

2. In order to derive the inverse transform, $L^{-1}[X(s)] = x(t)$, you could, of course, find partial-fraction expansions for each of the three terms $X(s)$, then use the inverse transform of the fundamental transform pair Equation 2.2.9, and, after much algebra, finally derive an acceptable final algebraic equation for $x(t)$. But this would be equivalent to reinventing the wheel, because people have been evaluating exactly the same inverse transforms for the two centuries since Laplace developed the method. To preclude the necessity for starting from scratch every time the same Laplace transform appears, the technical community has assembled tables of Laplace transform pairs in many handbooks and textbooks, in symbolic software, and even on the Internet. Your assignment is to invert $X(s)$ and derive the complete algebraic equation for $x(t)$, $t \geq 0$. You can find explicitly in Section 2-4 two of the three transform pairs in $X(s)$, but you are required to **look up** the third pair in some source other than Chapter 2.

15. Apply MATLAB's `residue` operation to determine the partial-fraction expansion of $F(s) = \frac{3s^3-2s^2+s}{-2s^4+3s^3+4s^2+5s+6}$. Submit a copy of your MATLAB session, and also write out the partial-fraction expansion in equation form (as in the example below). The following example illustrates the `residue` operation for the polynomial ratio of Problem 2.9.2, $F(s) = \frac{2(s+1)}{s(s+3)(s+4)} = \frac{2(s+1)}{s^3+7s^2+12s}$. On the MATLAB command window, enter as arrays the coefficients of the numerator and denominator polynomials in descending powers of s. Then enter the `residue` command to calculate the residues, poles, and *direct term* (a constant or a term proportional to s raised to some positive power, which appears only if $m \leq n$) of the expansion:

```
>> num=2*[1 1];den=[1 7 12 0]; [resids poles dterm]=residue(num,den)

resids =

-1.5000

1.3333

0.1667

poles =

-4

-3

0

dterm =

[]
```

In equation form, $F(s) = \frac{-1.5}{s-(-4)} + \frac{1.3333}{s-(-3)} + \frac{0.1667}{s-0} = -\frac{3}{2}\left(\frac{1}{s+4}\right) + \frac{4}{3}\left(\frac{1}{s+3}\right) + \frac{1}{6}\left(\frac{1}{s}\right)$.

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3: MECHANICAL UNITS, LOW-ORDER MECHANICAL SYSTEMS, AND SIMPLE TRANSIENT RESPONSES OF FIRST ORDER SYSTEMS

3.1: Common Mechanical Units

3.2: Calculation of Mass from Measured Weight

3.3: Reaction Wheel

3.4: Simples Transient Responses of First Order Systems, First Order Time Constants and Settling Times

3.5: Aileron-Induced Rolling of an Airplane or Missile

3.6: Translational Spring and Viscous Damper (Dashpot)

3.7: More Examples of Damped Mechanical Systems

3.8: Chapter 3 Homework

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3.1: Common Mechanical Units

We shall evaluate mechanical systems numerically using three different systems of units that are common in the United States: *International System*, *SI* (based on kilogram **mass**, meter, and second; kg-m-s), *traditional aeronautical* (based on pound **force**, foot, and second; lb-ft-s), and *traditional structural* (based on pound **force**, inch, and second; lb-inch-s). Table 3.1.1 summarizes the units of mechanical quantities that arise most often in this book.

Table 3.1.1: Common units of mechanical quantities

Quantity \ Unit System	International System SI (kg-m-s)	Traditional aeronautical (lb-ft-s)	Traditional structural (lb-inch-s)
Mass (translational inertia), m	kilogram mass (kg)	slug = lb-s^2 /f	lb-s^2 /inch
Length, translational motion	meter (m)	foot (ft)	inch (in.)
Time, t	second (s)	second (s)	second (s)
Force, translational action	newton (N) = kg-m/s^2	pound force (lb)	pound force (lb)
Translational stiffness constant, k	N/m	lb/ft	lb/inch
Translational damping constant, c	N/(m/s) = N-s/m	lb/(ft/s) = lb-s/ft	lb/(inch/s) = lb-s/inch
Angle, rotational motion	radian (rad), which is dimensionless	radian (rad), which is dimensionless	radian (rad), which is dimensionless
Rotational inertia, J	kg-m^2	slug-ft^2 = lb-s^2 -ft	lb-s^2 -inch
Moment or torque, rotational action	N-m	lb-ft	lb-inch
Rotational stiffness constant, k_θ	(N-m)/rad = N-m	(lb-ft)/rad = lb-ft	(lb-inch)/rad = lb-inch
Rotational damping constant, c_θ	(N-m)/(rad/s) = N-m-s	(lb-ft)/(rad/s) = lb-ft-s	(lb-inch)/(rad/s) = lb-inch-s

SI is called an *absolute* system of units, and the other two are called *gravitational* systems. Absolute and gravitational systems differ fundamentally in their primary and derived units as defined in the context of Newton's 2nd law. In any absolute system, mass is a primary unit, along with length, and time, but force is a unit derived from those of mass, length, and time. Thus, the SI force unit, the newton (N), is precisely defined from $F = ma$, as a kilogram-meter/second^2 (kg-m/s^2). In any gravitational unit system, force is considered to be a primary unit, and mass is a derived unit. Thus, from m = F/a, the mass unit in the traditional aeronautical system, commonly called a slug, is precisely defined as a pound-second^2 /foot (lb-s^2 /ft). In all of these systems, the *weight* of an object on Earth (in the force units) is defined as the mass times the standard sea-level gravitational acceleration, which is denoted as g, so $W = mg$. Table 3.1.2 includes the relevant values of g.

Table 3.1.2: Standard sea-level gravitational acceleration

Quantity \ Unit System	International System SI (kg-m-s)	Traditional aeronautical (lb-ft-s)	Traditional structural (lb-inch-s)
Standard acceleration of gravity, g	9.807 m/s^2	32.17 ft/s^2	386.1 inch/s^2

It is often convenient in technical notation to use prefixes that indicate powers of ten. For example, a force of 456 700 N can also be written as 456.7 $\times$ 10^3 N, or in more economical form as 456.7 kN, where kN denotes a kilo-newton = 10^3 N. Table 3.1.3 includes a standard set of prefixes used in dynamics of mechanical systems. We also use the "e" notation that is becoming

standard for input to and output from computer programs. Thus, for examples, the 456.7 kN force can be expressed as 456.7e3 (or e03, e+03, e+003, *etc.*) N, and a length of 4.321 mm can be expressed as 4.321e–3 (or e–03, e–003, etc.) m

Table 3.1.3*: Prefixes for units*

Multiple	Prefix	Letter Prefix
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h
10	deka	da
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n

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3.2: Calculation of Mass from Measured Weight

This practically important process is straightforward for the traditional aeronautical and structural unit systems, but we can be confused when working in the SI system if the particular sensor used measures weight in the kilogram **force** (kg_f) unit. Let us consider an example: suppose we have a laboratory scale with two different modes of weight measurement, pound force (lb) and kilogram force (kg_f). We place an object on the scale with the display set to the lb mode, and the reading is W = 15.50 lb. We next change the display to the kg_f mode, and the reading changes to W = 7.031 kg_f. Let us calculate the mass in the three different unit systems using $m = W/g$.

$$\text{lb-inch-sec system: } m = \frac{15.50\text{lb}}{386.1\text{inch}/\text{sec}^2} = 0.04015\frac{\text{lb}-\text{sec}^2}{\text{inch}} \equiv 4.015\text{e}-2\frac{\text{lb}-\text{sec}^2}{\text{inch}}$$

$$\text{lb-ft-sec system: } m = \frac{15.50\text{lb}}{32.17\text{ft}/\text{sec}^2} = 0.4818\text{slug} \equiv 4.818\text{e}-1\,\text{slug} \equiv 4.818\,\text{deci -slug}$$

For the SI calculation, we need to recognize that the kg_f is **not** a consistent SI force unit, even though many weight scales sold and used in the United States have been calibrated in this unit since the 19th century. The only consistent SI force unit is the newton (N). So let us ignore the W = 7.031 kg_f reading for now and use, instead, the W = 15.50 lb reading in association with the well-known conversion, 1 lb = 4.448 N:

$$\text{SI system: } m = \frac{15.50\text{lb} \times 4.448\frac{\text{N}}{\text{lb}}}{9.807\text{m}/\text{sec}^2} = \frac{68.95\text{N}}{9.807\text{m}/\text{sec}^2} = 7.031\text{kg}$$

So, **in the SI system, the *mass* in kg has the same numerical value as the *weight* in kg_f**, which, in the future, will relieve us from having to calculate a conversion. This equality is precisely true only at elevations on Earth where g is exactly 9.807 m/s^2, but it is close enough for most engineering calculations using weights measured anywhere on Earth's surface.

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3.3: Reaction Wheel

Reaction wheels are used for attitude control of spacecraft. The principal component of a reaction wheel is a rotor, the "wheel," which is normally either not spinning or spinning at a constant nonzero rotational velocity. When a motor increases or decreases the rotational velocity from the normal value, a control moment is imposed upon the vehicle to which the reaction wheel is attached.

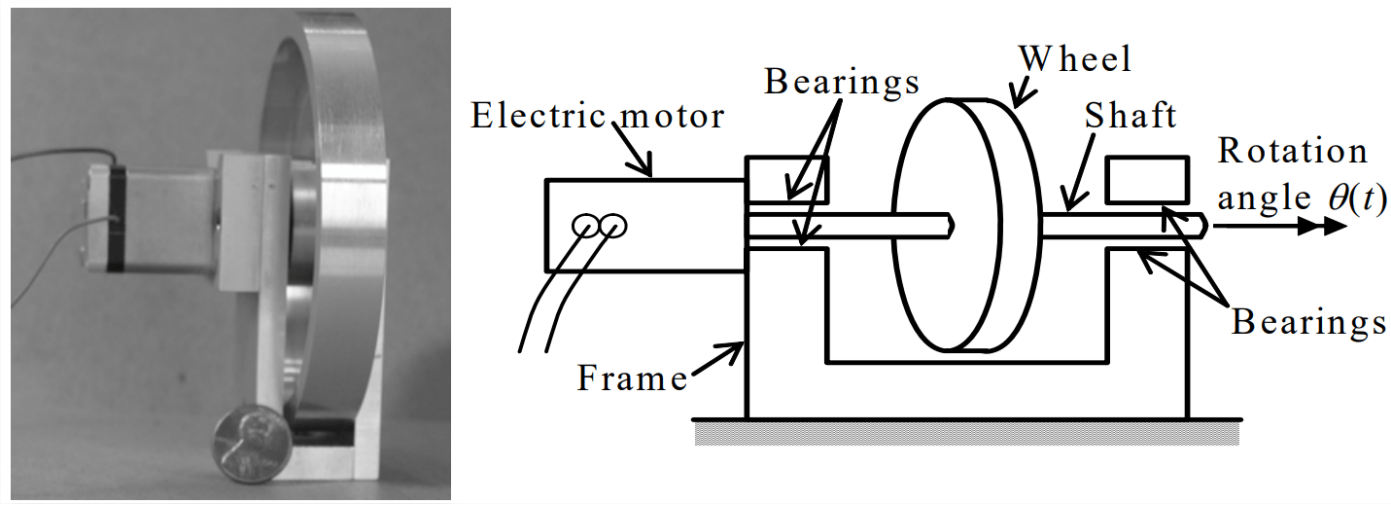


Figure 3.3.1: Laboratory reaction-wheel assembly: photograph at left, schematic drawing at right

The reaction-wheel assembly drawn in Figure 3.3.1 is a small laboratory device used in experimental research. Its basic functioning is the same as that of a flight-qualified reaction-wheel assembly. The rotating members are the wheel, the shaft, and the rotor of the motor. We denote their combined **rotational inertia** about the shaft axis as J, also called **moment of inertia**. The shaft is held in bearings, which are lubricated, but nevertheless impose a drag moment. The rotation angle of the rotating members is denoted $\theta(t)$, and it is represented graphically in Figure 3.3.1 by a double-headed arrow and the *right-hand rule* giving the polarity of rotation. The rotational velocity (angular velocity, spin rate) is denoted $\dot{\theta}(t) \equiv p(t)$, a symbol commonly used for angular velocity in aeronautical applications.

We will idealize the drag moment as having the linear viscous form $c_\theta \times p(t)$, where c_θ is the rotational viscous damping constant, which can be measured on real hardware. Also, we will denote as $M_m(t)$ the torque imposed on the rotating members by the electric motor, this torque being an independent input quantity.

The drawing below is the free-body diagram (FBD) of all moments (torques in this case) acting on the rotating members, with the bearing drag shown as a single torque, even though it is actually distributed over all bearing surfaces:

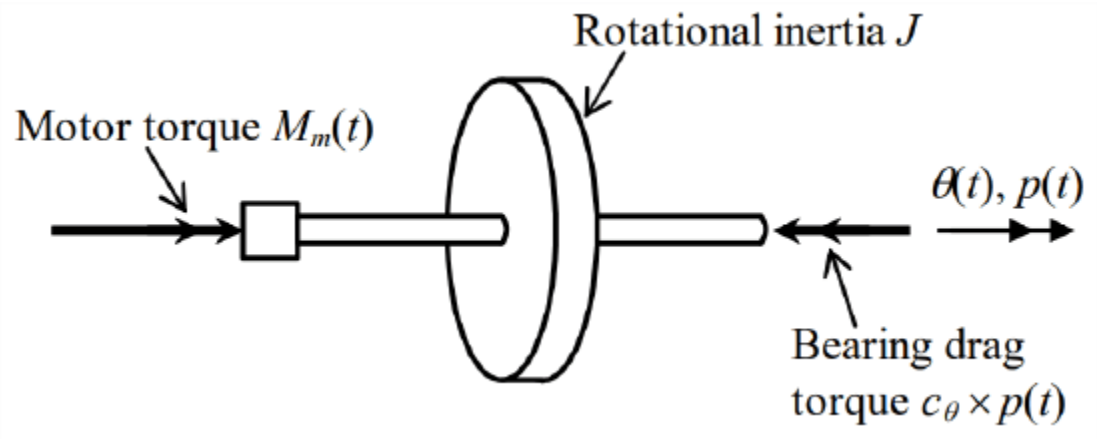


Figure 3.3.2

In order to derive the equation of motion, we write Newton's 2nd law for rotation about the shaft axis, based upon the FBD:

$$\Sigma(\text{ Moments })_{\text{about shaff axis}} = (\text{ rotational inertia }) \times (\text{ rotational acceleration })_{\text{about shaft axis}}$$

$$M_m(t) - c_\theta p = J\dot{p} \quad \Rightarrow \quad J\dot{p} + c_\theta p = M_m(t) \tag{3.3.1}$$

Converting Equation 3.3.1 into the standard 1st order form gives

$$\dot{p} - \left(-\frac{c_\theta}{J}\right) p = \frac{1}{J} M_m(t) \Rightarrow \dot{p} - ap = bM_m(t) \quad \text{where } a = -\frac{c_\theta}{J} \text{ and } b = \frac{1}{J} \tag{3.3.2}$$

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3.4: Simples Transient Responses of First Order Systems, First Order Time Constants and Settling Times

The adjective *transient* applies to system response that is dynamic for a finite time interval (often called the **settling time**), but is essentially static thereafter.

Consider the 1st order problem presented in Chapter 2,

$$\dot{x} - ax = bUe^{-wt}, x(0) = x_0, \text{ solve for } x(t), t > 0$$

with solution,

$$x(t) = x_0 e^{at} + \frac{bU}{a+w}\left(e^{at} - e^{-wt}\right), \text{ for } t \geq 0$$

If we let w = 0, then the input term becomes the step function, $U\left[e^{-wt}\right]_{w=0}$ = $UH(t) = U$ for t < 0, $H(t)$ being the Heaviside unit-step function defined in Section 2.4. So the problem and solution become:

$$\dot{x} - ax = bU, x(0) = x_0, \text{ solve for } x(t), t > 0$$

$$x(t) = \frac{bU}{(-a)}\left(1 - e^{-(-a)t}\right) + x_0 e^{-(-a)t}, \text{ for } t \geq 0 \tag{3.4.1}$$

In Equation 3.4.1, we write $a = -(-a)$ because usually a < 0 for engineering systems, as in Equation 3.3.3. We want to consider two special cases of solution Equation 3.4.1:

1. pure **initial condition response** for U = 0 and
2. pure **step response** for x_0 = 0.

Stable initial condition (IC) response

$$x(t) = x_0 e^{-(-a)t}, \text{ for } t \geq 0 \tag{3.4.2}$$

Define the 1st order-system *time constant*,

$$\tau_1 \equiv \frac{1}{(-a)} \tag{3.4.3}$$

For example, the time constant for the reaction wheel from Equation 3.3.3 is $\tau_1 \equiv \frac{J}{c_\theta}$. You should satisfy yourself, using Table 3.1.1 if necessary, that the quantity J/c_θ has the dimension of time (unit of second). With this definition of time constant τ_1, solution Equation 3.4.2 becomes:

$$x(t) = x_0 e^{-t/\tau_1} \equiv x_0 \exp\left(-t/\tau_1\right), \text{ for } t \geq 0 \tag{3.4.4}$$

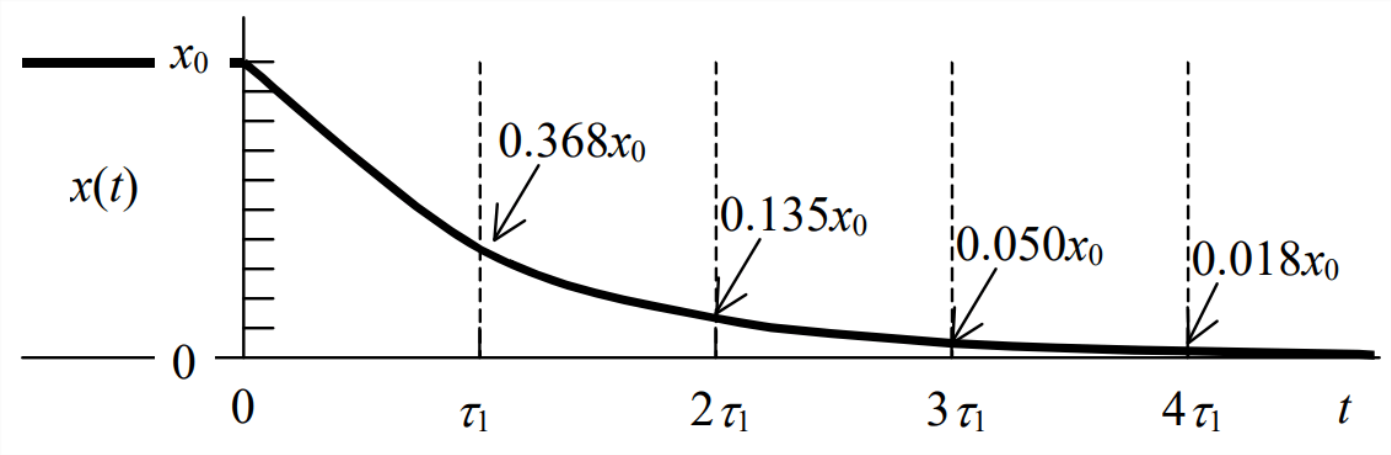


Figure 3.4.1: 1st order initial-condition response

At the *time constant*, $t = \tau_1$, the response has decayed to e^{-1} = 37% of the initial value. The other time "milestone" to which we shall often refer is $t = 4\tau_1$, which we call the *settling time*, at which time the response has decayed to e^{-4} = 2% of the initial value. For most practical engineering purposes, this settling time is considered to be the time required for the response essentially to reach its final steady-state value, which is x = 0 in this case of IC response. Mathematically, $x \to 0$ only as $t \to \infty$.

If the constant a is **positive**, then we write IC solution Equation 3.4.2 as $x(t) = x_0 e^{at}$. The mathematical response represented by this solution is unbounded: $x \to \infty$ as $t \to \infty$. In reality, no engineering variable will ever become infinite: as the variable becomes large, something in the system will fail or overload, or the system will become nonlinear, or the response will be limited by a governor, *etc*. Even though the actual response will not become infinite, an exponentially increasing linear mathematical response such as this is usually undesirable for practical purposes; an engineering system that exhibits this kind of response is considered to be engineering system with *unstable*. On the other hand, a **negative** constant a is *stable*.

The time constant τ_1 is such an important quantity for stable 1st order systems that we shall re-cast the standard 1st order system ODE in terms of τ_1, rather than constant a, using Equation 3.4.3. Thus, rather than analyzing Equation 1.2.1, $\dot{x} - ax = bu(t)$, hereafter we shall usually consider the following *standard **stable** form* for 1st order systems:

$$\dot{x} + \frac{1}{\tau_1} x = bu(t) \tag{3.4.5}$$

Stable step response

In Equation 3.4.1, we set x_0 = 0 and use time constant definition Equation 3.4.3 to obtain

$$x(t) = bU\tau_1 \left(1 - e^{-t/\tau_1}\right) \equiv bU\tau_1 \left[1 - \exp\left(-t/\tau_1\right)\right], \text{ for } t \geq 0 \tag{3.4.6}$$

Figure 3.4.2 is a graph of the *exponential rise* to a positive final value that is indicated in Equation 3.4.6.

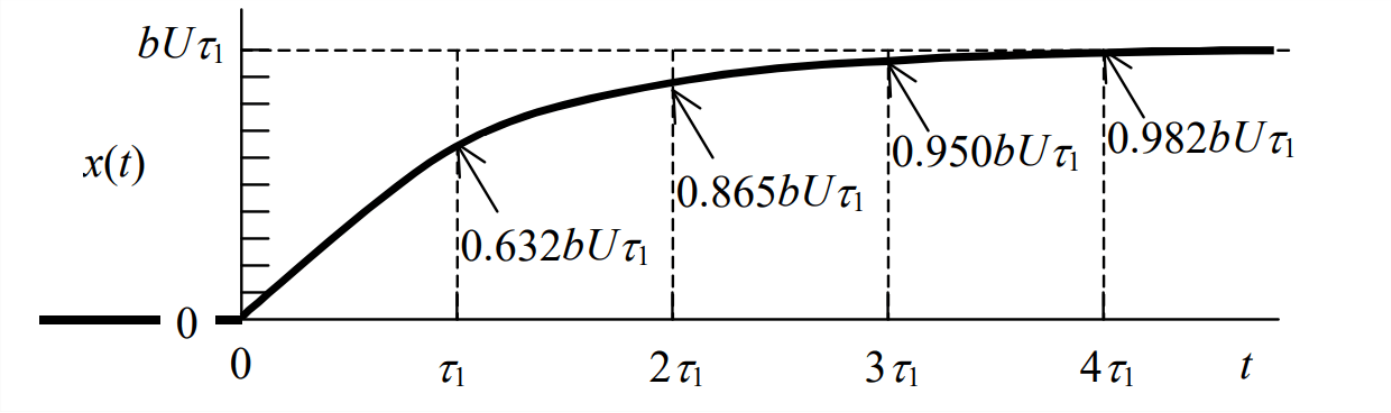


Figure 3.4.2: 1st order step response

The final value of unit-step response is $bU\tau_1$, and it is approached asymptotically as $t \to \infty$. At the *time constant*, $t = \tau_1$, the response has risen from the IC of zero to $1 - e^{-1}$ = 63%. At $t = 4\tau_1$, the settling time, the response has risen to $1 - e^{-4}$ = 98% of the final value.

Step response solutions such as Equation 3.4.6 are usually an approximation to the actual response since, in reality, a pure, discontinuous step change in a physical input quantity is rarely achievable. Nevertheless, step input is a sufficiently close approximation to many real physical inputs that step response solutions such as Equation 3.4.6 are close approximations to actual physical responses.

Consider again the reaction wheel of the previous section. Let us denote a step input from the motor as $M_m(t) = M \times H(t)$. Then ODE Equation 3.3.3 becomes

$$\dot{p} - ap = bM_m(t) = bMH(t) = bM \text{ for } t > 0, \text{ where } a = -\frac{c_\theta}{J} = -\frac{1}{\tau_1} \text{ and } b = \frac{1}{J} \tag{3.4.7}$$

So the time constant is $\tau_1 = -1/a = J/c_\theta$, and solution Equation 3.4.6 becomes

$$p(t) = bM\tau_1 \left(1 - e^{-t/\tau_1}\right) = \frac{1}{J} M \frac{J}{c_\theta} \left(1 - e^{-t/\tau_1}\right) = \frac{M}{c_\theta} \left(1 - e^{-t/\tau_1}\right) \text{rad/sec}, \text{ for } t \geq 0 \tag{3.4.8}$$

Note that in this case we adapt the “standard” mathematical solution Equation 3.4.6 to a particular physical problem. This approach is common in system dynamics. In other words, it is not always necessary to solve an ODE for every new physical problem; if a standard ODE solution has already been derived, you may just adapt that standard solution to the physical problem at hand, rather than re-deriving the ODE solution.

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3.5: Aileron-Induced Rolling of an Airplane or Missile

The principal source for the aerodynamic theory of this section is Nelson, 1989, pages 19-20, 153-156 and 250-260. The **aerodynamic rolling moment** on an airplane or missile is written in standard aeronautical notation as

$$L(t) = \frac{\partial L}{\partial \delta}\delta(t) + \frac{\partial L}{\partial p}p(t) \tag{3.5.1}$$

$$\equiv L_\delta \delta(t) + L_p p(t) \tag{3.5.2}$$

There is a notational ambiguity between moment $L(t)$ and the Laplace transform operator $L[f(t)]$ introduced in Chapter 2; such ambiguities are unavoidable in technical work, so you should become accustomed to them.

The roll (bank) angle (in radians) is $\theta(t)$, defined here as being positive clockwise as seen by an observer behind the vehicle, in other words, rightward rolling (Figure 3.5.1). In Equation 3.5.2, $\delta(t)$ is the input aileron deflection angle (in radians) to produce rolling, the right aileron being deflected upward and the left aileron being deflected downward to produce a positive roll rate, $p(t)$ $\equiv \dot{\theta}$ (rad/s).

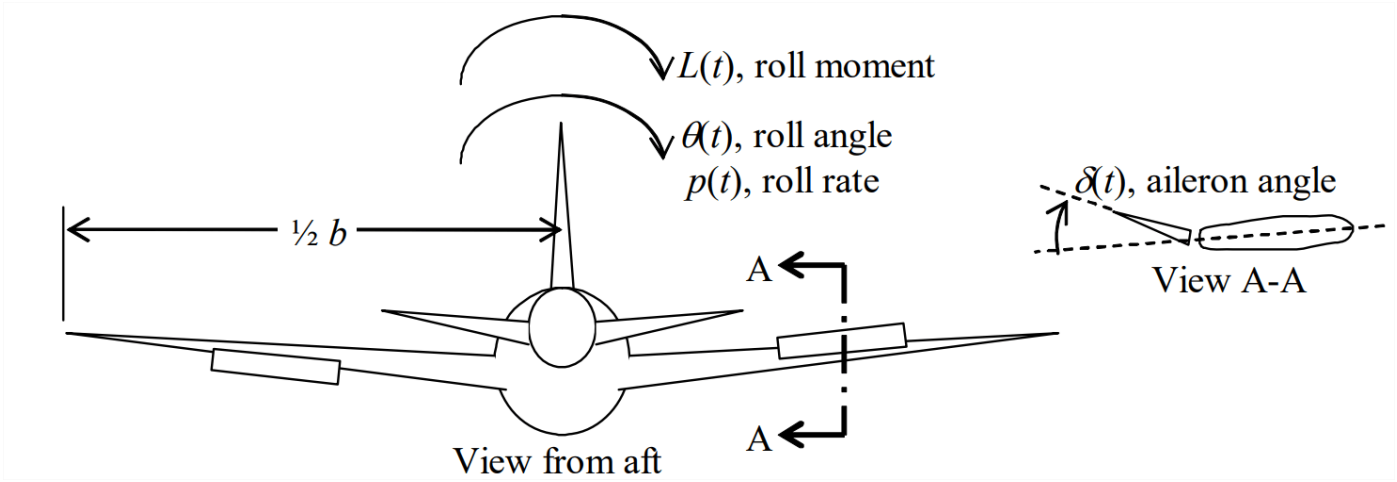


Figure 3.5.1: Aileron-induced rolling of an airplane

The dimensional aerodynamic derivatives in Equation 3.5.2 are related to dimensionless coefficients by:

$$L_\delta = \bar{q}SbC_\delta > 0 \quad \left(\frac{1b-ft}{rad} \text{ or } \frac{N-m}{rad}\right) \tag{3.5.3}$$

$$L_p = \bar{q}Sb\frac{b}{2V}C_p = \frac{\bar{q}Sb^2}{2V}C_p < 0 \quad \left(\frac{\text{1b}-\text{ft}}{\text{rad/sec}} \text{or } \frac{\text{N}-\text{m}}{\text{rad/sec}}\right) \tag{3.5.4}$$

The constants in Equations 3.5.3 and 3.5.4 are defined as:

- $\bar{q} = \frac{1}{2}\rho V^2$ is the free-stream dynamic pressure (lb/ft^2 or N/m^2)
- ρ is air density (slug/ft^3 or kg/m^3)
- V is the free-stream airspeed (ft/s or m/s)
- S is the wing planform area (ft^2 or m^2)
- b is the full wing span (ft or m)
- $C_\delta > 0$ is the dimensionless aerodynamic stability moment coefficient (rad^{-1}), rolling moment due to aileron deflection (a function of Mach number)
- $C_p < 0$ is the dimensionless aerodynamic stability moment coefficient (rad^{-1}), roll damping due to roll rate (a function of Mach number).

The term $b/2V$ in Equation 3.5.4, the damping moment due to roll rate, merits some explanation. The clockwise rolling velocity of the right wingtip is $p \times b/2$, so the additional angle of attack induced at the wingtip by the roll rate is $\tan^{-1}\left[\frac{pb/2}{V}\right] \approx \frac{pb}{2V}$, a small angle (the magnitude of the angle is exaggerated in Figure 3.5.2, just to improve clarity). Thus, $\frac{b}{2V}p(t)$ in Eqsuations 3.5.4 and 3.5.2 plays the same role as that of $\delta(t)$ in Eqs. 3.5.3 and 3.5.2.

Newton's 2nd law equation of motion for rolling is $J\ddot{\theta} \equiv J\dot{p} = L(t)$, in which J (slug-ft2 or kg-m2) is the rotational inertia of the vehicle about its rolling axis. This equation is based upon the assumption that the vehicle can only roll (i.e., no coupling with other degrees of freedom such as sideslip and yaw). Substituting Eqs. 3.5.2-3.5.4 into Newton's law gives:

$$J\dot{p} = L_\delta \delta + L_p p \Rightarrow J\dot{p} - L_p p = L_\delta \delta$$

$$\Rightarrow \quad \dot{p} + \left(\frac{-L_p}{J}\right) p = \left(\frac{-L_p}{J}\right)\left(\frac{L_\delta}{-L_p}\right)\delta(t) = \left(\frac{-L_p}{J}\right)\frac{2V}{b}\left(\frac{C_\delta}{-C_p}\right)\delta(t) \tag{3.5.5}$$

Writing Equation 3.5.5 in 1st order standard form Equation 3.4.8 gives [with b in Equation 3.4.8 temporarily replaced by B to reduce notational ambiguity]:

$$\dot{p} + \frac{1}{\tau_1}p = B\delta(t) \tag{3.5.6}$$

in which we have, respectively, time constant τ_1 and constant B:

$$\tau_1 = \frac{J}{-L_p} = \frac{2VJ}{\bar{q}Sb^2\,(-C_p)} = \frac{4J}{\rho VSb^2\,(-C_p)} > 0 \tag{3.5.7}$$

$$B = \left(\frac{-L_p}{J}\right)\frac{2V}{b}\left(\frac{C_\delta}{-C_p}\right) = \frac{1}{\tau_1}\frac{2V}{b}\left(\frac{C_\delta}{-C_p}\right) = \frac{\bar{q}SbC_\delta}{J} > 0 \tag{3.5.8}$$

Example 3.5.1

Use the results of this section and the general 1st order step response Equation 3.4.9 to calculate rolling response to a $\delta(t) = 2.5^\circ \mathrm{H}(t)$ step aileron input for a hypothetical medium-sized civilian transport airplane. The following are representative data: rotational inertia about the rolling axis is J = 4.0e5 slug-ft^2, wing planform area is S = 1 100 ft^2 , and wing span is b = 90 ft; for flight at 10 000 ft altitude and free-stream airspeed V = 350 ft/s, aerodynamic dimensionless rolling-moment coefficients are C_δ = 0.061 per radian and C_p = -0.34 per radian. At 10 000 ft altitude, standard air density is ρ = 0.001 755 slug/ft^3 (Nelson, 1989, p. 248).

Solution

$$\bar{q} = \frac{1}{2}\rho V^2 = \frac{1}{2} \times 1.755e-3\frac{\text{slug}}{\text{ft}^3} \times \left(350\frac{\text{ft}}{\text{s}}\right)^2 = 107.5\frac{\text{lb}}{\text{ft}^2}$$

$$L_p = \frac{\bar{q}Sb^2}{2V}C_p = \frac{107.5\text{lb/ft}^2 \times 1,100\text{ft}^2 \times (90\text{ft})^2}{2 \times 350\text{ft/s}} \times \left(-0.34\text{rad}^{-1}\right) = -4.65\text{e}5\frac{\text{lb}-\text{ft}}{\text{rad/s}}$$

$$\tau_1 = \frac{J}{-L_p} = \frac{4.0\text{e}5\text{lb}-\text{s}^2-\text{ft}}{4.65\text{e}5\text{lb}-\text{s}-\text{ft/rad}} = 0.86\text{s}$$

(**Comments regarding units**: Angle is a dimensionless quantity, so the radian, which is the fundamental metric of angle, is unitless. In the calculation above for L_p, all of the units shown, including radian, balance in the final result to give the correct resulting unit, $\frac{\text{lb}-\text{ft}}{\text{rad/s}}$; it is important to recognize this unit as opposed, for example, to $\frac{\text{lb}-ft}{\text{deg}/s}$, which would be the case if C_p were given per degree. However, in calculations that produce results which are not angular quantities, such as τ_1 above, the radian must be recognized as unitless, and it should not appear in the units of the is final numerical result. This a general characteristic that can confuse us when we check to make certain that units of numerical answers are correct.)

$$B = \frac{1}{\tau_1}\frac{2V}{b}\left(\frac{C_\delta}{-C_p}\right) = \frac{1}{0.86\text{s}} \times \frac{2 \times 350\text{ft/s}}{90\text{ft}} \times \frac{0.061\text{rad}^{-1}}{0.34\text{rad}^{-1}} = 1.62\text{s}^{-2}$$

From Equation 3.4.9, the roll-rate response to the 2.5° step input is

$$p(t) = BU\tau_1 \left[1 - \exp\left(-t/\tau_1\right)\right] = 1.62s^{-2} \times 2.5\text{deg} \times 0.86s \times \left(1 - e^{-t/(0.86\text{s})}\right)$$

$$p(t) = 3.5[1 - \exp(-1.16t)]\frac{\text{degrees}}{s}$$

The time constant is τ_1 = 0.86 s, and the steady-state roll rate of 3.5 deg/s is essentially achieved in the settling time of $4\tau_1$ = 3.44 s.

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3.6: Translational Spring and Viscous Damper (Dashpot)

The linear translation spring was introduced in Section 1.9 as the mechanical element connecting a mass to a rigid wall; for that simple case, the end of the spring attached to the wall was stationary. Now, let us consider the more general case in which both ends of the spring can move, as indicated in Figure 3.6.1. The force $f_x(t)$ required to deform the spring by the amount $x_b(t) - x_a(t)$, relative to the undeformed length of the spring, is proportional to the deformation. With the constant of proportionality denoted as k, the more general spring law is

$$f_x(t) = k\,[x_b(t) - x_a(t)] \tag{3.6.1}$$

The standard sign convention defines stretching of the spring as positive, $x_b - x_a$ > 0, corresponding to tensile force, $f_x(t) > 0$; accordingly, compression is negative in the standard convention. Equation 3.6.1 clearly accommodates both tension and compression of the spring. Occasionally, however, it is convenient to define compression as positive, as is illustrated in Example 3-4 in the next section.

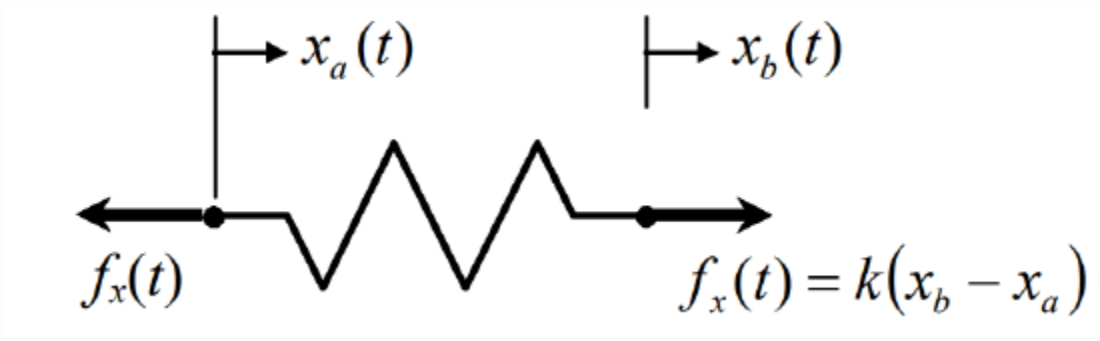


Figure 3.6.1: Ideal linear spring with stiffness constant k

With the spring end forces equal and opposite as shown in Figure 3.6.1, we have neglected the *inertial force* of the spring. Inertial forces are terms such as $m \times a$, mass times acceleration, in Newton's 2nd law. In this case, the inertial force of the spring is $m_s \times \ddot{x}_s$, in which m_s is the mass of the spring, and $\ddot{x}_s$ is acceleration of the spring's center of mass. So neglecting inertial forces means we assume they are negligibly small in comparison with the force of Equation 3.6.1. Notice that this does **not** strictly mean we are neglecting the mass, because even a tiny mass, when multiplied by a large acceleration, can produce a non-negligible inertial force. It is common to neglect the inertial forces of springs, but we should remember when doing so that we are really assuming both the inertias are small **and** the accelerations are sufficiently slow.

A dashpot (mechanical viscous damper) is a discrete component that resists translational velocity. An idealized linear, viscous damper is represented graphically by a piston-cylinder assembly, Figure 3.6.1. This device opposes relative velocity between piston and cylinder. With the forces on the dashpot defined to be positive in tension, as shown on Figure 3.6.1, the linear damping law is

$$f_x(t) = c\,[\dot{x}_c(t) - \dot{x}_p(t)] \tag{3.6.2}$$

in which c is the damping constant. The damping constant plays the same role for a dashpot here that it plays in Chapter 1 [Figure 1.3.1 and Equation 1.3.4] for a block sliding on a viscous liquid film. If we were to define the forces to be positive in compression, then we would have to reverse the signs in the damping law, Equation 3.6.2; it is sometimes appropriate to do so, as in Example 3-4 in the next section.

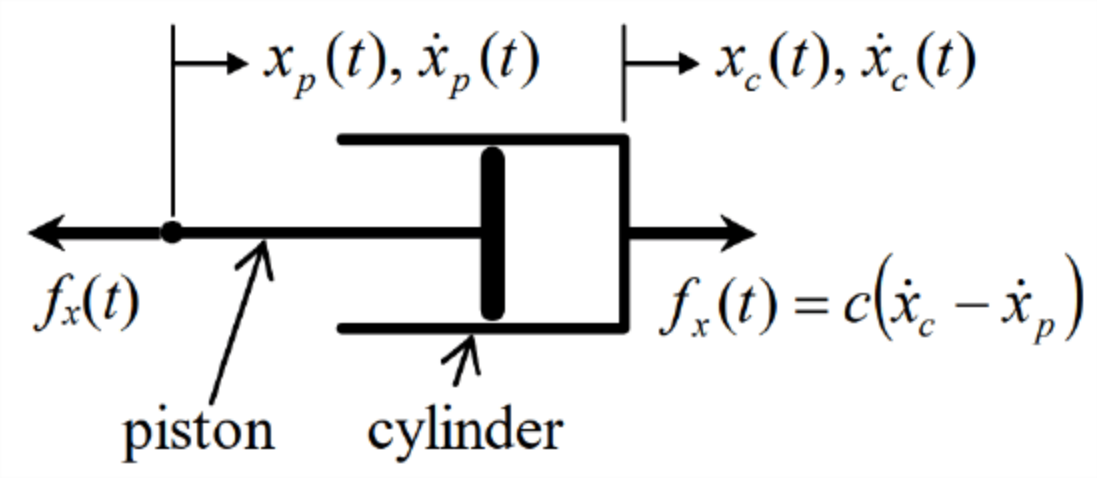


Figure 3.6.2: Ideal linear, viscous dashpot with damping constant c

With the cylinder and piston forces equal and opposite as shown in Figure 3.6.2, we have neglected the inertial forces of the cylinder and piston. It is common to neglect the *inertial forces* of dashpots, but we should remember when doing so that we are really assuming that both the inertias are small **and** the accelerations are sufficiently slow. On the other hand, in Example 3-4 of the next section, inertial forces of a piston-cylinder assembly are **not** neglected.

The idealized dashpot model of Figure 3.6.2 and Equation 3.6.2 is convenient for linear mathematical analysis because of its simplicity. It is assumed to resist motion between piston and cylinder by means of the drag force produced by a viscous fluid boundary layer between the two solid objects. We can use the simple dashpot to model approximately real devices such as door dampers and shock absorbers. However, it is **not** a precise model for most real damping devices. In most real translational dampers, a piston forces fluid through an orifice within a cylinder, and this generates a major portion of the resisting force. Unfortunately for the purposes of linear mathematical analysis, the resisting force is a complex and highly nonlinear function of the relative motion between piston and cylinder.

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3.7: More Examples of Damped Mechanical Systems

Let us consider next some combinations of dashpots with springs and masses. In each case, we will draw the physical device, draw appropriate free-body diagrams, and then derive the equations of motion. Force $f_x(t)$ is considered to be an independent input quantity in all of these examples.

Example 3.7.1

Newton's 2nd law for translation of the mass: $m\ddot{x} = f_x(t) - c\dot{x} - kx$

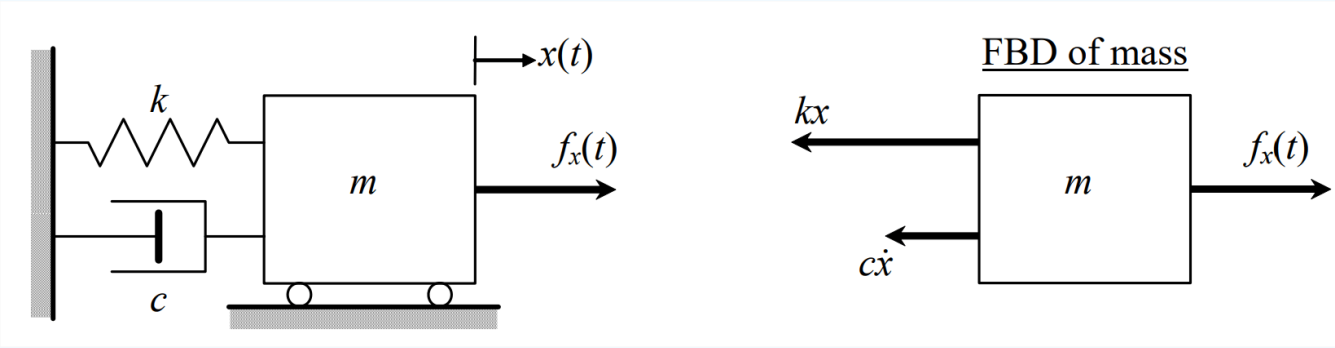


Figure 3.7.1: Mass-damper-spring system

Solution

$$\Rightarrow \quad \text{ODE} \ : \quad m\ddot{x} + c\dot{x} + kx = f_x(t) \tag{3.7.1}$$

Example 3.7.2

This combination is very similar to the mass-damper-spring system, except the inertial force of the mass (now considered just a rigid link) is assumed to be negligible.

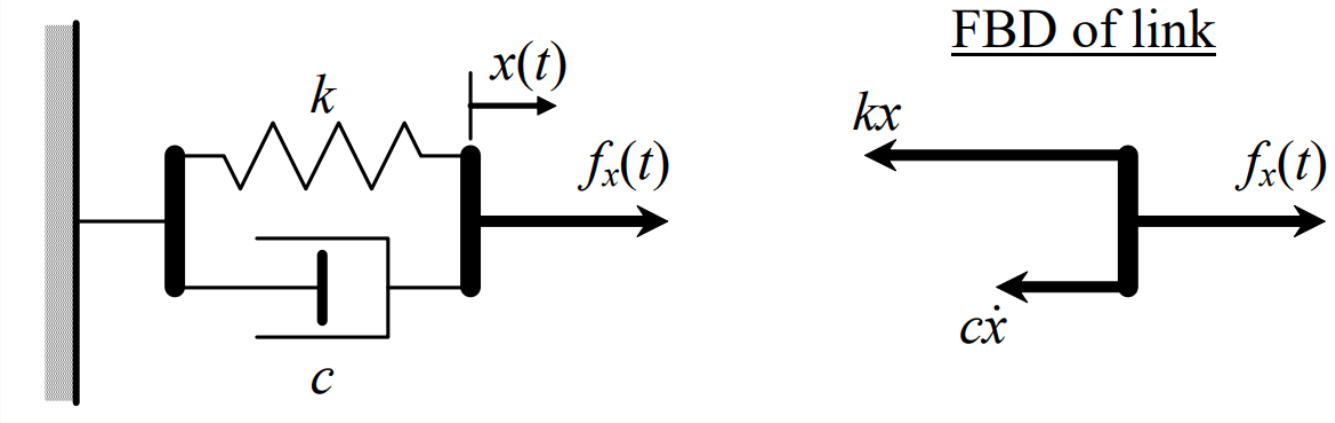


Figure 3.7.2: Parallel damper-spring system

Solution

Newton's 2nd law for translation of the link: $m\ddot{x} \cong 0 = f_x(t) - c\dot{x} - kx$

$$\Rightarrow \ \text{ODE} : c\dot{x} + kx = f_x(t) \tag{3.7.2}$$

Note that Equation 3.7.2 is a typical 1st order LTI ODE, but now the unknown quantity is translation (position) $x(t)$ of the rigid link.

Example 3.7.3

For the examples above, it is not necessary to draw separate FBDs of the spring and of the dashpot's piston and cylinder. However, it **is** necessary to draw separate FBDs for systems such as that of Figure 3.7.3, where dashpot and spring are arranged in *series*, not in *parallel*, and, for variety and generality, the inertial forces of the dashpot's piston and cylinder are considered to be significant, not negligible. Take notice of the component interaction forces f_c and f_k; note especially that piston-cylinder interaction force f_c is shown as an equal and opposite pair, as required by Newton's 3rd law.

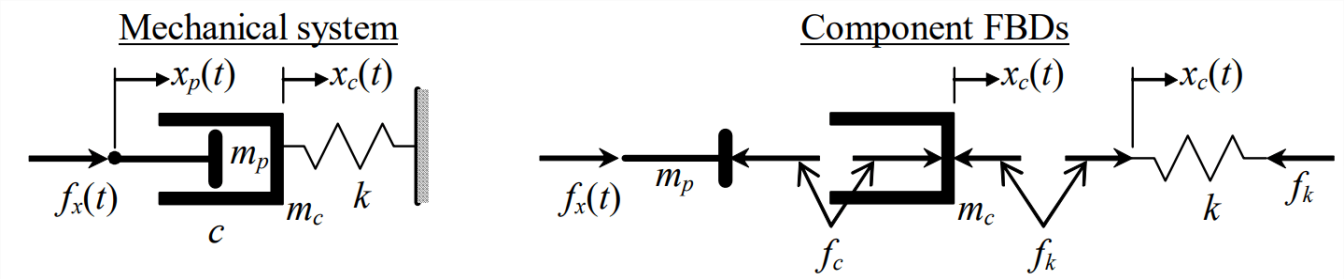


Figure 3.7.3: Example series mass-dashpot-spring

Newton's 2nd law for translation of the dashpot piston: $m_p\ddot{x}_p = f_x(t) - f_c$

Newton's 2nd law for translation of the dashpot cylinder: $m_c\ddot{x}_c = f_c - f_k$

Damper law: $f_c = c\,(\dot{x}_p - \dot{x}_c)$

Spring law: $f_k = kx_c$

Solution

Combining equations of motion and damper/spring laws gives:

$$m_p\ddot{x}_p + c\dot{x}_p - c\dot{x}_c = f_x(t) \tag{3.7.3}$$

$$m_c\ddot{x}_c - c\dot{x}_p + c\dot{x}_c + kx_c = 0 \tag{3.7.4}$$

Equations 3.7.3 and 3.7.4 are a pair of *coupled* 2nd order ODEs in the **two** dependent variables $x_p(t)$ and $x_c(t)$. Thus, the system of Figure 3.7.3 is a 4th order system. The dynamic response of a particular class of 4th order systems is discussed Chapter 12. However, this book does not address the dynamic response of **general** 4th order and higher-order systems; that is a subject of more advanced textbooks such as Ogata, 1998, Chapter 10.

Example 3.7.4

For both of the series damper-spring combinations in Figure 3.7.4, we specify the known input quantity to be translation $x_i(t)$, which is produced by an idealized motor that is capable of dictating a commanded displacement regardless of the force required. The output quantity in each case is translation $x_0(t)$. The damper and spring masses are assumed here to be negligible, which cannot be valid for all practical circumstances; therefore, these are idealized, not real, mechanical systems. But these idealized systems are worth studying because they have significant dynamic characteristics that will be the subjects of subsequent parts of this book.

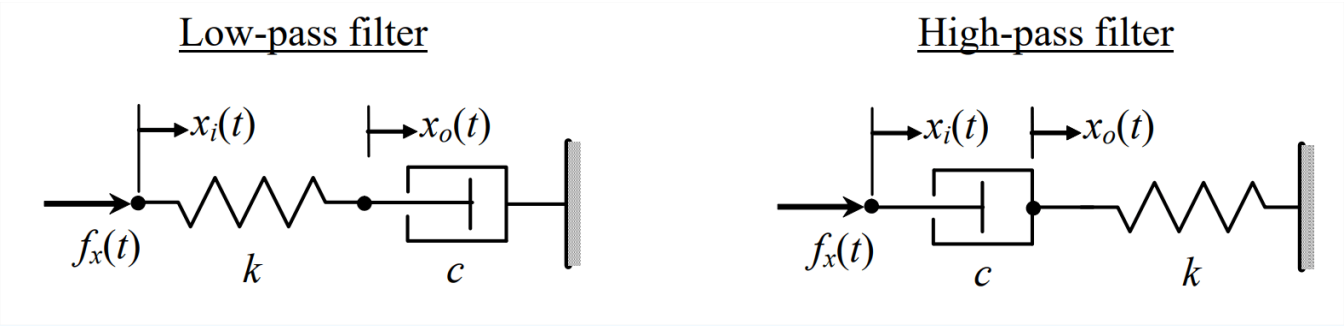


Figure 3.7.4: Series damper-spring idealized mechanical filters

Solution

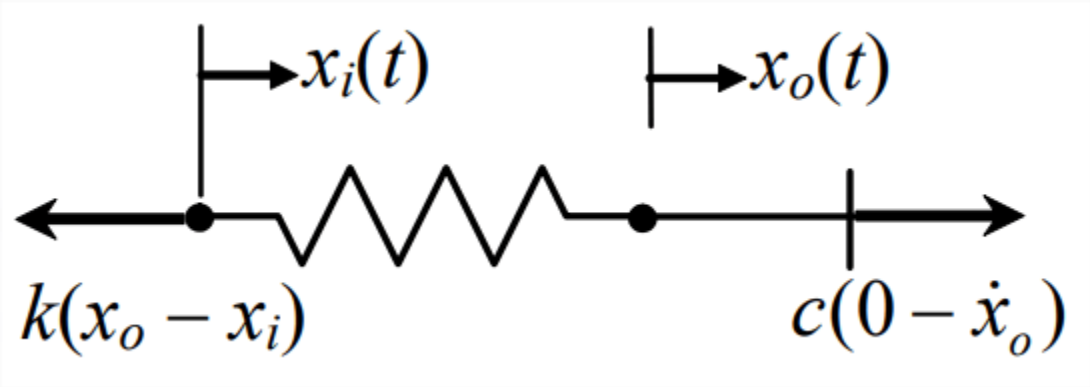


Figure 3.7.5: FBD of series damper-spring idealized mechanical filters

The governing ODE for the low-pass filter, with spring and damper masses idealized to be negligible, is derived easily from the FBD at right of the series combination consisting of the spring and the damper piston: $k\,(x_o - x_i) = c\,(0 - \dot{x}_o)$. Therefore the ODE for the low-pass filter is

$$\dot{x}_o + \frac{1}{\tau_1} x_o = \frac{1}{\tau_1} x_i(t), \quad \tau_1 \equiv \frac{c}{k} \tag{3.7.5}$$

The mathematical derivation and description of low-pass-filter characteristics will follow in later chapters, but it is appropriate here to describe the practical nature of the low-pass filter with use of the low-pass filter drawing in Figure 3.7.4 and physical intuition. Imagine that the input translation $x_i(t)$ is a steady oscillation at a fixed frequency. If that frequency is very low, so that translational velocity is low, then the damper resisting force $c\dot{x}_o$ will be very small and, consequently, both the spring and the damper piston will move slowly back and forth in unison by about the same distance as the input translation, $x_o(t) \approx x_i(t)$. On the other hand, if the frequency of oscillation is very high, then the damper resisting force $c\dot{x}_o$ will be great, so the input oscillatory translation $x_i(t)$ will compress and stretch the spring but the damper piston will act almost like a rigid wall, *i.e.*, $x_o(t) \approx 0$. For frequencies of oscillation between very low and very high, our intuition suggests that the output motion amplitude [the maximum of oscillatory $x_o(t)$] will be between the two extremes of the input motion amplitude and zero motion. The *low-pass filter* is so named because it "passes" through, without much alteration, low-frequency inputs, but it "blocks" high-frequency inputs.

Derivation of the governing ODE for the damper-spring high-pass filter is left for homework Problem 3.7. The result is:

$$\dot{x}_o + \frac{1}{\tau_1} x_o = \dot{x}_i(t), \quad \tau_1 \equiv \frac{c}{k} \tag{3.7.6}$$

As is done above for the damper-spring low-pass-filter, it is appropriate to describe the practical nature of the high-pass filter with use of the high-pass filter drawing in Figure 3.7.4 and physical intuition. Imagine that the input translation $x_i(t)$ is a steady oscillation at a fixed frequency. If that frequency is very low, so that translational velocity is low, then the damper resisting force $c\,(\dot{x}_i - \dot{x}_o)$ will be very small; consequently, the damper piston will move slowly back and forth within the cylinder by about the same distance as the input translation; but very little force will be imposed upon the cylinder and spring,

so the spring will be deformed hardly at all, $x_o(t) \approx 0$. On the other hand, if the frequency of oscillation is very high, then the damper resisting force $c\,(\dot{x}_i - \dot{x}_o)$ will be great, so the piston and cylinder will appear to become a single rigid body, and most of the input translation will be transmitted through the damper, $x_o(t) \approx x_i(t)$, to compress and stretch the spring. For frequencies of oscillation between very low and very high, our intuition suggests that the output motion amplitude will be between the two extremes of zero motion and the input motion amplitude. The *high-pass filter* is so named because it "blocks" low-frequency inputs, but it "passes" through, without much alteration, high-frequency inputs.

The damper-spring high-pass filter system is said to have *right-hand-side* (*RHS*) *dynamics* due to the explicit derivative of the input translation, $\dot{x}_i$, on the RHS of ODE Equation 3.7.6. Therefore, Equation 3.7.6 does not have the standard stable 1st order form of Equation 3.4.8, $\dot{x} + (1/\tau_1)\,x = bu(t)$, with the input $u(t)$ on the RHS, but none of its derivatives. For standard forms such as Equation 3.4.8, many readily available mathematical response solutions exist, e.g., step response Equation 3.4.9. But that is not the case for non-standard forms, so the non-standard character might be considered inconvenient. However, there is a process at converts non-standard ODE Equation 3.7.6 into standard form[1]. We define the difference variable $x_d(t)$:

$$x_d(t) = x_o(t) - x_i(t) \tag{3.7.7}$$

Differentiating Equation 3.7.7 and substituting the result into Equation 3.7.6 leads to the following standard-form ODE for $x_d(t)$ plus the auxiliary equation required to retrieve the desired output $x_0(t)$ from solution $x_d(t)$ of the ODE:

$$\dot{x}_d + \frac{1}{\tau_1} x_d = -\frac{1}{\tau_1} x_i(t), \quad x_o(t) = x_d(t) + x_i(t) \tag{3.7.8}$$

[1]The process shown here for a 1st order ODE is the simplest form of a more general process that can be applied to system ODEs of any order having RHS dynamics; see the textbook by Brogan, 1974, pages 174- 177.

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3.8: Chapter 3 Homework

1. Consider the reaction-wheel assembly described in Section 3.3. The rotor has been carefully machined to have rotational inertia J = 2.56e-3 lb-s^2-inch. We wish to determine the viscous damping constant c_θ of the bearings by indirect experimental measurement. We feed electric current into the motor and spin up the rotor to a high speed. Then we shut off the motor, allowing the wheel to spin down freely. With an optical tachometer, we measure the spin-down rotational speed at one instant to be 4 000 revolutions per minute (rpm). Exactly 20.0 seconds later, we measure the speed to be 1 010 rpm. From these data, calculate c_θ in consistent lb-s-inch units.
2. **Given**: an LTI reaction-wheel assembly, with unknown rotational inertia J and unknown rotational viscous damping constant c_θ associated with drag of the bearings. Your task is to calculate J and c_θ from experimental measurement. Starting with the wheel at rest (motionless), a motor torque in the form of a step is imposed upon the wheel, M_{m}(t)=M \times H(t), where M = 1.5 N-m. The resulting spin rate $p(t)$ of the wheel is measured, in radians/second, and its time history for the first 10 seconds of spin-up is recorded on the following graph.

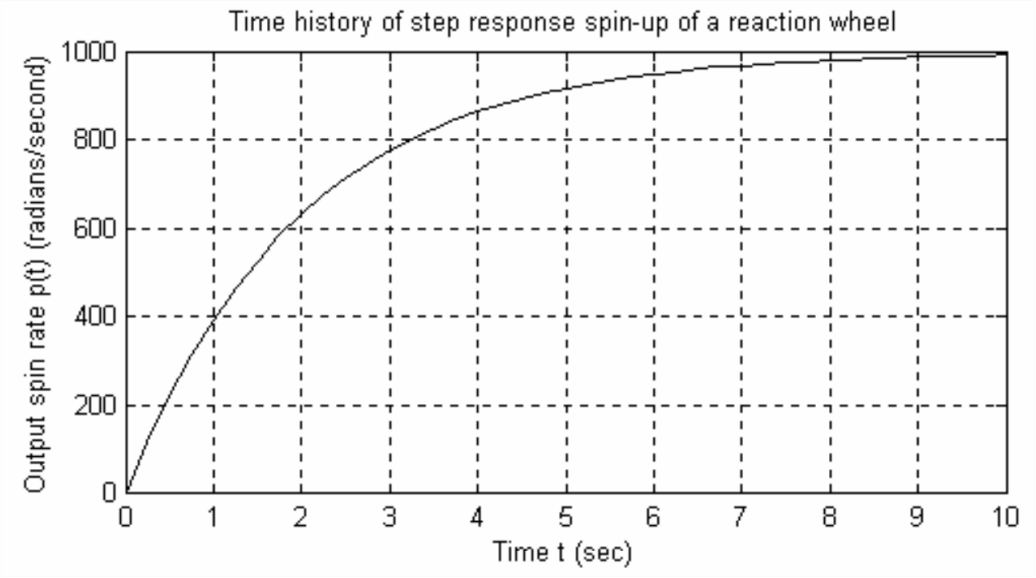


Figure 3.8.1

Use the data from the graph to calculate (with as much precision as possible, given the graphical nature of the data) the values of J and c_θ in consistent SI units.

3. The weight of an ocean surface ship is denoted as W, and the acceleration of gravity in consistent units is g. The resistance to low-velocity motion of the ship is modeled as being proportional to velocity, with viscous damping constant c. The ship is initially at rest when, at time t = 0, a tugboat begins pushing it with **constant** force F. Write algebraic equations (in terms of the given parameters, all assumed to be in consistent units) for velocity $v(t)$ and distance traveled $x(t)$.
4. Nominal data (Nelson, 1989, p. 260) for the Boeing 747 civilian transport airplane are: rotational inertia about the rolling axis J = 18.2e6 slug-ft^2, wing planform area S = 5 500 ft^2, and wing span b = 195.68 ft. For flight at Mach number 0.25 and at sea level altitude, nominal aerodynamic dimensionless coefficients relevant to uncoupled rolling are C_δ = 0.0461 per radian and C_p = –0.450 per radian. At sea level, air density is ρ = 0.002377 slug/ft^3, and Mach number 0.25 corresponds to free-stream airspeed V = 279.0 ft/s.
    1. Calculate the dimensional aerodynamic derivatives L_δ (in lb-ft/rad) and L_p (in lb-ft per rad/sec), then calculate the 1st order system time constant τ_1.
    2. Suppose that the 747 is in level flight with zero initial roll rate, $p(0)$ = 0, when the pilot cycles the ailerons through one complete sinusoid with amplitude 10° over a period of 5 seconds: $\delta(t) = 10^\circ \sin(0.4\pi t)$, $0 \leq t \leq 5$ s. Calculate the roll rate p (in degrees per s) at the end of the 5-second period. Do not develop any new theoretical solutions for this problem; just adapt to this problem the result of Problem 1.5.1. Your intuition might suggest that roll rate is zero at the end of the aileron cycling, since the total aileron “impulse” is zero, but you should find that roll rate is not zero.
5. The A-4 Skyhawk was a durable and versatile small (17.5-24.5 klb) fighter-bombertrainer that first flew in 1954 and was still used in military operations, with updated technology, in the 1990s (Lightbody et al., 1990, pp. 104-111). The relevant mass and geometry data for one model are: rolling rotational inertia J = 8 090 slug-ft^2; wing planform area \(S) = 260 ft^2; wing

span b = 27.5 ft. A flight test of this airplane is conducted at sea level and Mach number 0.4, for which the air density is ρ = 0.002377 slug/ft^3 and the free-stream velocity is V = 446.6 ft/s. Starting from straight and level flight, the pilot at time t = 0 sec abruptly actuates the ailerons to +5° deflection, producing a step input that rolls the airplane clockwise. The resulting roll rate is sensed by a rate gyroscope, digitized, and recorded. The data are analyzed, and it is found that the following equation fits the data very well: $\dot{\theta}(t) = p(t) = 50.0\,(1 - e^{-1.667t})$ degrees/s.

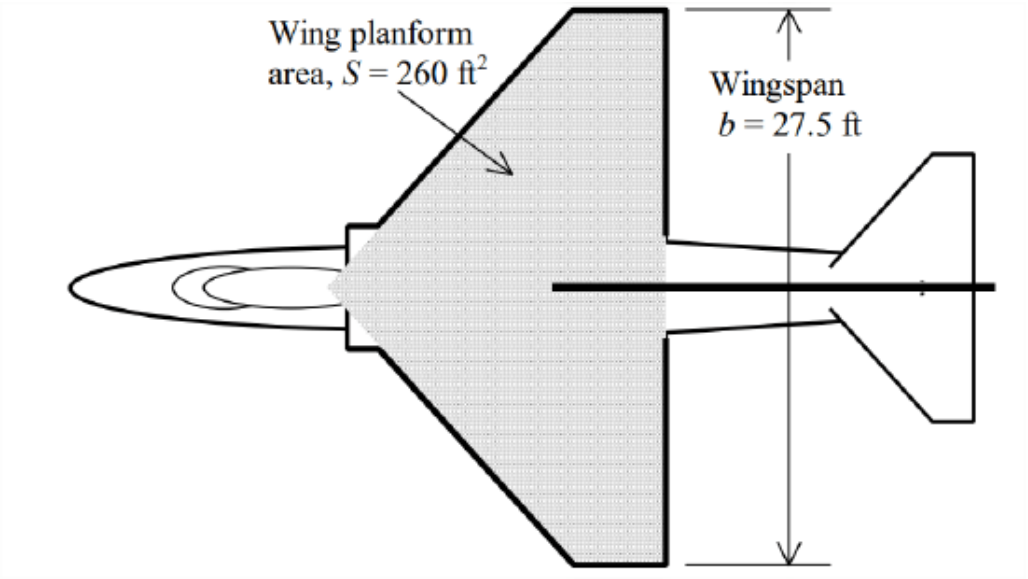


Figure 3.8.2: Planform sketch of A4D fighter airplane

1. Use the measured roll rate to perform *parameter identification*, specifically to determine values for the two roll aerodynamic stability coefficients C_δ and C_p, both dimensionless. Begin by calculating time constant τ_1 of the response, then use that to calculate aerodynamic derivative L_p, then use that to calculate C_p. Having these values, you can now use the steady-state roll rate to calculate C_δ. Check the units of your calculations to make certain that your final calculated coefficients are, in fact, dimensionless.
2. Given that $\theta(0)$ = 0 (level flight), derive an equation for $\theta(t)$ in degrees, $t \leq 0$. Evaluate that equation to calculate the bank angles at $t = \tau_1$ (one time constant) and at $t = 4\tau_1$ ($\approx$ steady-state roll rate).

6. Sketch (carefully) the appropriate free-body diagrams for the mechanical system drawn below. The dashpot's damping coefficient is c, its cylinder is fixed to the wall, and its piston has mass m_p. Next, use the FBDs and appropriate linear laws for the spring and dashpot to derive the ODEs that describe the motion of this system in terms of dependent variables $x_p(t)$ and $x(t)$. Note that the spring **stretch** is $x - x_p$, so the **tensile** force developed in the spring is $k\,(x - x_p)$. Do not neglect the inertial forces of the dashpot's piston and of the block with mass m, which rolls without friction. Force $f_x(t)$ is an independent input quantity.

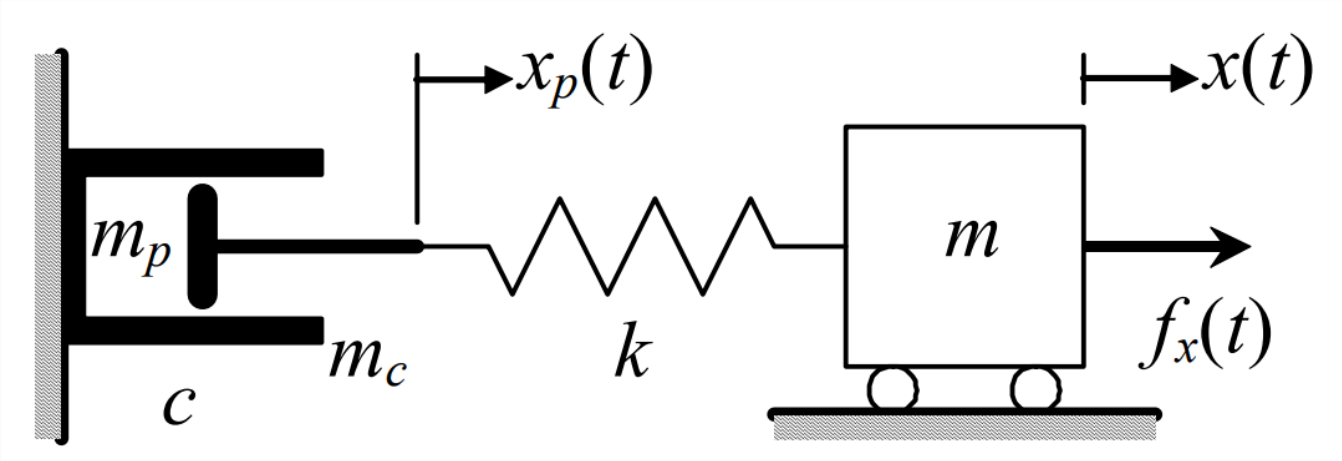


Figure 3.8.3: Series mass-dashpot-spring system

7. The LTI damper-spring high-pass filter of Figure 3.7.4 is repeated below. In this problem, neglect inertial forces of the damper and the spring. Input $x_i(t)$ is translation of the damper piston, generated by some displacement-controlling motor. Output $x_0(t)$ is translation of the damper cylinder and spring end. Sketch an appropriate free-body diagram (or more than

one), then write equilibrium equation(s) from which you derive the ODE $\tau_1 \dot{x}_o + x_o = \tau_1 \dot{x}_i$, which has right-hand-side (RHS) dynamics. Express time constant τ_1 in terms of damper constant c and spring constant k. (Hint: recognize that piston force $f_x(t)$ shown is a **dependent** variable, related to independent input $x_i(t)$ and output $x_0(t)$, but $f_x(t)$ should not appear in the equation of motion.)

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4: FREQUENCY RESPONSE OF FIRST ORDER SYSTEMS, TRANSFER FUNCTIONS, AND GENERAL METHOD FOR DERIVATION OF FREQUENCY RESPONSE

4.1: Definition of Frequency Response

4.2: Response of a First Order System to a Suddenly Applied Cosine

4.3: Frequency Response of the First Order Damper-Spring System

4.4: Period, Frequency, and Phase of Periodic Signals

4.5: Derivation of the Complex Frequency-Response Function - Easy derivation of the complex frequency-response function for standard stable first order systems.

4.6: Transfer Function - General Definition

4.7: Frequency-Response Function from Transfer Function

4.8: Chapter 4 Homework

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4.1: Definition of Frequency Response

We now consider system excitation that varies with time sinusoidally, as either $\sin \omega t$ or $\cos \omega t$, and persists for an indefinitely long duration. The frequency of excitation is ω radians/s, or $\omega/2\pi$ Hz (for hertz, which means cycles/s, named for German physicist and engineer Heinrich Rudolf Hertz, 1857-1894). After any transients due to initial conditions have decayed away, a stable linear system responds in the same sinusoidal fashion. That is, the *steady-state* response of a stable linear system to sinusoidal excitation also varies sinusoidally, as either $\sin(\omega t + \phi)$ or $\cos(\omega t + \phi)$, where the frequency ω is the same as the excitation frequency, and ϕ is a *phase angle*. This steady-state sinusoidal response is generally called *frequency response*. Although the frequency of response is the same as that of excitation, the **magnitude** of response can vary greatly for different excitation frequencies; therefore, in order to prevent the overloading of a system, it is important to know the frequencies of excitation to which the system is most sensitive.

Both mathematical and experimental analyses of system frequency response are common in engineering practice. We shall study the basic methods of mathematical analysis. The two primary unknowns in the analysis are the magnitude and phase of response as functions of excitation frequency. We can consider the excitation to vary as either $\sin \omega t$ or $\cos \omega t$; the steady-state frequency response magnitude and phase are the same in either case. For consistency, we will consider primarily $\cos \omega t$ excitation.

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4.2: Response of a First Order System to a Suddenly Applied Cosine

First, we derive a complete solution in the conventional manner for the original standard 1st order ODE $\dot{x} - ax = bu(t)$ [Equation 1.2.1], with IC $x(0) = x_0$, and with the suddenly applied (at t = 0) cosine input $u(t) = U\cos\omega t$, t > 0, where U is a constant amplitude [Problem 2.12.2].

$$\text{ODE} + \text{IC} : \dot{x} - ax = bU\cos\omega t = bU\frac{e^{j\omega t} + e^{-j\omega t}}{2}, x(0) = x_0, \text{ find } x(t) \text{ for } t > 0 \tag{4.2.1}$$

Laplace transformation of ODE + IC: $sX(s) - x_0 - aX(s) = \frac{bU}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)$

Solve for $X(s)$:

$$X(s) = \frac{x_0}{s-a} + \frac{bU}{2}\frac{1}{(s-a)}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)$$

Completed partial-fraction expansion (Problem 2.8):

$$X(s) = \frac{x_0}{s-a} + \frac{bU}{2}\frac{1}{a^2+\omega^2}\left(\frac{2a}{s-a} + \frac{-a-j\omega}{s-j\omega} + \frac{-a+j\omega}{s+j\omega}\right)$$

Inverse transform:

$$x(t) = x_0e^{at} + \frac{bU}{a^2+\omega^2}\left[ae^{at} + \frac{1}{2}\left((-a-j\omega)e^{j\omega t} + (-a+j\omega)e^{-j\omega t}\right)\right] \tag{4.2.2}$$

$$x(t) = x_0e^{at} + \frac{bU}{a^2+\omega^2}\left[ae^{at} + \frac{1}{2}\left((-a)\left(e^{j\omega t} + e^{-j\omega t}\right) + \omega\frac{e^{j\omega t} - e^{-j\omega t}}{j}\right)\right] \tag{4.2.3}$$

$$x(t) = x_0e^{at} + \frac{bU}{a^2+\omega^2}\left(ae^{at} - a\cos\omega t + \omega\sin\omega t\right), t \geq 0 \tag{4.2.4}$$

$$x(t) = x_0e^{at} + \left(\frac{b}{-a}\right)\frac{U}{\left[1 + \left(\frac{\omega}{-a}\right)^2\right]}\left[-e^{at} + \cos\omega t + \left(\frac{\omega}{-a}\right)\sin\omega t\right], t \geq 0 \tag{4.2.5}$$

Next, we adapt solution Equation 4.2.5 of general 1st order problem Equation 4.2.1 to the damper-spring system (ideal shock strut) of Figure 3.7.1 with suddenly applied cosine force input $f_x(t) = F\cos\omega t$, for which the comparable ODE + IC problem is

$$c\dot{x} + kx = F\cos\omega t, x(0) = x_0, \text{ find position } x(t) \text{ for } t > 0 \tag{4.2.6}$$

Comparing Equation 4.2.6 with Equation 4.2.1, we define $U \equiv F$, then the other constants of the standard equation become

$$a = -\frac{k}{c} \equiv -\frac{1}{\tau_1} \Rightarrow \text{ time constant } \tau_1 \equiv \frac{1}{-a} = \frac{c}{k} \quad \text{and} \quad b = \frac{1}{c} \text{ and } \frac{b}{-a} = \frac{1}{k}$$

$$\Rightarrow \quad x(t) = x_0e^{-t/\tau_1} + \frac{F}{k}\left(\frac{1}{1 + (\omega\tau_1)^2}\right)\left(-e^{-t/\tau_1} + \cos\omega t + \omega\tau_1\sin\omega t\right), t \geq 0 \tag{4.2.7}$$

It is instructive to study the physical nature of response Equation 4.2.7 in the context of a specific numerical example. Consider an ideal shock strut with the initial condition x_0 = -2 m, and with the system parameters $c = 1/\pi$ N/(m/s) = 0.3183 N-s/m and k = 1 N/m, so that time constant $\tau_1 = 1/\pi$ s = 0.3183 s. Let the magnitude of the cosine force be F = 1.5 N, and let the *period* of the cosine be T_p = 1 s/cycle. Therefore, the cyclic frequency is f = 1 Hz (cycle/s), and the circular frequency is $\omega = 2\pi$ radians/s. (Period, frequency, and phase of periodic signals are discussed more generally in Section 4.4.) The numerical solution of Equation 4.2.7 with these parameters for time $0 \leq t \leq 3$ s is calculated and graphed in the following MATLAB operations.

```
%MATLABdemo41.m
```

```
%Damper-spring ideal shock strut response to IC + cosine forcing
c=1/pi;k=1; %system viscous damping & stiffness constants, SI units
F=1.5;Tp=1; %cosine forcing: amplitude (N), period (sec)
xo=-2; %initial displacement (m)
w=2*pi/Tp; %circular frequency of cosine forcing (rad/sec)
t1=c/k;denom=1+(w*t1)^2;
t=0:0.01:3*Tp; %time instants for forced response
fx=F*cos(w*t);
x=(xo-(F/k)/denom)*exp(-t/t1)+(F/k)/denom*(cos(w*t)+w*t1*sin(w*t));
plot(t,fx,'k',t,x,'k.'),grid,xlabel('Time t (sec)'),... ylabel('Force input f_x(t) (N) and
Hello world!
```

MATLAB M-file script:

```
%MATLABdemo41.m
%Damper-spring ideal shock strut response to IC + cosine forcing
c=1/pi;k=1; %system viscous damping & stiffness constants, SI units
F=1.5;Tp=1; %cosine forcing: amplitude (N), period (sec)
xo=-2; %initial displacement (m)
w=2*pi/Tp; %circular frequency of cosine forcing (rad/sec)
t1=c/k;denom=1+(w*t1)^2;
t=0:0.01:3*Tp; %time instants for forced response
fx=F*cos(w*t);
x=(xo-(F/k)/denom)*exp(-t/t1)+(F/k)/denom*(cos(w*t)+w*t1*sin(w*t));
plot(t,fx,'k',t,x,'k.'),grid,xlabel('Time t (sec)'),... ylabel('Force input f_x(t) (N) and
displacement output x(t) (m)'),... title('Time response of an ideal shock strut to IC + cosine
input')
```

MATLAB command/responses:

```
>> MATLABdemo41
```

The graph of input $f_x(t)$ and response $x(t)$ is below (after figure editing in MATLAB,Version 6 or later, to flatten the aspect ratio and to distinguish the two curves).

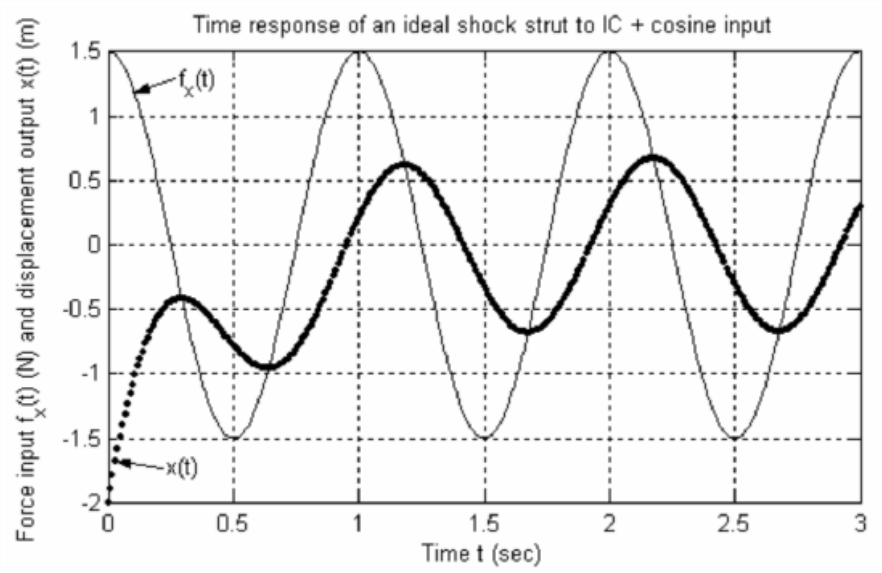


Figure 4.2.1

Let us observe from the graph some important features of the response:

- As calculated above, the time constant of the exponential decay terms in Equation 4.2.7 [the two terms involving $\exp(-t/\tau_1)$] is τ_1 = 0.3183 s, so the settling time for those terms is $4\tau_1$ = 1.273 s. In other words, the exponential decay terms in $x(t)$ have essentially vanished after about 1.3 s of response, and this is clear from the graph. Because the exponential decay terms are relatively short-lived, we often refer to them as the "transient" part of the total solution.
- After the exponential decay terms have vanished, only the $\cos\omega t$ and $\sin\omega t$ terms of Equation 4.2.7 remain. It appears from the graph that those two terms combine to make a single sinusoid at frequency ω, and that the remaining *steady-state* $x(t)$ sinusoid is displaced in time by a constant *time lag* from the $F_x(t)$ sinusoid. This steady-state sinusoidal response is what we call the frequency response, and we will derive equations that describe it explicitly in the remainder of the chapter.

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4.3: Frequency Response of the First Order Damper-Spring System

To extract the steady-state sinusoidal response $x_{ss}(t)$ from total response Equation 4.2.7, we drop the transient, exponential decay terms:

$$x_{ss}(t) = \frac{F}{k}\left(\frac{1}{1+(\omega\tau_1)^2}\right)(\cos\omega t + \omega\tau_1 \sin\omega t) \tag{4.3.1}$$

We can write Equation 4.3.1 in a form that is more physically expressive by using some algebraic manipulation and the general trigonometric identity,

$$\cos\theta \times \cos\phi - \sin\theta \times \sin\phi = \cos(\theta+\phi) \tag{4.3.2}$$

The algebraic manipulation involves the sinusoidally varying terms of Equation 4.3.1. For generality, let us assume that each term is multiplied by a general coefficient, C or $-S$, so that those terms can be written and manipulated as

$$C\cos\omega t - S\sin\omega t = \sqrt{C^2+S^2}\left(\cos\omega t\frac{C}{\sqrt{C^2+S^2}} - \sin\omega t\frac{S}{\sqrt{C^2+S^2}}\right)$$

Now we identify the coefficients of $\cos\omega t$ and $\sin\omega t$ inside parentheses as themselves representing trigonometric functions: $\frac{C}{\sqrt{C^2+S^2}} \equiv \cos\phi$ and $\frac{S}{\sqrt{C^2+S^2}} \equiv \sin\phi$, where angle ϕ is given by a **four-quadrant** tangent function,

$$\phi = \tan^{-1}\left(\frac{S}{C}\right) \tag{4.3.3}$$

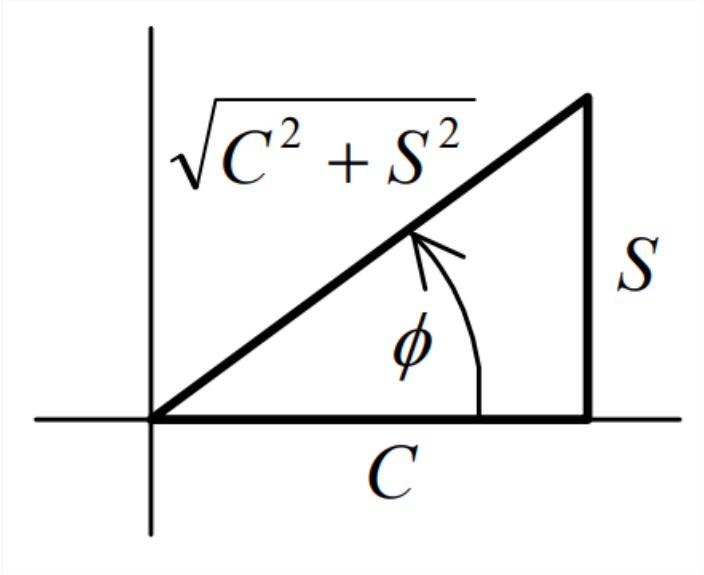


Figure 4.3.1

These trigonometric relations are illustrated at left on the drawing of a right triangle.

So, with identity Equation 4.3.2, the trigonometric sum becomes

$$C\cos\omega t - S\sin\omega t = \sqrt{C^2+S^2}(\cos\omega t\cos\phi - \sin\omega t\sin\phi) = \sqrt{C^2+S^2}\cos(\omega t+\phi) \tag{4.3.4}$$

Associating Equation 4.3.1 with Equation 4.3.4, we have $C=1$, $S=-\omega\tau_1$. Therefore, Equation 4.3.1 becomes

$$x_{ss}(t) = \frac{F}{k}\left(\frac{1}{1+(\omega\tau_1)^2}\right)\sqrt{1+(\omega\tau_1)^2}\cos(\omega t+\phi)$$

$$x_{ss}(t) = \frac{F}{k}\frac{1}{\sqrt{1+(\omega\tau_1)^2}}\cos(\omega t+\phi) \equiv X(\omega)\cos(\omega t+\phi(\omega)), \quad \phi(\omega) = \tan^{-1}\left(\frac{-\omega\tau_1}{1}\right) \tag{4.3.5}$$

Let us compare the steady-state response (output), $x_{ss}(t)$ of Equation 4.3.5, with the excitation (input), $f_x(t) = F\cos\omega t$. The *magnitude* of response, which is a function of excitation frequency ω, is

$$X(\omega) = \frac{F}{k}\frac{1}{\sqrt{1+(\omega\tau_1)^2}} \tag{4.3.6}$$

We will often deal with the *magnitude ratio*, defined as the magnitude of response divided by the magnitude of excitation,

$$\frac{X(\omega)}{F} = \frac{1}{k}\frac{1}{\sqrt{1+(\omega\tau_1)^2}} \tag{4.3.7}$$

The *phase* (or *phase angle*, since it is an angle in radians or degrees) of response relative to the phase of the excitation is also a function of excitation frequency ω:

$$\phi(\omega) = \tan^{-1}(-\omega\tau_1) \tag{4.3.8}$$

In general, if phase angle ϕ is positive, 0° < ϕ < 180°, it is called a phase *lead*, because the peaks and zeros of the response occur in time **before** those of the excitation. If phase angle ϕ is negative, –180° ≤ ϕ < 0°, it is called a phase *lag*, because the peaks and zeros of the response occur in time **after** those of the excitation. The response is said to be *in-phase* if ϕ = 0° exactly, and *out-of-phase* if ϕ = –180° exactly. For the standard stable 1st order system considered presently, $\phi(\omega) = \tan^{-1}(-\omega\tau_1)$, which is a phase lag, always negative in the 4th quadrant, –90° < ϕ < 0°, with $\phi \to$ 0° for very small ω and $\phi \to$-90° for very large ω.

The *frequency-response function* (abbreviated FRF) is considered to consist of **both** the magnitude ratio, Equation 4.3.7 in this case, **and** the phase angle, Equation 4.3.8 in this case. We will see later that both of these functions can be expressed together in a mathematical equation as a single **complex** function of frequency. It is common in engineering practice to plot these two real functions of frequency ω on a pair of over-and-under graphs in the format of Figure 4.3.2, which represents Equations 4.3.7 and 4.3.8 specifically. This format is a type of *Bode diagram* (after Hendrik Wade Bode, American mathematician, physicist, and control-system engineer, 1905-1982).

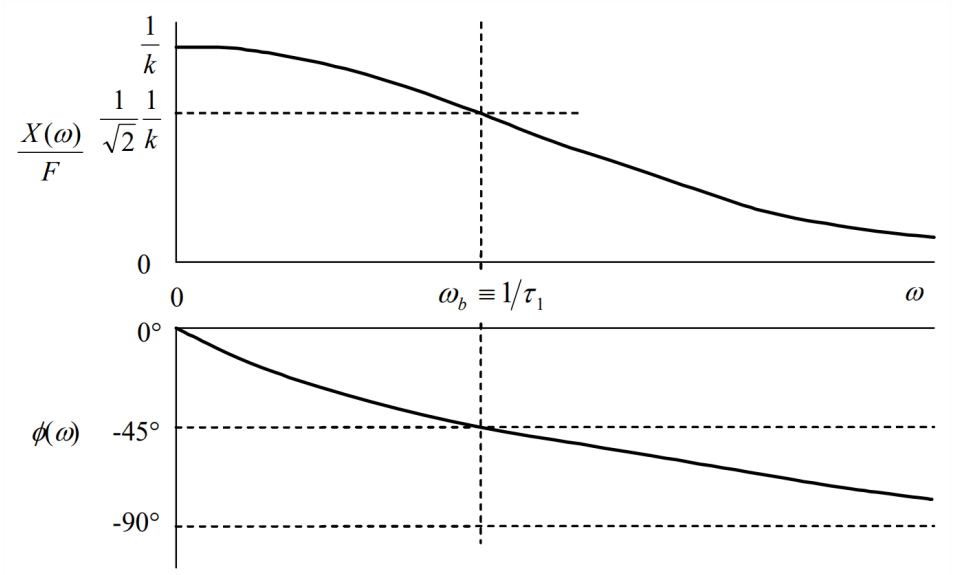


Figure 4.3.2: Frequency-response function of 1st order damper-spring system

The *break frequency* $\omega_b \equiv 1/\tau_1$ is shown explicitly on Figure 4.3.2. There is no obvious "break" in the curves of Figure 4.3.2; but a

break does indeed exist, the form of which will be revealed in the following discussion.

To represent frequency response over broad bands of frequency, the magnitude ratio and phase are often plotted versus the **logarithm** of frequency. Moreover, to allow the possibility of very large dynamic ranges of magnitude response, the magnitude ratio itself is often plotted on a logarithmic scale, making the magnitude ratio a log-log graph, where "log" denotes logarithm to the base 10. On such a graph, it is often possible to construct straight-line asymptotes that are useful for describing the variation with frequency of the magnitude. To illustrate this for the 1st order system, we re-write Equation 4.3.7 as a dimensionless ratio and, for clarity of expression, use the definition $\omega_b \equiv 1/\tau_1$, and then take the logarithm:

$$\frac{X(\omega)}{F/k} = \frac{1}{\sqrt{1+(\omega\tau_1)^2}} = \frac{1}{\sqrt{1+(\omega/\omega_b)^2}} = \frac{1}{\sqrt{1+(f/f_b)^2}} \tag{4.3.9}$$

$$\log\left(\frac{X(\omega)}{F/k}\right) = \log\left(\frac{1}{\sqrt{1+(\omega/\omega_b)^2}}\right) = -\frac{1}{2}\log\left[1+\left(\frac{\omega}{\omega_b}\right)^2\right] = -\frac{1}{2}\log\left[1+\left(\frac{f}{f_b}\right)^2\right] \tag{4.3.10}$$

In the last versions on the right-hand sides of Equations 4.3.9 and 4.3.10, we have used $\omega \equiv 2\pi f$ and $\omega_b \equiv 2\pi f_b$ in order to express the equation in terms of cyclic frequency. The *low-frequency asymptote* is the limit of Equation 4.3.10 as the frequency $\to$ 0 from above:

$$\lim_{\omega\to 0}\left[\log\left(\frac{X(\omega)}{F/k}\right)\right] = -\frac{1}{2}\log(1) = 0 \tag{4.3.11}$$

The *high-frequency asymptote* is the limit of Equation 4.3.10 as the frequency $\to \inf$ from below:

$$\lim_{\omega\to\infty}\left[\log\left(\frac{X(\omega)}{F/k}\right)\right] = -\frac{1}{2}\log\left[\left(\frac{\omega}{\omega_b}\right)^2\right] = -\log\left(\frac{\omega}{\omega_b}\right) = -\log\left(\frac{f}{f_b}\right) \tag{4.3.12}$$

Figure 4.3.3 on the next page is a log-log graph of Equations 4.3.9 and 4.3.10 that shows the asymptotes Equation 4.3.11 and Equation 4.3.12. Notice that on the log-log scale, the low-frequency asymptote is a good approximation for the actual function at frequencies \(ff_{b_{j}}\). The intersection of the two asymptotes is called the *break* (or *corner*) because it is a change in slope of the asymptotic approximations to the actual function. The *break* (or *corner*) frequency is denoted f_b. For this 1st order system at $f_b = \omega_b/2\pi = 1/(2\pi\tau_1)$, the magnitude of response is reduced to $1/\sqrt{2}$ of its static (f = 0) value, and the phase is -45° (see also Figure 4.3.2).

The high-frequency asymptote on Figure 4.3.2 merits some additional comment. Note that its slope on the log-log scale equals –1, *i.e.*, the magnitude shrinks one decade (order of magnitude) for each decade increase in the frequency. Frequency response magnitude ratios are often given in *decibels* (dB), especially in acoustics for sound pressure level. A magnitude ratio in *decibels* is defined to be 20 × log(magnitude ratio). If the magnitude ratio on Figure 4.3.2 had been plotted in decibels versus log(frequency), then the slope of the high-frequency asymptote would be –20; this is often referred to as a 20 dB/decade rolloff of the magnitude ratio.

Another observation worth making, especially from Figure 4.3.2, is that the standard 1st order system considered here behaves like a mild *low-pass filter*: the system responds most sensitively to excitation at frequencies below the break frequency, the magnitude of response being about equal to the static response. However, at excitation frequencies above the break frequency, the system responds much less sensitively, and the magnitude of response diminishes progressively as the excitation frequency increases.

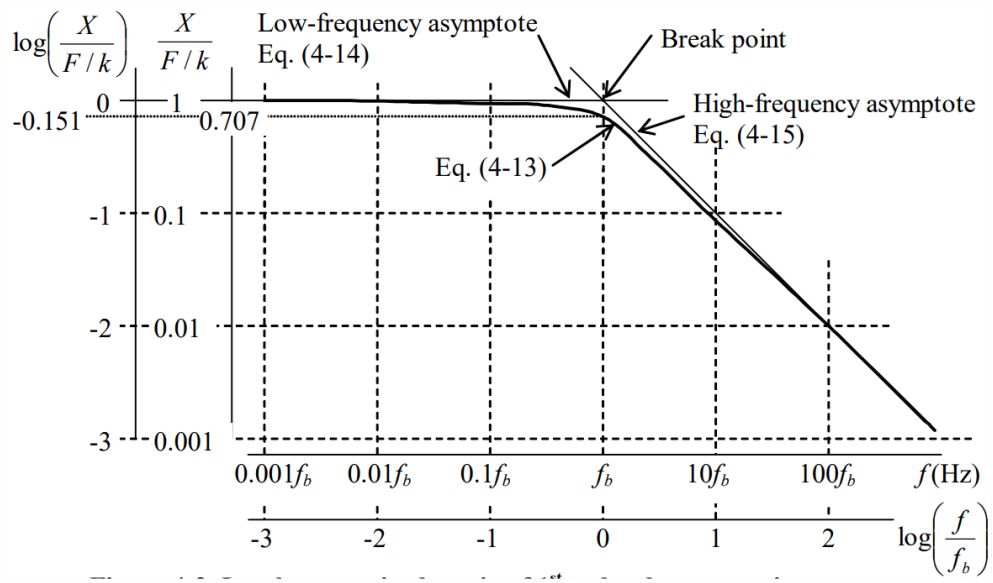


Figure 4.3.3: Log-log magnitude ratio of 1st order damper-spring system

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4.4: Period, Frequency, and Phase of Periodic Signals

Let us consider more generally the temporal quantities of periodic signals, represented in our applications by sinusoids. Period T_p is normally measured in seconds per cycle, so the **cyclic** frequency f in cycles per second is the inverse of period, $f = 1/T_p$. Also, the period is related to the **circular** frequency ω by $\omega T_p = 2\pi$ radians, so that

$$\text{circular frequency } \omega = \frac{2\pi}{T_p} \equiv 2\pi f \frac{\text{rad}}{\text{sec}}, \text{ and cyclicfrequency } f = \frac{\omega}{2\pi}\text{Hz}\left(\frac{\text{cycles}}{\text{sec}}\right) \tag{4.4.1}$$

These relationships between period and frequency are worth understanding completely and even memorizing, as we use them a great deal in system dynamics.

It is important also to understand how FRF **phase** is manifested in time history plots of input and output. This discussion is general, applicable for **any** LTI system. Suppose that there are plotted on the same graph steady-state sinusoidal time histories of both the input $u(t)$ and the output $x(t)$, as in Figure 4.4.1 on the next page. We want to calculate the FRF phase from measurements on the time-history plots. First, find the **nearest** positive $x(t)$ crest to the left of a reference positive $u(t)$ crest, as shown on Figure 4.4.1. Measure the positive time interval T_{lead} in seconds that $x(t)$ **leads** $u(t)$. (Note that you can measure T_{lead} also by comparing troughs or positive-going or negative-going zeros, as shown on Figure 4.4.1.)

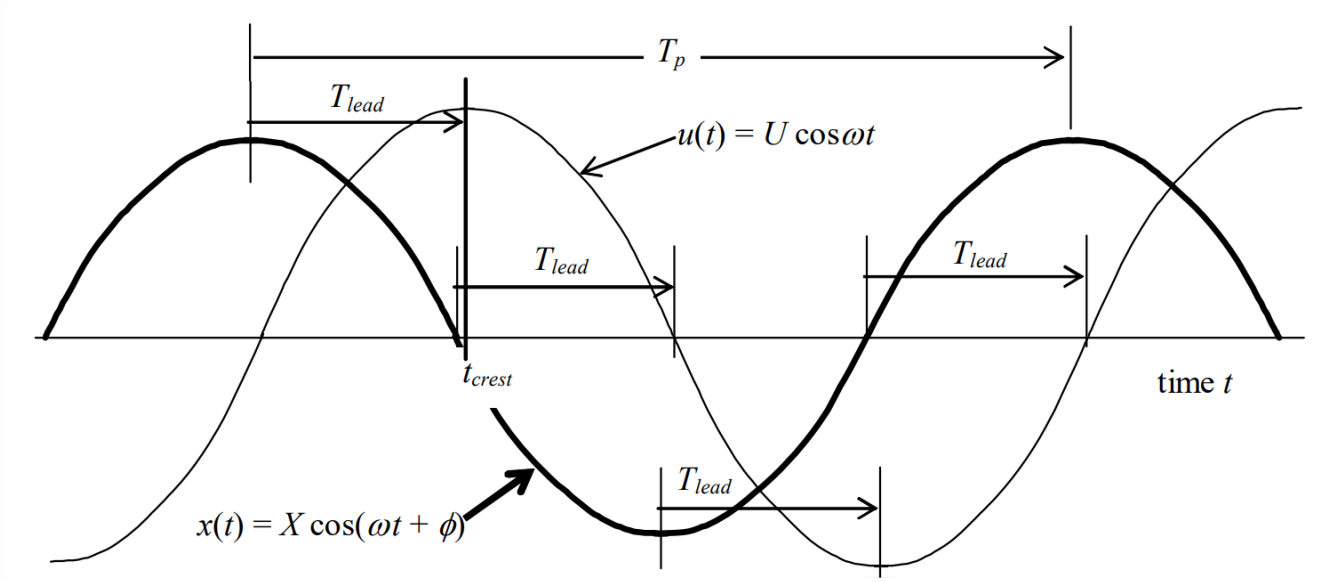


Figure 4.4.1: Frequency response phase measurement from time-history graphs

Now, referring to the positive crests, let us denote as t_{crest} the instant corresponding to a positive crest of input $u(t)$. Then we see from the drawing that

$$\cos\omega t_{crest} = +1 \quad \Rightarrow \quad \omega t_{crest} = 2\pi n, \text{ where } n \text{ is some integer}$$

$$\cos\left(\omega\left[t_{\text{crest}} - T_{\text{lead}}\right] + \phi\right) = +1 \quad \Rightarrow \quad \omega\left[t_{\text{crest}} - T_{\text{lead}}\right] + \phi = 2\pi n$$

Comparing these two equations shows that $\omega\left[-T_{lead}\right] + \phi = 0$, which gives the basic equation for phase angle (defined positive as a lead, negative as a lag):

$$\phi = \omega T_{\text{lead}} = \frac{2\pi}{T_p} T_{\text{lead}} = 2\pi \frac{T_{\text{lead}}}{T_p}\text{rad} \times \frac{360\text{deg}}{2\pi\text{rad}} = 360 \times \frac{T_{\text{lead}}}{T_p}\text{deg} \tag{4.4.2}$$

where $T_p (= 2\pi/\omega = 1/f)$ is the period of the input and the steady-state output, as shown on Figure 4.4.1.

If you choose to select the **nearest** positive $x(t)$ crest to the **right** of a reference $u(t)$ crest, then you are finding time **lag** T_{lag}, which is a negative lead. In this case, you can find the phase angle, with the correct sign, from

$$\phi = -\omega T_{lag} = -360 \times \frac{T_{lag}}{T_p} \text{deg} \tag{4.4.3}$$

Equations 4.4.2 and 4.4.3 give ϕ in the range 0° $\leq \phi <$ 360°, which is correct but not common. It is not common because the four-quadrant arctangent is conventionally expressed (in MATLAB, for example) in the range –180° $\leq \phi <$ +180°. Therefore, if you measure and calculate by the procedure above a phase $\phi \geq$ +180°, then it is conventional to replace that ϕ with (ϕ – 360°). For example, if you were to measure and calculate the phase ϕ = 262°, then it would be conventional to express it as a phase **lag**, φ = (262° – 360°) = –98°, rather than a phase **lead** of 262°.

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4.5: Derivation of the Complex Frequency-Response Function - Easy derivation of the complex frequency-response function for standard stable first order systems.

This section is an example of a much easier method (than that of Sections 4.2 and 4.3) for deriving the frequency-response function of a system. Let us find the frequency response of standard stable 1st order systems. From Equation 3.4.8 the standard stable ODE with sinusoidal excitation is

$$\dot{x} + (1/\tau_1)\, x = bu(t) = bU\cos\omega t$$

We seek steady-state sinusoidal response $x_{ss}(t) = X(\omega)\cos(\omega t + \phi(\omega))$, in which $X(\omega)$ and $\phi(\omega)$ are functions to be found. The first step in the method is to take the **general** [for arbitrary $u(t)$] Laplace transform, setting the IC to zero:

$$(s + 1/\tau_1)\, L[x(t)]|_{x_0=0} = bL[u(t)]$$

Next, we form the system general *transfer function*, $TF(s)$, defined as the ratio of the output transform to the input transform, with zero IC:

$$TF(s) \equiv \frac{L[x(t)]|_{x_0=0}}{L[u(t)]} = \frac{b}{s + 1/\tau_1}$$

The Laplace independent variable s is complex in general. However, in order to analyze frequency response, we let s in $TF(s)$ be purely imaginary, $S = j\omega$ (ω being the real circular frequency), producing the complex *frequency-response function* $FRF(\omega)$:

$$TF(s)|_{s=j\omega} \equiv TF(j\omega) \equiv FRF(\omega) = \frac{b}{1/\tau_1 + j\omega} = b\tau_1 \frac{1}{1 + j\omega\tau_1} \tag{4.5.1}$$

Next, with use of Equations 2.1.6, 2.1.7, 2.1.8, and 2.1.11, we convert $TF(j\omega) \equiv FRF(\omega)$ algebraically into polar form:

$$FRF(\omega) = b\tau_1 \frac{1}{1 + j\omega\tau_1} \times \frac{1 - j\omega\tau_1}{1 - j\omega\tau_1} \tag{4.5.2}$$

$$= b\tau_1 \frac{\sqrt{1 + (\omega\tau_1)^2}}{1 + (\omega\tau_1)^2} \exp\left[j\tan^{-1}\left(\frac{-\omega\tau_1}{1}\right)\right] \tag{4.5.3}$$

$$\equiv |FRF(\omega)| e^{j\angle FRF(\omega)} \tag{4.5.4}$$

$$= \frac{b\tau_1}{\sqrt{1 + (\omega\tau_1)^2}} e^{j\phi(\omega)} \tag{4.5.5}$$

in which phase angle $\phi(\omega) = \tan^{-1}(-\omega\tau_1/1) = \tan^{-1}(-\omega\tau_1)$.

Equations 4.5.1 and 4.5.5 define the complex frequency-response function, $FRF(\omega)$, of standard stable 1st order systems. It is proved in Sections 4.6 and 4.7 **for LTI systems in general** that the real magnitude $|FRF(\omega)|$ of function $FRF(\omega)$ is the magnitude **ratio** of system frequency response, and the phase angle $\phi(\omega)$ of function $FRF(\omega)$ is the phase angle of system frequency response. For example, let us adapt standard solution Equation 4.5.5 to the damper-spring system, for which [from Equation 4.2.5] $b = 1/c$ and $\tau_1 = c/k$. Thus, the magnitude of $FRF(\omega)$ from Equation 4.5.5 is

$$|FRF(\omega)| = \frac{X(\omega)}{U} = \frac{b\tau_1}{\sqrt{1 + (\omega\tau_1)^2}} = \frac{1}{k}\frac{1}{\sqrt{1 + (\omega\tau_1)^2}} \tag{4.5.6}$$

which is identical to damper-spring system FRF magnitude ratio $X(\omega)/F$ of Equation 4.3.8. Also, the phase angle of $FRF(\omega)$ from Equation 4.5.5 is

$$\angle FRF(\omega) = \phi(\omega) = \tan^{-1}(-\omega\tau_1) \tag{4.5.7}$$

which is identical to damper-spring system FRF phase $\phi(\omega)$ of Equation 4.3.9. Thus, with $FRF(\omega)$ of Equation 4.5.5, we have obtained here the same final results as before for the damper-spring system, but much more easily.

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4.6: Transfer Function - General Definition

For any LTI, single-input-single-output (SISO) physical system, we denote the input as $u(t)$ and the output as $x(t)$. For an nth order system, in general, the input and output are related by an ODE of the form

$$a_1\frac{d^n x}{dt^n}+a_2\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_{n+1}x=b_1\frac{d^m u}{dt^m}+b_2\frac{d^{m-1}u}{dt^{m-1}}+\ldots+b_{m+1}u \tag{4.6.1}$$

$a_1,\ldots,a_{n+1}$ and $b_1,\ldots,b_{m+1}$ are constants (with the numbering system keyed to MATLAB notation), and $m\leq n$. Also, we assume that the system is *stable*, which is defined more precisely in Section 4.7. Taking the Laplace transform of the ODE, with all ICs equal to zero, gives

$$\left(a_1 s^n+a_2 s^{n-1}+\ldots+a_{n+1}\right)L[x(t)]_{IC_s=0}=\left(b_1 s^m+b_2 s^{m-1}+\ldots+b_{m+1}\right)L[u(t)] \tag{4.6.2}$$

Then, from Equation 4.6.2, the system *transfer function*, defined to be the ratio of the output transform to the input transform, with zero ICs, is the ratio of two polynomials,

$$TF(s)\equiv\frac{L[x(t)]_{IC_s=0}}{L[u(t)]}=\frac{b_1 s^m+b_2 s^{m-1}+\ldots+b_{m+1}}{a_1 s^n+a_2 s^{n-1}+\ldots+a_{n+1}} \tag{4.6.3}$$

It is appropriate to state here (without proof) that the transfer function of any **physically realizable** system has $m\leq n$, *i.e.*, **the degree of the numerator polynomial is less than or equal to the degree of the denominator polynomial**. The condition $m\leq n$ makes the transfer function **causal**, which means that the current (in time) output of the system is dependent only upon past and present (**not** future) values of the input. In general, the future values of an input cannot be predicted, so it is logical that a real system and its transfer function must be causal. See Bélanger, 1995, page 440.

Note also from Equation 4.6.3 that, if given $TF(s)$ and input $u(t)$, we can express the transform of the output with zero initial conditions as

$$L[x(t)]|_{ICs=0}=TF(s)\times L[u(t)] \tag{4.6.4}$$

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4.7: Frequency-Response Function from Transfer Function

For frequency response of a general LTI SISO stable system, we define the input to be a time-varying cosine, with amplitude U and circular frequency ω,

$$u(t) = U\cos\omega t = \frac{U}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \tag{4.7.1}$$

in which we apply the complex exponential form for the cosine that is derived from Euler's equation (Problem 2.1). The Laplace transform of input Equation 4.7.1 is

$$L[u(t)] = \frac{U}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \tag{4.7.2}$$

Substituting Equation 4.6.3 and Equation 4.7.2 into Equation 4.6.4 gives

$$L[x(t)]_{IC_S=0} = \left(\frac{b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_{n+1}}\right)\frac{U}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \tag{4.7.3}$$

By expanding into partial fractions, we will usually be able to cast Equation 4.7.3 into the form

$$L[x(t)]_{IC_S=0} = \frac{A}{s-j\omega} + \frac{B}{s+j\omega} + \sum_{k=1}^{n}\frac{C_k}{s-p_k} \tag{4.7.4}$$

in which A, B, C_k, and ρ_k [the poles of $TF(s)$] all are constants. Taking the inverse transform of Equation 4.7.4 gives

$$x(t) = Ae^{j\omega t} + Be^{-j\omega t} + \sum_{k=1}^{n} C_k e^{p_k t} \tag{4.7.5}$$

The first two right-hand-side terms of Equation 4.7.5 are associated with steady-state forced sinusoidal response, and the third term is associated with response bounded by real exponential functions. The nature of system stability is determined by the poles p_k, in particular, by their real parts. If $\mathrm{Re}\,[p_k] < 0$ for all k = 1, 2, ..., n, then each of the $e^{p_k t}$ terms is bounded by a **decaying** exponential, that is, $\to 0$ as $t \to \infty$. A system for which for all k is said to be stable. Therefore, for steady-state sinusoidal response (after all exponentially bounded transients have decayed) of a stable system, only the first two right-hand-side terms of Equation 4.7.5 remain,

$$x_{ss}(t) = Ae^{j\omega t} + Be^{-j\omega t} \tag{4.7.6}$$

Our objective now is to find the complex constants A and B in Equation 4.7.6. The first step is to recognize that $x_{ss}(t)$ must be a real function mathematically; that is, in order to represent actual physical behavior, Equation 4.7.6 cannot have any imaginary component. From Problem 2.5, we conclude that B must be the complex conjugate of A, *i.e.*, $B = \bar{A}$. Therefore, Equation 4.7.6 becomes

$$x_{ss}(t) = Ae^{j\omega t} + \bar{A}e^{-j\omega t} \tag{4.7.7}$$

The Laplace transform of Equation 4.7.7 is

$$L\left[x_{ss}(t)\right] = \frac{A}{s-j\omega} + \frac{\bar{A}}{s+j\omega} \tag{4.7.8}$$

Recognizing that we seek only constant A for the steady-state sinusoidal response (not any of the C_k constants associated with transient response), we now combine Equation 4.7.8 and Equation 4.7.2 into Equation 4.6.4,

$$L\left[x_{ss}(t)\right] = \frac{A}{s-j\omega} + \frac{\bar{A}}{s+j\omega} = TF(s)\frac{U}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) \tag{4.7.9}$$

To find complex constant A and its conjugate, we use the labor-saving method for partial-fraction expansion from Chapter 2,

$$A = \left[(s - j\omega)TF(s)\frac{U}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right)\right]_{s=j\omega} = \frac{U}{2}TF(j\omega) \tag{4.7.10}$$

$$\bar{A} = \left[(s + j\omega)TF(s)\frac{U}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right)\right]_{s=-j\omega} = \frac{U}{2}TF(-j\omega) \tag{4.7.11}$$

Comparing Equation 4.7.10 and Equation 4.7.11 shows that

$$TF(-j\omega) = \overline{TF(j\omega)} \tag{4.7.12}$$

So now we can denote the transfer function terms in general polar form:

$$TF(j\omega) = |TF(j\omega)|e^{j\phi(\omega)} \tag{4.7.13}$$

where the phase is $\phi(\omega) = \angle[TF(j\omega)]$.

$$TF(-j\omega) = |TF(j\omega)|e^{-j\phi(\omega)} \tag{4.7.14}$$

Substituting Equations 4.7.13, 4.7.14, 4.7.10, and 4.7.11 back into Equation 4.7.7 gives

$$x_{ss}(t) = Ae^{j\omega t} + \bar{A}e^{-j\omega t} = \frac{U}{2}|TF(j\omega)|\left(e^{j(\omega t + \phi(\omega))} + e^{-j(\omega t + \phi(\omega))}\right) \tag{4.7.15}$$

Applying again the formula from Euler's equation that relates the cosine to complex exponentials gives the desired final result,

$$x_{ss}(t) = U|TF(j\omega)|\cos(\omega t + \phi(\omega)) \tag{4.7.16}$$

$$\equiv X(\omega)\cos(\omega t + \phi(\omega)) \tag{4.7.17}$$

in which $X(\omega)$ and $\phi(\omega)$ are, respectively, the amplitude (magnitude) and phase of the steady-state sinusoidal response. Note that $|TF(j\omega)| = X(\omega)/U$, the magnitude ratio.

Therefore, we define the complex frequency-response function $FRF(\omega)$ to be $TF(j\omega)$, Equation 4.7.13. Expressing $FRF(\omega)$ in polar form gives us the FRF magnitude ratio and phase directly, and relatively easily (without all the work of finding the particular solution of the ODE by the method of undetermined coefficients, or of finding the complete time response by forward and inverse Laplace transformation, *etc.*):

$$FRF(\omega) \equiv TF(j\omega) = \frac{X(\omega)}{U}e^{j\phi(\omega)} \tag{4.7.18}$$

This result is general for LTI SISO systems, it is valid for all of the systems considered in this book, and it is widely used in engineering practice.

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4.8: Chapter 4 Homework

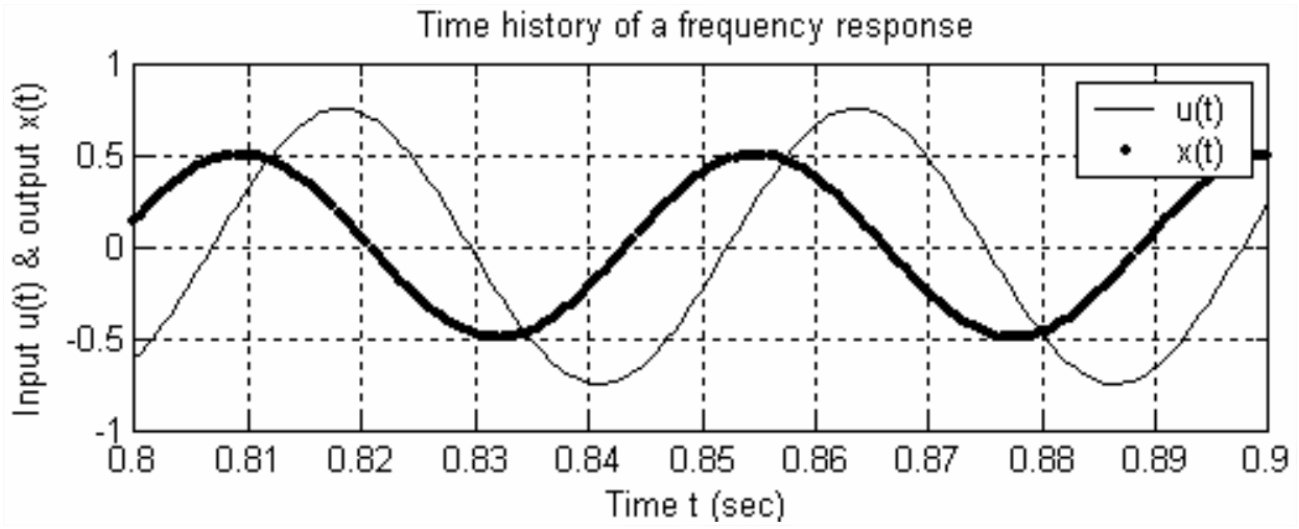
1. This problem involves the data and results for the 1st order damper-spring system that is analyzed in Section 4.2 with use of MATLAB program `MATLABdemo41`.
    1. Substitute the damper-spring data and the given values of F and ω into Equation 4.3.8 and 4.3.10 in order to calculate the magnitude X and phase angle φ (in degrees, in the range $-180° \leq \phi < +180°$) of the frequency response.
    2. Examine the time history graph generated by MATLAB program `MATLABdemo41`. The purpose of this problem is for you to evaluate that graph as if it were recorded from an experiment. Consider, in particular, the time after about t = 1.3 s, when the output has settled into steady-state sinusoidal response. Measure from the graph the steady-state output magnitude X. Measure whatever information is required from the graph, then calculate phase angle ϕ (in degrees, in the range $-180° \leq \phi < +180°$) of the steady-state state response. These simulated experimental results for X and ϕ should be equal to (or at least close to, since it is not possible to make precise measurements from a graph) the comparable results calculated in part 1.1 from theoretical formulas.
2. An experimental frequency response test is conducted on an LTI system (**not necessarily a 1st order system**), with input $u(t) = U\cos\omega t$ and output $x(t) = X\cos(\omega t + \phi)$. A simulated graph [2.1 or 2.2 or 2.3, whichever is assigned] of the steady-state input and output quantities at one particular frequency is shown below or on the next page. Calculate from the graph, with as much accuracy as the data permits, the following values: (i) the sinusoidal period T_p, and then the frequency (in both Hz and rad/s), (ii) the FRF magnitude ratio X/U (assume consistent physical units), and (iii) the FRF phase ϕ (in degrees, in the range $-180° \leq \phi < +180°$).
    1. 

Figure 4.8.1

2.

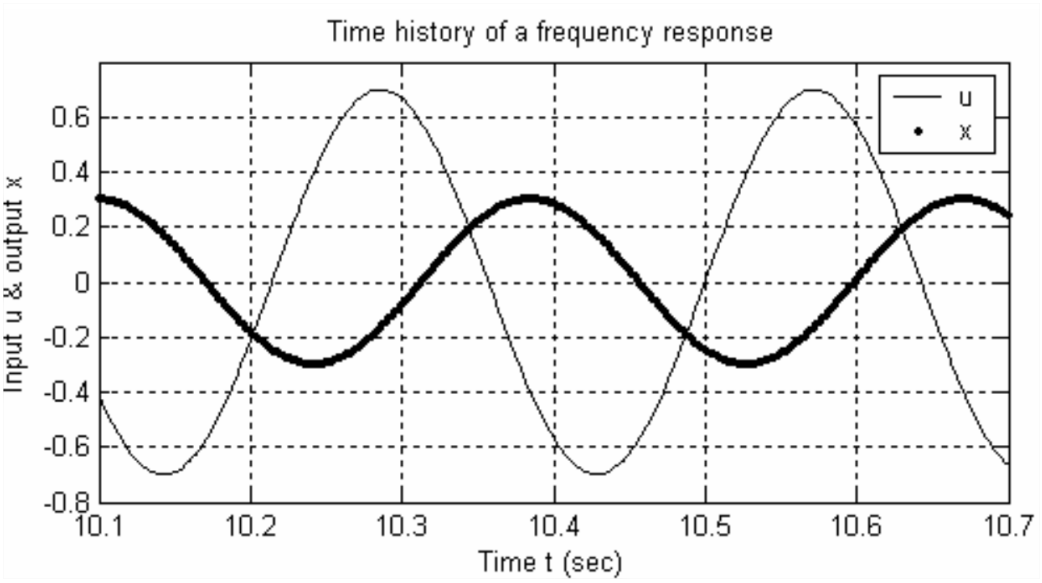


Figure 4.8.2

3.

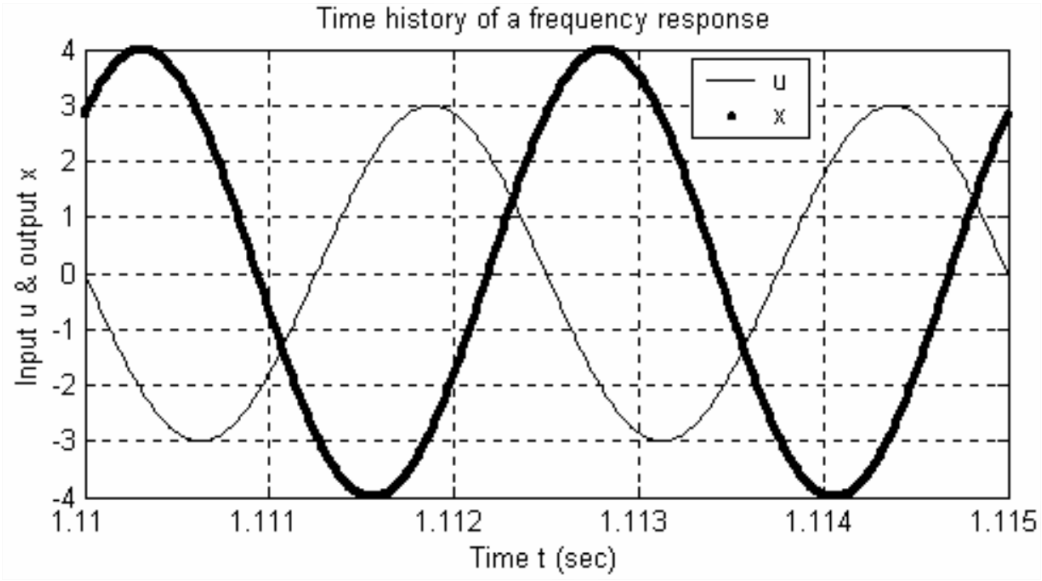


Figure 4.8.3

3. The governing ODE of the series damper-spring low-pass filter in Figure 3.7.4 is Equation 3.7.5, $\tau_1 \dot{x}_o + x_o = x_i(t)$. Show that the associated 1st order complex frequency-response function is $FRF(\omega) = TF(j\omega) = 1/(1 + j\omega\tau_1)$. The following MATLAB script graphs from this equation the FRF magnitude ratio and phase in one conventional format (log-log for magnitude ratio, semilog for phase in degrees, magnitude ratio graph directly over phase graph), for the case of time constant τ_1 = 0.0145 s, with excitation frequencies ranging from 1 Hz to 1 000 Hz. Note that MATLAB does most of the work for you if you start with the complex FRF, and then use MATLAB's capability for performing complex arithmetic. In particular, the MATLAB function `abs` calculates the absolute value (magnitude) of a complex number, and the MATLAB function `angle` calculates the angle in radians of a complex number.

```
t1=0.0145; % First-order system time constant (sec)

f=logspace(0,3,200);w=2*pi*f; % 200 pts. 1 Hz-1 kHz evenly spaced on log scale

FRF=1./(1+j*w*t1); % complex FRF; note necessity of array operation ./

magrat=abs(FRF);fazdeg=angle(FRF)*180/pi; % magnitude ratio and phase in deg

subplot(2,1,1),loglog(f,magrat,'k'),grid,ylabel('FRF magnitude ratio'), ...
```

```
title('FRF of 1st order system with time constant = 0.0145 sec')

subplot(2,1,2),semilogx(f,fazdeg,'k'),grid, ...

ylabel('FRF phase (deg)'),xlabel('Frequency (Hz)')
```

Run this script on your computer, and submit the graphs that it produces. **Additionally,**

1. calculate the break frequency $f_b = 1/(2\pi\tau_1)$ and show that it matches the break frequency inferred from the MATLAB graphs;
2. use complex arithmetic to evaluate **by hand** (and hand calculator) the complex FRF, $FRF(\omega) = \frac{1}{1+j\omega\tau_1}$, at frequency f = 30 Hz, and show that your hand-calculated magnitude ratio and phase match those of MATLAB. You might have occasion to do more graphing of FRFs, so, for your future reference, make sure that you understand the operations of all the MATLAB commands in the above script.

4. The governing ODE of the series damper-spring high-pass filter in Figure 3.7.4 is Equation 3.7.6, $\dot{x}_o + (1/\tau_1)\, x_o = \dot{x}_i(t)$ [see also homework Problem 3.7].
   1. Show that the associated 1st order high-pass filter complex frequency-response function is $FRF(\omega) = j\omega\tau_1/(1 + j\omega\tau_1)$.
   2. For frequency response, we are evaluating the input in the form $x_i(t) = X_i \cos\omega t$ and the output in the form $x_o(t) = X_o \cos(\omega t + \phi)$. Use the FRF of part 4.1 to show that the equations for FRF magnitude ratio and phase are
   $$\frac{X_o(\omega)}{X_i} = \frac{\omega\tau_1}{\sqrt{1 + (\omega\tau_1)^2}}, \quad \phi(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega\tau_1) = \tan^{-1}\left(\frac{1}{\omega\tau_1}\right)$$
   Use this magnitude ratio equation to derive equations for the low-frequency and high-frequency asymptotes, and sketch those asymptotes on a log-log graph such as Figure 4.3.3. You may use the definition of break frequency, $\omega_b = 1/\tau_1$, so that $\omega\tau_1 = \omega/\omega_b = f/f_b$. This system has the character of a mild high-pass filter, so your sketch of the asymptotes should have the appearance appropriate to that name.
   3. Let the time constant τ_1 = 1 s, and consider the excitation frequency range 0.01 $\leq \omega \leq$ 100 rad/s ($10^{-2} \leq \omega \leq 10^{+2}$ rad/s). Adapt the MATLAB program, and use the graphical format provided in homework Problem 4.3 to graph with MATLAB the FRF magnitude ratio and phase (in degrees). As is done in Problem 4.3, program directly the complex FRF, in this case $FRF(\omega) = j\omega\tau_1/(1 + j\omega\tau_1)$, **not** the equations of part 4.4.3.

5. A standard LTI mass-damper-spring system has the ordinary differential equation of motion $m\ddot{x} + c\dot{x} + kx = f_x(t)$, in which $f_x(t)$ is the input and $x(t)$ is the output.
   1. Use the ODE to derive the transfer function $TF(s) \equiv L[x(t)]_{ICs=0}/L[f_x(t)]$ for the m-c-k system. This transfer function is an algebraic equation in terms of independent variable s and constants m, c, and k.
   2. Use transfer function $TF(s)$ to derive the complex frequency-response function $FRF(\omega)$ for the m-c-k system in terms of independent variable frequency ω and constants m, c, and k.
   3. From $FRF(\omega)$ for the m-c-k system, write equations in terms of independent variable frequency ω and constants m, c, and k for the real magnitude ratio $|FRF(\omega)|$ and the real phase angle $\angle FRF(\omega) = \phi(\omega)$.

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5: BASIC ELECTRICAL COMPONENTS AND CIRCUITS

5.1: Introduction
5.2: Passive Components - Resistor, Capacitor, Inductor
5.3: Operational Amplifier (op-amp) and Op-amp Circuits
5.4: RC Band-Pass Filter
5.5: Chapter 5 Homework

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5.1: Introduction

Electrical circuits are systems that have many dynamic response characteristics in common with mechanical systems. But electrical variables (voltage, current, charge) and circuits are quite different physically than mechanical variables and systems, and the physical laws and methods for deriving ordinary differential equations that describe behaviors of electrical circuits also are entirely different. This chapter is an introduction to the theory of electrical circuits and basic analog electronics, and to some common but simple practical applications of the theory. The material presented here provides the basis for many examples of system dynamic behavior in later chapters.

Many engineering students in fields other than electrical engineering are exposed to electrical theory only in a required introductory physics course. This presentation is designed for such engineers who, nevertheless, have applications for simple circuits and electronics. For example, almost every engineer who works with electromechanical systems will at some time require at least the background in circuits and electronics that is presented in this chapter. The same is true for almost every engineer who uses sensors of any kind, or is involved with testing of prototypes or products in the laboratory or in the field. Even the process of logically evaluating for possible purchase the capabilities and specifications of modern electronic instrumentation requires more sophistication with circuits and electronics than a student can acquire from an introductory physics course.

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5.2: Passive Components - Resistor, Capacitor, Inductor

We denote the electrical potential, the *voltage* in volts (V) SI units, at a point in a circuit as $e(t)$, and the flow of positively charged particles, the *electrical current* in amps (A) SI units, as $i(t)$. These two electrical quantities are the principal variables that will appear in derivations of the ODEs describing the dynamic behavior of circuits.

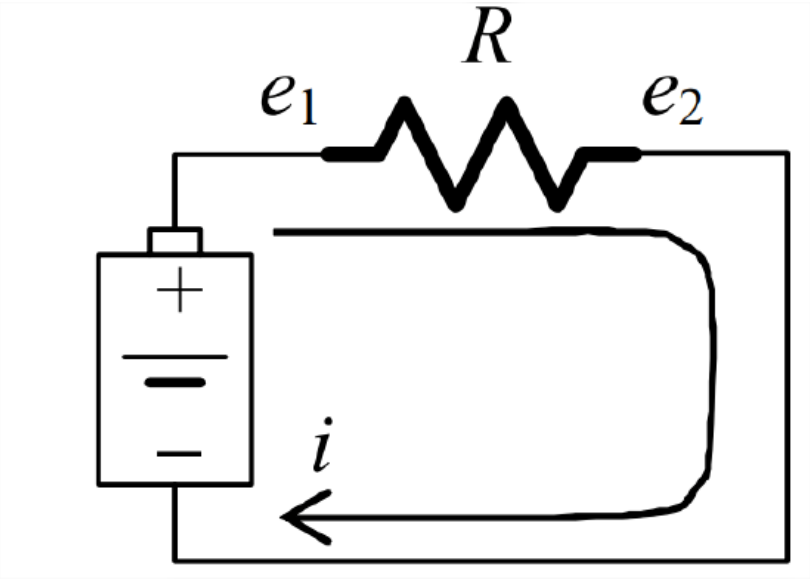


Figure 5.2.1: Resistor in a simple circuit

The circuit drawn in Figure 5.2.1 depicts an ideal linear *resistor*, with resistance R ohms (usually denoted by the upper-case omega, \(\text{\omega\)) in SI units. The voltage difference $e_1 - e_2$ between the positive and negative terminals of a battery causes current i to flow through the resistor. The material of the resistor is an electrical conductor, but a poor conductor that provides much more resistance to current flow than a good conductor such as copper wire. This resistance converts part of the electrical energy into heat energy, causing the resistor's temperature to rise slightly. For a standard, commercially produced resistor, the relationship between $e_1 - e_2$ and i is linear, with resistance R defined as the constant of proportionality (Halliday and Resnick, 1960, Sections 31-2 and 31-3). This relationship is known as *Ohm's law* (after German physicist Georg Simon Ohm, 1787-1854), and it is usually expressed in one of the following two forms:

$$e_1 - e_2 = iR \Rightarrow i = \frac{e_1 - e_2}{R} \tag{5.2.1}$$

Equation 5.2.1 applies at each instant, even if the voltages and the current are varying with time.

Note from Ohm's law the unit equivalence Ω = V/A; this relation giving the ohm in terms of the volt and the amp is useful for establishing correct units in calculations. Suppose, for example, that a 10-volt difference is imposed across a 5 kΩ resistor, typical numbers for instrumentation circuits. Then, from Equation 5.2.1, the current through the resistor is

$$i = \frac{10\text{V}}{5 \times 10^3 \text{V/A}} = 2\text{e} - 3\text{A} \equiv 2\text{mA}.$$

The following two examples serve to introduce a fundamental physical principle for electrical circuits, *Kirchhoff's current law* (abbreviated "KCL", after German physicist Gustav Robert Kirchhoff, 1824-1887), and to illustrate applications of KCL.

Example 5.2.1

Given resistors R_1 and R_2 arranged in series, as in the figure below, find the equivalent single resistance R_{eq}.

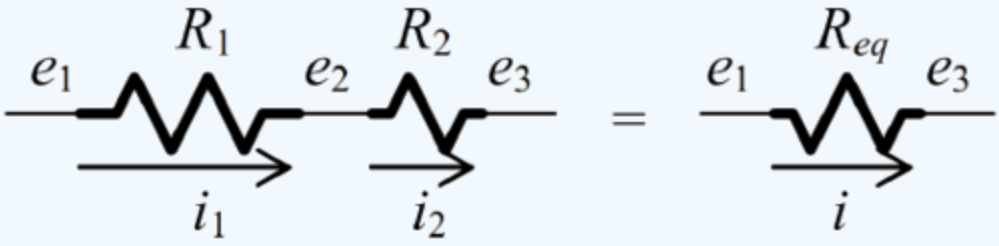


Figure 5.2.2

Solution

From the figure and Equation 5.2.1,

$$i_1 = \frac{e_1 - e_2}{R_1}, i_2 = \frac{e_2 - e_3}{R_2}, i = \frac{e_1 - e_3}{R_{eq}}$$

The form of KCL relevant to this situation is: *the quantity of current is continuous in a series arrangement*, so that $i_1 = i_2 \equiv i$. [This *continuity* condition for electrical current is directly analogous to the continuity condition for an incompressible fluid: the volume flow rate (average velocity $\times$ cross-sectional area) remains constant at all points along a flow tube or channel.] The total voltage change is the sum of the individual changes,

$$e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3) = i_1 R_1 + i_2 R_2 = i\,(R_1 + R_2)$$

$$\Rightarrow \quad i = \frac{e_1 - e_3}{R_1 + R_2} \Rightarrow R_{eq} = R_1 + R_2$$

The resistance of a series combination of resistors is the sum of the individual resistances.

Example 5.2.2

Given resistors R_1 and R_2 arranged in parallel, as in the figure below, find the equivalent single resistance R_{eq}.

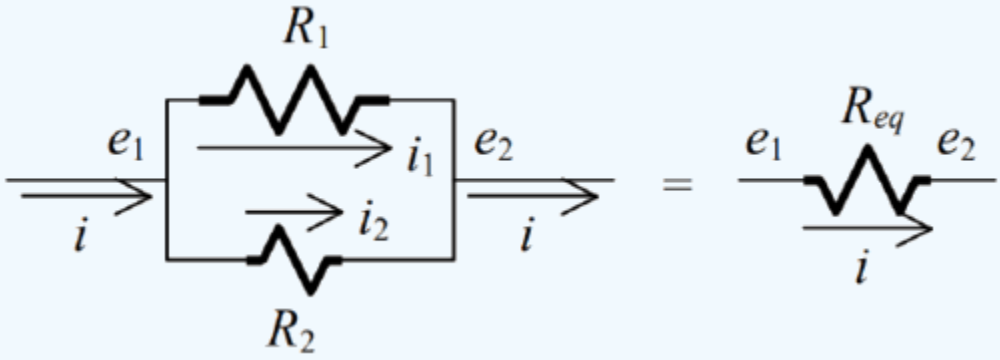


Figure 5.2.3

Solution

In this case, there are circuit junctions on both sides of the parallel resistors. The more general KCL applicable here is: *the quantity of current leaving a junction equals the quantity of current entering the junction*, so that $i_1 + i_2 = i$. The voltage difference is the same across each of the parallel resistors, so KCL in terms of voltage differences is

$$\frac{e_1 - e_2}{R_1} + \frac{e_1 - e_2}{R_2} = \frac{e_1 - e_2}{R_{eq}} \Rightarrow \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R_{eq}} = \frac{R_1 + R_2}{R_1 R_2} \Rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

The equivalent (effective) resistance is less than the smaller of the two parallel resistances.

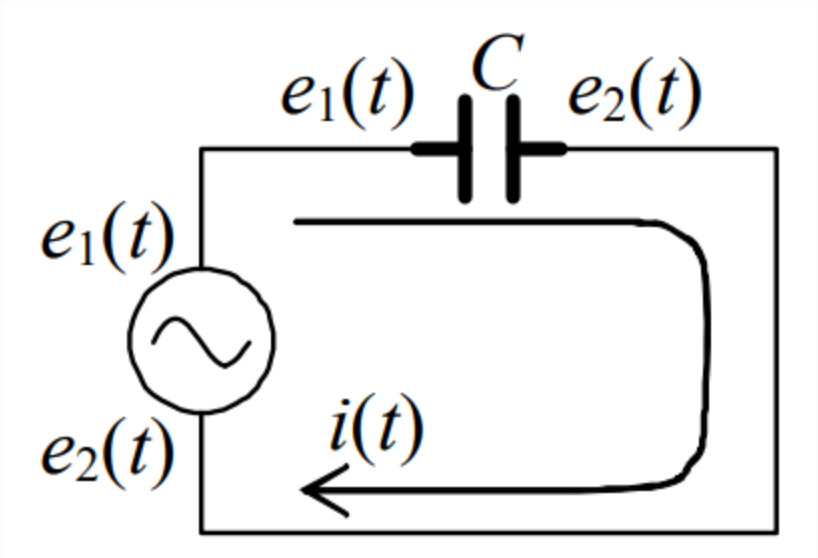


Figure 5.2.4: Capacitor in a simple circuit

The circuit drawn in Figure 5.2.4 depicts a linear *capacitor*, with capacitance C farad (F) in SI units. A voltage generator produces the possibly time-varying voltage difference $e_1 - e_2$ across the capacitor. The graphical symbol representing the capacitor depicts two plates separated by a dielectric (insulating) material. If there is a voltage difference between the plates of such a component, a positive electrical charge $+q$ coulombs (SI unit) appears on one plate, and a negative electrical charge $-q$ coulombs appears on the other plate (Halliday and Resnick, 1960, Section 30-2). The quantity of charge q is proportional to the voltage difference, with capacitance C defined as the constant of proportionality:

$$q = C\left(e_1 - e_2\right) \tag{5.2.2}$$

If the voltage difference is constant, then the charge on the plates remains constant, so that there is no flow of charged particles, *i.e.*, no current, which is the variation with time of the charge: $i(t) = \frac{dq}{dt}$. However, if the voltage varies with time, then current flows through the circuit in proportion with the derivative of the voltage difference:

$$i(t) = \frac{dq}{dt} = C\frac{d\left(e_1 - e_2\right)}{dt} = C\left(\dot{e}_1 - \dot{e}_2\right) \tag{5.2.3}$$

Conversely, the voltage difference across the capacitor can be expressed in terms of the current by integrating Equation 5.2.3 from initial time t_0 (when ICs are assumed to be known) to arbitrary time $t > t_0$:

$$\int_{\tau=t_0}^{\tau=t} i(\tau)d\tau = C\int_{\tau=t_0}^{\tau=t} \frac{d\left(e_1 - e_2\right)}{d\tau}d\tau = C\left[\left(e_1 - e_2\right)\big|_t - \left(e_1 - e_2\right)\big|_{t_0}\right]$$

$$\Rightarrow e_1(t) - e_2(t) = e_1\left(t_0\right) - e_2\left(t_0\right) + \frac{1}{C}\int_{\tau=t_0}^{\tau=t} i(\tau)d\tau \tag{5.2.4}$$

Note from Equation 5.2.3 the unit equivalence $A = F \times \frac{V}{s} \Rightarrow F = \frac{As}{V}$; this relation for the farad in terms of the amp, the volt, and the second is useful for clarifying units in calculations. Suppose, for example, that an instrumentation circuit contains a capacitor with C = 0.25 μF = 0.25e−6 F, a typical value. If, at a particular instant, the voltage across the capacitor is changing at the rate 6 000 V/s, then the current through the capacitor is, from Equation 5.2.3, $i = \left(0.25 \times 10^{-6}\,\frac{\text{A-sec}}{\text{V}}\right) \times \left(6 \times 10^{3}\,\frac{\text{V}}{\text{sec}}\right)$ $= 1.5 \times 10^{-3}\text{A} \equiv 1.5\text{mA}$.

Example 5.2.3 - 1st order, RC low-pass filter

This is a circuit containing both a resistor and a capacitor. The input voltage signal produced by some source is denoted $e_i(t)$, and the output, filtered signal is denoted $e_o(t)$. Figure 5.2.5 depicts this filter graphically in both the simple closed-circuit form on

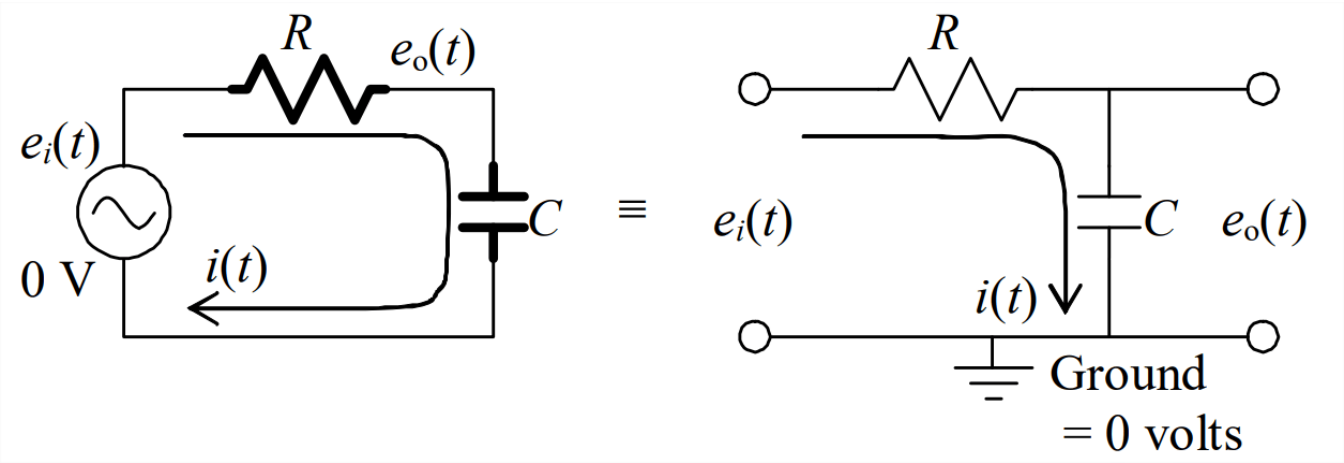


Figure 5.2.5: RC low-pass filter circuit

the left, and the more modern form on the right, in which the common ground (reference) potential is denoted by a special symbol and is assigned the value zero volts, to which all other circuit voltages are referenced. The input and output terminals are denoted by small circles. Note, in particular, that the output terminals are isolated away from the current-carrying portion of the circuit; this represents the realistic situation in which the filter output voltage is the input to some other circuit that has a very high input resistance, thereby essentially preventing any input current. This downstream circuit could be some measuring instrument (oscilloscope, voltmeter, *etc.*), or another stage of a larger circuit of which the RC circuit is just one part. Note also that this RC circuit, and any upstream circuit at its input, and any downstream circuit at its output, all must be referenced to the **same ground voltage**, which is constant and is the "zero" voltage relative to the other voltages in the circuit; this is an important requirement for practical circuits.

Solution

In order to derive the ODE describing the dynamics of this RC low-pass filter, we use Equation 5.2.1 for current i_R through the resistor, and Equation 5.2.3 for current i_C through the capacitor: $i_R = \frac{e_i - e_o}{R}, i_C = C\left(\dot{e}_o - \dot{0}\right) = C\dot{e}_o$. The resistor and capacitor are in series, so Kirchhoff's current law relevant to this situation is $i_C = i_R$:

$$\frac{e_i - e_o}{R} = C\dot{e}_o \Rightarrow \dot{e}_o + \frac{1}{RC}e_o = \frac{1}{RC}e_i \Rightarrow \dot{e}_o + \frac{1}{\tau_1}e_o = \frac{1}{\tau_1}e_i \tag{5.2.5}$$

in which $\tau_1 \equiv RC$ is the 1st order time constant.

Given ICs on output voltage $e_o(t)$ and given input voltage $e_i(t)$, we can in principle solve Equation 5.2.5 for $e_o(t)$ using the mathematical methods discussed previously. In particular, frequency response is the dynamic behavior that is of greatest practical interest for any circuit designed to be a filter. For frequency response, the input voltage is $e_i(t) = E_i \cos\omega t$ and the steady-state sinusoidal output voltage is $e_o(t) = E_o(\omega) \times \cos(\omega t + \phi(\omega))$. Let us find the frequency response simply by adapting a previously derived standard solution. First, we compare Equation 5.2.5 with Equation 3.4.8, $\dot{x} + (1/\tau_1)\, x = bu(t)$, which is the standard ODE for stable 1st order systems. If we define $u(t) = U\cos\omega t \equiv e_i(t) \Rightarrow U \equiv E_i$, then the other constant of the standard equation becomes $b = 1/\tau_1$. Therefore, the standard FRF magnitude ratio Equation 4.5.7 is adapted as

$$\frac{X(\omega)}{U} = \frac{b\tau_1}{\sqrt{1 + (\omega\tau_1)^2}} = \frac{E_o(\omega)}{E_i} = \frac{1}{\sqrt{1 + (\omega\tau_1)^2}} = \frac{1}{\sqrt{1 + (\omega/\omega_b)^2}} \tag{5.2.6}$$

and the standard FRF phase Equation 4.5.8 is adapted as $\phi(\omega) = \tan^{-1}(-\omega\tau_1) = \tan^{-1}(-\omega/\omega_b)$ in which the break frequency is $\omega_b = 1/\tau_1 = 1/RC = 2\pi f_b$. Figure 5.2.6 (adapted from Figure 4.3.3) is the log-log graph of magnitude ratio versus frequency, which clearly shows the low-pass character of the frequency response.

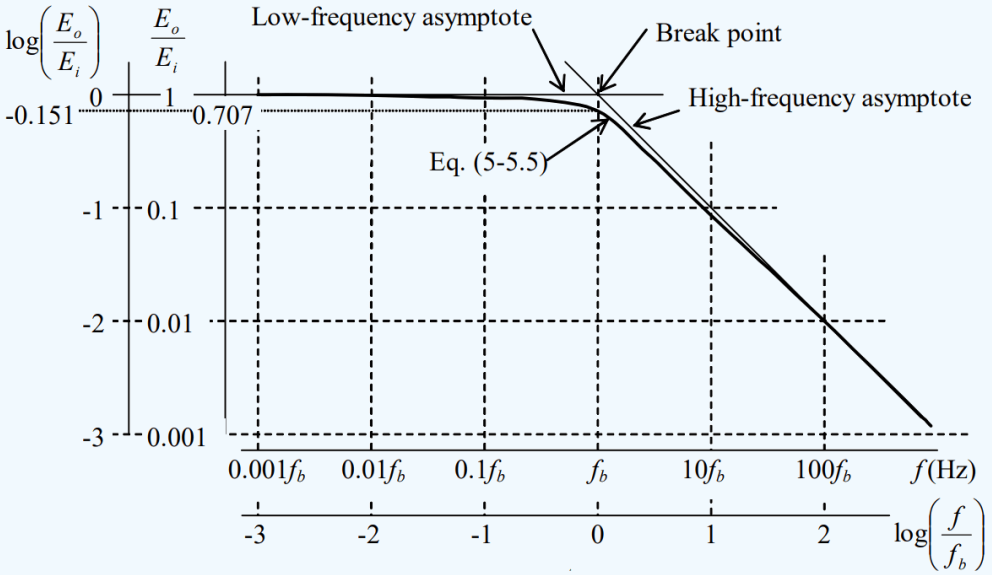


Figure 5.2.6: Log-log magnitude ratio of 1st order RC low-pass filter

The governing ODE of the mechanical series damper-spring low-pass filter in Figure 3.7.4 is Equation 3.7.5, $\tau_1 \dot{x}_o + x_o = x_i(t)$ (see also homework Problem 4.3), which has exactly the same form as Equation 5.2.5. Accordingly, the electrical low-pass filter is an exact electrical analog of the mechanical low-pass filter, input voltage $e_i(t)$ being directly analogous to input translation $x_i(t)$, output voltage $e_o(t)$ to output translation $x_o(t)$, and electrical time constant $\tau_1 = RC$ to mechanical time constant $\tau_1 = c/k$.

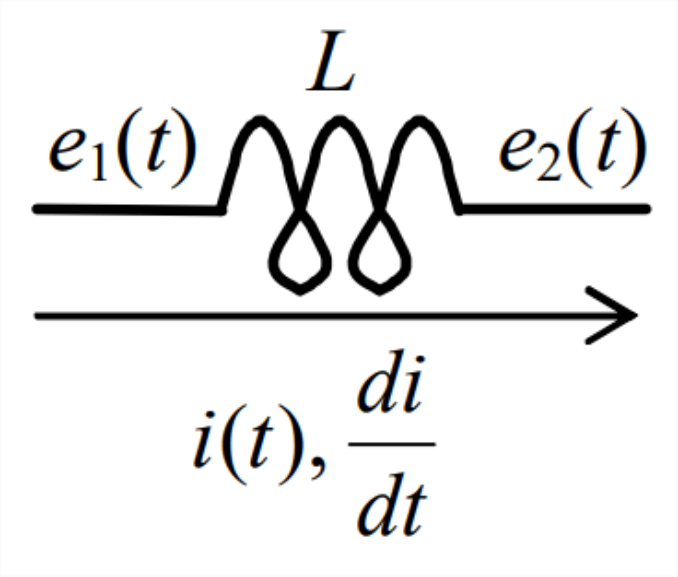


Figure 5.2.7: Ideal inductor

The passive electrical component drawn in Figure 5.2.7 represents an ideal linear *inductor*, with inductance L henry (H) in SI units. A time-varying current $i(t)$ is shown flowing through the inductor. The graphical symbol denoting the inductor depicts a coil of conducting wire. If the current is changing with time in such a coil, then a voltage difference appears across the coil, a voltage difference that **opposes** the current change (Halliday and Resnick, 1960, Sections 36-1 and 36-3). This voltage difference is called a *self-induced emf* (electromotive force); it is a manifestation of the interaction between electricity and magnetism that is described by Maxwell's equations of electromagnetism (after Scottish physicist James Clerk Maxwell, 1831-1879). The self-induced emf is proportional to the rate of change of current, with inductance L defined as the constant of proportionality:

$$e_1 - e_2 = L\frac{di}{dt} \tag{5.2.7}$$

You might find it difficult to perceive how Equation 5.2.7 represents physically a voltage difference that opposes current change di/dt. If so, then the following example should illustrate this characteristic more understandably.

Example 5.2.4 - 1st order LR circuit

Real inductors are not used in instrumentation circuits nearly as often as resistors and capacitors. Moreover, there is no such thing as the ideal inductor instrumentation component of Figure 5.2.7. Because an inductor component consists primarily of coiled wire, and because a considerable length of very fine wire accumulates resistance, a real inductor usually has non-negligible resistance as well as inductance. A common, simple, approximate circuit model for such a real inductor is a series combination of an ideal inductor and a resistor, L and R_L in the circuit of Figure 5.2.8. In this circuit, $e_i(t)$ is the input voltage (after switch S is closed to position c), and $e_o(t)$ is the output voltage. R is a resistor placed between the inductor and ground to permit sensing (by an oscilloscope, for example) of $e_o(t)$, which is directly related to the current by $e_o(t) = Ri(t)$.

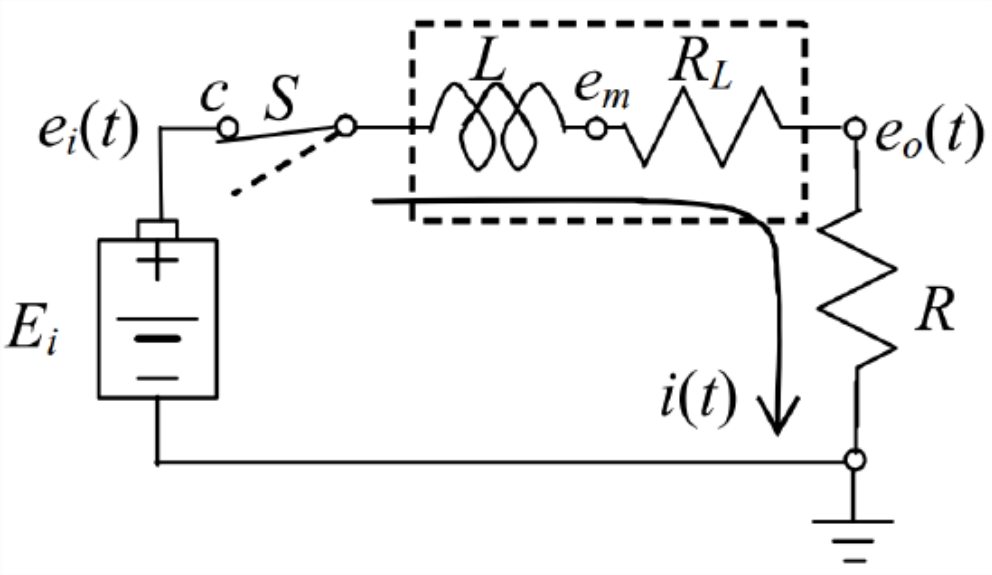


Figure 5.2.8: LR circuit

Solution

Next, we derive an ODE governing the dynamic behavior of the LR circuit in Figure 5.2.8. Observe that the **derivative** of current, di/dt, appears in Equation 5.2.7, unlike Equation 5.2.1 for a resistor and Equation 5.2.3 for capacitor, in which equations current $i(t)$ appears directly. Because of this, it is usually not convenient to apply Kirchhoff's current law for a circuit containing an inductor. It is usually better for such a circuit to apply *Kirchhoff's voltage law* (abbreviated "KVL"): *the sum of all voltage rises around a circuit loop is zero*. If we proceed around the circuit loop in the direction of defined positive current flow, then the "voltage rise" across any component (including the input voltage generator and every passive component) is defined to be the upstream voltage minus the downstream voltage. To write KVL for this LR circuit, we start at the input voltage generator and proceed clockwise:

$$(e_i - 0) + (e_m - e_i) + (e_o - e_m) + (0 - e_o) = 0 \tag{5.2.8}$$

Kirchhoff's voltage law is a fundamental law of circuits, but note that it also is just an algebraic identity. Next, we substitute Equation 5.2.7 and Ohm's law into Equation 5.2.8:

$$(e_i - 0) + \left(-L\frac{di}{dt}\right) + (-R_L i) + (-Ri) = 0 \Rightarrow L\frac{di}{dt} + (R_L + R)\, i = e_i(t) \tag{5.2.9}$$

Equation 5.2.9 is a stable 1st order LTI ODE solvable for arbitrary input voltage $e_i(t)$. For example, suppose that all ICs = 0 and that the input voltage is a step, $e_i(t) = E_i H(t)$, which is accomplished with use of a battery and a switch, as shown in Figure 5.2.8. Comparing Equation 5.2.9 with standard stable 1st order ODE Equation 3.4.8, we adapt the standard step-response solution Equation 3.4.9 with $e_i(t) \equiv u(t)$ so that $E_i \equiv U$, to write the electrical response quantities as

$$i(t) = \frac{E_i}{R_L + R}\left(1 - e^{-t/\tau_1}\right), 0 \le t \quad \text{in which} \quad \tau_1 = \frac{L}{R_L + R} \tag{5.2.10}$$

$$\Rightarrow e_o(t) = Ri(t) = \frac{R}{R_L + R} E_i \left(1 - e^{-t/\tau_1}\right) \equiv E_o \left(1 - e^{-t/\tau_t}\right), \quad E_o \equiv \frac{R}{R_L + R} E_i \tag{5.2.11}$$

Step response Equations 5.2.10 and 5.2.11 clearly show the effect of the inductance in opposing and delaying the rise of current flow. If the inductor were not present (*i.e.*, if L = 0) in the circuit of Figure 5.2.8, then the circuit would be a simple voltage divider (Problem 5.1), and the full step-response output voltage would be achieved instantly, $e_o(t) = E_o H(t)$, instead of rising gradually as in Equation 5.2.11.

The response of this circuit illustrates the physical effect of the inductor's self-induced emf, Equation 5.2.7. In this circuit, the inductor's upstream voltage is constrained to be the input, $e_i(t)$, so the self-induced emf must act in $e_m(t)$, the voltage at the inductor's downstream terminal. It is helpful to derive from Equations 5.2.8, 5.2.9, 5.2.10, and 5.2.11 the equation for that voltage, $e_m(t) = e_o(t) + R_L i(t) = E_i\left(1 - e^{-t/\tau_T}\right)$, and the equation for the rate of change of current, $di/dt = (E_i/L)\, e^{-t/\tau_1}$. At time t = 0^+ (just after switch S is closed to position c), we have $di/dt(0+) = E_i/L > 0$ but $e_m(0+) = 0$ and $i(0+) = 0$. Thus, at this instant, the inductor holds its downstream voltage at zero and prevents the current from rising instantaneously as a step function, which it would do if the inductor were not present. After time $t = 0+$, the current gradually rises, but at a declining rate, and the current increase is still opposed by the inductor downstream voltage, which also rises at the same declining rate.

From Equation 5.2.7, we have the unit equivalence $\mathrm{V} = \mathrm{H} \times \frac{\mathrm{A}}{\mathrm{s}} \Rightarrow \mathrm{H} = \frac{\mathrm{V-s}}{\mathrm{A}}$; this relation for the henry is useful for clarifying units in calculations, as shown in the following example. A particular "voice coil" is to be used in an electromagnetic force actuator; this is the type of wire coil found in the speakers of consumer sound systems. We wish to identify experimentally for this coil the parameters L and R_L (based upon the series model of Electricity Example 5.2.4) by measuring the step response of the circuit of Figure 5.2.8. For this particular circuit, the sensing resistor has R = 17.5 Ω, and the battery voltage is 9.00 V. We close switch S, then record the subsequent time-history graphical response onto the screen of a digital oscilloscope, which stores the data for analysis. The response graph (below) has the appearance of Figure 3.4.2, and we measure from it the time

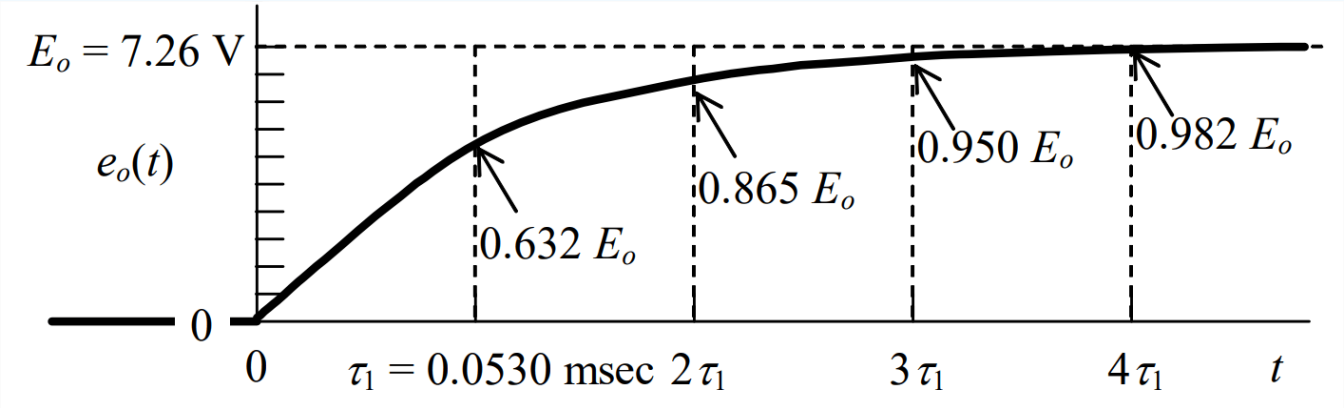


Figure 5.2.9

constant $\tau_1 = 0.0530$ ms and the final value of output voltage Eo = 7.26 V. From the equations for τ_1 in Equation 5.2.10 and in Equation 5.2.11, we derive the following equations for the required parameters L and R_L, and then the subsequent calculated values:

$$L = (R_L + R)\,\tau_1 = \left(\frac{R}{E_o} E_i\right)\tau_1$$
$$= \frac{E_i}{E_o} R\tau_1 = \left(\frac{9.00\mathrm{V}}{7.26\mathrm{V}}\right)\left(17.5\frac{\mathrm{V}}{\mathrm{A}}\right)\left(0.0530 \times 10^{-3}\mathrm{sec}\right) = 1.15 \times 10^{-3}\frac{\mathrm{V-sec}}{\mathrm{A}} \equiv 1.15\mathrm{mH}$$

$$R_L = \frac{R}{E_o}E_i - R = R\left(\frac{E_i}{E_o} - 1\right) = (17.5\Omega)\left(\frac{9.00\text{V}}{7.26\text{V}} - 1\right) = 4.19\Omega$$

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5.3: Operational Amplifier (op-amp) and Op-amp Circuits

Figure 5.3.1 is the standard graphical symbol for an operational amplifier (op-amp). An op-amp has a "positive" input port that accepts input voltage e_{ip}, and a "negative" input port that accepts input voltage e_{in}. The symbols e_{ip} and e_{in} are merely labels; they are not meant to restrict the polarities of these input voltages, each of which can be either positive or negative relative to the ground potential. An opamp has a single output voltage, labeled e_o on Figure 5.3.1. The fundamental ideal output-to-input relationship of an op-amp is (Horowitz and Hill, 1980, Chapter 3)

$$e_o = K\left(e_{ip} - e_{in}\right) \tag{5.3.1}$$

In Equation 5.3.1, *gain* K is a very large positive number, on the order of 10^5 to 10^6 for common, commercially available op-amps. The exact value of K varies gradually with the frequency of signals e_{ip} and e_{in}, but this variation is not important provided that frequency is below a known upper value; as we shall see, a very important characteristic is that K is large, $K \geq \mathrm{O}\left(10^5\right)$. (This equation is a common mathematical expression meaning "K is on the order of or greater than 10^5.") Another important characteristic of an op-amp is the extremely high resistance of the input ports, on the order of $10^6\ \Omega$ to $10^{12}\ \Omega$. The practical consequence of this high resistance is that essentially zero current can flow through the input ports.

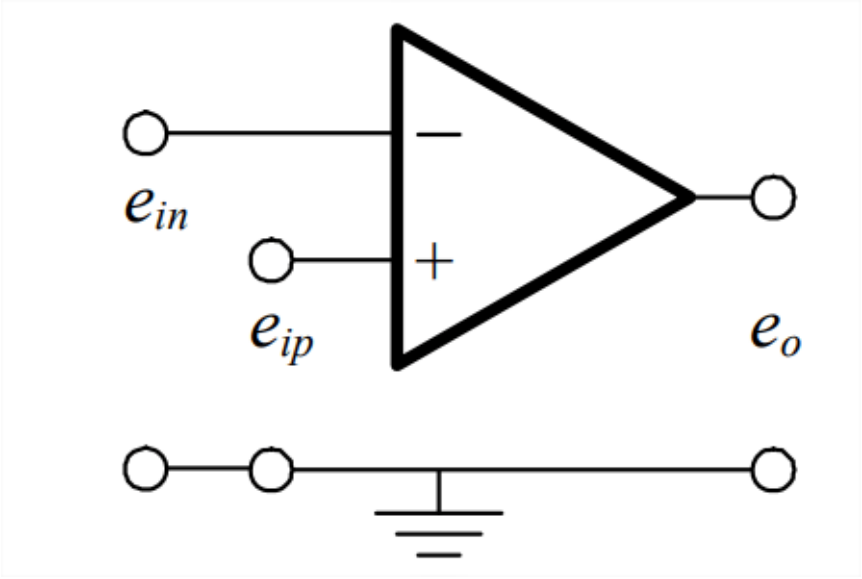


Figure 5.3.1: Operational amplifier (op-amp)

An op-amp is an active device, requiring external power to produce high gain, unlike the simple passive elements (resistor, capacitor, and inductor) of Section 5.2. An energy source (e.g., a $\pm$15-volt power supply, or a pair of 9-volt batteries) is usually connected to an op-amp, but this connection is normally not indicated on graphical representations such as Figure 5.3.1. An op-amp itself is a complex integrated circuit, full of miniaturized transistors and other electrical components. The physical form of op-amp seen most commonly on circuit boards is approximately the size and similar in appearance to a basement centipede with a black body (actually, more like a 8-legged or 16-legged insect, because it has on each side four or eight metal connectors that look like legs).

Op-amps are not often used in the *open-loop* configuration of Figure 5.3.1. Most op-amps can operate linearly according to Equation 5.3.1 only over a limited range of $\pm E_{lim}$ on output voltage e_o. The range depends upon the energy source, but typically E_{lim} is on the order of 10 V. If the input voltages e_{ip} and e_{in} are such that Equation 5.3.1 leads numerically to e_o greater than $+E_{lim}$ or less than $-E_{lim}$, then a real op-amp will limit and stick nonlinearly on either $+E_{lim}$ or $-E_{lim}$, respectively. When this happens, the op-amp is said to be *overloaded* or *saturated*. Since gain K in Equation 5.3.1 is so high, the input voltage difference $e_{ip} - e_{in}$ clearly must be very small in order for the op-amp to operate in the linear range for which it is primarily designed.

To operate an op-amp in its linear range, we almost always use *feedback*. When there is an electrical connection between the output port and the negative input port, the op-amp is said to be wired in a *closed-loop* manner, with feedback from output to

input, specifically in this case, *negative feedback*. This negative feedback acts to make the input voltage difference $e_{ip} - e_{in}$ so small that for practical purposes there is no difference, $e_{ip} \approx e_{in}$.

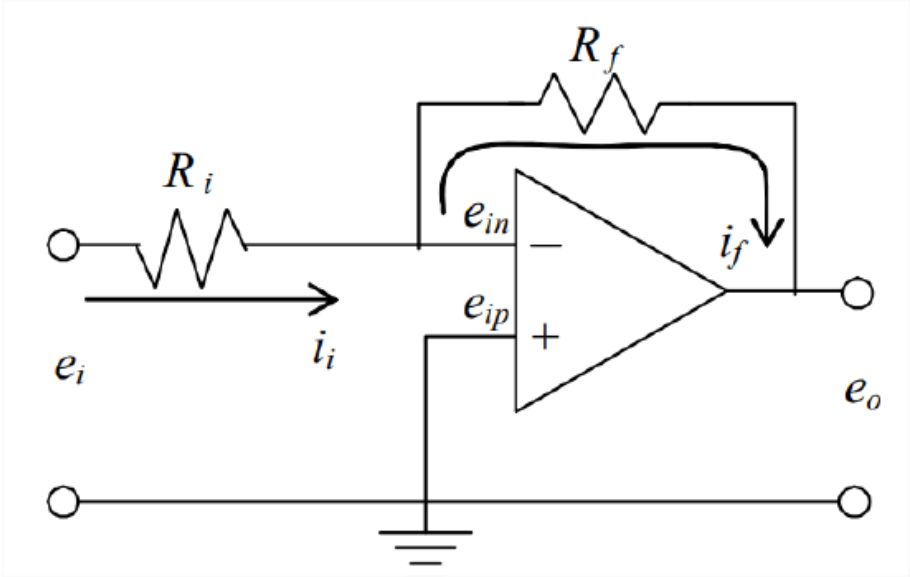


Figure 5.3.2: Inverting amplifier

We can illustrate the practical functioning of an op-amp by analyzing in detail what is probably the most common basic circuit consisting of an op-amp and resistors, the *inverting amplifier* depicted in Figure 5.3.2. Note that there is an input resistor R_i, and that there is negative feedback through feedback resistor R_f. These resistances are chosen to be on the order of 10^1 -10^4 Ω, at least two orders of magnitude less than the input resistances of the op-amp ports. Note also that the positive input port is grounded, e_{ip} = 0 V. Hence, from Equation 5.3.1,

$$e_o = K\,(0 - e_{in}) \Rightarrow e_{in} = -\frac{e_o}{K} \tag{5.3.2}$$

Due to the extremely high resistance of the negative input port relative to R_i and R_f, the current through that port is essentially zero, so Kirchhoff's current law in this case requires the feedback current to equal the input current, $i_i = i_f$. Using Ohm's law to write this condition of current continuity in terms of voltages gives

$$\frac{e_i - e_{in}}{R_i} = \frac{e_{in} - e_o}{R_f} \Rightarrow \frac{e_i - (-e_o/K)}{R_i} = \frac{(-e_o/K) - e_o}{R_f} \tag{5.3.3}$$

With a little algebra (which you should verify on your own), the solution of Equation 5.3.3 for circuit output voltage in terms of input voltage is found to be

$$e_o = -\frac{R_f}{R_i} e_i \frac{1}{1 + \frac{1}{K}\left(1 + \frac{R_f}{R_i}\right)} \tag{5.3.4}$$

Let us evaluate the denominator of the large fraction in Equation 5.3.4. In typical applications of this circuit, the resistance ratio R_f/R_i is on the order of 10^2 at most. Therefore, with gain $K = \mathrm{O}\left(10^5\right)$, the denominator is very, very close to 1: $1 + \mathrm{O}\left(10^{-3}\right) \approx 1$. (This validates the earlier statement that only the large magnitude of K matters, the exact value being unimportant.) So, the entire large fraction is essentially equal to one, and Equation 5.3.4 simplifies considerably to

$$e_o = -\frac{R_f}{R_i} e_i \tag{5.3.5}$$

The output voltage equals the input voltage amplified by the ratio R_f/R_i, and the sign is inverted; hence the name *inverting amplifier*.

Note also, from Equation 5.3.2, that the voltage at the negative input port is negligibly small relative to the output (and input) voltages:

$$e_{in} = -\frac{e_o}{K} \approx 0 = e_{ip}(\text{ the voltage of the grounded port }) \tag{5.3.6}$$

In other words, the op-amp's high gain drives the voltage e_{in} at the negative input port to be essentially equal to the voltage e_{ip} at the positive input port. Equation 5.3.6 is just a special case of the simplifying assumption that we can use in general, from Equation 5.3.1,

$$e_{ip} - e_{in} = \frac{e_o}{K} \approx 0 \Rightarrow e_{in} = e_{ip} \tag{5.3.7}$$

In circuit analysis, Equation 5.3.7 is considered a useful "rule" or "axiom" rather than just an assumption. In the future (unless directed otherwise, as for a homework or exam problem), you should always apply rule Equation 5.3.7 right from the very beginning of the derivation for an op-amp circuit with negative feedback, because it simplifies the derivation so much. For example, if we use Equation 5.3.7 from the beginning for the inverting amplifier, then the derivation becomes two easy steps [using $e_{in} = e_{ip} = 0$ in Equation 5.3.3]:

$$\frac{e_i - 0}{R_i} = \frac{0 - e_o}{R_f} \Rightarrow e_o = -\frac{R_f}{R_i} e_i$$

Figure 5.3.3 depicts an extremely simple, closed-loop op-amp circuit that is useful in practice. The input voltage signal is directed into the positive input port, and this port's high resistance prevents the flow of any current from the input source. Equation 5.3.7 in combination with the feedback connection states that the output voltage exactly equals the input voltage, $e_o = e_i$. This op-amp circuit functions as a current isolator and voltage transmitter, and it is usually called a *voltage follower*. Its main value is in providing a buffer between two different stages of a more complex circuit: the buffer allows the output voltage of the upstream stage to be the input voltage of the downstream stage, without permitting any current flow between the two stages. Any such interstage current flow would usually cause malfunctioning of both stages. The application of a voltage follower as a buffer between circuit stages is illustrated in the next section.

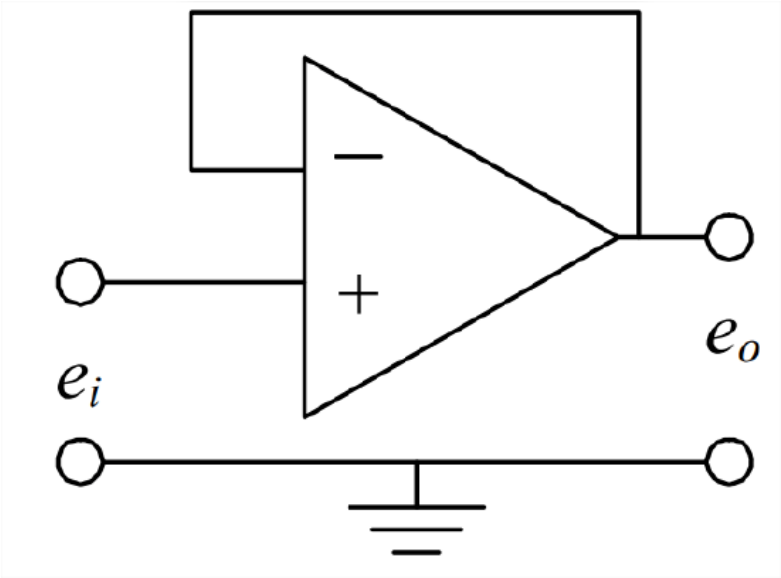


Figure 5.3.3: Voltage follower

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5.4: RC Band-Pass Filter

The circuit shown in Figure 5.4.1 consists of a low-pass filter stage to the left of the voltage follower, and a high-pass filter stage to the right. [The functioning of a high-pass filter is illustrated in homework Problem 5.4.2]. As indicated on Figure 5.4.1, the currents in the two stages are independent of each other, by virtue of the buffering due to the voltage follower. Therefore, the simple ODEs for each type of 1st order filter are still valid for the two-stage circuit of Figure 5.4.1, with mid-circuit voltage $e_m(t)$ being the quantity shared by the two stages, as the output from the low-pass stage and the input to the high-pass stage. These ODEs are Equation 5.2.7 for the low-pass filter,

$$\tau_L \dot{e}_m + e_m = e_i, \quad \tau_L = R_L C_L \tag{5.4.1}$$

and the ODE derived in homework Problem 5.4.1 for the high-pass filter,

$$\tau_H \dot{e}_o + e_o = \tau_H \dot{e}_m, \quad \tau_H = R_H C_H \tag{5.4.2}$$

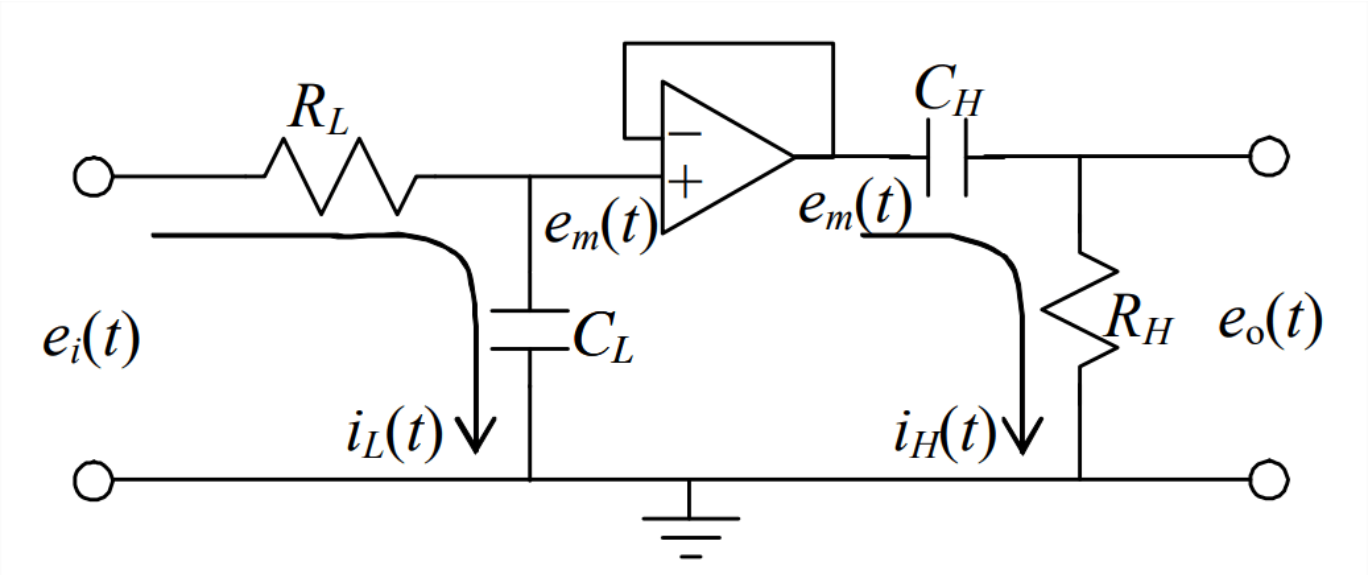


Figure 5.4.1: RC band-pass filter

An equation such as Equation 5.4.2 is described as having "right-hand-side (RHS) dynamics" because the right-hand-side includes a derivative of the input ($\dot{e}_m$ in this case), rather than just the undifferentiated input itself.

The combination of these two 1st order circuits turns out to be a 2nd order system, and we shall re-visit this subject in Sections 9.10 and 10.4. We consider the RC band-pass filter circuit now because it illustrates (1) application of a voltage follower, and (2) the important physical characteristic of op-amps that is described next.

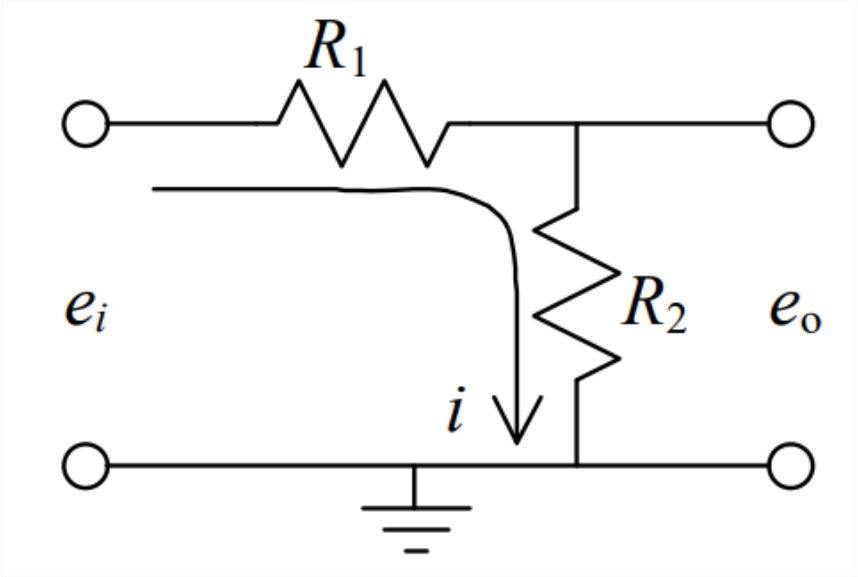
Note in Figure 5.4.1 that the feedback wire across the op-amp connects directly to the negative input port; this port has essentially infinite resistance, so there cannot be any current in the feedback wire. But Figure 5.4.1 also shows the non-zero, second-stage current $i_H(t)$ downstream of the op-amp; this appears to contradict the claim of zero feedback current in the op-amp, since the graphical representation of the op-amp suggests that the feedback wire is continuous electrically with the downstream circuit. **In fact, the standard graphical representation of an op-amp with negative feedback, such as that in Figure 5.4.1, is oversimplified to the point of being misleading. For an actual op-amp (as opposed to the standard graphical representation), the downstream current is *not continuous* with the feedback current, but instead, is completely independent.** In fact, the current downstream of an op-amp is determined only by the output voltage of the op-amp and the downstream electrical components, for example, the second-stage capacitor and resistor in Figure 5.4.1. The technical characteristic of an op-amp that permits this independence of currents is very low *output impedance* (Horowitz and Hill, 1980, pages 25, 92-95, and 105). For another example of the independence of feedback and downstream currents, see homework Problem 5.10.

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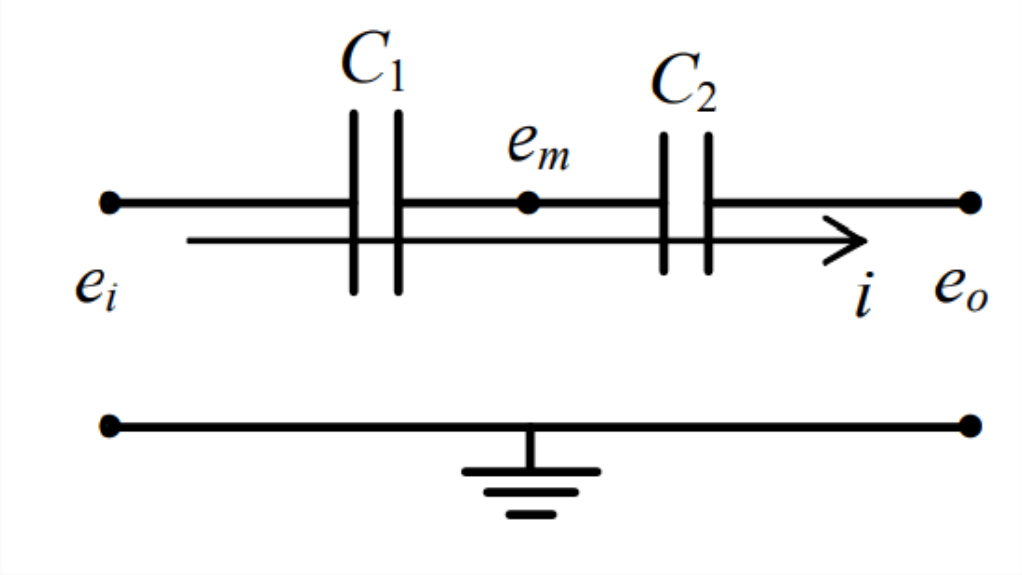
5.5: Chapter 5 Homework

1. The simple series circuit represented graphically below is a voltage divider, which is widely used in practical applications. Show that the voltage output-to-input ratio is

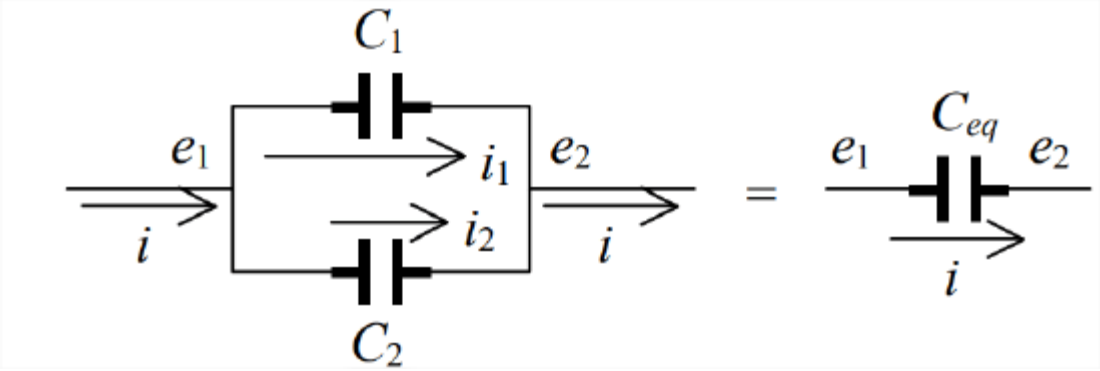
$$\frac{e_o}{e_i} = \frac{R_2}{R_1 + R_2}$$



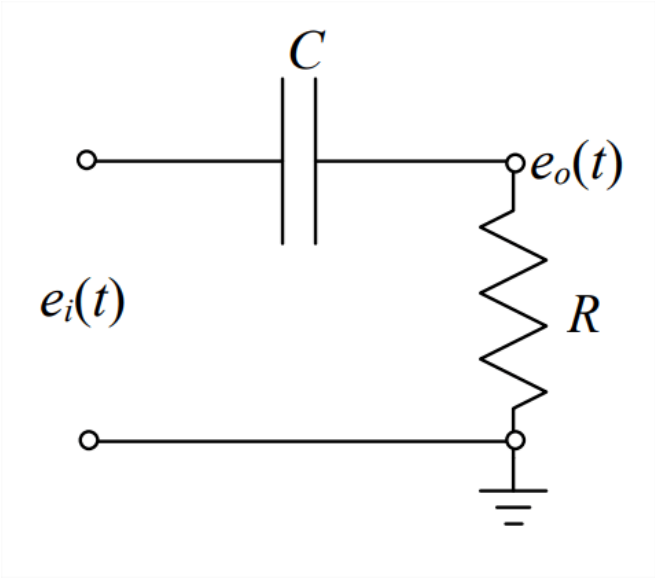
2. Two capacitors are arranged in series, as shown below.



    1. Use the current-voltage equation for capacitor C_1 to show that $\dot{e}_m = \dot{e}_i - \frac{i}{C_1}$, where the e's are the voltages indicated on the drawing and i is the current.
    2. Use the result of part 5.2.1 and the capacitor current-voltage equation for capacitor C_2 in order to derive an algebraic formula for the **equivalent** series capacitance C_{eq} (in terms of C_1 and C_2) in the equation $i = C_{eq}\,(\dot{e}_i - \dot{e}_o)$.
3. Given capacitors C_1 and C_2 arranged in parallel, as shown below, find the equivalent single capacitance C_{eq}.



4. The simple circuit represented graphically below is a 1st order RC **high**-pass filter.



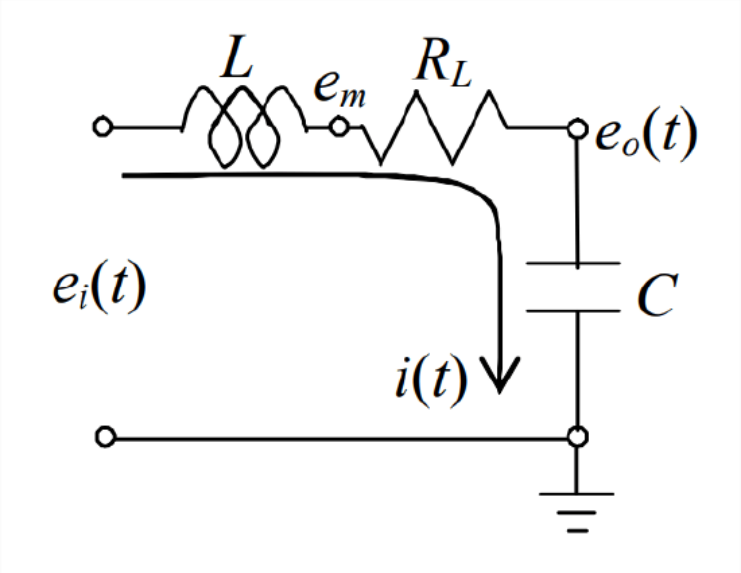
1. Use current continuity and the formulas relating current to voltage-difference for resistors and capacitors to derive in all detail the following ODE with right-hand-side (RHS) dynamics, which governs the output voltage $e_o(t)$ in terms of input voltage $e_i(t)$:

$$\dot{e}_o + \frac{1}{\tau_1} e_o = \dot{e}_i, \tau_1 = RC$$

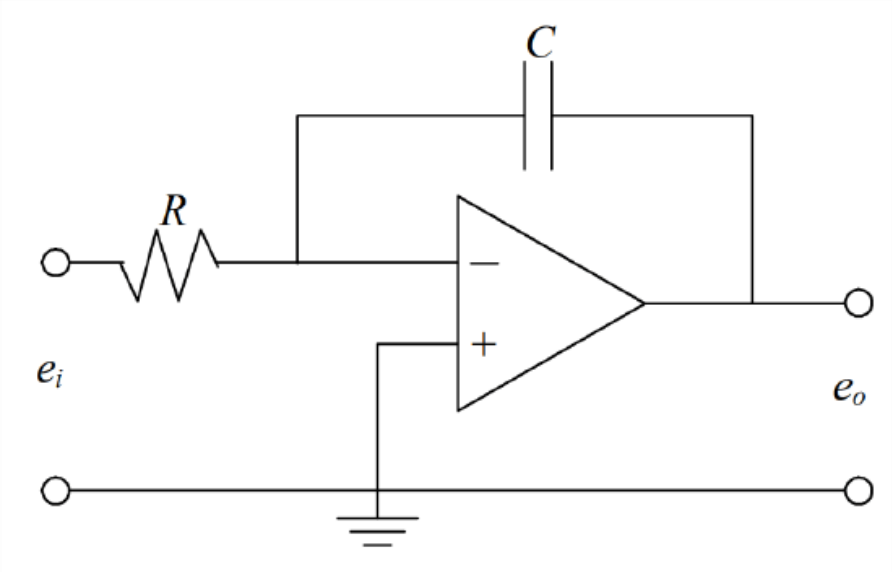
Explain why this circuit is considered to be an exact electrical analog of the mechanical series damper-spring high-pass filter described in Figure 3.7.4 and Equation 3.7.6.

2. For frequency response, the input voltage is $e_i(t) = E_i \cos \omega t$ and the steady-state output voltage is $e_o(t) = E_o \cos(\omega t + \phi)$. Use the governing equation from part 5.4.1 to derive the algebraic equations for FRF of the 1st order high-pass filter: the magnitude ratio $E_o(\omega)/E_i$ and the phase $\phi(\omega)$ (in radians) as functions of excitation frequency ω. (*partial answer*: $E_o(\omega)/E_i = \omega\tau_1/\sqrt{1+(\omega\tau_1)^2}$.) Suppose that R = 40 kΩ and C = 0.25 μF (recall that μ = 10^{-6}). Calculate the break frequency $f_b = 1/(2\pi\tau_1)$ in Hz. To get some feeling for the functioning of a high-pass filter, calculate the FRF magnitude ratio $E_o(\omega)/E_i$ for driving frequency ratios f/f_b = 0.01, 0.1, 1, 10, and 100. (See also homework Problem 4.4.)
3. Suppose that the input voltage is a ramp, $e_i(t) = e_r\,(t/t_r)\,H(t)$, where e_r is a reference voltage and t_r is a reference time such that $e_i\,(t_r) = e_r$. For IC $e_o(0)$ = 0 volt, solve the governing ODE from part 5.4.1 for the algebraic equation for $e_o(t)$ in terms of the given algebraic parameters. Note that the derivative of the input voltage in this case is a step function, $\dot{e}_i(t) = (e_r/t_r)\,H(t)$.

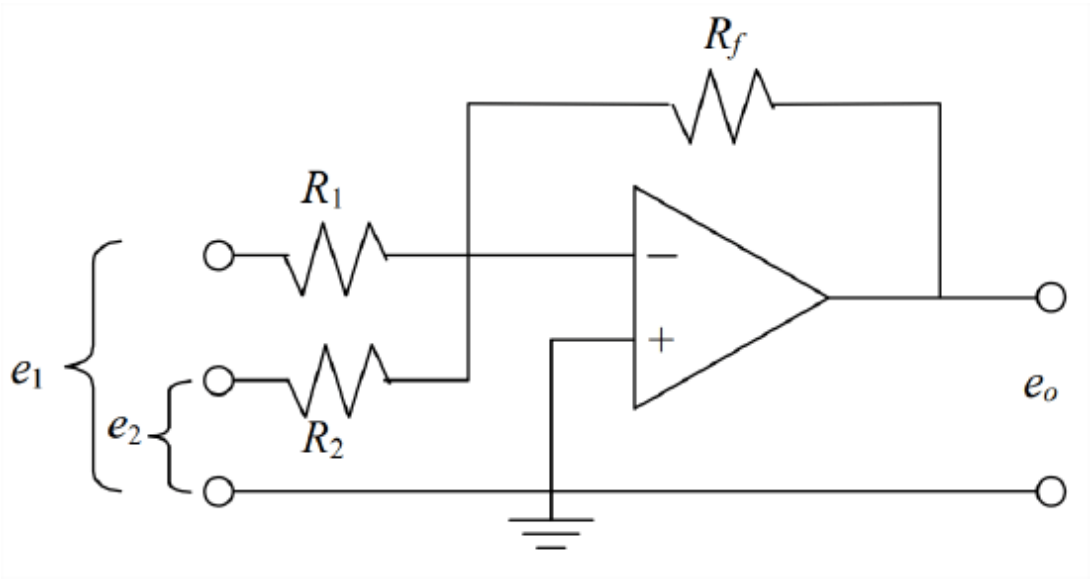
5. The circuit drawn below is a series combination of a voltage source $e_i(t)$, a coil (having both inductance L and resistance R_L), and a capacitor C; this is known as an LRC circuit. The input voltage is applied at time t = 0, at which time there is a non-zero voltage on the capacitor, . Apply Kirchhoff's voltage law, as in Electricity Example 5.2.4, and show that the equation for current $i(t)$ is $L\frac{di}{dt} + R_L i + \frac{1}{C}\int_{\tau=0}^{\tau=t} i(\tau)d\tau = e_i(t) - e_o(0)$.



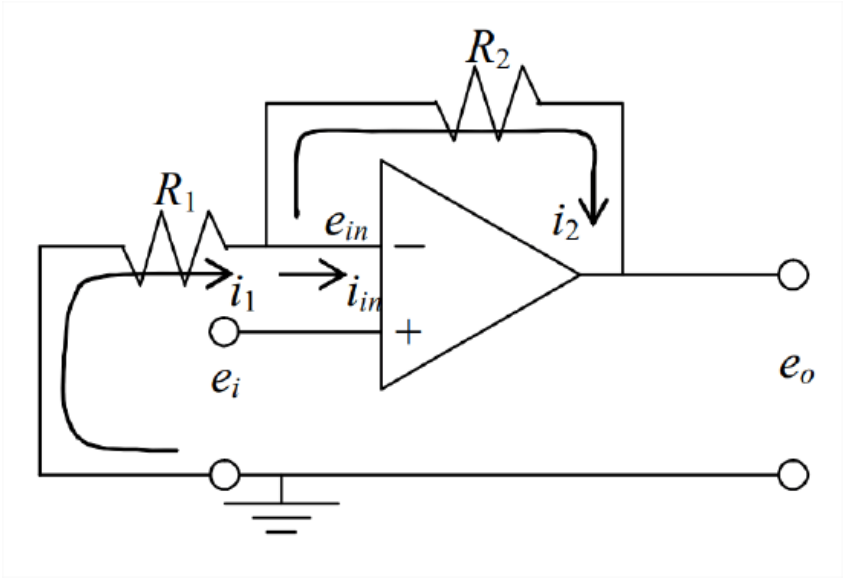
6. For the op-amp circuit represented graphically at right, an *integrator*, derive the algebraic equation for the transfer function between input and output voltages, $TF(s) = \frac{L[e_o(t)]}{L[e_i(t)]}$. Assume that there is no initial charge on the capacitor at time t = 0. To simplify the derivation, apply rule Equation 5.3.7.[1]



7. For the op-amp circuit represented graphically below, a *summing inverting amplifier*, derive the algebraic equation for output voltage $e_o(t)$ in terms of resistances R_1, R_2, and R_f, and input voltages $e_1(t)$ and $e_2(t)$. To simplify the derivation, apply rule Equation 5.3.7.



8. 1. For the op-amp circuit represented graphically below, a *non-inverting amplifier*, derive the algebraic equation for output voltage $e_o(t)$ in terms of resistances R_1 and R_2, and input voltage $e_i(t)$. To simplify the derivation, apply rule Equation 5.3.7.

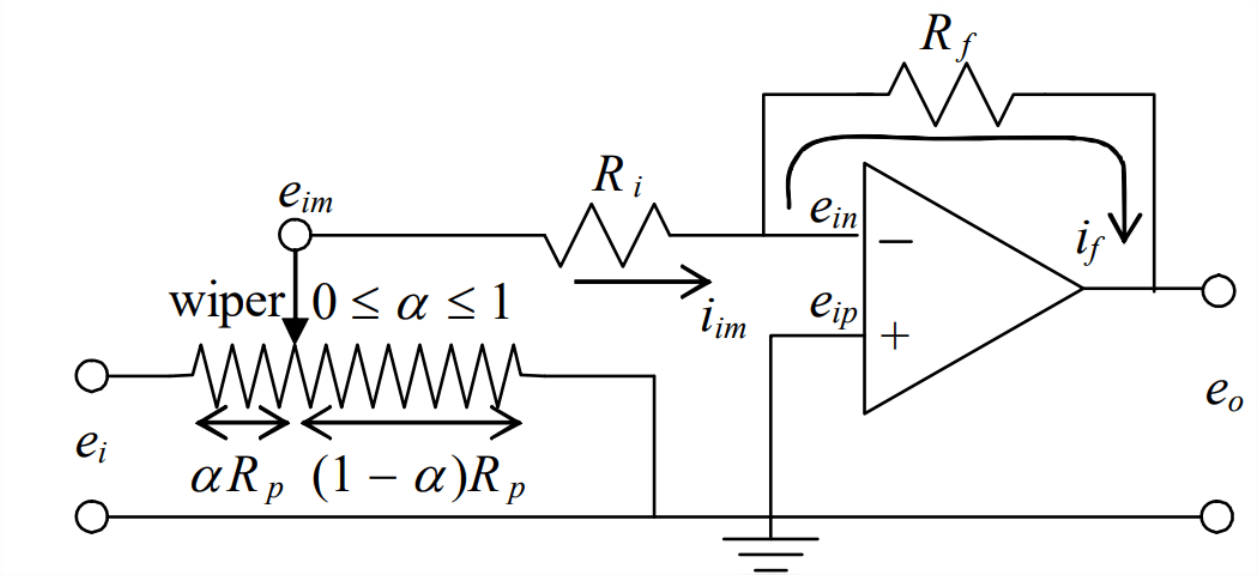


2. Suppose that e_i = 1.6 V, R_1 = 5 kΩ, and R_2 = 15 kΩ. Determine the values of e_{in} (in V), i_{in} (in milliamps, mA), i_1 (in mA), i_2 (in mA), and e_o (in V).

9. The circuit drawing below shows an inverting amplifier, from Figure 5.3.2, and, at the input of the amplifier, a *variable resistor* with a sliding contact, often called a wiper. The maximum useable resistance of the variable resistor is R_p; however, by moving the wiper along the surface of the resistor (adjusted by turning a dial or a screw), we can connect between e_i and e_{im} any portion αR_p, where $0 \leq \alpha \leq 1$. The purpose of this particular circuit configuration is to allow the gain constant of the amplifier to adjusted continuously to any value between zero and R_f/R_i, rather than being constrained to the value R_f/R_i of Equation 5.3.5. Your assignment is to derive the following voltage output-to-input equations:

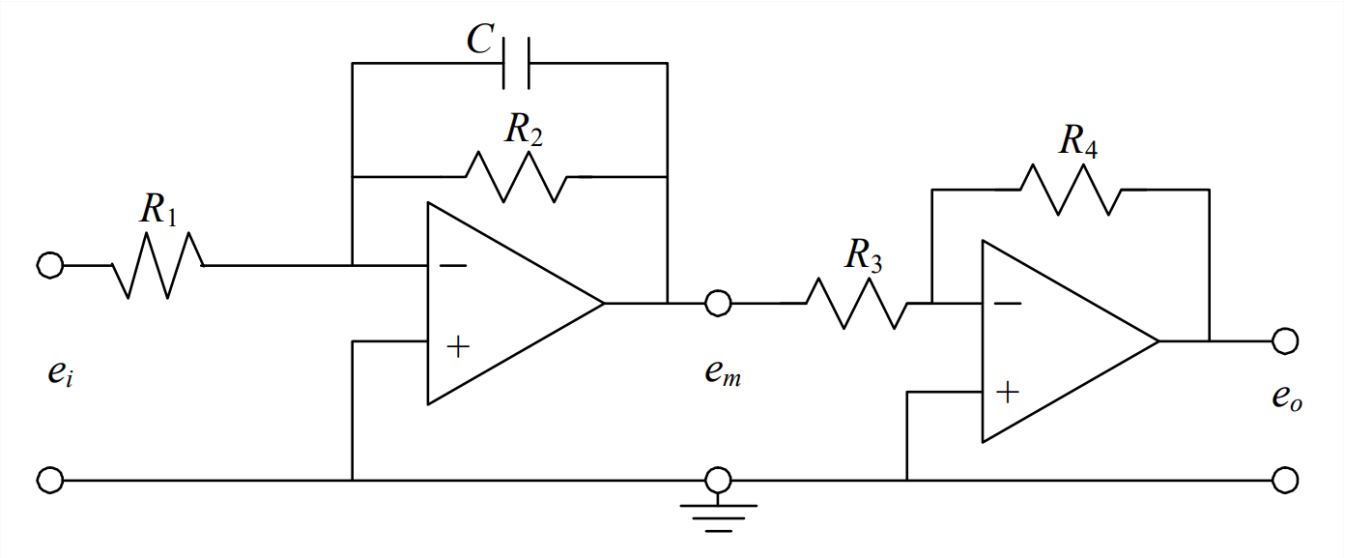
$$\frac{e_{im}}{e_i} = \frac{(1-\alpha)}{1+\alpha(1-\alpha)\frac{R_p}{R_i}} \Rightarrow \frac{e_o}{e_i} = -\frac{(1-\alpha)\frac{R_f}{R_i}}{1+\alpha(1-\alpha)\frac{R_p}{R_i}}$$

In your derivation, do not neglect the current to ground through the side of the variable resistor with resistance $(1-\alpha)R_p$. For applications in electronic analog computers (see the footnote to homework Problem 5.6), this arrangement of a variable resistor is called a *coefficient-setting potentiometer* or just a *pot* for short.



Inverting amplifier with coefficient-setting potentiometer

10. The circuit represented graphically below consists of two stages, each stage being built around an op-amp.

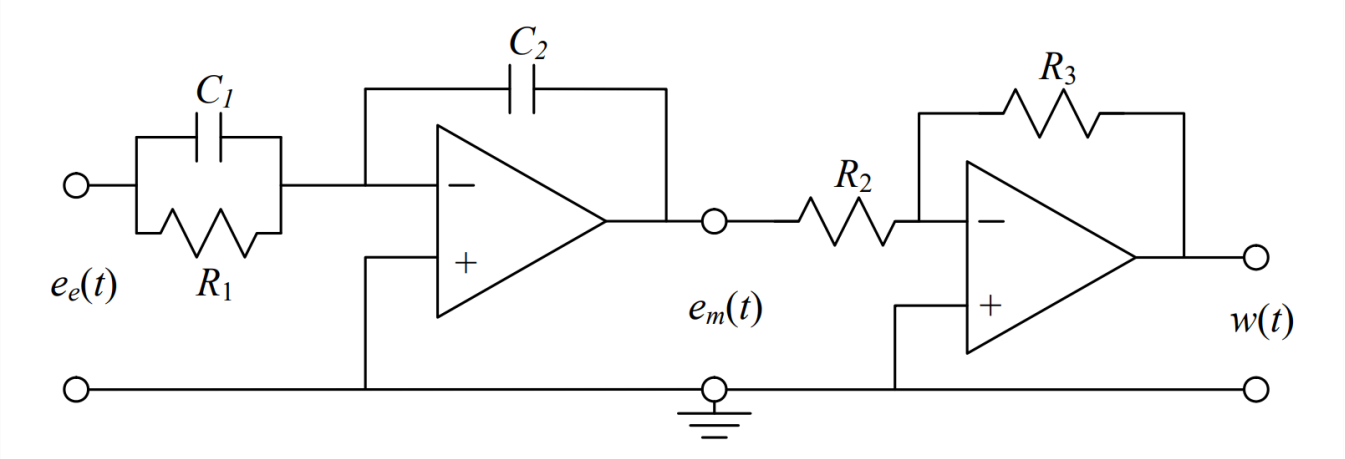


1. Derive from the left-hand stage the following ODE for interstage voltage $e_m(t)$:

$$\dot{e}_m + \frac{1}{R_2 C} e_m = -\frac{1}{R_1 C} e_i$$

2. Combine the result of part 5.10.1 with the effect of the right-hand stage to show that the ODE for output voltage $e_o(t)$ is $\dot{e}_o + \frac{1}{R_2 C} e_o = \frac{R_4}{R_3} \frac{1}{R_1 C} e_i$. (Note: This circuit is essentially the *electronic analog computer* (see the footnote to homework Problem 5.6) for solving the standard stable 1st order ODE Equation 3.4.8, $\dot{x} + (1/\tau_1)\, x = bu(t)$. Input voltage $e_i(t)$ is analogous to standard input $u(t)$, output voltage $e_o(t)$ is analogous to response , time constant $\tau_1 = R_2 C$, and constant $b = \frac{R_4}{R_3} \frac{1}{R_1 C}$.)

11. The circuit represented graphically on the next page is the basic, ideal form of electronic analog proportional-integral (PI) controller, one of the subjects of Chapter 15. (In practice, a great deal of electronic refinement and conditioning is required to produce a circuit that behaves even close to ideally.) The input voltage signal is $e_e(t)$, the intermediate voltage between the two stages is $e_m(t)$, and the output voltage from the PI controller is $w(t)$.



1. By applying the methods of circuit analysis described in Chapter 5, show that the ODE relating PI-controller output $w(t)$ to input $e_e(t)$, expressed in terms of the capacitor and resistor parameters of the circuit diagram, is:

$$\dot{w} = \frac{R_3 C_1}{R_2 C_2}\left(\dot{e}_e + \frac{1}{R_1 C_1} e_e\right)$$

2. Derive the PI-controller transfer function $TF(s) \equiv \frac{L[w(t)]|_{w_0=0}}{L[e_e(t)]|_{e_e=0}}$, expressed in terms of the capacitor and resistor parameters of the circuit diagram.

[1]This integrator seems simple on paper, but design of a real integrator that functions properly is difficult, requiring expensive, precision components and many subtle refinements of the basic circuit. Integrators based upon this circuit are essential components of the *electronic analog computer* (EAC), an elegant instrument dating from the 1940s for solving coupled ODEs, but one that has now almost completely been replaced by digital computers. Fifer's 1961 four-volume set is probably the most complete description of all types of analog computers; Peterson's 1967 textbook includes standard instructional material on EACs; and Lang's 2000 article is an interesting, more modern discussion of EACs.

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6: GENERAL TIME RESPONSE OF FIRST ORDER SYSTEMS BY APPLICATION OF THE CONVOLUTION INTEGRAL

Up to this point in the book, we have derived time response solutions of LTI systems only for relatively simple input functions $u(t)$. The convolution integral will permit us to derive time response solutions for **any** physically realistic input function $u(t)$, and even to compute time response solutions if $u(t)$ is given in numerical form rather than equation form. The general convolution transform and its inverse, the convolution integral, are defined and described in this chapter, and application of the convolution integral is illustrated specifically for 1st order systems.

6.1: The Convolution Transform and Its Inverse - the Convolution Integral

6.2: General Solution of the Standard Stable First Oder ODE and IC by Application of the Convolution Integral

6.3: Examples of First Order System Response

6.4: General Solution of the Standard First Order Problem - an Alternative Derivation

6.5: Numerical Algorithm for the General Solution of the Standard First Order Problem

6.6: Chapter 6 Homework

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6.1: The Convolution Transform and Its Inverse - the Convolution Integral

Suppose that we have two physically realistic functions of time, $f_1(t)$ and $f_2(t)$, that are zero for all time $t < 0$ and non-zero only for $t \geq 0$. The *convolution integral* is defined (Meirovitch, 1967, pp. 16-17, 534) to be another function of time in terms of a definite integral involving $f_1(t)$ and $f_2(t)$:

$$CI(t) \equiv \int_{r=0}^{t=t} f_1(\tau) f_2(t-\tau) d\tau$$

In this definite integral, τ is the dummy variable of integration, and time t appears both in the upper limit of the integral and in argument $(t-\tau)$ of the integrand. We can express the integral differently by making the change of integration variable $\lambda = t - \tau$, so that $\tau = t - \lambda$ and $d\tau = -d\lambda$, since t is regarded as a constant within the integration:

$$\int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\lambda=t}^{\lambda=0} f_1(t-\lambda) f_2(\lambda)(-d\lambda) = \int_{\lambda=0}^{\lambda=t} f_1(t-\lambda) f_2(\lambda) d\lambda$$

The final right-hand-side form results because exchanging the limits of a definite integral changes its sign. In the final right-hand-side integral, λ is just the dummy variable of integration, which might as well be τ, so we can write the convolution integral in either of the following forms:

$$CI(t) = \int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} f_1(t-\tau) f_2(\tau) d\tau \tag{6.1.1}$$

The Laplace transform $L[CI(t)]$ is called the *convolution transform*. Let us suppose that the Laplace transforms of functions $f_1(t)$ and $f_2(t)$ exist: $F_1(s) = L[f_1(t)]$ and $F_2(s) = L[f_2(t)]$. Then, as is derived in Appendix A, Section 18.5, the product of these two transforms equals the convolution transform:

$$F_1(s) \times F_2(s) = L[CI(t)] = L\left[\int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau\right] = L\left[\int_{\tau=0}^{\tau=t} f_1(t-\tau) f_2(\tau) d\tau\right] \tag{6.1.2}$$

It follows that the inverse transform of the product is the convolution integral:

$$L^{-1}[F_1(s) \times F_2(s)] = CI(t) = \int_{t=0}^{t=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} f_1(t-\tau) f_2(\tau) d\tau \tag{6.1.3}$$

For LTI systems in general, the convolution integral Equation 6.1.3 will permit us to derive time response solutions for **any** physically realistic input function $u(t)$.

It is of interest to note that in MATLAB, the most common definition of *convolution* is a type of multiplication of two vectors. If the vectors are the coefficients of two polynomials, then MATLAB convolution amounts to the multiplication of the two polynomials. Consider, for example, the following product of two polynomials in s:

$$F_1(s) \times F_2(s) = \left(2s^2 + 3s + 5\right) \times (4s + 6) = 8s^3 + 24s^2 + 38s + 30$$

The following are MATLAB operations that execute this multiplication, using the `conv` command, and the results[1]:

```
>> F1=[2 3 5];F2=[4 6];F3=conv(F1,F2)

F3 = 8 24 38 30
```

Similarly, MATLAB defines *deconvolution* as a type of division of two vectors. If the vectors are the coefficients of two polynomials, then deconvolution amounts to the division of one polynomial by the other. The MATLAB command is `deconv`.

[1]See also the description at the end of Section 8.11 of the relationship between the convolution sum and multiplication of polynomials.

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6.2: General Solution of the Standard Stable First Oder ODE and IC by Application of the Convolution Integral

From Equation 3.4.8, we have for stable 1st order systems:

$$\text{ODE} + \text{IC} : \dot{x} + (1/\tau_1)\, x = bu(t), \quad x(0) = x_0, \quad \text{find } x(t) \text{ for } t > 0 \tag{6.2.1}$$

Solve by first taking the Laplace transform,

$$L[\text{ODE}] : \quad sX(s) - x_0 + (1/\tau_1)\, X(s) = bU(s)$$

$$\Rightarrow \quad X(s) = \frac{x_0}{s + 1/\tau_1} + b \underbrace{\left(\frac{1}{s + 1/\tau_1} \right)}_{F_2(s)} U(s)$$

Take the inverse transform, using Equation 2.2.6 and convolution integral Equation 6.1.5:

$$x(t) = x_0 e^{-t/\tau_1} + b \int_{\tau=0}^{\tau=t} e^{-t/\tau} u(t-\tau) d\tau = x_0 e^{-t/\tau_1} + b \int_{t=0}^{\tau=t} e^{-(t-\tau)/\tau_1} u(\tau) d\tau \tag{6.2.2}$$

Equations 6.2.2 are general solutions of problem Equation 6.2.1 applicable for any input $u(t)$[1]. For both forms, the first term is obviously the initial-condition (IC) response and the second term is the forced response. In applications with specific $u(t)$ functions, the second form of the forced-response integral on the right-hand side of Equations 6.2.2 is used more commonly than the first. The first form is also valid, but the functional nature of $u(t-\tau)$ can sometimes be difficult to interpret correctly.

The forced-response integrals in Equations 6.2.2 are called convolution integrals, as in Equation 6.1.3. They are also sometimes known as *superposition* integrals, because, as is shown in Section 8.10, they can be derived as the linear superposition of responses to differentially small inputs.

In Equations 6.2.2, constants τ_1 and b should be expressed in terms of the physical constants of the actual system analyzed, such as mass m, damping constant c, *etc*.

[1]Note, however, that solutions Equations 6.2.2 are **not** valid for a non-standard 1st order ODE, *e.g.*, one with right-hand-side dynamics such as Equation 5.4.2.

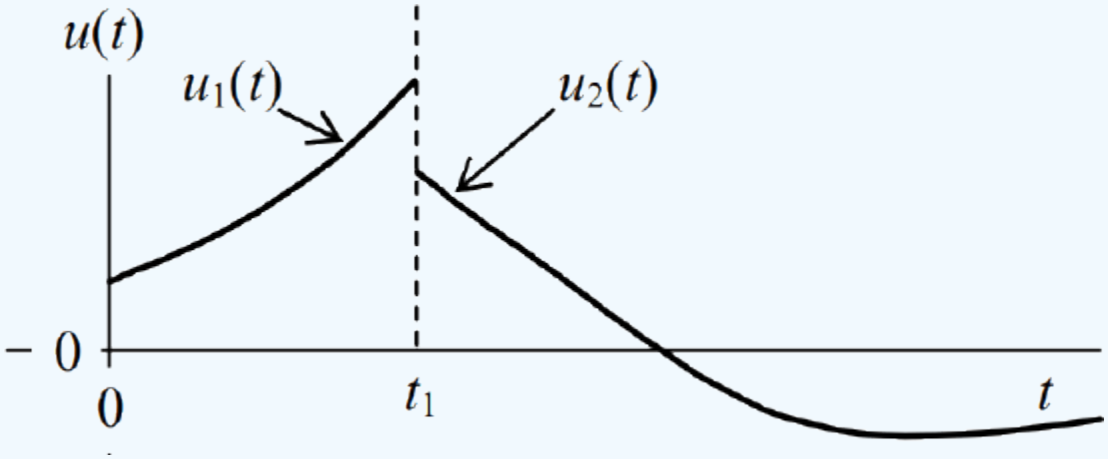
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6.3: Examples of First Order System Response

Example 6.3.1

It is often the case that the input to a system is described by different functions, each function in effect over a different time period for $t > 0$. For example, the following equation and drawing represent an input consisting of two different general functions:

\[u(t)=\left\{\begin{array}{c}
0, \quad t<0 \\
u_{1}(t), \quad 0 u_{2}(t), \quad t_{1} \end{array}\right. \nonumber \]



Solution

The general forced response to this type of input can be expressed nicely in terms of convolution integrals, but we must recognize that it requires two different equations, and we must consider carefully the limits of the definite integrals. First, for the time interval $0 \le t \le t_1$, it is clear that the second form of Equations 6.2.4 is directly applicable, with $u(t) \equiv u_1(t)$:

$$x(t) = x_0 e^{-t/\tau_1} + b \int_{\tau=0}^{\tau=t} e^{-(t-\tau)/\tau_1} u_1(\tau) d\tau \quad , \text{ for } 0 \le t \le t_1 \tag{6.3.1}$$

For the response during the second time interval, $t \ge t_1$, the definite integral in the second form of Equations 6.2.4 still must be evaluated over the limits $\tau = 0$ to $\tau = t > t_1$, which means that **both** u_1 **and** u_2 need to be integrated, but each only over the time interval for which it is defined:

$$x(t) = x_0 e^{-t/\tau_1} + b \left[\int_{\tau=0}^{\tau=t_1} e^{-(t-\tau)/\tau_1} u_1(\tau) d\tau + \int_{\tau=t_1}^{\tau=t} e^{-(t-\tau)/\tau_1} u_2(\tau) d\tau \right] \quad , \text{ for } t \ge t_1 \tag{6.3.2}$$

Note especially in Equation 6.3.2 that $\tau = t_1$ is both the **upper** limit of the integral that involves u_1 and the **lower** limit of the integral that involves u_2. Equations 6.3.1 and 6.3.2 are especially useful for response to a pulse of limited duration, as is illustrated in the next example.

Example 6.3.2

We consider again the problem for velocity of a mass moving on a viscous film, which is solved by basic ODE methods in Section 1.5:

$$\text{ODE} + \text{IC} : m\dot{v} + cv = f_x(t), \quad v(0) = v_0, \quad \text{find } v(t) \text{ for } t > 0 \tag{6.3.3}$$

\[f_{x}(t)=\left\{\begin{array}{c}
F \sin \left(\pi t / t_{d}\right) \equiv F \sin \omega t \quad \text { where } \omega=\pi / t_{d}, 0 \leq t \leq t_{d} \\
0, t_{d} \end{array}\right. \nonumber \]

Solution

Here we have $u(t) = f_x(t)$, $\tau_1 = m/c$, and $b = 1/m$. Relative to the notation of Example 6.3.1 above, u_1 is the half-sine pulse, and u_2 is zero.

While the pulse is active, $0 \leq t \leq t_d$, Equation 6.3.1 becomes

$$v(t) = v_0 e^{-t/\tau_1} + \frac{F}{m} e^{-t/\tau_1} \int_{\tau=0}^{\tau=t} e^{\tau/\tau_1} \sin \omega\tau d\tau \tag{6.3.4}$$

We can evaluate the definite integral by several different methods: find it in a table of **definite** integrals (or indefinite integrals, but do not forget the lower limit of integration); or integrate by parts twice; or evaluate it with software that does symbolic manipulation, such as Mathematica or recent versions of MATLAB. The result is

$$\int_{\tau=0}^{\tau=t} e^{\tau/\tau_1} \sin \omega\tau d\tau = \frac{\tau_1}{1+(\omega\tau_1)^2} \left[e^{t/\tau_1} (\sin \omega t - \omega\tau_1 \cos \omega t) + \omega\tau_1 \right] \tag{6.3.5}$$

Combining Equation 6.3.4 and Equation 6.3.5 gives the velocity during the pulse duration:

$$v(t) = \left[v_0 + \frac{F}{c} \frac{\omega\tau_1}{1+(\omega\tau_1)^2} \right] e^{-t/\tau_1} + \frac{F}{c} \frac{1}{1+(\omega\tau_1)^2} (\sin \omega t - \omega\tau_1 \cos \omega t), 0 \leq t \leq t_d \tag{6.3.6}$$

At the instant $t = t_d$ when the pulse goes to zero, $\omega t_d = \pi$, so the velocity then is

$$v(t_d) = \left[v_0 + \frac{F}{c} \frac{\omega\tau_1}{1+(\omega\tau_1)^2} \right] e^{-t_d/\tau_1} + \frac{F}{c} \frac{\omega\tau_1}{1+(\omega\tau_1)^2} \tag{6.3.7}$$

For the time after the pulse goes to zero, $t_d \leq t$, Equation 6.3.2 becomes

$$v(t) = v_0 e^{-t/\tau_1} + \frac{F}{m} e^{-t/\tau_1} \left(\int_{\tau=0}^{\tau=t_d} e^{\tau/\tau_1} \sin \omega\tau d\tau + \int_{\tau=t_d}^{\tau=t} e^{\tau/\tau_1} \times 0 d\tau \right) \tag{6.3.8}$$

With use again of Equation 6.3.5, and of some algebra, Equation 6.3.8 becomes

$$v(t) = v_0 e^{-t/\tau_1} + \frac{F}{c} e^{-t/\tau_1} \frac{\omega\tau_1}{1+(\omega\tau_1)^2} \left(1 + e^{t_d/\tau_1} \right) = v(t_d) e^{-(t-t_d)/\tau_1}, \text{ for } t \geq t_d \tag{6.3.9}$$

The final form on the right-hand side of Equation 6.3.9 uses Equation 6.3.7, and it shows that the post-pulse response is a pure exponential decay, which we would expect physically.

Example 6.3.3

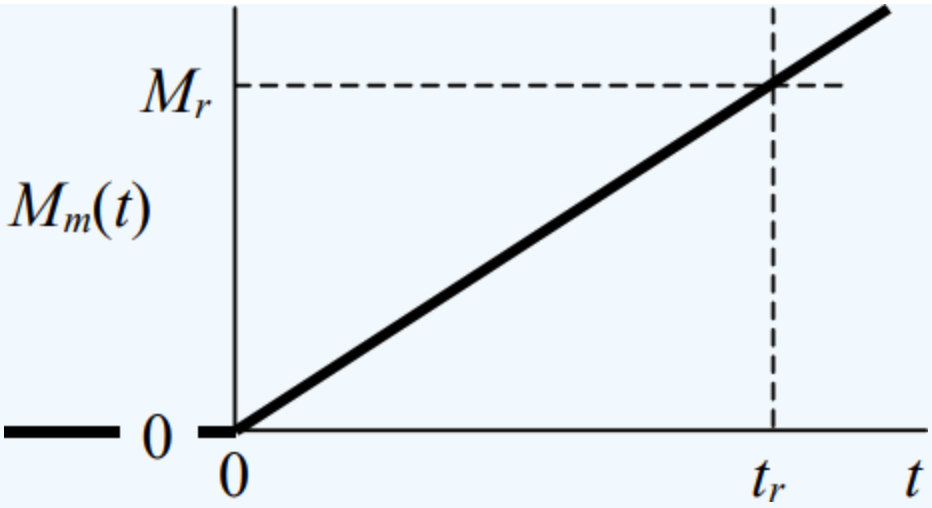
From Section 3.3, we have the equation of motion for spin velocity $p(t)$ of a reaction wheel Figure 3.3.1 with rotational inertia J, bearing viscous damping constant c_θ, and applied motor torque $M_m(t)$:

$$\dot{p} + \frac{c_\theta}{J} p = \frac{1}{J} M_m(t) \Rightarrow \dot{p} + \frac{1}{\tau_1} p = b M_m(t) \text{ where } \tau_1 = \frac{J}{c_\theta} \text{ and } b = \frac{1}{J}$$

Solution

Let the initial velocity be zero, $p(0) \equiv p_0 = 0$. Let the motor torque have the form of a *ramp*:

$$M_m(t) = \frac{M_r}{t_r} t, t \geq 0$$



Then the second form of general solution Equations 6.2.4 becomes

$$p(t) = p_0 e^{-t/\tau_1} + b\int_{\tau=0}^{\tau=t} e^{-(t-\tau)/\tau_1} M_m(\tau) d\tau = \frac{1}{J}\int_{\tau=0}^{\tau=t} e^{-(t-\tau)/\tau_1} \frac{M_r}{t_r} \tau d\tau$$

$$\Rightarrow \quad p(t) = \frac{1}{J}\frac{M_r}{t_r} e^{-t/\tau_1} \int_{\tau=0}^{\tau=t} e^{\tau/\tau_1} \tau d\tau$$

We can evaluate the integral easily using integration by parts:

$$\int_{\tau=0}^{\tau=t} \overset{u}{\tau}\overbrace{e^{\tau/\tau_1}}^{dv} d\tau = \left[\tau \times \tau_1 e^{\tau/\tau_1}\right]_{\tau=0}^{\tau=t} - \int_{\tau=0}^{\tau=t} \tau_1 e^{\tau/\tau_1} d\tau = \tau_1 t e^{t/\tau_1} - \tau_1^2\left(e^{t/\tau_1} - 1\right)$$

Therefore, the total solution is

$$p(t) = \frac{1}{J}\frac{M_r}{t_r} e^{-t/\tau_1}\left[\tau_1 t e^{t/\tau_1} - \tau_1^2\left(e^{t/\tau_1} - 1\right)\right] = \frac{M_r}{c_\theta t_r}\left[t - \tau_1\left(1 - e^{-t/\tau_1}\right)\right] \tag{6.3.10}$$

By sketching a plot versus time of the dimensionless spin velocity $\frac{p(t)}{M_r/c_\theta}$, you can easily show that, as $t \to \infty$, this quantity is asymptotic to the ramp function $(t - \tau_1)/t_r$.

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6.4: General Solution of the Standard First Order Problem - an Alternative Derivation

Consider again the general standard 1st order problem, Equation 1.2.1, in which $u(t)$ is the known input (excitation), $x(t)$ is the output (response) that we seek, and a and b are constants, with a not necessarily negative:

$$\dot{x} - ax = bu(t), \text{ with IC } x(t_1) \text{ assumed known, find } x(t) \text{ for } t_1 \leq t \tag{6.4.1}$$

(We revert to the more general form now, rather than our stable standard 1st order ODE, Equation 3.4.8 or Equation 6.2.1, in order to make this derivation more general, applicable for any physically realistic value or polarity of the constant a.) This problem is a bit more general than we have considered previously, since we allow the initial time t_1 to be different from zero. In order to expedite indexing of array quantities in computer algorithms, we use subscript 1, instead of 0, to denote the initial time and the initial value: $x(t_1) \equiv x_1$.

To find a general solution of this 1st order problem (ODE plus IC), we use the exponential function e^{at} and closely related functions, for which we have the following basic identities:

$$e^{at} \times e^{-at_1} = e^{a(t-t_1)} \text{ and } e^{at} \times e^{-at} = e^0 = 1 \tag{6.4.2}$$

We begin the general solution by multiplying the ODE by the *integrating factor* e^{-at}, recognizing that this will make the left-hand side a perfect derivative:

$$e^{-at}(\dot{x} - ax = bu) \Rightarrow e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu \Rightarrow \frac{d}{dt}\left(e^{-at}x\right) = e^{-at}bu$$

Now we integrate the multiplied equation from the initial time t_1 to an arbitrary time instant $t > t_1$, using τ as the variable of integration:

$$\int_{\tau=t_1}^{\tau=t} \frac{d}{d\tau}\left[e^{-a\tau}x(\tau)\right]d\tau = \int_{\tau=t_1}^{\tau=t} e^{-a\tau}bu(\tau)d\tau \Rightarrow e^{-at}x(t) - e^{-at_1}x(t_1) = \int_{\tau=t_1}^{\tau=t} e^{-a\tau}bu(\tau)d\tau$$

Finally, we multiply through by e^{at}, apply identities Equation 6.4.2, move the IC term to the right-hand side, and arrive at the exact, general solution:

$$x(t) = e^{a(t-t_1)}x(t_1) + \int_{t=t_1}^{\tau=t} e^{a(t-\tau)}bu(\tau)d\tau, t_1 \leq t \tag{6.4.3}$$

The term e^{at} need not be inside the integral, since it is not a function of the integration variable τ. Constant b also need not remain inside the integral. Solution Equation 6.4.3 is comparable to, but more general than, the second form of convolution solution Equation 6.2.4.

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6.5: Numerical Algorithm for the General Solution of the Standard First Order Problem

We seek solutions of Equation 6.4.5 in time-series form, that is, at discrete, equally spaced instants in time. Accordingly, we define the following notation that employs descriptive subscripts:

$t =$	t_1	$t_2 = t_1 + \Delta t$	$t_3 = t_2 + \Delta t$	...	$t_n = t_{n-1} + \Delta t$	$t_{n+1} = t_n + \Delta t$	...
$x(t) =$	x_1	x_2	x_3	...	x_n	x_{n+1}	...
$u(t) =$	u_1	u_2	u_3	...	u_n	u_{n+1}	...

Conceptually, we begin with the known values at $t = t_1$, then integrate Equation 6.4.5 from t_1 to $t_2 = t_1 + \Delta t$, in which we define Δt as the constant time step; we now have known values at $t = t_2$, so we can integrate again to go from t_2 to $t_3 = t_2 + \Delta t$. We proceed in this manner from one time to the next until we have determined values of $x(t)$ at discrete instants in time over the complete time interval of interest. From Equation 6.4.5, the exact, general equation for stepping from time instant t_{n-1} to the next instant t_n is

$$x_n = e^{a(t_n - t_{n-1})} x_{n-1} + \int_{\tau=t_{n-1}}^{\tau=t_n} e^{a(t_n-\tau)} b u(\tau) d\tau \tag{6.5.1}$$

By comparing Equation 6.5.1 with Equation 6.2.4, observe that the integral in Equation 6.5.1 is clearly a forced-response convolution integral.

Up to this point, the solution is exact. But now we introduce what, in general, is an approximation. We assume that $u(\tau)$ varies so little over the integration time step Δt that it introduces only small error to approximate $u(\tau)$ as being constant over Δt, with its value remaining that at the beginning of the integration time:

\[u(\tau) \approx u\left(t_{n-1}\right) \equiv u_{n-1} \text { for } t_{n-1} \leq \tau

Using approximation Equation ??? and $t_n = t_{n-1} + \Delta t$, we rewrite Equation 6.5.1 with the convolution integral expressed in a more easily integrable form:

$$x_n = e^{a\Delta t} x_{n-1} + \left[\int_{\tau=t_{n-1}}^{\tau=t_{n-1}-\Delta t} e^{a(t_{n-1}+\Delta t-\tau)} d\tau \right] b u_{n-1}$$

We change the variable of integration, $\xi = t_{n-1} + \Delta t - \tau$, so that the integral becomes

$$\int_{\xi=\Delta t}^{\xi=0} e^{a\xi}(-d\xi) = \int_{\xi=0}^{\xi=\Delta t} e^{a\xi} d\xi = \frac{1}{a}\left(e^{a\Delta t} - 1\right)$$

Finally, with approximation Equation ???, solution Equation 6.5.1 can be written as

$$x_n = e^{a\Delta t} x_{n-1} + \frac{1}{a}\left(e^{a\Delta t} - 1\right) b u_{n-1} \equiv \phi x_{n-1} + \gamma u_{n-1} \tag{6.5.2}$$

Equation 6.5.2 is a *recurrence formula* that is easy to evaluate numerically from one time instant to the next, especially since coefficients $\phi \equiv e^{a\Delta t}$ and $\gamma \equiv \left(e^{a\Delta t} - 1\right) b/a = (\phi - 1) b/a$ are invariant once Δt has been selected. Note that if the input $u(t)$ is a constant, as for a step function, then $u_n =$ constant for all $n = 1, 2, ...$, in which case Equation 6.5.2 is an **exact** solution because Equation ??? is exact, not an approximation. Furthermore, if the actual $u(t)$ is piecewise-constant, then Equation 6.5.2 can be applied to produce exact results, provided that Δt is chosen individually for each interval of constant $u(t)$ such that the interval is an integer multiple of its own Δt.

To illustrate the application of Equation 6.5.2, we re-visit the numerical solution of Section 1.6, the velocity response of a mass-damper system to a half-sine force pulse. M-file `MATLABdemo11.m` of Section 1.6, which calculates the response **exactly**, provides the benchmark for evaluating the approximate response calculation. The MATLAB script M-file to calculate and graph

the **approximate** response is:

```
%MATLABdemo61.m
%Mass-damper system approximate response to IC + half-sine pulse forcing
m=5;c=2; %system mass & viscous damping coefficient, SI units
a=-c/m;b=1/m;
F=18;td=7.5; %half-sine pulse, amplitude (N), pulse duration (s)
w=pi/td; %circular frequency of half-sine pulse (rad/s)
Dt=1.5; %time step for recurrence calculations
t=0:Dt:24;Lt=length(t); %array of time instants for recurrence calculations
phi=exp(a*Dt);gam=(phi-1)/a*b; %constants in recurrence formula
for n=1:Lt %time series array of input force pulse
    if t(n)<=7.5
        fx(n)=F*sin(pi*t(n)/td);
    else
        fx(n)=0;
    end
end
v=zeros(1,Lt);v(1)=3.3; %initialize velocity array, initial velocity (m/s)
for n=2:Lt
v(n)=phi*v(n-1)+gam*fx(n-1);
end
plot(t,v,'k'),bar(t+Dt/2,fx/10,1,'k')
```

Note that the implementation of recurrence formula Equation 6.5.2 in `MATLABdemo61.m` is a simple three-line `for`-loop.

The figure on the next page was produced by combining the results of M-files `MATLABdemo11.m` and `MATLABdemo61.m`, and then adding explanatory labels and editing the bar graph. For this example, time step Δt was intentionally chosen to be unreasonably large, Δt = 1.5 s (compared with system time constant τ_1 = 2.5 s), in order to produce clear distinctions between the exact and approximate dynamic variables. Nevertheless, the approximate calculation of velocity shows the correct trends qualitatively and is not highly inaccurate quantitatively. If the calculations were repeated with a more reasonable time step, $\Delta t \leq 0.1\tau_1$, then the results would be much more accurate (homework Problem 6.4).

Observe also in the figure on the next page the bar graph of approximate force $f_x(t)$ used in Equation 6.5.2, which is the graphical representation of approximation Equation ???. Because of its piecewise-constant character, this is sometimes called a *stairstep* approximation. This approximation introduces a *time delay* or *lag* on the order of Δt into the approximate input. This artificial time delay is obviously transmitted to the calculated approximate response.

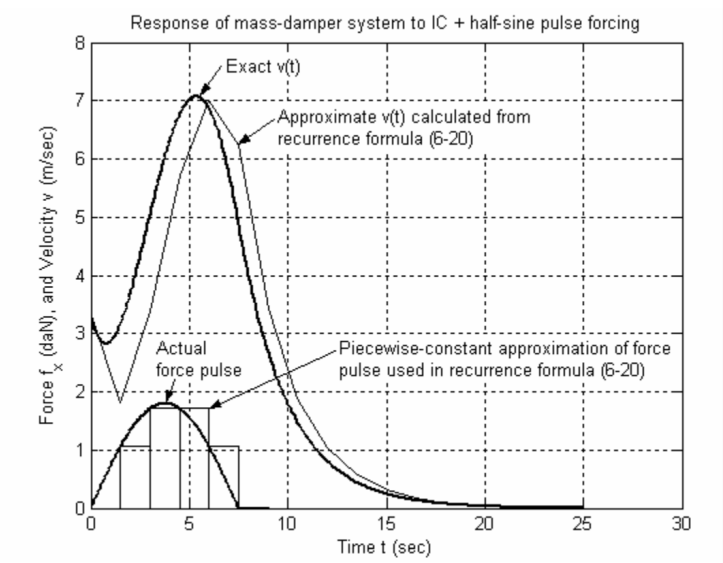


Figure 6.5.1

It is interesting and possibly useful to observe that, due to approximating $u(\tau)$ as being constant over each time step Δt, Equation 6.5.2 is essentially IC + step response over each Δt. Suppose that we were to seek an even more accurate recurrence formula than Equation 6.5.2 by approximating $u(\tau)$ as varying **linearly** with time over each time step. In that case, the solution would be essentially IC + step response + ramp response over each Δt. (Example 6.3 in Section 6.3 illustrates ramp response.) So the more refined approximate recurrence formula would be Equation 6.5.2 supplemented with an additional term that represents ramp response, and we would expect that additional term to include both u_{n-1} and u_n as a consequence of the approximated linear variation of $u(\tau)$; see homework Problem 6.5.

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6.6: Chapter 6 Homework

1. Consider a mass-damper system with a suddenly applied cosine forcing function beginning at t = 0, and let the mass have a known initial velocity. The complete problem for velocity $v(t)$ is described by the 1st order, LTI ODE $m\dot{v} + cv = F\cos\omega t$, t > 0, and the IC $v(0) = v_0$. Use the general solution Equation 6.2.4 to write an algebraic equation for the complete solution $v(t)$ of this problem. It will be necessary to evaluate the convolution integral. You may use integration by parts and/or published tables of integrals, which are highly recommended. Symbolic software (Mathematica, MATLAB, *etc.*) is also a potential source of assistance with difficult integrals.

$$\text{Answer: } v(t) = \left[v_0 - \frac{F}{c}\frac{1}{1+(\omega\tau_1)^2}\right]e^{-t/\tau_1} + \frac{F}{c}\frac{1}{1+(\omega\tau_1)^2}\left(\cos\omega t + \omega\tau_1\sin\omega t\right), t \geq 0$$

2. Consider the standard 1st order LTI ODE (of a stable physical system) for dependent variable $x(t)$: $\dot{x} + (1/\tau_1)\,x = bu(t)$, with IC $x(0)$ = 0. Let the input function be the following *flat pulse* of duration t_d: \[u(t)=\left\{\begin{array}{ll}
0, & t<0 \\
U, & 0 0, & t_{d} \end{array}\right. \nonumber \]
    1. Use solutions Equations 6.3.2 and 6.3.3 to write two algebraic equations for the complete response $x(t)$ of this problem. One equation should apply for the time during which the pulse is active (including initial and final times), $0 \leq t \leq t_d$, and the other equation should apply for the time after the pulse ceases, $t_d \leq t$. It will be necessary to evaluate the convolution integral.

$$\text{Answer: } x(t) = \begin{cases} bU\tau_1\left(1 - e^{-t/\tau_1}\right), & 0 \leq t \leq t_d \\ bU\tau_1\left(e^{-(t-t_d)/\tau_1} - e^{-t/\tau_1}\right), & t_d \leq t \end{cases}$$

       Note that the result for $t_d \leq t$ can be written also as

$$x(t) = bU\tau_1\left(1 - e^{-t_d/\tau_1}\right)e^{-(t-t_d)/\tau_1}, \quad t_d \leq t$$

    2. Suppose that $t_d = \tau_1$. Sketch by hand (not by computer) a reasonably accurate timehistory plot of the nondimensionalized output $\frac{x(t)}{bU\tau_1}$ versus time over the time interval $0 \leq t \leq 5\tau_1$.
3. Consider the equation of motion for spin velocity $p(t)$ of a reaction wheel (from Example 6.3 in Section 6.3): $\dot{p} + (c_\theta/J)\,p = (1/J)M_m(t)$, with IC $p(0) = 0$. Let the applied motor torque be the following *sawtooth pulse*: \[M_{m}(t)=\left\{\begin{array}{cc}
0, & t<0 \\
\frac{M_{r}}{t_{r}} t,& 0 \leq t 0, & t_{r} \end{array}\right. \nonumber \]

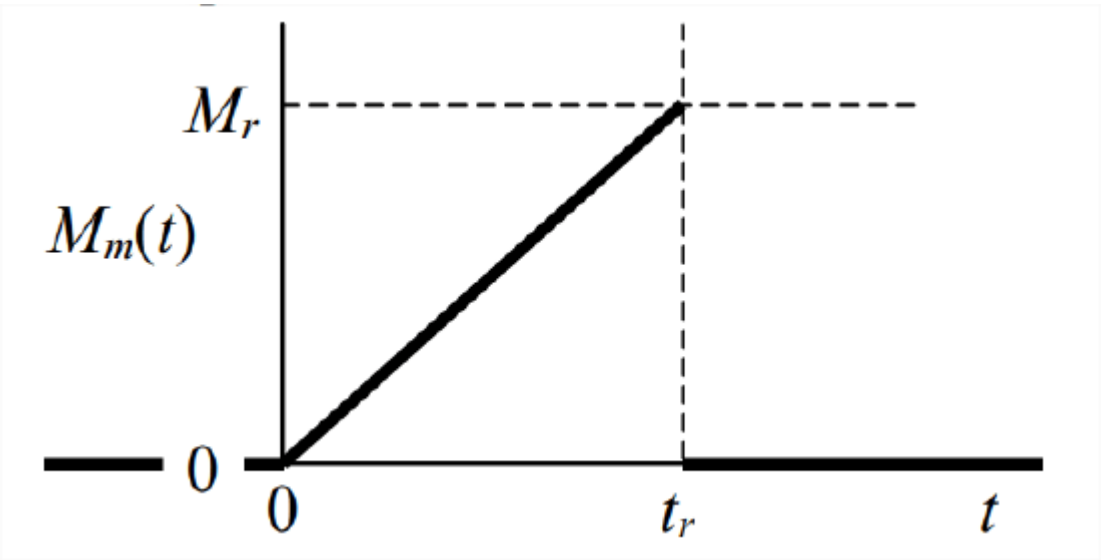


Figure 6.6.1

Use solutions Equations 6.3.2 and 6.3.3 to write two algebraic equations for the complete solution $p(t)$ of this problem. One equation should apply for the time during which the pulse is active (including initial and final times), $0 \leq t \leq t_r$, and the

other equation should apply for the time after the pulse ceases, $t_r \leq t$. You may and should use without re-derivation the appropriate results from Example 6.3 in Section 6.3, in which the applied moment is a permanently increasing ramp.

4. Use MATLABdemo61.m in Section 6.5 as a template (which must be revised and supplemented with labels, grids, *etc.*) to calculate approximately and graph the velocity response of the same mass-damper system with the same IC, but now specifying smaller (than in Section 6.5) calculation time steps: Δt = 0.5 s and Δt = 0.25 s. You should find that the calculated dynamic response becomes progressively more accurate as you reduce Δt.
5. 1. Derive a more accurate recurrence formula than Equation 6.5.5 by approximating $u(\tau)$ as varying **linearly** with time over each time step. In other words, use in Equation 6.5.1 the linear approximation

$$u(\tau) \approx u_{n-1} + \frac{\tau - t_{n-1}}{\Delta t}\left(u_n - u_{n-1}\right) \text{ for } t_{n-1} \leq \tau \leq t_n$$

   instead of the simpler approximation Equation 6.5.2. By completing the integration, show (in all detail, as if the answer were not given) that the refined version of Equation 6.5.5 is

$$x_n = \phi x_{n-1} + \gamma u_{n-1} + \beta\left(u_n - u_{n-1}\right)$$

   in which ϕ and γ are the constants defined in Equation 6.5.5, and $\beta \equiv \frac{\gamma}{a\Delta t} - \frac{b}{a}$.
   2. Revise M-file MATLABdemo61.m of Section 6.5 to implement the refined recurrence formula of part 6.5.1. Using exactly the same numerical data as in the original program, run the revised program and plot the approximate time history $v(t)$. The approximate time history calculated by the refined recurrence formula should be substantially more accurate than that calculated by Equation 6.5.5.
6. Consider the standard 1st order LTI-ODE of a stable system, Equation 3.4.8, for dependent variable $x(t)$: $\dot{x} + (1/\tau_1)\, x = bu(t)$. Let the initial condition be zero, $x(0) = 0$, and let the input function be a declining ramp, $u(t) = c\,(t_z - t)$, in which c, a dimensional constant, is the downward slope of the ramp, and t_z is the time at which the input passes through zero.
   1. Evaluate in all detail the convolution integral in Equation 6.2.4 to show that the **exact** response solution is $x(t) = bc\tau_1\left[(t_z + \tau_1)\left(1 - e^{-t/\tau_1}\right) - t\right] = bc\tau_1\left[t_z + \tau_1 - t - (t_z + \tau_1)\, e^{-t/\tau_1}\right]$.
   2. Let the numerical parameters be τ_1 = 2.5 s, t_z = 10 s, b = 3.5, and c = 1 (b and c in consistent units). Write a MATLAB program, or adapt the code in Section 6.5 that executes the recurrence formula, to calculate and plot an approximate numerical solution for $x(t)$ over the time interval $0 \leq t \leq 10$ s. Adjust the time-step size Δ and the number of time steps over the 10-s interval in your code until the graph of your approximate solution appears very similar to that of the corresponding exact solution from part 6.6.1. Submit your MATLAB code and your final graph of response.

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7: UNDAMPED SECOND ORDER SYSTEMS

The ideal 2nd order mass-damper-spring system was introduced in Section 1.9, and a theoretical response solution for an undamped system by elementary ODE methods was demonstrated in Section 1.10. In this chapter, we explore ideal undamped 2nd order systems in greater detail, deriving theoretical response solutions by means of Laplace transformation with application of the inverse convolution transform from Chapter 6. The final section of this chapter is an introduction to vibration of real structures, including actual laboratory data.

7.1: Standard Form for Undamped Second Order Systems and Natural Frequency

7.2: General Solution for Output of Undamped Second Order Systems

7.3: Simple IC Response and Step Response of Undamped Second Order Systems

7.4: Discussion of Physical Applicability of Step-Response Solutions

7.5: Dynamic Motion of a Mechanical System Relative to a Non-Trivial Static Equilibrium Position; Dynamic Free-Body Diagram

7.6: Introduction to Vibrations of Distributed-Parameter Systems

7.7: Chapter 7 Homework

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7.1: Standard Form for Undamped Second Order Systems and Natural Frequency

For the mass-damper-spring (m-c-k) system of Figure 7.1.1, the ODE of motion Equation 3.7.1 derived from Newton's 2nd law, with use of the FBD in Figure 3.7.1, is

$$m\ddot{x} + c\dot{x} + kx = f_x(t) \tag{7.1.1}$$

If we neglect damping by setting $c\dot{x} = 0$ in Equation 7.1.1, then we have an ideal (not real) undamped mass-spring (m-k) system. This undamped *passive* system is not fully realistic because every passive physical system is afflicted with some type and degree of energy dissipation, albeit very small in some systems[1]. Nevertheless, it is useful to study the undamped system because it reveals some important fundamental characteristics of 2nd order systems. From Equation 7.1.1, the ODE for the m-k system is:

$$m\ddot{x} + kx = f_x(t) \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{1}{m}f_x(t) = \frac{k}{m}\frac{f_x(t)}{k} \tag{7.1.2}$$

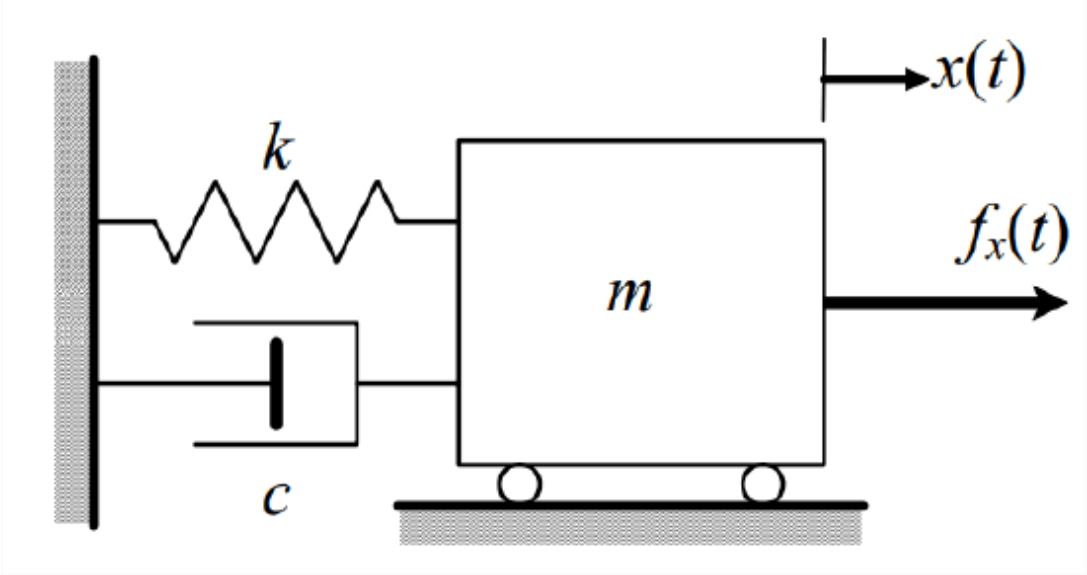


Figure 7.1.1: Mass-damper-spring system

Let us use Equation 7.1.2 to develop the standard form that will apply for undamped 2nd order systems more generally. First, we define algebraically the *natural frequency of undamped vibration* as the **positive** square root of the quotient of the stiffness and mass values (both of which are positive in most passive systems):

$$\omega_n \equiv \sqrt{\frac{k}{m}} \tag{7.1.3}$$

The natural frequency is very important physically: it is shown in the next section that an undamped 2nd order system tends to vibrate (oscillate, pulsate, shake, quiver, ...) periodically at circular frequency ω_n radians per second. Next, we define the standard input quantity

$$u(t) \equiv \frac{1}{k}f_x(t) \tag{7.1.4}$$

So now we define $x(t)$ as representing **any** appropriate output quantity (not necessarily just position as in Figure 7.1.1), and we re-write Equation 7.1.2 in *standard form* (standard in this book, at least, and mostly standard in engineering literature):

$$\ddot{x} + \omega_n^2 x = \omega_n^2 u(t) \tag{7.1.5}$$

From Equation 7.1.5 we see that input $u(t)$ has exactly the same physical dimensions as output $x(t)$. In fact, we can identify $u(t)$ as being the *pseudo-static* output, $x_{ps}(t)$; that is, if $x(t)$ varies slowly enough that $\ddot{x}$ is negligible in comparison with $\omega_n^2 x$, then ODE Equation 7.1.5 reduces to a simple algebraic equation, $\omega_n^2 x \approx \omega_n^2 u(t)$, the solution of which is the pseudo-static response, $x(t) = u(t) \equiv x_{ps}(t)$. This response is called pseudo-static because it is not necessarily static (constant over time), but it varies slowly enough that the second derivative of response is negligible.

Example 7.1.1

You have probably seen the pendulum of a grandfather clock. The ideal simple pendulum is very similar: there is a bob of mass m attached to the lower end of a rod of length ℓ, the upper end of the rod being supported at frictionless hinge H. We assume the rod to be rigid and to have negligible inertial moment. The pendulum hangs downward and swings side-to-side with angle $\theta(t)$ in the plane of the paper, subject to the acceleration of gravity, g. For completeness, we include an agent that can impose onto the pendulum an arbitrary applied moment $M(t)$ about H. Because we neglect the inertia of the rod, the rotational inertia of the pendulum about H is $J = m\ell^2$.

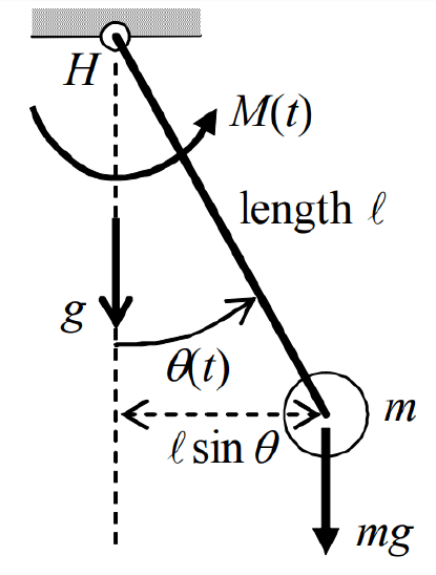


Figure 7.1.2

Newton's 2nd law for rotation about H is

$$\Sigma(\text{ all applied moments about } H) = J \times \text{ rotational acceleration } = m\ell^2\ddot{\theta}$$

$$\Rightarrow \quad M(t) - mg \times \ell \sin\theta = m\ell^2\ddot{\theta} \Rightarrow m\ell^2\ddot{\theta} + mg\ell\sin\theta = M(t)$$

Due to the $\sin\theta$ term, the exact ODE is nonlinear. However, let us consider only small rotational angles, $|\theta| <\approx 10^\circ$, for which $\sin\theta \cong \theta$ in radians. This restriction linearizes the ODE:

$$m\ell^2\ddot{\theta} + mg\ell\theta = M(t) \Rightarrow \ddot{\theta} + \frac{g}{\ell}\theta = \frac{g}{\ell}\frac{M(t)}{mg\ell} \Rightarrow \ddot{\theta} + \omega_n^2\theta = \omega_n^2 u(t)$$

In the standardized form of the linearized ODE, the natural frequency is $\omega_n = \sqrt{g/\ell}$, and the standard input quantity (also the pseudo-static response) is $u(t) = \theta_{ps}(t) = M(t)/mg\ell$. For the pendulum at small rotational angles, the term $mg\ell\theta$ is the restoring moment that opposes motion from the unforced static equilibrium position, θ = 0; essentially, gravity plays the role of a rotation spring. See homework Problem 7.4 for another interesting pendulous rotational system, a balloon carrying a basket.

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7.2: General Solution for Output of Undamped Second Order Systems

Let us solve Equation 7.1.5 for output $x(t)$, given **any** physically realistic input $u(t)$, for time $t > 0$, and given appropriate initial conditions at $t = 0$. We use Laplace transformation with application of the inverse convolution transform from Chapter 6. To simplify the notation, we denote $X(s) \equiv L[x(t)]$. Transforming Equation 7.1.5 with use of Equation 2.2.11 gives

$$s^2X(s) - sx(0) - \dot{x}(0) + \omega_n^2 X(s) = \omega_n^2 L[u(t)] \tag{7.2.1}$$

Equation 7.2.1 tells us that we need two initial conditions, one on the output and one on the derivative of the output, for this 2nd order ODE. Accordingly, we simplify the writing with the definitions

$$\text{ICs for } 2^{\text{nd}} \text{ order ODE: } x_0 \equiv x(0), \text{ initial "position"; } \dot{x}_0 \equiv \dot{x}(0), \text{ initial "velocity"} \tag{7.2.2}$$

Solving Equation 7.2.1 for $X(s)$ with the use of notation Equation 7.2.2 gives

$$X(s) = x_0 \frac{s}{s^2 + \omega_n^2} + \frac{\dot{x}_0}{\omega_n} \frac{\omega_n}{s^2 + \omega_n^2} + \omega_n \overbrace{\frac{\omega_n}{s^2 + \omega_n^2}}^{F_1(s)} \overbrace{L[u(t)]}^{F_2(s)} \tag{7.2.3}$$

We invert Equation 7.2.3 using transforms Equation 2.4.7 and Equation 2.4.8, and the inverse convolution transform Equation 6.1.5, to find the two equivalent general solution equations for the standard undamped 2nd order ODE, Equation 7.1.5, with ICs $x(0) \equiv x_0$ and $\dot{x}(0) \equiv \dot{x}_0$:

$$x(t) = \overbrace{x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}^{\text{IC response}} + \overbrace{\omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n \tau \times u(t-\tau) d\tau}^{\text{forced response}} \tag{7.2.4}$$

$$x(t) = \overbrace{x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t}^{\text{IC response}} + \overbrace{\omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n (t-\tau) \times u(\tau) d\tau}^{\text{forced response}} \tag{7.2.5}$$

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7.3: Simple IC Response and Step Response of Undamped Second Order Systems

We find in this section two important special case solutions of Equations 7.2.4 and 7.2.5:

1. pure **initial condition response** for zero input, $u(t) = 0$; and
2. pure **step response** with both initial conditions being zero.

For the case of pure **initial condition response** (also called free vibration), with $u(t) = 0$, Equations 7.2.4 and 7.2.5 become

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t, 0 \leq t \tag{7.3.1}$$

Let us use a trigonometric identity [see Equations 4.3.2, 4.3.4, and 4.3.5] to combine the two sinusoids of Equations 7.3.1 into a single term. The following definition will reduce the writing required and also will turn out to be physically meaningful:

$$x_{\max} \equiv \sqrt{x_0^2 + \left(\dot{x}_0/\omega_n\right)^2} \tag{7.3.2}$$

$$\Rightarrow \quad x(t) = x_{\max} \left[\cos \omega_n t \times \left(\frac{x_0}{x_{\max}} \right) - \sin \omega_n t \times \left(\frac{-\dot{x}_0/\omega_n}{x_{\max}} \right) \right]$$

$$\Rightarrow \quad x(t) = x_{\max} \cos\left(\omega_n t + \phi\right), 0 \leq t, \quad \phi = \tan^{-1} \left(\frac{-\dot{x}_0/\omega_n}{x_0} \right) \tag{7.3.3}$$

Figure 7.3.1 is an annotated graph of response Equation 7.3.2-Equation 7.3.3 for positive values of the ICs, $x_0 > 0$ and $\dot{x}_0 > 0$. Clearly, the output is a pure sinusoid of amplitude $x_{\max}$ and phase angle ϕ. This form of response is called free vibration, because it occurs without any forcing input, $u(t)$ = 0. The circular frequency of vibration is ω_n rad/s, the natural frequency defined in Equation 7.1.3. The cyclic natural frequency is $f_n = \omega_n/2\pi$ cycles/s (Hz), and the natural period annotated on Figure 7.3.1 is $T_n = 1/f_n = 2\pi/\omega_n$ s/cycle. Free vibration at the natural frequency is one of the most important characteristics of undamped systems and, more realistically, of lightly damped systems. Damped 2nd order systems are discussed in Chapter 9.

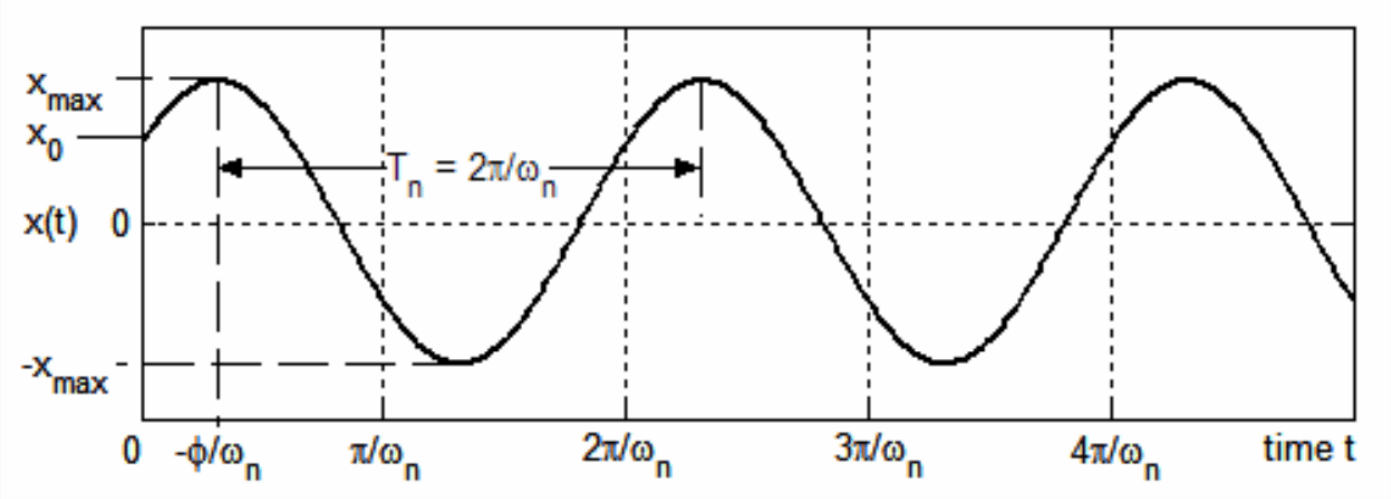


Figure 7.3.1: IC response of an undamped 2nd order system

For the case of **pure step response**, we set the ICs to zero, and we define the input to be a step function at time t = 0, with step magnitude U:

$$u(t) = UH(t) \tag{7.3.4}$$

The appropriate form of the general solution to use in this case is Equation 7.2.4,

$$x(t) = \omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n \tau \times u(t-\tau) d\tau = \omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n \tau \times UH(t-\tau) d\tau = \omega_n U \int_{\tau=0}^{\tau=t} \sin \omega_n \tau d\tau$$

Here we used the property of the unit-step function that $H(t-\tau) = 1$ for $t - \tau > 0$ (from Equation 2.4.2 and Figure 2.4.2); this inequality obviously is satisfied for τ over the limits of the definite integral. For most applications of the convolution integrals in Equations 7.2.4 and 7.2.5, the form in Equation 7.2.5 is preferable because the integrand term $u(t-\tau)$ in Equation 7.2.4 is usually difficult to interpret and/or awkward to handle in the integration. This case, however, is an exception since $u(t-\tau)$ is easy to interpret and is extremely simple. Completing the integration gives

$$x(t) = \omega_n U \int_{t=0}^{r=t} \sin \omega_n \tau d\tau = \omega_n U \int_{t=0}^{r=t} \left(\frac{-1}{\omega_n} \right) d(\cos \omega_n \tau) = -U \left[\cos \omega_n \tau \right]_0 = -U (\cos \omega_n t - 1)$$

$$\Rightarrow \quad x(t) = U (1 - \cos \omega_n t), 0 \leq t \tag{7.3.5}$$

Equation 7.3.5 is graphed for a few cycles of response in Figure 7.3.2. The response is sinusoidal and periodic with the system natural period T_n, as is the free-vibration IC response. However, Equation 7.3.5 is forced response, and the input $u(t)$ is non-zero and constant for $t > 0$, so this step response vibrates around a non-zero mean value (the pseudo-static output value $x_{ps} = U$), oscillating between 0 and $2U$. In contrast, the free-vibration IC response of Equation 7.3.3 and Figure 7.3.1 vibrates around the zero mean value, oscillating between $-x_{\max}$ and $+x_{\max}$.

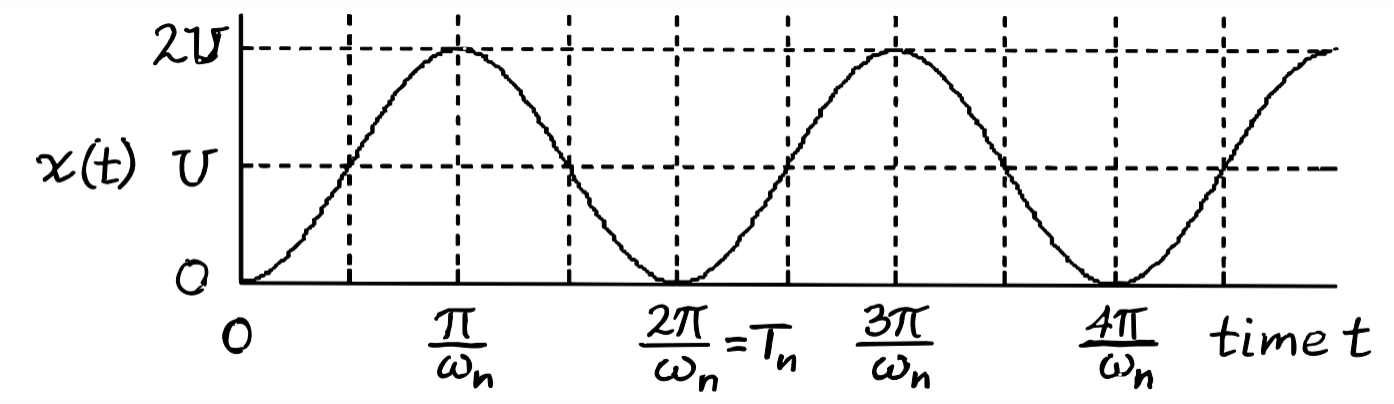


Figure 7.3.2: Step response of an undamped 2nd order system

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7.4: Discussion of Physical Applicability of Step-Response Solutions

The concept of step response in physical applications can be bothersome if one harbors the misconception that step response is valid only if the input is applied in an abrupt, discontinuous manner, as on the drawing at left. This misconception is natural and honest because of the mathematical definition of the step function, Equation 2.4.1 and Figure 2.4.1. If we apply that strict definition to the step input analyzed in Section 7.3, without any recognition of the nature of the initial conditions, then the time history of the input would have to be a discontinuous step function.

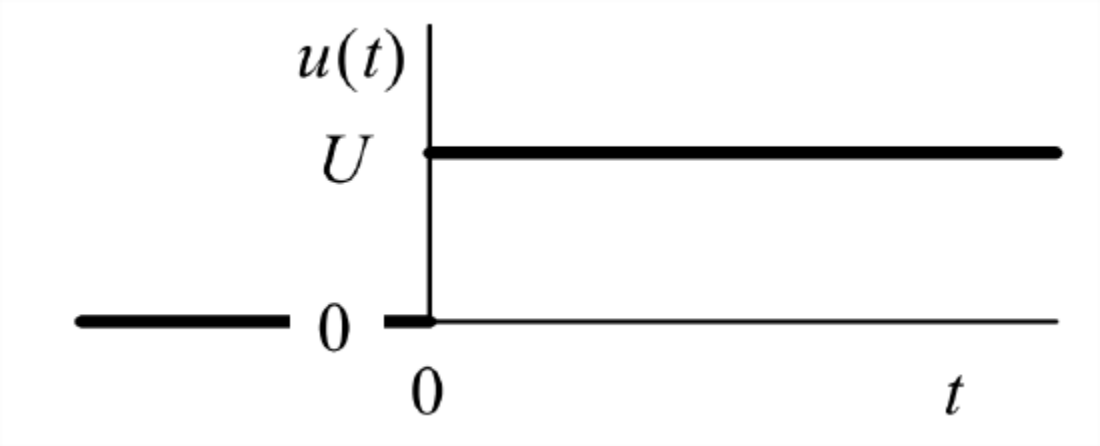


Figure 7.4.1

However, it is instructive to consider also the role of initial conditions in step response. Step response solution Equation 7.3.8 is derived for zero ICs, $x_0 = 0$ and $\dot{x}_0 = 0$. Imagine that some powerful external agent enforces these ICs for time t < 0, then at t = 0, the external agent abruptly releases its hold on the system. Under this circumstance, it does not matter whether the input $u(t)$ jumps discontinuously from 0 to U at t = 0, as in the drawing above, or the input has value U both after **and before** t = 0, as in the drawing at left. An input value of U before t = 0 could not produce any effect on output $x(t)$, because it could not overpower the external agent that enforces the initial conditions before and right up to the instant t = 0. A practical example to which we can relate is an airplane sitting on a runway, preparing to taxi for takeoff. With the brakes firmly engaged, the pilot increases engine thrust, but the airplane is held motionless by the brakes. In this example, the brakes play the role of “powerful external agent” that enforces the initial position and zero initial velocity, and the engine thrust plays the role of input quantity. After engine thrust has reached acceptable takeoff level, the pilot releases the brakes (at t = 0), and the airplane accelerates into takeoff taxi. The engine thrust exists before t = 0, but it produces no taxiing motion then because the brakes completely restrain the airplane.

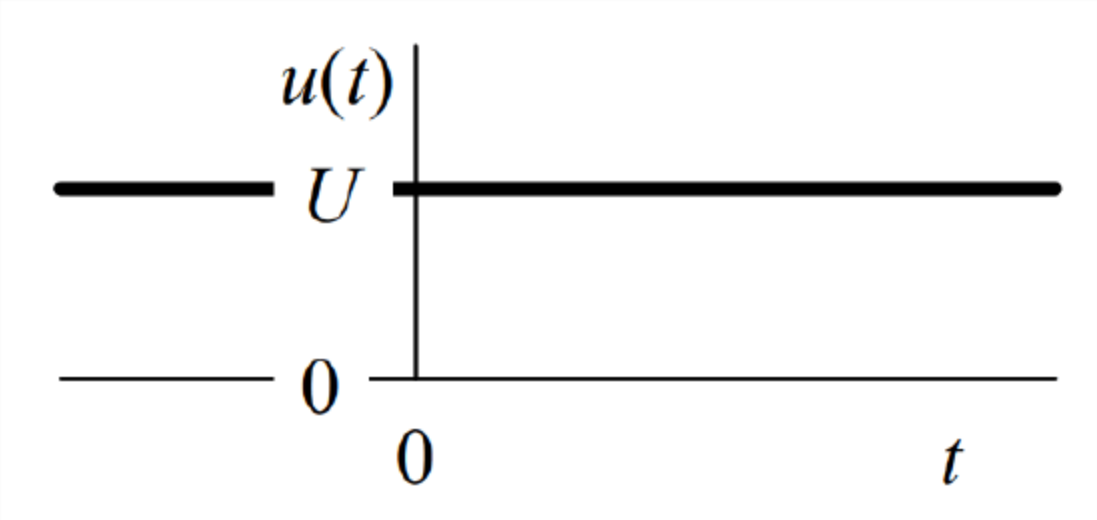


Figure 7.4.2

The conclusion of this discussion is that step response is a valid physical solution even if the input is constant before and after the initiation of motion (*i.e.*, is not applied strictly like a mathematical step function), provided that initial conditions enforce the motion prior to the instant (usually $t = 0$) when the input becomes effective.

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7.5: Dynamic Motion of a Mechanical System Relative to a Non-Trivial Static Equilibrium Position; Dynamic Free-Body Diagram

Consider the mass-spring system depicted in Figure 7.5.1 (on the next page), which is representative in many respects of some real mechanical systems. The system is oriented vertically in a gravity field with acceleration of gravity g. Moreover, let us suppose that the spring is *close-wound*: a helical extension spring which, in its unstretched state, is wound with its coils forced into contact with each other, such that an initial tension force F_i is required to break the coil contact and allow the spring to begin stretching (Shigley and Mitchell, 1983, p. 449). Figure 7.5.1 illustrates three possible states of such a system. The weight of mass $m$is $W = mg$, and it is a constant force vector acting downward through the center of gravity of the mass. The spring has stiffness constant k and initial tension F_i, so that the spring law is $f_y = F_i + ky$ for $f_y \geq F_i$, where y is spring stretch vertically downward. State 1 of Figure 7.5.1 shows the mass supported statically by an externally applied force W acting upward, with the spring unstretched and therefore exerting no force on the mass. We define $y_t(t)$ as the **total** vertical motion of the mass (positive downward in this example, but not necessarily always so) relative to the unstretched spring position. Now, if the externally applied upward force is slowly reduced, the weight of the mass is gradually transferred to the spring, and the spring stretches (provided that $W > F_i$), lowering the mass. State 2 of Figure 7.5.1 is the **static** equilibrium state that results after the externally applied upward force shrinks to zero, with the spring stretched by the amount y_s from the State 1 position. From the FBD of State 2, the equation of vertical static force equilibrium is

$$F_i + ky_s = W \quad \Rightarrow \quad y_s = \frac{W - F_i}{k}, \text{ for } W \geq F_i \tag{7.5.1}$$

State 3 of Figure 7.5.1 is a condition of dynamic response, with the dynamic force $f_y(t)$ acting on the mass. We define the **dynamic** position $y_d(t)$ as the motion relative to the static equilibrium position, y_s of Equation 7.5.1. We see from Figure 7.5.1 that the total translation (from the unstretched-spring position) and the associated acceleration are

$$y_t(t) = y_s + y_d(t) \Rightarrow \ddot{y}_t(t) = \ddot{y}_d(t) \tag{7.5.2}$$

The acceleration equation reflects the constancy of y_s. Let us use the FBD for State 3 to write Newton's 2nd law for vertical translation:

$$\Sigma(\text{ Forces })_y = m\ddot{y}_t = f_y(t) + W - (F_i + ky_t) \Rightarrow m\ddot{y}_t + ky_t = f_y(t) + W - F_i \tag{7.5.3}$$

Now use Equation 7.5.2 to eliminate $y_t(t)$ from Equation 7.5.3:

$$m\ddot{y}_d + k\left(y_s + y_d\right) = f_y(t) + W - F_i \tag{7.5.4}$$

Next, eliminate $ky_s = W - F_i$, Equation 7.5.1, from Equation 7.5.4 in order to express the ODE in terms of **the motion relative to the static equilibrium position**, the single dependent variable, $y_d(t)$:

$$m\ddot{y}_d + ky_d = f_y(t) \tag{7.5.5}$$

It is obvious physically that Equation 7.5.5 is valid provided the upward dynamic motion does not exceed the static spring stretch: $y_t(t) = y_s + y_d(t) > 0$, *i.e.*, $y_d(t) > -y_s$.

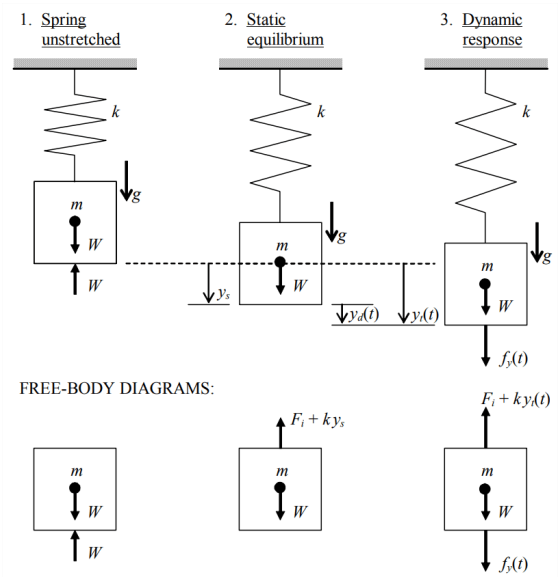


Figure 7.5.1: Static and dynamic positions of a hanging, spring-supported weight

Equation 7.5.5 is identical in form to Equation 7.1.2, which applies for a mass-spring system oriented horizontally, with no involvement of a close-wound spring or gravity. The only difference is that the unknown in Equation 7.5.5 is not the total motion, but, instead, the motion **relative** to the static equilibrium position established by gravity and the spring initial tension. Consider the ***dynamic*** *free-body diagram* (DFBD) shown in Figure 7.5.2 on the next page, which is similar to that of State 3 in Figure 7.5.1, but without weight W and with only the spring force relative to the static equilibrium position. These changes from the FBD of State 3 in Figure 7.5.1 are equivalent to eliminating $ky_s = W - F_i$ from Equation 7.5.4 in order to obtain Equation 7.5.5. Clearly, applying Newton's 2nd law to the DFBD Figure 7.5.2 leads again, and much more easily, to Equation 7.5.5. The important conclusion here: **if we want to solve only for the dynamic motion $y_d(t)$ relative to the static equilibrium position, not the total motion relative to the springundeformed position, then we should use a DFBD from which all static influences (weight and spring initial tension in this case) have been eliminated. The DFBD should include only dynamic forces (and/or moments) relative to the static equilibrium position.**

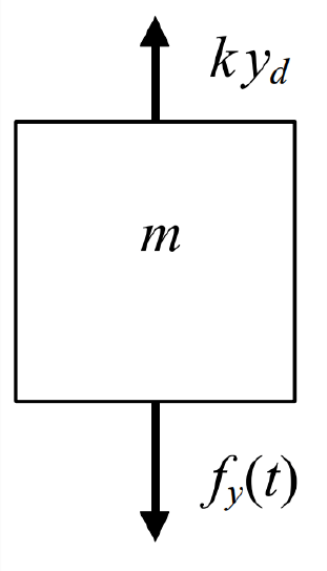


Figure 7.5.2: Dynamic freebody diagram

This conclusion is useful more generally because, in many situations, we are, indeed, interested primarily in the dynamic response relative to the static equilibrium position. Homework Problem 7.3 is one example. Moreover, this conclusion based upon the simple mass-spring system of Figure 7.5.1 has broad applicability to other mechanical systems. The subjects of Chapters 11 and 12 are more complex rotational systems and higher-order mechanical systems. It is usually convenient when analyzing those

systems to deal only with the simpler DFBDs and ODEs that we use when we solve only for the dynamic motion relative to the static equilibrium position.

However, there are at least two classes of mechanical systems for which we **must** include (in FBDs and ODEs of motion) the weights of massive components:

1. pendulous systems for which gravity provides a spring effect, such as the simple pendulum example in Section 7.1 and the balloon-basket vehicle of homework Problem 7.4;
2. systems for which initial conditions and/or dynamic inputs (forces, moments) are defined relative to the undeformed-spring state, for example, the airplane landing contact of homework Problem 7.5.

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7.6: Introduction to Vibrations of Distributed-Parameter Systems

A system such as the mechanical mass-dashpot-spring system of Figure 7.1.1 is often called a *lumped-parameter* system. This descriptive term is used because the essential physical features are idealized to be spatially concentrated: in Figure 7.1.1, all of the mass is idealized to reside in one physical element, all of the damping in another, and all of the stiffness in yet another. Some engineering systems can be modeled accurately using idealized lumped elements; but many real systems cannot, and these latter are called *distributed-parameter* systems. A structural beam, such as that shown in Figure 7-6 on the next page, is a prototypical distributed-parameter mechanical system. It is obvious that the mass of this beam is not concentrated at a single point in space, but instead is distributed throughout the beam's volume. Similarly, the flexibility of the beam does not reside in a discrete spring, but also is distributed over the entire spring. Finally, although there is very little damping in this particular system, it also is distributed throughout the beam in the forms of molecular "friction" within the deforming beam and fluid dynamic drag on the beam's surface if the beam is immersed in air, another gas, or liquid.

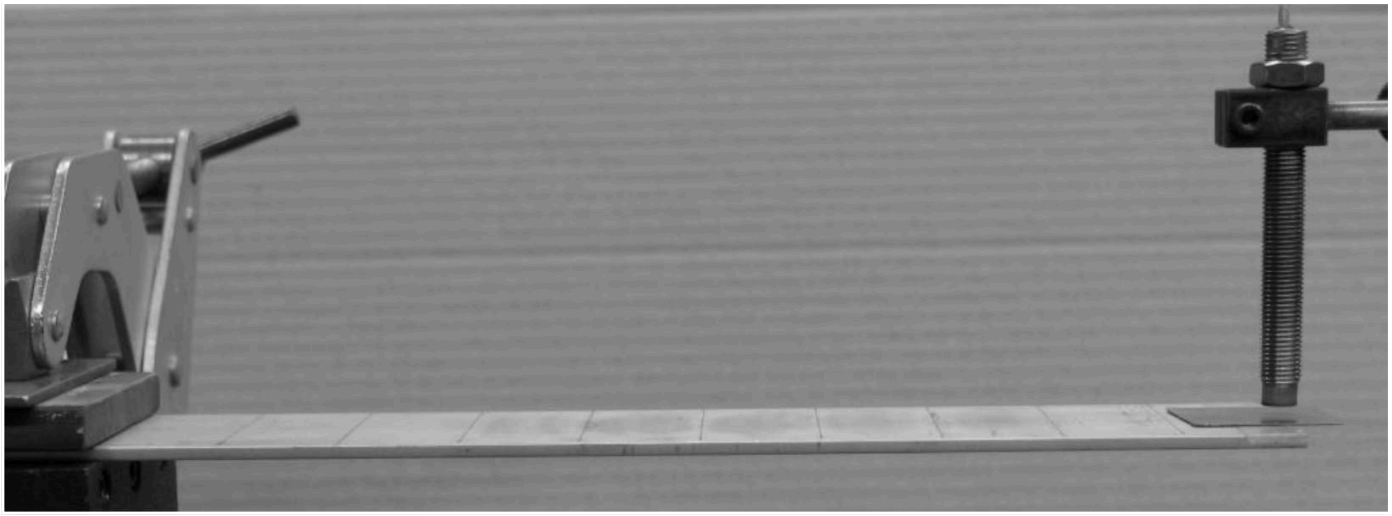


Figure 7.6.1: Clamped uniform aluminum beam with nominal cross section 2 inch $\times$ 81 inch. Test configuration: beam clamped at left between thick steel plates, leaving 10.0 inches of overhang; non-contacting proximity (displacement) sensor over right tip; bonded onto the beam's right tip is a small, thin, steel plate, the "target" in which the proximity sensor induces eddy currents.

Figure 7.6.2 is an actual record of motion of the beam in Figure 7.6.1 in response to a particular initial condition: the tip of the beam was bent upward statically, then released[1].

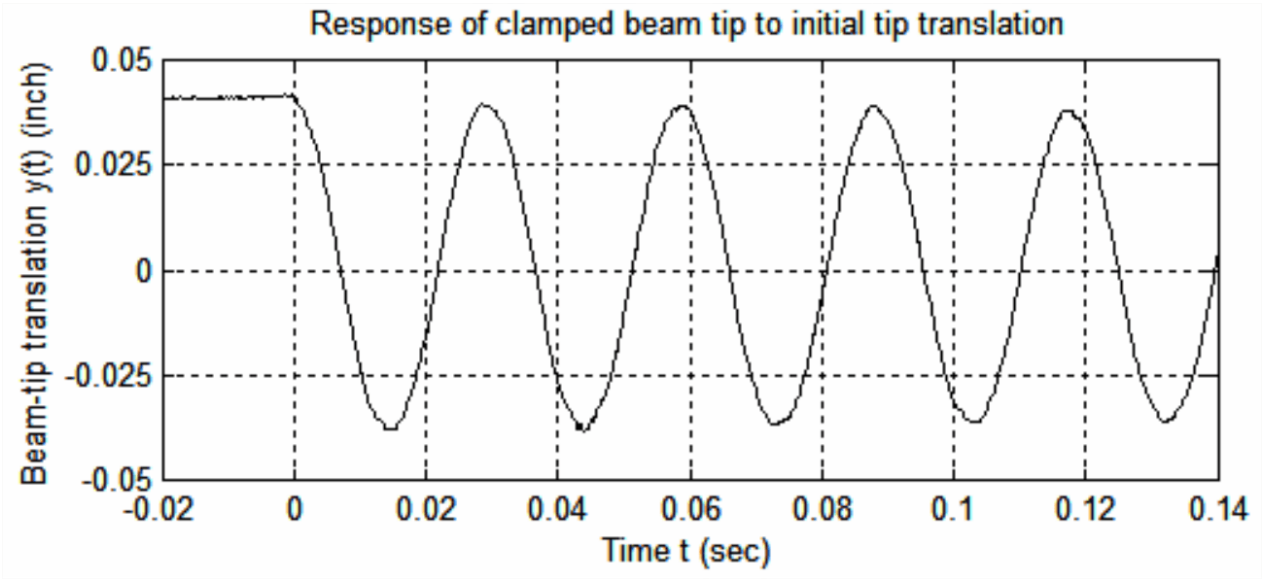


Figure 7.6.2: Initial-translation dynamic response of the beam in Figure 7.6.1, as detected by the proximity sensor.

The subsequent time response was directly analogous to the theoretical IC response shown in Figure 7.3.1, with positive initial deflection, $y_0 > 0$, but zero initial velocity, $\dot{y}_0 = 0$, $y(t)$ being vertical translation, positive upward, of the beam tip relative to the static equilibrium position under gravity loading. Note in Figure 7.6.2 that the character of the measured response appears very close to that predicted theoretically in Eq. (7-10) for an undamped 2nd order system, $y(t) = y_0 \cos \omega_n t$. However, we can detect by detailed analysis of the data in Figure 7.6.2 the following very small differences between actual and theoretical responses:

1. (i) slight reductions in amplitude from one cycle to the next, due to the very light system damping and to an imperfect release of the beam from its static initial deformation;
2. (ii) a slight eccentricity of the vibration relative to the zero-translation position, due to a small instrumentation error, which is almost unavoidable in experimental measurements; and
3. (iii) almost imperceptibly small distortion of the waveform shape relative to a perfect cosine.

(More easily observed distortion appears in Figure 7.6.4, which is discussed below.) This small waveform distortion occurs because the beam is **not** a perfect 2nd order, lumped-parameter system; in reality, the beam is, mathematically, a much more complicated distributed-parameter system. Nevertheless, it turns out that in many circumstances, including this particular one, we can approximately model distributed-parameter structures as 2nd order systems, with acceptable accuracy for many engineering purposes.

Careful study of the experimental data in Figure 7.6.2 shows that the time between any two successive peaks of the response signal, the "period" of damped vibration, is $T_d \approx$0.029 4 s. The system of Figure 7.6.1 is damped, so T_d is not a true undamped "natural" period. However, theoretical solutions for mathematical models predict that a measured period such as this for a very lightly damped system is almost equal, for all practical purposes, to the true natural period. Therefore, in the example problems below, we shall assume that the measured period is the system's natural period, and so the undamped natural frequency is $f_n \approx 1/T_d =$ 34.0 Hz.

Figure 7.6.3 illustrates the spatial distribution of the beam vibrating after the same type of initial condition as that of Figure 7.6.2 , but with a much greater magnitude of deformation. For this edge-on photo, the shutter of the digital camera was open for $\frac{1}{4}$ second, during which time the beam vibrated through 8+ complete cycles. The upper and lower limits of dynamic deformation appear as fairly distinct curved lines against the dark background in the photo, whereas the intermediate states of deformation appear as a blur. This occurs because the velocity of motion was zero at the extreme deformations; therefore the digital camera's photosites were exposed during disproportionately longer times to the extreme deformations than to the intermediate deformations.

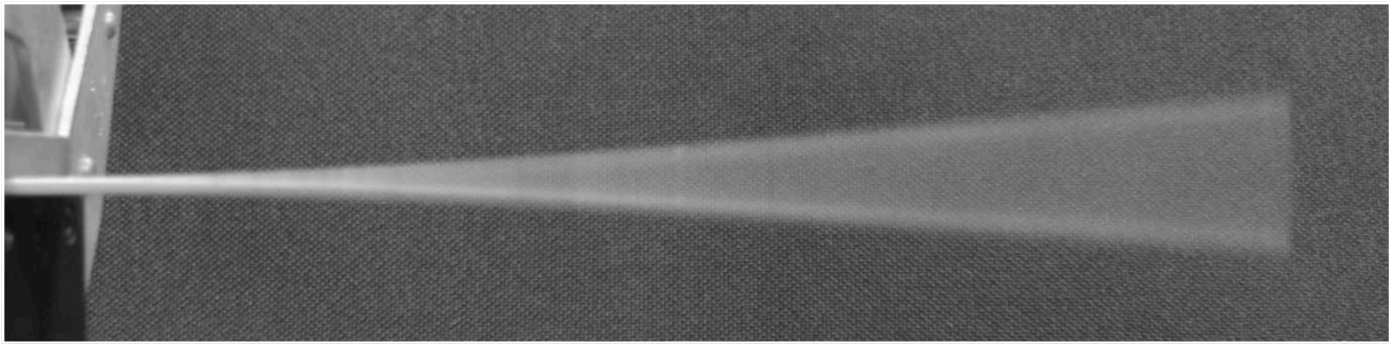


Figure 7.6.3: Edge-on, $\frac{1}{4}$-second exposure photo of the beam of Figure 7.6.1 vibrating after an initial static deformation and release of the beam tip.

Example 7.6.1: A Calculation of the "Effective" Tip Mass of the Vibrating Beam

Consider the system of Figure 7.6.1. By loading the beam statically with small calibrated weights hanging from the beam tip, and by measuring the resulting beam-tip downward vertical deflection with the proximity sensor, students determined a beam-tip average "effective" stiffness constant to be $k_E = 7.86$ lb/inch. With this stiffness and the measured natural frequency f_n = 34.0 Hz, we can infer a value for the "effective" tip mass from Equation 7.1.3 in the form

$$m_E = \frac{k_E}{\omega_n^2} = \frac{k_E}{(2\pi f_n)^2} = \frac{7.86\text{lb/inch}}{(2\pi \times 34.0\text{s}^{-1})^2} = 1.72\text{e}-4\text{lb}-\text{s}^2/\text{inch}$$

It is appropriate to conduct a plausibility check of this result, especially since most of us find it difficult to relate intuitively to values of mass that are expressed in the traditional structural system of units (see Section 3-1). Students measured the 10.00-inch overhang of the uniform aluminum beam in Fig. 7-6 to weigh Wb = 0.230 lb, and they measured the weight of the small steel target at the beam tip as 0.13 oz = 0.0081 lb. Therefore, the total **mass** of the beam plus the target is

$$m_T = W_T/g = (0.238\text{lb}) \div \left(386.1\text{inch/s}^2\right) = 6.16\text{e}-4\text{lb}-\text{s}^2/\text{inch}$$

The mass **ratio** is m_E/m_T = 0.28. In view of the photo of the vibrating beam, Figure 7.6.3, it seems physically plausible that roughly 28% of the total structural mass is effectively volved in the vibration. This ends Example Problem 7.6.1.

Example 7.6.2: Calculations Based Upon Ideal Theory, Normal DImensions, and Standard Material Constants

Let us calculate an estimate of the natural frequency of the beam system of Figure 7.6.1 using Equation 7.1.3 in the form $f_n = \frac{1}{2\pi}\sqrt{k_E/m_E}$. In this calculation, let us assume that we do not have measured data from actual hardware or instrumentation, *i.e.*, that we have only the type of information that would normally be available in the preliminary-design phase of an engineering project, before even a prototype is fabricated.

Solution

First, let us calculate the theoretical effective stiffness constant k_E. From your courses and textbooks on static structural behavior, you might be familiar with the deformation under loading of structural members such as uniform (prismatic) beams and shafts that are fabricated from homogeneous, isotropic, linearly elastic materials. The theory relevant to the present situation is that for a cantilever (ideally clamped-free) beam. From any textbook with the words "mechanics of materials" or "strength of materials" in its title[2], you can find that the effective stiffness at the tip of a uniform cantilever beam is $k_E = 3EI/L^3$, in which E is the material modulus of elasticity, I is the *2nd area moment* (also called *area moment of inertia*) of the beam cross section, and L is the overhanging length of the beam. For the beam of Figure 7.6.1, let us use the standard (in preliminary-design calculations) elastic modulus for aluminum: E = 10.0e+6 lb/inch^2. The nominal rectangular cross-

sectional dimensions of the beam are width b = 2.00 inches and depth h = 1/8 inch, for which the theoretical 2nd area moment is $I = bh^3/12$ = 3.255e–4 inch^4. Using the overhang length of 10.00 inches, we calculate the effective beam-tip stiffness to be

$$k_E = \frac{3 \times \left(10.00 \times 10^6 \text{lb/inch}^2\right) \times \left(3.255 \times 10^{-4} \text{inch}^4\right)}{(10.00 \text{inch})^3} = 9.766 \text{lb/inch}$$

Next, let us calculate an effective tip mass m_E. Consider first the vibration of an ideal uniform cantilever beam alone (*i.e.*, without a concentrated tip mass such as the steel target shown on Figure 7.6.1). More advanced theory[3] shows that, for such a bare beam, the effective tip mass vibrating at the fundamental natural frequency is 0.242 672 of the total beam mass, which we determine next. The weight of a uniform beam is $W_b = wAL$, in which w is the weight density (weight per unit volume), and the cross-sectional area of a rectangular beam is $A = bh$. Let us use the standard (in preliminary-design calculations) weight density of aluminum, w = 0.1000 lb/inch^3, so that the weight of our ideal cantilever beam is W_b = (0.1000 lb/inch^3) $\times$($\frac{1}{4}$ inch^2)$\times$ 0.2500 lb. Hence, the effective tip mass of the bare beam alone is m_{bE} = (0.242 672 $\times W_b/g$ = 0.242 672 $\times$ (0.2500 lb) $\div$ (386.1 inch/s^2) = 1.571e–4 lb-s^2/inch. To obtain an intuitively logical estimate of the total effective tip mass, we add to m_{bE} the mass of the steel target that is bonded to the beam tip on Figure 7.6.1: $m_E = m_{bE} + (0.0081\text{lb}) \div (386.1\text{inch/s}^2)$ $= 1.78\text{e} - 4\text{lb} - \text{s}^2/\text{inch}$.

Finally, the natural frequency based upon ideal theory, nominal dimensions, and standard material constants is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_E}{m_E}} = \frac{1}{2\pi}\sqrt{\frac{9.77\text{lb/inch}}{1.78 \times 10^{-4}\text{lb} \cdot \text{s}^2/\text{inch}}} = 37.3\text{Hz}$$

See homework Problem 7.8 for an analysis of the discrepancy between this calculated natural frequency and the measured value of 34.0 Hz. This ends Example Problem 7-2.

The clamped-beam system of Figure 7.6.1 appears to behave very much like a 2nd order system when it responds dynamically to a simple initial static deflection of the beam tip. However, when it is subjected to most other possible types of excitation, its dynamic response is more complicated. For example, when the beam tip was tapped lightly upward with a small hammer, the tip moved vertically as shown on Figure 7.6.4, next page.

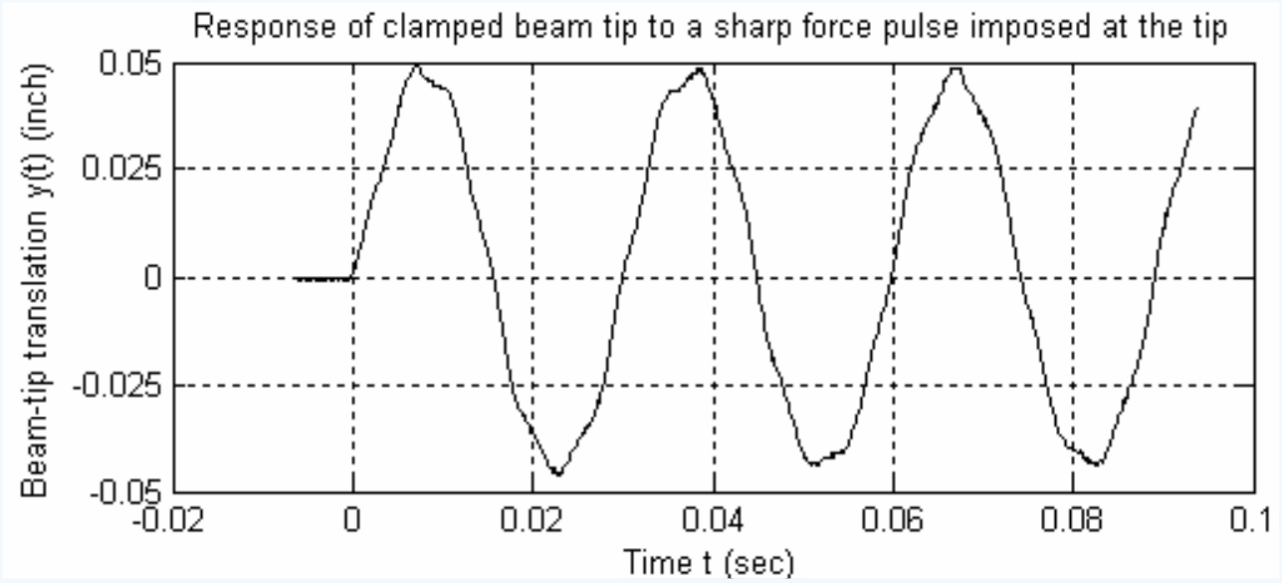


Figure 7.6.4: Dynamic response to a hammer tap at time t = 0 s of the beam on Figure 7.6.1, as detected by the proximity sensor.

The input force that stimulated the response of Figure 7.6.4 was approximately a half-sine pulse (as described in Sections 1.5 and 1.6) of very short duration relative to the lightly damped period of this system, $T_d \approx$ 0.029 4 s. In Chapter 8, theory is developed showing that, if the system were a true undamped 2nd order system, then the time response to such a pulse

would have almost a perfect sine waveform, starting at time t = 0 s. However, the waveform in Figure 7.6.4 is quite different from a perfect sine waveform; in fact, it is closer to being the sum of many sinusoids, in the general spirit of a Fourier series, but without the property of Fourier series that all higher series sinusoids have frequencies that are integer multiples of the frequency of the first (fundamental) sinusoid[4]. In the case of Figure 7.6.4, the total motion consists primarily of a first sinusoid with frequency 34.0 Hz, **plus** a second, lower-amplitude sinusoid with frequency approximately 215 Hz. The motion at 34.0 Hz represents the *first (fundamental) mode of vibration*, and the motion at 215 Hz represents the *second mode of vibration*.

In fact, the beam system of Figure 7.6.1 has many more than two modes of vibration, each with a characteristic frequency higher than that of the previous mode. The third and higher modes are not evident in Figure 7.6.4 only because the particular pulse stimulus applied in this case did not noticeably excite those higher modes; however, they could be excited by any number of other stimuli, such as sharp pulses applied at particular locations, high-frequency sound-pressure waves, *etc.* In reality, every distributed-parameter system is a higher-than-2nd-order system, and, if only lightly to moderately damped, has many modes of vibration. For example, every acoustic musical instrument is a fluid or structural distributed-parameter system whose distinctive sound quality is a direct consequence of its particular modes of vibration[5]. However, except for this short descriptive introduction, the subject of distributed-parameter systems is beyond the scope of this book. The subject of vibration modes appears again in Chapter 12, which deals with undamped, 4th order, lumped-parameter systems that have two modes of vibration.

Observe from Figure 7.6.4 that the translation response of the beam system of Figure 7.6.1 was still dominated by the first vibration mode, even though the tip-pulse stimulus also excited the second mode, but to a much lower amplitude than that of the first mode. Hereafter in this book, the only response of any distributed-parameter system that we shall consider is the contribution of the first mode, and we shall assume that it dominates the total response. Indeed, this is often a good engineering approximation for distributed parameter structural systems that include relatively large, spatially concentrated masses. Consider, for example, the system of Figure 7.6.5, which is the same as that of Figure 7.6.1 but with the addition at the beam tip of a concentrated-mass assembly that consists of two ceramic magnets and an aluminum spacer, and weighs 0.254 lb

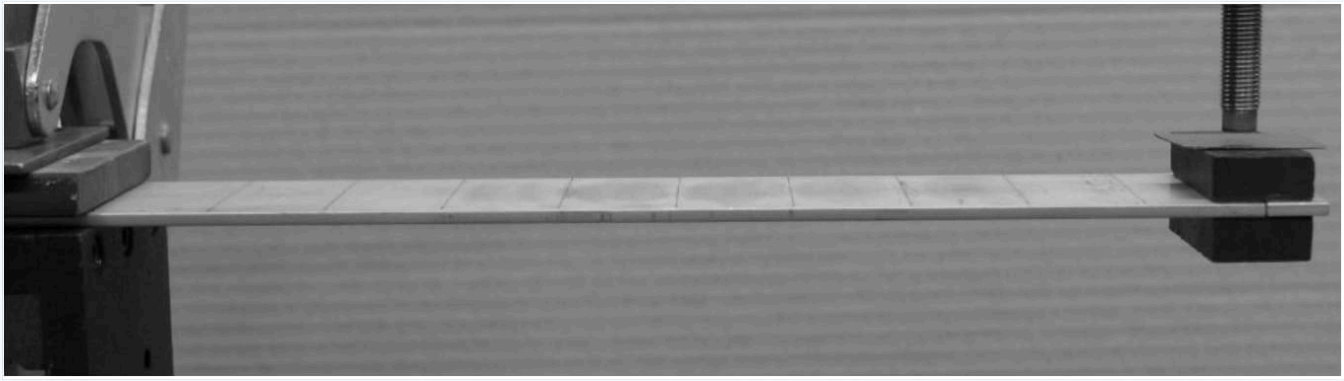


Figure 7.6.5: Structural system of Figure 7.6.1, with the addition at the beam tip of two ceramic magnets and an aluminum spacer. Note the same steel-plate tip-target for the proximity sensor as on Figure 7.6.1.

When the beam tip of Figure 7.6.5 was tapped lightly upward with a small hammer, the tip moved vertically as shown on Figure 7.6.6. The tip-translation response curve is a very clean sine waveform with almost no discernible distortion. The absence of significant distortion in the waveform means that, for our purposes, this beam-mass system behaves like a 2nd order system, even though it is a distributed-parameter system.

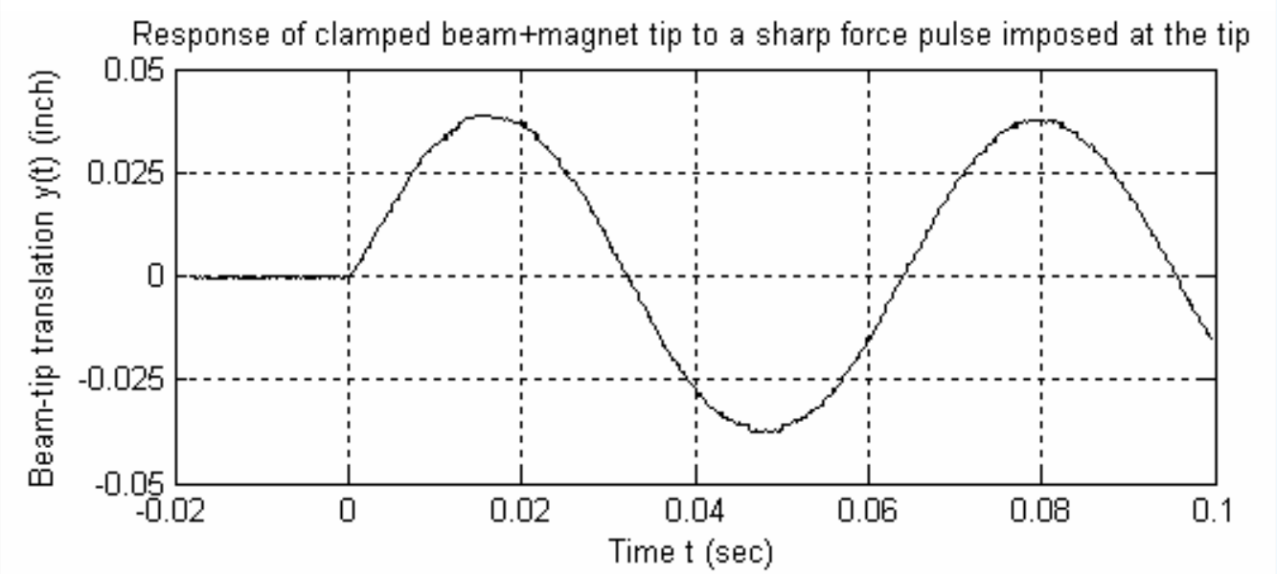


Figure 7.6.6: Dynamic response to a hammer tap at time t = 0 sec of the beam on Figure 7.6.5, as detected by the proximity sensor.

The addition of a relatively large quantity of mass at the beam tip reduces considerably the frequency of the waveform on Figure 7.6.6 relative to the 34.0-Hz frequency of the fundamental waveform on Figures 7.6.2 and 7.6.4; a careful analysis of the data in Figure 7.6.6 shows that the frequency of vibration is f = 15.85 Hz.

Example 7.6.3: a Predictive Calculation of the Vibrational Frequency of the Beam-Mass System of Figure 7.6.5

Suppose that we had wanted to predict the frequency of the beam-mass system of Figure 7.6.5 before actually assembling the system and measuring the frequency. The following is an intuitively logical approach, though not necessarily theoretically rigorous. Providing that the clamped beam deforms linearly under load, the addition of mass and weight to the beam tip should not alter the beam's stiffness constant, so we use the same measured beam-tip effective stiffness as that of Example Problem 7-1, k_E = 7.86 lb/inch. As stated previously, the concentrated-mass assembly that was added to the beam tip weighs 0.254 lb. To obtain the total effective vibrating tip mass, it seems reasonable to add together this value and the effective tip mass of the bare beam plus the steel target, as determined in Example Problem 7-1. Thus, we find the total effective mass m_E = 0.254 lb/(386.1 inch/s^2) + 1.72e–4 lb·s^2/inch = 8.30e–4 lb·s^2/inch. Finally, we calculate the predicted natural frequency to be

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_E}{m_E}} = \frac{1}{2\pi}\sqrt{\frac{7.86\text{lb/inch}}{8.30\times10^{-4}\text{lb}\cdot\text{s}^2/\text{inch}}} = 15.5\text{Hz}$$

This predicted natural frequency is lower than the actual measured value of 15.85 Hz by a margin that is unexpectedly large, presuming that all assumptions underlying the theory are valid. This suggests that the actual effective stiffness during the dynamic response of Figure 7.6.6 was greater than 7.86 lb/inch. This is quite possible, because the students measured the beam-tip **average** effective stiffness constant to be 7.86 lb/inch over a range of very small downward deformation of the beam, but they also observed that the **local** stiffness (slope of the curve of load versus deflection) increased progressively as the downward deformation increased. Observe on Figures 7.6.1 and 7.6.5 at the clamped end of the beam that there appears to be a small gap between the edge of the upper steel clamping plate and the beam top surface; although not clearly visible on these photos, there is a similar gap between the lower clamping steel plate and the beam lower surface. These gaps are widest when the beam is undeformed, but as the beam is deformed downward, the lower gap gradually closes, thus slightly reducing the beam's effective length L and thereby more significantly increasing its tip stiffness, $k_E = 3EI/L^3$, from Example 7.6.2. The static gravity load of the concentrated-mass assembly shown on Figure 7.6.5 closed the lower gap, relative to its width on Figure 7.6.1 without the concentrated-mass assembly. Thus, it is indeed likely that the average beam-tip stiffness

during the response recorded on Figure 7.6.6 was greater than that during the responses recorded on Figures 7.6.2 and 7.6.4 . The actual clamping support and restraint of the aluminum beam in this experiment is clearly not the ideal, perfectly sharp edge and completely rigid wall that the elementary theory assumes for a cantilever beam.

[1]In practice, this procedure is often called *twang* testing (also *snapback*, *step-relaxation*, and *pluck* testing)—see homework Problem 9.4 for a more detailed discussion.

[2]Examples of such textbooks, listed in the References section: Beer and Johnston, 1992; Hibbeler, 1997; and Timoshenko and Young, 1962.

[3]Theory from, among many possible sources, the following textbooks on structural dynamics that are listed in the References section: Bisplinghoff, *et al.*, 1955, Example 3-1; Craig, 1981, Example 10.3; Meirovitch, 1967, Section 5-10; and Meirovitch, 2001, Example 8.4.

[4]The subject of Fourier series is developed in most textbooks on advanced calculus (for example, Hildebrand, 1962, pages 216-226) and in most textbooks on elementary partial differential equations.

[5]If you are interested in the physics of musical instruments, see Cannon, 1967, Sections 13.2-13.6, for an excellent introduction written by an engineer for students of engineering.

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7.7: Chapter 7 Homework

1. A particular device is known to be an LTI mass-spring system having negligible damping. It is required that stiffness constant k and mass m be identified experimentally. First, a static force of 100 N is applied through a string tied to the mass, producing a static translation of the mass. Next, the string is cut cleanly, allowing the system to vibrate freely from the initial static translation (with zero initial velocity). The subsequent motion is measured by an *accelerometer*, a sensor that is attached to the system mass and measures, of course, translational acceleration of the mass. The measured acceleration is described with good accuracy by the equation \[a(t) \equiv \ddot{x}(t)=-4.93 \cos (10 \pi t) \frac{\mathrm{m}}{\mathrm{sec}^{2}}, \text { for } 0

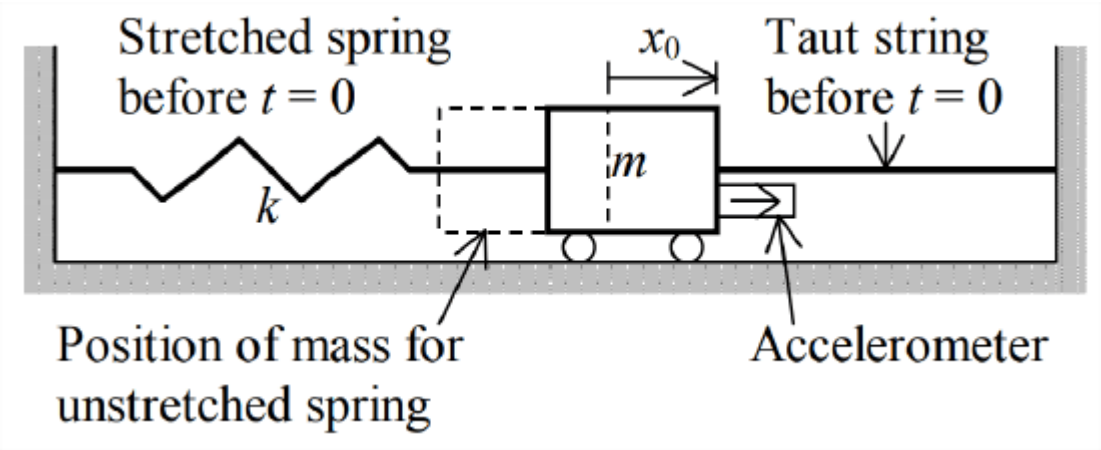


Figure 7.7.1

   1. Differentiate twice Equation 7.3.4 for displacement, $x(t) = x_{\max} \cos(\omega_n t + \phi)$: first, to derive the associated equation for velocity, $v(t) = \dot{x}(t)$; and second, to derive the associated equation for acceleration, $a(t) = \dot{v}(t) = \ddot{x}(t)$.
   2. From the experimental data given previously and your results in part 7.1.1, infer values (with units) for stiffness constant k and mass m.
2. Consider again the reaction wheel assembly introduced in Section 3.3 (Figure 3.3.1), but now suppose that there is a rotational spring with stiffness constant k_θ connecting the shaft to a rigid wall. Also, for this problem only, assume that the bearing viscous damping torque is negligible.

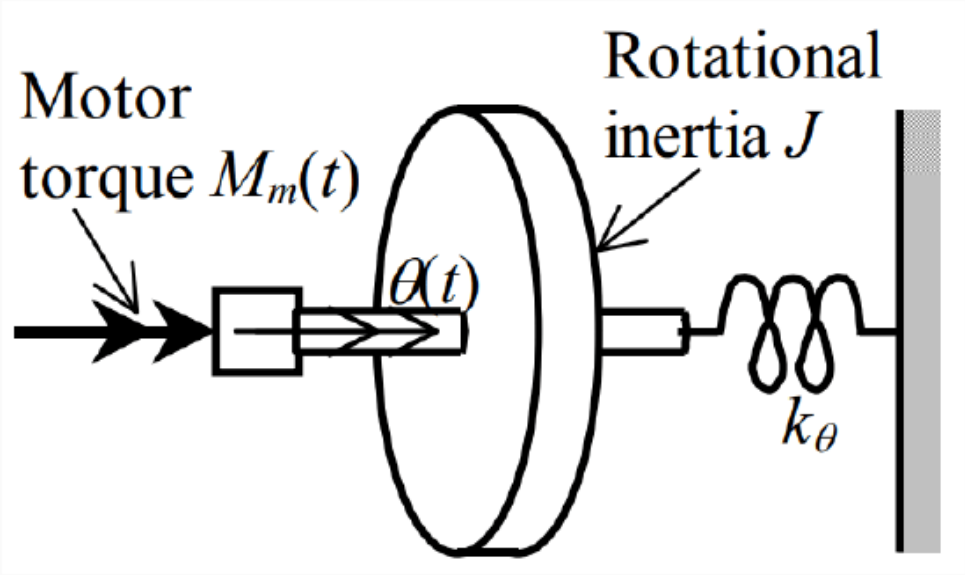


Figure 7.7.2

   1. Sketch a rotational FBD (similar to that in Section 3.3, but with appropriate differences), then apply Newton's 2nd law for rotation to derive the 2nd order ODE for wheel rotation $\theta(t)$. This ODE should have the form of Equation 7.1.2, but with the notation appropriate for this system.

2. Convert the ODE of part 7.2.1 into the standard form Equation 7.1.5, $\ddot{\theta} + \omega_n^2 \theta = \omega_n^2 u(t)$. Write specific equations (in terms of this system's parameters and notation) for natural frequency ω_n and input quantity $u(t)$.
3. The electric torque motor that drives the rotor generates 4.00 oz-inch of torque per amp of electrical current (1 lb = 16 oz). According to the motor manufacturer, the current should be limited to 5.00 A or less in order to avoid toasting the motor, so the maximum motor torque is 20.0 oz-inch. Suppose that this maximum torque is applied to the rotor as a step function. It is required that the subsequent rotation angle $\theta(t)$ of the rotor not exceed 45°. Calculate the **minimum** stiffness constant k_θ (units of lb-inch/rad) that will limit the step response $\theta(t)$ to 45° or less. Also, calculate the natural frequency in Hz of the system with this value of k_θ. The rotor has rotational inertia J = 2.56e–3 lb-s^2-inch.

3. In the test laboratory of a spacecraft manufacturer, a vibration-isolation platform is supported on a flexible foundation so that the vibration induced by traffic on a busy nearby freeway will not disturb functional testing of delicate space-qualified components. The platform weighs 907 kg$_f$, and the flexible foundation has nominal stiffness constant k = 3,440 kN/m.

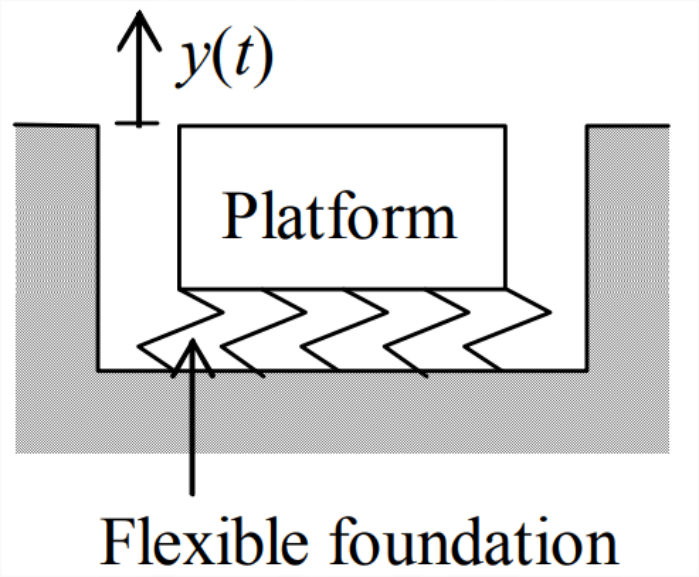


Figure 7.7.3

1. Calculate the predicted values of system natural frequency in rad/s and Hz, and system natural period T_n. Recall from Section 3.2 that the kg$_f$ is not a consistent unit in the SI system. (*partial answer*: ω_n = 61.6 rad/s)
2. In a test to determine precisely the dynamic characteristics of this mass-spring system (having negligible damping), the platform is to be given initial downward displacement $y(0)$ = –0.104 mm (relative to the static equilibrium position) and initial upward velocity $\dot{y}(0)$ = +7.20 mm/s. The platform will then be allowed to vibrate freely. Write the numerical algebraic equation for the predicted response $y(t)$ in mm.

4. Consider the rolling motion of a balloon carrying a basket, with the vehicle neither ascending nor descending[1]. Weight W of the vehicle acts downward through the vehicle's center of gravity G, and buoyancy force W acts upward through *center of buoyancy* M, which is the center of gravity of the volume of air displaced by the vehicle. We denote as R the separation of G and M in the vehicle's plane of symmetry. Note that with M above G, weight and buoyancy form a stabilizing couple moment, the moment arm being $R \sin\theta$. Suppose that wind, gusts, and possibly other forms of disturbance and control exert an externally applied rolling moment about G, which is denoted as $M_r(t)$. Denote the vehicle's rotational inertia about G as J_G. Assume that the vehicle rolls about point G, as if there were a frictionless hinge at G, and neglect all sources of damping.

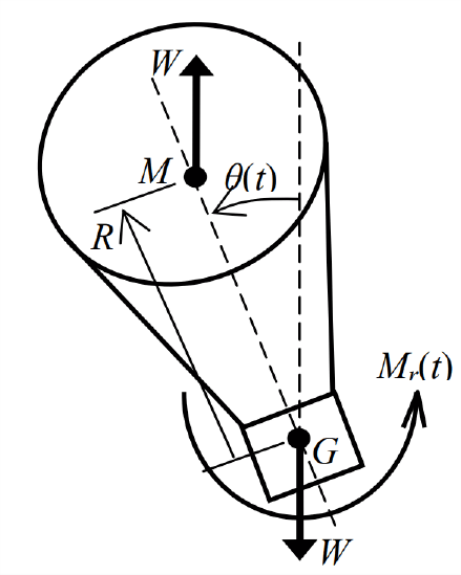


Figure 7.7.4

1. Apply Newton's 2nd law for rotation to derive the linearized 2nd order ODE for roll angle $\theta(t)$, assumed to be sufficiently small that $\sin\theta \approx \theta$ radians. This ODE should have the form of Equation 7.1.2, but with the notation appropriate for this system.
2. Convert the ODE of part 7.4.1 into the standard form Equation 7.1.5, $\ddot{\theta} + \omega_n^2\theta = \omega_n^2 u(t)$. Write specific equations (in terms of this system's parameters and notation) for rolling natural frequency ω_n and standard input variable $u(t)$. (*partial answer*: $\omega_n = \sqrt{WR/J_G}$)

5. (Adapted from Craig, 1981, Problem 5.1 on page 120) In single-engine, high-wing, general-aviation airplanes, the structures of the main landing gears are usually simple tilted cantilever beams, as illustrated in the drawing at right. When this type of airplane touches down in a landing, the beams, in combination with the flexible tires, constitute a structural spring that cushions the landing impact.

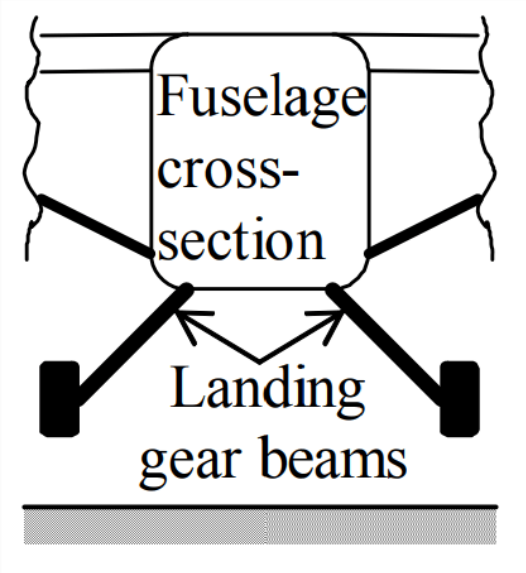


Figure 7.7.5

The drawings below show a simplified model used to study landing impact of a light airplane, with mass m being that of the airplane body, and stiffness k being that of the landing gear beams and tires. (The model also represents a person bouncing passively on a pogo stick.) The airplane's sinking speed just before touchdown is denoted as V, and g is the acceleration of gravity. Let us call the instant of tire contact $t = 0$ and measure the airplane's downward vertical motion $y(t)$ relative to its position at that instant, as indicated on the right-hand drawing. Hence, the initial conditions are $y(0) = 0$ and $\dot{y}(0) = V$.

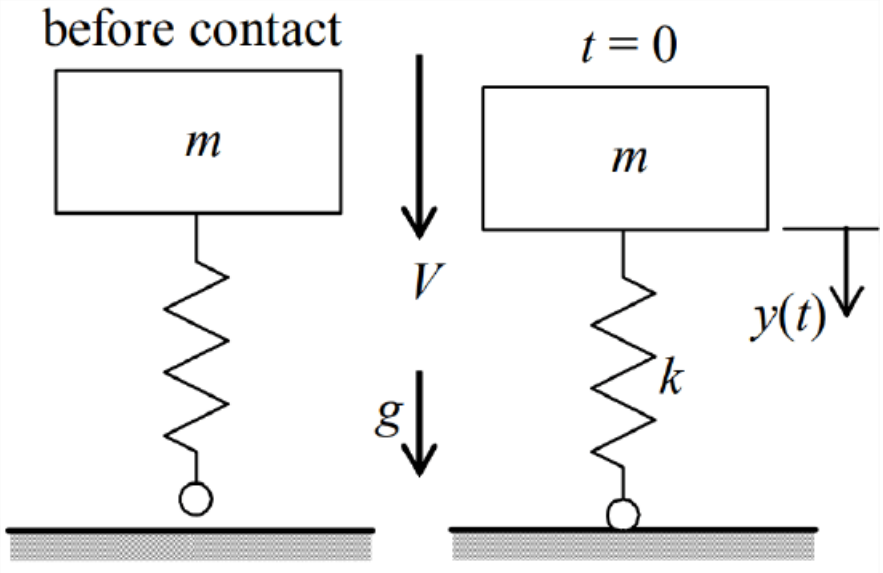


Figure 7.7.6

1. Sketch a FBD of the forces acting upon m during the time of tire contact with the ground, and use that FBD and Newton's 2nd law to write the 2nd order ODE of motion for $y(t)$. Note in this case that we measure motion relative to the spring-undeformed position, **not** relative to the static equilibrium position, so it is necessary to include the body weight $W = mg$ on the FBD.
2. Put the ODE of motion into standard form. Now combine **both** the IC solution (since $\dot{y}(0) = V > 0$) **and** the step-response solution (since W is essentially a step input) that are derived in Section 7.3 to write the algebraic equation for the response $y(t)$ during the time of tire contact. Express this equation in the form $y(t) = C_1 + C_2 \sin(\omega_n t - C_3)$, where the C_i and ω_n are positive constants that you should define in terms of the given algebraic parameters. You should find useful the trigonometric identity $\sin A \cos B \pm \cos A \sin B = \sin(A \pm B)$. If necessary, review the procedure in Section 4.3, Equations 4.3.2, 4.3.4, and 4.3.5, for combining sines and cosines.
3. Use the **correct** result from part 7.5.2 above to sketch a time history of the response $y(t)$ during the time of tire contact. To make it relatively easy to sketch, suppose that the landing is hard, with $V/\omega_n = \sqrt{3}W/k$, and show that $y(t) = (W/k)\,[1 + 2\sin(\omega_n t - 30^\circ)]$. From your $y(t)$ equation and sketch, infer general algebraic equations (not applicable only for $V/\omega_n = \sqrt{3}W/k$) for the maximum value $y_{\max}$, the time at which $y(t) = y_{\max}$, and the time at which the tires loses contact with the ground upon rebound. For what range of landing impact velocities V does this theory predict that the tires will lose ground contact upon rebound? Is this theory completely realistic?

6. The ideal LC circuit[2] drawn below is a series combination of a voltage source $e_i(t)$, an ideal inductor (having only inductance L, no resistance), and a capacitor with capacitance C. Recall that the current is the rate of change of charge $q(t)$ on the capacitor, $i = dq/dt$. Apply Kirchhoff's voltage law, as in Electricity Example 5.2.4 of Section 5.2, and show that the ODE for $q(t)$ is $L\ddot{q} + (1/C)q = e_i(t)$. Convert this ODE into the standard 2nd order form Equation 7.1.5, $\ddot{q} + \omega_n^2 q = \omega_n^2 u(t)$; write specific equations [in terms of L, C, and $e_i(t)$] for natural frequency ω_n and input quantity $u(t)$.

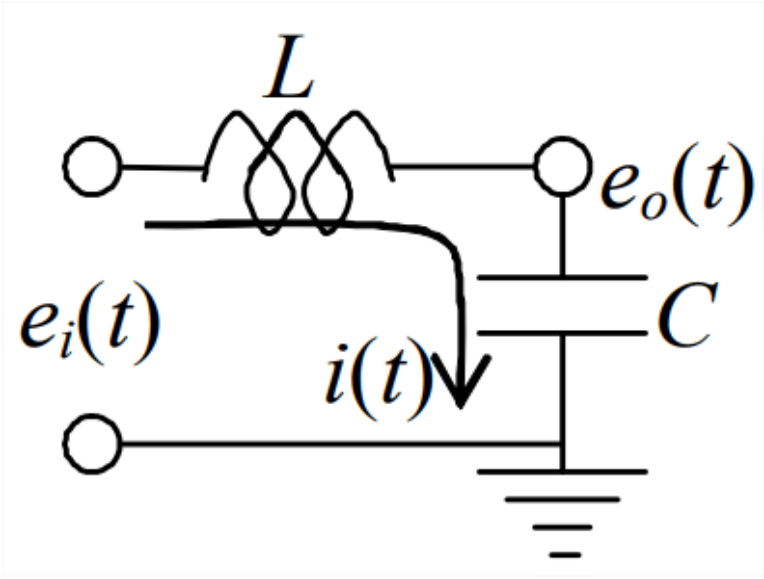


Figure 7.7.7

7. The circuit[3] drawn at right consists of three stages, each built around an op-amp, and there is *feedback* of voltage $e_f(t)$ from the last (righthand) stage to the first (lefthand) stage, where the input voltage $e_i(t)$ is applied. The circuit output voltage, $e_o(t)$, is the output of the middle stage. For the first stage, a *summing, inverting integrator*, use the methods of Chapter 5 to derive the first ODE, $e_f/R_{1f} + e_i/R_{1i} = -C_1\dot{e}_m$. Next, for the middle stage, an *inverting integrator*, derive the second ODE, $e_m/R_2 = -C_2\dot{e}_o$ (homework Problem 5.6). Now differentiate the second ODE and use the result to substitute for $\dot{e}_m$ in the first ODE. The last stage is a simple sign inverter, for which $e_f = -e_o$ from Equation 5.3.5. Substitute for e_f and show that the ODE relating output voltage $e_o(t)$ to input voltage $e_i(t)$ for the entire circuit is $\ddot{e}_o + \frac{1}{R_2C_2R_{1f}C_1}e_o = \frac{1}{R_2C_2R_{1f}C_1}\frac{R_{1f}}{R_{1i}}e_i$. Convert this ODE into the standard 2nd order form Equation 7.1.5, $\ddot{e}_o + \omega_n^2 e_o = \omega_n^2 u(t)$; write specific equations (in terms of this circuit's resistances, capacitances, and input voltage) for natural frequency ω_n and input quantity $u(t)$.

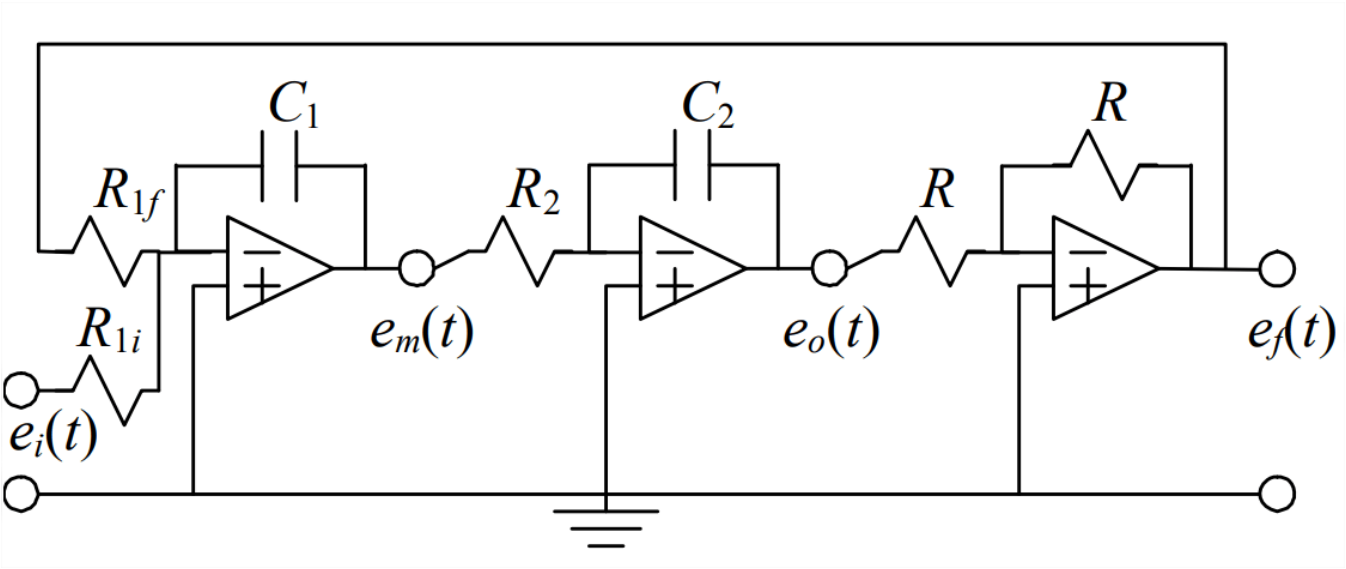


Figure 7.7.8

8. If you compare the experimental results for a beam that are presented in Example Problem 7.6.1 with the ideal theoretical results presented in Example Problem 7.6.2, you will observe that there are some significant differences between the two sets of results. In this problem, you will investigate some reasons for those differences.
   1. The beam of Figure 7.6.1 is an extruded aluminum "flat". Due to manufacturing imperfections, its cross section is not completely uniform (constant) along the length; however, students made the following careful measurements of average cross-sectional dimensions: width b = 2.00 inches, depth h = 0.120 inch. The students also measured the average weight **per unit length** of 0.0230 lb/inch. Using this information, calculate the "actual" average weight density w in lb/inch^3, and compare your value with the standard (in preliminary-design calculations) value for aluminum, 0.1000 lb/inch3 ,

which is used in Example Problem 7.6.2. Also, calculate the total weight of the beam, W_b in lb, based upon the lengthwise weight density 0.0230 lb/inch, and assuming L = 10.00 inches.

2. In a series of independent tests on aluminum flats from the same manufactured lot as the beam of Figure 7.6.1, students measured the average modulus of elasticity in bending E = 9.43e+6 lb/inch^2, which is about 6% lower than the standard (in preliminary-design calculations) value of 10.0e+6 lb/inch^2. Use this measured value, along with the measured dimensions from part 7.8.1 to calculate the effective beam-tip stiffness constant k_E =, assuming that the beam overhang length is precisely L = 10.00 inches. Also, compare this calculated value of with the measured value of 7.86 lb/inch from Example Problem 7.1 and with the ideal theoretical value of 9.766 lb/inch from Example Problem 7.6.2.[4]

   10 [NOTE: It can be shown, by evaluation of *propagation of error*[5], that a small error in the value of depth h produces **three times** that error in the calculated stiffness constant k_E (for example, a 2% error in h produces a 6% error in k_E); similarly, a small error in the value of length L produces three times that error in k_E.]

3. Use the weight W_b that you calculated in part 7.8.1 to calculate the effective tip mass of the bare beam alone from $m_{bE} = 0.242\ 672 \times W_b/g$, in which 0.242 762 is the theoretical factor introduced in Example Problem 7.6.2. Next, calculate the total effective tip mass, including the mass of the steel target that is bonded to the beam tip on Figure 7.6.1, from $m_E = m_{bE} + (0.0081\text{lb}) \div (386.1\text{inch/s}^2)$. Finally, use this m_E and the k_E that you calculated in part 7.8.2 to determine a corrected estimate (relative to that of Example Problem 7.6.2) for natural frequency f_n in Hz.

9. For the undamped (or lightly damped) mass-spring system drawn below at left, Equation 7.1.3 gives the natural frequency as $\omega_n = \sqrt{k/m}$. If we attach an additional mass m_a to the original mass, as drawn below at right, then the natural frequency of the modified system is $\omega_{an} \equiv \sqrt{k/(m + m_a)}$.

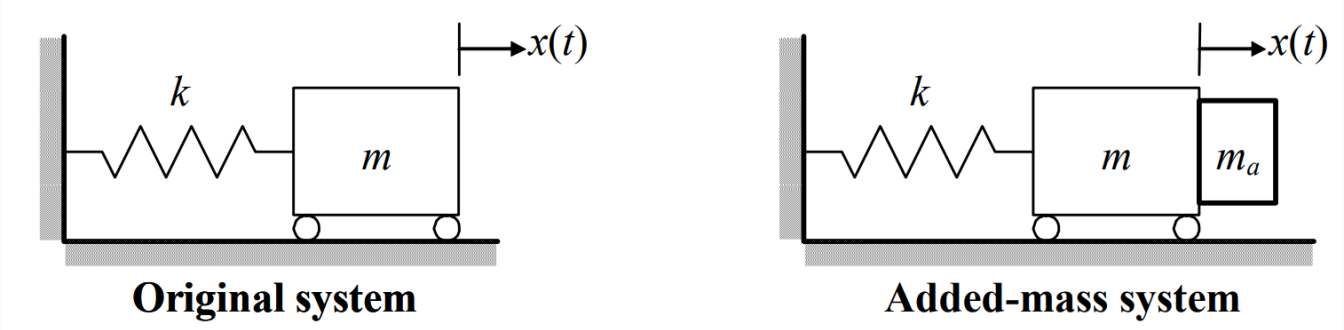


Figure 7.7.9

1. Suppose that the parameters m and k of the original system are unknown, but that we know the value of the added mass m_a, and that we are able to measure experimentally the natural frequencies ω_n and ω_{an}. Derive algebraically the following equation for the unknown mass m : $\quad m = \frac{m_a}{(\omega_n/\omega_{an})^2 - 1}$. The traditional method described here may be called the *added-mass* or *added-inertia* method.[6]

2. Apply the added-mass method of part 7.9.1 to calculate **both** the 2nd order effective tipmass quantity m_E **and** the 2nd order effective tip-stiffness constant k_E of the beam structural system shown on Figure 7.6.1. The data that you will need is stated in Example Problems 7.6.1 and 7.6.3: the measured natural frequency of the original system is f_n = 34.0 Hz; the concentrated-mass assembly that was added to the beam tip (Figure 7.6.5) weighs 0.254 lb; and the measured natural frequency of the added-mass system is f_{an} = 15.85 Hz.

10. A distributed-parameter structural system[7] is shown in the drawing; the "snapshot" of a dynamic deformation state (with exaggerated magnitude) is drawn in dashed lines. This system consists of an essentially rigid block of mass m and two parallel cantilever beams. The two flexible (in bending) beams are nominally uniform and identical to each other, with rectangular cross section of width b, depth h, and cross-sectional area $A = bh$. The beams are embedded into the mass, just as they are embedded into the foundation below. In-plane rotation of mass m is suppressed by the very high axial stiffness of the beams and the separation of the two beams. The bending slopes of the beams where they join mass m are essentially zero due to the rigidity of mass m and of the beam-to-mass joints. Provided that forcing [such as $f_x(t)$ on the drawing] and/or initial conditions are in the plane of the paper, motion of the mass is restricted to one-dimensional translation, $x(t)$. You should be able to verify from a textbook on mechanics of materials that the total effective stiffness constant of the two beams that restrain motion of mass m is $k_E \equiv (f_x/x)_{\text{static}} = 2 \times 12EI/L^3$. We can account for the contribution of the beams' mass to the approximately 2nd order dynamics of this system by using what is known in more advanced theory as the *consistent mass* of a distributed-parameter structure. We denote the weight per unit volume of the beam material as w and the mass as density $\rho \equiv w/g$. Then the total consistent mass from the two beams, which effectively moves with mass m through translation $x(t)$, is $2m_c = 2 \times (156/420)\rho AL$ (Craig, 1981, page 387); thus, the total effective mass of the structural system is $m_E = m + 2m_c$.

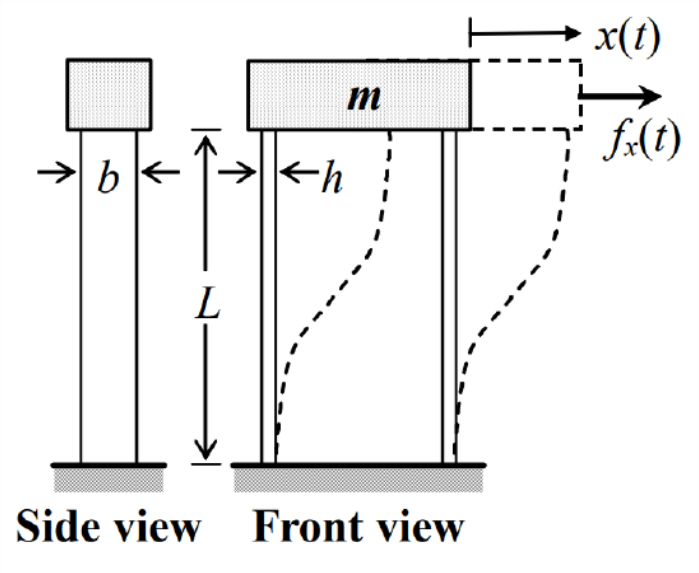


Figure 7.7.10

1. Write an algebraic equation for the natural frequency ω_n in rad/s of this structural system in terms of the system parameters E, I, L, m, w, A, and g.
2. Consider an actual fabricated version of this structural system, shown in the photograph, which is used in an instructional laboratory experiment[8]. The beams are extruded aluminum "flats", from which students made the following careful measurements of average dimensions and properties: L = 12.00 inches, b = 2.010 inches, h = 0.1259 inch; w = 0.0965 lb/inch^3; E = 9.40e+6 lb/inch^2. The relatively rigid mass m was machined from an aluminum block; the students measured its weight as 1.61 lb. Calculate the natural frequency f_n in Hz of this laboratory apparatus.

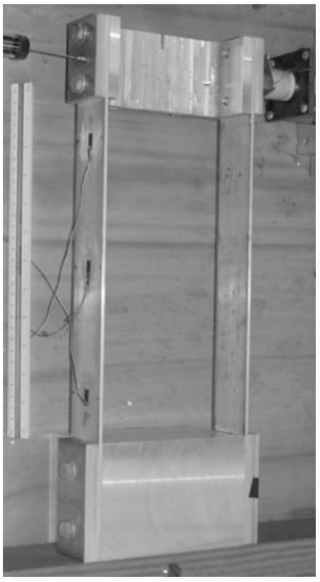


Figure 7.7.11

[1]The same basic principles of dynamics and fluid statics apply if the vehicle is a submerged submarine.

[2]The LC circuit is a simplified model for devices such as antennas and cavity resonators that transmit and receive electromagnetic energy (Halliday and Resnick, 1960, Chapters 38 and 39). Also, cascades of LC pairs are often used as passive filters in radio-frequency, 100 kHz and above, applications (Horowitz and Hill, 1980, pages 654-656).

[3]This circuit with op-amps, capacitors, and resistors is essentially the *electronic analog computer* (see the footnote to Problem 5.6) for solving ODE 7.1.5.

[4]Example Problem 7.6.3 describes how tiny gaps between the thick steel clamping plates and the aluminum beam surfaces might change the effective beam length L. It is worth observing also that another possible source of stiffness error, which is difficult to assess quantitatively, is some significant flexibility in the steel clamping plates. The theoretical equation for beam-tip stiffness is based on the assumption that the clamping medium is completely rigid.

[5]Propagation of error is discussed in textbooks on experimental instrumentation and measurements, for example, Dally, Riley, and McConnell, 1984, pages 544-545.

[6]The added-mass method is an elegantly simple technique for finding experimentally the unknown parameters of a 2nd order, or approximately 2nd order, linear mechanical system. The method requires only that we be able to measure accurately the free-vibration frequencies; otherwise, the motion sensors and other instrumentation can be uncalibrated. Use of the added-mass method is a simple example of *system identification*, which is discussed at greater length in Section 9.9.

[7]This structure is a rudimentary form of one-story *shear building* that is used for studying the structural dynamics of buildings (Craig, 1981, pages 42, 265, 346, etc.; Clough and Penzien, 1974, pages 226-227). The rigid mass represents a floor slab/girder, and the flexible beams represent structural columns.

[8]The principal plane of the actual laboratory apparatus is horizontal relative to gravity. The photograph is rotated 90° to make the plane appear vertical, in order to match the drawing on the previous page.

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8: PULSE INPUTS; DIRAC DELTA FUNCTION; IMPULSE RESPONSE; INITIAL VALUE THEOREM; CONVOLUTION SUM

We analyzed previously the response of systems to a half-sine pulse input (*e.g.*, in Sections 1.5 and 1.6). In this chapter, we will continue analysis of pulse excitation and subsequent dynamic response. Most importantly, we will extend the analysis to the concept of the *impulse*, which in its simplest form is $u \times t_{d_d}$, the product of excitation and the duration of excitation. In particular, the *Dirac delta* or *ideal unit-impulse* function is essential in the theory of linear systems and also useful in practical applications.

8.1: Flat Pulse

8.2: Impulse-Momentum Theorem for a Mass Particle Translating in One Direction

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8.1: Flat Pulse

This is probably the simplest form of limited-duration input. We define the *flat pulse* initially in the form of a general (standard) input quantity $u(t)$ that can be used with standard forms of system ODEs such as $\dot{x} + (1/\tau_1)\, x = bu(t)$ for a stable 1st order system and $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 u(t)$ for 2nd order systems. Figure 8.1.1 illustrates a flat pulse of duration t_d. It can be described mathematically with use of two unit-step functions, as defined in Section 2.4:

$$u(t) = U\left[H(t) - H\left(t - t_d\right)\right] \tag{8.1.1}$$

The Laplace transform of Equation 8.1.1, from Equations 2.4.4 and 2.4.5, is

$$L[u(t)] = U\left(\frac{1}{s} - \frac{e^{-st_d}}{s}\right) = \frac{U}{s}\left(1 - e^{-st_d}\right) \tag{8.1.2}$$

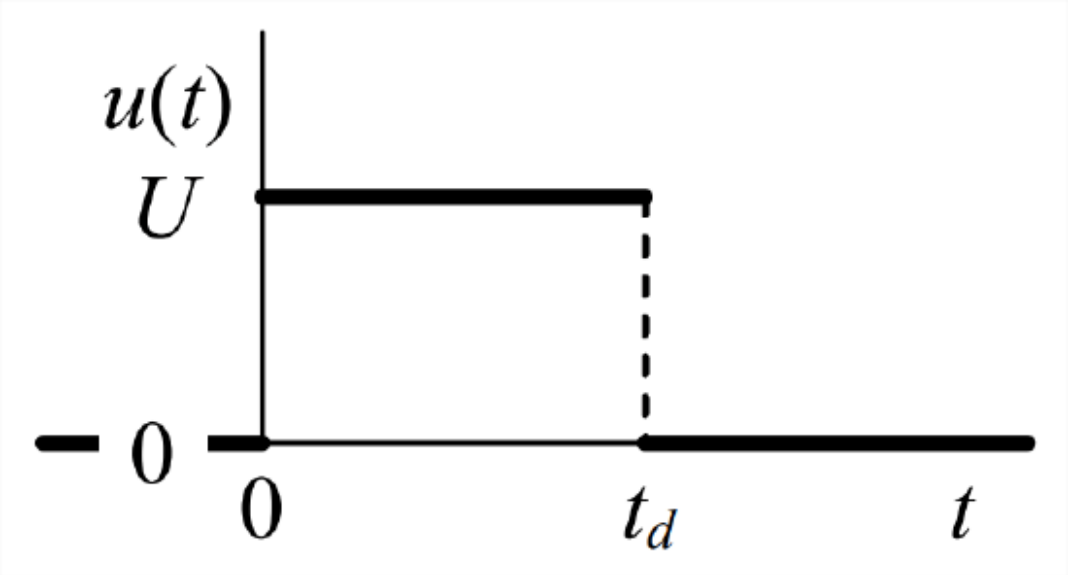


Figure 8.1.1: Flat pulse

Example 8.1.1: Response to a Flat Pulse of an Undamped 2nd Order System[1]

The ODE for an undamped 2nd order system is $\ddot{x} + \omega_n^2 x = \omega_n^2 u(t)$. Let the initial conditions be zero: $x_0 = 0$, $\dot{x}_0 = 0$. The general solution then is either of the two convolution integral forms, Equations 7.2.4 and 7.2.5. Of those two, the easier one to interpret for the discontinuous flat pulse is Equation 7.2.5, which with zero ICs becomes

$$x(t) = \omega_n \int_{\tau=0}^{\tau=t} \sin\omega_n(t-\tau) \times u(\tau)d\tau \tag{8.1.3}$$

Rather than use Equation 8.1.1 explicitly, we can observe the form of Figure 8.1.1 and write Equation 8.1.3 for two different time intervals:

$$x(t) = \begin{cases} \omega_n \int_{\tau=0}^{\tau=t} \sin\omega_n(t-\tau) \times U d\tau, & 0 \le t \le t_d \\ \omega_n \int_{\tau=0}^{\tau=t_d} \sin\omega_n(t-\tau) \times U d\tau + \omega_n \int_{\tau=t_d}^{\tau=t} \sin\omega_n(t-\tau) \times 0 d\tau, & t_d \le t \end{cases} \tag{8.1.4}$$

The second integral for $t_d \le t$ is obviously zero. This leaves us with a single integral for each time interval, each integral having the same integrand and lower limit. However, the integrals have different upper limits, so let us denote a general upper limit as T, and evaluate that common integral with use of the change of integration variable $\lambda = t - \tau \Rightarrow d\lambda = -d\tau$:

$$\int_{\tau=0}^{\tau=T} \sin\omega_n(t-\tau)d\tau = \int_{\lambda=t}^{\lambda=t-T} \sin\omega_n\lambda(-d\lambda) = \frac{1}{\omega_n}\int_{\lambda=t}^{\lambda=t-T} d\left(\cos\omega_n\lambda\right) = \frac{1}{\omega_n}\left[\cos\omega_n(t-T) - \cos\omega_n t\right]$$

Upper limit $T \equiv t$ for of Equation 8.1.4, and $T \equiv t_d$ for $t_d \leq t$, so substituting the integration result into Equation 8.1.4 gives the final equations,

$$x(t) = \begin{cases} U\left(1-\cos\omega_n t\right), & 0 \leq t \leq t_d \\ U\left[\cos\omega_n\left(t-t_d\right) - \cos\omega_n t\right], & t_d \leq t \end{cases} \tag{8.1.5}$$

Equation 8.1.5 for $0 \leq t \leq t_d$ is just the simple step response of Equation 7.3.8, as it logically should be. Inspection of Equation 8.1.5 shows that the response $x(t)$ is continuous at $t = t_d$; by differentiating both response equations, we could show that the velocity term $\dot{x}(t)$ also is continuous at $t = t_d$. Indeed, these response quantities should be continuous, because Equation 8.1.5 describes a physical response (*e.g.*, of a mass-spring system). In fact, we could solve this problem differently from the beginning by using the requirement of continuity along with step-response Equation 7.3.8 and IC response Equation 7.3.4. Step response Equation 7.3.8 is valid from $t = 0$ to $t = t_d$; then we could find $x\left(t_d\right)$ and $\dot{x}\left(t_d\right)$ from the step response and use those values as initial conditions for the IC solution to be valid for $t \geq t_d$.

[1]See also homework Problem 8.3 for a simple yet elegant direct Laplace-transform solution of this problem and a physically meaningful response equation.

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8.2: Impulse-Momentum Theorem for a Mass Particle Translating in One Direction

An unrestrained mass particle subjected to force acting only in the x direction is shown in the drawing at right. The equation of motion is

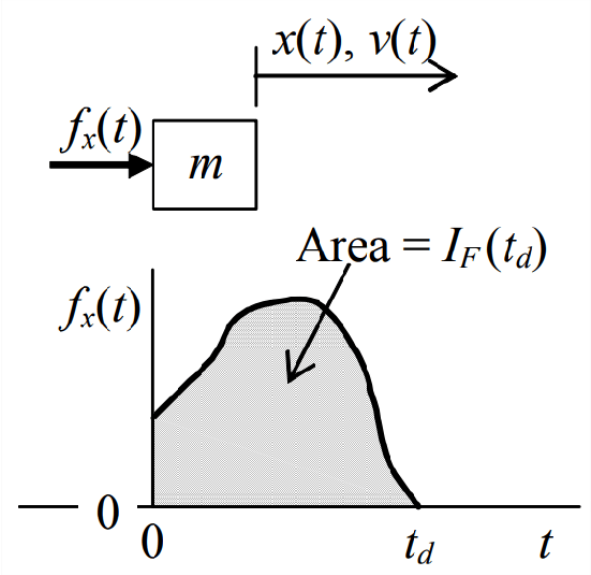


Figure 8.2.1

$$m\ddot{x} = f_x(t) \Rightarrow m\dot{v} = f_x(t)$$

Integrating the equation of motion from time zero to any time $t > 0$ gives

$$m\int_{\tau=0}^{\tau=t} \frac{dv}{d\tau} d\tau = mv(t) - mv(0) = \int_{\tau=0}^{\tau=t} f_x(\tau) d\tau \equiv I_F(t) \tag{8.2.1}$$

in which $I_F(t)$ is the area under the force time history and is called the *impulse* of the force, with dimensions of force $\times$ time. Equation \(8.2.1), the *impulse-momentum theorem* for a mass particle, states that the change of momentum equals the impulse. For a pulse, a force of limited duration as in the drawing on the previous page, we are generally interested in the total impulse $I_F\,(t_d)$, which remains unchanged for $t > t_d$.

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8.3: Flat Impulse

The *flat* ***impulse***, Figure 8.3.1, is defined almost identically with the flat **pulse**, the only difference being that we **specify** the magnitude of the generalized impulse, I_U, which is the area under the time history of the flat pulse, then we **define** the magnitude of the input function in terms of the impulse and duration, $U = I_U/t_d$. Thus, for the flat impulse, from Equation 8.1.1,

$$u(t) = \frac{I_U}{t_d}\left[H(t) - H\left(t - t_d\right)\right] \tag{8.3.1}$$

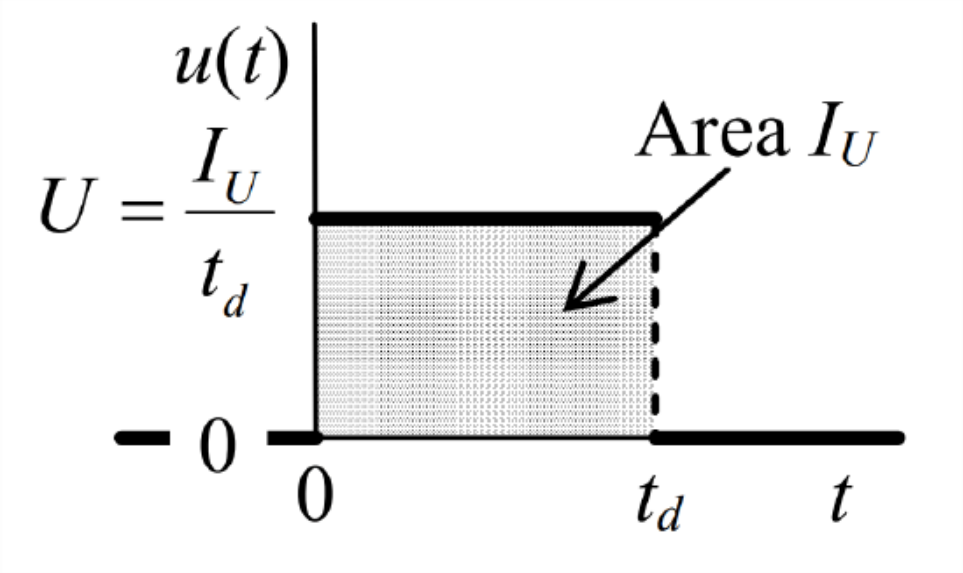


Figure 8.3.1: Flat impulse

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8.4: Dirac Delta Function

Consider a limit process with the flat impulse of Fig. 8.3.1, in which process we progressively shorten the pulse duration while maintaining constant the impulse magnitude (the shaded area), thereby progressively increasing the input magnitude, I_U/t_d. If we carry the process to the limit as $t_d \to 0$ while maintaining I_U constant, then magnitude $I_U/t_d \to \infty$. The limit process is illustrated on Figure 8.4.1. The function that results is called an *ideal impulse* with magnitude I_U, and it is denoted as $u(t) = I_U \times \delta(t)$, in which $\delta(t)$ is called the *Dirac delta function* (after English mathematical physicist Paul Dirac, 1902-1984) or the *unit-impulse function*. The ideal impulse function $I_U\delta(t)$ is usually depicted graphically by a thick picket at t = 0, as on Figure 8.4.1. With $I_U = 1$ in Equation 8.3.1, a formal mathematical definition of the unit-impulse function is

$$\delta(t) = \lim_{t_d \to 0} \frac{1}{t_d} \left[H(t) - H\left(t - t_d\right)\right] \tag{8.4.1}$$

Observe from Equation 8.4.1 that the dimension of $\delta(t)$ is time^{-1}, since the unit-step is dimensionless, so the typical unit of $\delta(t)$ is s^{-1}.

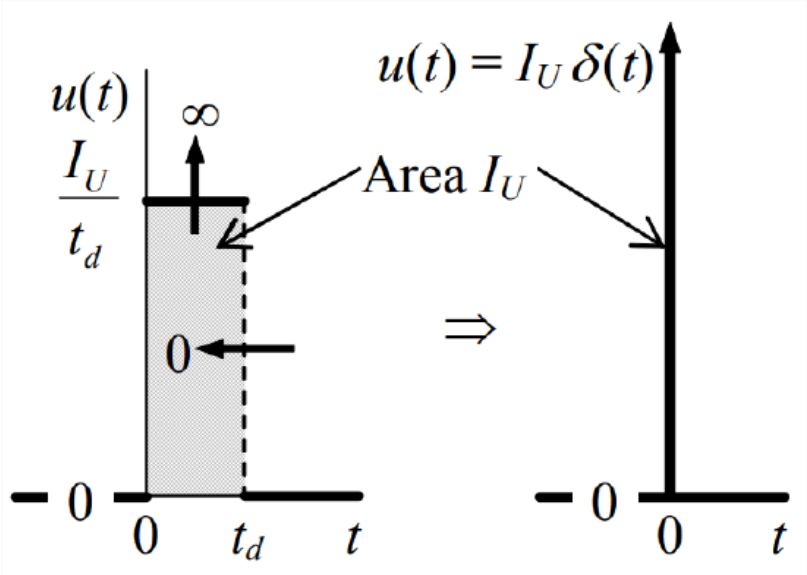


Figure 8.4.1: Ideal impulse

By definition, the area under $u(t) = I_U\delta(t)$ equals I_U, so the area under $\delta(t)$ equals 1 (hence the name **unit**-impulse function). This unit area under an infinitely short impulse suggests an important effect of the Dirac delta function in integrands. In order to describe this effect, we define a more general unit-impulse function, $\delta\left(t - t_\delta\right)$, which peaks infinitely at some arbitrary time t_δ that is not necessarily zero; otherwise, the nature of $\delta\left(t - t_\delta\right)$ is identical to that of $\delta(t)$ [in fact, $\delta(t) = \delta\left(t - t_\delta\right)$ for t_δ = 0]. Now suppose that we have some realistic physical function $f(t)$ that is defined over the time interval $t_1 \to t_2$, and suppose that time t_δ is within this interval, as in the figure below. Then the useful integration effect of $\delta\left(t - t_\delta\right)$ is:

$$\int_{t=t_1}^{t=t_2} f(t)\delta\left(t - t_\delta\right) dt = f\left(t_\delta\right) \tag{8.4.2}$$

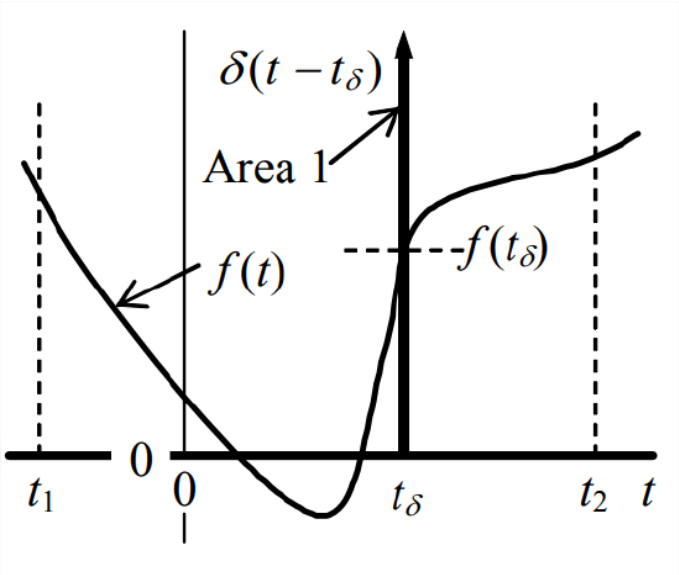


Figure 8.4.2

Equation 8.4.2 states that, as a multiplier within the integrand, $\delta\left(t-t_\delta\right)$ essentially "selects" the value of $f(t)$ at $t=t_\delta$. Result Equation 8.4.2 is intuitively clear, and it can be proved more rigorously with the theory of so-called *generalized functions* (Lighthill, 1958).

If the upper limit of integration in Equation 8.4.2 is an arbitrary time $t>t_\delta$, then the result is a step function:

$$\int_{\tau=t_1}^{\tau=t>t_\delta} f(\tau)\delta\left(\tau-t_\delta\right)d\tau = f\left(t_\delta\right)H\left(t-t_\delta\right) \tag{8.4.3}$$

Setting $f(\tau)=1$ in Equation 8.4.3 leads to a fundamental integral relationship between the unit-impulse function and the unit-step function:

$$\int_{\tau=t_1}^{\tau=t>t_\delta} \delta\left(\tau-t_\delta\right)d\tau = H\left(t-t_\delta\right) \tag{8.4.4}$$

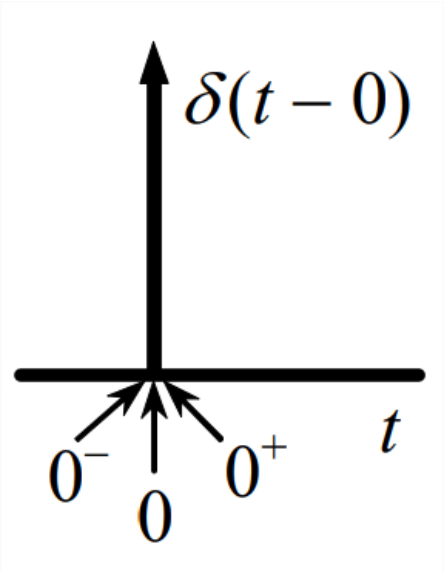


Figure 8.4.3

Next, we use Equation 8.4.2 to derive the Laplace transform of delta function $\delta(t)=\delta(t-0)$. According to the definition of Equation 2.2.5, this transform would be written as $L[\delta(t)]=\int_{t=0}^{t=\infty} e^{-st}\delta(t-0)dt$; however, there is a problem with this particular definition: at

t = 0, which is the lower limit of the integrand (and the initial-value instant for most ODEs that we solve using Laplace transforms), the function $\delta(t-0)$ is nominally infinite, so the meaning of the integral is uncertain. We choose to remove the uncertainty by specifying that $\delta(t-0)$ must lie within the limits of the integration. In order to indicate this clearly in notation, we now define three different reference instants, as depicted on the drawing at right:

1. $t = 0^-$, the instant **just before** activity of the ideal impulse function, at and before which $\delta(t-0) = 0$;
2. $t = 0$, the instant when $\delta(t-0)$ acts and is nominally infinite; and
3. $t = 0^+$, the instant **just after** activity of the ideal impulse function, at and after which $\delta(t-0) = 0$.

Accordingly, we re-define the Laplace transform, more generally than in Equation 2.2.5, as

$$L[f(t)] = \int_{t=0^-}^{t=\infty} e^{-st} f(t)dt \tag{8.4.5}$$

The distinction between $t = 0$ and $t = 0^-$ in the lower limits of Equations 2.2.5 and 8.4.5, respectively, is meaningless for all problems considered in this book except those in which $f(t)$ involves an ideal impulse function. Application of Equations 8.4.2 and 8.4.5 now leads to the Laplace transform of the basic ideal impulse function:

$$L[\delta(t)] = \int_{t=0^-}^{t=\infty} e^{-st}\delta(t-0)dt = e^{-s\times 0} = 1 \tag{8.4.6}$$

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8.5: Ideal Impulse Response of a Standard Stable First Order System

From Equation 3.4.8, the problem statement for a standard stable 1st order system is

$$\text{ODE} + \text{IC}: \quad \dot{x} + (1/\tau_1)\, x = bu(t), \quad x(0) = x_0, \quad \text{find } x(t) \text{ for } t > 0 \tag{8.5.1}$$

Let the input function be the ideal impulse, $u(t) = I_U \delta(t)$. Although there are several methods for finding response to an ideal impulse, the conventional Laplace-transform approach is relatively simple and probably the most instructive, so we will use this method. With $L[\delta(t)] = 1$ from Equation 8.4.6, the steps of the solution are:

$$L[\text{ODE}]: \quad sX(s) - x_0 + (1/\tau_1)\, X(s) = bU(s) = bI_U \times 1$$

$$\Rightarrow \quad X(s) = \frac{x_0 + bI_U}{s + 1/\tau_1} \tag{8.5.2}$$

$$\Rightarrow \quad x(t) = (x_0 + bI_U)\, e^{-t/\tau} \tag{8.5.3}$$

As a check, we evaluate the solution at t = 0:

$$x(0) = (x_0 + bI_U)\, e^{-0/\tau_1} = x_0 + bI_U \neq x_0 \tag{8.5.4}$$

It appears that something is wrong with response solution Equation 8.5.3, because Equation 8.5.4 for $x(0)$ contradicts the original IC, $x(0) = x_0$. In order to explain this discrepancy, we must account mo e carefully for the nature of response to an ideal impulse, and for the distinction between the three different reference instants that are defined in the previous section:

1. $t = 0^-$, the instant **just before** activity of the ideal impulse function;
2. $t = 0$, the instant when $\delta(t - 0)$ acts; and
3. $t = 0^+$, the instant **just after** activity of the ideal impulse function.

The more detailed analysis follows.

We begin by integrating the basic ODE, $\dot{x} + (1/\tau_1)\, x = bI_U \delta(t)$, across the ideal impulse function, just from $t = 0^-$ to $t = 0^+$:

$$\int_{\tau=0^-}^{\tau=0^+} \frac{dx}{d\tau} d\tau + \frac{1}{\tau_1} \int_{\tau=0^-}^{\tau=0^+} x d\tau = b \int_{\tau=0^-}^{\tau=0^+} I_U \delta(\tau - 0) d\tau \tag{8.5.5}$$

The fi left-hand-side term of Equation 8.5.5 is identically equal to $x\,(0^+) - x\,(0^-)$; this result introduces the new quantity $x\,(0^+)$, which apparently is the *post-impulse initial value* at $t = 0^+$; the *pre-impulse initial value*, $x\,(0^-) \equiv x_0$, is the original IC specified in Equation 8.5.1, but now understood to exist at $t = 0^-$. The right-hand-side integral of Equation 8.5.5 gives the **finite** area I_U under the infinite ideal impulse function. In contrast, the second left-hand-side term of Equation 8.5.5 has a **finite** integrand; we can use the trapezoid rule to approximate the value of the integral and find that it is **zero:**

$$\int_{\tau=0^-}^{\tau=0^+} x d\tau \approx \lim_{\Delta t \to 0} \left\{ \frac{1}{2} \left[x\,(0^+) + x\,(0^-) \right] \Delta t \right\} = 0 \tag{8.5.6}$$

In Equation 8.5.6, $\Delta t \equiv 0^+ - 0^- \to 0$, in the spirit of the idealization that the ideal impulse function acts over an infinitesimal time interval. Therefore, Equation 8.5.5 gives $x\,(0^+) - x_0 + 1/\tau_1 \times 0 = b \times I_U$, or

$$x\,(0^+) = x_0 + bI_U \tag{8.5.7}$$

Comparing Equations 8.5.4 and 8.5.7 shows that the term $x(0)$ in the former equation really should be the **post**-impulse initial value, $x\,(0^+)$.

Equation 8.5.7 can also be found directly from the Laplace transform $X(s)$ by application of the *initial-value theorem*:

$$\lim_{t \to 0^+ \text{from } t>0} f(t) \equiv f\,(0^+) = \lim_{s \to \infty} [sF(s)] \tag{8.5.8}$$

We present only the relevant application of the initial-value theorem here; derivation of the theorem itself is postponed to Section 8.6. Applying Equation 8.5.8 to Laplace transform solution Equation 8.5.2 gives

$$x\left(0^{+}\right)=\lim_{s\to\infty}[sX(s)]=\lim_{s\to\infty}\left(s\times\frac{x_0+bI_U}{s+1/\tau_1}\right)=\lim_{s\to\infty}\left(\frac{s}{s}\times\left(x_0+bI_U\right)\right)=x_0+bI_U$$

This consequence of the initial-value theorem is identical to Equation 8.5.7, which was derived by a different method.

Finally, we present an application of Equation 8.5.7 to a mechanical system, and we show that the results conforms with a principle of mechanics. Consider a mass-damper system with mass m and viscous damping constant c, as drawn at right. The mass is initially, moving with velocity $v\left(0^{-}\right)=v_0$ when, at $t=0^{-}$, the mass is disturbed by an ideal force impulse, $f_x(t)=I_F\delta(t)$. The equation of motion is $m\dot{v}+cv=f_x(t)$, or in terms of a standard stable 1st order system, $\dot{v}+\left(1/\tau_1\right)v=bu(t)$, in which $\tau_1=m/c$, $b=1/m$, and $u(t)=f_x(t)=I_F\delta(t)$, so that $I_U=I_F$. The initial (pre-impulse) momentum of the mass is mv_0. According to Equation 8.5.7, the post-impulse velocity of the mass is $v\left(0^{+}\right)=v_0+(1/m)\times I_F=v_0+I_F/m$, so that the post-impulse momentum of the mass is $mv\left(0^{+}\right)=mv_0+I_F$. In words, the momentum of the mass is increased exactly by the magnitude of the ideal impulse, in agreement with the impulse-momentum theorem, Equation 8.2.2. Force impulse $I_F\delta(t)$ is a mathematically ideal impulse, not a physically realizable excitation, so the mathematical change in momentum occurs instantly, and the viscous dashpot has no influence on this instantaneous response. From Equation 8.5.3, the post-impulse response of the mass is $v(t)=\left(v_0+I_F/m\right)e^{-t/\tau_1}$, $t>0$.

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8.6: Derivation of the Initial-Value Theorem

Consider a physical function $f(t)$, with derivative df/dt, and with Laplace transform $L[f(t)] = F(s)$. The initial-value theorem is:

$$\lim_{t\to 0^+ \text{from } t>0} f(t) \equiv f\left(0^+\right) = \lim_{s\to\infty}[sF(s)] \tag{8.6.1}$$

In general, Equation 8.6.1 gives the initial value $f\left(0^+\right)$ of a time function $f(t)$ based only on the Laplace transform $L[f(t)] = F(s)$, without requiring that the equation for $f(t)$ be available. If $f(t)$ is dynamic response to excitation that involves the ideal unit-impulse function $\delta(t-0)$, then $f\left(0^+\right)$ is the post-impulse initial value, as defined in Section 8.5; otherwise, $f\left(0^+\right) \equiv f(0) \equiv f\left(0^-\right)$, which is the standard initial value known to exist before excitation occurs.

Our derivation of the initial-value theorem (from a more detailed proof in Cannon, 1967, p. 569) is based upon the form of Laplace transform that can accommodate the ideal impulse function $\delta(t-0)$:

$$L[f(t)] = \int_{t=0^-}^{t=\infty} e^{-st} f(t)dt \tag{8.6.2}$$

First, we need the following Laplace transform of a derivative, the transform that is associated with definition Equation 8.6.2:

$$L\left[\frac{d}{dt}f(t)\right] = \int_{t=0^-}^{t=\infty} e^{-st}\frac{df}{dt}dt = sF(s) - f\left(0^-\right) \tag{8.6.3}$$

The derivation of Equation 8.6.3 using integration by parts is almost identical to the derivation shown in Equation 2.2.8, the only difference being the lower limit of the integral at $t = 0^-$ instead of $t = 0$. We do not need the corresponding formula for higher-order derivatives right now, but it is appropriate here to state that the initial conditions at $t = 0$ in general formula Equation 2.2.10 may similarly be replaced by values at $t = 0^-$1:

$$L\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f\left(0^-\right) - s^{n-2}\dot{f}\left(0^-\right) - \cdots - \overset{(n-1)}{f}\left(0^-\right) \tag{8.6.4}$$

Next, taking the limit of all terms in Equation 8.6.3 as $s \to \infty$ gives

$$\lim_{s\to\infty} L\left[\frac{df}{dt}\right] = \lim_{s\to\infty}\left(\int_{t=0^-}^{t=0^+} 1\times\frac{df}{dt}dt + \int_{t=0^+}^{t=\infty} e^{-st}\frac{df}{dt}dt\right) = \lim_{s\to\infty}[sF(s)] - f\left(0^-\right) \tag{8.6.5}$$

In Equation 8.6.5, we separate the definite integral into two parts:

1. a part over the interval from $t = 0^-$ to $t = 0^+$, during which we set $e^{-st} = 1$ (and during which an ideal impulse including $\delta(t-0)$ could be acting); and
2. a part over the interval from $t = 0^+$ to $t = \infty$. The integrand of the second part includes e^{-st}, and since $s \to \infty$, we set this integral to zero: $\lim_{s\to\infty}\int_{t=0^+}^{t=\infty} e^{-st}(df/dt)dt = 0$. Furthermore, the first integral, which is now independent of s, is evaluated indentically as $\int_{t=0^-}^{t=0^+}(df/dt)dt = f\left(0^+\right) - f\left(0^-\right)$. Therefore, Equation 8.6.5 becomes

$$\lim_{s\to\infty} L\left[\frac{df}{dt}\right] = f\left(0^+\right) - f\left(0^-\right) = \lim_{s\to\infty}[sF(s)] - f\left(0^-\right) \tag{8.6.6}$$

$$\Rightarrow \quad f\left(0^+\right) = \lim_{s\to\infty}[sF(s)]$$

This is the version of initial-value theorem that was applied in Section 8.5 to re-derive result Equation 8.5.8.

If $f(t)$ is dynamic response to excitation that does **not** involves an ideal unit-impulse function $\delta(t-0)$, then there is no discontinuous jump at $t = 0$, *i.e.* $f\left(0^+\right) - f\left(0^-\right) = 0$. For this case, therefore, Equation 8.6.6 gives the more common (but less general) version of initial-value theorem:

$$f\left(0^-\right) = \lim_{s\to\infty}[sF(s)]$$

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8.7: Ideal Impulse Response of an Undamped Second Order System

The ODE for a standard undamped 2nd order system is $\ddot{x} + \omega_n^2 x = \omega_n^2 u(t)$. Let the input function be the ideal impulse, $u(t) = I_U \delta(t)$, and let the pre-impulse initial conditions be $x(0^-) \equiv x_0$, $\dot{x}(0^-) \equiv \dot{x}_0$. Just as for the 1st order system with ideal-impulse excitation in Section 8.5, we will solve the present problem by the conventional Laplace-transform approach. Note that we will use the second derivative form of Equation 8.6.4: $L[\ddot{f}] = s^2 F(s) - sf(0^-) - \dot{f}(0^-)$. With $L[\delta(t)] = 1$ from Equation 8.4.6, the steps of the solution are:

$$L[\text{ODE}]: \quad s^2 X(s) - sx_0 - \dot{x}_0 + \omega_n^2 X(s) = bU(s) = \omega_n^2 I_U \times 1$$

$$\Rightarrow \quad X(s) = \frac{\omega_n^2 I_U + \dot{x}_0 + sx_0}{s^2 + \omega_n^2} = \left(\omega_n I_U + \frac{\dot{x}_0}{\omega_n}\right) \frac{\omega_n}{s^2 + \omega_n^2} + x_0 \frac{s}{s^2 + \omega_n^2} \tag{8.7.1}$$

$$\Rightarrow \quad x(t) = \left(\omega_n I_U + \frac{\dot{x}_0}{\omega_n}\right) \sin \omega_n t + x_0 \cos \omega_n t, t \geq 0^+ \tag{8.7.2}$$

At the risk of being excessively repetitive, we emphasize that the discussions in Sections 8.5 and 8.6 show that response Equation 8.7.2 is the **post-impulse** solution, certainly valid mathematically for $t \geq 0^+$, but not necessarily correct, in particular, for the pre-impulse initial conditions at $t = 0^-$.

Solution Equation 8.7.2 holds for a general undamped 2nd order system; let us adapt it specifically to a mass-spring system, with zero initial conditions, that is hit by an ideal impulse of force having magnitude I_F, with units, for example, of N-s or lb-s. From Equation 7.1.4, we find

$$u(t) = f_x(t)/k \Rightarrow I_U \delta(t) = I_F \delta(t)/k \Rightarrow I_U = I_F/k \tag{8.7.3}$$

So, for translational motion $x(t)$ of the mass, Equation 8.7.2 with zero initial conditions becomes (recall $k = m\omega_n^2$)

$$x(t) = \omega_n \frac{I_F}{k} \sin \omega_n t = \frac{I_F}{m\omega_n} \sin \omega_n t \tag{8.7.4}$$

Figure 8.7.1 shows a few cycles of response Equation 8.7.4.

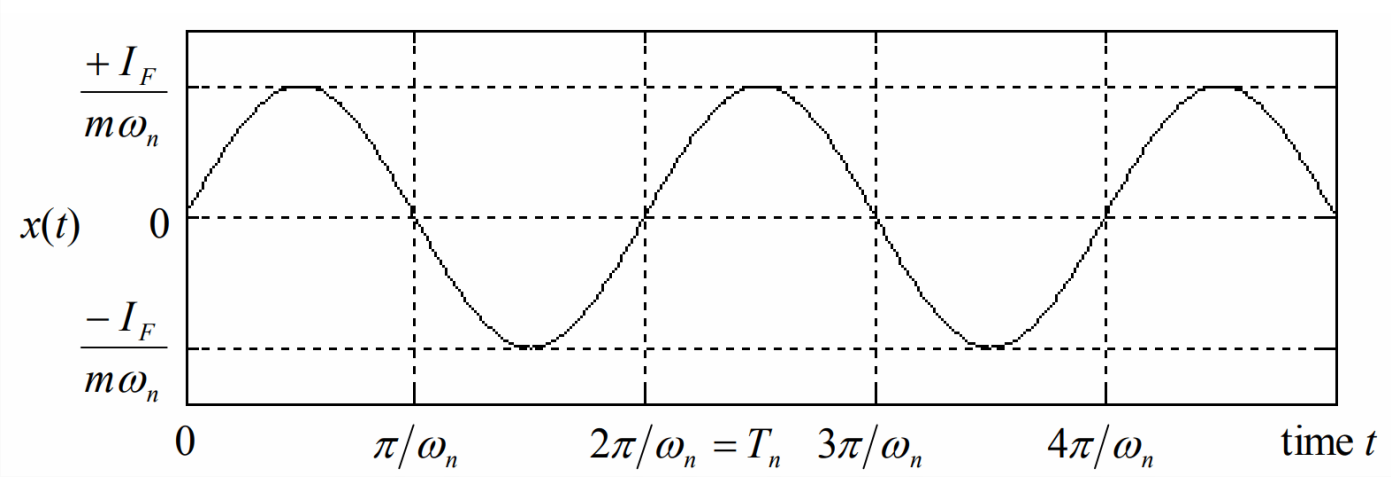


Figure 8.7.1: Ideal impulse response of a mass-spring system

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8.8: Ideal Impulse Response Versus Real Response of Systems

Section 8.5 shows that, for a 1st order system subjected to excitation by an ideal impulse function, the post-impulse initial value at $t = 0^+$ of output $x(t)$ differs from the pre-impulse initial value at $t = 0^-$. Let us now investigate the initial values in $x(t)$ and $\dot{x}(t)$ that are associated with the 2nd order system of Section 8.7. The output itself is given by Equation 8.7.3: $x(t) = (\omega_n I_U + \dot{x}_0/\omega_n) \sin \omega_n t + x_0 \cos \omega_n t, t \geq 0^+$, $t \geq 0^+$. Evaluating this equation at $t = 0^+$ shows that the post-impulse value is $x\left(0^+\right) = x_0$; this is exactly the same value as the given initial value at $t = 0^-$, so there is no discontinuous change in the initial value of $x(t)$ itself for the 2nd order system. Differentiating Equation 8.7.3 gives the "velocity" function:

$$\dot{x}(t) = \left(\omega_n^2 I_U + \dot{x}_0\right) \cos \omega_n t - \omega_n x_0 \sin \omega_n t, t \geq 0^+ \tag{8.8.1}$$

Evaluating Equation 8.8.1 at $t = 0^+$ gives

$$\dot{x}\left(0^+\right) = \omega_n^2 I_U + \dot{x}_0 \neq \dot{x}_0 = \dot{x}\left(0^-\right) \tag{8.8.2}$$

Thus, for the 2nd order system, there is a discontinuous change in the initial value of the "velocity" function.

The discontinuous changes that we observe in initial values for both 1st and 2nd order systems violate physical laws governing real systems, so ideal impulse response solutions appear to be defective and perhaps not applicable to real systems. The reason for this defectiveness is the ideal, not real, nature of the Dirac delta function $\delta(t)$. In physical reality, there is no such thing as a pulse of infinitely short duration and infinitely great magnitude; so $\delta(t)$ and $I_U \delta(t)$ are mathematical functions that do not represent exactly any real physical quantities.

However, the ideal impulse responses that we find can still be useful in applications to real systems, because the ideal impulse function $I_U \delta(t)$ can **approximate** the effect of a real, time-limited pulse **that has the same impulse magnitude**, I_U; therefore, the ideal impulse response can approximate the actual physical response. Why would we use the approximate ideal impulse response rather than just determining the more precise response solution based upon the actual pulse? We would do this because it is always much easier to derive and compute the ideal impulse response than the actual pulse response; moreover, the equation for the ideal impulse response is always simpler and more amenable to practical purposes (such as *system identification*, a form of which is discussed in Section 9.9) than the equation(s) for the actual pulse response.

When is ideal impulse response a good approximation of real pulse response, and when is it not? It depends on the duration t_d of the actual input pulse relative to the shortest *characteristic time* T_c of the system that is being analyzed. The characteristic time is defined loosely as the time interval required for system response to change substantially. The system characteristic times that we know from previous chapters are *time constant* τ_1 for a 1st order system and about one quarter of a *natural period* T_n for a 2nd order system. If t_d is **brief** relative to the system characteristic time, \(t_{d}<st order system, Equation 8.7.3 for a standard undamped 2nd order system, other equations for other types of systems] will be a good approximation to the actual pulse response; on the other hand, if t_d is on the order of or greater than the system characteristic time, $\mathrm{O}\left(t_d\right) \geq \mathrm{O}\left(T_c\right)$, then the ideal impulse response would be a poor approximation, and we should solve for the response using the actual physical pulse. There is no hard and fast rule as to how small the ratio t_d/T_c should be to justify use of the ideal impulse response solution.

In what manner would ideal impulse response differ from the actual pulse response? Let us consider, for example, an undamped 2nd order system with zero ICs. Suppose that the duration of the actual pulse satisfies $t_d << 1/4T_n$, so that, according to the discussion above, the ideal impulse response should be a good approximation to the actual response. But we already know from Equation 8.8.2 that the initial velocity of the ideal impulse response will be wrong, with value $\omega_n^2 I_U$ when it ought to be zero. How will the remainder of the ideal impulse response compare with the actual response? Figure 8.8.1 depicts conceptually the nature of both ideal impulse response and the comparable real response for a 2nd order system during the initial portion of response. The real response shows the correct initial zero slope, $\dot{x}_0 = 0$. For a brief period after $t = 0$, the slope of real response increases rapidly until both the response x and the slope $\dot{x}$ essentially match the comparable values of the ideal response, but with a short time lag relative to the ideal response. In other words, the ideal response has a short, artificial time lead relative to the real response. After the pulse ends, \(t_{d}

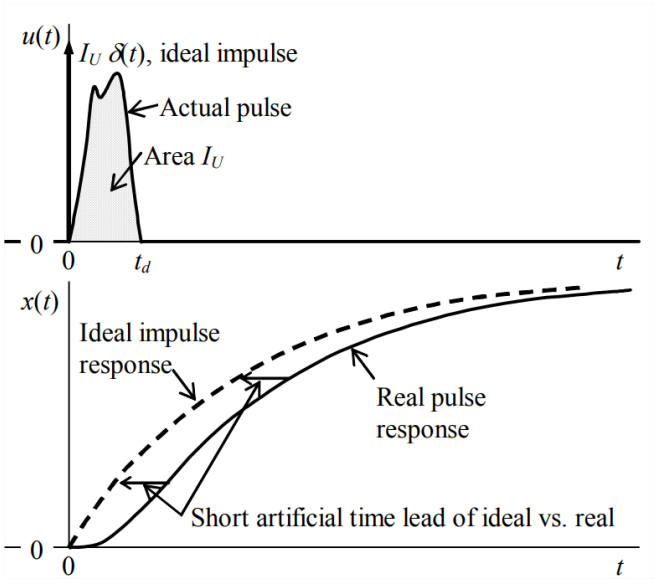


Figure 8.8.1: Conceptual ideal impulse response and real pulse response for an undamped 2nd order system

Note that the previous paragraph applies specifically for a 2nd order system, but not necessarily for other system types. In particular, for a 1st order system, the differences between a real pulse response and the approximating ideal impulse response have a different character (see, for example, homework Problem 8.5).

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8.9: Unit-Step-Response Function and IRF

From Chapter 4, Equation 4.6.4, the general equation that relates the output Laplace transform to the input transform through the system transfer function is

$$L[x(t)]|_{ICs=0} = TF(s) \times L[u(t)] \tag{8.9.1}$$

Equation 8.9.1 is valid for **any** LTI SISO system. Suppose that the input to such a system is the **unit**-step function, $u(t) = UH(t)$ with step magnitude U = 1:

$$u(t) = H(t) \quad \Rightarrow \quad \text{from Equation 2.4.5,} \quad L[u(t)] = L[H(t)] = \frac{1}{s} \tag{8.9.2}$$

We denote the *unit-step-response function* as $x_H(t)$: as usual for unit-step response and unit-impulse response, we specify zero ICs. Substituting Equation 8.9.2 into Equation 8.9.1 gives

$$L\left[x_H(t)\right] = TF(s) \times \frac{1}{s} \tag{8.9.3}$$

Next, suppose that the input is the ideal **unit**-impulse function, $u(t) = I_U\delta(t)$ with impulse magnitude $I_U = 1$:

$$u(t) = \delta(t) \quad \Rightarrow \quad \text{from Eq. (8.4.6),} \quad L[u(t)] = L[\delta(t)] = 1 \tag{8.9.4}$$

We denote the *unit-impulse-response function*, with zero ICs, as $h(t)$, and we abbreviate this important function in text as IRF[1]. Substituting Equation 8.9.4 into Equation 8.9.1 gives

$$L[h(t)] = TF(s) \times 1 \equiv TF(s) \tag{8.9.5}$$

Equation 8.9.5 is an important relationship in linear-system theory: The Laplace transform of the unit-impulse-response function (IRF) equals the transfer function (TF).

Comparing Equations 8.9.2 and 8.9.4 shows that

$$sL[H(t)] = L[\delta(t)] \tag{8.9.6}$$

Next, applying Equation 8.6.3, with $H\left(0^-\right) = 0$, then Equation 8.9.6, gives

$$sL[H(t)] = L\left[\frac{dH}{dt}(t)\right] + H\left(0^-\right) = L\left[\frac{dH}{dt}(t)\right] = L[\delta(t)] \tag{8.9.7}$$

Thus, we infer from Equation 8.9.7 that

$$\frac{dH}{dt}(t) = \delta(t) \tag{8.9.8}$$

We can also derive Equation 8.9.8 formally by differentiating with respect to time t the following specific form of Equation 8.4.4: $H(t-0) = \int_{\tau=0^-}^{\tau=t>0} \delta(\tau - 0)d\tau$. Expressed in words, the time derivative of the unit-step function is the unit-impulse function. These derivations of Equation 8.9.8 are not mathematically rigorous, and the result might seem implausible since both $H(t)$ and $\delta(t)$ are strongly discontinuous functions. However, Equation 8.9.8 can be proved with the theory of generalized functions (Lighthill, 1958), and it can also be demonstrated plausibility with the use of limiting processes on functions other than the flat impulse of Section 8.3 (*e.g.*, homework Problem 8.6).

Comparing Equations 8.9.3 and 8.9.5 shows that

$$TF(s) = sL\left[x_H(t)\right] = L[h(t)] \tag{8.9.9}$$

Applying Equation 8.6.3 again, with $x_H\left(0^-\right) = 0$ by definition, gives

$$\frac{dx_H}{dt}(t) = h(t) \tag{8.9.10}$$

In words, the velocity of unit-step response equals the unit-impulse response. Equation 8.9.10 for responses (outputs) is directly analogous to Equation 8.9.8 for excitations (inputs). The identical form of the two equations is a consequence of system linearity. Equation 8.9.10 is another important relationship in linear-system theory, for which we shall have a convenient application in Section 9.8.

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8.10: The Convolution Integral as a Superposition of Ideal Impulse Responses

Suppose that an LTI system has zero ICs and that the input is an arbitrary physically realistic function, $u(t)$ for $t > 0$. Let us apply directly the principle of superposition to derive an equation for response $x(t)$ at some arbitrary instant of time $t > 0$. At any instant τ less than t, \(0<\tau\tau\) due to this impulse at τ is $dx(t) = dI_U \times h(t-\tau) = u(\tau)h(t-\tau)d\tau$, in which $h(t-\tau)$ is the IRF due to a unit impulse acting at instant τ. Because this system is linear, we can express the total response as the superposition of responses due to all separate inputs, as is stated in Section 1.2. In this case, there are an infinite number of instants τ between time zero and time t, so the superposition, or summation, becomes a definite integral:

$$x(t)|_{ICs=0} = \int_{\tau=0}^{\tau=t} dx(t) = \int_{\tau=0}^{\tau=t} u(\tau)h(t-\tau)d\tau \tag{8.10.1}$$

Equations such as Equation 8.10.1 and its versions for specific systems [with explicit functions for $h(t-\tau)$] are often called *Duhamel integrals* (after Jean-Marie Duhamel, 1797-1872, rench mathematician and physicist), especially in the literature of structural dynamics (e.g., C F raig, 1981, p. 124).

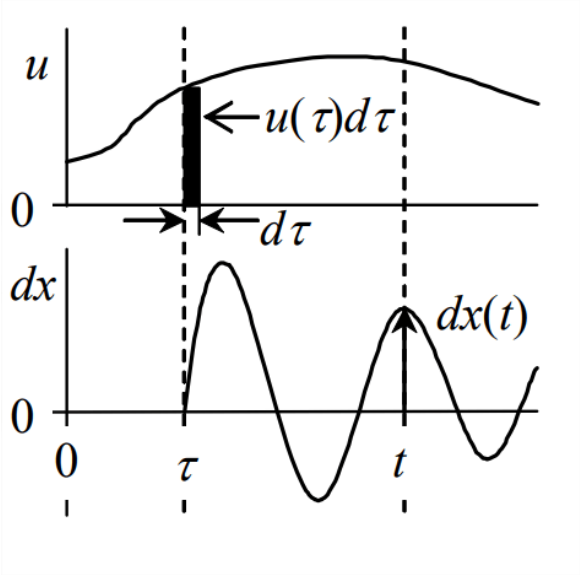


Figure 8.10.1

Equation (8-41) is general, valid for **any** LTI system. To see a specific application, let us re-visit the standard undamped 2nd order system. With $I_U = 1$, Equation 8.7.3 gives the IRF as $h(t) = \omega_n \sin \omega_n t$, so Equation 8.10.1 becomes

$$x(t)|_{ICs=0} = \omega_n \int_{\tau=0}^{\tau=t} u(\tau) \sin \omega_n (t-\tau) d\tau \tag{8.10.2}$$

Equation 8.10.2 is the Duhamel integral response solution for the standard undamped 2nd order system, and it is identical to convolution integral response solution Equation 7.2.5 with zero ICs. Recall that Equation 7.2.5 was derived mostly from the mathematics of convolution integrals and transforms in Chapter 6. On the other hand, the derivation of Equation 8.10.2 above is primarily physical, based upon the IRF and the principle of superposition for linear systems.

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8.11: Approximate Numerical Solutions Based on the Convolution Sum

In Section 6.5, we developed a recurrence formula for the approximate numerical solution of an LTI 1st order ODE with any IC and any physically plausible input function $u(t)$. It is possible to generalize the recurrence-formula approach to 2nd and higher order LTI ODEs, but it would require development of theory and detail that are beyond the objectives of this book (see, for examples: Craig, 1982, Section 7.1; and Meirovitch, 2001, Section 4.10). There is, however, an alternative, relatively simple approach based upon the convolution integral, and this approach is readily applicable to calculating approximate numerical solutions for forced response of 1st and 2nd order LTI systems.

To describe this approach, we begin with the general convolution (Duhamel) integral for forced response, which is valid for any LTI system:

$$x(t)|_{ICs=0} = \int_{\tau=0}^{\tau=t} u(\tau)h(t-\tau)d\tau \tag{8.11.1}$$

If initial conditions (ICs) are not zero, then appropriate IC-response solutions must be added to the convolution integral, which represents only the forced response. Examples of such IC-response solutions are given explicitly in Eqs. (6-5), (7-9), and, later, (9-20). Hereafter in this section, we will consider only the forced response, so it will be convenient to omit the "ICs = 0" subscript from the output variable and denote it simply as $x(t)$, and later as x with subscripts, x_5 for example.

We seek approximate evaluations of Equation 8.11.1 in time-series form, that is, at discrete, equally spaced instants in time. Accordingly, just as in Section 6.5, we define the following notation that employs descriptive subscripts, with the initial instant defined as time zero:

$t =$	$t_1 = 0$	$t_2 = t_1 + \Delta t = \Delta t$	$t_3 = t_2 + \Delta t = 2\Delta t$	...	$t_n = t_{n-1} + \Delta t$ $= (n-1)\Delta t$	...
$x(t) =$	x_1	x_2	x_3	...	x_n	...
$u(t) =$	u_1	u_2	u_3	...	u_n	...
$h(t) =$	h_1	h_2	h_3	...	h_n	...

Let us designate as a *sequence of length* N any series of N numbers such as $t_1, t_2, \ldots, t_N$, or $x_1, x_2, \ldots, x_N$ and let us denote the entire sequence as $\{t\}_N$, or $\{x\}_N$. If time t is the independent variable, and quantify $f(t)$ is any time-dependent function, let us denote the values of $f(t)$ at times $t_1, t_2, \ldots, t_N$ as $\{f(t)\}_N$.

Forced-response solution Equation 8.11.1 is exact. But now we introduce what, in general, is an approximation. We assume that the integrand product $u(\tau)h(t-\tau)$ varies so little over the integration time step Δt that it introduces only small error to approximate $u(\tau)h(t-\tau)$ as being constant over Δt, with its value remaining that at the beginning of the time step:

\[u(\tau) h(t-\tau) \approx u\left(t_{k-1}\right) h\left(t-t_{k-1}\right) \text { for } t_{k-1} \leq \tau

In Equation ???, k is an index that varies, just as τ varies over the integration limits in Equation 8.11.1.

Suppose that we have matching sequences of time $\{t\}_N$, input $\{u(t)\}_N$, and IRF $\{h(t)\}_N$, and we wish to calculate the corresponding forced-response output sequence, $\{x(t)\}_N$. Then, using approximation Equation ???, we express Equation 8.11.1 as follows, with the convolution integral approximated as a summation:

$$\left.\begin{array}{l} x\left(t_1\right) = 0 \\ x\left(t_n\right) = \sum_{k=1}^{n-1} u\left(t_k\right) h\left(t_n - t_k\right) \Delta t, \text{ for } 2 \leq n \leq N \end{array}\right\} \tag{8.11.2}$$

Additional explanation of Equation 8.11.2 might be helpful, because the extrapolation of Equation 8.11.1 from continuous time to discrete time is not completely obvious. Physically, the forced response must be zero at the instant $t = t_1 = 0$ when forcing is

begun, even if the initial input value $u(0)$ is nonzero; also, the exact convolution integral Equation 8.11.1 gives the same result for the upper limit $\tau = t_1 = 0$; hence, $x(t_1) = 0$. Regarding summation only over $n-1$ values of product $u(t_k)\,h(t_n - t_k)\,\Delta t$ in order to obtain $x(t_n)$, consider again approximation Equation ???. Equation ??? means that the very last term in the summation of Equation 8.11.2 that approximates the integral for $x(t_n)$ is the product $u(t_{n-1})\,h(t_n - t_{n-1})\,\Delta t$.

Equation 8.11.2 can be written in the following simpler form, using subscript notation and the constancy of time step Δt:

$$\left.\begin{aligned} x_1 &= 0 \\ x_n &= \Delta t \sum_{k=1}^{n-1} u_k h_{n-k}, \text{ for } 2 \le n \le N \end{aligned}\right\} \tag{8.11.3}$$

The summation in Equation 8.11.3 is called a *convolution sum*.

It is instructive to write out explicitly a few terms of Equation 8.11.3, and then to study a graphical interpretation of the process. Suppose that we have sequences of input values $\{u(t)\}_N$ and IRF values $\{h(t)\}_N$, with length $N \ge 4$, and we wish to find the corresponding sequence of responses $\{x(t)\}_N$. Then the first five terms in the response sequence, from Equation 8.11.3, are:

$$x_1 = 0 \tag{8.11.4}$$

$$x_2 = \Delta t\,(u_1 h_1) \tag{8.11.5}$$

$$x_3 = \Delta t\,(u_1 h_2 + u_2 h_1) \tag{8.11.6}$$

$$x_4 = \Delta t\,(u_1 h_3 + u_2 h_2 + u_3 h_1) \tag{8.11.7}$$

$$x_5 = \Delta t\,(u_1 h_4 + u_2 h_3 + u_3 h_2 + u_4 h_1) \tag{8.11.8}$$

Figure 8.11.1 on the next page is a graphical representation of many aspects of the previous discussion. The figure consists of three graphs of conceptual time-dependent functions, with the three time axes aligned. The top graph is a typical input function, $u(t)$; and one plot on the middle graph is a unit-impulse response function (IRF), $h(t)$, which was chosen to resemble the exponential IRF of a 1st order system, with, in particular, $h(0) \neq 0$. The second (dashed) plot on the middle graph is the function $h(t_4 - t)$, and it was chosen to illustrate Equation 8.11.8 for output quantity x_5.

Observe on the middle graph of Figure 8.11.1 that $h(t_4 - t)$ is derived from $h(t)$ by "folding" $h(t)$ about a vertical line through time $t_4/2$ and then setting to zero the folded function for all times $t < 0$. One of the dictionary definitions for the word "convolution" as a noun is "a fold", and this is apparently the reason for the use of "convolution" as the name of a transform, integral, or sum (Evans, 1954, pp. 199-204). It seems that "convolution sum", for example, sounds more elegant in English than "folding sum"; however, the German noun for mathematical convolution is *Faltung*, meaning "folding" or "bending", and *Faltung* was often used interchangeably with "convolution" in older English-language textbooks and other technical literature.

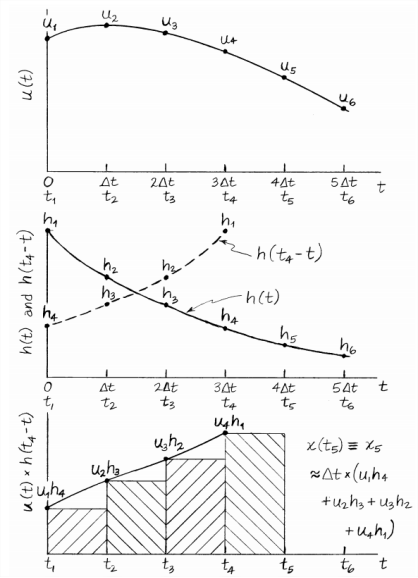


Figure 8.11.1: Graphical representation of the convolution sum for a conceptual problem with input function $u(t)$ and impulse-response function $h(t)$, with $h(0) \neq 0$.

The plot on the bottom graph of Figure 8.11.1 is the product $u(t)h\,(t_4 - t)$. At the times t_1, t_2, t_3, and t_4, the discrete products $u_k h_{5-k}$ are marked, and approximation Equation ??? of piecewise constancy is illustrated by straight horizontal lines, each representing the time step Δt. This is a *stairstep* approximation similar to that used in Section 6.5. The bottom graph shows that Equation 8.11.8 for x_5 is the area under all the cross-hatched rectangles (the entire "staircase"); that area is an approximation of the integral from Equation 8.11.1, $\int_{\tau=0}^{\tau=t_5} u(\tau)h\,(t_5 - \tau)\,d\tau$. It is obvious from this graphical interpretation of the convolution sum that smaller values of time step Δt will produce more accurate numerical integrations, at the expense of longer calculations.

For the conceptual example of Figure 8.11.1, which resembles a 1st order system that has initial IRF value $h(0) = h_1 \neq 0$, and also initial input value $u(0) = u_1 \neq 0$, Equation 8.11.5 gives $x_2 = \Delta t\,(u_1 h_1) \neq 0$. This is noted because it is easy to make the mistake of calculating approximate time response by applying directly the MATLAB function `conv`, as in the command line `X = conv(u,h)`, and then by taking the first N elements of the resulting sequence $\{X\}_{2N-1}$ to be the desired forced response, $\{x(t)\}_N$; this process executes the **incorrect** equation

$$x\,(t_n) = \sum_{k=1}^{n} u\,(t_k)\,h\,(t_{n+1} - t_k)\,\Delta t = x_n = \Delta t \sum_{k=1}^{n} u_k h_{n+1-k}$$

for $1 \leq n \leq N$.

This incorrect equation calculates a sequence with $N - 1$ elements the same as those of the correct forced-response sequence, but the incorrect sequence is shifted backward in time by a single time step, Δt. In particular, the incorrect process sets $x_1 = \Delta t\,(u_1 h_1)$, which would be nonzero for the conceptual problem of Figure 8.11.1, and might alert the analyst that something is wrong. (Recall from the discussion above that the initial forced-response value x_1 must be zero.) But for most systems other than standard 1st order systems, the initial value of the IRF is zero, $h_1 = 0$, so the error in $x_1 = \Delta t\,(u_1 h_1)$ would not be obvious in the calculated results. The lessons from this discussion are:

1. use of the incorrect convolution sum might produce results that are not obviously incorrect; in fact, for very small Δt, the incorrect and correct plots of approximate response versus time might be almost indistinguishable visually; and
2. if you apply directly MATLAB's function `conv`, then, in order to obtain the correct approximate forcedresponse sequence, be sure to shift the sequence calculated by `conv` forward in time by a single time step, Δt, and to set $x_1 = 0$.

The following are two numerical examples of the application of Equation 8.11.3, first for a 1st order ODE, next for an undamped 2nd order ODE.

Convolution-sum Example 8.11.1: 1st order systems

Consider the standard 1st order LTI-ODE of a stable system, Equation 3.4.8, for dependent variable $x(t)$: $\dot{x} + (1/\tau_1)\,x = bu(t)$. From Section 8.5 and homework Problem 8.5.1, the unit-impulse-response function (IRF) is $h(t) = be^{-t/\tau_1}$. Letus set the input function to be a declining ramp, $u(t) = c(a - t)$, in which c, a dimensional constant, is the downward slope of the ramp, and a is the time at which the input passes through zero. For reference, the **exact** solution for this problem is derived from Equations 6.2.4 or 8.11.1[1]:

$$x(t) = be^{-t/\tau_1} \int_{\tau=0}^{\tau=t} e^{\tau/\tau_1} u(\tau) d\tau = be^{-t/\tau_1} \int_{\tau=0}^{\tau=t} e^{\tau/\tau_1} c(a - \tau) d\tau$$

$$= bc\tau_1 \left[(a + \tau_1)\left(1 - e^{-t/\tau_1}\right) - t\right]$$

With parameters τ_1 = 2.5 s, a = 10 s, b = 3.5, and c = 1 (b and c in consistent units), the MATLAB commands to calculate and plot the exact solution $x(t)$, and the approximate solution from Equation 8.11.3, for the relatively large time step Δt = 1 s, are given next, followed by the resulting graph, which was edited later to add labels, title, and legend. Observe in the MATLAB code that the declining-ramp is entered into the input sequence $\{u(t)\}_N$, as is the IRF sequence $\{h(t)\}_N$. Note that the simple code to calculate the convolution sums over the **entire** forced-response sequence, $\sum \frac{x_n}{\Delta t} = \sum_{k=1}^{n-1} u_k h_{n-k}$ for $2 \leq n \leq N$, consists of a `for` loop over index k nested within a `for` loop over index n.

```
>> a=10;b=3.5;c=1;tau=2.5;

dt=1;t=0:dt:10;
```

```
N=length(t); %sequence length
u=c*(a*ones(1,N)-t); %declining-ramp excitation
plot(t,tau*b*u,'k--'),grid %pseudo-static response
h=b*exp(-t/tau); %IRF, impulse-response function
xc(1)=0; %zero initial value
for n=2:N
    sum=0;
    for k=1:(n-1)
        sum=sum+u(k)*h(n-k);
    end
    xc(n)=sum;
end
xc=dt*xc;
hold,plot(t,xc,'ko-')
te=0:0.05:10;
xe=c*b*tau*((a+tau)*(1-exp(-te/tau))-te); %exact solution
plot(te,xe,'k')
```

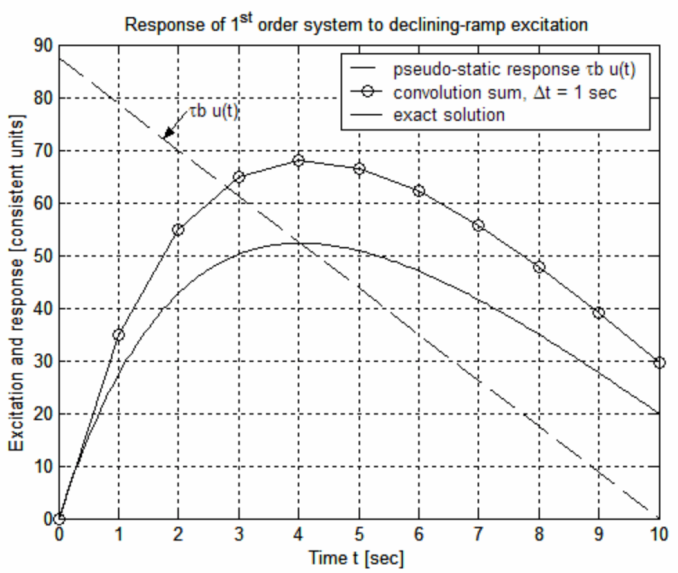


Figure 8.11.1

The graph above shows that Equation 8.11.3 with the relatively large time step $\Delta t = 1$ s produces a poor approximation to the exact solution of the 1st order ODE in this problem. The following MATLAB command lines evaluate Equation 8.11.3 and plot the results for three progressively smaller values of Δt.

```
>> a=10;b=3.5;c=1;tau=2.5;
tend=10;
m=[2 8 16]; %inverses of time steps for convolution
maxln=tend*max(m)+1;
t=zeros(3,maxln);xc=zeros(3,maxln); %initialize oversized arrays
for j=1:3
    dt=1/m(j);tj=0:dt:tend;N(j)=length(tj);t(j,1:N(j))=tj;
    u=c*(a*ones(1,N(j))- tj); %excitation
    h=b*exp(-tj/tau); %IRF
    xc(j,1)=0; %zero initial value
    for n=2:N(j)
        sum=0;
        for k=1:(n-1)
            sum=sum+u(k)*h(n-k);
        end
    xc(j,n)=dt*sum;
    end
end
plot(t(1,1:N(1)),xc(1,1:N(1)),'ko',t(2,1:N(2)),xc(2,1:N(2)),'ks')
hold,plot(t(3,1:N(3)),xc(3,1:N(3)),'k.')
te=0:0.05:tend;
xe=c*b*tau*((a+tau)*(1-exp(-te/tau))-te); %exact solution
plot(te,xe,'k'),grid
```

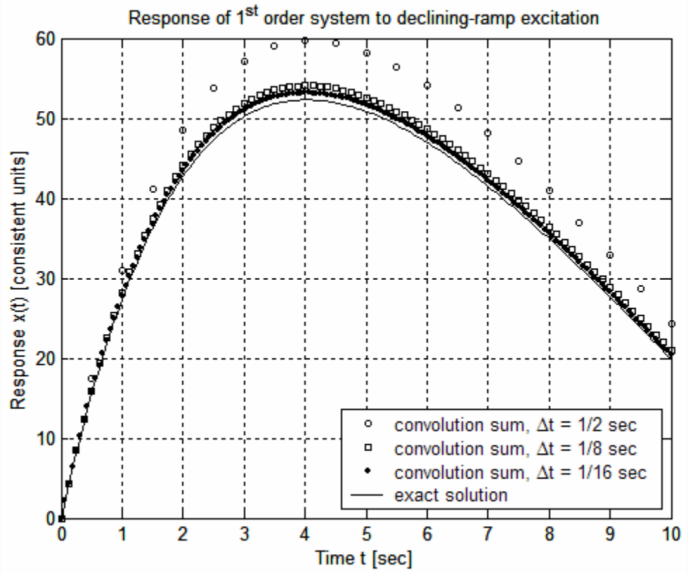


Figure 8.11.2

The results above demonstrate for this 1st order system that the approximate solution calculated from the convolution sum approaches the exact solution as $\Delta t \to 0$. This concludes Convolution-sum Example 1.

Convolution-sum Example 8.11.2: undamped 2nd order system

Consider the LTI-ODE Equation 7.1.5 for a standard undamped 2nd order system, $\ddot{x} + \omega_n^2 x = \omega_n^2 u(t)$, with ICs $x(0) = 0$ and $\dot{x}(0) = 0$. From Sections 8.7 and 8.10 [Equations 8.7.3 and 8.10.2], the unit-impulse-response function (IRF) is $h(t) = \omega_n \sin \omega_n t$. Let us set the input function to be a declining-ramp pulse,

\[u(t)=\left\{\begin{array}{c}
c(a-t), 0 \leq t 0, t_{d} \end{array}\right. \nonumber \]

in which c, a dimensional constant, is the downward slope of the ramp, ca is the input at $t = 0$, and t_d is the pulse duration. For reference, the **exact** solution equations for this problem are derived next from Equations 7.2.4 and 7.2.5, without details of the integration processes. For the interval during which the pulse is active, $0 \leq t \leq t_d$, we apply Equation 7.2.4, although Equation 7.2.5 would serve just as well:

$$x(t) = \omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n \tau \times u(t-\tau) d\tau = \omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n \tau \times c[a-(t-\tau)] d\tau$$

$$= c\left[a - t - a\cos \omega_n t + (1/\omega_n) \sin \omega_n t\right], \text{ valid for } 0 \leq t \leq t_d$$

For the time after the pulse ceases, \(t_{d}

$$x(t) = \omega_n \int_{\tau=0}^{\tau=t} \sin \omega_n (t-\tau) \times u(\tau) d\tau = \omega_n \int_{\tau=0}^{\tau=t_d} \sin \omega_n (t-\tau) \times c(a-\tau) d\tau$$

$$= \omega_n c \left[\sin \omega_n t \int_{\tau=0}^{\tau=t_d} \cos \omega_n \tau \times (a-\tau) d\tau - \cos \omega_n t \int_{\tau=0}^{\tau=t_d} \sin \omega_n \tau \times (a-\tau) d\tau\right]$$

\[\left.

$$= c\left\{\sin \omega_n t \left[a \sin \omega_n t_d - (1/\omega_n)\left(\omega_n t_d \sin \omega_n t_d + \cos \omega_n t_d - 1\right)\right]\right. \tag{8.11.9}$$

.

\right\}, \text { valid for } t_{d}

Let the parameters be: undamped natural frequency $\omega_n = \pi/4.5$ rad/s (making the associated period 9 s), 2 = 10 s, c = 1 (in consistent units), and pulse period t_d = 10 s^2. The MATLAB commands to calculate and plot the exact solution $x(t)$, and the approximate solution from Equation 8.11.3, for the relatively large time step Δt = 1 s, are given next, followed by the resulting graph of response, which was edited later to add labels, title, and legend. Observe in the MATLAB code that the entire declining-ramp pulse, including zeros, is entered into the input sequence $\{u(t)\}_N$, as is the IRF sequence $\{h(t)\}_N$. Also, just as in Convolution-sum Example 1, the simple code to calculate the convolution sums over the **entire** forced-response sequence, $\frac{x_n}{\Delta t} = \sum_{k=1}^{n-1} u_k h_{n-k}$ for $2 \le n \le N$, consists of a `for` loop over index k nested within a `for` loop over index n.

```
>> a=10;c=1;wn=pi/4.5;td=10;

tend=25; %time range (sec)

dt=1;t=0:dt:tend;

nd=td/dt+1; %warning: not necessarily an integer for arbitrary td

N=length(t); %sequence length

u=zeros(1,N);

u(1:nd)=c*(a*ones(1,nd)-t(1:nd)); %declining-ramp pulse

plot(t(1:nd),u(1:nd),'k--'),grid %pseudo-static response

hold,plot(t(nd:N),zeros(1,(N-nd+1)),'k--')

h=wn*sin(wn*t); %IRF, impulse-response function

xc(1)=0; %zero initial value

for n=2:N

    sum=0;

    for k=1:(n-1)

        sum=sum+u(k)*h(n-k);

    end

    xc(n)=sum;

end

xc=dt*xc;

plot(t,xc,'ko-')

t1=0:0.1:td;t2=(td+0.1):0.1:tend;te=[t1 t2];

wt1=wn*t1;wt2=wn*t2;

x1=c*(a-t1-a*cos(wt1)+sin(wt1)/wn);

arg=wn*td;sn=sin(arg);cn=cos(arg);

S=a*sn-(arg*sn+cn-1)/wn;C=a*(1-cn)-(-arg*cn+sn)/wn;

x2=c*(S*sin(wt2)-C*cos(wt2));
```

```
xe=[x1 x2]; %exact solution

plot(te,xe,'k')
```

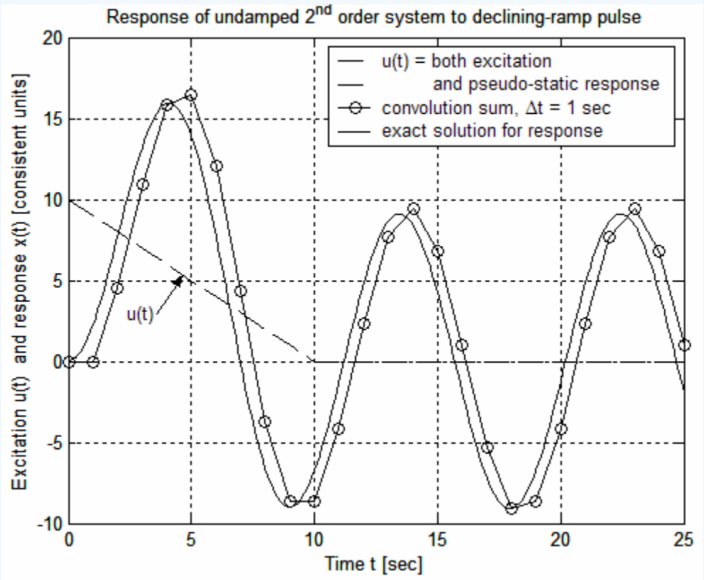


Figure 8.11.3

The graph above shows that Equation 8.11.3 with the relatively large time step Δt = 1 s produces only a mediocre approximation to the exact solution of the 2nd order ODE in this problem. But the plot of the convolution-sum solution also illustrates an important point: not only is x_1 = 0, the requirement discussed previously, but also $x_2 = u_1 h_1$ = 0 because the initial IRF value h_1 = 0 for the 2nd order ODE, regardless of the initial value u_1 of the input. The appropriate approximation in this case for the initial time derivative of response is $\frac{dx}{dt}(t=0) \approx \frac{x_2 - x_1}{\Delta t} = 0$. The necessary initial conditions (ICs) for a 2nd order ODE are both $x(0)$ **and** $\dot{x}(0)$. As is stated at the beginning of this section, any nonzero ICs must be accounted for by separate IC-response solutions, as these ICs cannot be included in the forced-response solution, which is the convolution sum. This means that we must have $x_1 = x_2 = 0$ for this, or, in fact, any correct convolution-sum approximate solution of a standard 2nd order ODE. The plot above of the convolution-sum solution also demonstrates a negative (relative to numerical accuracy) consequence of the necessary condition $x_2 = 0$: a consistent time lag, of the approximate solution relative to the exact solution, on the order of Δt. The following MATLAB command lines evaluate Equation 8.11.3 and plot the results (on the next page) for two progressively smaller values of Δt.

```
>> a=10;c=1;wn=pi/4.5;td=10;

tend=25; %time range (sec)

m=[2 8]; %inverses of time steps for convolution

maxln=tend*max(m)+1;

t=zeros(2,maxln);xc=zeros(2,maxln); %initialize oversized arrays

for j=1:2

    dt=1/m(j);tj=0:dt:tend;N(j)=length(tj);t(j,1:N(j))=tj;

    nd=td/dt+1; %warning: not an integer for arbitrary td

    u=zeros(1,N(j));

    u(1:nd)=c*(a*ones(1,nd)-tj(1:nd)); %declining-ramp pulse
```

```
    h=wn*sin(wn*tj); %IRF, impulse-response function
    xc(j,1)=0; %zero initial value
    for n=2:N(j)
        sum=0;
        for k=1:(n-1)
            sum=sum+u(k)*h(n-k);
        end
    xc(j,n)=dt*sum;
    end
end
plot(t(1,1:N(1)),xc(1,1:N(1)),'ks')
hold,plot(t(2,1:N(2)),xc(2,1:N(2)),'k.'),grid
t1=0:0.1:td;t2=(td+0.1):0.1:tend;te=[t1 t2];
wt1=wn*t1;wt2=wn*t2;
x1=c*(a-t1-a*cos(wt1)+sin(wt1)/wn);
arg=wn*td;sn=sin(arg);cn=cos(arg);
S=a*sn-(arg*sn+cn-1)/wn;C=a*(1-cn)-(-arg*cn+sn)/wn;
x2=c*(S*sin(wt2)-C*cos(wt2));
xe=[x1 x2]; %exact solution
plot(te,xe,'k')
```

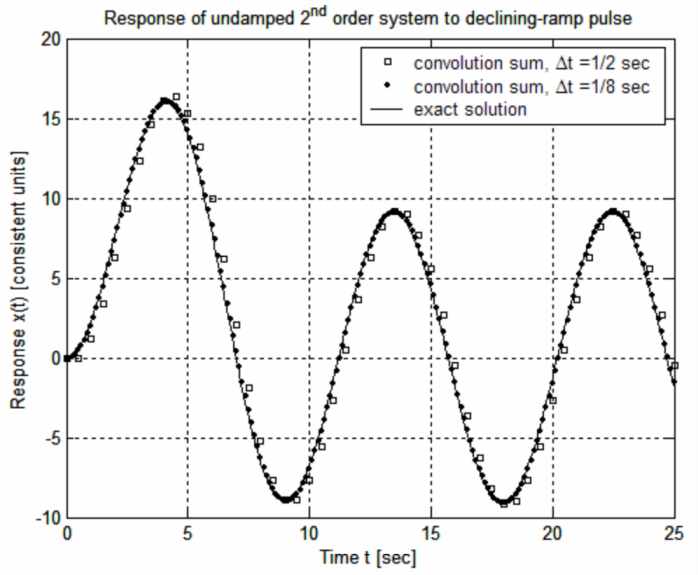


Figure 8.11.4

The numerical solution for time step $\Delta t = 1/2$ s is a fair approximate solution, but still with a noticeable time lag, relative to the exact solution, on the order of Δt. However, the numerical solution for time step $\Delta t = 1/8$ s is almost indistinguishable from the exact solution by eye on the graph. This concludes Convolution-sum Example 2.

Finally, we note a curious intersection of two seemingly unrelated mathematical operations, convolution sum and multiplication of polynomials. Suppose that we have two arbitrary sequences of numbers, $\{u\}_N$ and $\{h\}_M$. Consider the convolution sum in the form applied by the MATLAB command line `X = conv(u,h)`, which produces the sequence $X_n = \sum_{k=1}^{n} u_k h_{n+1-k}$ for $1 \leq n \leq (N + M - 1)$: $X_1 = u_1 h_1$, $X_2 = u_1 h_2 + u_2 h_1$, $X_3 = u_1 h_3 + u_2 h_2 + u_3 h_1$, $X_4 = u_1 h_4 + u_2 h_3 + u_3 h_2 + u_4 h_1$, *etc.* Evans (1954, pp. 201-203) observed that the convolution-sum process can be executed with use of tables in a "process similar to multiplication." First, we define the table of sequences $\{h\}_M$ and $\{u\}_N$:

$\{h\}_M =$	h_1	h_2	h_3	h_4	...
$\{u\}_N =$	u_1	u_2	u_3	u_4	...

In order to execute the operation $X_n = \sum_{k=1}^{n} u_k h_{n+1-k}$ using the sequences $\{h\}_M$ and $\{u\}_N$, we construct the table of products below. The first row of the table consists of the products $u_1 \times \{h\}_M$, the second row consists of the products $u_2 \times \{h\}_M$ and this sequence is offset by one column toward the right, the third row consists of the products $u_3 \times \{h\}_M$ and this sequence is offset by two columns toward the right, *etc.*

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	...
$k = 1$	$u_1 h_1$	$u_1 h_2$	$u_1 h_3$	$u_1 h_4$	...
$k = 2$		$u_2 h_1$	$u_2 h_2$	$u_2 h_3$	...
$k = 3$			$u_3 h_1$	$u_3 h_2$	...
$k = 4$				$u_4 h_1$	...
...					...

Then each element X_n of the sequence $\{X\}_{N+M-1}$ is the sum of the products in the corresponding n^{th} column of the table of products above:

$\{X\}_{N+M-1} =$	$X_1 = u_1 h_1$	$X_2 = u_1 h_2 +$ $u_2 h_1$	$X_3 = u_1 h_3 +$ $u_2 h_2 + u_3 h_1$	$X_4 = u_1 h_4 +$ $u_2 h_3 + u_3 h_2 +$ $u_4 h_1$	*dots*

A numerical example applying MATLAB's `conv` function to multiplication of two polynomials was given in Section 6.1. Here is another example of the process using the notation in the tables above, the product of a 1st degree polynomial, P_1, with a 2nd degree polynomial, P_2:

$$P_1 \times P_2 = (u_1 s + u_2) \times \left(h_1 s^2 + h_2 s + h_3\right) = u_1 h_1 s^3 + (u_1 h_2 + u_2 h_1)\, s^2 + (u_1 h_3 + u_2 h_2)\, s + u_2 h_3$$

With zero input polynomial coefficients $u_i = 0$ for $i > 2$ and $h_j = 0$ for $j > 3$, the coefficients of the product cubic polynomial (in descending order of powers of s) are the coefficients X_1, X_2, X_3, and X_4 from the last table above.

If you wished to execute product $P_1 \times P_2$, you would probably use the basic algebra below for multiplication of two polynomials, which, although less general, is essentially the same as the tabular process described above.

Polynomial P_2 =	$h_1 s^2 +$	$h_2 s +$	h_3	
Polynomial P_1 =	$u_1 s +$	u_2		
$u_1 s \times P_2$ =	$u_1 h_1 s^3 +$	$u_1 h_2 s^2 +$	$u_1 h_3 s$	
$u_2 \times P_2$ (offset) =		$u_2 h_1 s^2 +$	$u_2 h_2 s +$	$u_2 h_3$
$P_1 \times P_2$ = column sums =	$u_1 h_1 s^3 +$	$(u_1 h_2 + u_2 h_1)\, s^2 +$	$(u_1 h_3 + u_1 h_3)\, s +$	$u_2 h_3$

[1]Details of the integration process are not shown. Note, however, that you can easily verify the validity of the given solution by substituting it into the ODE and IC and finding that they are satisfied.

[2]With $t_d = a$, the pulse is continuous at zero when it ends.

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8.12: Chapter 8 Homework

1. A “shaped”, 10-millisecond force pulse in pounds is described by the equation

$$f_x(t) = \left[50 \sin^2(100\pi t)\right] \text{ lb} \times [H(t) - H(t - 0.01\text{sec})]$$

which is graphed at right. In a calculation of system response to this input force, we wish to approximate the force as an ideal impulse function, $f_x(t) \cong I_F \delta(t)$. What is the value (with units) of I_F? (Hint: before embarking on a long integration, examine the graph carefully for symmetries that will make the calculation almost trivial.)

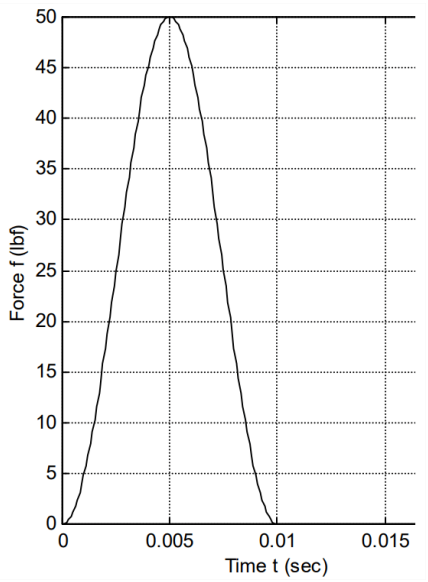


Figure 8.12.1

2. Consider an undamped mass-spring system. The mass weighs 20.0 lb including an air-jet thruster that is attached to it, and the spring is calibrated to stretch 0.279 inch when subjected to 100 lb of tension. The air-jet thruster has been designed and calibrated to provide 0.767 lb of thrust in a nearly perfect flat pulse. Let us study the dynamic position response $x(t)$ of the mass, starting from zero ICs, to this force input from the thruster.
    1. Calculate the natural frequency of the mass-spring system in rad/s and Hz, and the natural period of the system in seconds and milliseconds. Calculate the impulse magnitude I_F delivered by the air-jet thruster in a blast of duration t_d = 4.00 milliseconds. (Partial answers: T_n = 75.53 ms, I_F = 3.068 × 10-3 lb-s)
    2. Model the 4.00-ms air-jet blast as a perfect flat pulse of force. For this input, write equations that describe the real pulse response $x(t)$ (in inches); for help, see Equation 8.1.6 and Equation 7.1.4; do not re-derive anything that is already available. Use MATLAB to make an accurate time history plot of this real pulse response $x(t)$ (in inches) over the time interval $0 \leq t \leq 40$ ms.
    3. Approximate the air-jet force input as an ideal impulse with the same impulse magnitude I_F as calculated in part 8.2.1. Use Equation 8.7.5 to write the numerical equation for the ideal impulse response $x(t)$ in inches.
    4. Use MATLAB to make two accurate time history plots, **on the same graph**, of $x(t)$ (in inches) over the time span $0 \leq t \leq 40$ ms: the real pulse response of part 8.2.2 and the ideal impulse response of part 8.2.3.
3. Consider the mass-spring system drawn below with m = 8.03 kg and k = 317 N/m. This system is initially at rest in the static equilibrium position when it is hit by a flat force pulse $f_y(t) = F\,[H(t) - H\,(t - t_d)]$ of magnitude $F = 4.50$ N and short duration t_d = 0.05 s. In this problem, we analyze the dynamic displacement of the mass relative to the initial static equilibrium position.

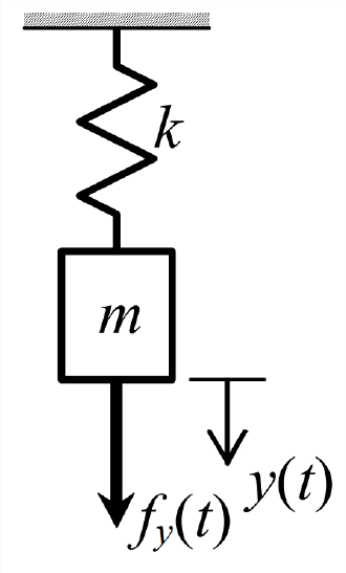


Figure 8.12.2

1. The following succinct response equation can be derived easily with use of transform Equation 8.1.2 and the general Laplace transform of a function that is translated in time: $L\left[f\left(t-t_d\right)H\left(t-t_d\right)\right]=e^{-st_d}F(s)$.

$$y(t)=\frac{F}{k}\left[\left(1-\cos\omega_n t\right)H(t)-\left(1-\cos\omega_n\left(t-t_d\right)\right)H\left(t-t_d\right)\right]$$

   Show that this equation is fully equivalent to the response equations in the two-part solution Equation 8.1.6. Note from Equation 7.1.4 that $U=F/k$.

   Next, use MATLAB to plot the response curve of $y(t)$ versus t from t = 0 to t = 1 s, the period of the mass-spring system.

2. In order to compare with the real response, plot [on the same graph as part 8.3.1] the ideal impulse response of the system, $y(t)=(I_F/m\omega_n)\sin\omega_n t$, Equation 8.7.5, using for I_F the value of the actual impulse from given data. Label clearly which curve is the real response and which is the ideal impulse response; title and label your graph appropriately.

4. In this exercise, let us apply the initial-value theorem to the problem of Section 8.7: the standard undamped 2nd order system that has non-zero initial conditions and is disturbed by an ideal impulse, $u(t)=I_U\delta(t)$.
   1. Determine the post-impulse initial value, $x\left(0^+\right)$, by using Equation 8.7.2 for in the initial-value theorem. Is your result the same as that found directly from Equation 8.7.3?
   2. Use Equation 8.7.2 for $L[x(t)]$ to find the Laplace transform $L[\dot{x}(t)]$ of the "velocity" function; be sure to include **both** terms of Equation 8.6.3 for the transform of a derivative.
   3. Determine the post-impulse initial "velocity", $\dot{x}\left(0^+\right)$, by using your $L[\dot{x}(t)]$ from part 8.4.2 in the initial-value theorem. Is your result the same as that of Equation 8.8.2?
5. Consider the standard 1st order LTI-ODE (of a stable physical system) for dependent variable $x(t)$: $\dot{x}+(1/\tau_1)\,x=bu(t)$, with IC $x(0)=0$, and with excitation by an ideal impulse, $u(t)=I_U\delta(t)$.
   1. Infer from the results of Section 8.5 the unit-impulse-response function (IRF), $h(t)$.
   2. Use the result from part 8.5.1 to write the Duhamel integral response solution for the standard stable 1st order system, which is comparable to Equation 8.11.2 for the standard undamped 2nd order system.
   3. The object of this part is to compare numerically the ideal impulse response derived in Section 8.5 with the response to a real, flat pulse that has the same impulse magnitude as the ideal impulse and a pulse duration that is short relative to the system time constant. Use MATLAB to make two accurate time history plots, on the same graph, of the dimensionless output $x(t)/bI_U$ versus dimensionless time t/τ_1 over the time interval $0\le t/\tau_1\le 1$. One plot will be the response to the ideal impulse. The other plot will be the response to a flat pulse having dimensionless duration $t_d/\tau_1=0.2$; for this plot, use the response equations derived in homework Problem 6.2. You should find that the ideal impulse response is much easier to calculate than the real-pulse response. However, is the ideal impulse response a sufficiently accurate approximation to the real response, and, if so, is it accurate over all time, or over just some portion of the response time?
6. If you compare Figure 2.4.1 for the Heaviside unit-step function $H(t)$ with Figure 3.4.2 for step response of a stable 1st order system, you can see that as time constant τ_1 becomes progressively smaller, the 1st order step response looks progressively

more like the unit-step function. Using Equation 2.4.1 for $H(t)$ as a model, let us define the "exponential-unit-step" function $H_e(t)$:

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \Rightarrow H_e(t) \equiv \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t/\tau_1} & \text{for } t > 0 \end{cases}$$

It is clear that $H(t)$ can be defined as a limit of $H_e(t)$: $H(t) = \lim_{\tau_1 \to 0} H_e(t)$. Now, let us define the "exponential-unit-impulse" function $\delta_e(t)$:

$$\delta_e(t) \equiv \begin{cases} 0 & \text{for } t < 0 \\ \frac{d}{dt}\left(1 - e^{-t/\tau_1}\right) = \frac{1}{\tau_1} e^{-t/\tau_1} & \text{for } t > 0 \end{cases}$$

Sketch by hand an over-and-under pair of graphs, the upper graph being $H_e(t)$ vs. t and the lower graph being $\delta_e(t)$ vs. t. Sketch conceptually the two exponential functions for a few values of time constant τ_1; show in particular how $H_e(t)$ and $\delta_e(t)$ evolve as τ_1 becomes progressively smaller. Describe the character of $\delta_e(t)$ as $\tau_1 \to 0$. Is it plausible physically to define the Dirac delta function as $\delta(t) = \lim_{\tau_1 \to 0} \delta_e(t)$, which, with the definition $H(t) = \lim_{\tau_1 \to 0} H_e(t)$, is equivalent to $\delta(t) = dH/dt$? [This is one of many possible limit-process definitions of $\delta(t)$; Equation 8.4.1 is a more commonly used definition.]

7. Consider the mass-spring system (with damping neglected) of Figure 1.10.1, for which the ODE of motion is $m\ddot{x} + kx = f_x(t)$, and the system parameters are m = 2.20 kg and k = 770 N/m. Initial conditions are zero: $x(0)$ = 0 and $\dot{x}(0)$ = 0. The excitation is a ramped exponential force pulse, $f_x(t) = F_m\,(t/t_m) \exp\,(1 - t/t_m)$, with F_m = 6.0 N and t_m = 1/12 s. Write a MATLAB program, or adapt the code of Convolution-sum Example 2 in Section 8.11, to calculate and plot an approximate numerical solution for $x(t)$ over the time interval $0 \le t \le 1$ s. Adjust the time-step size Δt and number of time steps over the 1 s interval in your code until the graph of your approximate solution appears very similar to that on Figure 1.10.2 of the corresponding exact solution. Submit your MATLAB code and your final graph of response.

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9: DAMPED SECOND ORDER SYSTEMS

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9.1: Homogenous Solutions

For the mass-dashpot-spring (m-c-k) system of Figure 9.1.1, the equation of motion Equation 3.7.1 derived from Newton's 2nd law, with use of the FBD in Figure 3.7.1, is

$$m\ddot{x} + c\dot{x} + kx = f_x(t) \tag{9.1.1}$$

Although 2nd order systems appear in many different mechanical and electrical forms, the m-c-k system of Figure 9.1.1 is generally considered to be the prototype.

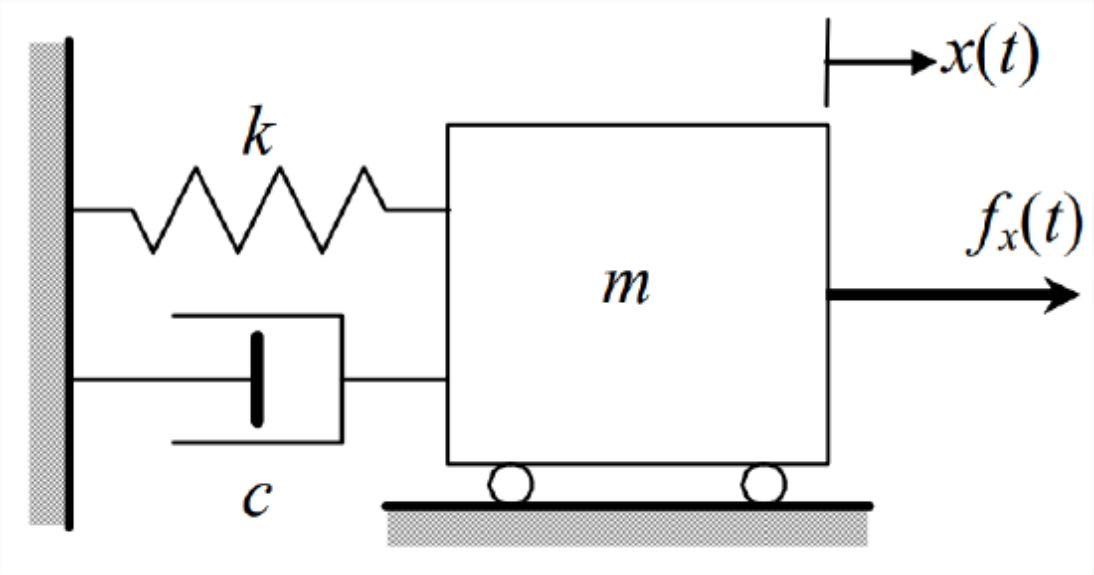


Figure 9.1.1: Mass-dashpot-spring system

To provide direction for developing a standard form of damped 2nd order ODE, let us consider the homogeneous form of Equation 9.1.1 and seek a homogeneous solution $x_h(t)$ using the conventional method that is described in Section 1.5 [Equation 1.5.3 and the subsequent discussion]:

$$m\ddot{x}_h + c\dot{x}_h + kx_h = 0 \tag{9.1.2}$$

We seek $x_h(t) = Ce^{\lambda t}$, in which C and λ are unknown:

$$\Rightarrow \left(m\lambda^2 + c\lambda + k\right) Ce^{\lambda t} = 0$$

Now $x_h(t) = Ce^{\lambda t} \neq 0$ for a non-trivial solution, leading to the characteristic equation:

$$m\lambda^2 + c\lambda + k = 0 \tag{9.1.3}$$

We solve Equation 9.1.3 for characteristic values λ using the quadratic formula, recognizing $k/m = \omega_n^2$ from Equation 7.1.3:

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

The dimensionless *viscous damping ratio* ζ and the *critical* viscous damping constant c_c are defined as follows:

$$\zeta \equiv \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}} \equiv \frac{c}{c_c} \quad \text{where} \quad c_c \equiv 2m\omega_n = 2\sqrt{mk} \tag{9.1.4}$$

Recall that constants m, c, and k are normally positive for a passive, stable system, so that ζ and c_c also are normally positive. With definitions Equation 9.1.4, the characteristic values take the form

$$\lambda = -\left(\frac{c}{2m\omega_n}\right)\omega_n \pm \sqrt{\left(\frac{c}{2m\omega_n}\right)^2 \omega_n^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{9.1.5}$$

Depending upon the value of ζ relative to 1, there are three possible distinct solution types for characteristic value λ and the associated homogeneous solution $x_h(t)$:

1. Both roots are real and negative:

   $$\lambda_{I,II} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{9.1.6}$$

   The corresponding homogeneous solution is an exponentially decaying response with two distinct time constants:

   $$x_h(t) = C_I e^{\lambda_I t} + C_{II} e^{\lambda_{II} t} = C_I e^{-t/\tau_I} + C_{II} e^{-t/\tau_{II}} \quad \text{where} \quad \tau_{I,II} \equiv -1/\lambda_{I,II} \tag{9.1.7}$$

   If we were given initial conditions $x_h(0)$ and $\dot{x}_h(0)$, we could now solve for constants and in terms of the ICs, and thus obtain an IC solution for the overdamped m-c-k system. We will obtain such a solution in Section 9.10, using different methods.
2. The two roots Equation 9.1.5 are equal and negative:

   $$\lambda_{I,II} = -\omega_n \tag{9.1.8}$$

   Due to the existence of repeated roots, the homogeneous solution involves both a pure exponential decay term and a term multiplied by time t that also decays, but more slowly than exponential:

   $$x_h(t) = C_I e^{-\omega_n t} + C_{II} t e^{-\omega_n t} \tag{9.1.9}$$

   The case of critical damping is more of academic than practical interest, since it is rare in practice that physical parameters lead to the exact value $\zeta = 1$. (see homework Problem 9.11)
3. In this case, the terms in Equation 9.1.5 within the square root are negative, so we have two complex characteristic values, each with a negative real part:

   $$\lambda_{I,II} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \equiv -\zeta\omega_n \pm j\omega_d \quad \text{where} \quad \omega_d \equiv \omega_n\sqrt{1 - \zeta^2} \tag{9.1.10}$$

   Frequency ω_d is called the *damped natural frequency*. The corresponding homogeneous solution is

   $$x_h(t) = C_I e^{\lambda_I t} + C_{II} e^{\lambda_{II} t} = C_I e^{(-\zeta\omega_n + j\omega_d)t} + C_{II} e^{(-\zeta\omega_n - j\omega_d)t} = e^{-\zeta\omega_n t}\left(C_I e^{j\omega_d t} + C_{II} e^{-j\omega_d t}\right)$$

   By applying Euler's Equation 2.1.12 and combining constants and appropriately to form two new constants D_I and D_{II}, we can re-write this homogeneous solution in the form of an exponentially decaying sinusoid:

   $$x_h(t) = e^{-\zeta\omega_n t}\left(D_I \cos\omega_d t + D_{II} \sin\omega_d t\right) \tag{9.1.11}$$

   If we were given initial conditions $x_h(0)$ and $\dot{x}_h(0)$, we could now solve for real constants D_I and D_{II} in terms of the ICs, and thus obtain an IC solution for the underdamped m-c-k system. We will obtain this solution in Section 9.4, using the general solution for $x(t)$ obtained from Laplace transformation. Underdamped systems are the most important and common in practical applications and the most interesting, so we will devote more attention to them than to the other two types of damped 2nd order systems.

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9.2: Standard Form ODE

Using the concepts and notation developed in the previous section, we now derive the "standard" form of ODE governing response of damped 2nd order systems, beginning with ODE Equation 9.1.1 for an m-c-k system:

$$m\ddot{x}+c\dot{x}+kx=f_x(t)\Rightarrow \ddot{x}+2\frac{c}{2\sqrt{mk}}\sqrt{\frac{k}{m}}\dot{x}+\frac{k}{m}x=\frac{1}{m}f_x(t)=\frac{k}{m}\frac{f_x(t)}{k} \tag{9.2.1}$$

Using the definitions from the previous section and the standard input quantity Equation 7.1.4, $u(t)\equiv f_x(t)/k$, we re-write Equation 9.2.1 in the standard form:

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=\omega_n^2u(t) \tag{9.2.2}$$

In Equation 9.2.2, $x(t)$ represents any appropriate output quantity (not necessarily just position as in Figure 9.1.1 for a damped 2nd order system. Recall from Chapter 7 that we can identify $u(t)$ as being the pseudo-static output, $x_{ps}(t)$; if $x(t)$ varies slowly enough that the terms $\ddot{x}$ and $2\zeta\omega_n\dot{x}$ are negligible in comparison with ω_n^2x, then ODE Equation 9.2.2 reduces to a simple algebraic equation, $\omega_n^2x\approx\omega_n^2u(t)$, the solution of which is the pseudo-static response, $x(t)=u(t)\equiv x_{ps}(t)$.

Example 9.2.1: the LRC circuit, a 2nd order electrical system

Let us derive an ODE governing the dynamic behavior of the LRC circuit in the figure below. To write Kirchhoff's voltage law for this circuit, we start at the input voltage generator and proceed clockwise [see Equation 5.2.10]:

$$(e_i-0)+(e_m-e_i)+(e_o-e_m)+(0-e_o)=0$$

Next, we substitute in Equation 5.2.9 for the inductor and Ohm's law Equation 5.2.1 for the resistor:

$$e_i+\left(-L\frac{di}{dt}\right)+(-Ri)-e_o=0\Rightarrow L\frac{di}{dt}+Ri+e_o=e_i(t)$$

For the capacitor, we use Equation 5.2.6, then differentiate the result and substitute into the ODE:

$$i=C\frac{d\left(e_o-0\right)}{dt}=C\dot{e}_o\Rightarrow\frac{di}{dt}=C\ddot{e}_o\Rightarrow LC\ddot{e}_o+RC\dot{e}_o+e_o=e_i(t)$$

Therefore, we can write the ODE in the standard 2nd order form Equation 9.2.2 as[1]

$$\ddot{e}_o+\frac{R}{L}\dot{e}_o+\frac{1}{LC}e_o=\frac{1}{LC}e_i(t)$$

From this standard form, we see that the undamped natural frequency is $\omega_n=1/\sqrt{LC}$, and the viscous damping ratio is $\zeta=(1/2\omega_n)\,(R/L)=\frac{1}{2}R\sqrt{C/L}$.

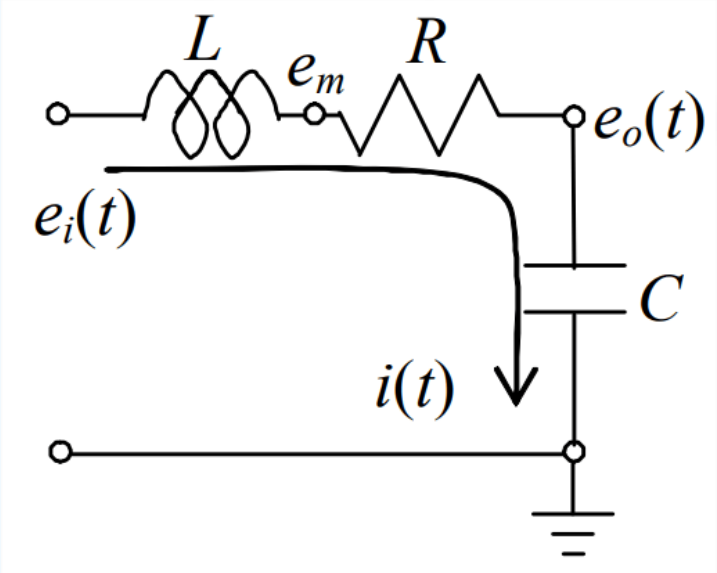


Figure 9.2.1

Example 9.2.2: the rate gyroscope, a 2nd order mechanical system

The schematic three-view engineering sketch on the next page represents the basic functional form of a single-axis *rate gyroscope* (gyro), a sensor of rotational velocity[2]. The supporting turntable in the sketch could be a fixture in a laboratory setup for calibrating the sensor; the turntable rotates clockwise (as viewed from above) with rotational velocity (rate) $p(t)$. At the heart of the rate gyro is a spinning rotor with polar rotational inertia J_r about its spin axis; it is driven by a motor to spin counterclockwise (as viewed from the front) at the high constant spin rate (rotational speed) Ω_r, which is usually orders of magnitude greater than $|p(t)|$. The motor and spinning rotor are attached to a gimbal (rotating frame) and shaft segments that fit into bearings within brackets projecting from the turntable. The gimbal-shaft assembly (including the motor and spinning rotor) can rotate through **small** angle $\theta(t)$ about the so-called "gimbal axis". This $\theta(t)$ rotation is resisted by a rotational spring with constant k_θ, and by a rotational viscous damper with constant c_θ. (Drag is imposed by viscous liquid within a gap between the outer surface of the shaft segment and the inner surface of the sleeve, which is attached to the bracket.) The polar rotational inertia of the gimbal-shaft assembly about the gimbal axis is J_θ. Due to the inertia J_r and high speed Ω_r of the spinning rotor, turntable rotation $p(t)$ induces an inertial moment about the gimbal axis, $M_\theta(t) = J_r\Omega_r p(t)\cos\theta(t)$ [as derived from Newton's laws of rigid-body dynamics by, *e.g.*, Cannon, 1967, pages 152-163]; we assume that $\theta(t)$ is small enough that $\cos\theta(t) \approx 1$, so that $M_\theta(t) \approx J_r\Omega_r p(t)$, as labeled on the sketch. From Equation 3.3.2, Newton's 2nd law for rotation of the gimbal-shaft assembly about the gimbal axis is

$$\Sigma(\text{ Moments })_{\text{about } \textit{gimbal axis}} = (\text{ rotational inertia}) \times (\text{ rotational acceleration})_{\text{about gimbal axis}}$$

$$M_\theta(t) - c_\theta\dot{\theta} - k_\theta\theta = J_\theta\ddot{\theta} \quad \Rightarrow \quad J_\theta\ddot{\theta} + c_\theta\dot{\theta} + k_\theta\theta = J_r\Omega_r p(t)$$

Therefore, we can write the ODE in the standard 2nd order form Equation 9.2.2 as

$$\ddot{\theta} + \frac{c_\theta}{J_\theta}\dot{\theta} + \frac{k_\theta}{J_\theta}\theta = \frac{J_r\Omega_r}{J_\theta}p(t) = \frac{k_\theta}{J_\theta}\frac{J_r\Omega_r}{k_\theta}p(t)$$

From this standard form, we see that the undamped natural frequency is $\omega_n = \sqrt{k_\theta/J_\theta}$, and the viscous damping ratio is $\zeta = (c_\theta/J_\theta)\,/\,(2\omega_n)$, and the standard input quantity, with the same dimensions as rotation-angle output $\theta(t)$, is $u(t) = (J_r\Omega_r/k_\theta)\,p(t) \equiv \theta_{ps}(t)$, the pseudo-static output.

In practical application of a rate gyro, a transducer detects the rotation of the gimbal-shaft assembly and generates an electrical signal proportional to $\theta(t)$, which might be displayed and/or recorded by a data-acquisition-and-processing system, and might also serve as an input to a control system. Spinning-rotor rate gyros come in various sizes and shapes; typical units are around the size of a one- or two-pound can of vegetables, and their cases can be cylindrical or box-shaped[3].

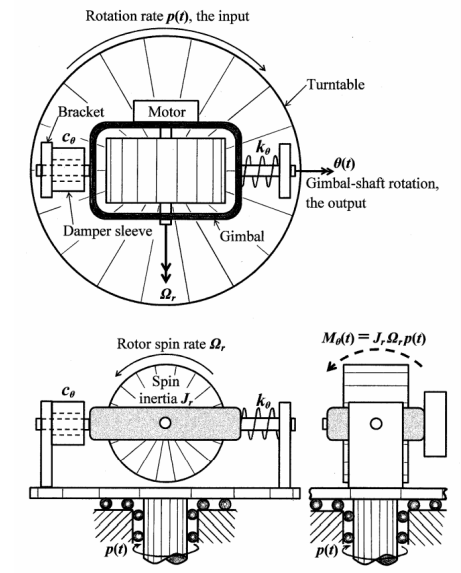


Figure 9.2.2

[1]In Appendix B, Section 19.3, this ODE is derived by an alternative method using energy and power. The rate of change of system energy is equated with the power supplied to the system.

[2]Gyroscopes have been used in sensors and actuators for both aerospace vehicles and water-borne vehicles. Some examples are described by Cannon, 1967, pages 159-163, 617-626, 696-697, and by Den Hartog, 1956, pages 108-112.

[3]This type of spinning-rotor gyro can be considered a “legacy” design, not necessarily the most modern or the best for current applications. Rotation sensors using laser optics and microelectromechanics (MEMS) have been developed more recently.

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9.3: General Solution for Underdamped Second Order Systems

We want to solve Equation 9.2.2 for output $x(t)$ during positive time $t > 0$, given **any** input $u(t)$, and given appropriate initial conditions at $t = 0$. We will use Laplace transformation with application of the inverse convolution transform. To simplify the notation, let us denote $X(s) \equiv L[x(t)]$. Transforming Equation 9.2.2 with use of Equation 2.2.11 gives

$$s^2X(s) - sx(0) - \dot{x}(0) + 2\zeta\omega_n[sX(s) - x(0)] + \omega_n^2X(s) = \omega_n^2L[u(t)]$$

We denote the two initial conditions, $x_0 \equiv x(0)$ initial "position" and $\dot{x}_0 \equiv \dot{x}(0)$ initial "velocity". Collecting terms algebraically and rearranging the equation gives

$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) X(s) = (s + 2\zeta\omega_n)\, x_0 + \dot{x}_0 + \omega_n^2 L[u(t)] \tag{9.3.1}$$

Note that up to this point in the derivation, no restriction has been placed on the value of damping ratio ζ.

To cast Equation 9.3.1 into an easily solvable form, we use two algebraic tricks that are not obvious *a priori*. The first trick is to re-write the left-hand-side quadratic term [which, essentially, is the same as the quadratic term in characteristic Equation 9.1.4]:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 + \omega_n^2 - (\zeta\omega_n)^2 = (s + \zeta\omega_n)^2 + \omega_d^2 \tag{9.3.2}$$

We shall regard ω_d^2 as a positive parameter in the following, so Equation 9.3.2 is nominally valid only for an underdamped system ($0 \le \zeta < 1$). The second trick is to split into a particular form the IC terms on the right-hand side of Equation 9.3.1:

$$(s + 2\zeta\omega_n)\, x_0 + \dot{x}_0 = (s + \zeta\omega_n)\, x_0 + (\dot{x}_0 + \zeta\omega_n x_0) \tag{9.3.3}$$

Applying Equations 9.3.2 and 9.3.3 to Equation 9.3.1, then solving for $X(s)$ gives:

$$X(s) = \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} x_0 + \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2} \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right) + \frac{\omega_n^2}{\omega_d} \overbrace{\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}}^{F_1(s)} \overbrace{L[u(t)]}^{F_2(s)} \tag{9.3.4}$$

The motivation for casting the solution into form Equation 9.3.4 is a relevant general Laplace transform pair that has not appeared previously in this book: given a function $f(t)$, its Laplace transform $F(s)$, and the exponential function $e^{\sigma t}$, where σ is a constant, then the Laplace transform of the product $e^{\sigma t}f(t)$ is

$$L\left[e^{\sigma t}f(t)\right] = \int_{t=0}^{t=\infty} e^{-st}e^{\sigma t}f(t)dt = \int_{t=0}^{t=\infty} e^{-(s-\sigma)t}f(t)dt \equiv F(s-\sigma) \tag{9.3.5}$$

The associated inverse transform is

$$L^{-1}[F(s-\sigma)] = e^{\sigma t}f(t) \tag{9.3.6}$$

Returning to Equation 9.3.4, we identify $\sigma = -\zeta\omega_n$ and invert the two IC-response terms using Equation 9.3.6 in conjunction with sine and cosine transforms Equation 2.4.7 and Equation 2.4.8. To invert the forced-response term, we apply both Equation 9.3.6 and inverse convolution transform Equation 6.1.5. This leads to the two equivalent general equations for output $x(t)$ of an **underdamped** 2nd order system:

$$x(t) = \overbrace{e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)\sin\omega_d t\right]}^{\text{IC response}} + \overbrace{\frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau \times u(t-\tau)d\tau}^{\text{forced response}} \tag{9.3.7}$$

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)\sin\omega_d t\right] + \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau) \times u(\tau)d\tau \tag{9.3.8}$$

Note the emphasis that Equations 9.3.7 and 9.3.8 are valid for underdamped systems. This solution is nominally **not** valid for overdamped systems, although we will see in Section 9.10 that it can be converted easily. The nominal restriction to underdamped

systems stems from the use of sinusoidal transforms Equation 2.4.7 and Equation 2.4.8, which are valid in this case only for **positive** ω_d^2, which holds only if $0 \leq \zeta < 1$, from Equation 9.1.12. For example, we used the inverse transform $L^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin \omega t$, valid for $\omega^2 > 0$, in order to find Equations 9.3.7 and 9.3.8, but the following transform holds for a negative term in the denominator: $L^{-1}\left[\frac{\omega}{s^2-\omega^2}\right] = \sinh \omega t$, a hyperbolic sine.

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9.4: IC Transient Response of Underdamped Second Order Systems

One form of transient, free vibration is response from non-zero initial conditions, with input $u(t) = 0$ for all time. With $u(t) = 0$, response Equations 9.3.8 and 9.3.9 becomes

$$x(t) = e^{-\zeta\omega_n t}\left[x_0 \cos\omega_d t + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)\sin\omega_d t\right], \text{ for } 0 \le t \text{ and } 0 \le \zeta < 1 \tag{9.4.1}$$

By following the procedure of Equations 7.3.2-7.3.4 for combining the two sinusoids of Equation 9.4.1, we can also express the IC, free-vibration response as

$$x(t) = x_{\max} e^{-\zeta\omega_n t}\cos\left(\omega_d t + \phi\right), \text{ for } 0 \le t \text{ and } 0 \le \zeta < 1 \tag{9.4.2}$$

where $x_{\max} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$ and $\phi = \tan^{-1}\left[-\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)/x_0\right]$

Note that for $\zeta = 0$, response Equations 9.4.1 and 9.4.2 reduce to the results Equation 7.3.1 and Equation 7.3.4 derived for an undamped system in Chapter 7. In this regard, it is useful to keep in mind the definition of the damped natural frequency, $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$.

The equation $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$ also shows that damping reduces the frequency of free vibration, and increases the period, $T_d \equiv 2\pi/\omega_d$. However, the words "frequency" and "period" are used loosely in this case, because the damped response is not truly periodic. More correctly, T_d is defined as the time between successive local crests or troughs of the response, and between successive positive-going or negative-going zeros.

Figure 9.4.2 on the next page is an annotated sketch of response Equations 9.4.1 and 9.4.2 for positive values of the ICs, $x_0 > 0$ and $\dot{x}_0 > 0$, and for the small damping ratio $\zeta = 0.11$. The output is a sinusoid, $\cos\left(\omega_d t + \phi\right)$, modulated by a decaying *exponential envelope*, $\pm x_{\max} e^{-\zeta\omega_n t} \equiv \pm x_{\max} e^{-t/\tau_2}$. In this equation for the exponential envelope, we define the *time constant* τ_2 appropriate for underdamped 2nd order systems as

$$\tau_2 \equiv \frac{1}{\zeta\omega_n} \tag{9.4.3}$$

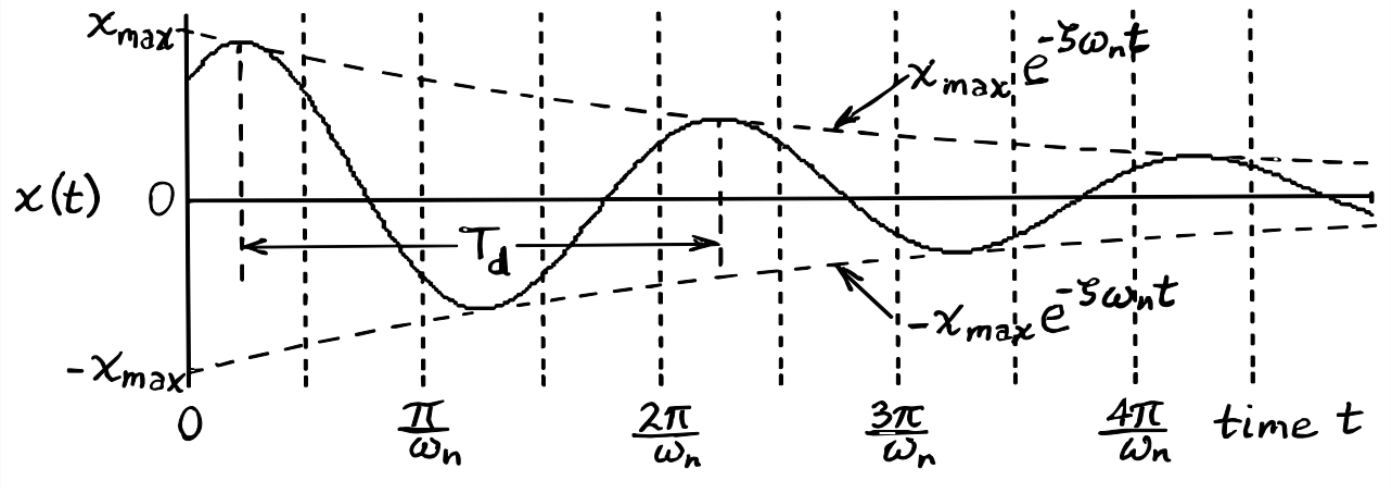


Figure 9.4.1: IC response of a damped 2nd order system, $\zeta = 0.11$

It is often necessary to measure experimentally the dynamic response of mechanical systems, which requires sensors (transducers) that sense a response quantity and convert it into an electrical voltage. The sensors most abundant and relatively inexpensive for mechanical systems are translational accelerometers (see homework Problem 10.12). Velocity sensors and

translation (displacement) sensors are also used, but less commonly. It is appropriate, therefore, that we find from "displacement" Equations 9.4.1 and 9.4.2 the corresponding equations for free-vibration "velocity" and "acceleration." One differentiation of Equation 9.4.2 gives

$$\dot{x}(t) = x_{\max} e^{-\zeta\omega_n t} \left[-\zeta\omega_n \cos\left(\omega_d t + \phi\right) - \omega_d \sin\left(\omega_d t + \phi\right)\right]$$

Next, we combine the two sinusoidal terms with use of the trigonometric identity $\sin A \cos B + \cos A \sin B = \sin(A + B)$ and the equation $\omega_d \equiv \omega_n \sqrt{1 - \zeta^2}$, leading to

$$\dot{x}(t) = -\omega_n x_{\max} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \phi + \sin^{-1}\zeta\right), \text{ for } 0 \leq t \text{ and } 0 \leq \zeta < 1 \tag{9.4.4}$$

One more differentiation of Equation 9.4.4 followed by a similar combination procedure gives

$$\ddot{x}(t) = -\omega_n^2 x_{\max} e^{-\zeta\omega_n t} \cos\left(\omega_d t + \phi + 2\sin^{-1}\zeta\right), \text{ for } 0 \leq t \text{ and } 0 \leq \zeta < 1 \tag{9.4.5}$$

We see that displacement, velocity, and acceleration all have the same damped sinusoidal form. Therefore, experimental measurements of any of the three can be used for identification of system parameters such as ζ and ω_d, a subject addressed in subsequent sections.

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9.5: Calculation of Viscous Damping Ratio

Consider an underdamped 2nd order system in a state of free vibration, *i.e.*, with zero input, $u(t) = 0$. This free vibration can be an initial-condition response or the residual response after input excitation has ceased, *e.g.*, for $t > t_d$ if the input is a pulse. Then the free-decay will have the form of Figure $9.5.1$. On the figure, a reference local extreme value $x(t_0) \equiv x_0$ is annotated (at a crest on Figure $9.5.1$, but it could just as well be at a trough), and subsequent local extreme values (both crests and troughs) also are annotated. We wish to calculate viscous damping ratio ζ from the graph of response. Note also that the exponential envelope is indicated on Figure $9.5.1$ with dashed lines.

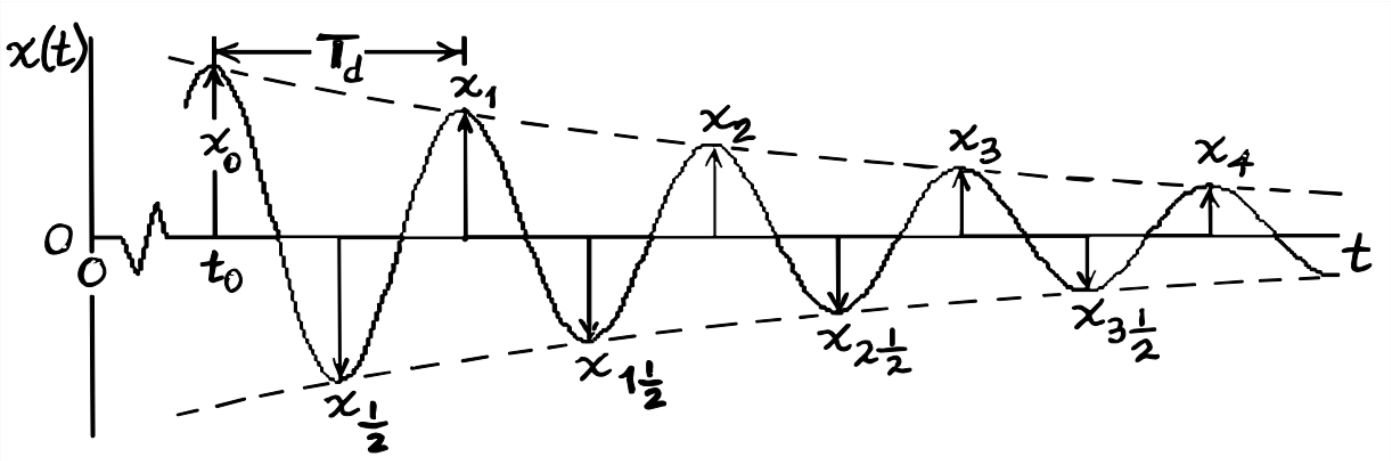


Figure $9.5.1$: Free-vibration response of a damped 2nd order system

We use Equation $9.5.1$ for free-vibration response:

$$x(t) = x_{\max} e^{-\zeta\omega_n t} \cos(\omega_d t + \phi) \tag{9.5.1}$$

From this equation, we form the ratio x_0/x_r of the reference extreme absolute value $|x(t_0)|$ divided by the rth crest or trough absolute value, where r as shown on Figure $9.5.1$ can have either integer or half-integer values. Note that $|\cos(\omega_d t + \phi)| = 1$ at each of the local extreme values, and that the time of the rth extreme value is $t_r = t_0 + rT_d$, where $T_d = 2\pi/\omega_d$ is the damped natural period. So the required ratio is

$$\frac{x_0}{x_r} = \frac{e^{-\zeta\omega_n t_0}}{e^{-\zeta\omega_n (t_0 + rT_d)}} = e^{\zeta\omega_n r T_d} = \exp\left(\zeta\omega_n r \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}\right) = \exp\left(2\pi r \frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{9.5.2}$$

Taking the natural logarithm of Equation $9.5.2$ gives the so-called *logarithmic decrement*:

$$\ln\left(\frac{x_0}{x_r}\right) = 2\pi r \frac{\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\ln(x_0/x_r)}{2\pi r} \equiv \zeta_s \tag{9.5.3}$$

In the last term of Equation $9.5.3$, we define ζ_s as being the approximate for "small" damping ratio $zeta$. It is very common for a system to have positive, but *small damping*. We define damping to be small if $\sqrt{1-\zeta^2} \approx 1$, which simplifies considerably equations such as Equation $9.5.3$. For $\zeta = 0.2$, $\sqrt{1-\zeta^2} = 0.980$, so this is a reasonable upper limit for "smallness." We can find the **exact** equation for ζ by squaring Equation $9.5.3$ and then proceeding algebraically to derive

$$\zeta = \frac{\zeta_s}{\sqrt{1+\zeta_s^2}}, \text{ for } 0 \le \zeta < 1 \tag{9.5.4}$$

It might appear that the preceding derivation requires the values of x_0 and x_r to be at crests and troughs of the response plot, and that these should be zero-to-peak values; but neither of these restrictions is necessary. The values of x_0 and x_r can be at **any** convenient instants along the time history (zeros of the response, as well as extremes), provided that we interpret x_0 and x_r as being the magnitudes of the **exponential envelope** at the chosen instants. Normally, only the free-vibration response plot

is available (from a storage oscilloscope, strip-chart recorder, *etc.*), so we should sketch in the exponential envelope to aid in measuring x_0, x_r and r. Moreover, rather than measuring zero-to-peak values, it is more accurate and easier to measure x_0 and x_r as peak-to-peak values, from the lower exponential boundary to the upper exponential boundary.

Let us summarize the procedure for measuring/calculating ζ from a plot of free-vibration response. First, sketch in the exponential envelope. Next, choose time instants along the graph at which you can measure with reasonable accuracy the number of periods r (usually an integer, half-integer, or quarter-integer) and the magnitudes x_0 and x_r between the exponential boundaries. Next, substitute the measured values of r, x_0, and x_r into Equation 9.5.3 and calculate ζ_s. If this ζ_s is ≤ 0.2, then $\zeta \approx \zeta_s$ with sufficient engineering accuracy. However, if $0.2 < \zeta_s < 1$, then calculate ζ more accurately from Equation 9.5.4.

There is a simplified version of Equation 9.5.3 that is often used for quick calculation of small ζ. If possible, we find the reference magnitude and the rth magnitude such that $x_0/x_r = 2$, and we label the number of periods as $r_{1/2}$. Hence, $\ln(x_0/x_r)/2\pi \equiv 0.110$, which leads us from Equation 9.5.3 to the *half-amplitude formula* for small ζ:

$$\zeta_s = \frac{0.110}{r_{1/2}} \approx \zeta \tag{9.5.5}$$

Finally, the preceding derivation was based upon Equation 9.5.1 for "displacement" response $x(t)$. However, the formulas derived for ζ are equally valid if the measurements are made from graphs of "velocity" $\dot{x}(t)$ or "acceleration" $\ddot{x}(t)$. This is so because, from Equations 9.4.5 and 9.4.6, the derivatives of $x(t)$ have the same exponentially-bounded sinusoidal form as $x(t)$ itself.

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9.6: Step Response of Underdamped Second Order Systems

For step response, we set the ICs to zero, and we define the input to be a step function at time $t = 0$, with step magnitude U: $u(t) = UH(t)$. The appropriate form of the general solution to use is Equation 9.3.8, which becomes [with $H(t - \tau) = 1$ for \(\tau

$$x(t) = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau \times u(t-\tau)d\tau = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau \times UH(t-\tau)d\tau$$

$$= U\frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau d\tau$$

This integral can be evaluated by hand (*e.g.*, using integration by parts), but the process is tedious. The following evaluation was completed with use of a table of integrals:

$$x(t) = U\frac{\omega_n^2}{\omega_d}\frac{1}{(-\zeta\omega_n)^2+\omega_d^2}\left[e^{-\zeta\omega_n\tau}\left(-\zeta\omega_n\sin\omega_d\tau - \omega_d\cos\omega_d\tau\right)\right]_{\tau=0}^{\tau=t}$$

$$= U\frac{1}{\omega_d}\left[e^{-\zeta\omega_n t}\left(-\zeta\omega_n\sin\omega_d t - \omega_d\cos\omega_d t\right) - 1\left(-\omega_d\right)\right]$$

$$x(t) = U\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right],\ \text{for } 0 \le t \text{ and } 0 \le \zeta < 1 \tag{9.6.1}$$

Note that the coefficient of $\sin\omega_d t$ in Equation 9.6.1 is dependent only on the damping ratio:

$$\frac{\zeta\omega_n}{\omega_d} = \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{\zeta}{\sqrt{1-\zeta^2}} = \zeta_s$$

Step response Equation 9.6.1 for small damping ratio $\zeta = 0.11$ is plotted over a few cycles of response on Figure 9.6.1. Relative to the pseudo-static response, $x_{ps} = U$, the actual step response of a damped system initially overshoots, then undershoots, then overshoots again, then undershoots again, *etc.*, *etc.* But damping dissipates the energy of vibration, causing the response eventually (not shown on Figure 9.6.1) to settle statically at $\lim_{t\to\infty} x(t) = x_{ps} = U$. Several step-response characteristics (called *specifications*, or *specs* in engineering jargon) of a system can be quantified and often are of great interest in practice. For example, the *rise time* is the time required for the response first to reach U, which on Figure 9.6.1 is just a bit longer than $\frac{1}{2}\pi/\omega_n$. However, before studying those characteristics in more detail, it is appropriate that we first consider impulse response.

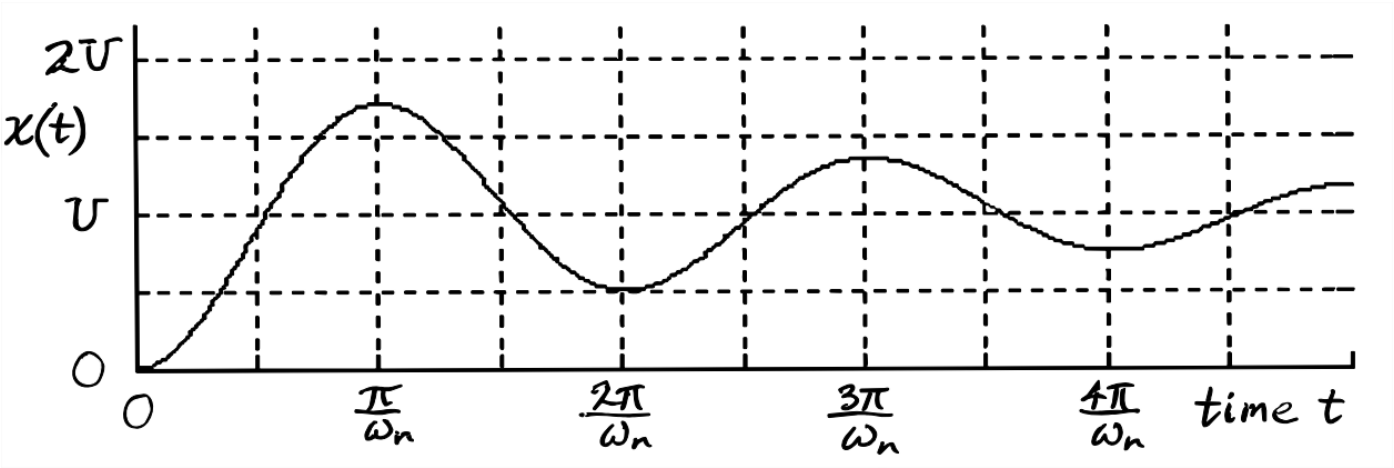


Figure 9.6.1: Step response of a damped 2nd order system, $\zeta = 0.11$

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9.7: Ideal Impulse Response of Underdamped Second Order Systems

For impulse response, we set the ICs to zero, and we define the input to be an ideal impulse at time $t = 0$, with impulse magnitude I_U: $u(t) = I_U\delta(t)$. The more appropriate form of the general solution to use is Equation 9.3.9, which becomes

$$x(t) = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\times u(\tau)d\tau = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\times I_U\delta(\tau)d\tau$$

Using the integration property of $\delta(\tau)$, Equation 8.4.4, we find the relatively simple result:

\[x(t)=I_{U} \frac{\omega_{n}^{2}}{\omega_{d}} e^{-\zeta \omega_{n} t} \sin \omega_{d} t, \text { for } 0

Ideal impulse response Equation ??? for small damping ratio $\zeta = 0.11$ is plotted over a few cycles of response on Figure 9.7.1. Observe that this ideal response violates the specified initial condition $\dot{x}(0) = 0$; thus, the solution is defective in this respect, as is discussed in Section 8.7. Figure 9.7.1 shows the exponential envelope; note, in particular, the values at $t = 0$ of the exponential envelope, $\pm I_U\omega_n^2/\omega_d$. This magnitude is easily measurable from a graph, after we have sketched in the exponential envelope, if necessary, and it is essential in the process of estimating mechanical system parameters from an experimental response to a short force pulse (Section 9.9).

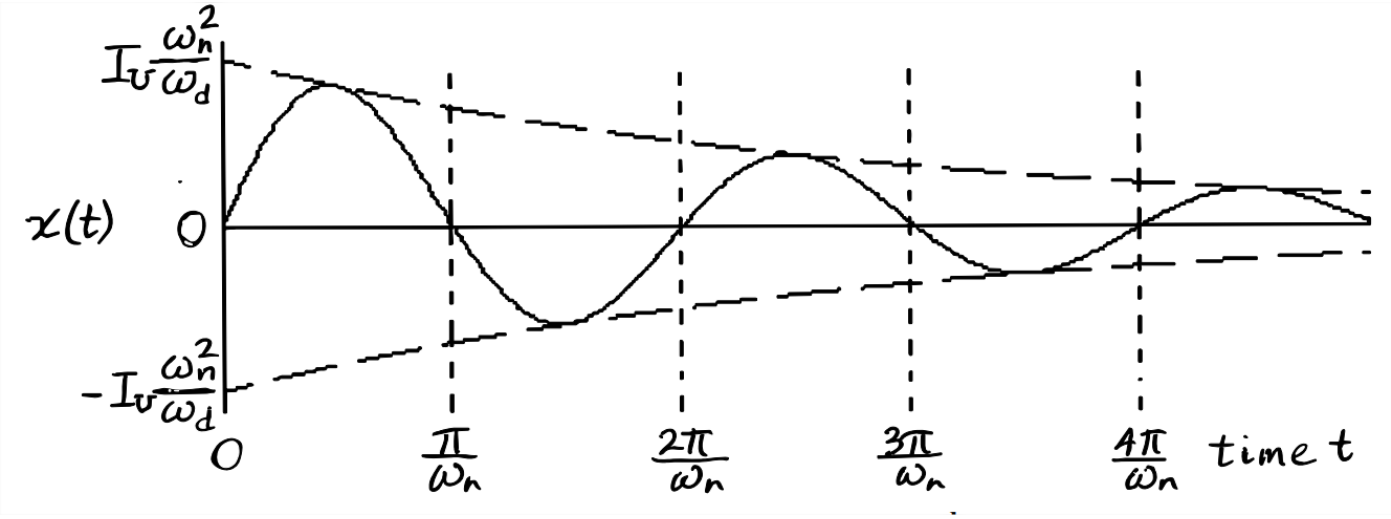


Figure 9.7.1: Ideal impulse response of a damped 2nd order system, $\zeta = 0.11$

From Equation Equation ???, we find the unit-impulse-response function (IRF, as defined in Section 8.7) for underdamped 2nd order systems:

\[\left.h(t) \equiv x(t)\right|_{I_{U}=1}=\frac{\omega_{n}^{2}}{\omega_{d}} e^{-\zeta \omega_{n} t} \sin \omega_{d} t, \text { for } 0

Further, from Equation 8.7.2, we find the Duhamel integral giving general response to input $u(t)$, with zero ICs, for underdamped 2nd order systems:

$$x(t)|_{ICs=0} = \int_{\tau=0}^{\tau=t} u(\tau)h(t-\tau)d\tau = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} u(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)d\tau \tag{9.7.1}$$

Duhamel integral Equation 9.7.1 is identical to convolution-integral response solution Equation 9.3.9 with zero ICs.

Finally, it is worthy of mention that convolution sum Equation 8.11.4 for approximate numerical forced response applies just as well to damped 2nd order systems as to 1st order and undamped 2nd order systems. Therefore, to calculate approximate forced response of an underdamped 2nd order system, we would apply exactly the same procedure described in Convolution-sum

Example 2 of Section 8.11, but instead of calculating in Equation 8.11.4 the IRF $h(t) = \omega_n \sin \omega_n t$ for an undamped system, we would calculate Equation ???.

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9.8: Step-Response Specifications for Underdamped Systems

Engineering systems are often designed so that the time history of an output quantity will mimic as closely as possible the time history of the input quantity. An example, with reference to Section 3.5, is the aileron-induced rolling of an airplane, for which the original input is the control wheel angle set manually ("commanded") by the pilot, and the ultimate output is the airplane roll rate. Another example is the modern automobile; we usually describe a car as being "responsive" if the steering (or the acceleration, or the braking) mimics quickly and precisely the driver's commands set by hand or foot.

Step response of a system is often used for measuring and quantifying dynamic "responsiveness." Ideally, step response would mimic exactly the step input, but system characteristics such as inertia and damping prevent such instantaneous response. The degree to which step response fails to mimic step input is quantified in the following *four step-response specifications*: rise time, t_r; peak time, t_p, maximum overshoot ratio, $\bar{x}_p$, and settling time, t_s. These step-response quantities are illustrated on Figure 9.6 on the next page. They are called "specifications" or "specs" because it is common in the beginning of a project to specify them as design targets; later, these step-response quantities are measured experimentally on prototype and/or production test articles.

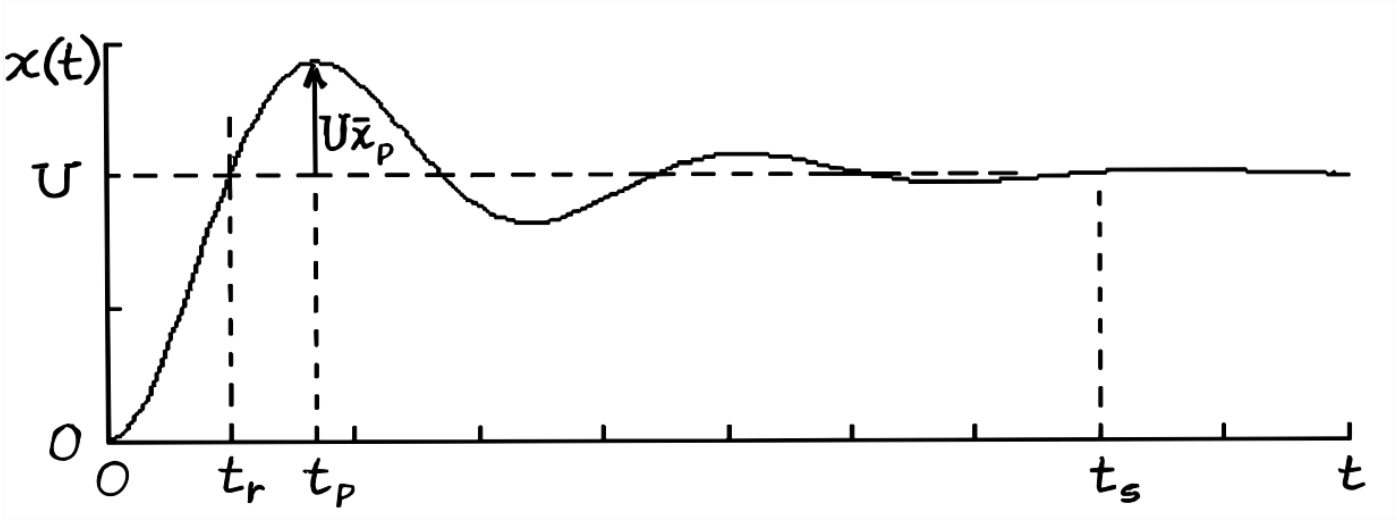


Figure 9.8.1**: Step-response specifications of an underdamped system**

For underdamped 2nd order systems, we can apply step-response solution Equation 9.6.5 and impulse-response solution Equation 9.7.2 to derive specific equations for the step-response specifications:

- From Figure 9.8.1, we evaluate Equation 9.6.5 at $t = t_r$, the **first** time when $x(t) = U$:

$$x\left(t_r\right) = U = U\left[1 - e^{-\zeta\omega_n t_r}\left(\cos\omega_d t_r + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_r\right)\right] \Rightarrow \cos\omega_d t_r + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_r = 0$$

$$\Rightarrow \tan\omega_d t_r = -\frac{\omega_d}{\zeta\omega_n} = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{1}{\zeta_s}$$

  This last equation shows that as $\zeta \to 0$ from positive values, then $\tan\omega_d t_r \to -\infty$ from negative values; therefore, $\omega_d t_r \to \frac{1}{2}\pi \times n$ from higher values, where $n = 1, 5, 9, ...$; and the **first** time when $x(t) = U$ corresponds to $n = 1$: $\omega_d t_r \to \frac{1}{2}\pi$ from higher values. We can see this in Figure 9.6.1, where t_r is just a bit after $\frac{1}{2}\pi/\omega_n$ (keep in mind that $\omega_d \leq \omega_n$). This leads to the conclusion that the rise time is given by

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right) \tag{9.8.1}$$

  In Equation 9.8.1, we take the value of the four-quadrant inverse tangent that is between $\frac{1}{2}\pi$ and π, as shown on the drawing below.

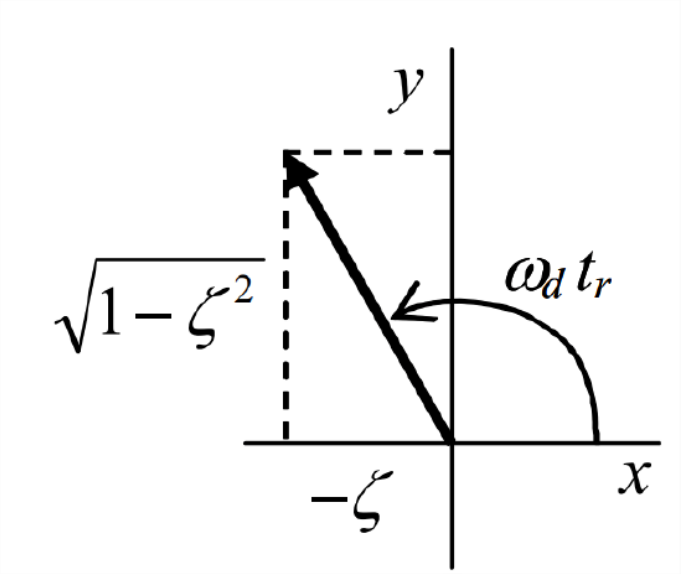


Figure 9.8.2

- From Figure 9.8.1, t_p is the time at which $x(t)$ is maximum and the first time after $t = 0$ that $t = 0$. So we need to differentiate Equation 9.6.5, set it to zero, and solve for t_p from the resulting equation. From the appearance of Equation 9.6.5, though, that differentiation will be long and tedious. But that drudgery will not be necessary, because a fundamental relationship derived in Section 8.7 will come to our rescue. We identify the unit-step response as Equation 9.6.5 with step magnitude $U = 1$:

$$x_H(t) = 1 \times \left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right] = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right) \tag{9.8.2}$$

Clearly, t_p is independent of step magnitude U. Now, Equation 8.6.6 gives the derivative of Equation 9.8.2 as $dx_H/dt = h(t)$, where the IRF $h(t)$ in this case is Equation 9.7.3. Therefore, the equation leading to that we seek is

$$\frac{dx_H}{dt} = h(t) = \frac{\omega_n^2}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t = 0 \text{ at } t = t_P \tag{9.8.3}$$

The required solution of Equation ??? is the lowest positive value of t that satisfies $\sin\omega_d t = 0$, which is

$$t_p = \frac{\pi}{\omega_d} \tag{9.8.4}$$

The graphical equivalent of this mathematical derivation of Equation 9.8.4 is evident in an examination of Figures 9.6.1 and 9.7.1, where we can see that both the peak of the step response and the first zero of the ideal impulse response occur at an instant t just a bit after π/ω_n (recall that $\omega_d \le \omega_n$).

- From Figure 9.8.1, Equation 9.6.5, and Equation 9.8.4, the maximum overshoot ratio is

$$\bar{x}_p = \frac{x(t_p) - U}{U} = -e^{-\zeta\omega_n t_p}\left(\cos\omega_d t_p + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t_p\right) = -\exp\left(-\zeta\omega_n\frac{\pi}{\omega_d}\right)\left(\cos\pi + \frac{\zeta\omega_n}{\omega_d}\sin\pi\right)$$

$$\Rightarrow \quad \bar{x}_p = \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi\right) \le 1, \text{ valid for } 0 \le \zeta < 1 \tag{9.8.5}$$

So the maximum overshoot ratio is a function only of viscous damping ratio ζ. Conversely, ζ can be determined from a measurement of $\bar{x}_p$ by taking the natural logarithm of Equation 9.8.5:

$$\ln\bar{x}_p = -\frac{\zeta}{\sqrt{1-\zeta^2}}\pi \Rightarrow -\frac{\ln\bar{x}_p}{\pi} = \frac{\zeta}{\sqrt{1-\zeta^2}} \equiv \zeta_s, \text{ valid for } 0 < \bar{x}_p \le 1 \tag{9.8.6}$$

Note the similarity of Equation 9.8.6 to Equation 9.5.3 for the logarithmic decrement. Therefore, we arrive again at Equation 9.5.4, the exact equation giving ζ for any overshoot in the range $0 < \bar{x}_p \le 1$: $\zeta = \zeta_s/\sqrt{1+\zeta_s^2}$, for $0 \le \zeta < 1$, now with $\zeta_s = -\ln\bar{x}_p/\pi$.

If damping is small such that $\sqrt{1-\zeta^2} \approx 1$, *i.e.*, $0 \le \zeta \le 0.2$, then Equations 9.8.5 and 9.8.6 are approximated as:

$$\bar{x}_p \approx e^{-\zeta\pi} \quad \text{and} \quad \zeta \approx -\frac{\ln\bar{x}_p}{\pi} \quad \text{valid for } 0 \le \zeta \le 0.2 \text{ and } 0.534 \le \bar{x}_p \le 1 \tag{9.8.7}$$

- This specification is defined as the time required for response $x(t)$ to settle to within ± 2 of the final steady-state (pseudo-static) value, U. For this underdamped 2nd order system, the time constant of the exponential envelope is Equation 9.4.3, $\tau_2 \equiv 1/\zeta\omega_n$. From Chapter 3, we have $1 - e^{-4} = 0.982$ (see Figure 3.4.2), so the settling time is defined as

$$t_s = 4\tau_2 \equiv \frac{4}{\zeta\omega_n} \tag{9.8.8}$$

It is appropriate here to evaluate the variation in step response of a standard 2nd order system as damping varies. Figure $9.8.3$ on the next page displays 2nd order step responses for a range of viscous damping ratios ζ. [The response curves of Figure $9.8.3$ are calculated as follows: for underdamping, $0 \leq \zeta < 1$, from Equation 9.6.5; for critical damping, $\zeta = 1$, from the result of homework Problem 9.11; for overdamping, $\zeta = 1.2$, from the result of homework Problem 9.16. The time reference is undamped natural period $T_n = 2\pi/\omega_n$.] As engineers designing a system, we might wish to design into the system a quantity of damping ζ that makes the time history of an output quantity mimic as closely as possible the time history of the input quantity. This means, in the context of step response, we would want both to make rise time as fast as possible and to minimize overshoot. However, Figure $9.8.3$ shows that we cannot simultaneously do both for a standard 2nd order system: rise time is fastest for small ζ, but overshoot is minimized or eliminated with larger ζ. Therefore, we would have to compromise and select a value of ζ that produces practically acceptable values of both rise time and overshoot, even though neither response parameter would be the best possible. Observe from Figure $9.8.3$ that overshoot exists only for underdamping.

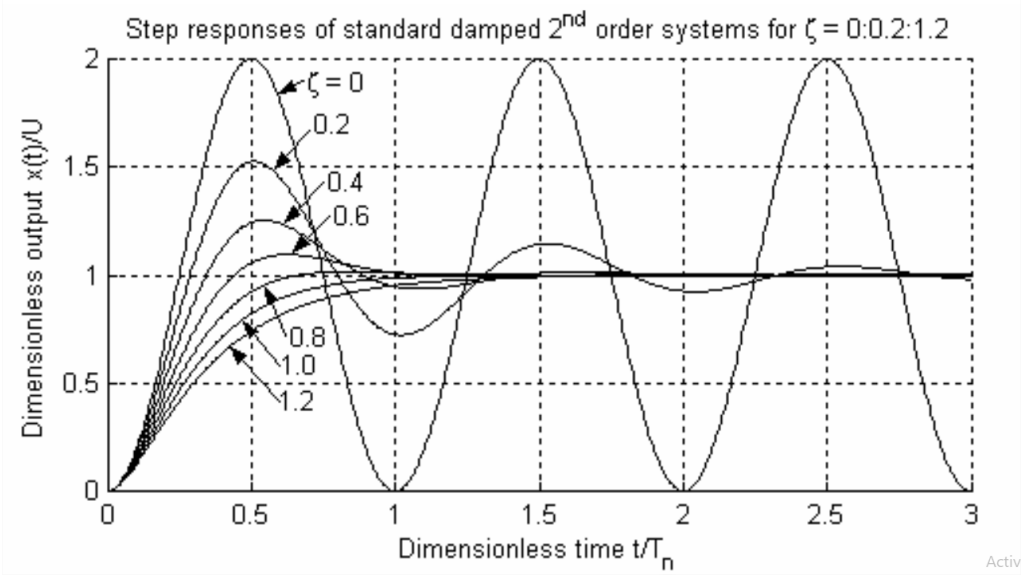


Figure $9.8.3$: Step responses of standard 2nd order systems as viscous damping varies

The transient response of engineered control systems is very important in practice, so there is more analysis and discussion of subjects such as rise time and overshoot later in the book, beginning in Chapter 14.

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9.9: Identification of a Mass-Damper-Spring System

Suppose we have a mechanical system that is known to be an m-c-k system (or a close enough approximation thereto, for engineering purposes), such as that of Figure 9.1.1, and suppose we need to estimate from experimental measurements the system parameters: mass m, effective viscous damping constant c, and stiffness constant k. This is a form of the process known generally as *system identification* (ID). There are many methods of system ID using both transient response and frequency response. In this section, we illustrate one common method based upon pulse response.

The theoretical basis for system ID by pulse testing of a mechanical m-c-k system is the ideal impulse response given by Equation 9.7.2, with use of Equation 8.4.6, $I_U = I_F/k$, and the equation for natural frequency, $\omega_n^2 = k/m$:

$$x(t) = \frac{I_F}{k}\frac{k/m}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t = \frac{I_F}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{9.9.1}$$

Similarly, the values at $t = 0$ of the upper and lower boundaries of the exponential envelope (for guidance, see Figure 9.7.1) are

$$\pm I_U\frac{\omega_n^2}{\omega_d} = \pm\frac{I_F}{m\omega_d} \tag{9.9.2}$$

The practical implementation of these equations is based upon using them to approximate the real response from a **short** force pulse, as is discussed extensively in Section 8.6.

The experimental procedure is as follows. An engineer or technician strikes the mass lightly but sharply with a specially designed hammer. A force sensor mounted in the head of the hammer measures the pulse. A displacement sensor measures motion of the mass due to the force pulse. **For accuracy of the system ID, it is essential that the pulse duration t_d of the hammer strike be very short relative to a quarter of the system natural period:** $t_d << 1/4T_n$; achieving this might require some experimental iteration, testing hammer contact tips of different degrees of hardness. The time history $t_f(x)$ of force input to the mass is recorded; this force pulse might be somewhat irregular, such as Figure 8.5, or it might appear to have a more regular form, such as a half-sine. The time history of displacement response, $x(t)$, also is recorded; with a properly short pulse duration, response $x(t)$ should look very much like the damped sinusoid of Fig. 9.7.1, ideal impulse response. The steps of the subsequent calculation algorithm are:

1. Calculate from measurements on the $x(t)$ graph the **damped** natural frequency f_d (Hz) and the viscous damping ratio ζ. In order to obtain reasonably accurate values of f_d and ζ, be sure to average over as many cycles as possible of the damped sinusoid. To aid in the calculation of ζ, first sketch in the exponential envelope. Next, use the values of f_d and ζ to calculate the two circular natural frequencies in rad/s: $\omega_d = 2\pi f_d$ and $\omega_n = \omega_d/\sqrt{1-\zeta^2}$. For small ζ, these two frequencies will be essentially identical.
2. From the $f_x(t)$ graph, use graphical or approximate theoretical integration to calculate the actual force impulse I_F. From the $x(t)$ graph, find the value at $t = 0$ of the upper and lower boundaries of the exponential envelope, which are approximately $\pm I_F/m\omega_d$ from Equation 9.9.2. It is important in this step to be very careful with the units of these quantities measured from experimental data. Now, you have the data required to calculate the mass from the identity
$$m = \left[\omega_d \times \frac{1}{I_F} \times \frac{I_F}{m\omega_d}\right]^{-1} \tag{9.9.3}$$
3. Finally, calculate the stiffness constant using $k = m\omega_n^2$, and calculate the effective viscous damping constant from Equation 9.1.6, $c = 2\zeta m\omega_n = 2\zeta\sqrt{mk}$.

It is always essential in engineering practice to check your calculations as much as is practical. After you have calculated m-c-k from the procedure described above, you can check the validity of your values by using MATLAB (or some similar calculation software) to graph the ideal impulse response of your calculated system, Equation 9.9.1; then compare the calculated graph with the recorded experimental response. If the two graphs are very similar, then your system ID is probably correct, **provided that** you calculated I_F correctly. However, suppose that you use an incorrect value of I_F, then calculate wrong values of m-c-k based upon this wrong I_F; if you then graph the ideal impulse response, using these wrong values in Equation 9.9.1, the result will look very much like the recorded experimental response, giving you a false indication that your system ID is correct. Therefore, be sure to calculate correctly the force impulse I_F.

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9.10: Deriving Response Equations for Overdamped Second Order Systems

For $\zeta > 1$, we can consider the damped natural frequency to be an imaginary number:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = j\omega_n\sqrt{\zeta^2-1} \equiv j\mu_d \quad \text{where} \quad \mu_d \equiv \omega_n\sqrt{\zeta^2-1} \text{ is real} \tag{9.10.1}$$

The general method of deriving transient response equations for the overdamped case is to substitute Equation 9.10.1 into the Laplace transform Equation 9.3.5, and then proceed to invert the resulting equation, leading to general expressions that include IC response terms and convolution integrals, analogous to Equations 9.3.8 and 9.3.9.

There is an easier method for finding overdamped-system response equations if the comparable underdamped-system equations have already been derived. The method is to use Equation 9.10.1 in order to convert trigonometric terms of the $\zeta < 1$ equations into hyperbolic terms for the $\zeta > 1$ equations. From homework Problem 2.13, we have the following conversions valid for $\zeta > 1$:

$$\cos\omega_d t = \cosh\mu_d t \quad \text{and} \quad \frac{\sin\omega_d t}{\omega_d} = \frac{\sinh\mu_d t}{\mu_d} \tag{9.10.2}$$

An example of applying Equations 9.10.1 and 9.10.2 is conversion of IC response Equation 9.4.1 from the underdamped ($\zeta < 1$) form into the overdamped ($\zeta > 1$) form:

$$x(t) = e^{-\zeta\omega_n t}\left[x_0\cos\omega_d t + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)\sin\omega_d t\right] \text{ for } 0 \le \zeta < 1$$

$$\Rightarrow \quad x(t) = e^{-\zeta\omega_n t}\left[x_0\cosh\mu_d t + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\mu_d}\right)\sinh\mu_d t\right] \text{ for } \zeta > 1 \tag{9.10.3}$$

in which $\mu_d = \omega_n\sqrt{\zeta^2-1}$. This IC-response equation is valid for $0 \le t$ and $\zeta > 1$. The hyperbolic functions are defined in terms of exponential functions as

$$\cosh\mu_d t = \frac{e^{\mu_d t} + e^{-\mu_d t}}{2} \quad \text{and} \quad \sinh\mu_d t = \frac{e^{\mu_d t} - e^{-\mu_d t}}{2}$$

Therefore, IC-response Equation 9.10.3 can be written a bit more clearly as

$$x(t) = \frac{1}{2}e^{-\zeta\omega_n t}\left[e^{\mu_d t}\left(x_0 + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\mu_d}\right) + e^{-\mu_d t}\left(x_0 - \frac{\dot{x}_0 + \zeta\omega_n x_0}{\mu_d}\right)\right] \tag{9.10.4}$$

All terms in Equation 9.10.4 are exponentially damped. Even the first term within the square brackets decays away exponentially because $-\zeta\omega_n + \mu_d = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} = -(\zeta - \sqrt{\zeta^2-1})\omega_n < 0$ for $\zeta > 1$.

Example 9.10.1: RC band-pass filter, an overdamped 2nd order system

We re-visit Section 5.4, where the input voltage to the RC band-pass filter is defined as $e_i(t)$, the mid-circuit voltage between low-pass and high-pass stages is $e_m(t)$, and the output voltage is $e_o(t)$; also, the 1st order time constants of the low-pass and high-pass stages are (τ_L and τ_H, respectively, defined in terms of resistance and capacitance values in the circuit. The two coupled 1st order ODEs derived in Section 5.4 are

$$\text{for the low-pass filter stage, } \tau_L\dot{e}_m + e_m = e_i \tag{9.10.5}$$

$$\text{for the high-pass filter stage, } \tau_H\dot{e}_o + e_o = \tau_H\dot{e}_m \tag{9.10.6}$$

We combine these coupled 1st order ODEs into a single 2nd order ODE with the following operations: differentiate Equations 9.10.5 and 9.10.6; in the differentiated Equation 9.10.5, replace $\dot{e}_m$ using the original Equation 9.10.6, and replace $\ddot{e}_m$ using the differentiated Equation 9.10.6; rearrange and collect terms to find the ODE relating output $e_o(t)$ to input $e_i(t)$:

$$\tau_L \tau_H \ddot{e}_o + (\tau_L + \tau_H)\, \dot{e}_o + e_o = \tau_H \dot{e}_i \tag{9.10.7}$$

Due to the right-hand-side dynamics (the presence of $\dot{e}_i$ rather than just e_i), we cannot cast this entire equation into the standard form Equation 9.2.2; see homework Problem 9.15 for a modified standard form of ODE and for convolution-integral response solutions. However, the order of the circuit is determined only by the left-hand-side terms, and it is clearly 2nd order, with natural frequency and damping ratio defined as

$$\omega_n = \frac{1}{\sqrt{\tau_L \tau_H}} \quad \text{and} \quad \zeta = \frac{1}{2\omega_n} \frac{\tau_L + \tau_H}{\tau_L \tau_H} = \frac{1}{2} \frac{\tau_L + \tau_H}{\sqrt{\tau_L \tau_H}}$$

Suppose, for example, that a particular circuit has the high-pass and low-pass break frequencies f_H = 10 Hz and f_L = 500 Hz, respectively. Then the time constants are $\tau_H = 1/\omega_H = 1/(2\pi f_H) = 1.592\text{e}-2$ s, and $\tau_L = 1/\omega_L = 1/(2\pi f_L) = 3.183\text{e}-4$ s, and the undamped natural frequency is $\omega_n = 4.443\text{e}2$ rad/s ($f_n = \omega_n/2\pi = 70.71$ Hz). Finally, the damping ratio is $\zeta = 3.606$, which means that this is a strongly overdamped system.

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9.11: Chapter 9 Homework

1. In the system drawn below, mass m is attached to a movable support by viscous dashpot c_1 and spring k, and the mass also slides on a lubricated surface, which causes viscous damping with constant c_2. There are **two** independent **input** quantities: The output quantity is displacement $x(t)$ of the mass. Sketch and label clearly appropriate free-body diagrams, then use your FBD to derive the differential equation of motion for $x(t)$ in terms of constants m, c_1, c_2 and k, and variable inputs $x_i(t)$ and $f_x(t)$. Your ODE of motion should include some non-standard right-handside dynamics.

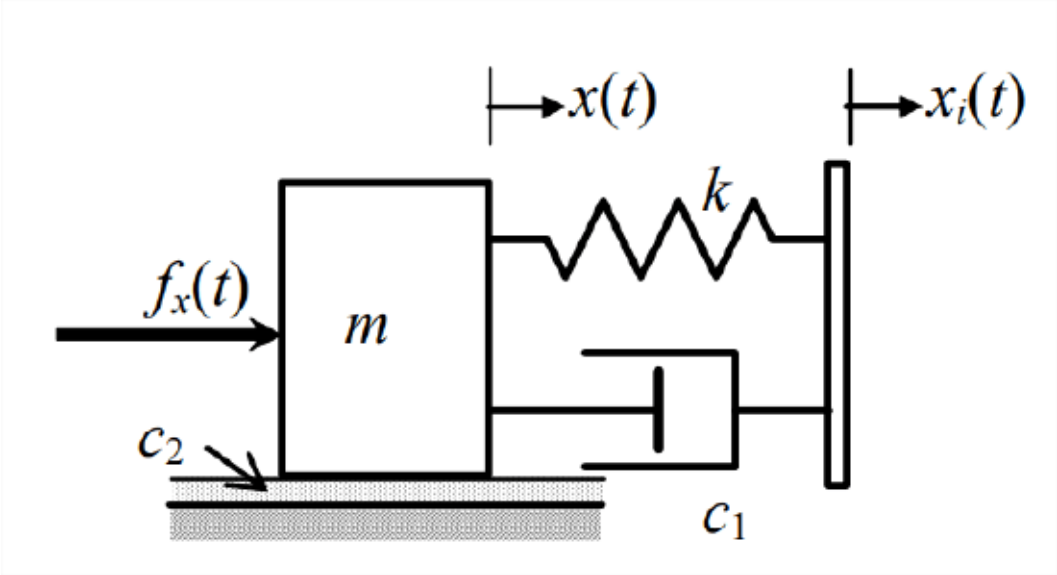


Figure 9.11.1

    1. support displacement $x_i(t)$ shown (displacement control, *e.g.*, by a cam), which is often called base excitation[1], and
    2. force $f_x(t)$ shown.
2. Given, a standard m-c-k system, Figure 9.1.1, with the following parameters: weight of the mass $W = 0.386$ lb, so that $m = \frac{W}{g} = \frac{0.386\text{lb}}{386\text{inch/s}^2} = 0.001\frac{\text{lb}-\text{s}^2}{\text{inch}}$; damper viscous constant $c = 1.1 \times 10^{-3}$ lb/inch/s; spring stiffness constant $k = 3.025$ lb/inch.
    1. Calculate the following quantities, all to at least 3 significant figures: ω_n in rad/s, f_n in Hz, ζ, ω_d in rad/s, f_d in Hz, and $T_d \equiv 1/f_d$. You may calculate these quantities in MATLAB, if you wish, using the command `format short e`.
    2. Given initial conditions $x(0) = 0.5$ inch and $\dot{x}(0) = 35$ inch/s, calculate and graph with MATLAB the free-vibration response due to the ICs over about 20 complete decaying cycles. Since you are multiplying sinusoidal time functions by an exponential time function, it is necessary that you use array multiplication (.*) in MATLAB. Draw grids on your graph and add an appropriate title and appropriate axis labels.
    3. Sketch by hand the exponential envelope on your graph, then use the grids and/or a graduated straight edge to measure amplitudes of local peaks of the decaying sinusoid. In particular, find (approximately) the quantity $r_{1/2} \equiv$ the number of decaying cycles to decay to half amplitude (not necessarily an integer). Now calculate the dimensionless ratio $0.110/r_{1/2}$; the value that you calculate should match closely with your previously calculated value of ζ.
3. The free-vibration response to initial conditions of a standard damped 2nd order system is given by $x(t) = 2.24e^{-0.300t}\cos(12.00t - 1.11)$ mm, where t is in seconds. Calculate
    1. the damped natural frequency f_d in Hz,
    2. the damping time constant τ_2 in seconds, and
    3. the viscous damping ratio ζ (use the small-ζ approximation). Show your calculations.
4. A simple type of vibration testing often used for underdamped mechanical systems, such as the m-c-k system of Figure 9.1.1, is called twang testing (named after the sharp, ringing sound made by plucking a string of a musical instrument; Section 7.6 demonstrates twang testing of a flexible beam): the mass is displaced from its static equilibrium position by amount x_0 (with zero initial velocity, $\dot{x}_0 = 0$), then it is released. From the record of subsequent transient response, we can calculate directly the damped natural frequency f_d and the viscous damping ratio ζ. If we know initially either the mass m or the stiffness constant k, then we can use the measured f_d and ζ to calculate the other constant and the effective viscous damping constant c. Use Equation 9.4.2 to show that twang test response is

$$x(t) = x_{\max}e^{-\zeta\omega_n t}\cos\left(\omega_d t + \phi\right), \text{ for } 0 \le t \text{ and } 0 \le \zeta < 1$$

where $x_{\max} = x_0/\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}(-\zeta/\sqrt{1-\zeta^2})$ For a very lightly damped system, the preferred simple approximation of this twang response is

$$x(t) \approx x_0 e^{-\zeta\omega_n t}\cos(\omega_n t)\,, \text{ with } \phi \approx 0$$

How small must ζ be so that phase lag ϕ is so small (say, $|\phi| \leq 5°$) that it can not be distinguished from zero phase lag, $\phi \approx 0$? [NOTE: the measurement resolution on a typical analog graphical record of transient-decay vibration response (for example, from a small oscilloscope screen) is so limited that the resolution of calculated phase is probably only around $\pm 5°$.

5. A laboratory mass-damper-spring system was set into motion and then allowed to vibrate freely. The time-history acceleration on the next page was sensed by a lightweight accelerometer attached to the mass. The graph was copied from the screen of a digital oscilloscope. Calculate from measurements on a photocopy of the graph the **damped** natural frequency f_d (Hz) = $\omega_d/2\pi$ and the viscous damping ratio ζ. In order to obtain reasonably accurate values for f_d and ζ, be sure to average over many cycles of the damped sinusoid. To calculate ζ, first sketch in the exponential envelope. Also, note that the zero level of the acceleration signal is offset from the center horizontal grid line; therefore, peak-to-peak amplitude measurements are more appropriate than zero-to-peak measurements, as well as more accurate. Annotate your measurements on the graph (submit the annotated graph), and show your calculations clearly. From your values for f_d and ζ , calculate the **undamped** natural frequency f_n (Hz). For a lightly damped system such as this one, is the difference between f_d and f_n numerically significant, given the limited precision of the measured quantities?

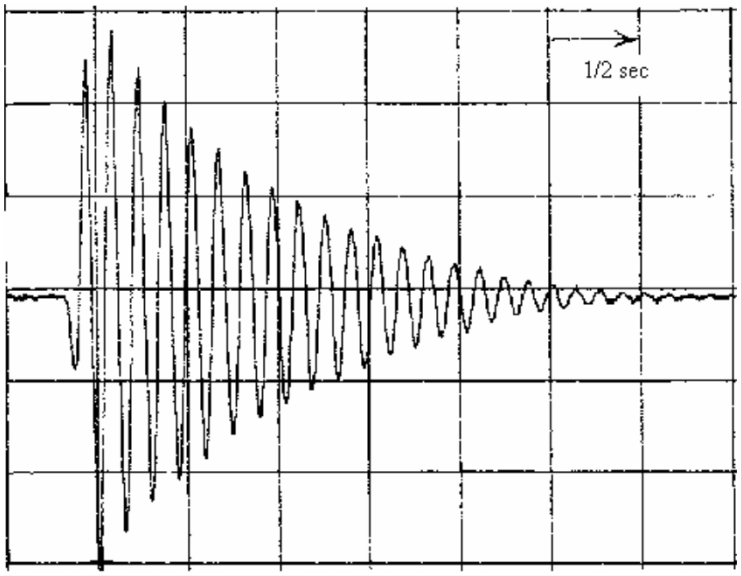


Figure 9.11.2

6. A standard m-c-k system is initially at rest. Then an engineer strikes the mass with a **soft**-tipped hammer that produces a **relatively slow** half-sine force pulse of amplitude 25 lb and duration 0.1 s [$f_x(t) = 25\text{lb} \times \sin(10\pi t)$ for $0 \leq t \leq 0.1$ s]. Both the force pulse $f_x(t)$ and the displacement response $x(t)$ of the mass are recorded on the graph below. To support the calculations assigned (next page), annotate your measurements on a photocopy of the graph (hand in the annotated graph), and show your calculations clearly.
    1. Calculate from the $f_x(t)$ equation the impulse produced by the force. However, you should **not** use this I_F in Equation 9.9.3 to determine m. Explain why not.
    2. Calculate from the graph, with as much accuracy as the data permits, the damped natural frequency f_d (Hz), and the viscous damping ratio ζ of the system.
    3. The system's mass m is known to be 0.0652 lb-s^2/inch. Calculate stiffness constant k and damping constant c.
7. A large mass-damper-spring (m-c-k) system is initially at rest. Then a machine imposes upon the mass a **slowly** varying force in the form of a sawtooth pulse of amplitude 800 N and pulse duration 1.5 s, $f_x(t) = \frac{800\text{N}}{1.5\text{s}} \times t$ for $0 \leq t \leq 1.5$ s. Both the force pulse $f_x(t)$ and the displacement response $x(t)$ of the mass are recorded on the graph below. The unusual displacement scale " (mm * 100)" is employed in order to accommodate the ranges of both $f_x(t)$ and $x(t)$ on the same graph; this scale simply means that the range of displacements on the graph is $-6.00\text{mm} \leq x(t) \leq +8.00\text{mm}$. To support the calculations assigned (next page), annotate your measurements on a photocopy of the graph (hand in the annotated graph), and show your calculations clearly.

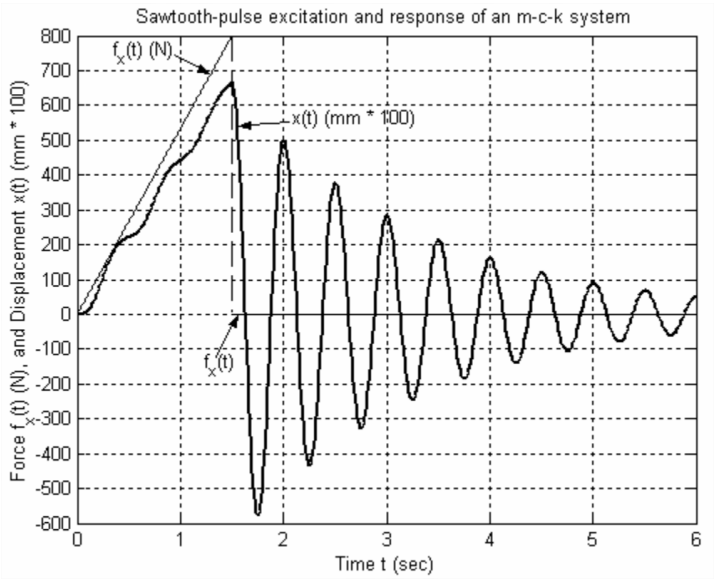


Figure 9.11.3

1. Calculate the total impulse I_F produced by the force. However, you should **not** use this I_F in Equation 9.9.3 to determine m. Explain why not.
2. Calculate from the graph, with as much accuracy as the data permits, the damped natural frequency f_d (Hz), and the viscous damping ratio ζ of the system.
3. The system's stiffness constant k is known from static testing to be 120.0 kN/m. Calculate mass m **and** damping constant c.
4. Find the algebraic/numerical equation for the **pseudo-static** response $x_{ps}(t)$ while the pulse is active ($0 \le t \le 1.5$ s), and compare your $x_{ps}(t)$ with the actual response $x(t)$ on the graph.

8. Consider the mass-damper-spring system shown in the figure below, with: The system is initially at rest (the left-hand figure) in the static equilibrium position for weight $m_0 g$. At t = 0, mass m_1 is gently placed upon m_0 and then released. This imposes an $m_1 g$ step force (the weight, due to gravity) onto the system. **Note also** that mass m_1 changes the system, since the total vibrating mass now will be $m_0 + m_1$, not just m_0. Use Equation 9.6.5 for underdamped 2nd order system step response to write the **numerical** equation for response $y(t)$ **in inches** of combined mass $m_0 + m_1$ (relative to the static equilibrium position for the weight of m_0 alone, as indicated on the figure). Then use MATLAB to plot the response curve of $y(t)$ versus t from t = 0 until the mass has settled to within at least 2% of its final steady-state position. This means plot to at least four time constants, $t = 4\tau_2$, where $\tau_2 = (\zeta\omega_n)^{-1}$. Since you are multiplying sinusoidal time functions by an exponential time function, it is necessary that you use array multiplication (.*) in MATLAB. Provide appropriate grids, titles, and labels on your graph. It is not required, but you can check from your graph that the response has the values of frequency, steady-state (final static, pseudo-static) displacement, and viscous damping ratio that you can calculate from the original data for mass, damping, stiffness, and applied force.

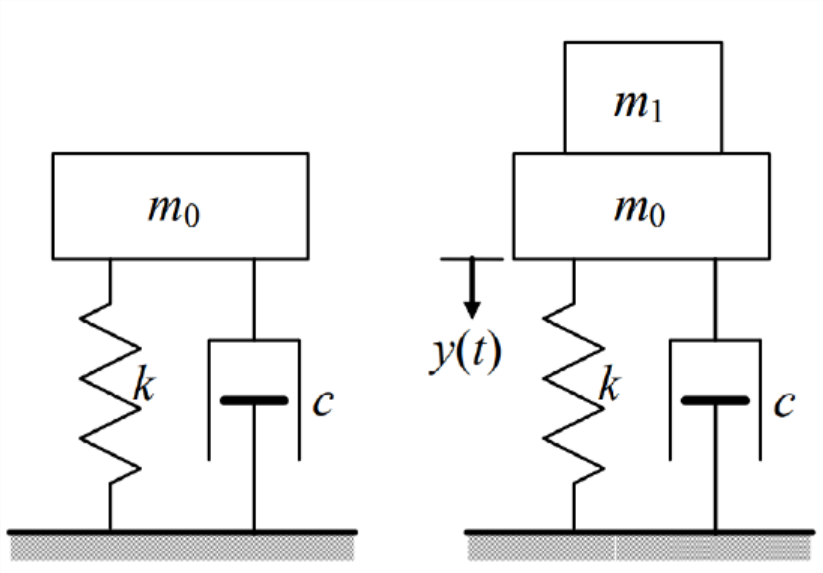


Figure 9.11.4

1. m_0 = 10.0 kg, c = 40.0 N-s/m, k = 400 N/m, and m_1 = 2.00 kg;
2. $m_0 g$ = 225 lb, c = 2.10 lb-s/inch, k = 30.5 lb/inch, and $m_1 g$ = 105 lb.

9. For the m-c-k system and step input/response of homework Problem 9.8 [part 9.8.1 or 9.8.2, whichever is assigned], use the equations of Section 9.8 to calculate rise time t_r, peak time t_p, maximum overshoot ratio $\bar{y}_p$, and settling time t_s. These calculations should agree with the comparable results that you can measure from your MATLAB step-response graph for homework Problem 9.8, assuming that your graph is correct.
10. Weight W_2 = 98.1 N (mass m_2 = 10.0 kg) is suspended by a string from weight W_1 = 196.2 N (mass m_1 = 20.0 kg), which is attached to the ceiling by spring k and viscous dashpot c, as shown below. The system is in static equilibrium when, at time t = 0, the string is cut cleanly, dropping W_2. (This simulates the release of a bomb from a flexible airplane wing.) Consider the **subsequent** motion $x(t)$, $t \geq 0$, of m_1. With the $y(t)$ datum shown on the drawing, you can think of this as step-response, with the release of W_2 being equivalent to imposing upon m_1 an upward step force equal to W_2.

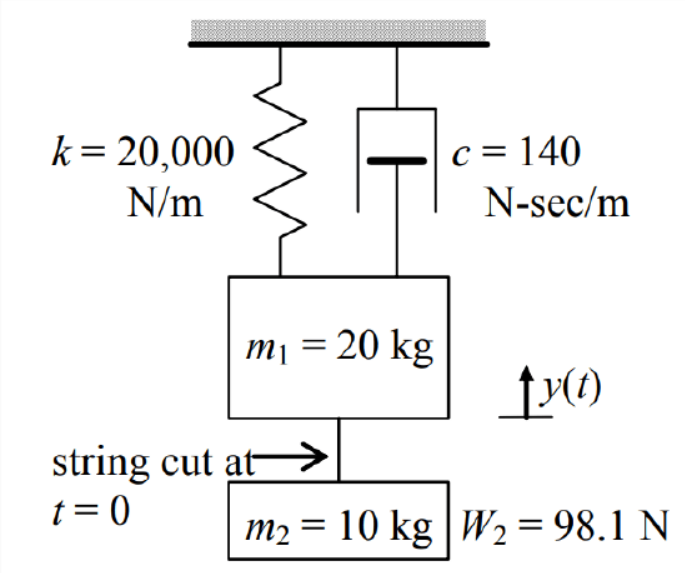


Figure 9.11.5

1. Calculate the undamped natural frequency f_n (Hz), the damping ratio ζ, the damped natural frequency f_d (Hz), and the damped period of oscillation T_d (s).
2. Calculate the final (as $t \to \infty$) static value of y and the settling time t_s required for the motion to settle within 2% of that final static value.
3. Sketch a plot of the response $y(t)$ versus t from $t = 0$ until about $t = t_s$. Do not carry out extensive calculations; just use the results of parts 9.10.1 and 9.10.2 to sketch an engineer's back-of-the-envelope type of quick graph that clearly shows the principal features of the response, but is not precise like a computer-generated graph.

11. Derive the algebraic equation for step response $x(t)$ with zero ICs of **critically** damped 2nd order systems: $\ddot{x} + 2\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 U$, for $t \geq 0$, with $x(0) = 0$ and $\dot{x}(0) = 0$. One easy method of solution, the conventional approach of Section 1.5, is to use homogeneous solution Equation 9.1.11, add to it an appropriate particular solution, then enforce the ICs to determine the unknown constants. (*Answer:* $x(t) = U\left[1 - (1 + \omega_n t)\, e^{-\omega_n t}\right]$, $t \geq 0$)
12. Having solved by inverse convolution transform for step and impulse responses of underdamped 2nd order systems, we can now use those results to obtain rather easily two messy Laplace transform pairs. Use Equation 9.3.5 first with Equation 9.6.5 and next with Equation 9.7.3 to obtain $f_H(t) = L^{-1}\left[\frac{\omega_n^2}{S\left[(s+\zeta\omega_n)^2+\omega_d^2\right]}\right]$ and $f_\delta(t) = L^{-1}\left[\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$, both for $0 \leq t$, $|\zeta| < 1$, and $\omega_d^2 > 0$. Write the algebraic equations for $f_H(t)$ and $f_\delta(t)$.
13. Consider the *LRC* circuit drawn below. The ODE describing output voltage $e_o(t)$ in terms of input voltage $e_i(t)$ is derived in Example 9.2.1 of Section 9.2. Because an inductor consists primarily of coiled wire, and because coiled wire accumulates resistance, a real inductor has resistance as well as inductance. A common, simple, approximate circuit model for such a real inductor is a series combination of an ideal inductor and a resistor, L and R_L, as shown in the drawing. Suppose that one small circuit component has the values L = 1.4 H and R_L = 210 Ω, and that it is in series with a capacitance of C = 0.25 μF (micro-farad), as shown in the drawing. Calculate the natural frequency f_n (Hz) and damping ratio ζ of this circuit. Suppose further that the input voltage is $e_i(t) = 1.50H(t)$ volts (a 1.5-volt battery is connected suddenly by switch S), and that all ICs are zero (in particular, there is no initial charge on the capacitor). Use MATLAB to plot output voltage $e_o(t)$ over a time interval from zero until $e_o(t)$ settles to within at least 2% of its final steady-state value.
14. A particular device is known to be a mass-damper-spring system. It is required that the mass m, the viscous damping constant c, and the stiffness constant k be identified experimentally from a transient-response test. In this test, an engineer strikes the mass sharply with a specially designed hammer that is instrumented with a force sensor. The force input to the mass is recorded, and examination of the actual force time history shows that it is closely approximated by a particular half-sine force pulse that lasts only 20 milliseconds, $f_x(t) = 58.9 \sin 50\pi t \times [H(t) - H(t - 0.02\text{s})]$ lb. The measured displacement response of the mass is shown on the graph below. Use this information to calculate m, c, and k in consistent units and with as much accuracy as the data permits. To support these calculations, annotate your measurements on the graph (hand in a photocopy of the annotated graph), and show your calculations clearly. The input force pulse is **very short** relative to the oscillatory period of the system. Therefore, for the purpose of approximate parameter identification, you should assume that the response is very close to ideal impulse response of the system to an ideal impulse, $f_x(t) = I_F \delta(t)$, **with I_F having the value of the impulse of the actual half-sine force pulse**. [NOTE: after you calculate m, c, and k, you can easily check the validity of your values by graphing (with MATLAB) the ideal impulse response of your system and then comparing that graph with the actual time history of response. However, this check is valid only if you use the correct value of impulse I_F, so first be 100% certain that your I_F is correct.]

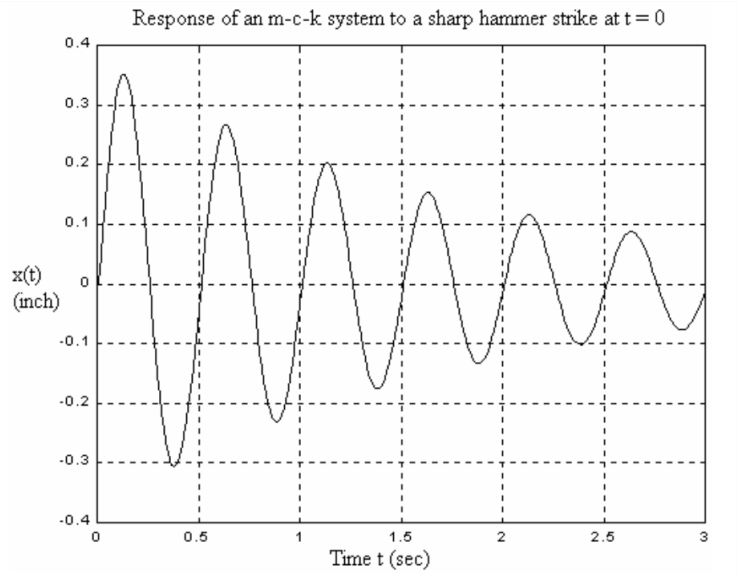


Figure 9.11.6

15. For systems with the particular type of right-hand-side dynamics of Equation 9.10.9, we can define a physical constant T having dimensions of time, then write a modified form of standard ODE Equation 9.2.2 as $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \omega_n^2 T \dot{u}(t)$. [You can easily verify, for example, that $T = \tau_H$ for the band-pass filter, Equation 9.10.9.] The pure IC-response of this equation is the same as Equations 9.4.1 and 9.4.2, so let us focus on the forced response and set all ICs = 0. Also, let us assume initially that $|\zeta| < 1$, so that $\omega_d^2 = \omega_n^2 (1 - \zeta^2) > 0)$.
    1. For this modified ODE, follow the steps from Equations 9.3.2 to 9.3.5, and assume also that $u(0) = 0$. to find $L[x(t)]$

$= \frac{\omega_n^2 T}{\omega_d} \frac{s\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2} L[u(t)]$. Next, use a Laplace transform pair developed, in effect, between Equations 9.3.5 and 9.3.9 to solve for $x(t)$ if the input is step function $u(t) = UH(t)$. Which of the zero ICs is or is not preserved in this solution?

2. Use the inverse transform for step response of part 9.15.1, and the general transform $L[\dot{f}(t)] = sF(s) - f(0)$ to derive the following inverse transform:

$$L^{-1}\left[\frac{s\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t}\left(\omega_d \cos\omega_d t - \zeta\omega_n \sin\omega_d t\right)$$

3. Use the result of part 9.15.2 and the inverse convolution transform to write $x(t)|_{ICs=0}$ equations, which involve convolution integrals, that apply for **arbitrary** input function $u(t)$.
4. Use the result of part 9.15.2 and Equation 8.6.6 to show that the algebraic equation for the unit-impulse-response function (IRF) valid for the modified 2nd order ODE with righthand side dynamics is $h(t) = \omega_n^2 T e^{-\zeta\omega_n t}\left[\cos\omega_d t - (\zeta\omega_n/\omega_d)\sin\omega_d t\right]$. Now use this IRF to write a Duhamel integral equation, Equation 8.7.2, for $x(t)|_{ICs=0}$ that applies for **arbitrary** input function $u(t)$.
5. The results of parts 9.15.1-9.15.4 apply for an underdamped 2nd order system, $|\zeta| < 1$. But Example 9.10.1 in Section 9.10 shows that an *RC* band-pass filter, for one, is an overdamped 2nd order system, so we need to convert the previous results to obtain results valid for $|\zeta| > 1$. Use the methods of Section 9.10 to write an algebraic equation for IRF $h(t)$ and an integral equation for $x(t)|_{ICs=0}$, both valid for the modified overdamped 2nd order ODE with right-hand side dynamics and the latter valid for **arbitrary** input function $u(t)$.

16. Figure 9.7 includes a curve that is step response of a standard **over**damped 2nd order system. Use the methods of Section 9.10 to derive the algebraic equation from which that particular curve was calculated. *Answer*:

$$x(t) = U\left[1 - e^{-\zeta\omega_n t}\left(\cosh\mu_d t + \frac{\zeta\omega_n}{\mu_d}\sinh\mu_d t\right)\right], \text{ for } 0 \le t \text{ and } \zeta > 1, \mu_d \equiv \omega_n\sqrt{\zeta^2-1}$$

17. The circuit drawn at right[2] consists of three stages, each built around an op-amp, and there is *feedback* of voltage $e_f(t)$ from the last (right-hand) stage to the first (left-hand) stage, where the input voltage $e_i(t)$ is applied. The circuit output voltage, $e_o(t)$, is the output of the middle stage. The first stage is a *summing, inverting integrator*; use the methods of Chapter 5 to derive the first ODE, $e_m/R_c + e_f/R_b + e_i/R_a = -C\dot{e}_m$. Next, for the middle stage, an *inverting integrator*, derive the second ODE, $e_m/R_b = -C\dot{e}_o$ (homework Problem 5.6); now use the second ODE to substitute for e_m in the first ODE, and differentiate the second ODE and use the result to substitute for $\dot{e}_m$ in the first ODE. The last stage of the circuit is a simple sign inverter, for which $e_f = -e_o$ from Equation 5.3.5; substitute for e_f and show that the ODE relating output voltage $e_o(t)$ to input voltage $e_i(t)$ for the entire circuit is $\ddot{e}_o + \frac{1}{R_c C}\dot{e}_o + \left(\frac{1}{R_b C}\right)^2 e_o = \left(\frac{1}{R_b C}\right)^2 \frac{R_b}{R_a} e_i$. Convert this ODE into the standard damped 2nd order form Equation 9.2.2, $\ddot{e}_o + 2\zeta\omega_n\dot{e}_o + \omega_n^2 e_o = \omega_n^2 u(t)$; write specific equations (in terms of this circuit's resistances, capacitances, and input voltage) for undamped natural frequency ω_n, viscous damping ratio ζ, and input quantity $u(t)$.
18. One rate gyroscope (Example 9.2 in Section 9.2) has the following rotor parameters: polar rotational inertia $J_r = 2.9 \times 10^{-4}\text{kg} - \text{m}^2$, and spin rate $\Omega_r = 2,100$ revolutions per minute (rpm); also, it has the following gimbal-shaft parameters: polar rotational inertia $J_\theta = 2.2 \times 10^{-4}\text{kg} - \text{m}^2$, $J_\theta = 2.2 \times 10^{-4}\text{kg} - \text{m}^2$, and damping constant $c_\theta = 0.0046\text{N} - \text{m}/(\text{rad}/\text{s})$, and stiffness constant $k_\theta = 0.096\text{N} - \text{m}/\text{rad}$. Consider output gimbal-shaft rotation $\theta(t)$ in response to an input turntable rotational rate that **approximates** step function $p(t) \approx P \times H(t)$. (Only a physically impossible ideal impulse moment acting on the turntable could produce an **exact** step change in rotational rate; therefore, imagine for this problem that a very short but very large moment pulse produces the approximate step function, but use the equations that apply for an exact step function.) For the step magnitude $P = 15$ degrees/s, calculate the following response quantities: pseudo-static gimbal-shaft rotation, θ_{ps} (in degrees); and rise time, t_r; and peak time, t_p; and maximum overshoot ratio, $\bar{x}_p$. (*Partial answer*: $t_p = 0.174$ s) The theory developed in Example 9.2.2 is accurate only if gimbalshaft rotation $\theta(t)$ is sufficiently small that $\cos\theta(t) \approx 1$, say, $\cos\theta(t) > 0.95$. For the parameters in this problem, is this criterion for accuracy satisfied?
19. Another type of single-axis gyroscopic sensor is similar to the rate gyro described in Example 9.2.2 of Section 9.2 **except** that there is no spring to restrain the rotation of the gimbal-shaft assembly, so $k_\theta = 0$ in the equations derived in Example 9.2.2. Therefore, the equation of motion becomes effectively a 1st order ODE in dependent variable $\dot{\theta}(t)$, the rate of gimbal-shaft rotation:

$$J_\theta\ddot{\theta} + c_\theta\dot{\theta} = J_r\Omega_r p(t)$$

Observe from the ODE that the pseudo-static response is $(\dot{\theta})_{ps} = (J_r\Omega_r/c_\theta)\,p(t)$. But the electrical transducer of this device detects $\theta(t)$, **not** $\dot{\theta}(t)$, which means that the sensed quantity has the form $\theta(t) = \int_t \dot{\theta}(\tau)d\tau = (J_r\Omega_r/c_\theta)\int_t p(\tau)d\tau$. Therefore, this type of sensor is called a *rate-integrating gyroscope*; its output is proportional, not to the rotational rate, but to the actual rotational angle of the turntable, $\psi(t) = \int_t p(\tau)d\tau$, where $p \equiv \dot{\psi}$. Consider output gimbal-shaft rotational rate $\dot{\theta}(t)$ **and** rotation $\theta(t)$ in response to an input turntable rotational rate that **approximates** the following *flat pulse* of duration t_d

:\[p(t)=\left\{\begin{array}{l}
0, \quad t<0 \\
P, 0 0, \quad t_{d} \end{array}\right. \nonumber \](Only physically impossible ideal impulse moments acting on the turntable could produce **exact** step changes in rotational rate; therefore, imagine for this problem that very short but very large moment pulses produce the approximate step changes, but use the equations that apply for the exact flat pulse.)

1. The mathematical solution for response of a standard stable 1st order system to a flat-pulse input is given in homework Problem 6.2.1. Adapt this solution to write response equations for the rate-integrating gyro's gimbal-shaft rotational rate, $\dot{\theta}(t)$, given the IC $\dot{\theta}(0)=0$. Next, integrate appropriately (show all details!) the $\dot{\theta}(t)$ equations, given the IC $\theta(0)=0$, to derive the following equations for gimbal-shaft rotation:

$$\theta(t)=\begin{cases} bP\tau_1\left[t-\tau_1\left(1-e^{-t/\tau_1}\right)\right], 0\le t\le t_d \\ \theta\left(t_d\right)+bP\tau_1^2\left(1-e^{-t_d/\tau_1}\right)\left(1-e^{-(t-t_d)/\tau_1}\right), t_d\le t \end{cases}$$

   where $b=J_r\Omega_r/J_\theta$ and $\tau_1=J_\theta/c_\theta$.
2. One rate-integrating gyro has the following rotor parameters: polar rotational inertia $J_r=2.9\times10^{-4}\text{kg}-\text{m}^2$, and spin rate $\Omega_r=420$ revolutions per minute (rpm); also, it has the following gimbal-shaft parameters: polar rotational inertia $J_\theta=8.5\times10^{-4}\text{kg}-\text{m}^2$, and damping constant $c_\theta=9.2\times10^{-3}$ N-m/(rad/s). For the flat-pulse rotational-rate magnitude $P=5$ degrees/s and pulse duration $t_d=2.5$ s, calculate and plot from the equations of part 9.19.1 the gimbal-shaft rotation $\theta(t)$ (in degrees) over the time interval $0\le t\le5$. The theory developed in Example 9-2 is accurate only if gimbal-shaft rotation $\theta(t)$ is sufficiently small that $\cos\theta(t)\approx1$, say, $\cos\theta(t)>0.95$. For the parameters in this problem, is this criterion for accuracy satisfied?
3. Since $(\dot{\theta})_{ps}=\left(J_r\Omega_r/c_\theta\right)p(t)=\left(J_r\Omega_r/c_\theta\right)\dot{\psi}(t)$, it is clear the rate-integrating gyro's output is designed to be proportional, as closely as its dynamics permit, to the actual rotational angle of the turntable, *i.e.*, $\theta(t)\approx\left(J_r\Omega_r/c_\theta\right)\psi(t)$; therefore, the equation that estimates $\psi(t)$ from the measured gimbal-shaft rotation is $\psi(t)\approx\left[c_\theta/\left(J_r\Omega_r\right)\right]\theta(t)$. Use your calculated $\theta(t)$ from part 9.19.2 to calculate and plot the quantity $\left[c_\theta/\left(J_r\Omega_r\right)\right]\theta(t)$ over the same time interval, $0\le t\le5$ s. Next, calculate the actual turntable rotational angle from $\psi(t)=\int_0^t p(\tau)d\tau$, given the IC $\psi(0)=0$, and plot it on the same graph for comparison with $\left[c_\theta/\left(J_r\Omega_r\right)\right]\theta(t)$. In this case, how well does the rate-integrating gyro follow the actual turntable rotation?

[1]Base excitation is displacement input imposed at the "ground" or "base" side of the spring instead of directly upon the mass. It is very common to consider base excitation in analyses of earthquake-induced response of buildings, automobile tires rolling over bumpy roads, and vibration isolation of sensitive devices such as delicate payloads in rocket boosters.

[2]This circuit is essentially the *electronic analog computer* (see the footnote to homework Problem 5.6) for solving ODE Equation 9.2.2.

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10: SECOND ORDER SYSTEMS

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10.1: Frequency Response of Undamped Second Order Systems and Resonance

As discussed in Section 7.1, a truly undamped 2nd order passive system is physically unrealistic. Nevertheless, it is useful to derive the frequency response for this ideal type of system because the results reveal important characteristics of the resonating behavior of general underdamped 2nd order systems. Moreover, the frequency response of an undamped 2nd order system is, for many practical purposes, a simple yet satisfactory engineering approximation to the frequency response of the corresponding lightly damped 2nd order system.

Consider the standard ODE for idealized undamped 2nd order systems:

$$\ddot{x} + \omega_n^2 x = \omega_n^2 u(t) \tag{10.1.1}$$

To derive frequency response, we let $u(t) = U\cos\omega t$ and seek the steady-state sinusoidal response in the form $x(t) = X_S(\omega)\cos\omega t$, in which $X_S(\omega)$ is the unknown quantity. We find an equation for $X_S(\omega)$ by substituting into Equation 10.1.1:

$$\left(-\omega^2 + \omega_n^2\right) X_S(\omega)\cos\omega t = \omega_n^2 U\cos\omega t$$

$$\Rightarrow \frac{X_S(\omega)}{U} = \frac{\omega_n^2}{\omega_n^2 - \omega^2} = \frac{1}{1-(\omega/\omega_n)^2} \tag{10.1.2}$$

Note from Equation 10.1.2 that $X_S(\omega)$ is a **signed** quantity; it can be positive or negative depending upon the value of frequency ratio ω/ω_n relative to 1. But our conventional notation for a frequency response function is $FRF(\omega) = \frac{X(\omega)}{U}e^{j\phi(\omega)}$, Equation 7.7.18, in which magnitude $X(\omega) \geq 0$. Therefore, we infer from Equation 10.1.2 the following magnitude-ratio and phase functions in the conventional notation:

$$\frac{X(\omega)}{U} = \left|\frac{1}{1-(\omega/\omega_n)^2}\right|, \phi(\omega) = \begin{cases} 0^\circ \text{ for } 0 \leq \omega/\omega_n < 1 \\ -180^\circ \text{ for } 1 < \omega/\omega_n \end{cases} \tag{10.1.3}$$

Equations 10.1.3 give the frequency response function (FRF) of an undamped 2nd order system. The magnitude ratio $X(\omega)/U$ and phase $\phi(\omega)$ are plotted on Figure 10.1.1 as functions of excitation frequency ratio ω/ω_n.

The most important feature of FRF Equations 10.1.3 is **the infinite magnitude of steady-state response when excitation frequency ω exactly equals natural frequency ω_n**. This is the extreme version of the phenomenon called *resonance*. In reality, any damping, no matter how small, reduces the infinite resonance magnitude of Fig. 10-1 to some finite value, as is demonstrated in the next section; but that finite value will still generally be much greater than the static or pseudo-static response, which is $X(\omega \approx 0) = U$. Figure 10.1.1 shows also that the response magnitude is greatly amplified relative to static response even if the excitation frequency is close (but not equal) to the natural frequency; note, for example, that the *dynamic amplification* exceeds five (5) whenever excitation frequency ω is within about $\pm$10% of natural frequency ω_n. This high sensitivity to excitation near the natural frequency is a very important characteristic of lightly damped 2nd order systems. There is additional discussion of resonance in the next section.

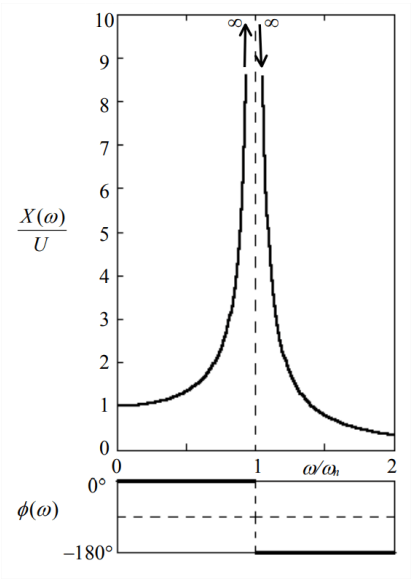


Figure 10.1.1: Frequency response function of an undamped 2nd order system

FRF Equations 10.1.3 contains not only the magnitude **but also** the phase ϕ: for $\omega < \omega_n$, the phase is 0°, so the response is exactly in phase with the excitation; however, for $\omega > \omega_n$, the phase is –180°, so the response is exactly out of phase with the excitation.

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10.2: Frequency Response of Damped Second Order Systems

From Section 9.2, the standard form of the 2nd order ODE is:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 u(t) \tag{10.2.1}$$

To derive frequency response, we take the Laplace transform of Equation 10.2.1, with zero ICs:

$$\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right) L[x(t)]\big|_{ICs=0} = \omega_n^2 L[u(t)] \tag{10.2.2}$$

Definition Equation 4.6.3 for the transfer function gives

$$TF(s) = \frac{L[x(t)]|_{ICs=0}}{L[u(t)]} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10.2.3}$$

Then the fundamental relationship Equation 4.7.18 between the transfer function and the complex frequency response function gives

$$FRF(\omega) = TF(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2} = \frac{1}{(1-\omega^2/\omega_n^2) + j2\zeta\omega/\omega_n} \tag{10.2.4}$$

To simplify notation, we define the dimensionless excitation frequency ratio, the excitation frequency relative to the system undamped natural frequency:

$$\beta \equiv \frac{\omega}{\omega_n} \tag{10.2.5}$$

With notation Equation 10.2.5, the relationship Equation 4.7.18 between $FRF(\omega)$ and the magnitude ratio $X(\omega)/U$ and phase angle $\phi(\omega)$ of the frequency response gives

$$FRF(\omega) = \frac{1}{(1-\beta^2) + j2\zeta\beta} = \frac{X(\omega)}{U} e^{j\phi(\omega)} \tag{10.2.6}$$

After the standard manipulation of the complex fraction in Equation 10.2.6, we find the following equations for magnitude ratio and phase angle [homework Problem 2.6.1]:

$$\frac{X(\omega)}{U} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}, \quad \phi(\omega) = \tan^{-1}\left(\frac{-2\zeta\beta}{1-\beta^2}\right), \quad \beta \equiv \frac{\omega}{\omega_n} \tag{10.2.7}$$

Note that results Equations 10.2.7 are valid for **any** non-negative value of viscous damping ratio, $\zeta \geq 0$; unlike most of the time-response equations derived in Chapter 9, Equations 10.2.7 alone apply for underdamped, critically damped, and overdamped 2nd order systems.

Results Equations 10.2.7 are plotted on Figure 10.2.1 for viscous damping ratios varying from 0 to 1. The graphs of Figure 10.2.1 were produced with use of MATLAB (Version 6 or later). First, an M-file was programmed to calculate and graph FRFs using the method described in the statement of homework Problem 4.3 [starting with complex FRF Equation 10.2.6), not the real equations Equations 10.2.7]:

```
%frf2orddamped.m

%Fig. 10-2 Frequency response, damped 2nd order

%Magnitude ratio for zero damping, zeta=0

bt0a=logspace(-1,log10(0.998),100);

frf0a=(1)./(1-bt0a.^2);
```

```
loglog(bt0a,frf0a,'k'),hold on
bt0b=logspace(log10(1.002),1,100);
frf0b=(1)./(bt0b.^2-1);
loglog(bt0b,frf0b,'k')
%Magnitude ratio for several cases of zeta > 0
bt1=logspace(-1,1,400);
zt=[ 0.05 0.1 0.2 0.5 1/sqrt(2) 1];nzts=length(zt);
for i=1:nzts
    frf=(1)./(1-bt1.^2+j*2*zt(i).*bt1);
    magrat=abs(frf);
    loglog(bt1,magrat,'k')
end
hold off
%Phase angle in degrees for zero damping, zeta=0
bt0a=[0.1 1];faz0a=zeros(1,2);
figure,semilogx(bt0a,faz0a,'k'),hold on
bt0b=[1 10];faz0b=-180*ones(1,2);
semilogx(bt0b,faz0b,'k')
%Phase angle in degrees for several cases of zeta > 0
for i=1:nzts
    frf=(1)./(1-bt1.^2+j*2*zt(i).*bt1);
    faz=angle(frf)*180/pi;
    semilogx(bt1,faz,'k')
end
```

After the M-file was executed, the extensive figure editing features of MATLAB were used to set axis limits and to generate titles, labels, and curve annotations.

Even though Figure $10.2.1$ was produced by computer-aided graphics, it is still important that you understand how to evaluate Equations $10.2.7$ with a hand calculator. These equations appear simple, but they can be tricky, particularly for the special case $\zeta = 0$. See homework Problem 2.6.2 for help with understanding how to evaluate Equations $10.2.7$ numerically.

There are several noteworthy characteristics of the frequency response of damped 2nd order systems, from Equations $10.2.7$ and Figure $10.2.1$:

- ***Resonance*: maximum (peak) magnitude ratio** The frequency at which the peak magnitude ratio occurs is called the *resonance frequency*, denoted ω_r, and this frequency is lower than natural frequency ω_n if damping is positive, $\zeta > 0$. To find

the maximum, we differentiate the magnitude ratio of Equations 10.2.7, set the result to zero, and solve for the associated frequency (you can verify the result yourself, or see Meirovitch, 2001, pp. 115-116 for the details of calculus and algebra; also, there are instructive alternative approaches in Ogata, 1998, pp. 480-481 and in Clark, 1962, pp. 316-322):

$$\frac{d}{d\beta}\left(\frac{X(\beta)}{U}\right)=0 \Rightarrow \beta_r=\sqrt{1-2\zeta^2} \Rightarrow \omega_r=\omega_n\sqrt{1-2\zeta^2} \tag{10.2.8}$$

Subscript r in Equation 10.2.8 denotes a resonance quantity corresponding to the peak magnitude ratio X_r/U. The excitation frequency must be real and greater than or equal to zero, so $\omega_r=\omega_n\sqrt{1-2\zeta^2}$ is valid only for $0\le\zeta\le 1/\sqrt{2}=0.707$. If ζ is positive but very small, then X_r/U occurs almost at the natural frequency, $\omega_r\approx\omega_n$. As ζ increases from very small positive values, ω_r decreases until, for $\zeta=1/\sqrt{2}$, $\omega_r=0$. However, dynamic amplification is insignificant for values of ζ greater than about 0.5. Substituting Equation 10.2.8 back into $X(\omega)/U$ of Equations 10.2.7 gives (after more algebra) the resonance value:

$$\frac{X\left(\beta_r\right)}{U}=\frac{X_r}{U}=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{10.2.9}$$

Note from Equation 10.2.9 that for $\zeta=1/\sqrt{2}$ [the maximum permissible value for validity of Equation 10.2.8], we have $X_r/U=1$ at $\omega_r=0$. As indicated by the trend on Figure 10.2.1, for higher damping ratios, $\zeta\ge 1/\sqrt{2}$, the magnitude $X(\omega)/U$ is maximum at $\omega=0$, and decreases monotonically as ω increases.

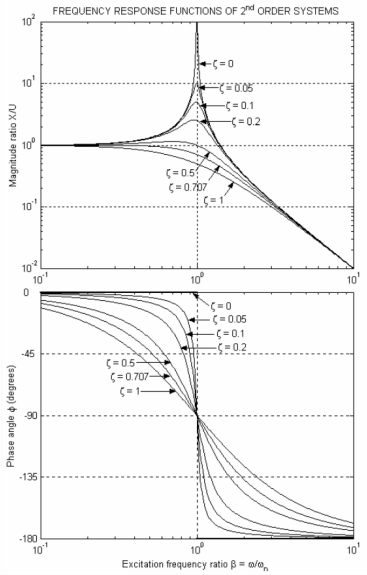


Figure 10.2.1: Frequency response functions for standard 2nd order systems with viscous damping ratios ζ varying from 0 to 1

- **Response at the natural frequency** The frequency response at $\omega=\omega_n$, $\beta=1$, consists of phase angle $\phi\left(\omega_n\right)=-90^{\circ}$ **regardless** of the value of viscous damping ratio ζ, and magnitude ratio $X\left(\omega_n\right)/U=1/(2\zeta)$. Also, the peak magnitude response occurs essentially at $\omega=\omega_n$ if damping is small, as discussed next.
- **Small viscous damping, *small-*ζ *approximation*** If damping is so small that $\sqrt{1-2\zeta^2}\approx 1$, then we can use the following accurate approximations of the values associated with response at resonance, from Equations 10.2.8 and 10.2.9:

$$\omega_r=\omega_n\sqrt{1-2\zeta^2}\approx\omega_n \tag{10.2.10}$$

$$\frac{X_r}{U}=\frac{1}{2\zeta\sqrt{1-\zeta^2}}\approx\frac{X\left(\omega_n\right)}{U}=\frac{1}{2\zeta} \tag{10.2.11}$$

The upper limit of viscous damping ratio for which the small-ζ approximation is practically useful is about $\zeta=0.2$, which we can see on Figure 10.2.1. At this upper limit, the approximation is in error on the order of 4%, since $\sqrt{1-2(0.2)^2}=0.959$. The small-ζ resonance amplification factor Equation 10.2.11 is sometimes called the *quality factor*, and it is denoted $Q=1/(2\zeta)$. The "quality" reference is to tuning, especially of a receiving device such as a radio (Hammond, 1961, pp. 106-114). When you turn the analog tuning dial of an old radio, you are varying the value of capacitance in an LRC electrical tuning circuit in order to produce resonance with the electromagnetic carrier signal, which gives the strongest possible reception. Clearly, a high Q is desirable in the tuning circuit of a radio. On the other hand, high Q is often undesirable in mechanical systems, because it can lead to structural failure or other negative consequences. For example, if the system consisting of a car's tire, suspension spring, and damping strut happens to resonate with the waves in a washboard dirt road, then the resulting vibration will produce poor ride quality and might even break car parts.

Note on the Figure $10.2.1$ graph of magnitude ratio that the $\zeta = 0$ curve bounds the curves for all $\zeta > 0$, and that it provides a good approximation for small ζ at all frequencies except those close to ω_n; for small damping, therefore, the simple equation (10-3a) for magnitude ratio will, for some calculations, be sufficiently accurate.

- **Phase angles of frequency response** Figure $10.2.1$ shows that the more lightly damped a system is, the closer its response is to being in phase with excitation below the natural frequency, and out of phase with excitation above the natural frequency. Furthermore, for excitation at the natural frequency, $\omega = \omega_n$, response lags excitation by exactly 90°, regardless of the level of viscous damping; this so-called *quadrature phase* is an important characteristic often used to help determine natural frequencies in vibration testing of machines and structures.
- **2nd order low-pass filter** A moderately damped $(0.5 \leq \zeta \leq 1)$ 2nd order system can function as a low-pass filter, with natural frequency ω_n being the break (corner) frequency. Compare the magnitude ratio curve for $\zeta = 1/\sqrt{2} = 0.707$ on Figure $10.2.1$ with the magnitude ratio curve for a 1st order low-pass filter, Figure 4.3.3. The most significant difference between the two curves is in their high-frequency asymptotes: the 2nd order magnitude ratio rolls off at the rate of two decades for each decade increase of frequency (40 dB/decade rolloff), twice as steeply as the 1st order magnitude ratio. Thus, the 2nd order filter functions much more effectively than the 1st order filter. However, an important practical deficiency (in some potential applications) of both types of low-pass filters is that they impose on the low-passed output large phase lags relative to the input.

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10.3: Frequency Response of Mass-Damper-Spring Systems

Reviewing the basic 2nd order mechanical system from Figure 9.1.1 and Section 9.2, we have the m-c-k and standard 2nd order ODEs:

$$m\ddot{x} + c\dot{x} + kx = f_x(t) \Rightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 u(t) \tag{10.3.1}$$

with the definitions

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta \equiv \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}} \equiv \frac{c}{c_c}, \quad u(t) \equiv \frac{1}{k} f_x(t) \tag{10.3.2}$$

So we can use the correspondence $U = F/k$ to adapt FRF (10-10) directly for m-c-k systems:

$$\frac{X(\omega)}{F/k} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}, \quad \phi(\omega) = \tan^{-1}\left(\frac{-2\zeta\beta}{1-\beta^2}\right), \quad \beta \equiv \frac{\omega}{\sqrt{k/m}} \tag{10.3.3}$$

In the conceptually simplest form of forced-vibration testing of a 2nd order, linear mechanical system, a force-generating *shaker* (an electromagnetic or hydraulic translational motor) imposes upon the system's mass a sinusoidally varying force at cyclic frequency f, $f_x(t) = F\cos(2\pi ft)$. The resulting steady-state sinusoidal translation of the mass is $x(t) = X\cos(2\pi ft + \phi)$. Calibrated sensors detect and $x(t)$, and then F, X, f and ϕ are measured from the electrical signals of the sensors. In principle, the testing involves a *stepped-sine sweep*: measurements are made first at a lower-bound frequency in a steady-state *dwell*, then the frequency is stepped upward by some small increment and steady-state measurements are made again; this frequency stepping is repeated again and again until the desired frequency band has been covered and smooth plots of X/F and ϕ versus frequency f can be drawn.

For system identification (ID) of 2nd order, linear mechanical systems, it is common to write the frequency-response magnitude ratio of Equation 10.3.3 in the form of a dimensional magnitude of *dynamic flexibility*[1]:

$$\frac{X(\omega)}{F} = \frac{1}{k}\frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}} = \frac{1}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}} \tag{10.3.4}$$

Also, in terms of the basic m-c-k parameters, the phase angle of Equation 10.3.3 is

$$\phi(\omega) = \tan^{-1}\left(\frac{-c\omega}{k-m\omega^2}\right) \tag{10.3.5}$$

Note that if $\omega \to 0$, dynamic flexibility Equation 10.3.4 reduces just to the *static flexibility* (the inverse of the stiffness constant), $X(0)/F = 1/k$, which makes sense physically. In fact, the first step in the system ID process is to determine the stiffness constant. Electromagnetic shakers are not very effective as static loading machines, so a static test independent of the vibration testing might be required. In principle, static force F imposed on the mass by a loading machine causes the mass to translate an amount $X(0)$, and the stiffness constant is computed from

$$k = \frac{F}{X(0)} \tag{10.3.6}$$

However, suppose that it is more convenient to shake the mass at a relatively low frequency (that is compatible with the shaker's capabilities) than to conduct an independent static test. Note from Figure 10.2.1 that if the excitation frequency is less than about 25% of natural frequency ω_n, then the magnitude of dynamic flexibility is essentially the same as the static flexibility, so a good approximation to the stiffness constant is

$$k \approx \left(\frac{X\left(\omega \le 0.25\omega_n\right)}{F}\right)^{-1} \tag{10.3.7}$$

Even if it is possible to generate frequency response data at frequencies only as low as 60-70% of ω_n, one can still knowledgeably

extrapolate the dynamic flexibility curve down to very low frequency and apply Equation 10.3.7 to obtain an estimate of k that is probably sufficiently accurate for most engineering purposes.

Assuming that all necessary experimental data have been collected, and assuming that the system can be modeled reasonably as an LTI, SISO, m-c-k system with viscous damping, then the steps of the subsequent system ID calculation algorithm are:

1. Calculate k from Equation 10.3.6 and/or Equation 10.3.7, preferably both, in order to check that both static and dynamic testing lead to the same result.
2. Determine natural frequency ω_n from the frequency response curves. The frequency at which the phase angle is −90° is the natural frequency, regardless of the level of damping. Also, if viscous damping ratio ζ is small, less than about 0.2, then the frequency at which the dynamic flexibility peaks is essentially the natural frequency. With ω_n and k known, calculate the mass: $m = k/\omega_n^2$.
3. Measure the resonance (peak) dynamic flexibility, X_r/F. Then the maximum dynamic amplification equation Equation 10.2.9 gives the following equation from which any viscous damping ratio $\zeta \leq 1/\sqrt{2}$ can be calculated:

$$\frac{X_r}{F/k} = k\left(\frac{X_r}{F}\right) = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{10.3.8}$$

   The calculation is simplified considerably if ζ is small, less than about 0.2:

$$\frac{1}{2\zeta} \approx k\left(\frac{X_r}{F}\right) \Rightarrow \zeta \approx \frac{1}{2}\left[k\left(\frac{X_r}{F}\right)\right]^{-1} \tag{10.3.9}$$

   Finally, the viscous damping constant is calculated from Equation 10.3.2:

$$c = c_c \times \zeta = 2\zeta\sqrt{mk} = 2\zeta m\omega_n \tag{10.3.10}$$

[1]However, see homework Problem 10.16 for the practical reasons why it might often be better to measure *dynamic stiffness*, Eq. (10-31), rather than dynamic flexibility.

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10.4: Frequency-Response Function of an RC Band-Pass Filter

We re-visit once again Section 5.4 and the example of Section 9.10, where it is demonstrated that a typical RC band-pass filter, as depicted below, is an overdamped 2nd order system with right-hand-side dynamics. Our goal now is to derive the FRF of this filter circuit. Rather than analyze 2nd order ODE 9.10.9 directly, it is instructive to start with the coupled 1st order ODEs of the two stages: Equation 5.4.1, $\tau_L \dot{e}_m + e_m = e_i$, with $\tau_L = R_L C_L$, for the upstream, low-pass filter; and Equation 5.4.2, $\tau_H \dot{e}_o + e_o = \tau_H \dot{e}_m$, with $\tau_H = R_H C_H$, for the downstream, high-pass filter. In this approach, we represent a system (the band-pass filter, in this case) as an assemblage of stages or lower-order sub-systems (the low-pass and high-pass filters, in this case). This section is a preview of the more general discussion beginning in Chapter 13.

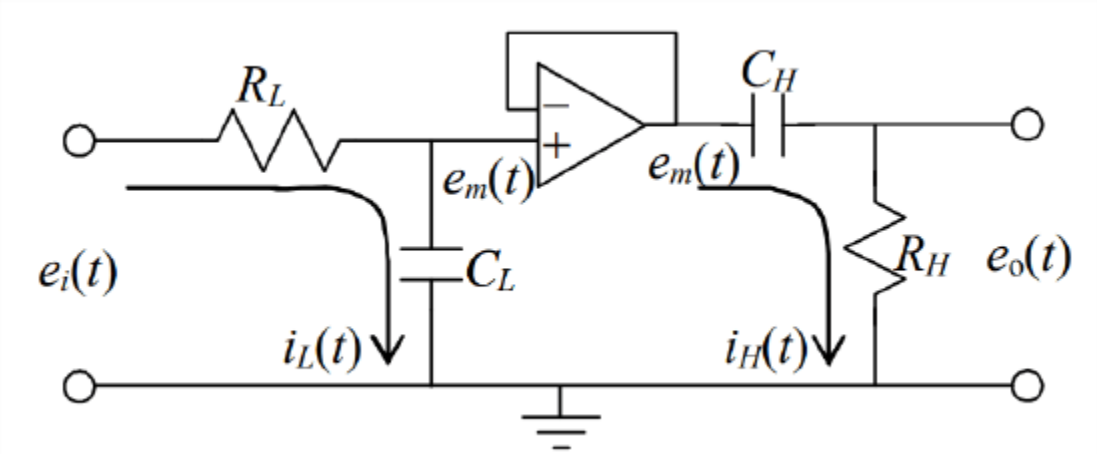


Figure 10.4.1: RC band-pass filter

Taking the Laplace transform of the ODE for the low-pass stage, with the assumption that all ICs are zero, gives

$$(\tau_L s + 1)\, L\left[e_m\right]|_{ICs=0} = L\left[e_i\right] \Rightarrow L\left[e_m\right]|_{/Cs=0} = \frac{1}{\tau_L s + 1} L\left[e_i\right] \tag{10.4.1}$$

Next, we take the Laplace transform of the ODE for the high-pass stage, still with the assumption that all ICs are zero, and then substitute Equation 10.4.1 into the result:

$$(\tau_H s + 1)\, L\left[e_o\right]_{/Cs=0} = \tau_H s L\left[e_m\right]|_{ICs=0}$$

$$\Rightarrow \quad L\left[e_o\right]|_{ICs=0} = \frac{\tau_H S}{\tau_H S + 1} L\left[e_m\right]\Big|_{ICs=0} = \frac{\tau_H s}{\tau_H s + 1} \times \frac{1}{\tau_L s + 1} L\left[e_i\right] \tag{10.4.2}$$

So the transfer function of the entire RC band-pass filter circuit, clearly displaying right-hand-side dynamics, is

$$TF(s) \equiv \frac{L\left[e_o\right]|_{IC_{s=0}}}{L\left[e_i\right]} = \frac{\tau_H s}{(\tau_H s + 1)(\tau_L s + 1)} \tag{10.4.3}$$

Finally, the system complex frequency-response function is

$$FRF(\omega) = TF(j\omega) = \frac{j\omega\tau_H}{(j\omega\tau_H + 1)(j\omega\tau_L + 1)} \tag{10.4.4}$$

See homework Problem 10.11 for a numerical evaluation of this frequency response that demonstrates the practical function for which a band-pass filter is designed.

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10.5: Common Frequency-Response Functions

The term *frequency-response function* (FRF) is general, meaning physically the magnitude and phase in steady-state sinusoidal variation with time of some subject variable, relative to the magnitude and phase of some other reference variable. Often, but not always, the **subject** variable is clearly a physical response (output), and the **reference** variable is clearly a physical excitation (input). During over a century of development, engineers have found uses for many different types of FRF subject and reference variables, and names for these types are widely recognized. Many of the most common of these FRF definitions, their names, and examples are presented in this section.

Diagrams such as Figure 4.3.1 and Figures 10.1.1 and 10.2.1 display FRFs graphically, and there are many different forms of these diagrams. One of the most well-known and useful forms is the *Bode diagram*, on which relative magnitude and phase are plotted separately against the sinusoidal frequency, which is on a logarithmic scale (as on Figure 10.2.1). Relative magnitude is usually plotted on the *decibel scale* [$\mathrm{dB} = 20 \times \log_{10}(\text{relative magnitude})$], and relative phase is usually expressed in degrees. Another useful graphical format for FRFs is the *Nyquist diagram* or *plot* (after Harry Nyquist, Swedish-born American electrical engineer and physicist, 1889-1976), which is described at length in Section 17.2. Application of Bode diagrams and Nyquist diagrams to analysis of control systems is discussed in Chapter 17.

The use of FRFs apparently began during the 1880s with the definitions of *admittance* and *impedance* in the context of electrical engineering. Electrical *admittance* is traditionally assigned the symbol $Y(\omega)$. With $\Delta e(t)$ being a specified voltage difference and $i(t)$ being a specified current, admittance is defined by the equation

$$Y(\omega) = \left\{ \frac{L[i(t)]}{L[\Delta e(t)]} \right\}_{s=j\omega} \tag{10.5.1}$$

Keep in mind that the Laplace transforms in Equation 10.5.1 and all other definitions of FRFs are for zero initial conditions. Electrical *impedance* is traditionally assigned the symbol $Z(\omega)$, and it is defined to be the inverse of admittance: $Z(\omega) = 1/Y(\omega)$. In general, admittance is the degree to which an electrical system **admits** (or **permits**, if you prefer) the flow of current, and impedance is the degree to which the system **impedes** the flow of current. The simplest examples of electrical admittance and impedance are for an ideal linear resistor of resistance R, with voltage difference $\Delta e(t)$ across the resistor and current $i(t)$ flowing through it. Ohm's law, Equation 5.2.1, is $\Delta e(t) = R \times i(t)$, so $Y(\omega) = 1/R$ and $Z(\omega) = R$, the same for all frequencies[1].

The electrical FRFs for an ideal linear capacitor with capacitance C are more complicated. The relationship between voltage difference $\Delta e(t)$ across a capacitor and current $i(t)$ flowing through it is Equation 5.2.4: $i(t) = C \times d[\Delta e(t)]/dt$. Taking the Laplace transform of this equation gives $L[i(t)] = C \times sL[\Delta e(t)]$, so, from Equation 10.5.1, a capacitor's admittance is $Y(\omega) = j\omega C$, and its impedance is $Z(\omega) = 1/(j\omega C) = -j \times 1/(\omega C)$. The magnitude of this impedance is called *capacitive reactance* (Hammond, 1961, pp. 97-100), and is usually written as $X_C = 1/(\omega C) = 1/(2\pi f C)$. Physically, for a given magnitude of voltage difference and a given capacitance C, the magnitude of current permitted by a capacitor is directly proportional to frequency ω or f, and the current leads the voltage difference by 90°.

The relationship between voltage difference $\Delta e(t)$ across an ideal linear inductor and current $i(t)$ flowing through it is Equation 5.2.9: $\Delta e(t) = \mathcal{L} \times di(t)/dt$ denoting inductance $\equiv \mathcal{L}$ **in this paragraph only**, to distinguish it from Laplace transform operator L). You can show easily that the associated admittance and impedance are, respectively, $Y(\omega) = 1/(j\omega\mathcal{L}) = -j \times 1/(\omega\mathcal{L})$ and $Z(\omega) = j\omega\mathcal{L}$. The magnitude of this impedance is called *inductive reactance*, and is usually written as $X_\ell = \omega\mathcal{L} = 2\pi f\mathcal{L}$. Physically, for a given magnitude of voltage difference and a given inductance $\mathcal{L}$, the magnitude of current permitted by an inductor is inversely proportional to frequency ω or f, and the current lags the voltage difference by 90°.

The subject and reference variables of FRFs commonly used for mechanical and structural systems are, of course, different than those used for electrical systems. During much of the 20th century, conflicting and ambiguous names for these FRFs appeared in the engineering literature, but the names in English have been more uniformly standardized in recent years, as described by Ewins, 1984, pp. 26-27, and by Maia and Silva, 1997, pp. 38-39. In the following, we will derive theoretical FRF equations for an m-c-k system; however, in engineering practice, these FRF definitions are also applied both theoretically and experimentally for much more general and more complicated mechanical and structural systems.

For *mechanical admittance*, the subject variable is a physical displacement (translation or rotation) at some point and in some direction on the mechanical or structural system, and the reference variable is a physical action (force or moment) imposed at some point and in some direction onto the system (Bisplinghoff *et al.*, 1955, pp. 663- 665). Thus, for an m-c-k system, from

Laplace transformation of the ODE $m\ddot{x} + c\dot{x} + kx = f_x(t)$, and with use of notation defined in Equations 10.2.5 and 10.3.2, the equation for complex mechanical admittance is

$$\left\{\frac{L[x(t)]}{L\left[f_x(t)\right]}\right\}_{s=j\omega} = \frac{1}{(k-\omega^2 m)+j\omega c} = \frac{1}{k}\left[\frac{1}{(1-\beta^2)+j2\zeta\beta}\right] \tag{10.5.2}$$

The magnitude ratio and phase angle associated with Equation 10.5.2 were written earlier in Equations 10.3.4 and 10.3.5. Some synonyms in English for mechanical admittance are *dynamic flexibility*, *dynamic compliance*, and *receptance*.

The inverse of mechanical admittance is known as *dynamic stiffness* (occasionally also *mechanical impedance*, but this name usually denotes a different FRF, as described below); thus, the dynamic stiffness of an m-c-k system is

$$\left\{\frac{L\left[f_x(t)\right]}{L[x(t)]}\right\}_{s=j\omega} = (k-\omega^2 m)+j\omega c = k\left[(1-\beta^2)+j2\zeta\beta\right] \tag{10.5.3}$$

The definition of Equation 10.5.3, with the Laplace transform of action in the numerator and that of displacement in the denominator, might seem unorthodox, since an action (force or moment) is usually the physical input and a displacement is usually the physical output. However, this type of FRF is useful in practice for experimental measurements, as is described in homework Problem 10.16. Dynamic stiffness is also the basis of an advanced method for analyzing theoretically the dynamics of distributed-parameter structures (Clough and Penzien, 1975, Chapter 20; Fergusson and Pilkey, 1991).

For mechanical and structural systems, the first and second time derivatives of displacement are also of interest, primarily because velocity sensors and, especially, accelerometers are often used to measure motion. Accordingly, appropriate FRFs have been defined. For *mobility*, the subject variable is a velocity, and the reference variable is an action. Since $L[\dot{x}(t)] = s \times L[x(t)]$, the mobility of an m-c-k system, from Equation 10.5.2, is

$$\left\{\frac{L[\dot{x}(t)]}{L\left[f_x(t)\right]}\right\}_{s=j\omega} = \frac{j\omega}{(k-\omega^2 m)+j\omega c} = \frac{1}{c}\left[\frac{j2\zeta\beta}{(1-\beta^2)+j2\zeta\beta}\right] \tag{10.5.4}$$

The inverse of mobility is usually given the name *mechanical impedance*. Accordingly, the mechanical impedance of an m-c-k system is

$$\left\{\frac{L\left[f_x(t)\right]}{L[\dot{x}(t)]}\right\}_{s=j\omega} = \frac{(k-\omega^2 m)+j\omega c}{j\omega} = c\left[1-j\left(\frac{1-\beta^2}{2\zeta\beta}\right)\right] \tag{10.5.5}$$

For *accelerance* (also known as *inertance*), the subject variable is an acceleration, and the reference variable is an action. Since $L[\ddot{x}(t)] = s^2 \times L[x(t)]$, the accelerance of an m-c-k system, from Equation 10.5.2, is

Label 'eqn:10.32' multiply defined

It is worth noting that for static loading ($\omega = 0$), mechanical admittance equals a nonzero static flexibility, but both mobility and accelerance are zero. The inverse of accelerance is called *apparent mass*, as motivated by the following expression for an m-c-k system:

$$\left\{\frac{L\left[f_x(t)\right]}{L[\ddot{x}(t)]}\right\}_{s=j\omega} = \frac{(k-\omega^2 m)+j\omega c}{(j\omega)^2} = m\left[\left(1-\frac{1}{\beta^2}\right)-j\frac{2\zeta}{\beta}\right] \tag{10.5.6}$$

All of the mechanical and structural FRFs described in this section have dimensional units. For example, common units of accelerance used in the dynamic testing of structures are *G*'s or meters/s^2 (of acceleration) per pound or newton (of force).

[1]Clearly, the SI unit for impedance is the ohm (Ω), that of resistance; apparently since admittance is the inverse of impedance, the SI unit for admittance was given the whimsical name *mho* (Ω^{-1}).

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10.6: Beating Response of Second Order Systems to Suddenly Applied Sinusoidal Excitation

Thus far in Chapter 10, we have considered only the pure frequency response of 2nd order systems, *i.e.*, the steady-state sinusoidal response to sinusoidal excitation. In frequency-response analysis based upon Equation 4.7.18, initial conditions are ignored, as is any non-steady-sinusoidal start-up portion of the excitation; in other words, both pure sinusoidal excitation and response are idealized to start at some indefinite time in the past. But any real excitation must start at some definite time, at which time any real dynamic system has some initial conditions. After the initiation of sinusoidal excitation, there will be a transitional interval of response between the initial conditions and, provided the system is stable, the achievement of steady-state sinusoidal response. Section 4.2 illustrates such a transitional interval for a stable 1st order system, which does not exhibit natural oscillatory behavior. In this section, we consider a form of transitional response, *beating*, that is often observed in lightly damped vibrating systems, the simplest forms of which are underdamped 2nd order systems.

The basic character of beating response to sinusoidal excitation is best illustrated theoretically by the idealized undamped 2nd order system. Consider again the standard ODE for undamped 2nd order systems, Equation 10.1.1. Suppose now that $u(t) = 0$ for $t < 0$, and let us define the suddenly applied sinusoidal (SAS) excitation as $u(t) = U \sin \omega t$ for $t \geq 0$. (We use $\sin \omega t$ here rather than the customary $\cos \omega t$ because it is more natural, especially for mechanical systems, that the excitation begins continuously from zero at t = 0, rather than with a discontinuity.) Thus, the ODE Equation 10.1.1 becomes

$$\ddot{x} + \omega_n^2 x = \omega_n^2 u(t) = \omega_n^2 U \sin \omega t, \text{ for } t \geq 0 \tag{10.6.1}$$

For simplicity, we let the initial conditions be zero: $\dot{x}(0) = 0$ and $x(0) = 0$. The complete algebraic solution of ODE Equation 10.6.1 with these rest ICs is found in homework Problem 1.12 by the method of undetermined coefficients:

$$x(t) = \frac{U}{1 - (\omega/\omega_n)^2} \left[\sin \omega t - (\omega/\omega_n) \sin \omega_n t\right], \text{ valid for } \omega \neq \omega_n, t \geq 0 \tag{10.6.2}$$

Equation 10.6.2 is valid only if the excitation frequency is different than the natural frequency, $\omega \neq \omega_n$; however, this is not a serious practical restriction, because it is almost impossible in reality to excite a system exactly at its natural frequency. [But see homework Problem 1.12.4 for the theoretical solution valid if $\omega = \omega_n$.]

It is appropriate to write the response solution Equation 10.6.2 in a slightly different algebraic form:

$$x(t) = \frac{U}{1 - (\omega/\omega_n)^2} \left[\sin \omega t - \sin \omega_n t + (1 - \omega/\omega_n) \sin \omega_n t\right] \tag{10.6.3}$$

This form of $x(t)$ allows us to invoke a useful trigonometric identity:

$$\sin \omega t - \sin \omega_n t = 2 \cos \frac{1}{2} (\omega t + \omega_n t) \times \sin \frac{1}{2} (\omega t - \omega_n t) \tag{10.6.4}$$

Now we can express the solution Equation 10.6.3 in an equation that more clearly displays the characteristics of beating:

$$x(t) = \frac{U}{1 - (\omega/\omega_n)^2} \left[2 \cos \frac{1}{2} (\omega + \omega_n) t \times \sin \frac{1}{2} (\omega - \omega_n) t + (1 - \omega/\omega_n) \sin \omega_n t\right] \tag{10.6.5}$$

For computation, with definitions of driving frequency ratio $\beta \equiv \omega/\omega_n$ and natural period $T_n \equiv 2\pi/\omega_n$, we express Equation 10.6.5 in the dimensionless form:

$$\frac{x(t)}{U} = \frac{1}{1 - \beta^2} \left[2 \cos \left((\beta + 1)\pi \frac{t}{T_n}\right) \times \sin \left((\beta - 1)\pi \frac{t}{T_n}\right) + (1 - \beta) \sin \left(2\pi \frac{t}{T_n}\right)\right] \tag{10.6.6}$$

Let us use numerical evaluation and graphical display to help us examine the parts of Equations 10.6.5 and 10.6.6; Figure 10.6.1 on the next page includes plots of both suddenly applied sinusoidal excitation and the associated response calculated from Equation 10.6.6, with the excitation frequency set at 90% of the natural frequency, $\omega = 0.9\omega_n$ or $\beta = 0.9$.

With use of Figure 10.6.1, we can identify the roles of the terms within square brackets of Equations 10.6.5 and 10.6.6 for any case in which the excitation frequency ω is somewhat close to the system natural frequency ω_n. The dominant (in magnitude) term that closely resembles the excitation is $2\cos\frac{1}{2}(\omega+\omega_n)t$, a sinusoid whose frequency is the average of the excitation and natural frequencies. But this dominant term is multiplied by $\sin\frac{1}{2}(\omega-\omega_n)t$, which is a slowly varying amplitude modulator whose frequency is half the difference between the excitation and natural frequencies. For the case of Figure 10.6.1 with $\omega/\omega_n = 0.9$, we have $\left|\frac{1}{2}(\omega-\omega_n)\right| = 0.05\omega_n$. Therefore, the full period of the amplitude-modulating term is $2\pi/0.05\omega_n = 20T_n$; from Figure 10.6.1, this period is the total interval between time $t = 0$ s and the **second** subsequent minimum of the amplitude-modulating envelope. Furthermore, the beating terms are minimal in amplitude at every half-period of the amplitude-modulating term, in this case $0, 10T_n, 20T_n, ...$, and the beating terms are maximal in amplitude at instants between the half-period instants, in this case $5T_n, 15T_n, 25T_n, ...$

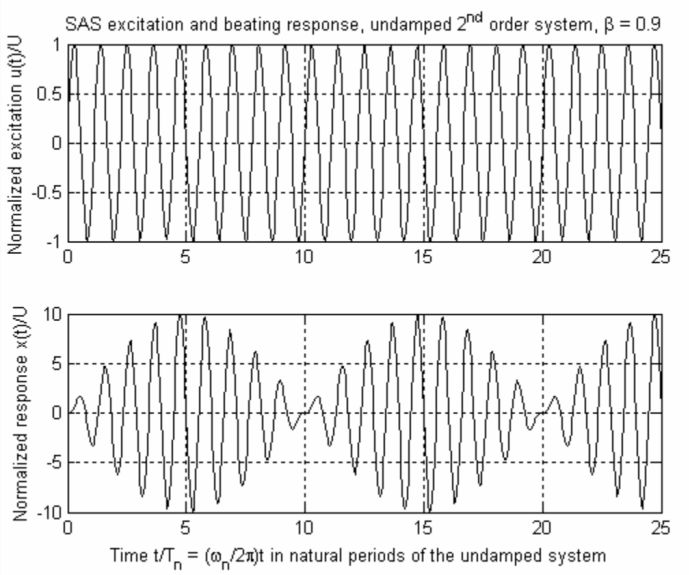


Figure 10.6.1: Upper graph: suddenly applied (at $t = 0$) sinusoidal (SAS) excitation with frequency $\omega = 0.9\omega_n$; lower graph: response from zero initial conditions of a standard undamped 2nd order system having natural frequency ω_n.

The *apparent* ***period*** *of beating* is the interval between successive minima or successive maxima of response, $10T_n$ in the case of Figure 10.6.1, which is half the period of the amplitude-modulating term. Therefore, the *apparent* ***frequency*** *of beating* is $|\omega-\omega_n|$, which is twice the frequency of the amplitude-modulating term. Similarly, when we hear two musical tones of close but not identical pitches (frequencies), the frequency of beating that we perceive is the difference between the two tonal frequencies.

Let us consider now the remaining term, $(1-\omega/\omega_n)\sin\omega_n t$, within the brackets of Equations 10.6.5 and 10.6.6. Pure undamped beating, in general, is the combination of two sinusoids that have different but closely spaced frequencies, as the two sinusoids pass into and out of phase with each other. The beating of the undamped 2nd order system shown on Figure 10.6.1 is the combination of two sinusoids in the term $\sin\omega t - \sin\omega_n t$ of Equation 10.6.3. These two sinusoids represent

1. the driven response of the system at the excitation frequency ω, and
2. most of the free-vibration response of the system at its natural frequency ω_n.

When these two sinusoids are in phase, they combine and, together with the smaller contribution of the term $(1-\omega/\omega_n)\sin\omega_n t$, form the maximal response; when they are out of phase, they nullify each other completely, leaving only the small remaining part of the free-vibration response due to the term $(1-\omega/\omega_n)\sin\omega_n t$.

Thus far, we have examined response to suddenly applied sinusoidal excitation of a physically unrealistic undamped 2nd order system. Let us consider next the influence of more realistic subcritical damping by solving for complete time response of an underdamped 2nd order system the ODE $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = \omega_n^2 u(t)$ with $u(t) = 0$ for $t < 0$ and $u(t) = U\sin\omega t$ for $t \geq 0$, and with rest ICs $x(0) = 0$ and $\dot{x}(0) = 0$. An appropriate convolution-integral solution is Equation 9.3.8:

$$x(t) = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau \times u(t-\tau)d\tau = \frac{\omega_n^2}{\omega_d}\int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau}\sin\omega_d\tau \times U\sin\omega(t-\tau)d\tau$$

The damped frequency is $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$, so we can express the response integral in dimensionless form as

$$\frac{x(t)}{U} = \frac{\omega_n}{\sqrt{1-\zeta^2}} \int_{\tau=0}^{\tau=t} e^{-\zeta\omega_n\tau} \sin\omega_n\sqrt{1-\zeta^2}\tau \times \sin\omega(t-\tau)d\tau \tag{10.6.7}$$

After evaluation of the formidable integral in Equation 10.6.7, the solution is[1]

$$\frac{x(t)}{U} = \frac{e^{-\zeta\omega_n t}\beta\left[2\zeta\cos\omega_d t - \frac{(1-\beta^2-2\zeta^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] + (1-\beta^2)\sin\omega t - 2\zeta\beta\cos\omega t}{(1-\beta^2)^2 + (2\zeta\beta)^2} \tag{10.6.8}$$

The equation for response of an **undamped** 2nd order system that is most similar in algebraic form to Equation 10.6.8 is Equation 10.6.2, and you can easily show that for $\zeta = 0$ Equation 10.6.8 reduces to Equation 10.6.2. However, unlike Equation 10.6.2, Equation 10.6.8 apparently cannot be reconfigured into a form comparable to Equation 10.6.5 that is amenable to physical interpretation relative to beating behavior. To learn the physical character of Equation 10.6.8, it seems that the best we can do is to evaluate it computationally with relevant numerical parameters and then interpret the results. For computation of Equation 10.6.8, with definitions of driving frequency ratio $\beta \equiv \omega/\omega_n$ and natural period $T_n \equiv 2\pi/\omega_n$, we can express the time-dependent function arguments of Equation 10.6.8 in the following forms in terms of dimensionless time t/T_n: $-\zeta\omega_n t = -2\pi\zeta\,(t/T_n)\,, \omega_d t = 2\pi\sqrt{1-\zeta^2}\,(t/T_n)$, and $\omega t = 2\pi\beta\,(t/T_n)$. Figure 10.6.2 includes plots of SAS response calculated from Equation 10.6.8 for zero damping and for light viscous damping ratios ζ = 0.01 and 0.02, with the excitation frequency set at 90% of the natural frequency, $\omega = 0.9\omega_n$ or $\beta = 0.9$, the same as for Figure 10.6.1.

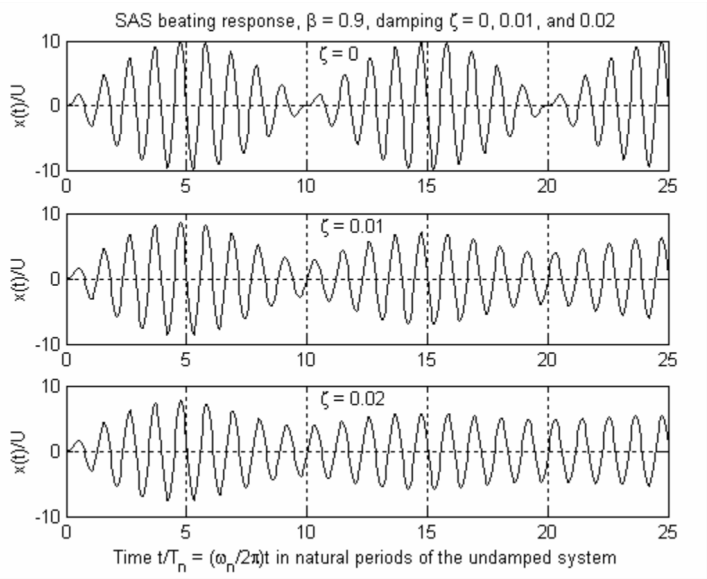


Figure 10.6.2: Calculations of Equation 10.6.8 for a standard 2nd order system having undamped natural frequency ω_n, with zero damping (top) and two positive values of viscous damping ratio, ζ = 0.01 and 0.02; response from rest initial conditions to a suddenly applied (at t = 0) sinusoidal (SAS) excitation with frequency $\omega = 0.9\omega_n$.

The response in Figure 10.6.2 for $\zeta = 0$ is, of course, the same as that in Figure 10.6.1, with very significant beating. However, the responses for ζ = 0.01 and 0.02 show that even these relatively small quantities of damping suppress the beating, at least for this case of $\omega = 0.9\omega_n$. Although less obvious than for zero damping, the apparent period of beating for $\zeta = 0.01$, $\sim 10T_n$, is still quite evident. However, for $\zeta = 0.02$, the response in Figure 10.6.2 after 25 natural periods has almost settled into its steady-state condition, for which you can calculate from Equation 10.2.7 that the steady-state frequency-response magnitude ratio is $X/U = 5.171$. To investigate the nature of beating for any other case of damping ratio $\zeta > 0$ and excitation frequency ratio $\beta \equiv \omega/\omega_n$, we would need again to evaluate Equation 10.6.8 numerically; see, for example, homework Problem 10.17.

[1]The author derived Equation 10.6.8 first with MATLAB by applying the `syms`, `int`, `simple`, and `pretty` symbolic operations, and then by implementing the human touch with pencil and paper to simplify further the MATLAB result. See homework Problems 1.6 and 1.9.

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10.7: Chapter 10 Homework

Problems 10.1–10.4 are frequency response problems (inspired by Myklestad, 1944) about standard 2nd order mass-damper-spring systems. In each case, the mass is excited by sinusoidal force $f_x(t) = F\cos\omega t$, and it responds with steady-state sinusoidal displacement $x(t) = X\cos(\omega t + \phi)$, where excitation frequency in Hz is $f = \omega/2\pi$. Some partial answers are given to help you see if you are on the right track; you still must calculate these values as if the answers were not given. To derive maximum benefit from these problems, use the appropriate equations for calculations, but also figure out how the results are represented in the FRF magnitude and phase graphs, Figures 10.1.1 and 10.2.1. Be sure to use the appropriate value for g, either 386.1 inch/s^2 or 9.807 m/s^2.

1. (Myklestad, 1944, p. 61) Suppose that damping is negligibly small. The measured weight of the mass is W = 19.0 lb, and the spring stiffness is k = 13.0 lb/inch. With F = 5.00 lb at a certain frequency of excitation, the response amplitude is X = 0.750 inch. Calculate the dimensionless dynamic amplification $X/(F/k)$. There are **two** different excitation frequencies, \(f_{1}partial answers: ϕ_1 = 0°, f_2 = 3.18 Hz)
2. Suppose that damping is negligibly small. The measured weight of the mass is W = 8.50 kgf, and the spring stiffness is k = 2.50 kN/m. With F = 5.00 N at a certain frequency of excitation, the response amplitude is X = 20.0 mm. Calculate the dimensionless dynamic amplification $X/(F/k)$. There are **two** different excitation frequencies, \(f_{1}partial answer: f_2 = 2.86 Hz)
3. The mass weighs W = 1.26 kN, and the viscous damping and spring constants are, respectively, c = 2,400 N-s/m and k = 1.12 MN/m. Calculate the undamped natural frequency f_n in Hz and the viscous damping ratio ζ. Suppose that the response amplitude is X = 2.50 mm when the phase ϕ = −90.0°; calculate the corresponding amplitude F of the excitation force. (*partial answer*: F = 560 N)
4. (Myklestad, 1944, p. 113) The mass weighs W = 2.36 lb, and the spring and viscous damping constants are, respectively, k = 37.8 lb/inch, and c = 0.0169 lb-s/inch. The force amplitude is F = 0.785 lb. Calculate the critical damping constant c_c, the damping ratio ζ, the maximum possible amplitude of response X_r, and the amplitude X and phase ϕ (in degrees) for excitation at 13.0 Hz. (Note: to calculate ϕ correctly, you must account carefully for the quadrant of the fraction in the arctangent argument; if you just calculate the fraction and use the arctan function on your calculator, you will get the wrong answer; if you need some help, see the `atan2` command in MATLAB). Is the small-ζ approximation appropriate for this system? (*partial answers*: X_r = 0.591 inch, ϕ = −155°)
5. A frequency response measurement is conducted on a standard 2nd order m-c-k system with known stiffness constant k, so that the input pseudo-static displacement is calculated from the measured input force, $u(t) = f_x(t)/k$. A measured graph of the input and output displacements at one particular excitation frequency is shown below. Calculate from the graph, with as much accuracy as the data permits, the following quantities: the frequency (in Hz); the FRF magnitude ratio X/U; the FRF phase ϕ (in degrees). What are the natural frequency f_n (in Hz) and the damping ratio ζ of this system?

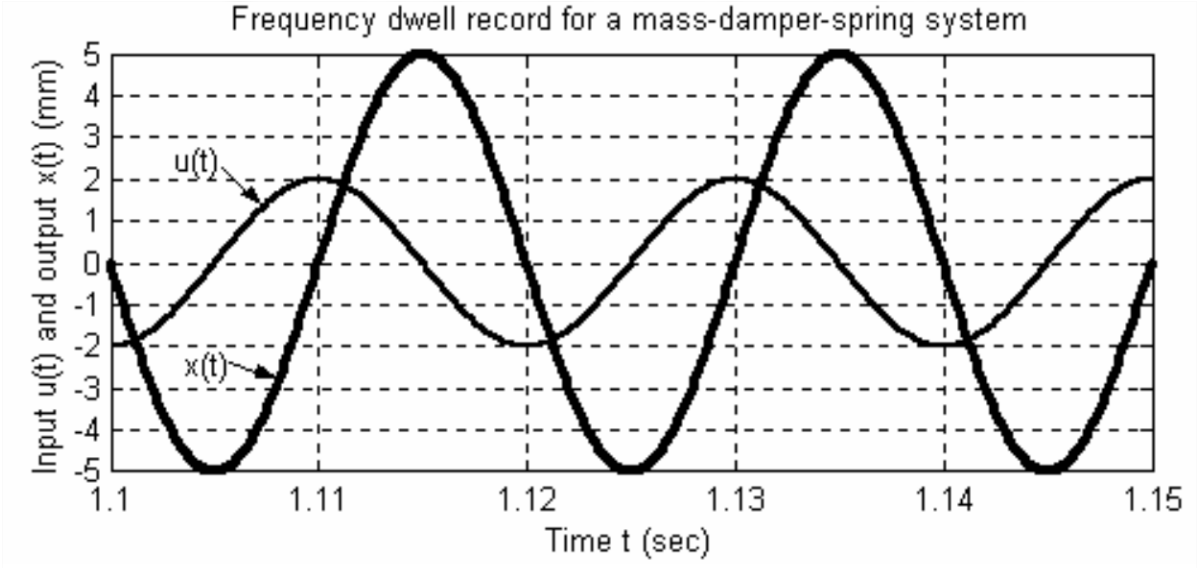


Figure 10.7.1

6. The mechanical system of homework Problem 9.1 [with $f_x(t) = 0$] has *base excitation* input $x_i(t)$ and output $x(t)$. The ODE of motion is $m\ddot{x} + (c_1 + c_2)\,\dot{x} + kx = c_1\dot{x}_i + kx_i$, which includes right-hand-side (RHS) dynamics, where m is the mass, c_1 and c_2 are viscous damping constants, and k is the stiffness constant.

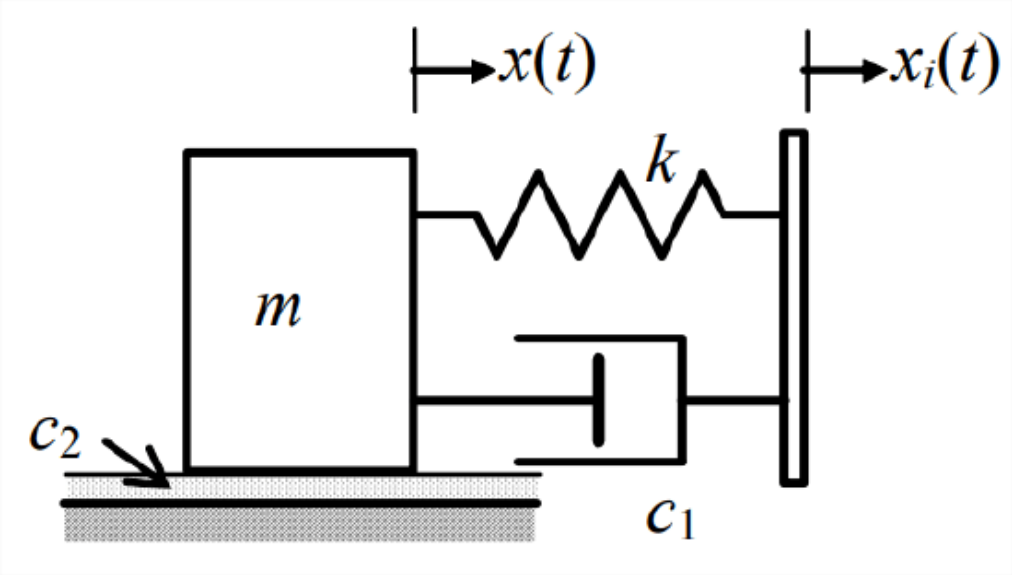


Figure 10.7.2

   1. Derive for this system the transfer function $\mathrm{TF}(s) = L[x(t)]|_{ICs=0} / L[x_i(t)]$.
   2. Use the fundamental result $TF(j\omega) = FRF(\omega) = [X(\omega)/X_i]\, e^{j\phi(\omega)}$ and the result from part 10.6.1 to derive the relatively simple algebraic equation for **real** magnitude ratio $X(\omega)/X_i$. This equation should be in terms of physical parameters m, c_1, c_2, and k, and excitation frequency ω. [NOTE: You should implement complex division by the **polar** method described in Section 2.1; do **not** use the rectangular method, because it will produce an awful algebraic mess!] If you wish, you may derive the associated relatively simple algebraic/trigonometric equation for the **real** phase angle $\phi(\omega)$ (in radians), but that is not required.
   3. Let the system constants be m = 128.5 kg, c_1 = 1,600 N-s/m, c_2 = 800 N-s/m, and k = 1.12 MN/m. Calculate and plot the frequency response (magnitude ratio X/X_i, and phase ϕ in **degrees**) over the range of excitation frequencies 0-30 Hz. Use the over-under graphical format of Figure 10.2.1, except plot both graphs on **linear** (not log) scales. As in the script that produced Figure 10.2.1, let MATLAB do most of the work for you by starting with the complex frequency response function $FRF(\omega)$ from part 10.6.2, then using MATLAB's capability for performing complex arithmetic with the `abs` and `angle` functions. If you need some help with the array multiplications required for this FRF, see the "`frf=`..." command lines in the script that produced Figure 10.2.1. Submit your MATLAB script along with your FRF graphs.

7. The drawing represents a very simplified model of a land vehicle driving over a wavy (washboard) road with constant forward velocity V. The vehicle mass is m, and it is connected to the rolling wheel through the shock strut that has viscous damping constant c and stiffness constant k. The deviation of the road surface from flatness is denoted as $y_s(x) = Y_s \cos(2\pi x/\lambda)$, in which x is horizontal distance traveled from a reference position, Y_s is the amplitude (wave height at the crests), and λ is the wave length. We make the following assumptions for this model: the wheel itself is rigid, it rolls without slipping, and it maintains contact with the road at all times due to the weight of mass m; also, wave height Y_s is small relative to wheel radius, so that, in effect, the contact point is always directly beneath the wheel axle, which means that y_s plays the role of *base excitation*, and essentially is a function of time: $y_s(x) = Y_s \cos \omega t$.
   1. Assume for the moment that $y_s(t)$ is **arbitrary** base excitation. Draw a dynamic freebody diagram of mass m, and derive from your DFBD the ODE of vertical dynamic motion $y(t)$, measured relative to the static equilibrium position for a flat road surface. **First**, write the ODE in terms of constants m, c, and k; **second**, use the standard definitions from Equations 7.1.3 and 9.1.6 to write the ODE in terms of natural frequency ω_n and viscous damping ratio ζ.
   2. Use the second ODE of part 10.7.1 to derive the complex frequency response function for arbitrary input $y_s(t)$ and output $y(t)$ as s$FRF(\omega) = \frac{1+j2\zeta\beta}{(1-\beta^2)+j2\zeta\beta} \equiv \frac{Y(\omega)}{Y_s} e^{j\phi(\omega)}$, in which $\beta \equiv \omega/\omega_n$. Even though this system has right-hand-side dynamics that influence response, let us consider the condition $\omega = \omega_n$. Derive equations for FRF magnitude ratio $Y(\omega_n)/Y_s$ and phase $\phi(\omega_n)$, in terms of ζ. (NOTE: To obtain the most manageable algebra, implement complex division by the **polar** method described in Section 2.1, **not** by the rectangular method.) If ζ is small, *e.g.*, $\zeta = 0.05$, how close are your values of $Y(\omega_n)/Y_s$ and $\phi(\omega_n)$ to the values at $\omega = \omega_n$ that would prevail if there were no righthand-side dynamics?
   3. For the washboard road surface, we use the kinematic relation $x = Vt$ to express the base excitation as a sinusoidal function: $y_s(t) = Y_s \cos(2\pi Vt/\lambda) \equiv Y_s \cos \omega t$, in which $\omega \equiv 2\pi V/\lambda$. Write the algebraic equation, in terms of m, k, and λ, for the velocity V_n at which base excitation frequency equals the undamped natural frequency, $\omega = \omega_n$.

8. A particular device is known to be an LTI mass-damper-spring system. You (the engineer) are required to identify experimentally the mass m, the viscous damping constant c, and the stiffness constant k. First, you apply a static force of 162 lb, and you observe that the mass deflects statically by 0.108 inch. Next, you run a stepped-sine frequency-response test, applying sinusoidal force onto the mass, with the frequency increasing in small increments from 8 to 20 Hz. You measure at each frequency the steady-state input force magnitude F (in lbs), the output translation magnitude X (in inches) and the phase of translation relative to force. The frequency response is plotted below. Use this information to calculate m, c, and k in consistent units and with as much accuracy as the data permits. Show all calculations.

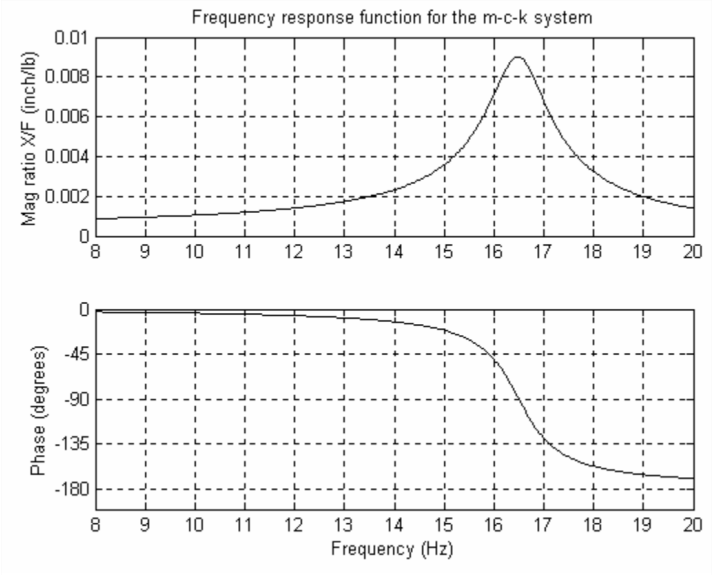


Figure 10.7.3

9. A particular device is known to be an LTI mass-damper-spring system, but the mass m, viscous damping constant c, and stiffness constant k are unknown. It is required that the system parameters be identified experimentally with a combination of static and vibration testing. First, a static force of 52.0 N is applied, and the mass is measured to deflect statically by 1.13 mm. Next, sinusoidal force is applied to the mass by an electromagnetic shaker, with the frequency increased in small increments from 1 to 10 Hz The steady-state input force magnitude F and output translation magnitude X are measured at each frequency, as is the phase ϕ of translation relative to force. The frequency-response translation-to-force magnitude ratio and phase are plotted at left. Use this information to calculate m, c, and k in consistent SI units and with as much accuracy as the data permits. Show all calculations.

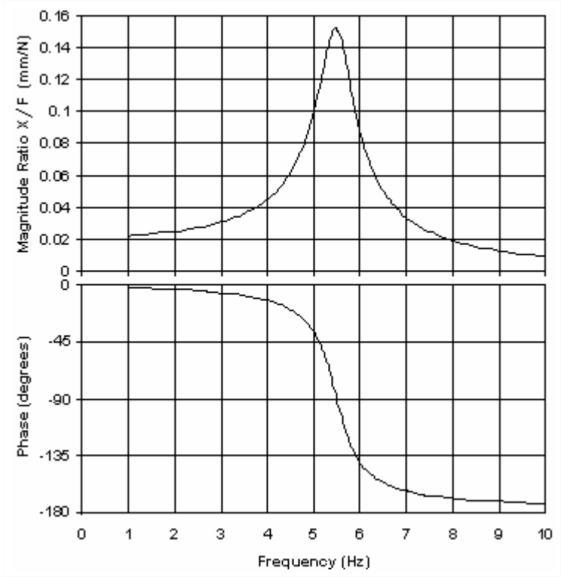


Figure 10.7.4

10. You, the engineer, are asked to determine experimentally the electrical constants (inductance L and resistance R_L) of a small coil, based upon the simple model of inductor and resistor in series, to see if these constants match the design

specifications. You elect to infer these constants from a frequency response test on an LRC circuit driven by a sinewave voltage generator, with the coil connected in series to a calibrated capacitor with capacitance C = 0.500 μF. The circuit is shown in the drawing (from the example in Section 9.2), and its ODE in standard 2nd order form is $\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i(t)$. You run a stepped-sine frequency-response test, applying sinusoidal input voltage of constant magnitude $E_i = 1.00$ V, with the frequency increasing in small increments from 100 to 1,000 Hz. You measure at each frequency the steady-state output voltage magnitude E_o, and the phase ϕ of output voltage relative to input voltage. The frequency response is plotted on the next page. Use this information to calculate L and R_L in consistent units and with as much accuracy as the data permits. Show all calculations.

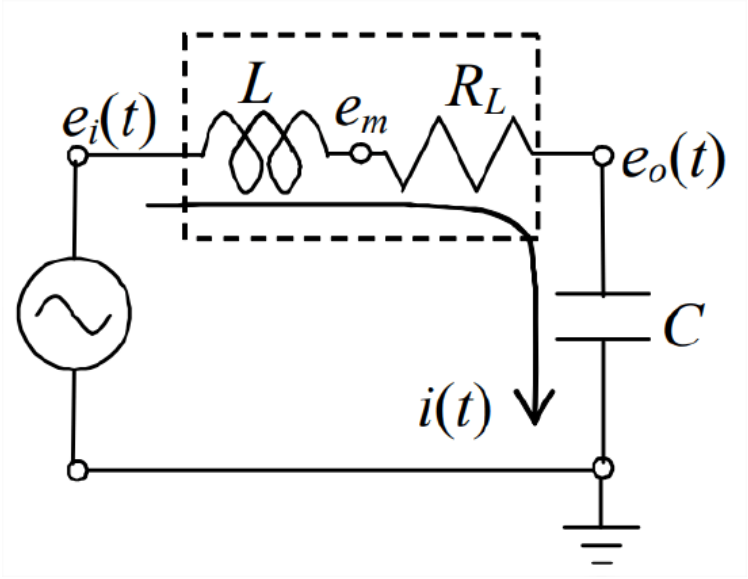


Figure 10.7.5

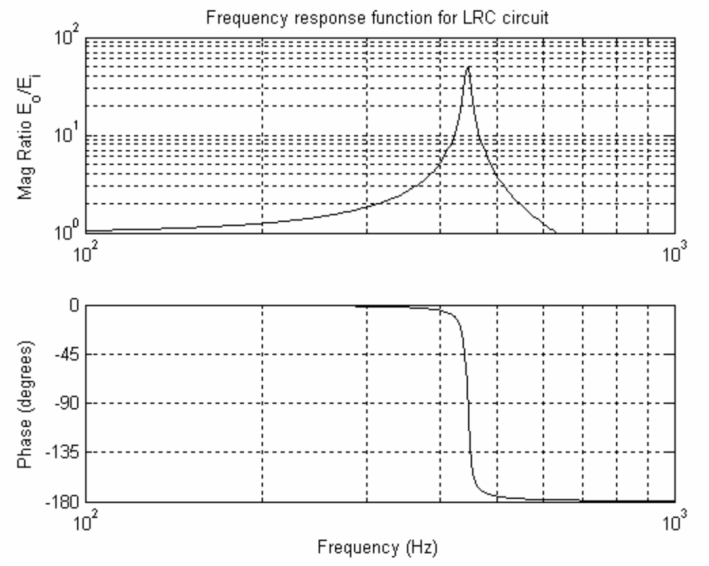


Figure 10.7.6

11. Consider the RC band-pass filter described in Section 10.4, for which the frequency response function is $FRF(\omega) = TF(j\omega) = \frac{j\omega\tau_H}{(j\omega\tau_H+1)(j\omega\tau_L+1)}$. The purpose of this problem is to demonstrate the practical function for which that filter is designed, by means of FRF graphs. Let $\tau_H = \frac{1}{2\pi\times 10}$ s and $\tau_L = \frac{1}{2\pi\times 500}$ s to make the high-pass and low-pass break frequencies be, respectively, 10 Hz and 500 Hz. Now, use MATLAB to calculate and plot the FRF magnitude ratio and phase (in degrees) over the frequency range 0.1 to 10,000 Hz. Use the graphical format described in homework Problem 4.3 (log-log for magnitude ratio, semilog for phase in **degrees**, magnitude ratio graph directly over phase graph). As in Problem 4.3, let MATLAB do most of the work for you by starting with the complex FRF representation, and then using MATLAB's capability for performing complex arithmetic. Recall, in particular, that the MATLAB function abs calculates the absolute value (magnitude) of a complex number, and the MATLAB function `angle` calculates the angle in **radians** of a complex number.

12. An idealized mechanical model for motion sensors of the **seismic** type is shown in the figure. Such a sensor is entirely mounted on a moving body (is said to be "structure-borne"), unlike, for example, the proximity displacement sensor shown in Figures 7.6.1 and 7.6.5. The mechanical part of the sensor is a mass-damper-spring system sealed within a sturdy case, which is attached firmly to the body whose motion is sensed. The stimulus to the sensor is *base excitation*: translation $x_i(t)$ of the body and the case. The **absolute** translation of seismic mass m is $x(t)$. A transducer within the case detects the **relative** translation $z(t) \equiv x(t) - x_i(t)$ and generates an electrical signal[1] proportional to $z(t)$, which typically is displayed and/or recorded by a data-acquisition-and-processing system, and may also serve as an input to a control system.
    1. Sketch and label appropriate free-body diagrams, then use your FBDs to derive an ODE of motion for $z(t)$ [not $x(t)$] in terms of constants m, c, and k, and variable input quantity $x_i(t)$ or its derivatives. Convert your ODE into the standard form Equation 10.2.1, except with $z(t)$ [not $x(t)$] as the dependent variable: $\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \omega_n^2 u(t)$. Write an explicit equation for standard input quantity $u(t)$ in terms of the appropriate constants and $x_i(t)$ or its derivatives.
    2. Evaluate the use of this seismic sensor as a translational *accelerometer*, a sensor of the body's acceleration in the direction defined by $x(t)$ and $x_i(t)$. First, use the results of part 10.12.1 to show that the pseudo-static response is $z_{ps}(t) = -(m/k)\ddot{x}_i(t)$. This means that if the quantity $\ddot{z} + 2\zeta\omega_n\dot{z}$ is small (or zero) in comparison with $\omega_n^2 z$, then the seismic sensor acts as an accelerometer, since then $z(t)$ and its corresponding electrical signal are directly proportional to $\ddot{x}_i(t)$[2]. This pseudo-static response is reflected clearly in the FRF graphs of Figure 10.2.1: If the frequencies of motion within base excitation $x_i(t)$ are substantially below the natural frequency of the sensor ($\beta \equiv \omega/\omega_n = 1$), then both the magnitude and phase of response follow closely those of excitation. *An accelerometer is a seismic sensor intended to measure frequencies that are considerably **lower** than the sensor's own natural frequency*. Suppose you are designing an accelerometer that will measure with reasonable engineering accuracy body accelerations whose frequencies are no higher than 25% of the sensor's natural frequency. What is the **highest** value of damping ratio ζ (to three significant digits) that your sensor can have in order that phase error will be less than 5°, and what is the maximum magnitude **error** for that ζ? Use Figure 10.2.1 to find an estimate, then make trial-and-error iterative calculations (preferably with MATLAB) to determine more precisely the required ζ and magnitude ratio.
    3. Evaluate the use of this seismic sensor as a translational *seismometer*, a sensor of the body's translation $x_i(t)$. Now, due to the right-hand-side second derivative $\ddot{x}_i$, your first ODE from part 10.12.1 is **not** in the standard form Equation 10.2.1, so it is necessary to derive new response equations. The appropriate transfer function is defined as $TF(s) = L[z(t)]|_{IC_{S=0}} \div L[x_i(t)]_{ICs=0}$. Derive this $TF(s)$, then use it to derive the corresponding FRF:

$$FRF(\omega) \equiv \frac{Z(\omega)}{X_i} e^{j\phi(\omega)} = \frac{\beta^2}{(1-\beta^2) + j2\zeta\beta}, \text{ with } \beta = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{k/m}}, \text{ and } \zeta = \frac{c}{2m\omega_n}$$

    Suppose that a seismometer has damping ratio $\zeta = 0.22$, high enough to dampen resonant response quickly. Plot over the range $0.1 \le \beta \le 10$ the frequency response magnitude ratio $Z(\omega)/X_i$ and, separately, phase angle $\varphi(\omega)$ in degrees. (If necessary, see for guidance the description of producing Figure 10.2.1 with MATLAB.) From these graphs, and if you wish from low- and high-frequency asymptotes of $Z(\omega)/X_i$, you should recognize that: *A translational seismometer is a seismic sensor intended to measure frequencies that are considerably **higher** than the sensor's own natural frequency*. Suppose this seismometer is intended to measure with reasonable engineering accuracy body translations whose frequencies are no lower than five (5) times that of the sensor's natural frequency. Calculate the greatest values of sensor error, percent of magnitude ratio and degrees of phase angle[3]? Describe physically the motion of the seismic mass in response to base-motion frequencies that are considerably higher than the seismometer's natural frequency.

13. The drawing at right is the prototypical idealized model for the subject of *vibration isolation*. Suppose that mass m in the drawing houses a reciprocating internal combustion engine with a single vertically oriented piston-cylinder pair, and that the damper-spring-base assembly attaches the mass to the floor. The engine exerts an oscillatory vertical force $f(t)$ on mass m , and the mass-damper-spring system transmits force through the rigid base to the stiff floor, which reacts the transmitted force with oscillatory vertical force $f_R(t)$[4]. For the benefit of the floor and its superstructure (building, vehicle, *etc.*) and for the comfort of the occupants, it is usually desirable to design the damper-spring support so that $f_R(t)$ is as small as practical, in other words, to isolate the engine from the floor. In practice, the damper-spring support often is an elastomeric (rubber-like) padding.
    1. Use a dynamic free-body diagram (DFBD), if necessary, to write the ODE of dynamic motion $y(t)$ relative to the static equilibrium position, in terms of excitation force $f(t)$ and constants m, c, and k. Take the Laplace transform of the ODE (assuming zero ICs) to obtain an equation for transform $L[y(t)] \equiv Y(s)$, which equation will include as the input quantity transform $L[y(t)] \equiv F(s)$. Next, write an equation for dynamic reaction force $f_R(t)$ in terms of $y(t)$ and its derivative and the appropriate constants, then take its Laplace transform to find an equation for transform $\backslash L[f_R(t)] \equiv F_R(s)).Next, combinethetwotransformequationsappropriatelytofindthetransferfunction\backslash(TF(s) = F_R(s)/F(s)$ relating the floor-reaction dynamic force to the excitation force. Finally, use the standard definitions of ω_n, ζ, and β to show that the corresponding complex frequency-response function, known as *transmissibility*, is $TF(j\omega) \equiv FRF(\omega) \equiv \frac{F_R(\omega)}{F} e^{j\phi(\omega)} = -\frac{1+j2\zeta\beta}{(1-\beta^2)+j2\zeta\beta}$[5].

2. Plot on a single graph the transmissibility magnitude ratio $F_R(\omega)/F$ for the cases of zero damping, $\zeta = 0$, and the relatively high damping ratio $\zeta = 0.25$; plot these curves on linear (not logarithmic) scales over the frequency ratios $0 \leq \beta \leq 4$, and display the magnitude ratio only over the range $0 \leq F_R(\omega)/F \leq 2.5$. (If necessary, see for guidance the description of producing Figure 10.2.1 with MATLAB.) Explain how your graph shows (do not bother with any theory) that there is effective vibration isolation only if $\beta > \sqrt{2}$. Suppose that you know mass m and excitation frequency ω; what is the range of spring constant k for which there is effective vibration isolation? Physically, should the springs be stiff or soft, and how stiff or soft? Does positive damping either increase or decrease the effectiveness of vibration isolation at all excitation frequencies, or is damping a mixed blessing, helpful in some range of frequencies, but not so much in others?

14. The translational velocity $v(t)$ of a point on an object is sometimes measured with use of a structure-borne accelerometer (see homework Problem 10.12). An accelerometer is a transducer that senses translational acceleration in one direction, *e.g.* $a(t) = \ddot{x}(t)$, and converts the motion into an electrical voltage signal, $e_a(t)$, which can be displayed, or recorded, or processed, or used in a control system. The transducer's output signal is nominally $e_a(t) = C_{ea}a(t) + n(t) + \varepsilon$: C_{ea} is the sensor's calibration factor, with units such as volts per meter/s; $n(t)$ is small, randomly varying electrical "noise"; and \(\mathcal{E}) is a small, **constant** *offset voltage*. Error voltage $n(t) + \varepsilon$ is unrelated to the sensed motion; it is an unwanted but practically unavoidable by-product of a transducer's circuitry. Acceleration is the rate of change of velocity, $\dot{v}(t) = a(t)$, so the accelerometer's output signal must be integrated to produce an electrical signal $e_v(t)$ that is proportional to velocity. Thus, the basic ODE that an &(tv) **exact integrator** (in analog circuitry or a digital algorithm) would solve is $\dot{e}_v(t) = (1/T)e_a(t)$, where T is a physical constant having dimensions of time (*e.g.*, $T = -RC$ for the op-amp-circuit integrator of homework Problem 5.6). However, in practice exact integration of an accelerometer's signal is **not** desirable due to constant error $\mathcal{E}), because\(\int e_a(t)dt = \int [C_{ea}a(t) + n(t) + \varepsilon]\, dt = \int C_{ea}a(t)dt + \int n(t)dt + \varepsilon t$. The term $\int n(t)dt$ typically is negligible because random $n(t)$ has average value of zero. On the other hand, the error term $\mathcal{E}t$ is artificial drift that grows with time and distorts the exact integrator's output signal, regardless of how small \(\mathcal{E}) is. Therefore, it is necessary to use **approximate** integration that is not vulnerable to artificial drift but is still sufficiently accurate for practical purposes. Perhaps the simplest approximate integrator, which we name the **low-pass approximate integrator**, is defined by the 1st order ODE $\dot{e}_v + \Omega e_v = (1/T)e_a(t)$; the static response to non-zero constant e_a is clearly non-zero, $e_v = e_a/(\Omega T)$. This non-zero static response makes the low-pass approximate integrator unsuitable for applications that require the device's output to have zero offset voltage. To avoid such an artificial constant static response, we have the **band-pass approximate integrator** defined by the 2nd order ODE $\ddot{e}_v + \Omega\dot{e}_v + \Omega^2 e_v = (1/T)\dot{e}_a(t)$, which has right-hand-side dynamics.

    1. Using the definition of dimensionless excitation frequency $\beta \equiv \omega/\Omega$, derive and show that the complex frequency-response functions of the exact integrator, the low-pass approximate integrator, and the band-pass approximate integrator can be written in the forms, respectively, $[\Omega T \times FRF(\omega)]_{\text{exact}} = 1/(j\beta)$, and $[\Omega T \times FRF(\omega)]_{\text{band-pass}} = j\beta/\left(1 - \beta^2 + j\beta\right)$.
    2. Plot on one graph the magnitudes versus β (at least over the range $0.1 \leq \beta \leq 10$) of $[\Omega T \times FRF(\omega)]_{\text{exact}}$, and $[\Omega T \times FRF(\omega)]_{\text{low-pass}}$, and $\[\Omega T \times FRF(\omega)]_{\text{band-pass}}$).$Plotonanothergraphthephasesversus\(\beta$ (over the same range) of these three frequency-response functions. Your graphs should clearly display features noted earlier such as the static responses of the low-pass and band-pass approximate integrators, and they should show that for $\beta \equiv \omega/\Omega > 6$ both approximate integrators have negligible magnitude error, relative to the exact integrator, and phase error less than 10°.

15. Consider the *reaction-mass actuator* (RMA) drawn schematically below, which includes an m-c-k system that is augmented with an internal force generator, $f_i(t)$. An external dynamic voltage signal $e_i(t)$ commands the force generator to impose equal and opposite dynamic forces $f_i(t)$ upon reaction mass m and the rigid interface, forces $f_i(t)$ that are proportional to input $e_i(t)$. The intended function of the RMA is to "react" against m in order to impose a dynamic force $f_a(t)$ through the connecting rod onto the rigid wall drawn at left. However, due to the dynamics of the m-c-k system, the actuation force $f_a(t)$ actually transmitted through the connecting rod is different than internal force $f_i(t)$.

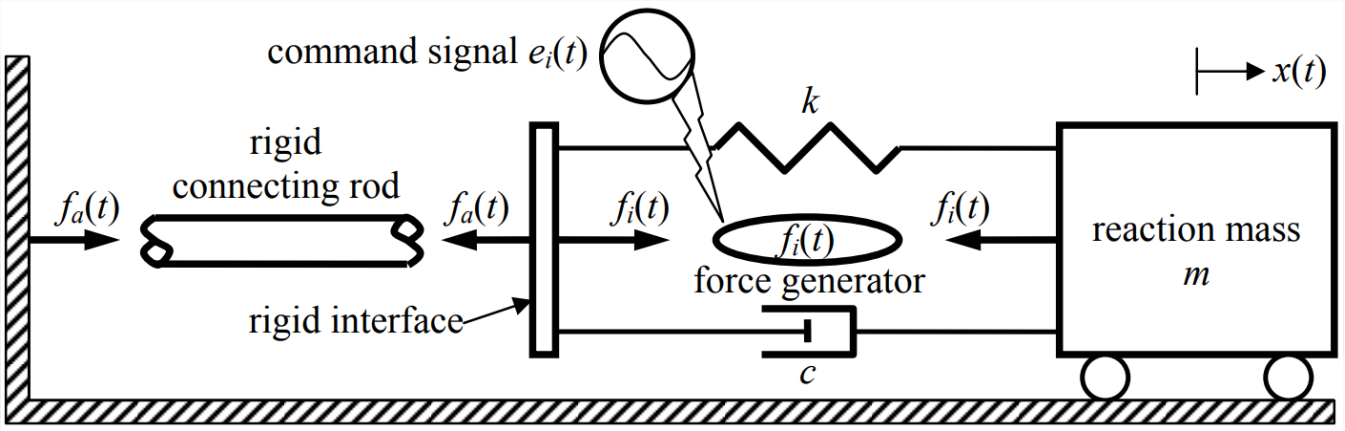


Figure 10.7.7: Reaction-Mass Actuator (RMA) in a calibration setup

Your tasks in this problem are to derive the mathematical relationship between the output actuation force $f_a(t)$ and the input voltage signal $e_i(t)$, and then to calculate and plot frequency response that illustrates the character of RMA functioning.

1. Derive the **very simple** ODE relating translation $x(t)$ of reaction mass m to the actuation force $f_a(t)$ shown acting onto the rigid interface. (The masses of the internal components—spring, damper, and force generator—are negligible relative to m.) Then derive from that ODE the transfer function $F_a(s)/X(s)$, where $F_a(s)$ and $X(s)$ are the Laplace transforms, respectively, of $f_a(t)$ and $x(t)$.
2. The linear relationship between generated internal force $f_i(t)$ and input voltage signal $e_i(t)$ is $f_i(t) = Ge_i(t)$, where gain (calibration) constant G has units such as newtons/volt. Derive the ODE relating the translation $x(t)$ of reaction mass m to the internal force $f_i(t)$ shown acting onto m, and then to input signal $e_i(t)$. Then derive from that ODE the transfer function $X(s)/E_i(s)), where\backslash(E_i(s)$ is the Laplace transform of $e_i(t)$.
3. Multiply the transfer functions of parts 10.15.1 and 10.15.2 in order to show that the required transfer function relating the output actuation force $f_a(t)$ to the input voltage signal $e_i(t)$ is $F_a(s)E_i(s) = Gs^2/\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)$, where, as usual, $\omega_n^2 = k/m$ and $\zeta = c/\left(2m\omega_n\right)$.
4. During the 1990s, a team of aerospace companies designed a specialized RMA and fabricated several units of the device for use as vibration-control actuators in experiments on a test bed that simulated a space-satellite laser-beam-director structure [Dettmer, 1995, which uses another common name, *proof-mass actuator* (PMA) for the device][6]. Imagine that these RMAs are restored for a new project and that you, as an instrumentation engineer, are assigned to perform system identification and calibration tests on one of them. First, you run free-vibration tests with the force generator turned off, $f_i(t) = 0$. You excite vibration by displacing reaction mass m from $x = 0$, then releasing it (twang testing, see homework Problem 9.4). You measure the frequency of the resulting free vibration of m to be 5.00 Hz, with very light inherent damping that reduces the amplitude of vibration in half in exactly 9 full cycles [in Equation 9.5.5, $r_{1/2} = 9.00$]. The RMA has a flange allowing you to bolt a calibrated mass of 3.00 kg firmly to reaction mass m; so you run a second twang test, from which you measure the frequency of free vibration of reaction mass plus added mass, m + 3.00 kg, to be 4.23 Hz (see homework Problem 7.9). Next, you remove the 3.00-kg added mass and turn on the force actuator. In static testing, you measure a linear relationship between applied voltage e_i and translation x, with maximum voltage $e_i = +10.0)Vmoving\backslash(m$ the distance $x = -6.00$ mm (and, conversely, –10.0 V producing +6.00 mm). Use your measured data to infer values for the components m, c, k, and G of the RMA system (*Partial answer*: G = 4.47 N/V). Finally, calculate the frequency-response function corresponding to transfer function $F_a(s)/E_i(s)$ of part 10.15.3, and plot it in the format of Figure 10.2.1 [log(magnitude ratio) vs. log(frequency in Hz), linear(phase in degrees) vs. log(frequency in Hz)] over the excitation-frequency range 1-100 Hz. These graphs should indicate that the RMA produces output actuation force $f_a(t)$ nicely representative of input voltage signal $e_i(t)$ for frequencies above about twice the RMA's natural frequency, but that $f_a(t)$ is not at all representative of $e_i(t)$ for frequencies near and below the RMA's natural frequency.

16. A distributed-parameter *shear-building* structural model used in an instructional laboratory is shown in the drawing and the photograph, and is described in detail in

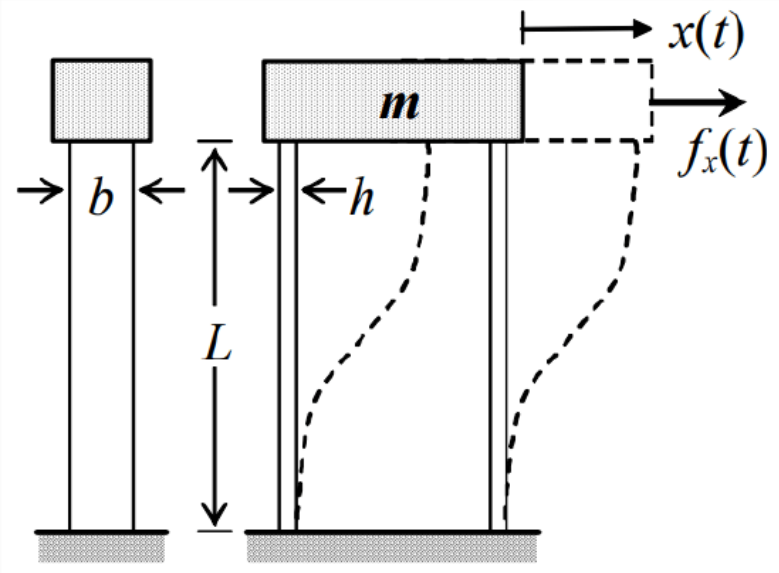


Figure 10.7.8: Side and Front Views

homework Problem 7.10. Measurements by students give the effective vibrating mass as $m_E = 0.00473$ lb-s^2/inch, and the lateral stiffness due to the two $L = 12$-inch aluminum beams as $k_E = 43.6$ lb/inch. The inherent structural and aeroacoustic damping of the vibrating beams and mass is very, very small, but a supplementary dashpot (near top left in the photo) attached between the mass and ground increases the total measured effective viscous damping constant to $c_E = 0.0309$ lb-s/inch

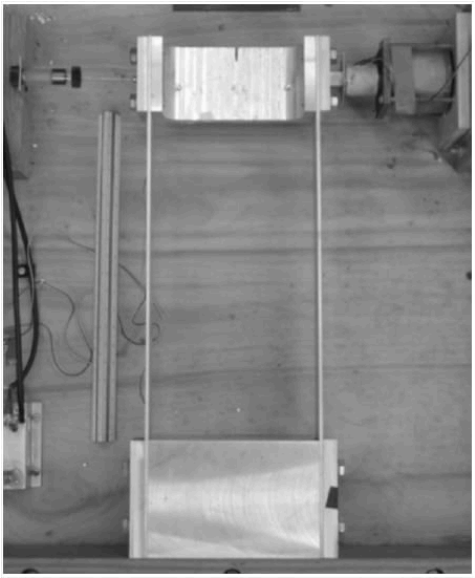


Figure 10.7.9

1. Calculate the undamped natural frequency f_n in Hz and the damping ratio ζ. (*Answers*: $f_n = 15.3$ Hz, $\zeta = 0.0340$)
2. A small electromagnetic shaker[7] is attached to the right-hand side of the mass, and it produces the time-varying force $f_x(t)$ labeled on the drawing. Any real force actuator has a maximum *stroke*, the range of motion of the device's moving parts, over which:
   1. the output force $f_x(t)$ is proportional to the input electrical command signal, and/or
   2. the moving parts of the device can deflect without damaging themselves or other parts within the device.

   It is important to recognize that the moving parts of the shaker are attached directly to the structure, so they experience the same deflection as the attachment point on the structure. Hence, the structural vibration can actually cause the maximum shaker stroke to be exceeded, especially at or near resonance. Suppose you want to run an experiment to measure an FRF of the shear-building model, in a stepped-sine sweep of excitation frequency f in Hz from $0.5f_n$ to $1.5f_n$. The shaker will drive the mass with sinusoidal force $f_x(t) = F\cos(2\pi ft)$, and the mass will respond with steady-state

sinusoidal translation $x(t) = X\cos(2\pi ft + \varphi)$. Suppose the shaker in this case has the maximum stroke of $\pm$0.05 inch and maximum force magnitude F_{max} = 2 lb (typical values for a small commercial off-the-shelf shaker). If you were to run the stepped-sine sweep so as to measure directly the *mechanical admittance* (also known as *dynamic flexibility*) by holding the force magnitude constant at 80% of its capability, $F = 0.8F_{max}$, what are the theoretically predicted magnitudes X of structural (and shaker moving parts) translation at the frequencies $0.5f_n$, $1.0f_n$, and $1.5f_n$? (*Partial answer*: $X\,(f_n) = 0.539$ inch) Why would this dynamic-flexibility approach not work in real life?

3. Suppose you run a stepped-sine sweep, with the same shaker properties as defined in part 10.16.2, in such a way as to measure directly the *dynamic stiffness*, instead of the dynamic flexibility. For example, imagine that at each discrete excitation frequency in the sweep you adjust the shaker force magnitude so that the response magnitude equals 80%, $X = 0.04$ inch, of the maximum shaker stroke. Calculate the theoretically predicted magnitudes F of required shaker force at the excitation frequencies $f = 0$ (the static condition), $0.5f_n$, $1.0f_n$, and $1.5f_n$. (*Partial answer*: $F\,(f_n) = 0.119$ lb) For what (if any) range of excitation frequencies would this dynamic-stiffness approach work in real life?
4. Calculate and plot the theoretically predicted dynamic stiffness (magnitude and phase) of this structural system for excitation frequency over the range $0 \le f \le 1.5f_n$.

17. Evaluate Equation 10.6.9 numerically (using MATLAB or your choice of software) to calculate and plot the time response from rest ICs to a suddenly applied sinusoidal (SAS) input into the electronic analog computer circuit at right. The ODE relating output voltage $e_o(t)$ to input voltage $e_i(t)$ derived in homework Problem 9.17, is, in standard 2nd order form, $\ddot{e}_o + \frac{1}{R_cC}\dot{e}_o + \left(\frac{1}{R_bC}\right)^2 e_o = \left(\frac{1}{R_bC}\right)^2 \frac{R_b}{R_a} e_i(t)$. The input voltage is zero for time $t < 0$ and $e_i(t) = E_i \sin(2\pi ft)$ for $t \ge 0$, in which $E_i = 3.20$ V and $f = 14.2$ Hz. The circuit component values are: $R_a = R_d = 50.0$ $k\Omega$, $R_c = 500$ $k\Omega$, and $C = 2.04\mu$F. Calculate and plot both excitation and response over the time interval $-0.1 \le t \le 2.4$ s, at increments of 0.001 s (to produce suitable graphical resolution). This circuit was "patched" (programmed with plug-in wires) on a Comdyna GP-10 analog computer, and the given input voltage was produced by a sine-wave generator; graphs of the actual measured time-varying input excitation and output response are on the next page. Do the numerically simulated excitation and response that you calculate look very similar to the measured excitation and response? (If not, they should!) Although the circuit is lightly damped, there is still clear beating behavior. Show that the period and frequency of beating evident in the time-response graph are what you would expect, based upon Equation 10.6.5 for beating of an undamped 2nd order system.

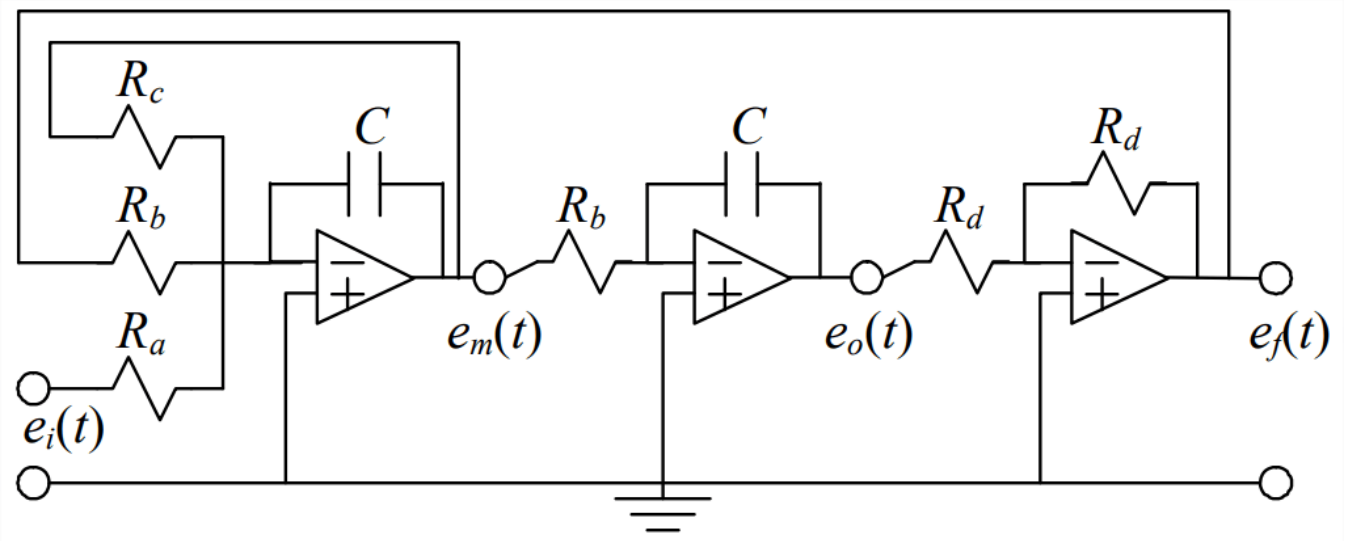


Figure 10.7.10

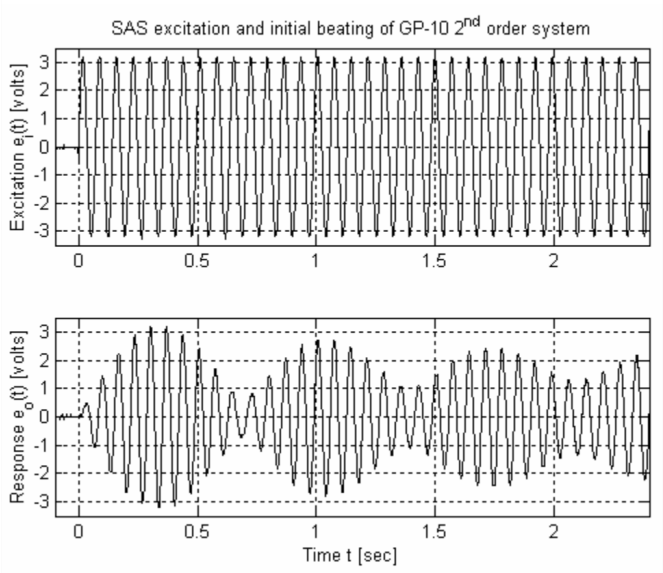


Figure 10.7.11

[1]For most commercially available sensors, the electrical signal passes through wires connected to the sensor, but some modern sensors send the signal wirelessly. The size of such a sensor ranges typically from that of an aspirin tablet to that of a large soda bottle.

[2]Although the theoretical constant of proportionality is $-m/k$, the true constant, including polarity, of any real sensor is always measured by laboratory calibration.

[3]For the phase-angle error, regard a 180° phase difference as being equivalent to 0°, because it is simply a sign difference that is corrected in calibration and/or data processing.

[4]The floor's true reaction is distributed force, *i.e.* stress, acting against the base over the area of contact. Discrete force $f_R(t)$ in the drawing is the equivalent *resultant* of the distributed force, that is, fR(t) has the magnitude of the total force distribution and its point of application is the center of the distribution.

[5]This FRF applies for a vibration isolator with viscous damping. However, the *internal-friction* mathematical model of structural damping is more realistic than viscous damping. In Appendix B, Section 19.5, the internal-friction model is described and applied for vibration isolation.

[6]The schematic drawing for this problem shows the connecting rod attached to a rigid wall, which is a configuration suitable for calibrating an RMA; however, in any end-user practical application, the connecting rod would be attached to a flexible structure. RMAs are applied in practice to impose dynamic forces upon structures, both to provide excitation for vibration testing, and to suppress unwanted vibration as actuators in control systems. An RMA can be structure-borne, meaning it can be completely supported by the structure onto which it imposes dynamic force. For example, large RMAs have been installed in the top floors of tall buildings in order to reduce bending vibration excited by wind and ground tremors; an RMA in this civil engineering application is often called an *active tuned-mass damper* (ATMD). The internal force generators of ATMDs are hydraulic motors, whereas those of small RMAs (having total weight on or under the order of 100 lb) are electromagnetic linear motors.

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11: MECHANICAL SYSTEMS WITH RIGID-BODY PLANE TRANSLATION AND ROTATION

Up to this point, we have considered mechanical systems with motion consisting of either translation in only one direction or rotation about one axis, but not both together. Simple rotational systems have appeared in previous chapters (for example, in Sections 3.3, 3.5, and 7.1), but now we will treat rigid-body plane motion more generally, as consisting of **both** translation and rotation, and with the two forms of motion possibly coupled together by system components and system geometry. The focus in this chapter is on **deriving** correctly the equations of motion, which generally are higher-order, coupled sets of ODEs. Chapter 12 introduces some methods for **solving** such equations, leading to fundamental characteristics of an important class of higher-order systems.

11.1: Equations of Motion for a Rigid Body in General Plane Motion

11.2: Equation of Motion for a Rigid Body in Pure Plane Rotation

11.3: Examples

11.4: Chapter 4 Homework

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11.1: Equations of Motion for a Rigid Body in General Plane Motion

Consider a rigid body that is restricted to motion in the xy plane. Figure 11.1.1 shows the body in its reference static equilibrium position (solid lines) and in a position of motion (dashed lines) relative to the reference position. The xy axis system shown is an inertial reference with its origin at point O, which is either motionless or translating at a **constant** velocity. We will use vectors and vector operations, so we need to recognize that the inertial reference system is really the three-dimensional Cartesian xyz axis system, with the z axis being perpendicular to the xy plane and pointed toward us, in the sense of the right-hand rule. We denote the fixed unit vectors in the reference Cartesian directions as $\mathbf{1}_\mathrm{x}$, $\mathbf{1}_\mathrm{y}$, and $\mathbf{1}_\mathrm{z}$. From the background of your previous dynamics and physics courses, you should be familiar with vectors, vector algebra and calculus, and vector kinematics and dynamics; if necessary, you should review these concepts in your engineering dynamics textbook.

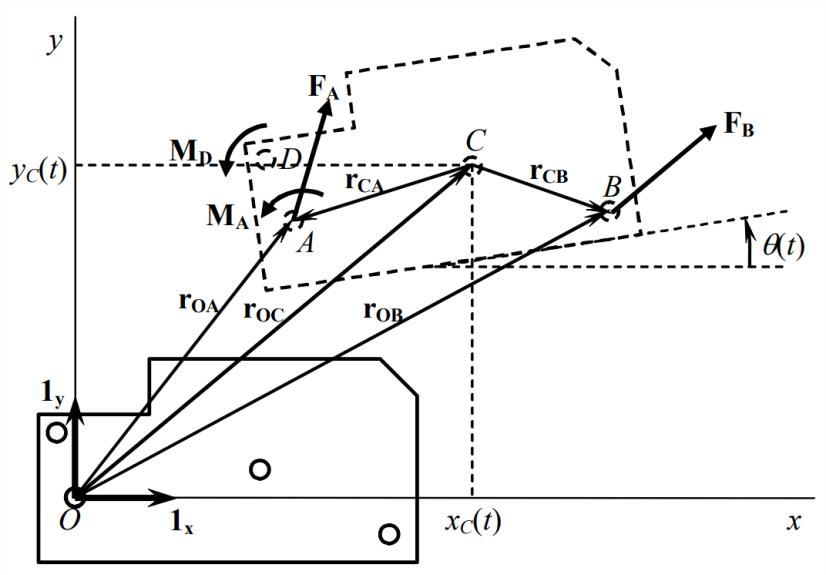


Figure 11.1.1: Rigid body in general plane motion

Of particular interest is the position and orientation of the rigid body's center of mass C. At any instant of time t, C has relative position coordinates $[x_C(t), y_C(t)]$. Thus, the vector position of C relative to some **absolutely fixed** point P is $\mathbf{r}_\mathrm{C}(t) = \mathbf{r}_\mathrm{o}(t) + \mathbf{r}_\mathrm{oC}(t) = \mathbf{r}_\mathrm{o}(t) + x_C(t)\mathbf{1}_\mathrm{x} + y_C(t)\mathbf{1}_\mathrm{y}$, where is the known instantaneous position of O relative to P. Similarly, the vector velocity of C relative to absolutely fixed point P is $\mathbf{v}_\mathrm{C}(t) = \mathbf{v}_\mathrm{o}(t) + \mathbf{v}_\mathrm{OC}(t) = \mathbf{v}_\mathrm{o}(t) + \dot{x}_C(t)\mathbf{1}_\mathrm{x} + \dot{y}_C(t)\mathbf{1}_\mathrm{y}$, where $\mathbf{v}_\mathrm{o}(t)$ is the **constant** velocity of O relative to P. Because $\mathbf{v}_\mathrm{o}(t)$ is constant, the absolute vector acceleration of C is $\mathbf{a}_\mathrm{C}(t) = \mathbf{a}_\mathrm{oc}(t) = \ddot{x}_C(t)\mathbf{1}_\mathrm{x} + \ddot{y}_C(t)\mathbf{1}_\mathrm{y}$. We denote as $\theta(t)$ the positive-counterclockwise rotational orientation of mass center C (indeed, of the entire body, since it is rigid). Position $\mathbf{r}_\mathrm{O}(t)$ of O is known, so if we can determine at any instant the three motion quantities $x_C(t)$, $y_C(t)$, and $\theta(t)$, then we can calculate from simple geometry the position of any point on the rigid body. These three quantities are independent in general plane motion, and only these three [plus known $\mathbf{r}_\mathrm{O}(t)$] are required to describe completely the position in space of the rigid body. Therefore, $x_C(t)$, $y_C(t)$, and $\theta(t)$ are called *degrees of freedom*, and general plane rigid-body motion is called a *three-degrees-of-freedom* (3-DOF) *problem*.

For generality, we identify three other points fixed to the rigid body, at which point actions (forces or moments) are applied by sources external to the body. (In general, there can be more or fewer points of externally applied actions; analyzing such situations will require only minor modifications to the present derivation.) Point A is a connection to the external environment, possibly through springs, dampers, *etc.*; actions applied at Point A are vector reaction force $\mathbf{F}_\mathrm{A} = F_{Ax}\mathbf{1}_\mathrm{x} + F_{Ay}\mathbf{1}_\mathrm{y}$ and vector reaction moment $\mathbf{M}_\mathrm{A} = M_A\mathbf{1}_\mathrm{z}$. It is useful to express the vector position of point A in terms of that of mass center C: $\mathbf{r}_\mathrm{OA} = \mathbf{r}_\mathrm{OC} + \mathbf{r}_\mathrm{CA}$ (see Figure 11.1.1). An arbitrary vector force $\mathbf{F}_\mathrm{B} = F_{Bx}\mathbf{1}_\mathrm{x} + F_{By}\mathbf{1}_\mathrm{y}$ is applied at point B, and that point's vector position is expressed as $\mathbf{r}_\mathrm{OB} = \mathbf{r}_\mathrm{OC} + \mathbf{r}_\mathrm{CB}$. Finally, an arbitrary vector moment $\mathbf{M}_\mathrm{D} = M_D\mathbf{1}_\mathrm{z}$ acts at point D.

We denote as m the mass of the rigid body. Newton's 2nd law for forces, written in vector form, is

$$m\mathbf{a}_{\mathrm{C}} = \mathbf{F}_{\mathrm{A}} + \mathbf{F}_{\mathrm{B}} \tag{11.1.1}$$

Expressing Equation 11.1.1 in terms of scalar components gives the two scalar ODEs:

$$m\ddot{x}_C = F_{Ax} + F_{Bx} \tag{11.1.2}$$

$$m\ddot{y}_C = F_{Ay} + F_{By} \tag{11.1.3}$$

We denote as J_C the rotational inertia of the rigid body about an axis normal to the plane of motion and **passing through mass center** C. Newton's 2nd law for moments relevant to this plane-motion case states that the inertial moment about mass center C equals the sum of all applied moments about mass center C; it is expressed as follows in vector notation, including vector cross products:

$$J_C\ddot{\theta}\mathbf{1}_{\mathbf{z}} = \mathbf{r}_{\mathrm{CA}} \times \mathbf{F}_{\mathbf{A}} + \mathbf{r}_{\mathrm{CB}} \times \mathbf{F}_{\mathbf{B}} + (M_A + M_D)\,\mathbf{1}_{\mathbf{z}} \tag{11.1.4}$$

Despite the vector notation, Equation 11.1.4 is actually a scalar equation, because the vector cross products obviously produce vectors in the z direction; the use of these cross products just provides economy of notation[1].

We shall see an application of general-plane-motion Equations 11.1.2, 11.1.3, and 11.1.4 to a specific aeroelastic system in Section 11.3 (Example 11-3); but, first, it is appropriate to develop in Section 11.2 and to illustrate in Section 11.3 (Examples 11-1 and 11-2) the special case of pure plane rotational motion about a fixed point.

[1]For Equations 11.1.2, 11.1.3, and 11.1.4, the reference xyz axis system is inertial, either stationary in space or moving at a constant translational velocity, and motion of the rigid body is measured relative to this inertial system. However, for many problems in dynamics, it is preferable to derive equations of motion using a reference axis system that is fixed in the body and moves with it. A plane-motion problem of this type is the leadinglagging rotation of a hinged helicopter rotor blade, as analyzed by Bramwell, 1976, pp. 51-52.

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11.2: Equation of Motion for a Rigid Body in Pure Plane Rotation

An important special case of general plane motion is pure rotation about a fixed pivot or hinge, as depicted in Figure 11.2.1. To analyze this case, let us suppose that reaction point A of Figure 11.1.1 is the hinge point, and let us position the xyz origin O at point A. To emphasize the character of this hinge point, we re-label it as H; in other words, both points A and O on Figure 11.1.1 now become hinge point H on Figure 11.2.1. We can use the previous equations simply by setting position vector $\mathbf{r}_{\mathrm{OA}}$ on Figure 11.1.1 equal to zero: $\mathbf{r}_{\mathrm{OA}} = \mathbf{0} = \mathbf{r}_{\mathrm{oc}} + \mathbf{r}_{\mathrm{CA}} \Rightarrow \mathbf{r}_{\mathrm{CA}} = -\mathbf{r}_{\mathrm{OC}} \equiv -\mathbf{r}_{\mathrm{HC}}$. It is helpful also to express $\mathbf{r}_{\mathrm{CB}}$ on Figure 11.1.1 in terms of $\mathbf{r}_{\mathrm{oC}}$: $\mathbf{r}_{\mathrm{CB}} = \mathbf{r}_{\mathrm{OB}} - \mathbf{r}_{\mathrm{OC}}$. Using these substitutions in Equation 11.1.4 gives

$$J_C\ddot{\theta}\mathbf{1}_{\mathbf{z}} = -\mathbf{r}_{\mathbf{OC}} \times \mathbf{F}_{\mathbf{A}} + (\mathbf{r}_{\mathrm{OB}} - \mathbf{r}_{\mathbf{OC}}) \times \mathbf{F}_{\mathrm{B}} + (M_A + M_D)\,\mathbf{1}_{\mathbf{z}} \tag{11.2.1}$$

After we replace subscripts O and A with H and re-arrange terms, Equation 11.2.1 becomes

$$J_C\ddot{\theta}\mathbf{1}_{\mathrm{z}} + \mathbf{r}_{\mathbf{HC}} \times (\mathbf{F}_{\mathbf{H}} + \mathbf{F}_{\mathrm{B}}) = \mathbf{r}_{\mathrm{HB}} \times \mathbf{F}_{\mathrm{B}} + (M_H + M_D)\,\mathbf{1}_{\mathrm{z}} \tag{11.2.2}$$

Next, substituting Equation 11.1.1 into Equation 11.2.2 gives

$$J_C\ddot{\theta}\mathbf{1}_{\mathbf{z}} + \mathbf{r}_{\mathbf{HC}} \times m\mathbf{a}_{\mathbf{C}} = \mathbf{r}_{\mathrm{HB}} \times \mathbf{F}_{\mathrm{B}} + (M_H + M_D)\,\mathbf{1}_{\mathbf{z}} \tag{11.2.3}$$

Since hinge point H is fixed, let us simplify notation by writing $\mathbf{r}_{\mathbf{HC}} \equiv \mathbf{r}_{\mathbf{C}}$ and $\mathbf{r}_{\mathrm{HB}} \equiv \mathbf{r}_{\mathrm{B}}$. Also, we can express the acceleration as $\mathbf{a}_{\mathrm{C}} \equiv \ddot{\mathbf{r}}_{\mathrm{C}}$, so Equation 11.2.3 becomes

$$J_C\ddot{\theta}\mathbf{1}_{\mathrm{z}} + \mathbf{r}_{\mathrm{C}} \times m\ddot{\mathbf{r}}_{\mathrm{C}} = \mathbf{r}_{\mathrm{B}} \times \mathbf{F}_{\mathrm{B}} + (M_H + M_D)\,\mathbf{1}_{\mathrm{z}} \tag{11.2.4}$$

We can simplify the $\ddot{\mathbf{r}}_{\mathrm{C}}$ term on the left-hand side of Equation 11.2.4 by using the kinematics of rigid-body, plane, pure rotation about point H. For this task, it is convenient to work in polar rather than Cartesian coordinates, so we define on Figure 11.2.1 the orthogonal rotating unit vectors $\mathbf{1}_{\mathrm{r}}$ and $\mathbf{1}_{\theta}$; thus, we can express the position vector to mass center C as $\mathbf{r}_{\mathrm{C}} = r_C\mathbf{1}_{\mathrm{r}}$, where r_C is the constant radius from hinge point H to mass center C. We express the rotational velocity vector as $\boldsymbol{\omega} = \dot{\theta}\mathbf{1}_{\mathbf{z}}$. (In this context, the vector symbol ω denotes rotational velocity, which is standard in dynamics textbooks, whereas everywhere else in this book the scalar symbol ω denotes frequency of vibration.) Therefore, the first derivative of the mass-center position vector is

$$\dot{\mathbf{r}}_{\mathrm{C}} = \boldsymbol{\omega} \times \mathbf{r}_{\mathrm{C}} = \dot{\theta}\mathbf{1}_{\mathrm{z}} \times r_C\mathbf{1}_{\mathrm{r}} = r_C\dot{\theta}\mathbf{1}_{\theta} \tag{11.2.5}$$

To obtain the second derivative, we differentiate product Equation 11.2.5, recognizing that r_C is constant, but that $\mathbf{1}_{\theta}$ is rotating:

$$\ddot{\mathbf{r}}_{\mathrm{C}} = r_C\left[\ddot{\theta}\mathbf{1}_{\mathrm{e}} + \dot{\theta}\dot{\mathbf{i}}_{\theta}\right] = r_C\left[\ddot{\theta}\mathbf{1}_{\theta} + \dot{\theta}\left(\dot{\theta}\mathbf{1}_{\mathrm{z}} \times \mathbf{1}_{\theta}\right)\right] = r_C\left[\ddot{\theta}\mathbf{1}_{\mathrm{e}} - \dot{\theta}^2\mathbf{1}_{\mathrm{r}}\right] \tag{11.2.6}$$

With Equation 11.2.6, the vector cross product on the left-hand side of Equation 11.2.4 becomes

$$\mathbf{r}_{\mathrm{C}} \times \ddot{\mathbf{r}}_{\mathrm{C}} = r_C\mathbf{1}_{\mathrm{r}} \times r_C\left[\ddot{\theta}\mathbf{1}_{\theta} - \dot{\theta}^2\mathbf{1}_{\mathrm{r}}\right] = r_C^2\ddot{\theta}\mathbf{1}_{\mathbf{z}} \tag{11.2.7}$$

With Equation 11.2.7, the left-hand side of Equation 11.2.4 becomes

$$J_C\ddot{\theta}\mathbf{1}_{\mathbf{z}} + \mathbf{r}_{\mathbf{C}} \times m\ddot{\mathbf{r}}_{\mathrm{C}} = \left(J_C + mr_C^2\right)\ddot{\theta}\mathbf{1}_{\mathbf{z}} \equiv J_H\ddot{\theta}\mathbf{1}_{\mathbf{z}} \tag{11.2.8}$$

In Equation 11.2.8, we use the parallel-axis theorem to define the rotational inertia of the rigid body about center of rotation H, $J_H = J_C + mr_C^2$. Thus, Equation 11.2.4 becomes

$$J_H\ddot{\theta}\mathbf{1}_{\mathbf{z}} = \mathbf{r}_{\mathrm{B}} \times \mathbf{F}_{\mathrm{B}} + (M_H + M_D)\,\mathbf{1}_{\mathbf{z}} \equiv \Sigma(\text{ all active and reactivemoments about } H)\mathbf{1}_{\mathbf{z}} \tag{11.2.9}$$

To summarize, for a rigid-body in pure plane rotation about point H, the only degree of freedom is rotation $\theta(t)$, and the single, relatively simple ODE of motion is

$$J_H\ddot{\theta} = \Sigma(\text{ allactive and reactivemoments about } H) \tag{11.2.10}$$

in which J_H is the rotational inertia of the rigid body about center of rotation H.

Note that the reaction **force** at hinge H does not appear in ODE Equation 11.2.10. In fact, to find this reaction force, you would have to solve Equation 11.2.10 for $\theta(t)$, then find the acceleration of C, then substitute the results back into Equations 11.1.1 or 11.1.2 and 11.1.3.

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11.3: Examples

Example 11.3.1: damped, spring-supported rigid bar, hinged at point H

Let us derive the specific ODE of motion for the system of Figure 11.3.1, assuming small angles of rotation. The bar has rotational inertia J_H about hinge H, which we assume to have negligible friction. The bar is shown in the reference static equilibrium position, subject to static gravity but **before** applied dynamic force $F(t)$ becomes active. Dynamic degree of freedom $\theta(t)$ shown with dashed lines is the **small** rotation **relative to the static equilibrium position**. (It is this book's graphical convention for this and all subsequent systems of this type that the rigid body is drawn in its static equilibrium position, and dynamic motion relative to this reference position is indicated with dashed lines and annotation.) Note the rotation spring with spring constant k_θ at hinge H; this spring resists rotation by generating an opposing moment of magnitude $k_\theta \times \theta$. Typical units for k_θ are lb-inch/radian and N-m/radian.

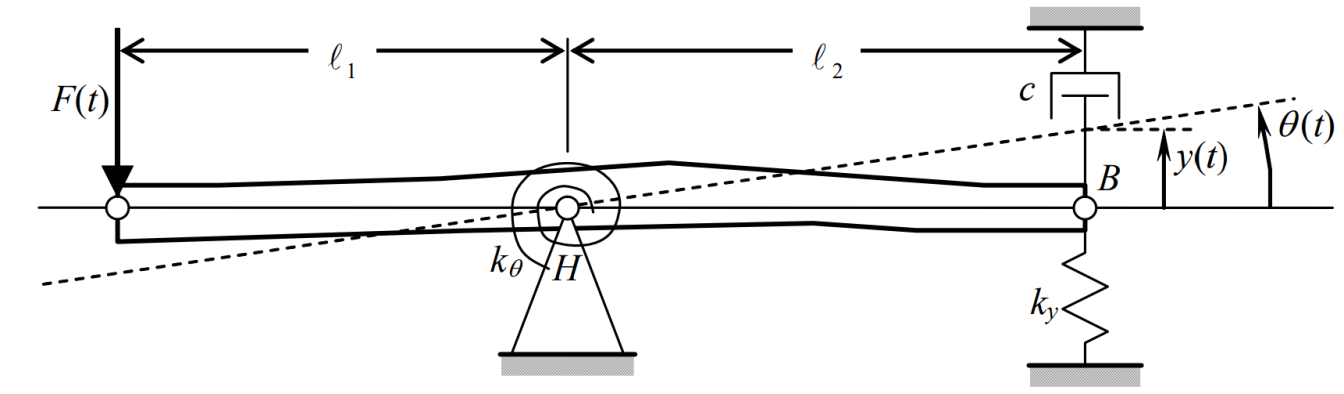


Figure 11.3.1: Example 11.3.1: damped, spring-supported, hinged rigid bar

Before drawing a dynamic free-body diagram (DFBD, as defined in Section 7.5), we need to express appropriately the reaction forces on the bar at point B from the translational spring ($k_y \times y$) and the translational viscous damper ($c \times \dot{y}$). Observe that point B moves in a circular arc of radius ℓ_2, so that $y = \ell_2 \sin\theta$. For arbitrarily large θ, the term $\sin\theta$ and its time derivative would make equation of motion Equation 11.2.10 nonlinear. However, the assumption of small rotation, $|\theta(t)| <\approx 10^\circ$, linearizes that ODE; the **geometry of small rotation** gives approximate linear equations in terms of θ for stretch y of the translational spring due to motion of point B, and for damper-piston velocity $\dot{y}$:

$$y = \ell_2 \sin\theta \approx \ell_2\theta(\theta \text{ in radians }), \quad \text{and} \quad \dot{y} = \ell_2 \cos\theta \times \dot{\theta} \approx \ell_2\dot{\theta}$$

The geometry of small rotation was used previously in the example of Section 7.1 to linearize the pendulum equation of motion. For the system of Figure 11.3.1, the assumption of small rotation is physically plausible as well as mathematically expedient, because the translational and rotational springs most likely would restrict rotation of the bar to small angles relative to the static equilibrium position.

The appropriate DFBD is drawn below. Note that we do **not** include bar weight among the applied forces. We omit bar weight because we are interested only in dynamic motion relative to the static equilibrium position, and because this bar is approximately horizontal so that gravity has no pendulous effect on it; if you are not sure why we neglect weight in this case, then review Section 7.5.

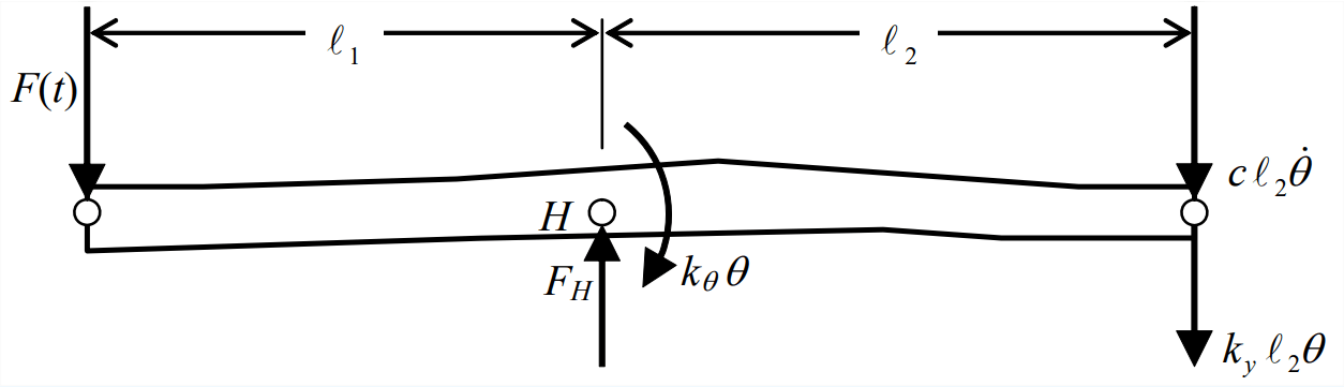


Figure 11.3.2

From the DFBD and Equation 11.2.10, the ordinary differential equation of motion is

$$J_H\ddot{\theta} = \ell_1 F(t) - k_\theta\theta - \ell_2\left(c\ell_2\dot{\theta} + k_y\ell_2\theta\right)$$

$$\Rightarrow \quad J_H\ddot{\theta} + c\ell_2^2\dot{\theta} + \left(k_\theta + k_y\ell_2^2\right)\theta = \ell_1 F(t) \tag{11.3.1}$$

Equation 11.3.1 can be put into the standard form for a 2nd order damped system, and all of the relevant results of Chapters 9 and 10 can be expressed in terms of the physical parameters of this particular mechanical system. For example, we can immediately find equations for the system undamped natural frequency and viscous damping ratio:

$$\omega_n = \sqrt{\frac{k_\theta + k_y\ell_2^2}{J_H}}, \quad 2\zeta\omega_n = \frac{c\ell_2^2}{J_H} \Rightarrow \zeta = \frac{1}{2}\frac{c\ell_2^2}{\sqrt{J_H\left(k_\theta + k_y\ell_2^2\right)}} \tag{11.3.2}$$

NOTE: It is easy to make mistakes in algebra when we derive results such as Equations 11.3.2. One good, easy type of **partial** check that you can use is to evaluate the physical dimensions of algebraic results. (It is only a partial check because dimensional consistency is a necessary condition, but it does not guarantee the correctness of an algebraic equation.) Do Equations 11.3.2 have the correct physical dimensions? You should be able to show that the equation for ω_n has the dimension (time)$^{-1}$ and that the equation for ζ is dimensionless. Do not forget that rotation θ is dimensionless and that the radian, the natural metric of rotation, is unitless. If you are more comfortable working with units than dimensions, it is satisfactory to check units instead. Table 3.1.1 should be helpful relative to mechanical units.

Example 11.3.2: 1-DOF *typical-section* model for study of aerolastic wing twist

Since the beginning of modern aeronautical engineering in the late 1800s and early 1900s, the interaction of aerodynamic pressure with airplane structural flexibility has often produced unexpected and sometimes disastrous consequences. The subject of these phenomena became known as *aeroelasticity*. During and after World War I, highaspect-ratio, cantilevered monoplane wings tended to be susceptible both to aeroelastic *flutter*, undamped vibration that can escalate to destruction of wing and airplane, and to aeroelastic *divergence*, the gradual twisting-off of a wing in, for example, pull-up from a dive[1]. In their attempts to analyze these aeroelastic phenomena, early aeronautical engineers used simplified, low-degrees-of-freedom dynamic models of wing structures. The basis of these models is called a *typical section*: one "typical" cross-section of a high-aspect-ratio, straight, unswept wing is treated aerodynamically as an airfoil in twodimensional flow (without spanwise flow), and structurally as a spring-supported rigid body (without spanwise beam bending and twisting).

The simplest typical-section model is the one-degree-of-freedom (1-DOF) system drawn in Figure 11.3.3. It is helpful to visualize this as a physical model mounted for testing in a wind tunnel. *Elastic axis EA* (the hinge) is assumed to be frictionless, and rotation spring k_θ represents wing torsional stiffness. (The term *elastic axis* refers to the line along the span of a high-aspect-ratio, straight, unswept wing about which the wing chordwise sections rotate if a pure twisting moment is imposed upon the wing.) This model is intended primarily for the study of aeroelastic divergence. Figure 11.3.3 shows the typical

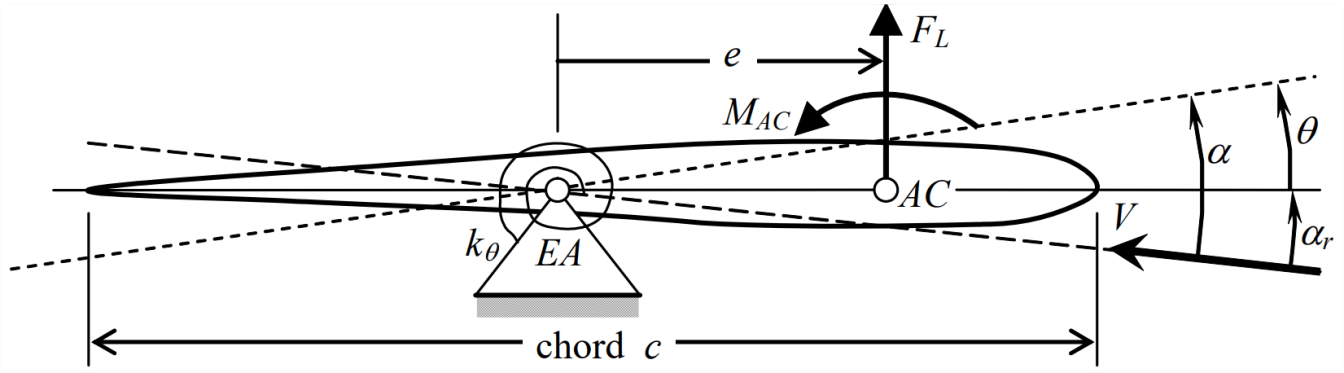


Figure 11.3.3: Example 11.3.2: 1-DOF typical section in an airstream

section in the reference static equilibrium position **before** the wind tunnel fan is turned on. Turning on the wind tunnel fan then produces an airstream of steady free-stream velocity V flowing over the typical section at small angle of incidence α_r relative to the reference chordline of the typical section. The airstream creates a resultant lifting force F_L that acts through *aerodynamic center* AC, and a resultant pitching moment M_{AC} that acts about AC. For most cases of practical interest, AC is **forward** of EA by some positive distance e, as shown on Figure 11.3.3; the polarity $e > 0$ is very important relative to divergence. Aerodynamic actions F_L and M_{AC} impose moments upon rotation spring k_θ, and the spring flexibility permits structural rotation θ relative to the reference position.

The DFBD associated with Figure 11.3.3 is drawn below. The equation of motion comes directly from the DFBD and Equation 11.2.10, with the assumption of small angles of incidence and rotation, $|\alpha_r| <\approx 10^\circ$ and $|\theta(t)| <\approx 10^\circ$, and with J_{EA} being the rotational inertia of the typical section about EA:

$$J_{EA}\ddot{\theta} = F_L e + M_{AC} - k_\theta\theta \Rightarrow J_{EA}\ddot{\theta} + k_\theta\theta = F_L e + M_{AC} \tag{11.3.3}$$

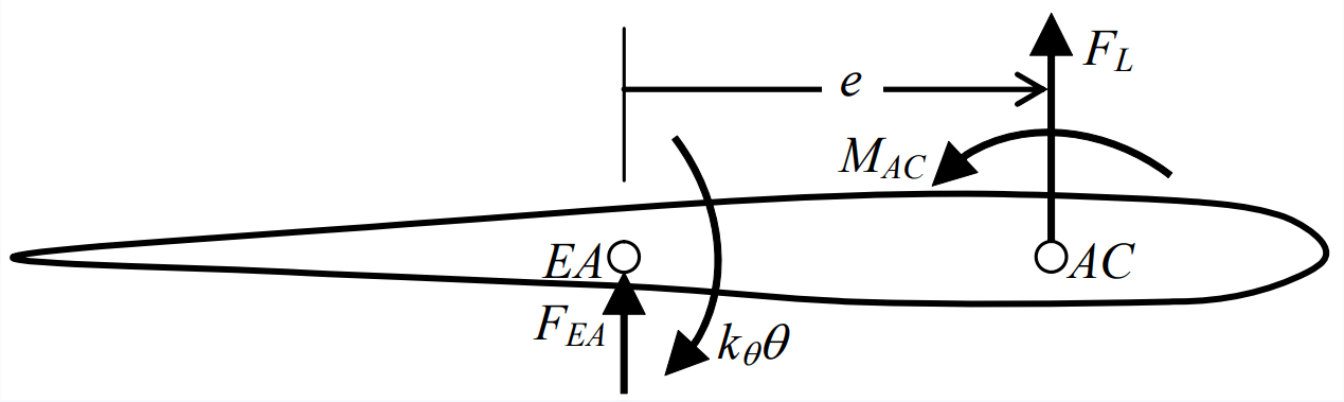


Figure 11.3.4

We can cast Equation 11.3.3 into a more interesting and useful form by relating the aerodynamic actions to the structural rotation. We assume that the airstream direction coincides with the zero-lift attitude of the airfoil, *i.e.*, that F_L would be zero for any value of airspeed V if the airfoil incidence were fixed at $\alpha_r = 0$. Consequently, the total *angle of attack* of the airfoil relative to the zero-lift attitude is $\alpha = \alpha_r + \theta$. Then aerodynamic lift and moment can be expressed as

$$F_L = \bar{q}SC_{L\alpha}\alpha = \bar{q}SC_{L\alpha}\left(\alpha_r + \theta\right), \quad \text{and} \quad M_{AC} = \bar{q}ScC_{MAC} \tag{11.3.4}$$

Note especially in Equations 11.3.4 that an aerodynamic action (in this case, lift F_L) is dependent upon structural deformation (in this case, rotation θ); this type of interaction between aerodynamics and structural flexibility is an important characteristic of all aeroelastic phenomena. The symbols used in Equations 11.3.4 beyond those previously defined are:

$\bar{q} = \frac{1}{2}\rho V^2$ is the free-stream dynamic pressure (lb/ft^2 or N/m^2).

ρ is air density (slug/ft^3 or kg/m^3).

$S = bc$ is the planform area of the typical section (ft^2 or m^2).

b is the span of the typical section into the plane of the paper (ft or m).

$C_{L\alpha} > 0$ is the slope of the curve of section lift coefficient ($F_L/\bar{q}S$) versus angle of attack (rad^{-1}), which we assume here to be constant for small angles of attack.

C_{MAC} is the dimensionless coefficient of pitching moment about AC, normally positive for a positively cambered airfoil, but zero for an uncambered thin airfoil.

Wind tunnel experiments and aerodynamic theory show that aerodynamic action Equations 11.3.4 are valid only for steady flow, for which angle of attack α does not vary with time. However, we will approximate a bit here and assume that Equations 11.3.4 are at least qualitatively valid for slowly-time-varying angle of attack, $\alpha(t) = \alpha_r + \theta(t)$. This approximation is sometimes called *quasi-static* or *quasi-steady* aerodynamics. Substituting Equations 11.3.4 into Equation 11.3.3 gives

$$J_{EA}\ddot{\theta} + k_\theta\theta = \bar{q}SC_{L\alpha}\left(\alpha_r + \theta\right)e + \bar{q}ScC_{MAC}$$

$$\Rightarrow \quad J_{EA}\ddot{\theta} + \left(k_\theta - \bar{q}SC_{L\alpha}e\right)\theta = \bar{q}SC_{L\alpha}\alpha_r e + \bar{q}ScC_{MAC} \tag{11.3.5}$$

Note in Equation 11.3.5 that the total stiffness constant, $k_\theta - \bar{q}SC_{L\alpha}e$ (structural stiffness plus aerodynamic "stiffness") can, for $e > 0$, become zero or even negative if dynamic pressure $\bar{q}$ is sufficiently large. This is critically important relative to the stability of the system. Consider steady flow, for example, for which $\ddot{\theta} = 0$; in this case we can solve Equation 11.3.5 algebraically for the static structural rotation:

$$\theta = \frac{\bar{q}S\left(C_{L\alpha}\alpha_r e + cC_{MAC}\right)}{k_\theta - \bar{q}SC_{L\alpha}e} \equiv \frac{\bar{q}\left(C_{L\alpha}\alpha_r e + cC_{MAC}\right)}{C_{L\alpha}e\left(\bar{q}_D - \bar{q}\right)} \tag{11.3.6}$$

In Equation 11.3.6, we define the *divergence dynamic pressure* in terms of other fixed system parameters as

$$\bar{q}_D \equiv \frac{k_\theta}{SC_{L\alpha}e} \tag{11.3.7}$$

If it is possible for a given set of physical parameters to increase the wind-tunnel airspeed V up to the point that $\bar{q} \to \bar{q}_D = k_\theta/SC_{L\alpha}e$, then, from Equation 11.3.6, clearly something very interesting will happen. From your knowledge of aerodynamics and structures, what types of physical behavior do you think are possible? Keep in mind that Equation 11.3.6 is predicated on the assumptions of small angles and complete linearity of both aerodynamics and structures (in this system, the rotation spring).

A conventional airplane control surface is mounted on the trailing edge of a major lifting surface, with the leading edge of the control surface hinged to the lifting surface (*e.g.*, an aileron on a wing). In this case, the aerodynamic center of the control surface is **aft** of the elastic axis, making the moment arm e **negative** (see Figure 11.3.3). From Equation 11.3.6, this produces the negative moment $\bar{q}SC_{L\alpha}\alpha_r e$, which is called a *blowdown* moment, and it also **augments** the structural stiffness in the total stiffness constant $k_\theta - \bar{q}SC_{L\alpha}e$. Therefore, a conventional trailing-edge control surface cannot diverge aeroelastically.

Example 11.3.3: 2-DOF *typical-section* model for coupled wing being and torsion

We now return to general-plane-motion Equations 11.1.2, 11.1.3, and 11.1.4, which are three ODEs in three unknown DOFs: $x_C(t)$, $y_C(t)$, and $\theta(t)$. In practice, these ODEs are often coupled by system geometry and the nature of the active and reactive actions. Moreover, if motion is allowed to have arbitrarily large magnitude, then the equations are usually nonlinear. In general, these arbitrary-plane-motion equations can be solved only in numerical form, not in the form of algebraic equations. However, the mechanical systems considered in this book are limited, in most cases, to those for which center-of-mass translation is small relative to rigid-body dimensions, and for which rotation is small, $|\theta(t)| <\approx 10^\circ$ (*e.g.*, Examples 11.3.1 and 11.3.2 in this section); these limitations allow us to linearize the ODEs into more easily solvable forms.

One of the first forms of aeroelastic wing flutter that was observed and carefully analyzed during the early 1900s involves coupled bending and torsional vibration (Bisplinghoff *et al.*, 1955, Section 9.2). The two-degrees-of-freedom (2-DOF) typical section depicted in Figure 11.3.5 is a very simplified structural model used to study wing bending-torsion flutter[2]. The airfoil

section is hinged without friction at elastic axis EA to a vertical bar (assumed rigid and of negligible inertial force), which is restricted to vertical translation by frictionless linear bearings. Rotational spring k_θ simulates wing torsional stiffness, and translational spring k_y simulates wing bending stiffness. The theory of unsteady aerodynamics, even for two-dimensional flow, is very complicated, so we shall simply represent the aerodynamic resultant actions generally as lift $F_L(t)$ and pitching moment about the aerodynamic center $M_{AC}(t)$, and we shall focus on the structural dynamics.

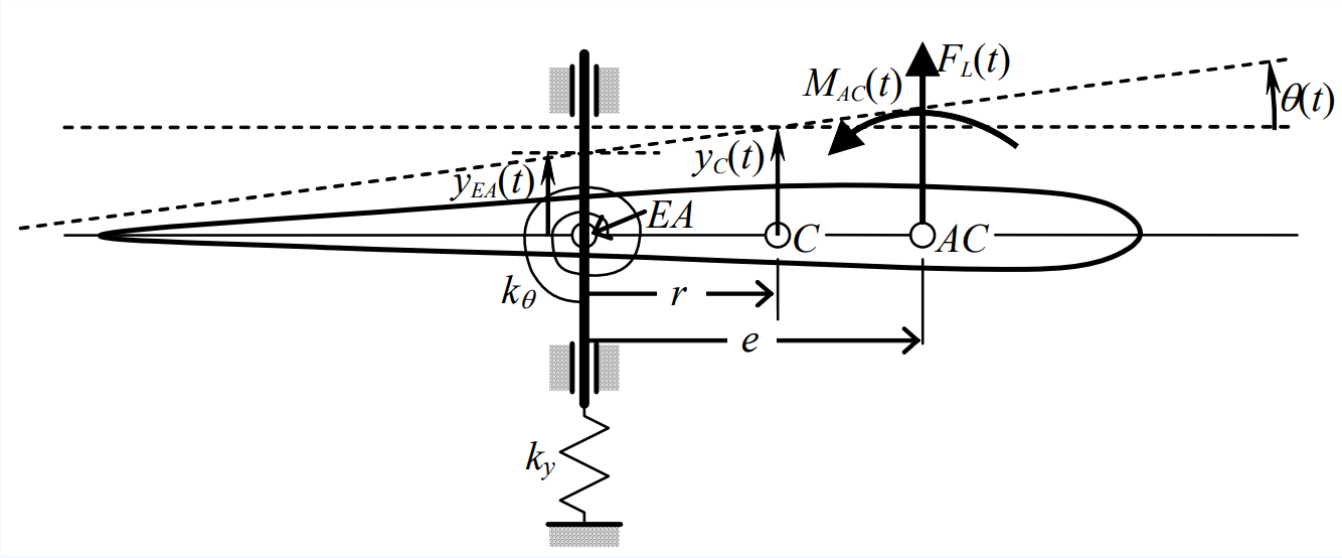


Figure 11.3.5: Example 11.3.3: 2-DOF typical section

In order to write the equations of motion for this typical section, we select as degrees of freedom the vertical translation $y_C(t)$ of mass center C, and pitching rotation $\theta(t)$, both relative to the reference static equilibrium position. The geometry of small rotation is represented by the relationship between $y_C(t)$ and $y_{EA}(t)$:

$$y_C(t) = y_{EA}(t) + r\sin\theta(t) \approx y_{EA}(t) + r\theta(t) \Rightarrow y_{EA}(t) \approx y_C(t) - r\theta(t) \tag{11.3.8}$$

The DFBD associated with Figure 11.3.5 is drawn below, with approximation Equation 11.3.8 used to annotate the force due to translation spring k_y.

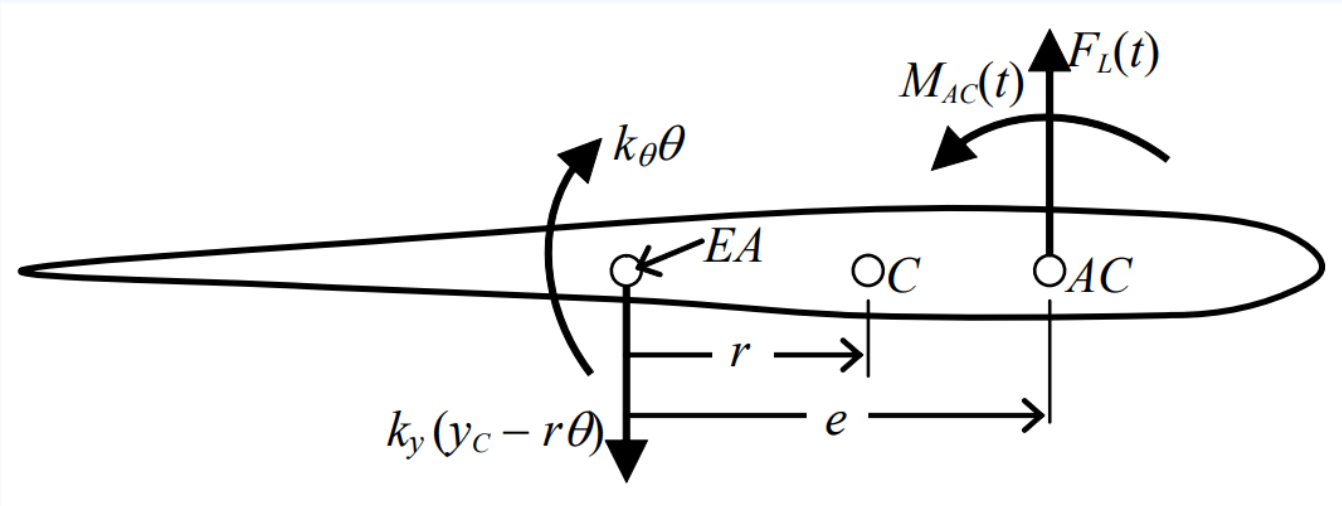


Figure 11.3.6

Using this DFBD, we write the two relevant equations of motion, Equations 11.1.3 and 11.1.4 in scalar form, as

$$m\ddot{y}_C = F_L(t) - k_y(y_C - r\theta) \tag{11.3.9}$$

$$J_C\ddot{\theta} = (e-r)F_L(t) + M_{AC}(t) - k_\theta\theta + rk_y\left(y_C - r\theta\right) \tag{11.3.10}$$

It is appropriate to express Equations 11.3.9 11.3.10 with all terms involving degrees of freedom $y_C(t)$ and $\theta(t)$ transposed to the left-hand sides:

$$m\ddot{y}_C + k_y y_C - rk_y\theta = F_L(t) \tag{11.3.11}$$

$$J_C\ddot{\theta} - rk_y y_C + \left(k_\theta + r^2 k_y\right)\theta = (e-r)F_L(t) + M_{AC}(t) \tag{11.3.12}$$

Equations 11.3.11 and 11.3.12 are a pair of 2nd order, *coupled* ODEs in unknowns $y_C(t)$ and $\theta(t)$. Just as Equations 11.3.4 , Equations 11.3.11 and 11.3.12 are described as being coupled because each equation contains both dependent variables and neither equation can be solved independently of the other. It is possible, though we will not bother doing it, to combine these two equations into a single 4th order ODE in a single unknown function of time; therefore, this 2-DOF typical section is actually a 4th order system. It is appropriate and common also to express the set of ODEs Equations 11.3.11 and 11.3.12 in matrix form:

$$\overbrace{\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix}}^{\text{inertia matrix}} \begin{bmatrix} \ddot{y}_C \\ \ddot{\theta} \end{bmatrix} + \overbrace{\begin{bmatrix} k_y & -rk_y \\ -rk_y & k_\theta + r^2 k_y \end{bmatrix}}^{\text{structural stiffness matrix}} \begin{bmatrix} y_C \\ \theta \end{bmatrix} = \begin{bmatrix} F_L(t) \\ (e-r)F_L(t) + M_{AC}(t) \end{bmatrix} \tag{11.3.13}$$

The labeled left-hand-side coefficient matrices of Equation 11.3.13 are typical for structural dynamic systems, and they exhibit some important general properties. The *inertia matrix* in this case is diagonal; more generally, it will be non-diagonal but symmetric. The inertia matrix will always be *positive definite* because all matter has positive mass; therefore, the determinant of the inertia matrix will always be positive. In Equation 11.3.13, that determinant is simply the product $mJ_C > 0$. The *structural stiffness* matrix in Equation 11.3.13 is symmetric, and this is a general property resulting from linearity between applied actions and structural deformations. Moreover, if a structure is restrained by grounded supports, such as the rotational spring k_θ and translational spring k_y of the typical section, then each of the diagonal elements of the stiffness matrix will be positive, and the stiffness matrix itself will be positive definite. You can check easily that the determinant of the stiffness matrix in Equation 11.3.13 is the product $k_y k_\theta > 0$. The coupling between vertical translation $y_C(t)$ and pitching rotation $\theta(t)$ of the typical section is both produced by and displayed by the non-zero off-diagonal terms of the stiffness matrix, each being $-rk_y$.

[1]For the early history of aeroelastic divergence, see Bisplinghoff *et al*., 1955, pp. 3-7, and Gordon, 1978, pp. 259-270.

[2]Physical typical-section airfoils were designed, fabricated, and tested in recent wind-tunnel experimental research studies of unsteady aerodynamics (as summarized by Bennett, 2000), and of feedback control to suppress flutter (DeMarqui, *et al*., 2005 and 2006). Homework Problem 16.11 applies an approximate theory of unsteady aerodynamics in a simulation of one such experimental project.

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11.4: Chapter 4 Homework

1. The trimming tab of a small airplane is attached to the elevator by a hinge at axis H and a rotational spring (the actuator linkage) with constant k_θ, as shown in the drawing below. You are required to determine experimentally both rotational inertia J_H of the elevator about the hinge, and rotational stiffness k_θ. The field experiment consists of two separate *twang* tests. (See homework Problem 9.4 for a detailed description of twang testing and relevant equations.) First, the rotational spring is disconnected from the tab, and a vertical translational support spring of known stiffness $k_y = 15.6$ lb/inch (and low mass, producing negligible inertial force) is attached near the trailing edge, as shown. Then the tab is twanged (displaced statically from the static equilibrium position, then released). A non-contacting displacement sensor picks up the dynamic motion at a point near the support spring: $y(t) = 0.107e^{-0.144t}\cos(59.9t)$ inch (t in seconds). (There is slight damping due to friction at the hinge, internal structural damping in the spring, and drag of the surrounding air.) From this first twang test, calculate J_H (in lb-s^2 -inch). Next, translational spring k_y is removed, rotational spring k_θ is re-connected, and the tab is again twanged. The same displacement sensor as in the first twang test now measures dynamic motion $y(t) = 0.132e^{-0.112t}\cos(66.0t)$ inch. From the previous results and this second twang test, calculate k_θ (in lb-inch/radian).

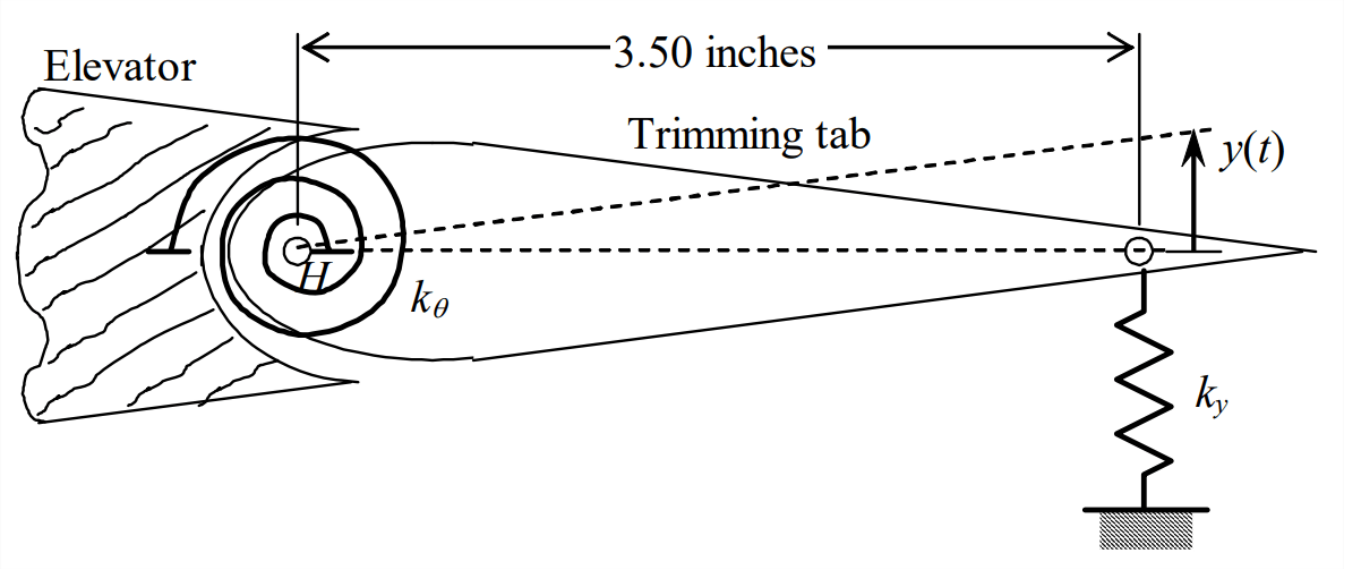


Figure 11.4.1

2. The system depicted on the next page consists of a rigid bar, a mass, and springs. The rigid bar is supported at frictionless hinge H, where it is elastically restrained by rotational spring k_θ, and the rotational inertia of the bar about point H is J_H. Mass m is connected to the bar by translational spring k_y at point B, and the mass is restrained to move only vertically by frictionless walls. An externally applied vertical force $F(t)$ acts on the bar at point B. Degrees of freedom $y(t)$ and $\theta(t)$ are the **small** motions relative to the static equilibrium position. Sketch and label a DFBD for the bar and a DFBD for mass m , then use these diagrams to derive the two coupled differential equations of motion for this system. (Hints: Show that for small bar rotation, the stretch of spring k_y is $a\theta - y$. Do **not** include mg or bar weight among the applied forces, because we are interested only in dynamic motions relative to the static equilibrium position; if you are not sure why we neglect weights in these equations of motion, then review Section 7.5.)

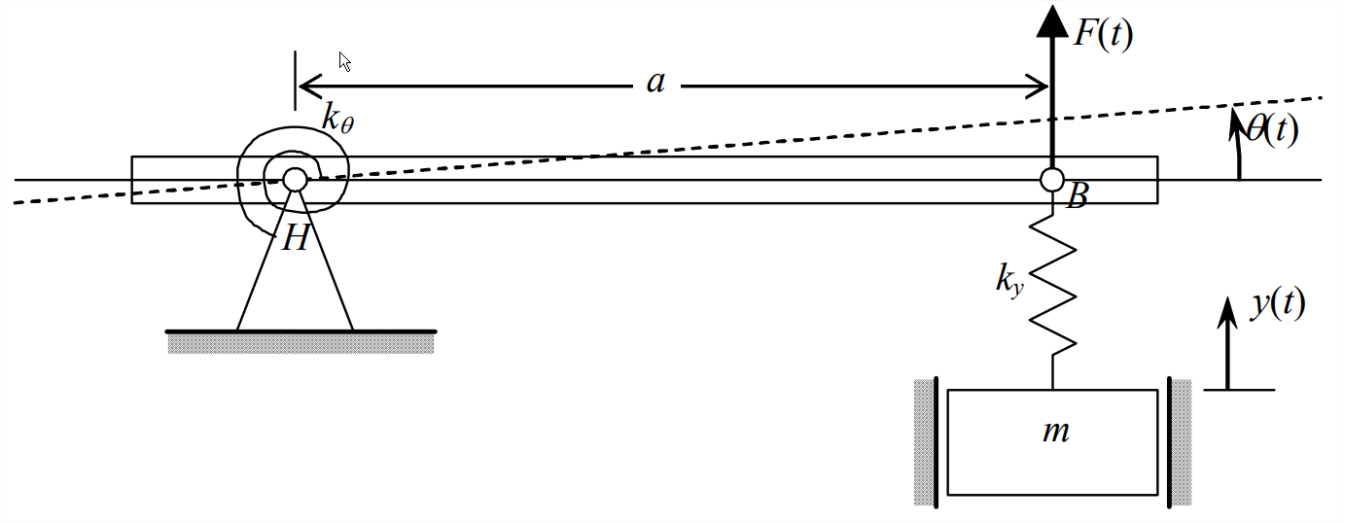


Figure 11.4.2

3. The two-degrees-of-freedom system drawn below consists of a pendulum supported at frictionless hinge H, and a mass-spring connected by a viscous damper to the pendulum rod. The inertial moment of the lightweight pendulum rod is negligible, so pendulum rotational inertia about H is $J_H = m_p\ell^2$. Degrees of freedom $x(t)$ and $\theta(t)$ are m small motions **relative to** the static equilibrium position. Show that the velocity of the damper cylinder relative to the damper piston is $\dot{x} - a\dot{\theta}$. Both gravity and the externally applied horizontal force $F(t)$ act on mass m_p of the pendulum bob. Sketch and label an FBD for the pendulum and an FBD for mass m, then use these diagrams to derive the two coupled differential equations of motion for this system. In addition to the motion and force variables and the constant parameters shown on the drawing, the acceleration of gravity acting vertically downward is denoted g. For this problem, unlike for Problems 11.1 and 11.2 and the examples in Section 11.3, you should sketch your pendulum FBD with the pendulum rotated through a small angle, so that you can indicate the restraining effect of weight $m_p g$ on pendulum rotation; see the pendulum example in Section 7.1.

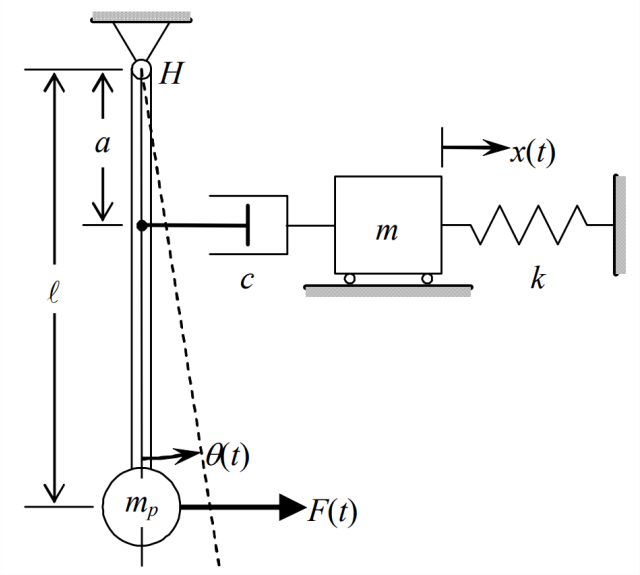


Figure 11.4.3

4. The form of the typical-section ODEs of motion represented by Equations 11.3.14, 11.3.15, and 11.3.16 is not unique. Suppose, for example, you want to define $y_{EA}(t)$ as the vertical translation degree of freedom, instead of $y_c(t)$. With use of geometry, inertia, and algebraic manipulation, you can cast the ODEs into a cosmetically different, but mathematically equivalent form. First, substitute Equation 11.3.11 into Equations 11.3.14 and 11.3.15 in order to replace $y_c(t)$ with appropriate terms that include $y_{EA}(t)$ and small-rotation $\theta(t)$; next, use the parallel-axis theorem to replace J_C with the rotational inertia about the elastic axis, J_{EA}, and a term including mass m; finally, perform any additional algebraic substitution required to write the ODEs of motion in the following relatively simple matrix form:

$$\overbrace{\begin{bmatrix} m & mr \\ mr & J_{EA} \end{bmatrix}}^{\text{inertia matrix}} \begin{bmatrix} \ddot{y}_{EA} \\ \ddot{\theta} \end{bmatrix} + \overbrace{\begin{bmatrix} k_y & 0 \\ 0 & k_\theta \end{bmatrix}}^{\text{structural stiffness matrix}} \begin{bmatrix} y_{EA} \\ \theta \end{bmatrix} = \begin{bmatrix} F_L(t) \\ eF_L(t) + M_{AC}(t) \end{bmatrix}$$

Which terms in this matrix equation produce and display the coupling between vertical translation $y_{EA}(t)$ and pitching rotation $\theta(t)$? Will this coupling still exist if the applied aerodynamic actions $F_L(t)$ and $M_{AC}(t)$ are either constant or vary slowly enough that the response is pseudo-static? Prove the positive definiteness of these inertia and structural stiffness matrices by finding the values and polarities of their determinants.

5. Consider the pendulum-spring system depicted below. The inertial moments of the lightweight pendulum rods are negligible, so pendulum rotational inertias about their hinges are $J_i = m_i \ell^2$, $i = 1, 2$. Degrees of freedom $\theta_1(t)$ and $\theta_2(t)$ are **small** rotations relative to the vertical hanging static equilibrium positions. A horizontal translational spring with stiffness constant k connects the two pendulums, and it is undeformed when the pendulums are in the vertical hanging static equilibrium positions. Show that the stretch of the spring is given by $a\,(\theta_2 - \theta_1)$. Externally applied moment $M_2(t)$ acts on pendulum 2 at its hinge (applied, for example, by a motor). Sketch and label an FBD for each pendulum, then use these diagrams to derive the two coupled differential equations of motion for this system. In addition to the motion and applied action variables and the constant parameters shown on the drawing, the acceleration of gravity acting vertically downward is denoted g. In this problem, you should sketch your FBDs with the pendulums rotated through small angles, so that you can indicate the restraining effect of the weights $m_i g, i = 1, 2$, on the rotations; see the pendulum example in Section 7.1.

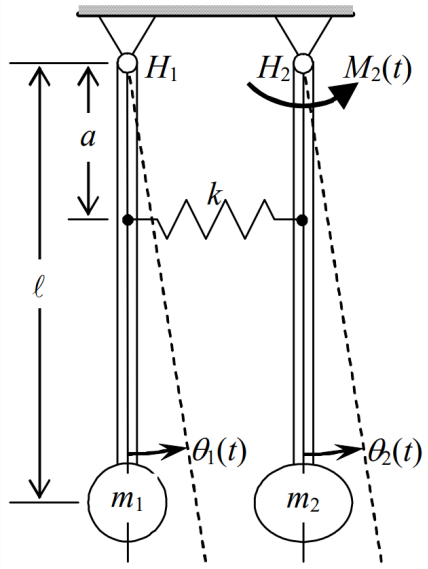


Figure 11.4.4

6. The drawing below depicts a **very** simplified model for the pitch-translation dynamics of an automotive vehicle. The body and frame are considered to be rigid, with center of mass C located as shown, and having mass m and rotational inertia J_C about C. The wheel-axle assemblies are considered to have negligible inertial forces. The flexibilities of tires and suspension systems are lumped together into a single rear-axle translational spring k_1 and a single front-axle translational spring k_2 . Damping is provided by shock struts and rubber tires, and it is lumped into viscous dashpots c_1 and c_2 arranged in parallel with the respective springs. Sketch and label a DFBD for this vehicle model, then use it to derive the two coupled differential equations of motion for this system, in terms of the given parameters. Use as degrees of freedom the small vertical translation $y_C(t)$ and the small pitching rotation $\theta(t)$, both relative to the static equilibrium position.

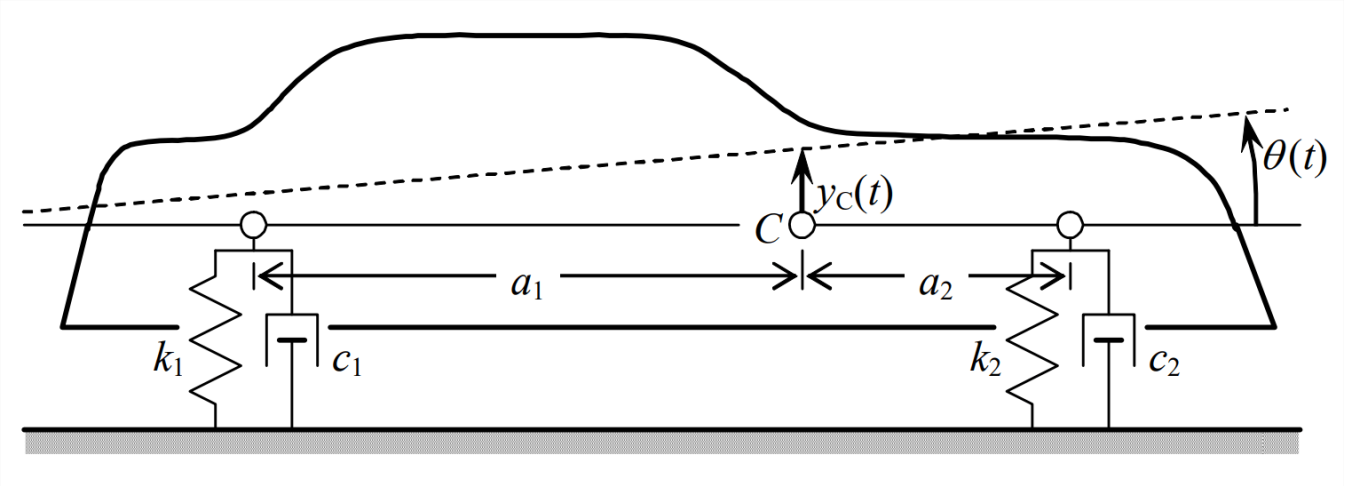


Figure 11.4.5

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12: VIBRATION MODES OF UNDAMPED MECHANICAL SYSTEMS WITH TWO DEGREES OF FREEDOM

12.1: Undamped Mass-Spring System
12.2: Undamped Two-Mass-Two-Spring System
12.3: Vibration Modes of an Undamped Typical Model of a Wing
12.4: Chapter 12 Homework

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12.1: Undamped Mass-Spring System

The subject of this chapter is free vibration of undamped mechanical systems, beginning with the one-degree-of-freedom (1-DOF) mass-spring system of Fig 12.1.1. We evaluated the m-k system more generally in Chapter 7, but we will re-visit it here using a different approach. Free vibration can arise from initial translation of the mass (relative to the static equilibrium position) and/or initial velocity, and also from excitation that has ceased, which might provide the mass with both initial translation and initial velocity at the instant when excitation ceases. However, it is easiest conceptually to visualize this motion as response to only non-zero initial translation relative to the static equilibrium position. Doing so sacrifices a little generality because we cannot also account for a non-zero initial velocity, but the essential characteristics of the free vibration are revealed by this approach. Consider an undamped mass-spring system with nonzero initial translation, $y(0) = y_0$, and zero initial velocity, $\dot{y}(0) = 0$, but without forcing excitation, so that $f_y(t) = 0$. From Equation 7.1.2, the ODE of free vibration is

$$m\ddot{y} + ky = f_y(t) = 0 \tag{12.1.1}$$

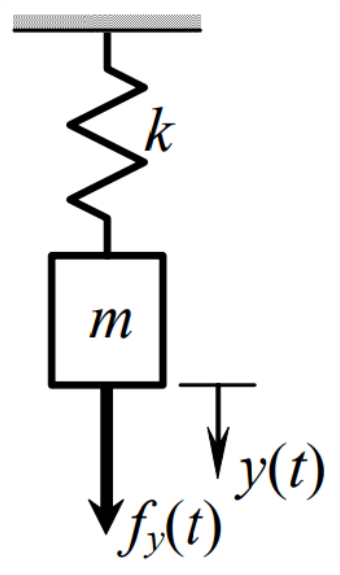


Figure 12.1.1

We now ask the questions: Can motion exist in the form $y(t) = Y\cos\omega t$, and, if so, what are the constants Y and ω? To find the answers, we substitute the presumed form of motion into ODE Equation 12.1.1, perform the differentiations and algebra, and see where the process leads:

$$m\left(-\omega^2 Y\cos\omega t\right) + kY\cos\omega t = \left(-\omega^2 m + k\right)Y\cos\omega t = 0$$

This equation can be satisfied in general, for a non-trivial solution, only if

$$\omega^2 = k/m \equiv \omega_n^2 \Rightarrow \omega = \omega_n = \sqrt{k/m}$$

Moreover, we can use the initial condition to find Y:

$$y(0) = y_0 = Y\cos(\omega \times 0) = Y$$

$$\Rightarrow \quad y(t) = y_0\cos\omega_n t \tag{12.1.2}$$

This simple result was already derived in Equation 7.3.1; we re-derive it differently here just to establish the approach that will be useful next for higher-order systems.

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12.2: Undamped Two-Mass-Two-Spring System

Next, we analyze the two-degrees-of freedom (2-DOF) undamped mass-spring system of Figure 12.2.1. Dynamic translations $y_1(t)$ and $y_2(t)$ shown are relative to the static equilibrium positions. As usual for the purpose of drawing forces on dynamic freebody diagrams (DFBDs, as defined in Section 7.5), we let the translational springs be stretched at the instant depicted, so that, in particular, the dynamic **tension** in the lower spring $k_2\,(y_2 - y_1)$.

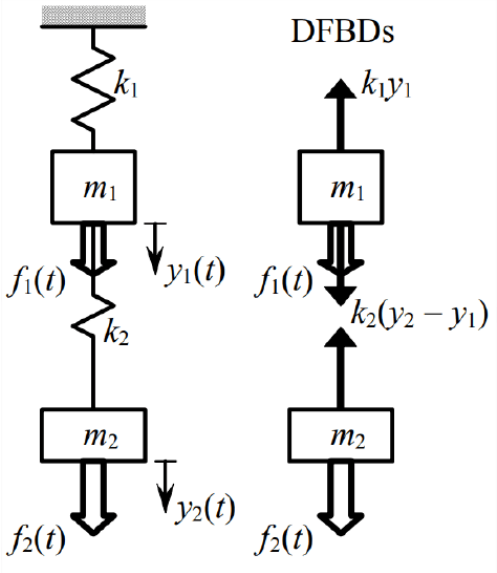


Figure 12.2.1: Two-mass-twospring mechanical system

From the DFBDs, Newton's 2nd law gives the ODEs of motion:

$$\begin{aligned} m_1\ddot{y}_1 &= f_1(t) + k_2\,(y_2 - y_1) - k_1y_1 \\ m_2\ddot{y}_2 &= f_2(t) - k_2\,(y_2 - y_1) \end{aligned} \tag{12.2.1}$$

Transposing dependent-variable terms to the lefthand sides, and collecting terms gives

$$\begin{aligned} m_1\ddot{y}_1 + (k_1 + k_2)\,y_1 - k_2y_2 &= f_1(t) \\ m_2\ddot{y}_2 - k_2y_1 + k_2y_2 &= f_2(t) \end{aligned} \tag{12.2.2}$$

Now, expressing Equations 12.2.2 in the same matrix form as Equation 11.3.16 gives

$$\overbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}^{\text{inertia matrix}} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \overbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}^{\text{structural stiffness matrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \tag{12.2.3}$$

You can verify easily that, just as in Equation 11.3.16, the inertia and structural stiffness matrices of Equation 12.2.3 are positive definite, and all diagonal elements of both matrices also are positive definite.

Next, we want to solve matrix Equation 12.2.3 for free vibrations, so we set to zero the applied forces, $f_1(t) = f_2(t) = 0$. Following the procedure in Section 12.1 for the 1-DOF system, let us seek response to non-zero initial translations, with zero initial velocities. Accordingly, we assume motion solutions of the unforced 2-DOF system in the form $y_1(t) = Y_1 \cos\omega t$ and $y_2(t) = Y_2 \cos\omega t$, in which ω, Y_1, and Y_2 are the unknown quantities. It is efficient and appropriate to express the assumed solution in matrix form:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \cos\omega t \tag{12.2.4}$$

Substituting Equation 12.2.4 into Equation 12.2.3 with zero forcing gives

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \left(-\omega^2\right) \cos \omega t + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \cos \omega t = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12.2.5}$$

The common multiple $\cos \omega t$ in Equation 12.2.5 cannot be zero in general, so we cancel it out of the equation. Further, we consolidate all of the coefficients of Y_1 and Y_2 into a single matrix, giving

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12.2.6}$$

Before dealing with Equation 12.2.6, let us review a little from your mathematics background about solving two linear algebraic equations in two unknowns. In general, the equations are expressed in matrix notation as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \text{ with } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \equiv [\mathbf{a}], \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv \{\mathbf{x}\}, \text{ and } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \equiv \{\mathbf{b}\} \tag{12.2.7}$$

in which the a_{ij}'s are known constant coefficients, the x_i's are the unknowns, and the b_i's are known right-hand-side constants. It is useful to express the matrix equation symbolically by using bold fonts, brackets, and parentheses: $[\mathbf{a}]\{\mathbf{x}\} = \{\mathbf{b}\}$. Then the *determinant* and the *adjoint matrix* of the coefficient matrix are, respectively,

$$\det[\mathbf{a}] = a_{11}a_{22} - a_{21}a_{12} \tag{12.2.8}$$

$$\text{adj}[\mathbf{a}] = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \tag{12.2.9}$$

The solution of Equation 12.2.7 for the unknowns involves the inverse $[\mathbf{a}]^{-1}$ of the coefficient matrix, which is defined in terms of the determinant and the adjoint matrix:

$$\{\mathbf{x}\} = [\mathbf{a}]^{-1}\{\mathbf{b}\} = \frac{\text{adj}[\mathbf{a}]}{\det[\mathbf{a}]}\{\mathbf{b}\} \tag{12.2.10}$$

If $\{\mathbf{b}\} = 0$, which is the case for Equation 12.2.6, then there are two possible types of solutions:

1. if $\{\mathbf{b}\} = 0$ and $\det[\mathbf{a}] \neq 0$, then the solution must be the trivial result $\{x\} = 0$;
2. however, if $\{\mathbf{b}\} = 0$ **and** $\det[\mathbf{a}] = 0$, then the right-hand side of Equation 12.2.10 has the indeterminate form $0/0$, so that there exists, and we can solve for a non-trivial $\{\mathbf{x}\} \neq \mathbf{0}$, but any constant multiple of $\{\mathbf{x}\}$ is also a solution since $\{\mathbf{b}\} = 0$ in Equation 12.2.7.

Therefore, for our current application, Equation 12.2.6, we must seek a type 2 solution.

To find the same type 2 result as in the last paragraph, but without using matrices or theory from linear algebra, let us write Equation 12.2.6 as two separate scalar equations, and then attempt to solve them algebraically for Y_1 and Y_2:

$$\left(k_1 + k_2 - \omega^2 m_1\right) Y_1 - k_2 Y_2 = 0 \Rightarrow Y_2 = \frac{1}{k_2}\left(k_1 + k_2 - \omega^2 m_1\right) Y_1 \tag{12.2.11}$$

$$-k_2 Y_1 + \left(k_2 - \omega^2 m_2\right) Y_2 = 0 = -k_2 Y_1 + \left(k_2 - \omega^2 m_2\right) \frac{1}{k_2}\left(k_1 + k_2 - \omega^2 m_1\right) Y_1 \tag{12.2.12}$$

Re-arranging the last part of Equation 12.2.12 a bit differently gives

$$\left[\left(k_1 + k_2 - \omega^2 m_1\right)\left(k_2 - \omega^2 m_2\right) - k_2^2\right] \frac{Y_1}{k_2} = 0 \tag{12.2.13}$$

We want to solve Equation 12.2.13 for a non-zero , which leads us to conclude that:

1. the bracketed term multiplying Y_1/k_2 **must be zero** [Note that the bracketed term is the determinant of the coefficient matrix in Equation 12.2.6.]; and
2. we cannot yet assign any specific value to Y_1 (but we will be able to do so later if we chose to regard this problem as being time response to specific initial translations).

The bracketed term in Equation 12.2.13 must be zero; upon carrying out the multiplications, we see that it leads to a quadratic

equation in the unknown ω^2:

$$m_1 m_2 \left(\omega^2\right)^2 + \left[-m_1 k_2 - m_2 \left(k_1 + k_2\right)\right] \omega^2 + k_1 k_2 = 0 \tag{12.2.14}$$

This type of polynomial equation is generally called the *characteristic equation* of the free-vibration problem. For any realistic values of the mass and stiffness constants, Equation 12.2.14 has two real, positive roots, which we denote as ω_1^2 and ω_2^2.

Let us consider a numerically simplified case for which the masses are equal, $m_1 = m_2 \equiv m$, and the stiffness constants also are equal, $k_1 = k_2 \equiv k$. In this case, the quadratic equation Equation 12.2.14 becomes

$$m^2 \left(\omega^2\right)^2 - 3mk\omega^2 + k^2 = 0 \tag{12.2.15}$$

Using the standard solution of quadratic equations, Equation 2.1.2, we find the roots of Equation 12.2.15 to be

$$\omega^2 = \frac{3 \pm \sqrt{5}}{2} \frac{k}{m} = (0.382, 2.618) \frac{k}{m} \tag{12.2.16}$$

For future reference, note that the **lower** of the $\pm$ signs gives the smaller (the "first") of the two ω^2 roots, and that the **upper** sign gives the larger (the "second") of the two ω^2 roots. Only positive circular frequencies are physically meaningful, and they are:

$$\omega = (0.618, 1.618)\sqrt{k/m} \Rightarrow \omega_1 = 0.618\sqrt{k/m} \text{ and } \omega_2 = 1.618\sqrt{k/m} \tag{12.2.17}$$

Of the two circular frequencies in Equations 12.2.17, the smaller value ω_1 is called the *first* or *fundamental natural frequency*, and the larger value ω_2 is called the *second natural frequency*. By convention, natural frequencies are always numbered in ascending order for all higher-order systems that have two or more natural frequencies. It is mathematically systematic to define an integer subscript n that is either 1 or 2, and to identify the natural frequencies more generally as ω_n, n = 1, 2. (In this case, the symbol n performs double-duty, because it both implies *natural* frequency of vibration and takes the values 1 or 2 to distinguish the **two** natural frequencies from each other.)

Next, let us seek values of Y_1 and Y_2 associated with each of the natural frequencies. It is appropriate and efficient to extend use of the subscript n to these motion magnitude values by labeling as \(Y_{1n\) and Y_{2n} the values associated with frequency ω_n, n = 1, 2. Let us re-write Equations 12.2.11 and 12.2.12 using this notation and the numerically simplified case of equal masses and equal stiffness constants:

$$\left(2k - \omega_n^2 m\right) Y_{1n} - k Y_{2n} = 0 \Rightarrow Y_{2n} = \frac{1}{k}\left(2k - \omega_n^2 m\right) Y_{1n} \tag{12.2.18}$$

$$-k Y_{1n} + \left(k - \omega_n^2 m\right) Y_{2n} = 0 \Rightarrow Y_{2n} = \frac{k}{k - \omega_n^2 m} Y_{1n} \tag{12.2.19}$$

As is stated in the discussion of Equation 12.2.13, we cannot yet assign any specific value to Y_{1n}. If we leave Y_{1n} arbitrary (but not zero) for now, then Equations 12.2.18 and 12.2.19 appear to give two different ways of solving for Y_{2n} in term of Y_{1n}. Let us use natural frequency Equation 12.2.16 to evaluate both equations:

$$Y_{2n} = \frac{1}{k}\left(2k - \omega_n^2 m\right) Y_{1n} = \left(2 - \omega_n^2 \frac{m}{k}\right) Y_{1n} = \left(2 - \frac{3 \pm \sqrt{5}}{2}\right) Y_{1n} = \frac{1 \mp \sqrt{5}}{2} Y_{1n} \tag{12.2.20}$$

$$Y_{2n} = \frac{k}{k - \omega_n^2 m} Y_{1n} = \frac{1}{1 - \omega_n^2 \frac{m}{k}} Y_{1n} = \frac{1}{1 - \frac{3 \pm \sqrt{5}}{2}} Y_{1n} = \frac{2}{-1 \mp \sqrt{5}} Y_{1n}$$

$$Y_{2n} = \frac{2}{-1 \mp \sqrt{5}} \times \frac{1 \mp \sqrt{5}}{1 \mp \sqrt{5}} Y_{1n} = \frac{1 \mp \sqrt{5}}{2} Y_{1n} \tag{12.2.21}$$

Both equations Equations 12.2.20 and 12.2.21 give the same answer! Mathematically, this result is related to the zero of the determinant of the coefficient matrix in Equation 12.2.6, but the detailed theory is best left to a more advanced book (for example, Craig, 1981, Section 13.1). For our purposes, it is enough to recognize from this example that, in this type of problem, both equations have the same solution, so it is necessary to solve only one of them.

Next, let us collect together from Equations 12.2.6, 12.2.20, and 12.2.21 all the numerical results for each value of the index n into *modes of vibration*, the physical significance of which will become clear later. We have observed already that Y_{1n} is some arbitrary

non-zero value, so nothing is lost if we set $Y_{1n} = 9$ (with dimension of translation, *i.e.*, length) for both modes of vibration.

n **= 1,** ***first (fundamental) mode of vibration***: [the **lower** sign in Equations 12.2.20 and 12.2.21]

$$\omega_1 = 0.618\sqrt{\frac{k}{m}}, \quad Y_{21} = \frac{1+\sqrt{5}}{2}Y_{11} = 1.618 \Rightarrow \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} \tag{12.2.22}$$

n **= 2, second mode of vibration**: [the **upper** sign in 12.2.20 and 12.2.21]

$$\omega_2 = 1.618\sqrt{\frac{k}{m}}, \quad Y_{22} = \frac{1-\sqrt{5}}{2}Y_{12} = -0.618 \Rightarrow \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} \tag{12.2.23}$$

For reasons that will become clear later, the column matrices $\begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix}$ and $\begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix}$ are called the *mode shapes* of, respectively, the first and second modes of vibration. **NOTE**: In Equations 12.2.22 and 12.2.23 we set $Y_{11} = Y_{12} = 1$ with dimension of length, but that does not necessarily imply that the magnitudes of physical responses $y_1(t)$ and $y_2(t)$ are one meter or one foot or one inch (depending upon the unit system in use). The actual physical response magnitudes are determined by initial conditions, as is discussed in the next paragraph.

Recall that we are seeking response to non-zero initial deformation, with zero initial velocity, and so we chose the solution in the form $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \cos \omega t$, Equation 12.2.4. In view of the free-vibration results Equations 12.2.22 and 12.2.23, it is quite plausible physically (and it can be proved rigorously) that the general initial-translation response can be written as the **linear sum** of two terms like Equation 12.2.4, one for each of the vibration modes:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = C_1 \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} \cos \omega_1 t + C_2 \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} \cos \omega_2 t \tag{12.2.24}$$

In Equation 12.2.24, the dimensionless multiplying constants C_1 and C_2 show the contribution to the total response due to each mode of vibration; they are found from given initial translations $y_1(0)$ and $y_2(0)$ by writing Equation 12.2.24 for t = 0:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = C_1 \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} + C_2 \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{12.2.25}$$

The determinant of the 2×2 *mode-shape matrix* $[\mathbf{Y}]$ in Equation 12.2.25 is non-zero, so that matrix can be inverted and the equation can be solved by application of Equation 12.2.10, for arbitrary known $y_1(0)$ and $y_2(0)$, to give mode contribution constants C_1 and C_2:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\det[\mathbf{Y}]} \begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \tag{12.2.26}$$

Now, suppose that the initial translation is chosen to have the shape of the first mode, in other words, to be proportional to $\begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix}$ with dimensionless constant of proportionality D_1: $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = D_1 \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix}$. Substituting this initial translation into Equation 12.2.26 gives

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\det[\mathbf{Y}]} \begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix} D_1 \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} = \frac{D_1}{\det[\mathbf{Y}]} \begin{bmatrix} Y_{22}Y_{11} - Y_{12}Y_{21} \\ -Y_{21}Y_{11} + Y_{11}Y_{21} \end{bmatrix} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}$$

Therefore, the total response Equation 12.2.24 is $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = D_1 \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} \cos \omega_1 t$ for $t \geq 0$. Let us express this mathematical result in words: if we impose the initial translation in the shape of the first mode of vibration, then the subsequent free vibration of the system will be pure sinusoidal motion at the first natural frequency, and the relative positions of the two masses at each instant of time will remain in the first mode shape. In other words, both masses will vibrate with frequency ω_1, and at each instant $t \geq 0$ the ratio $y_2(t)/y_1(t)$ will have the same value, Y_{21}/Y_{11}, which is 1.618 for the numerical example of Equations 12.2.15, 12.2.16, and 12.2.17 and Equation 12.2.20, 12.2.21, 12.2.22, and 12.2.23. In this case, the two masses move in the same direction at each instant, so they are said to move *in phase* with each other.

Suppose next that the initial translation is chosen to have the shape of the second mode, to be proportional to $\begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix}$ with dimensionless constant of proportionality D_2. By following the procedure of the last paragraph, we can find that the total response Equation 12.2.24 is $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = D_2 \begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix} \cos \omega_2 t$ for $t \geq 0$. In words, the subsequent free vibration of the system will be pure sinusoidal motion at the second natural frequency, and the relative positions of the two masses at each instant of time will remain in the second mode shape: both masses will vibrate with frequency ω_2, and at each instant $t \geq 0$ the ratio $y_2(t)/y_1(t)$ will have the same value, Y_{22}/Y_{12}, which is −0.618 for the numerical example of Equations 12.2.15, 12.2.16, and 12.2.17 and Equation 12.2.20 , 12.2.21, 12.2.22, and 12.2.23. In this case, the two masses move in opposite directions at each instant, so they are said to move *out of phase*.

The modes of vibration of this 2-DOF mass-spring system, the numerical example of Equations 12.2.15, 12.2.16, and 12.2.17 and Equation 12.2.20, 12.2.21, 12.2.22, and 12.2.23, are depicted graphically in Figure 12.2.2. The system is shown in its static equilibrium position for reference, and in deformed positions corresponding to each of the modes of vibration. We can not animate on the printed page the dynamic motion of a vibration mode, so we imagine that we are viewing a snapshot of the masses taken at one instant of time when the deformations are non-zero during any cycle of vibration.

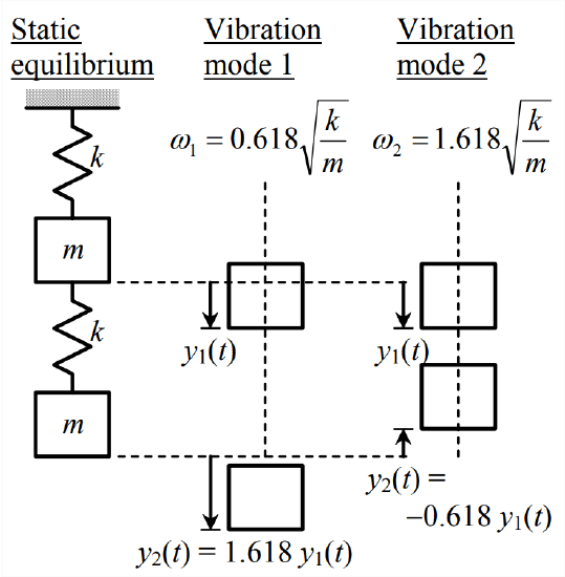


Figure 12.2.2: Vibration modes of the 2-DOF mass-spring system that has equal masses and equal springs

The object in this chapter is to derive and illustrate the physical character of modes of vibration of undamped 2-DOF systems. The clearest and most direct way to do this is to restrict attention to response resulting from simple initial deformation, with zero initial velocity and zero input action. We will not go further in this book, but you should be informed that the theory started here can be extended to account for arbitrary non-zero initial conditions (both deformation and velocity) and arbitrary non-zero input actions imposed on damped 2-DOF systems. Indeed, the theory can be extended to damped higher-order systems having any number of degrees of freedom, which are often called *N-DOF systems*, with $N \geq 2$. The general method of analysis is called *modal analysis*, and it is used extensively in engineering practice and research for both mathematical and experimental studies. This book just scratches the surface of modal analysis; the theory is extended to general response of LTI mechanical systems by more advanced books on linear-systems analysis and structural dynamics, two examples being Meirovitch, 2001, Chapter 7 and Craig, 1981, Chapters 13-15.

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12.3: Vibration Modes of an Undamped Typical Model of a Wing

From Example 11.3.3 of Section 11.3, the system is depicted again in Figure 12.3.1:

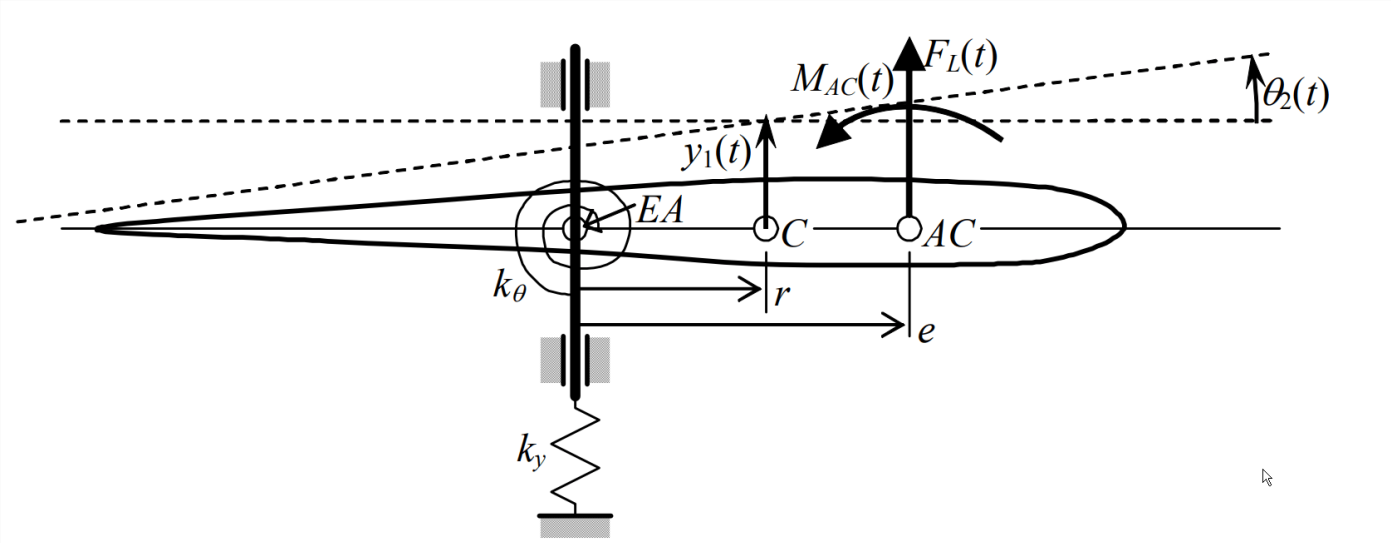


Figure 12.3.1: 2-DOF typical section

The degree-of-freedom symbols $y_C(t)$ and $\theta(t)$ used in Chapter 11 are changed here, respectively, to $y_1(t)$ and $\theta_2(t)$, in order to denote clearly that these motion quantities are, respectively, degrees of freedom (DOFs) #1 and #2.

The matrix equation of motion in DOFs $y_1(t)$ and $\theta_2(t)$, from Equation 11.3.16 is

$$\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_y & -rk_y \\ -rk_y & k_\theta + r^2 k_y \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} F_L(t) \\ (e-r)F_L(t) + M_{AC}(t) \end{bmatrix} \tag{12.3.1}$$

To analyze Equation 12.3.1 for modes of free vibration following the procedure described in the previous section, we set to zero the applied aerodynamic actions, and we seek motion solutions of the unforced system in the form:

$$\begin{bmatrix} y_1(t) \\ \theta_2(t) \end{bmatrix} = \begin{bmatrix} Y_1 \\ \Theta_2 \end{bmatrix} \cos \omega t \tag{12.3.2}$$

Substituting Equation 12.3.2 into Equation 12.3.1, canceling the common $\cos \omega t$ term, and collecting all coefficients in a single square matrix leads to

$$\begin{bmatrix} k_y - \omega^2 m & -rk_y \\ -rk_y & k_\theta + r^2 k_y - \omega^2 J_C \end{bmatrix} \begin{bmatrix} Y_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12.3.3}$$

Setting to zero the determinant of the coefficient matrix gives the quadratic equation for the natural frequencies (the characteristic equation of the free-vibration problem),

$$mJ_C \left(\omega^2\right)^2 + \left[-m\left(k_\theta + r^2 k_y\right) - J_C k_y\right] \omega^2 + k_y k_\theta = 0 \tag{12.3.4}$$

For physically realistic values of the parameters in Equation 12.3.4, there are two positive real roots, $\omega_1^2 < \omega_2^2$, one for each of the vibration modes, $n = 1, 2$. After these roots have been found, they can be substituted back into either scalar equation of matrix Equation 12.3.3 to give equations for the mode shapes. We choose to use the first scalar equation (because it has fewer terms than the second), and we solve for pitch mode shape component Θ_2 in terms of vertical translation component Y_1:

$$\left(k_y - \omega_n^2 m\right) Y_{1n} - r k_y \Theta_{2n} = 0 \Rightarrow \Theta_{2n} = \frac{k_y - \omega_n^2 m}{r k_y} Y_{1n}, n = 1, 2 \tag{12.3.5}$$

There is an interesting physical feature of motion in a pure vibration mode that is sometimes exhibited by bodies such as the typical-section model of Figure 12.3.1, bodies that both translate and rotate: whereas most of the body is vibrating at a natural frequency ω_n, there might be a particular point on the body that is completely motionless. Such a point is called a *node* or *nodal point* of the vibration mode. We can determine the location of a nodal point on the typical section by referring to Figure 12.3.2 , which shows both the airfoil in the static equilibrium position and the displaced chord line at some arbitrary instant during a cycle of pure modal vibration. We define on the diagram a stationary x axis with its origin at mass center C of the airfoil in the static equilibrium position. The

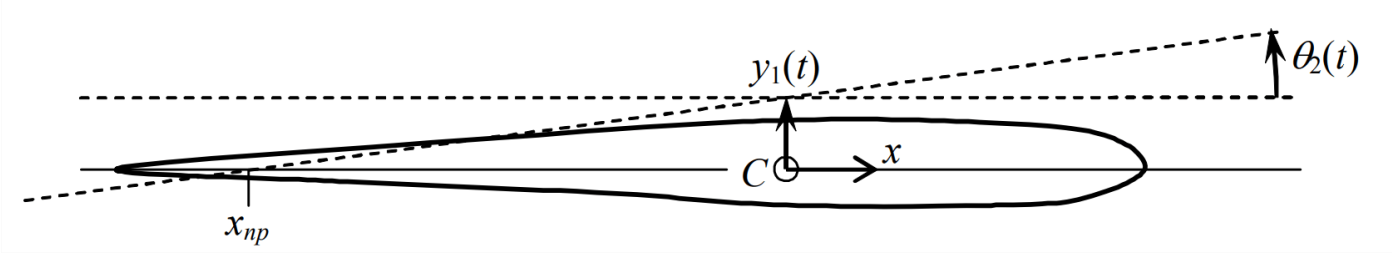


Figure 12.3.2: Nodal point of the 2-DOF typical section in a mode of vibration

coordinate xnp shown at the intersection of the static equilibrium chord line and the displaced chord line is the nodal point. Using the geometry of small rotation and the notation on Figure 12.3.2, we have $y_1(t) \approx -x_{np}\theta_2(t)$; the minus sign is necessary because x_{np} as shown is negative (aft of point C), but the values of $y_1(t)$ and $\theta_2(t)$ shown are positive. Therefore, the location of the nodal point for vibration mode n is given by

$$\left(x_{np}\right)_n = -\frac{y_1(t)}{\theta_2(t)} = -\frac{Y_{1n}}{\Theta_{2n}} \tag{12.3.6}$$

If mode n is mostly translation, with little rotation, then the coordinate might have such a large magnitude that there is no motionless point along the chord of the airfoil. However, if there is a nodal point on a typical section in a vibration test, one can easily observe it by sprinkling a little sand on the airfoil in the vicinity of the nodal point. As the body vibrates, the sand will migrate away from oscillating surfaces and accumulate on the motionless nodal point.

Rather than deal with an excessive number of algebraic symbols in equations such as Equation 12.3.4, it is convenient now to consider a numerical example. A certain wind-tunnel typical section has a chord of 12 inches, weighs mg = 5.00 lb, has rotational "weight" about its mass center J_{CG} = 90.0 lb-inch^2, and has its mass center located r = 3.00 inches forward of the axis at which it is supported by lightweight flexible beams that function as springs. The spring constants provided by the beams are k_y = 51.0 lb/inch and k_θ = 920 lb-inch/radian. The MATLAB session printed below evaluates Equations 12.3.4 through 12.3.6 for this numerical case.

```
» mg=5.00;Jcg=90.0;r=3.00;g=386.1; %mass and inertia constants

» ky=51.0;kt=920;format short e %stiffness constants

» p=[mg*Jcg/g^2 -(mg*(kt+ky*r^2)+Jcg*ky)/g ky*kt] %coeffs of Eq.(12-28)

p =

3.0187e-003 -2.9746e+001 4.6920e+004

» w_sqrd=roots(p) %roots w^2 of quadratic Eq.(12-28)

w_sqrd =

7.8822e+003

1.9720e+003
```

```
» wn=[sqrt(w_sqrd(2)) sqrt(w_sqrd(1))] %circular natural frequencies (in rad/s) arranged in ascending order

wn =

4.4407e+001 8.8782e+001

» fn=wn/2/pi %cyclic natural frequencies (in Hz)

fn =

7.0676e+000 1.4130e+001

» Th2n=(ky-mg/g*wn.^2)/(r*ky) %Eq.(12-29) for mode shape components \(\Theta\)2n, with Y1n = 1 inch

Th2n =

1.6642e-001 -3.3382e-001

» XNPn=(-1)./Th2n %Eq.(12-30) for locations of the nodal points
```

The numerical results of the MATLAB calculations are summarized in the following table:

Mode of vibration n	1	2
Natural frequency ω_n and $f_n = \omega_n/2\pi$	ω_1 = 44.41 rad/s f_1 = 7.068 Hz	ω_2 = 44.41 88.78 rad/s f_2 = 14.13 Hz
Mode shape	$\begin{bmatrix} Y_{11} \\ \Theta_{21} \end{bmatrix} = \begin{bmatrix} 1 \text{ inch} \\ 0.1664\text{rad} \end{bmatrix}$	$\begin{bmatrix} Y_{12} \\ \Theta_{22} \end{bmatrix} = \begin{bmatrix} 1 \text{ inch} \\ -0.3338\text{rad} \end{bmatrix}$
Nodal point location $(x_{np})_n$	−6.01 inch	+3.00 inch

The vibration modes of this typical section are also displayed effectively in a graphical format, Figure 12.3.3, which is comparable to Figure 12.2.2 for the mass-spring system. Figure 12.3.3 indicates the airfoil in its reference position with a solid chord line, and in a modal displaced position with a dashed chord line. Let us suppose that mass center C is located 5 inches aft of leading edge LE along the 12-inch chord. In this case, the nodal points of both modes (listed in the table above) are located on the airfoil as shown on Fig. 12-6, not forward of LE or aft of trailing edge TE. For graphical clarity, the magnitudes of motions $y_1(t)$ and $\theta_2(t)$ are shown greatly exaggerated relative to the 12-inch chord. Actual motions of a realistic typical section would be tens or hundreds of times smaller.

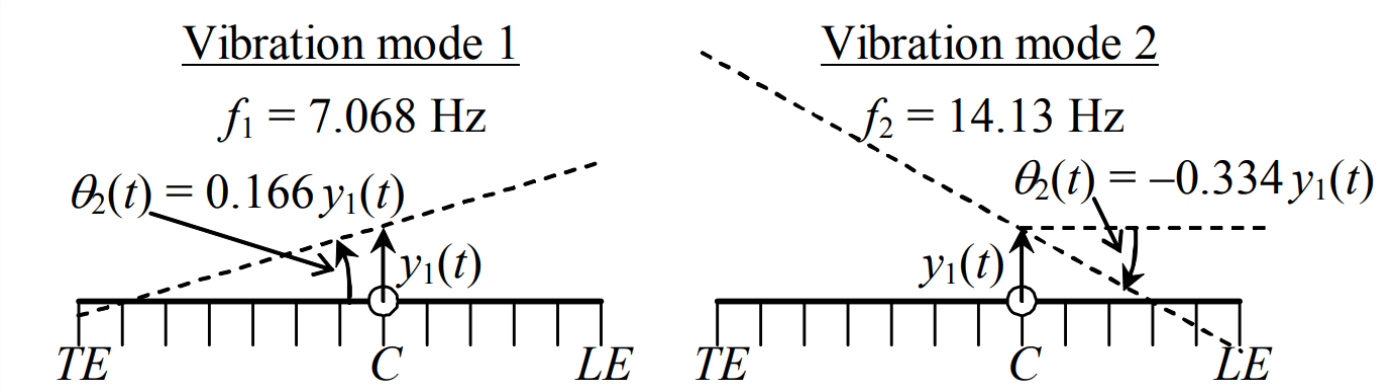


Figure 12.3.3: Vibration modes of the example typical section

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12.4: Chapter 12 Homework

1. Consider again the 2-DOF system of homework Problem 11.2, depicted below.

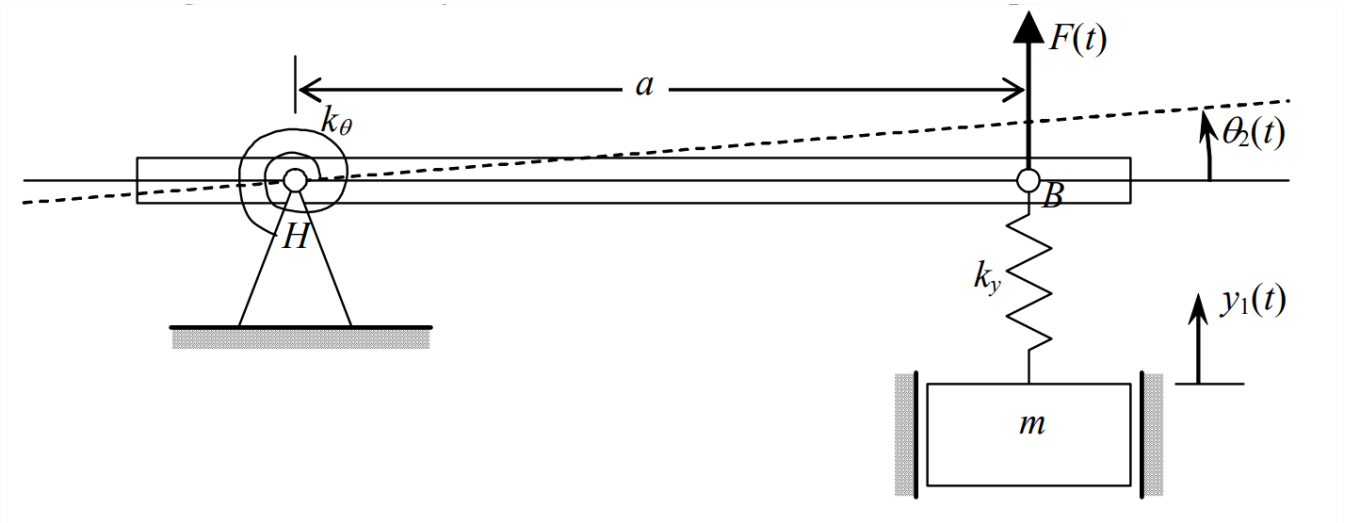


Figure 12.4.1

The two coupled differential equations of motion for this system, as derived for homework Problem 11.2 and now expressed in matrix form, are

$$\begin{bmatrix} m & 0 \\ 0 & J_H \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_y & -ak_y \\ -ak_y & k_\theta + a^2 k_y \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ aF(t) \end{bmatrix}$$

All the parameters in these equations are defined on the figure except J_H, which is the rotational inertia of the bar about frictionless hinge H. Consider the free-vibration problem, with $F(t) = 0$. First, show that the characteristic equation in general is

$$mJ_H \left(\omega^2\right)^2 + \left[-m\left(k_\theta + a^2 k_y\right) - J_H k_y\right] \omega^2 + k_y k_\theta = 0$$

Now, consider the following situation. Suppose that you are a design engineer working on a new satellite project for an aerospace company. You have designed a sensor package that includes a small mechanical device in the form of the subject 2-DOF system. The satellite has been fabricated, and is now being subjected to functional ground testing. The primary attitude-control actuator for the satellite is a *control-moment gyroscope* (CMG), the rotor of which spins at 11,400 rpm in normal operation. The spin-up time for this CMG is about four hours. During the first functional test, at the very end of the CMG spin-up, test engineers observe a short period of loud buzzing coming from the vicinity of your sensor package. The buzzing suddenly stops, and nobody thinks anything more about it until functional tests begin on your sensor. Unfortunately, the readings from your sensor are gibberish. The sensor package is removed and opened up, and examination shows that the 2-DOF device is completely destroyed: both springs are fractured, the bar is badly bent, the mass is detached, and the bar and mass have banged against near-by circuit boards within the package. It is now your responsibility to determine what went wrong. One experienced test engineer speculates that the 2-DOF device might have resonated due to base excitation caused by the CMG. Although the rotors of space-qualified CMGs are always carefully balanced, it is impossible to remove all imbalance. Consequently, rotor-spin imbalance imposes small periodic forces at the spin rate onto the CMG supports, and these forces produce vibration that is transmitted in waves throughout the spacecraft structure. With the assistance of calibration technicians, you determine that the numerical parameters of the 2-DOF device are: m = 19.2 g = 0.0192 kg, J_H = 37.3e−6 kg-m^2, k_y = 16.9 N/mm = 16.9 kN/m, k_θ = 12.8 N-m/rad, and a = 30.0 mm = 0.0300 m. Calculate the vibration modes of this system (natural frequencies, mode shapes, and sketches of the mode shapes), and discuss in a few sentences the probable cause of the structural failure within your satellite sensor package.

2. Consider again the 2-DOF pendulum-spring system of homework Problem 11.5, depicted below. The two coupled differential equations of motion for this system, as derived for homework Problem 11.5 and now expressed in matrix form, are

$$\begin{bmatrix} m_1\ell^2 & 0 \\ 0 & m_2\ell^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1 g\ell + ka^2 & -ka^2 \\ -ka^2 & m_2 g\ell + ka^2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ M_2(t) \end{bmatrix}$$

Consider the free-vibration problem, with $M_2(t) = 0$. It will simplify the writing if, initially, you will use the following notational abbreviations: $J_1 = m_1\ell^2$, $J_2 = m_2\ell^2$, $k_1 = m_1 g\ell + ka^2$, $k_2 = m_2 g\ell + ka^2$, and $k_c = ka^2$ (the "coupling" coefficient). First, show that the characteristic equation in general is

$$J_1 J_2 \left(\omega^2\right)^2 + \left(-J_2 k_1 - J_1 k_2\right)\omega^2 + k_1 k_2 - k_c^2 = 0$$

Now consider the special case with $m_1 = \frac{1}{2}m_2$ and $k_c = ka^2 = m_2 g\ell$. Express all the coefficients of the characteristic equation in terms of m_2, g, and ℓ, and then show that this quadratic equation can be put into the form $\left(\omega^2\right)^2 - 5(g/\ell)\omega^2 + 4(g/\ell)^2 = 0$. For this special case, solve for the natural frequencies and mode shapes, and sketch the mode shapes.

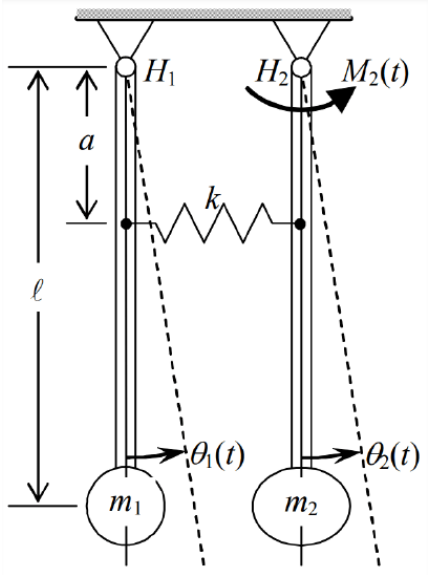


Figure 12.4.2

3. Consider again from homework Problem 11.6 the simplified model for pitching-translation dynamics of an automotive vehicle, depicted below. The rigid body frame has mass m and rotational inertia J_c about center of mass C.

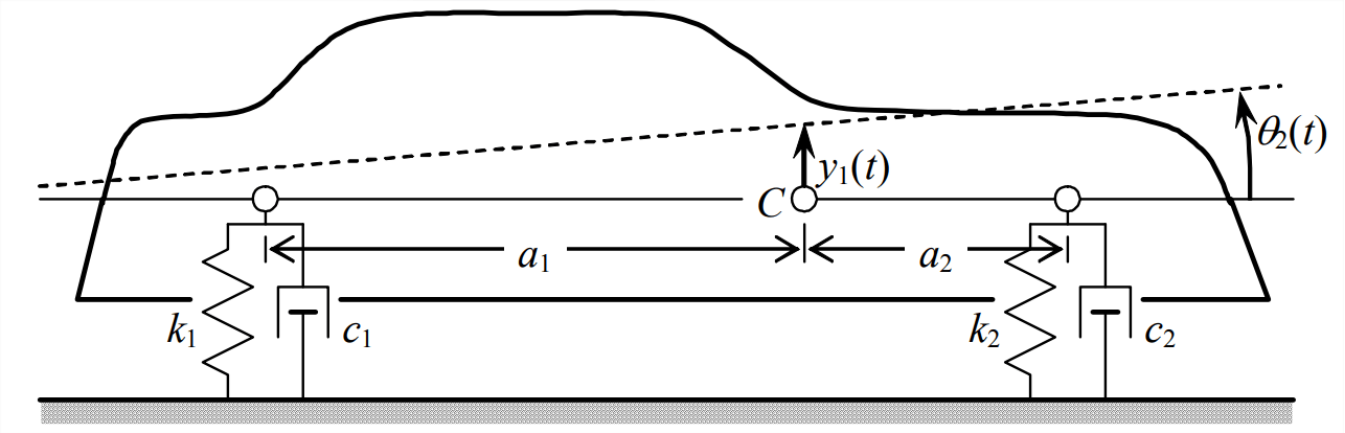


Figure 12.4.3

The two coupled differential equations of motion for this system, as derived for homework Problem 11.6 and now expressed in matrix form, are

$$\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & c_2 a_2 - c_1 a_1 \\ c_2 a_2 - c_1 a_1 & c_1 a_1^2 + c_2 a_2^2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 a_2 - k_1 a_1 \\ k_2 a_2 - k_1 a_1 & k_1 a_1^2 + k_2 a_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To obtain the vibration modes of the **undamped** system, we set $c_1 = c_2 = 0$, which deletes the entire second term of the matrix equation. The following data are representative for a compact four-door sedan: mg = 2,150 lb, $J_c g$ = 5.47e6 lb-inch^2, $k_1 = k_2 = 155$ lb/inch, $a_1 = 82.7$ inches, and $a_2 = 35.4$ inches. Calculate the natural frequencies, mode shapes, and nodal point

locations, and sketch the mode shapes.

4. Consider again Figure 12.2.1, but now suppose that the entire undamped mass-spring system consists of a primary sub-system, with mass m_1 and spring k_1, plus a smaller auxiliary sub-system, with mass m_2 and spring k_2; also, that a steady sinusoidal force $f_1(t) = F_1 \cos \omega t$ acts upon m_1, but no externally applied force acts upon m_2, $f_2(t) = 0$. Your assignment is to solve for and evaluate the steady-state **forced** sinusoidal responses $y_1(t) = Y_1(\omega) \cos \omega t$ and $y_2(t) = Y_2(\omega) \cos \omega t$. To solve, use the theory developed in Section 12.2, **not** the more general procedure described in Section 4.7 for finding frequency-response functions.

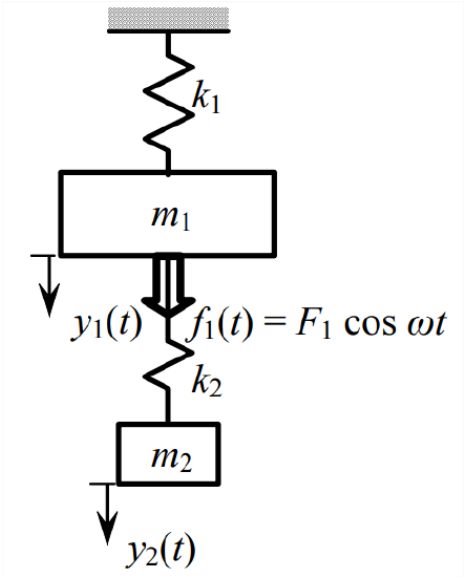


Figure 12.4.4

1. Determine algebraic equations for $Y_1(\omega)$ and $Y_2(\omega)$ in terms of m_1, k_1, m_2, k_2, F_1, and ω. Also, write the characteristic equation of the free-vibration problem, and explain what happens if its roots are substituted into the equations for $Y_1(\omega)$ and $Y_2(\omega)$.

   Partial answer: $Y_2(\omega) = \frac{k_2 F_1}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$

2. Suppose that this is a system for which *vibration isolation* is required: primary mass m_1 includes an unbalanced reciprocating motor or machine that oscillates with specific frequency $\omega = \Omega$, which tends to cause m_1 to vibrate vertically at frequency ω; but it is desirable to protect from that vibration the supporting structure to which primary spring k_1 is attached. (See homework Problem 10.13 for additional discussion of vibration isolation.) According to your solution in part 12.4.1, what happens to motion amplitudes $Y_1(\Omega)$ of m_1 and $Y_2(\Omega)$ of m_2 if you tune the single-DOF frequency of the auxiliary mass and spring, m_2 and k_2, to the excitation frequency, *i.e.*, if you design the auxiliary sub-system so that $k_2/m_2 \equiv \Omega^2$? With $k_2/m_2 \equiv \Omega^2$, what is the sinusoidal force $k_2 [y_2(t) - y_1(t)]$ that spring k_2 imposes on primary mass m_1 ? With the correct answers to these questions, you will have explained the functioning of an ideal *undamped dynamic vibration absorber*.
3. An ideal undamped dynamic vibration absorber is unattainable in reality because systems always have some damping. Moreover, even if we could magically eliminate all damping, the ideal absorber would still have a serious deficiency: if the excitation frequency were to differ from the nominal value, which usually happens in practice during motor start-up or off-nominal operation, then the system could experience a 2-DOF resonance failure. Suppose, for example, we have the following numerical parameters: m_1 = 25.3 kg, k_1 = 2.62 MN/m, $\Omega/2/\pi$ = 50 Hz, and the auxiliary sub-system is tuned to 50 Hz with m_2 = 5.066 kg and k_2 = 0.5 MN/m. First, solve the characteristic equation to calculate the two system natural frequencies in Hz, $f_1 = \omega_1/2/\pi < f_2 = \omega_2 2 = \prime\prime\pi$;="" if="" you="" wish,="" may="" use="" the="" roots="" matlab="" function="" for="" this="" task.="" next,="" calculate="" and="" plot="" versus="" excitation="" frequency="" (in="" hz)="" frequency-response="" dynamic="" flexibility,="" $y_1(\omega) = \prime\prime f_1$,="" of="" primary="" sub-system.="" range="" frequencies="" over="" which="" response="" must="" include="" at="" least="" all="" three="" frequencies,="" f_1,="" ω / = ″ ″2 / = ″ ″π,="" f_2;="" it="" probably="" should="" even="" extend="" somewhat="" below="" lowest="" these="" above="" highest.="" be="" sure="" to="" appropriate="" axis="" labels="" on="" your="" graph.="" graph="" display="" clearly="" both="" strengths="" weaknesses="" ideal="" undamped="" vibration="" absorber="" particular="" numerical="" parameters.<sup="">1 </f_{2}=\omega_{2}>
5. The drawing at right represents a rotational mechanical system that is directly analogous to the translational mechanical

system of Figure 12.2.1. Rotational inertias J_1 and J_2 are analogous, respectively, to masses m_1 and m_2 on Figure 12.2.1; rotational springs K_1 and K_2 are analogous to translational springs k_1 and k_2; applied moments $M_1(t)$ and $M_2(t)$ are analogous to applied forces $f_1(t)$ and $f_2(t)$; and rotations $\theta_1(t)$ and $\theta_2(t)$ are analogous to translations $y_1(t)$ and $y_2(t)$. Therefore, the matrix equation of motion for this rotational system is directly analogous to Equation 12.2.3 for the translational system:

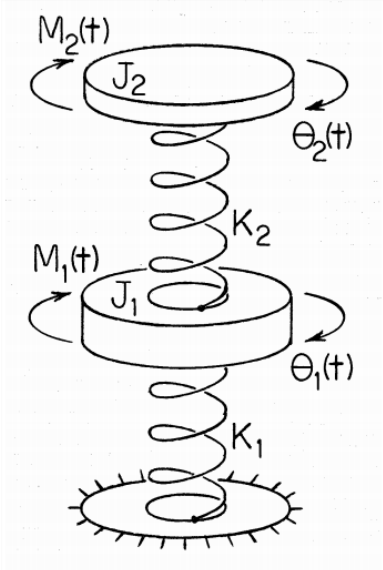


Figure 12.4.5

$$\overbrace{\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}}^{\text{inertia matrix}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \overbrace{\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}}^{\text{structural stiffness matrix}} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}$$

Consider a 2-DOF inertia-spring system with rotational inertias $J_1 = J_2 = 0.01042$ kg-m^2 and rotational stiffness constants $K_1 = K_2 = 2.77$ N-m/rad. Your assignment is to calculate and plot, over the time interval $0 \le t \le 5$ seconds, the **unforced** initial-condition (IC) responses, rotations $\theta_1(t)$ and $\theta_2(t)$, of this system if the initial rotations are $\theta_1(0) = \theta_2(0) = 14.58$ degrees, and the initial rotational velocities are zero. Use the theory (and even the appropriate calculations, if you wish) of Section 12.2. The given inertias and stiffness constants are those of a real structural system, which is described on the next two pages, and the given initial conditions were imposed on that system, producing the measured responses shown on the graph following this paragraph. Your theoretical responses should appear somewhat similar to the measured responses, but the theoretical and measured responses should differ in one significant respect. Read the description below of the actual structural system, then identify the property of all real mechanical dynamic systems that is **not** modeled by the theory of Chapter 12, and explain how this modeling deficiency is illustrated in the differences between the theoretical and measured responses shown on the graphs.

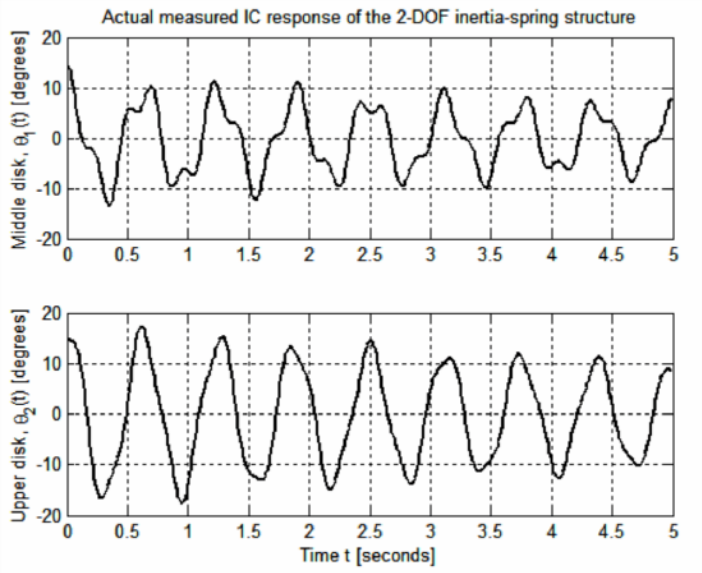


Figure 12.4.6

The photograph on the next page shows the relevant structural portion of the actual laboratory system[2] that is represented by the drawing on the previous page, and from which the IC-response graph above was measured. Three steel disks are attached to a continuous steel shaft of diameter about 3.2 mm. Each disk is 20 cm in diameter and 0.5 cm thick, with four radial slots to accommodate added masses. The total length of shaft between lower and upper disks is about 68 cm. The shaft is supported near the middle and upper disks by bearings, which impose some damping on the structure. The lower disk is clamped to the frame, so the shaft is fixed at the bottom but is free to twist everywhere else along its length. Two metal cylinders, each of nominal mass 0.5 kg, are bolted symmetrically at 9-cm radii to each of the middle and upper disks. The rotational inertias of the two bulked-up disks are orders of magnitude greater than that of the shaft, so this system is effectively a 2-DOF rotational structure, those DOFs being rotations $\theta_1(t)$ and $\theta_2(t)$ of the middle and upper disks, respectively. $\theta_1(t)$ and $\theta_2(t)$ are sensed by the optical encoders shown on the photograph; each encoder shaft is connected to the structural shaft by a lightweight belt and pulleys, which add to the system some damping but negligible inertia.

A clever student **determined experimentally** the inertias and stiffness constants of this system, rather than relying on purely theoretical calculations, mostly by using the added-inertia method (from homework Problem 7.9). To determine J_2 and K_2, the student clamped the middle disk to the frame, and then measured free-vibration frequencies of the upper disk [$\theta_2(t)$ with $\theta_1(t) = 0$], both bare and with symmetrically added combinations of the metal cylinders. The student determined J_1 and $K_1 + K_2$ by clamping the upper disk, and then measuring free-vibration frequencies of the middle disk [$\theta_1(t)$ with $\theta_2(t) = 0$]. The values of the parameters determined by this process are $J_1 = J_2 = 0.01042$ (± 0.00003) kg-m^2 and $K_1 = K_2 = 2.77$ (± 0.04) N-m /rad.

The initial rotations, $\theta_1(0) = \theta_2(0) = 14.58°$, were imposed on the middle and upper disks by twang, also known as snapback, excitation: a fine thread was tied around the front cylinder of the middle disk, the thread was pulled laterally and the opposite end was anchored, thus imposing onto the middle disk the static initial rotation, then the thread was cut with scissors, abruptly releasing the middle and upper disks to free vibration. The structural shaft was fixed only at the bottom, so the initial rotation of the upper disk was the same as that of the middle disk.



Figure 12.4.7

6. A structure's fundamental mode of vibration is represented in the drawing below by mass m_1 and spring k_1, with the translation of m_1 denoted as $x_1(t)$. Suppose we wish to exercise active control over that vibration by use of an attachment to the structure: a reaction-mass actuator (RMA), represented in the diagram by reaction mass m_2, spring k_2, and internally generated force $f_i(t)$ acting equally and oppositely on m_2 and m_1, with the translation of m_2 denoted as $x_2(t)$. (To simplify the equations here, we neglect the normally small natural passive damping inherent in both structures and RMAs.) Use Equation 12.2.3 to write the matrix equation of motion for this combination of structure and RMA. Next, revise this equation to include the particular type of **feedback** vibration control described in the following. Suppose that reaction mass m_2 carries a transducer that senses the translational velocity of m_2 **relative** to that of m_1 and generates an electrical voltage signal proportional to that relative velocity, $e_v(t) = V(\dot{x}_2 - \dot{x}_1)$, where V is a calibration constant with units such as volt per meter/s. Homework Problem 10.15 explains that an external voltage signal $e_i(t)$ [not shown on this drawing] commands internal force $f_i(t)$ according to the linear relationship $f_i(t) = Ge_i(t)$, G being a constant with units such as newton per volt. Suppose that we multiply the relative velocity signal $e_v(t)$ by a variable gain factor F (using, *e.g.*, a non-inverting amplifier, homework Problem 5.8) and then **feed** the multiplied signal **back** to serve as the force-command signal, $e_i(t) = Fe_v(t) = FV(\dot{x}_2 - \dot{x}_1)$. Therefore, the generated internal force is $f_i(t) = Ge_i(t) = GFV(\dot{x}_2 - \dot{x}_1) \equiv C(\dot{x}_2 - \dot{x}_1)$, where we define the overall feedback constant $C \equiv GFV$ with units such as newton per meter/second. Revise your matrix equation of motion to reflect the feedback vibration control described here; in particular, show that the feedback produces an *artificial viscous damping matrix*, $C\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. With sufficiently large feedback gain factor F and sufficient authority of the RMA's internal force generator, this form of control can impose significant positive damping upon a structure's vibration.

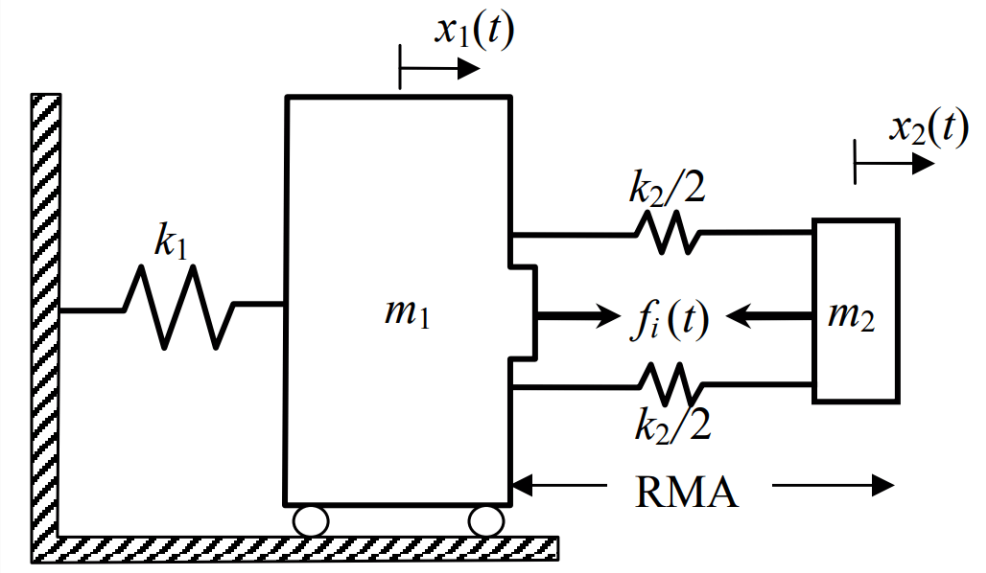


Figure 12.4.8

NOTE: It is not in required in this problem, but it might be of interest to you that replacing dependent variable $x_2(t)$ with the **relative** translation of reaction mass m_2, $z(t) = x_2(t) - x_1(t)$ can transform the matrix equation of motion into the following form, in which the inertia matrix becomes full and symmetric, but the artificial damping and stiffness matrices become simpler and diagonal:

$$\begin{bmatrix} m_1+m_2 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{z} \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[1]Homework Problem 12.4 is only an introduction to the subject of dynamic vibration absorbers. See Den Hartog, 1956, Sections 3.2 and 3.3, for a more advanced and extensive treatment of this subject.

[2]This is a Model 205a Torsional Plant designed and fabricated by Educational Control Products of Bell Canyon, California, USA.

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13: LAPLACE BLOCK DIAGRAMS AND FEEDBACK-CONTROL SYSTEMS BACKGROUND

We will consider in Chapter 13 a graphical format, the Laplace block diagram, which is useful in the analysis of complicated systems that consist of more than one subsystem. While developing closed-loop block diagrams, we will also analyze the equation of motion and the time response of a base-excited mass-damper-spring system, which turns out to be an instructive prototype for output feedback control. All of this is preparation for the more general study of feedback-control systems beginning in Chapter 14.

13.1: Laplace Block Diagrams for an RC Band-Pass Filter

13.2: Laplace Block Diagram with Feedback Branches

13.3: Forced Response of an m-c-k System with Base Excitation

13.4: Chapter 13 Homework

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13.1: Laplace Block Diagrams for an RC Band-Pass Filter

To introduce the Laplace block diagram for a simple case, we re-visit again the RC band-pass filter considered previously in Sections 5.4, 9.10, and 10.4. As shown in Fig. 5.4.1, this circuit consists of a low-pass-filter stage to the left of the voltage follower, and a high-pass-filter stage to the right. The Laplace transforms derived for these stages in Section 10.4, assuming zero ICs, are: for the low-pass stage, with $\tau_L = R_L C_L$,

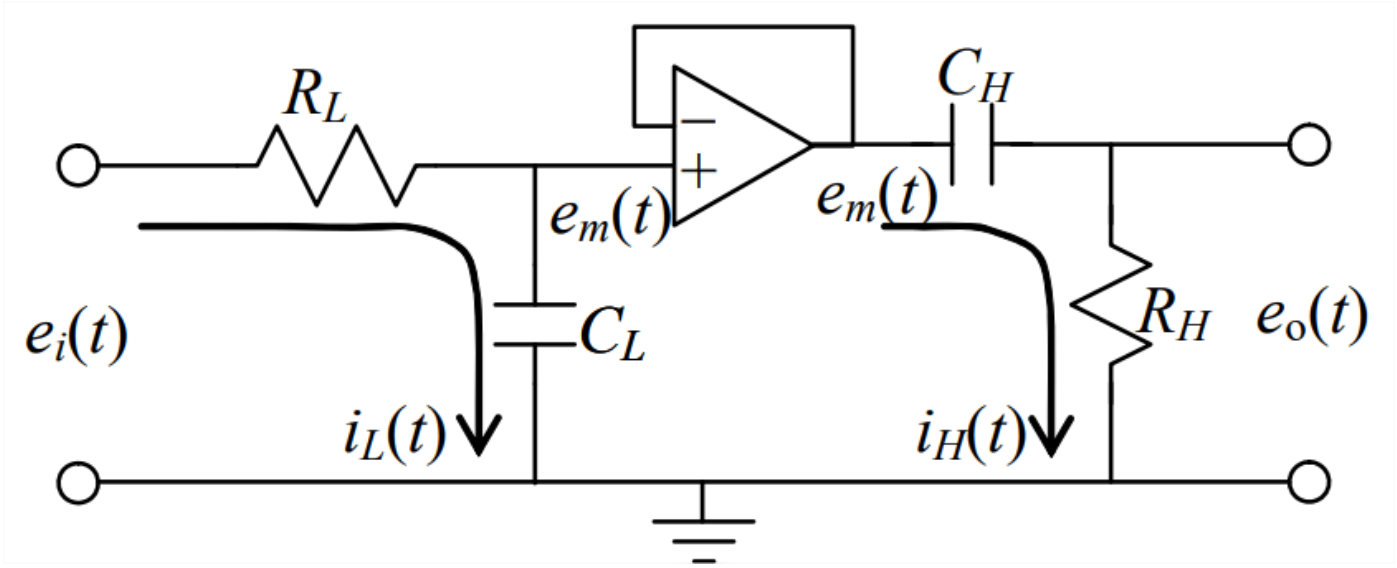


Figure 13.1.1: (repeated) RC band-pass filter. (Copyright; author via source)

$$L[e_m] = \frac{1}{\tau_L s + 1} L[e_i] \tag{13.1.1}$$

and for the high-pass stage, with $\tau_H = R_H C_H$,

$$L[e_o] = \frac{\tau_H S}{\tau_H s + 1} L[e_m] \tag{13.1.2}$$

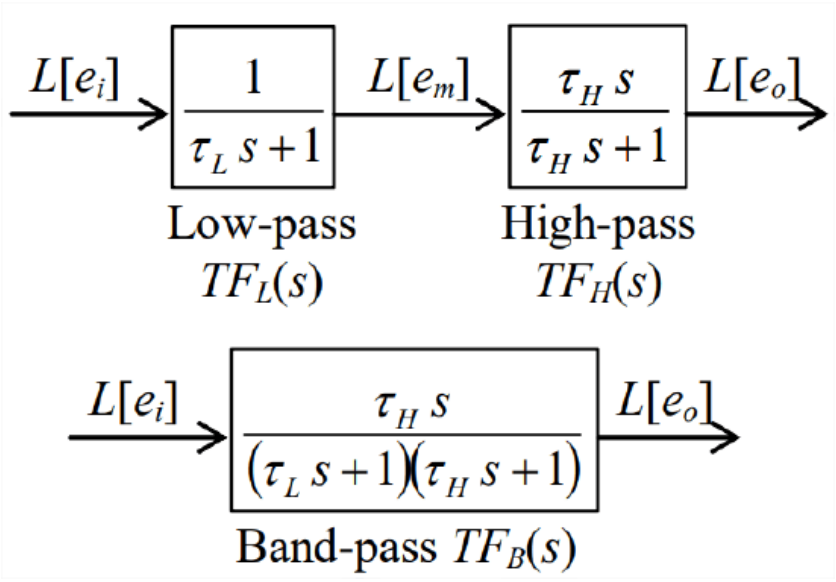


Figure 13.1.2: Laplace block diagrams of the RC band-pass filter. (Copyright; author via source)

Figure 13.1.2 shows a convenient graphical format, the *block diagram*, for representing these Laplace-transform output-to-input relationships. From Equations 13.1.1 and 13.1.2, we have the individual transfer functions of the low-pass and high-pass stages:

$$TF_L(s) = \frac{L\left[e_m\right]}{L\left[e_i\right]} = \frac{1}{\tau_L s + 1} \tag{13.1.3}$$

$$TF_H(s) = \frac{L\left[e_o\right]}{L\left[e_m\right]} = \frac{\tau_H s}{\tau_H s + 1} \tag{13.1.4}$$

In the upper row of Figure 13.1.2, transfer functions Equations 13.1.3 and 13.1.4 are shown as individual blocks, and the Laplace transforms are shown as input and output "signals" relative to the blocks. The most basic rule of "block-diagram algebra" is that the input signal (transform) multiplied by the block transfer function equals the output signal (transform), which gives Equations 13.1.1 and 13.1.2.

One common objective in construction of Laplace block diagrams is to simplify the process of deriving the transfer function of the overall system from the transfer functions of individual sub-systems. To help achieve this objective, we have the second basic rule of block-diagram algebra: multiply the transfer functions of two adjacent blocks in order to obtain the more general transfer function relating the output signal from the downstream block to the input signal into the upstream block. The result of this operation in the present case is shown in the lower row of Figure 13.1.2, where transfer function Equation 10.4.4 of the entire RC band-pass filter circuit is derived once again:

$$TF_B(s) = \frac{L\left[e_o\right]}{L\left[e_i\right]} = \frac{\tau_H s}{\left(\tau_H s + 1\right)\left(\tau_L s + 1\right)} \tag{13.1.5}$$

Each of the block diagrams in Figure 13.1.2 has a single *forward* (left-to-right) *path* or *branch*. A more general type of block diagram has a forward branch and also one or more *backward paths* denoting signal flow in the opposite direction; such a right-to-left path is called a *feedback branch*. A forward branch combined with a feedback branch forms a *closed loop*, and a block diagram that includes closed loops is called a closed-loop block diagram. Accordingly, a block diagram with only a forward branch is called an *open-loop* block diagram. The next section develops the concept of closed-loop block diagrams in the context of a familiar mechanical system.

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13.2: Laplace Block Diagram with Feedback Branches

Let us proceed beyond the most basic aspects of Laplace block diagrams by analyzing the m-c-k system with base excitation shown on Figure 13.2.1. The input is base translation $x_i(t)$, and the output is mass translation $x(t)$. The ODE of motion for this system (from homework Problem 9.1) is

$$m\ddot{x} + c\dot{x} + kx = kx_i(t) \tag{13.2.1}$$

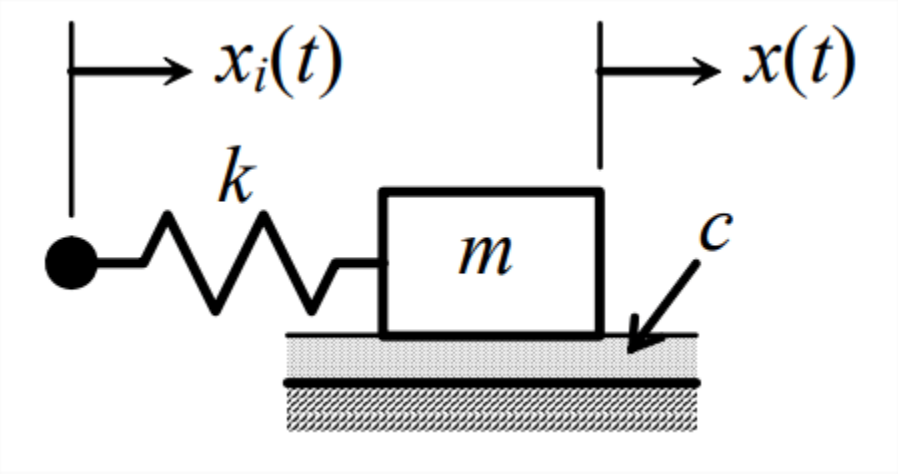


Figure 13.2.1: m-c-k system with base excitation. (Copyright; author via source)

It is convenient to denote the input transform as $L\left[x_i(t)\right] = X_i(s)$ and the output transform as $L[x(t)] = X(s)$. Taking the Laplace transform of Equation 13.2.1 with zero ICs, for the purpose of deriving the transfer function, gives

$$ms^2X(s) + csX(s) + kX(s) = kX_i(s) \tag{13.2.2}$$

Equation 13.2.2 clearly leads to the transfer function,

$$TF(s) \equiv \frac{X(s)}{X_i(s)} = \frac{k}{ms^2 + cs + k} \tag{13.2.3}$$

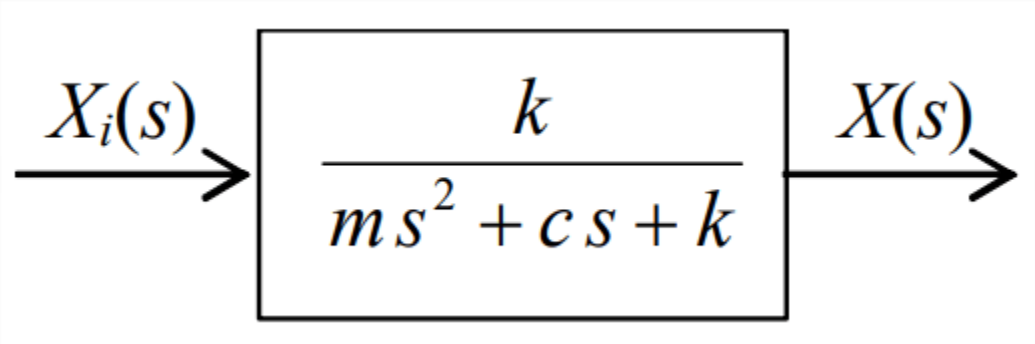


Figure 13.2.2 (Copyright; author via source)

Equation 13.2.3 is represented in the simple open-loop Laplace block diagram just above.

It will be instructive for our later study of feedback control to develop now an alternative but equivalent block diagram from Equation 13.2.2. First, we transpose to the righthand side all terms other than the $X(s)$ term of highest order:

$$ms^2X(s) = kX_i(s) - csX(s) - kX(s) \tag{13.2.4}$$

Now we "solve" algebraically for $X(s)$ on the left-hand side of Equation 13.2.4 simply by dividing through by the coefficient of the

left-hand-side $X(s)$:

$$X(s) = \frac{1}{s}\left\{\frac{1}{ms}\left[kX_i(s) - kX(s) - csX(s)\right]\right\} \tag{13.2.5}$$

Observe in Equation 13.2.5 that output transform $X(s)$ appears in the right-hand-side terms as well as on the left-hand side; these right-hand-side terms imply *feedback*. Using Equation 13.2.5 leads to the following block diagram with several different transfer-function blocks and with two feedback branches:

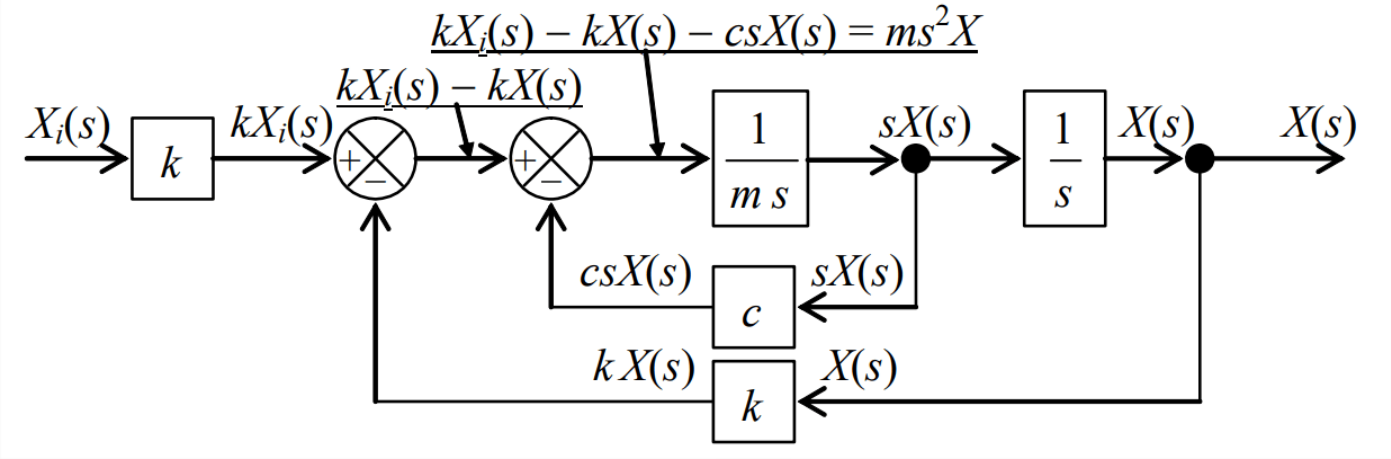


Figure 13.2.3: Laplace block diagram with feedback branches for a base-excited m-c-k system (Copyright; author via source)

The input and output signals for every block are labeled on Figure 13.2.3 in order to help you relate the block diagram to Equation 13.2.5, and you should carefully compare the equation and the diagram until you are certain that you understand.

Also, block diagram Figure 13.2.3 introduces two kinds of *junctions*. The first kind is a simple *branch point* (black circle), from which the same blockoutput "signal" travels onward on two different branches, as is indicated by the signal labels on Figure 13.2.3. The second kind is a *summing junction*, shown at right with input and output signals labeled. This summing junction is configured for negative feedback, so it differences the two input signals, $\mathrm{Out}(s) = \mathrm{In}_1(s) - \mathrm{In}_2(s)$.

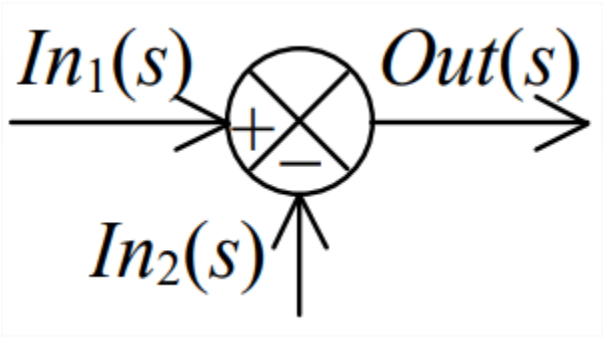


Figure 13.2.4 (Copyright; author via source)

In this section, we started with ODE Equation 13.2.1 and developed from it Laplace block diagram Figure 13.2.3. This approach is primarily for the purpose of presenting some important features of block diagrams. This approach is actually the reverse of the standard procedure for analysis of feedback-control systems, which we will consider in later chapters. There, we will **start** with the block diagram of a system, something like Figure 13.2.3, and **derive from it** the *closed-loop transfer function*, CLTF(s)), which in this case is just Equation 13.2.3. The process of deriving $CLTF(s)$ from the system block diagram will require some new operations in block-diagram algebra; it is expedient to postpone the descriptions of these operations until we need them in later chapters.

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13.3: Forced Response of an m-c-k System with Base Excitation

It will be helpful for our introduction to feedback control in Chapter 14 if we now examine the relationship between input motion $x_i(t)$ and output motion $x(t)$ for the mechanical system of Figure 13.2.1 and Equation 13.2.1. Let us adopt the perspective here that we want to use $x_i(t)$ in order to **control** $x(t)$; in particular for this system, let us suppose that we want to make motion $x(t)$ of mass m be as close as possible to input motion $x_i(t)$. You can visualize intuitively from Figure 13.2.1 that if $x_i(t)$ is imposed very slowly, then $x(t)$ will be essentially the same as $x_i(t)$; in this case, the inertial force of mass m will be negligible, and spring k will deform only slightly, therefore acting essentially as a rigid link. (You can, of course, prove theoretically this qualitative analysis, for example, by finding frequency response at a very low input frequency.)

But can output motion $x(t)$ be made close to input $x_i(t)$ if $x_i(t)$ is **not** imposed slowly? Your intuition probably fails to answer this question because, in this case, spring k might deform significantly, and the inertial force of mass m might not be negligible. We will have to analyze this situation theoretically. Let us do so specifying that input motion $x_i(t)$ is a step function of magnitude X_H imposed at time t = 0. An ideal step input is not merely fast; it is in fact the **fastest** imaginable (though not physically realizable) base excitation, because base position changes with infinite velocity at t = 0 from $x_i = 0$ to $x_i = X_H$. So this ideal input poses a formidable challenge to the control objective: can output translation $x(t)$ approximate in some reasonable manner the input translation $x_i(t)$, even though initially the input translation is infinitely fast?

In order to analyze this case, let us first adapt ODE Equation 13.2.1 to the standard ODE for damped 2nd order systems as it is defined in Chapter 9. Dividing Equation 13.2.1 through by m, and then applying the standard definitions Equation 7.1.3 for undamped natural frequency, $\omega_n = \sqrt{k/m}$, and Equation 9.1.6 for viscous damping ratio, $\zeta = c/(2\sqrt{mk})$, casts the ODE into the form $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 x_i(t)$. Comparing this ODE with Equation 9.2.2 shows that input $x_i(t)$ for this particular base-excited m-c-k system is exactly equal to the standard input quantity: $x_i(t) \equiv u(t)$. We define step input $x_i(t) = X_H H(t)$. Therefore, $X_H \equiv U$, which is the general step-input magnitude of Section 9.6, and we can write the step-response solution for $|\zeta| < 1$ directly from Equation 9.6.5:

$$x(t) = X_H \left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right], \text{ for } 0 \le t \tag{13.3.1}$$

In Equation 13.3.1, the damped natural frequency is $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$. Also, for use in Figure 13.3.1, the period of the undamped system is $T_n = 2\pi/\omega_n$. Recall that our control objective is to make the output motion at least approximately equal to the input motion. At $t = 0$, input motion $x_i(t)$ jumps discontinuously from 0 to X_H, then x_i remains constant thereafter, so we would like the output motion $x(t)$ to approximate that step input as closely as possible. In particular, we want the final, steady-state value of $x(t)$ to be X_H.

It is instructive to calculate and graph Equation 13.3.1 for a few different values of ζ, assuming that the system is underdamped, $0 \le \zeta < 1$. The step-response graphs are shown on Figure 13.3.1. If the system is undamped, $c = 0$, then the mass oscillates forever (in theory) **about** the desired final position, $x = X_H$; in this ideal (with zero damping) case, the control objective cannot be achieved.

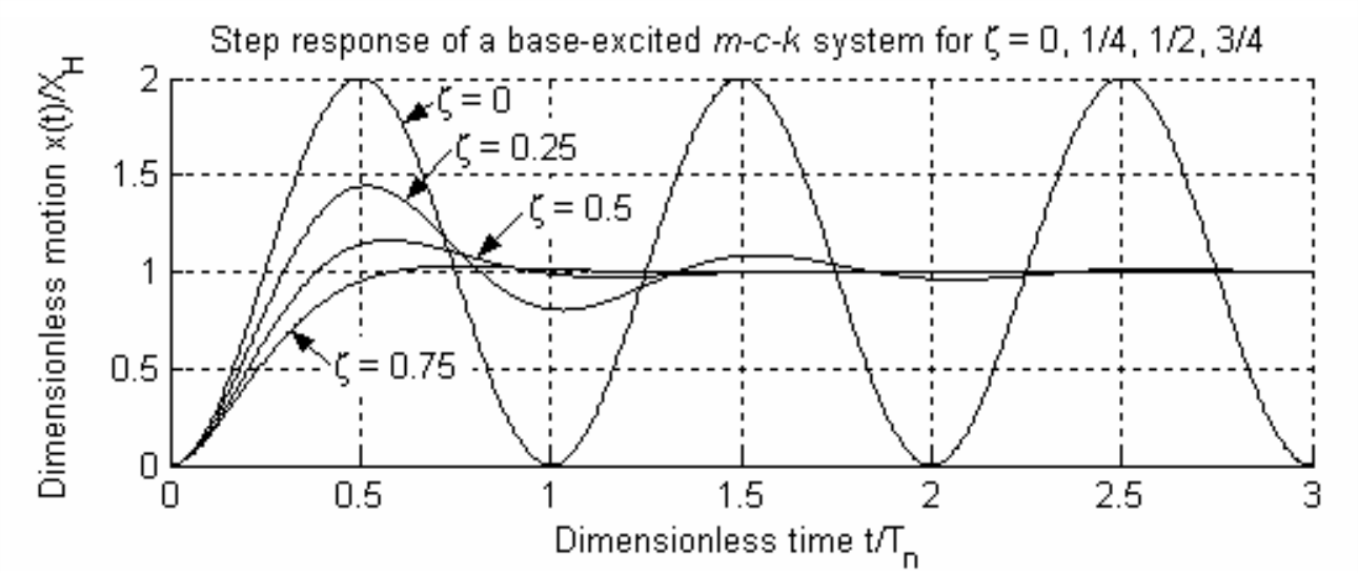


Figure 13.3.1: Base-excitation control of mass position for an m-c-k system. (Copyright; author via source)

However, Figure 13.3.1 also demonstrates that, if there is positive damping, $\zeta > 0$, then the motion output $x(t)$ does, indeed, **approximate** the step input, $x_i(t) = X_H H(t)$: $x(t)$ rises from 0, though with finite velocity, and it eventually settles at the desired final value, $x = X_H$.

Note that the equations for step-response specifications (rise time, peak time, maximum overshoot ratio, and settling time) derived in Section 9.8 are applicable to this case of excitation of an m-c-k system by base motion.

The base-excited mass-damper-spring system of Figure 13.2.1 is an instructive prototype for output feedback control of mechanical position, which is the subject of Chapter 14. Therefore, we shall refer back to this system and to the results of Sections 13.2 and 13.3 in order to clarify and justify some of the control-system developments presented in Chapter 14.

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13.4: Chapter 13 Homework

1. Consider the standard ODE Equation 3.4.8 for a stable 1st order system, $\dot{x} + (1/\tau_1)\, x = bu(t)$, in which $u(t)$ is the input quantity, $x(t)$ is the output quantity, τ_1 is the positive time constant, and b is a constant. Denote the Laplace transforms as $L[u(t)] = U(s)$ and $L[x(t)] = X(s)$. Derive and draw for this system a Laplace block diagram with feedback, a block diagram of the same type as Figure 13.2.3. The input signal on the left side of your block diagram is $U(s)$, and the output signal on the right side is $X(s)$.
2. With input and output voltages $e_i(t)$ and $e_o(t)$, respectively, the governing ODE for the electrical LRC series circuit is $L\ddot{e}_o + R\dot{e}_o + (1/C)e_o = (1/C)e_i(t)$, as derived in Section 9.2. Denote the Laplace transforms as $L\,[e_i(t)] = E_i(s)$ and $L\,[e_o(t)] = E_o(s)$, then derive and draw for this system a Laplace block diagram with feedback, a block diagram of the same type as Figure 13.2.3. The input signal on the left side of your block diagram is $E_i(s)$, and the output signal on the right side is $E_o(s)$.
3. A certain 3rd order system has excitation $u(t)$ and response $x(t)$, and its governing ODE is $a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = bu(t)$, in which all a_i and b are constants. Denote the Laplace transforms as $L[u(t)] = U(s)$ and $L[x(t)] = X(s)$. Derive and draw for this system a Laplace block diagram with feedback, a block diagram of the same type as Figure 13.2.3, but somewhat more complicated. The input signal on the left side of your block diagram is $U(s)$, and the output signal on the right side is $X(s)$. Your block diagram should have three feedback branches.
4. Consider the higher-order (4th order in this case), 2-DOF mechanical system of homework Problems 11.2 and 12.1, for which the coupled matrix equation of motion is

$$\begin{bmatrix} m & 0 \\ 0 & J_H \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_y & -ak_y \\ -ak_y & k_\theta + a^2 k_y \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ aF(t) \end{bmatrix}$$

    1. From the matrix equation, write the two separate scalar ODEs that are coupled by the term $-ak_y\theta_2$ in the "$y_1(t)$ ODE" (*i.e.*, the ODE that includes acceleration $\ddot{y}_1$), and the term $-ak_y y_1$ in the "$\theta_2(t)$ ODE." This system has the single input, force $F(t)$, but **two** output quantities, translation $y_1(t)$ of the mass, and rotation $\theta_2(t)$ of the rigid bar.
    2. Denote the Laplace transforms as $L[F(t)]$, $L\,[y_1(t)] = Y(s)$, and $L\,[\theta_2(t)] = \Theta(s)$. Take the Laplace transforms of both ODEs, with zero ICs. Now, derive and draw for this system a Laplace block diagram (in the form of Figure 13.2.3, but with differences) that has feedback **and also** "feed-across" between the forward-flowing branch that develops from the $y_1(t)$ ODE and the forward-flowing branch that develops from the $\theta_2(t)$ ODE. The input signal on the left side of your block diagram is $L[F(t)]$, and the **two** output signals on the right side are $Y(s)$ and $\Theta(s)$, one from each of the forward-flowing branches.

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14: INTRODUCTION TO FEEDBACK CONTROL

14.1: Initial Definitions and Terminology
14.2: Definitions and Examples of Open-Loop Control Systems
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14.1: Initial Definitions and Terminology

As an introduction to feedback control and related matters, this chapter considers a common task for many engineering systems: *remote* (*indirect*) *control* of the rotational position of some solid body; let us refer to this solid body generically as the *rotor*. The input to a rotation-control system is typically a *command* issued by a human operator using a device such as a steering wheel, control column, dial, computer keyboard, *etc.*; the input can also be a computer-generated signal that carries previously programmed instructions. The output is the rotational position of the rotor. The modifiers "remote" and "indirect" of "control" mean that there is no direct, rigid, mechanical linkage between the input device and the rotor; instead, for a remote control system, there might exist between the input device and the rotor such engineering equipment as sensors, electrical circuits, signal processors (analog and/or digital), flexible links or cables, electromagnetic or acoustic transmitters and receivers, and actuators of many types (motors, thrusters, *etc.*).

The following are a few of the many "rotors" in modern engineering practice with which you are probably familiar: aerodynamic and hydrodynamic control surfaces, such as rudders, elevators, and ailerons; the steering mechanisms that guide land vehicles and taxiing airplanes; nozzles of rocket engines or vectored-thrust jet engines; any entire vehicle itself, the three-dimensional rotational orientation (roll, pitch, yaw) of which is generally called the vehicle's *attitude*. You can probably add other rotors to this list from your own experience.

In general, the mass center of a solid body can have three degrees of rotational freedom. However, in order to keep the rotational dynamics simple here and to focus on feedback control, we will restrict our attention to rigid rotors that rotate in a plane about a fixed point, so that there is only one degree of rotational freedom.

Consider the simple generic rotor shown in Figure 14.1.1. Let us specify that it is restricted to rotating about the shaft axis, that its rotation angle relative to some reference position is $\theta(t)$, and that its angular velocity is $p(t) \equiv \dot{\theta}$. The rotational inertia of the rotor about the axis of rotation is denoted as J, and the total moment applied onto the rotor by all sources is denoted as $M(t)$. This is the physical object that we wish to control; in the terminology of control, the object or process to be controlled is often called the *plant*.

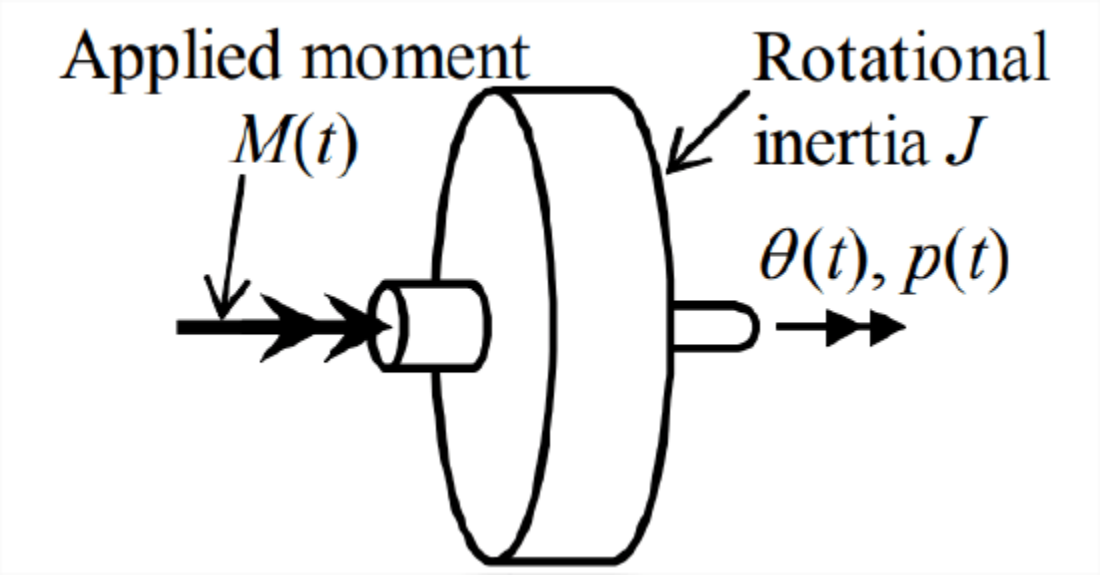


Figure 14.1.1: Generic rotor, the *plant*. (Copyright; author via source)

Provided that there is no damping moment proportional to $p(t)$ imposed upon the rotor, the equation of motion for the plant of Figure 14.1.1 is a simplified version of Equation 3.3.2:

$$J\dot{p} = J\ddot{\theta} = M(t) \tag{14.1.1}$$

Let us derive the transfer function of the plant, $PTF(s)$, by taking the Laplace transform of Equation 14.1.1 with zero initial conditions:

$$Js^2 L[\theta(t)] = L[M(t)] \Rightarrow \mathrm{PTF}(s) \equiv \frac{L[\theta(t)]}{L[M(t)]} = \frac{1}{Js^2} \tag{14.1.2}$$

In our previous work, we usually assumed that $M(t)$ is known, and we then solved for the resulting $p(t)$ and/or $\theta(t)$. Now, our objective is different: **for control, we want to *command* a desired rotor-position time history**. The command usually is something like a steering-wheel position or dial setting, or a signal carrying computer-generated instructions. In order to distinguish the input command from the output rotor angle, we introduce a different symbol for the command:

$$r(t) \equiv \text{ operator setting intended to produce the desired } \theta(t) \tag{14.1.3}$$

Operator setting $r(t)$ is the *reference* input quantity, which is transmitted to the system by electrical circuitry, mechanical linkages or cables, electromagnetic or acoustic waves, *etc*.

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14.2: Definitions and Examples of Open-Loop Control Systems

As an initial attempt to produce the desired motion history $\theta(t)$, it is probably most natural simply to seek the input setting $r(t)$ that will do the job completely, without the need for any further design refinement. This type of control is called open-loop control, for reasons that will be given later. Figure $14.2.1$ depicts an open-loop control system.

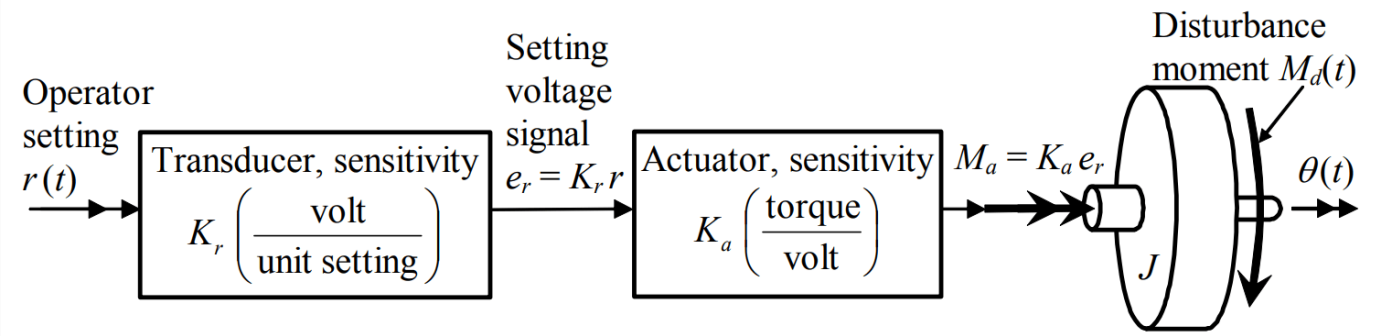


Figure $14.2.1$: Functional diagram of open-loop rotor-position control system. (Copyright; author via source)

Operator setting $r(t)$ is shown in Figure $14.2.1$ as a general quantity, and the input transducer is assumed to be a general linear device that produces voltage signal $e_r(t) = K_r r(t)$; if the input device were a rotary dial, for example, the operator setting might be a rotational position in degrees, and the transducer might be a variable resistor in a voltage divider circuit (see homework Problems 5.1 and 5.9), with accurately calibrated sensitivity of K_r volts per degree. The actuator also is assumed to be a general linear device that produces control moment $M_a(t) = K_a e_r(t) = K_a K_r r(t)$; if it were an electromagnetic actuator, for example, it would probably consist of a power amplifier and a motor, both of which are designed to function linearly. In practice, device sensitivity constants do not have to be positive and often are negative, but let's assume here that all are positive ($K_a > 0$, $K_r > 0$, *etc.*) just to avoid unnecessary complications.

Also shown on Figure $14.2.1$ is *disturbance* moment $M_d(t)$ acting upon the rotor. In the terminology of control, a disturbance is some extraneous action upon the plant, not the intended control action. In the system of Figure $14.2.1$, for example, an actuator moment produced by unwanted electrical noise in the circuitry would be an *internal* disturbance, and a moment produced by wind or impact from solid objects onto the rotor would be an *external* disturbance. Disturbances are often random, or at least unpredictable, so that we usually cannot write equations to describe their variation with time. In most cases, disturbances tend to influence adversely the performance of a control system.

Let us analyze the control system of Figure $14.2.1$. The total moment acting upon the rotor is $M_a(t) + M_d(t)$, so that Equation 14.1.1 becomes

$$J\ddot{\theta} = M_a(t) + M_d(t) = K_a K_r r(t) + M_d(t) \tag{14.2.1}$$

Disturbance $M_d(t)$ in Equation $14.2.1$ is the nemesis of open-loop control, so let us consider initially the case with zero disturbance, $M_d(t) = 0$. If we have an equation for the desired motion history $\theta(t)$, then we can solve Equation $14.2.1$ directly for the required operator setting:

$$r(t) = \frac{J}{K_a K_r}\ddot{\theta} \tag{14.2.2}$$

This method for finding $r(t)$ is easy, and open-loop control appears to be a feasible engineering tool. However, open-loop control has some serious deficiencies. A more detailed example will help to reveal these deficiencies, so we consider next an application of open-loop attitude control for a spacecraft.

Suppose that a capsule carrying astronauts is re-entering Earth's atmosphere, Figure $14.2.2$, and that pitch angle $\theta(t)$ must be controlled in order to position the heat shield for optimal functioning and to prevent unstable tumbling. Two collinear pairs of gas thrusters (usually fueled by liquid chemicals) are separated by moment arm e so that they actuate rotation by a couple (pure

moment), without causing mass center C to translate. Firing of the two thrusters shown active on Figure 14.2.2 produces a positive couple, and firing of the two opposite collinear thrusters would produce a negative couple. To simplify the analysis, let us **assume** that the thrusters are continuously throttleable, so that thrust force is related linearly to the operator setting by

$$f_v(t) = Kr(t) \tag{14.2.3}$$

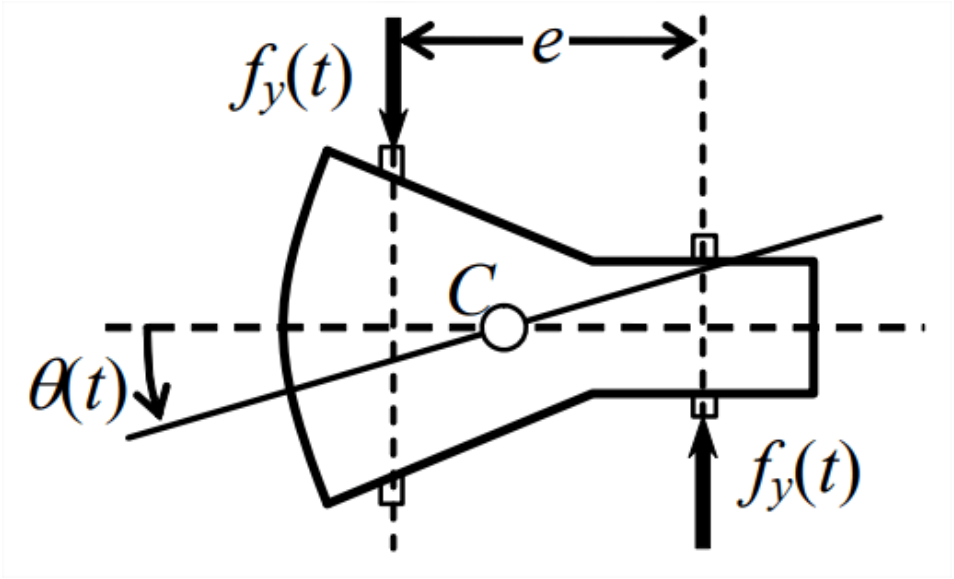


Figure 14.2.2: Pitch control of a re-entry capsule. (Copyright; author via source)

In Equation 14.2.3, K is a positive sensitivity constant, with appropriate dimensions depending upon the specific character of $r(t)$. Then the control moment is $M_a(t) = ef_y(t) = eKr(t)$, so at the equation of motion equivalent to Equation 14.2.1 is

$$J\ddot{\theta} = M_a(t) + M_d(t) = eKr(t) + M_d(t) \tag{14.2.4}$$

in which J is the capsule's rotational inertia about mass center C. If we neglect all possible disturbances, $M_d(t) = 0$, then we can solve Equation 14.2.4 for the required operator setting in terms of the desired pitching maneuver:

$$r(t) = \frac{J}{eK}\ddot{\theta} \tag{14.2.5}$$

For example, let us require that the spacecraft rotate smoothly from $\theta = 0$ to final pitch angle $\theta = \theta_f$ during t_f seconds, according to the lower time history graphed on Figure 14.2.3, which is expressed by the equation[1]

$$\theta(t) = \frac{\theta_f}{\pi}\left[\frac{\pi}{t_f}t - \frac{1}{2}\sin\left(\frac{2\pi}{t_f}t\right)\right] \tag{14.2.6}$$

for $0 \le t \le t_f$. Differentiating Equation 14.2.6 twice and then substituting the result into Equation 14.2.5 leads to

$$r(t) = r_A \sin\left(\frac{2\pi}{t_f}t\right), \quad r_A \equiv \frac{2\pi J\theta_f}{eKt_f^2}, \quad \text{for } 0 \le t \le t_f \tag{14.2.7}$$

This reference input setting $r(t)$ is shown on the upper graph of Figure 14.2.3.

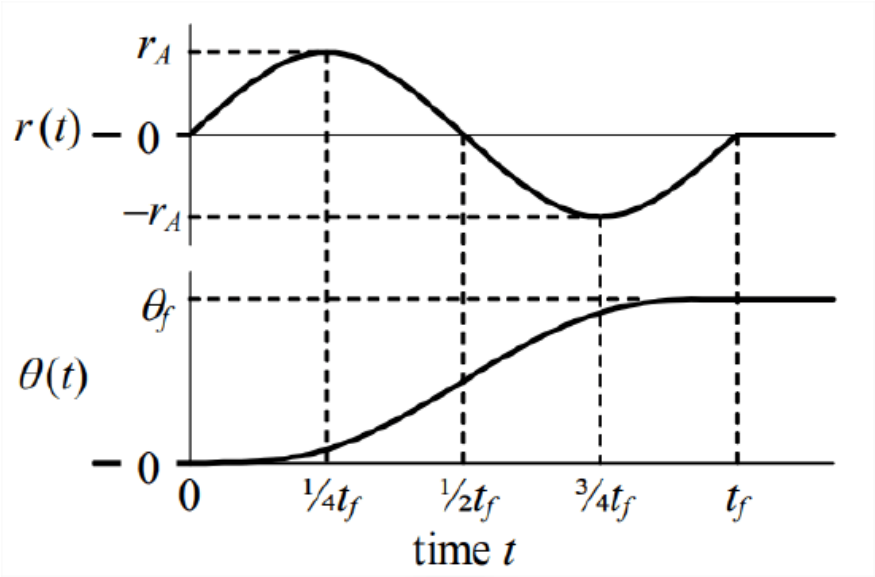


Figure 14.2.3: Example operator setting and resulting pitch maneuver for re-entry capsule. (Copyright; author via source)

Note that Equations 14.2.6 and 14.2.7 and Figure 14.2.3 illustrate the impulse-momentum theorem for 1-DOF rotation, which is directly analogous to the impulse-momentum theorem for 1-DOF translation (Section 8.2). The control moment is proportional to $r(t)$, $M_a(t) = eKr(t)$, so the $r(t)$ time history shows that the total moment impulse (area under the moment curve) in this maneuver is zero; since the total impulse is zero, the final ($t \geq t_f$) angular momentum also is zero, $J\dot{\theta} = 0$, so the spacecraft is rotated to the final position $\theta = \theta_f$ with zero final rotational velocity, $\dot{\theta} = 0$.

Let us pause for a reality check. One feature of the control equipment described for this spacecraft maneuver is not practically realistic: space qualified gas thrusters in common use are nonlinear on-off (approximately) force generators; they are **not** continuously throttleable as assumed in this example, Equation 14.2.3. Gas thrusters are, in fact, used to actuate spacecraft attitude and position control, but their on-off character produces rougher motion (abrupt changes in acceleration) than that of Equation 14.2.6; see Sarafin, 1995, pp. 636-637. A smooth maneuver such as Equation 14.2.6 would be actuated in practice by a space-qualified reaction wheel or control-moment gyroscope. We have assumed the hypothetical throttleable gas thrusters for this example just because it is easy to visualize this situation and to describe it in equations.

Now, with reference to the example above of spacecraft attitude control, let us evaluate open-loop control as represented generally by Figure 14.2.1, with particular interest in identifying the following deficiencies:

1. Note from Figure 14.2.3 that output $\theta(t)$ is not a single-valued, monotonic function of operator setting $r(t)$. For example, to $r = 0$ there correspond three different output values: $\theta = 0$ **before** the maneuver, $\theta = 1/2\theta_f$ **during** the maneuver, and $\theta = \theta_f$ **after** the maneuver. This type of complicated functional relationship is not necessarily an operational deficiency, but it certainly is counter-intuitive to the natural engineering expectation of a unique, monotonic, often linear relationship between cause and effect.
2. Equations 14.2.2, 14.2.5, and 14.2.7 show that the validity of derived operator setting $r(t)$ is directly dependent upon the accuracy of the plant parameters and control system sensitivities, and upon the fidelity of the theoretical model of the system. If, for example, we use an inaccurate value of rotational inertia J (a distinct possibility for most vehicles, since burning of fuel changes inertias), or if the thrusters fail to function exactly as we model them, then the derived $r(t)$ clearly will not lead to the exact desired rotation time history, $\theta(t)$.
3. The $r(t)$ Equations 14.2.2 and 14.2.5 are predicated on the absence of disturbances; but if $M_d(t) \neq 0$, then the derived $r(t)$ will fail to produce the exact desired $\theta(t)$. In practice, unpredictable disturbances are common, and they should not be ignored. For example, moments that can act upon a re-entering spacecraft might be produced by atmospheric drag, impact from space debris, astronauts shifting position within the vehicle, *etc*.

Deficiencies 2 and 3 above can be described in a more general manner: the performance of an open-loop control system is strongly dependent upon the quality of the theoretical model of the system and its environment; in other words, the control performance is highly sensitive to imperfections in the theoretical model, which are inevitable for most real engineering situations.[2]

It is a fundamental characteristic and weakness of open-loop control that the actual output is not measured and compared with

the desired output, for the purposes of assessing the output error and attempting to reduce the error. A simple and natural remedy for this weakness is to “close the loop” with feedback, which is described in the next section. Note that the absence of a feedback branch from Figure 14.2.1 leads to the designation “open-loop.” But even this designation is at least a little misleading because there is no “loop” at all; there is simply a single branch that carries forward-flowing signals!

[1]Maneuver time history Equation 14.2.6 comes from Sarafin, 1995, pages 644-645, a book that includes additional interesting discussions of spacecraft attitude control.

[2]In the language of control engineering, a control system is described as *robust* if its performance and stability are insensitive to potentially detrimental influences such as disturbances, deficiencies in the theoretical model of the plant, and inaccurate response measurements due to sensor dynamics.

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14.3: Closed-Loop Control of Rotor Position (1)

Let us modify the open-loop control system of Figure 14.2.1 in the following manner: measure the output with a rotational-position sensor, and send the voltage output from this sensor back to a summing junction that forms the *error signal*, which is the difference between the signal representing the operator setting and the signal representing the actual output. The feedback branch and summing junction convert the open-loop system into the *closed-loop* system depicted on Figure 14.3.1.

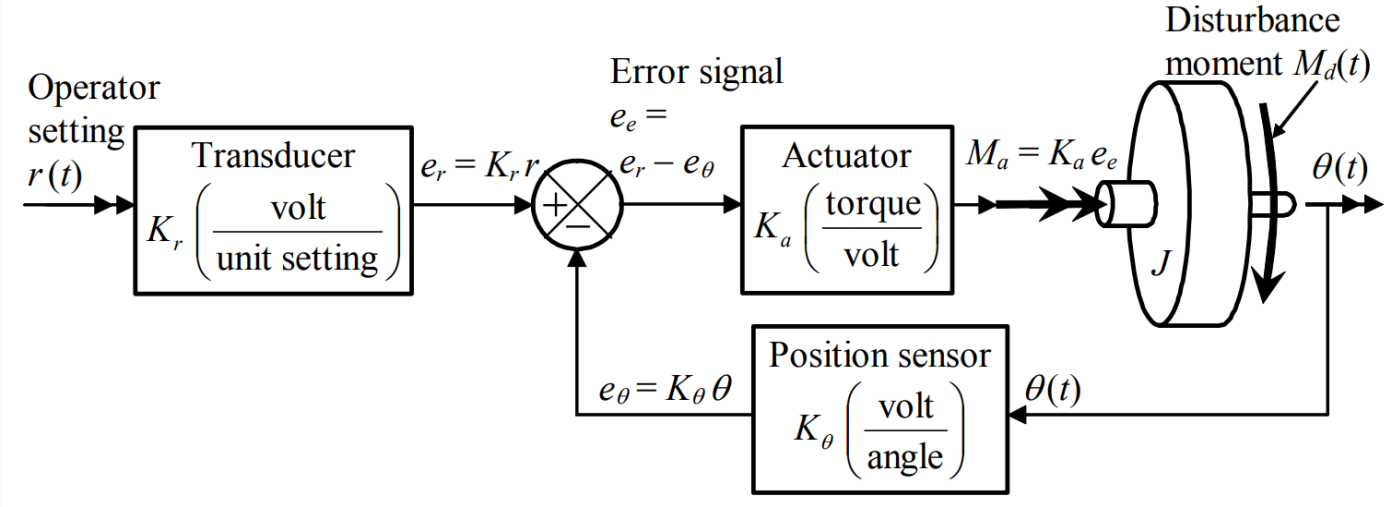


Figure 14.3.1: Functional diagram of position-feedback control system. (Copyright; author via source)

The elements of the closed-loop system are the same as those of the open-loop system, Figure 14.2.1, but with the addition of the output sensor and the summing junction (defined in Section 13.2). We assume that the sensor is a device such as an optical encoder, with accurately calibrated positive sensitivity K_θ (volt per degree or radian). The summing junction could be a simple circuit consisting of an inverting amplifier (Figure 5.3.2) and a summing inverting amplifier (homework Problem 5.7).

This closed-loop system tends to be *self-correcting*. We can understand this tendency by writing in detail the equation for the control moment, from the signals labeled on Figure 14.3.1:

$$M_a(t) = K_a e_e(t) = K_a \left[e_r(t) - e_\theta(t) \right] = K_a \left[K_r r(t) - K_\theta \theta(t) \right] \tag{14.3.1}$$

From Equation 14.3.1, the control moment is zero when $K_r r(t) - K_\theta \theta(t) = 0$, so that

$$\theta(t) = \frac{K_r}{K_\theta} r(t) \tag{14.3.2}$$

Equation 14.3.2 clearly is the desired instantaneous linear relation of output $\theta(t)$ to input $r(t)$. At instants when Equation 14.3.2 is **not** satisfied, the control actuator imposes a corrective moment. Suppose, for example, that at a certain instant both $\theta(t)$ and $r(t)$ are positive, but that the output is less than the desired value, $\theta(t) < [K_r / K_\theta]\, r(t)$; at this instant, then, the error signal is $K_r r(t) - K_\theta \theta(t) = e_e(t) > 0$, and $M_a(t) = K_a e_e(t) > 0$ thus, when the output is less than it ought to be, the closed-loop control system imposes a positive moment to increase the output. Moreover, by reversing the inequality signs above, you can show that the control system imposes a negative moment to decrease the output when the output is greater than it ought to be.

Because the corrective moment is proportional to the output error and is independent of the theoretical model, the self-correcting tendency of this closed-loop control system is in effect regardless of how well the theoretical model represents the system and its environment.

To infer more information about the closed-loop system of Figure 14.3.1, let us derive the ODE of motion by substituting Equation 14.3.1 into Equation 14.2.1:

$$J\ddot{\theta} = M_a(t) + M_d(t) = K_a \left[K_r r(t) - K_\theta \theta(t) \right] + M_d(t)$$

$$\Rightarrow \quad J\ddot{\theta} + K_a K_\theta \theta = K_a K_r r(t) + M_d(t) \tag{14.3.3}$$

From Equation 14.3.3, we make the following observations:

1. the feedback control in Figure 14.3.1 has **the effect** of attaching between the operator setting and the rotor inertia an **artificial** restoring spring with stiffness constant $K_a K_\theta$;
2. if only operator input $r(t)$ acts, *i.e.*, if $M_d(t) = 0$, then the desired output Equation 14.3.2 is actually the pseudo-static response of the system (as defined in Section 7.1),

$$K_a K_\theta \theta_{ps} = K_a K_r r(t) \Rightarrow \theta_{ps}(t) = \frac{K_r}{K_\theta} r(t) \tag{14.3.4}$$

3. if only operator input $r(t)$ acts, and if the input-transducer sensitivity equals the output-sensor sensitivity, $K_r = K_\theta$, then the closed-loop system is directly analogous to an undamped, base-excited mass-spring system [as described by Equation 13.2.1 with $c = 0$];
4. the self-correcting tendency even counteracts the adverse influence of many types of disturbance, which we can conclude from our previous experience with forced-response solutions for 2nd order systems.

If, for example, the disturbance is time-limited, such as a pulse, then post-pulse response due to the pulse will oscillate about zero but will not contain any constant steady-state component. If, for another example, the disturbance is sinusoidal, then the resulting response will oscillate about zero but will not contain any constant steady-state component; however, we can also see that if the frequency of the sinusoidal disturbance is in the neighborhood of the closed-loop system natural frequency, $\omega_n = \sqrt{K_a K_\theta / J}$, then the disturbance could produce resonance, which this particular control system could not prevent.

To establish the method for analysis later of more complicated control systems, let us draw and analyze the Laplace block diagram corresponding to functional diagram Figure 14.3.1. For this task, we define the following Laplace transforms: $L[r(t)] \equiv R(s)$, $L[\theta(t)] \equiv \Theta(s)$, $L\,[e_r(t)] \equiv E_r(s)$, $L\,[e_\theta(t)] \equiv E_\theta(s)$, $L\,[e_e(t)] \equiv E_e(s)$, $L\,[M_a(t)] \equiv \mathrm{M_a}(s)$, $L\,[M_d(t)] \equiv \mathrm{M_d}(s)$, and $L[M(t)] \equiv \mathrm{M}(s)$. The Laplace block diagram, Figure 14.3.2, is similar to the functional diagram, except that we replace the rotor in Figure 14.3.1 with its transfer function, $PTF(s)$ of Equation 14.1.2, and we use a summing junction to denote the actions upon the rotor of both control moment $M_a(t)$ and disturbance moment $M_d(t)$.

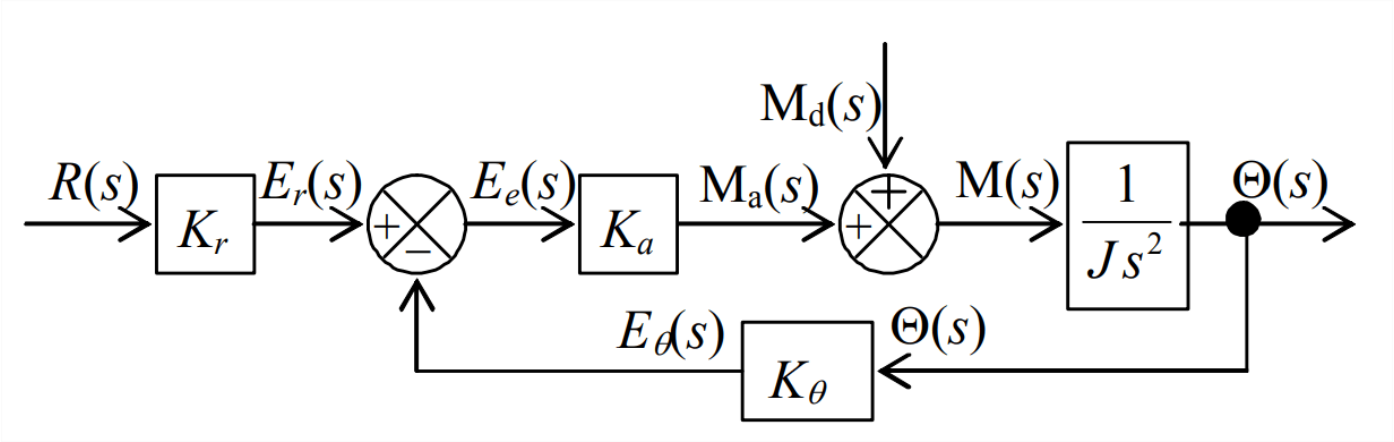


Figure 14.3.2: Laplace block diagram, including disturbance input, of position-feedback control system. (Copyright; author via source)

This system actually has two independent inputs, $r(t)$ and $M_d(t)$, so it is not strictly a single-input-single-output (SISO) system. However, the most important fundamental relationship for this control system is that between the reference input and output $\theta(t)$, so we now will focus on that relationship by setting $M_d(t) = 0$. Therefore, $\mathrm{M_d}(s) = 0$, and we re-draw Figure 14.3.2 as the SISO block diagram at right, with the objective of deriving from it the closed-loop transfer function, $\mathrm{CLTF}(s) \equiv \Theta(s)/R(s)$.

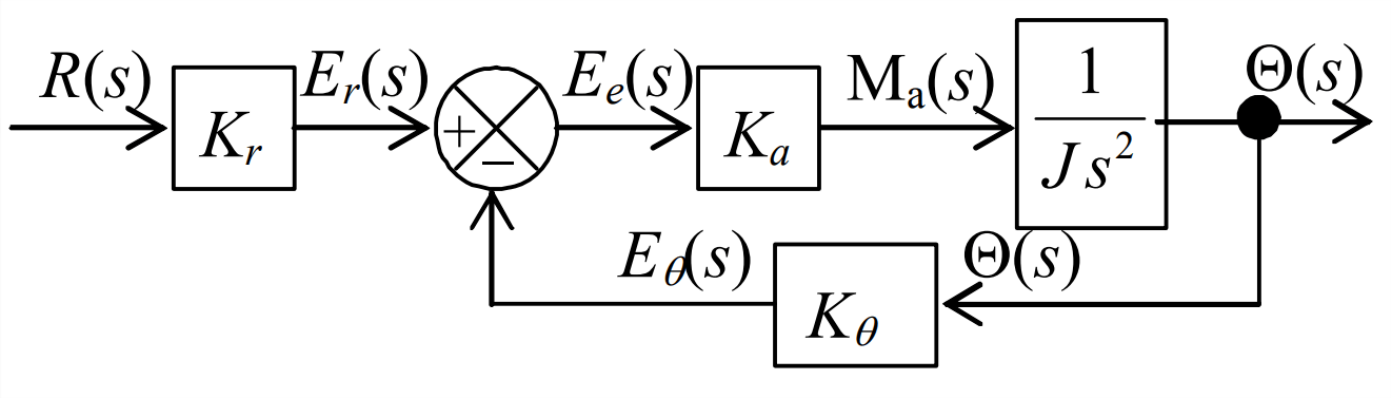


Figure 14.3.3: Laplace block diagram of SISO position-feedback control system. (Copyright; author via source)

Three distinct steps of block-diagram algebra are required to reduce Figure 14.3.3 to a single block, the $\mathrm{CLTF}(s)$, that separates input and output signals:

1. Combine (multiply) the two block transfer functions within the forward branch of the loop, as shown below. This multiplication operation is derived in Section 13.1.

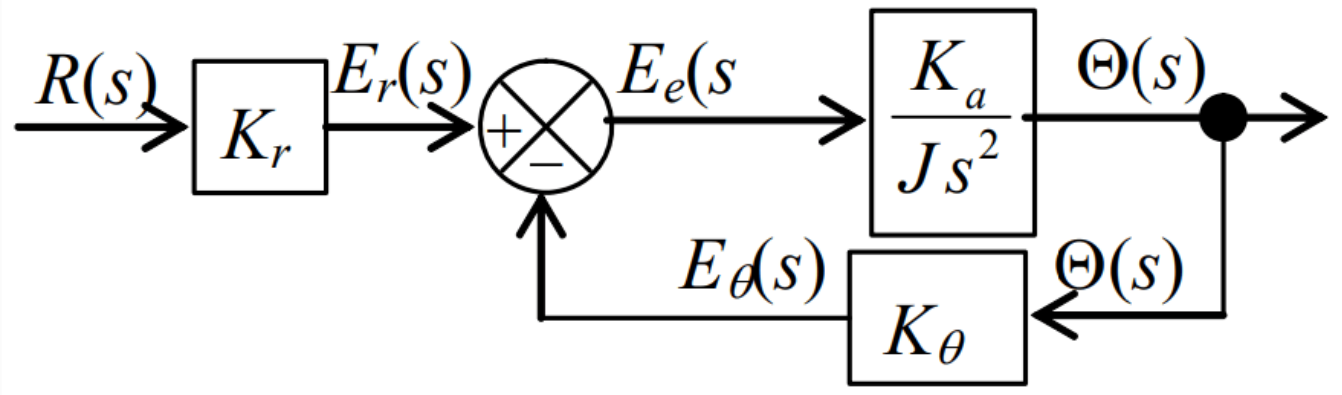


Figure 14.3.4: Step 1. (Copyright; author via source)

2. Resolve the forward-branch and feedback-branch transfer functions of the closed loop into the single equivalent transfer function shown below. The block-diagram algebra used in this step is derived in the next section; it is an important basic tool for the analysis of control systems.

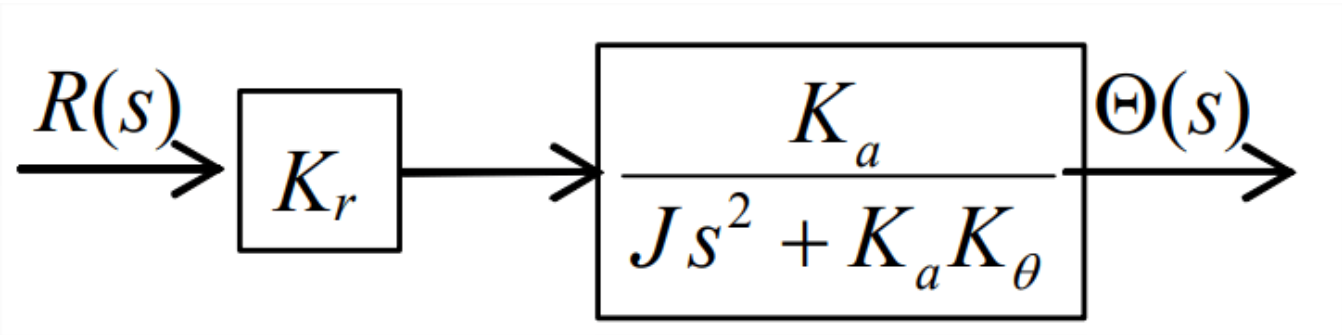


Figure 14.3.5: Step 2. (Copyright; author via source)

3. Finally, and obviously, just multiply the remaining two blocks to produce the required result shown below.

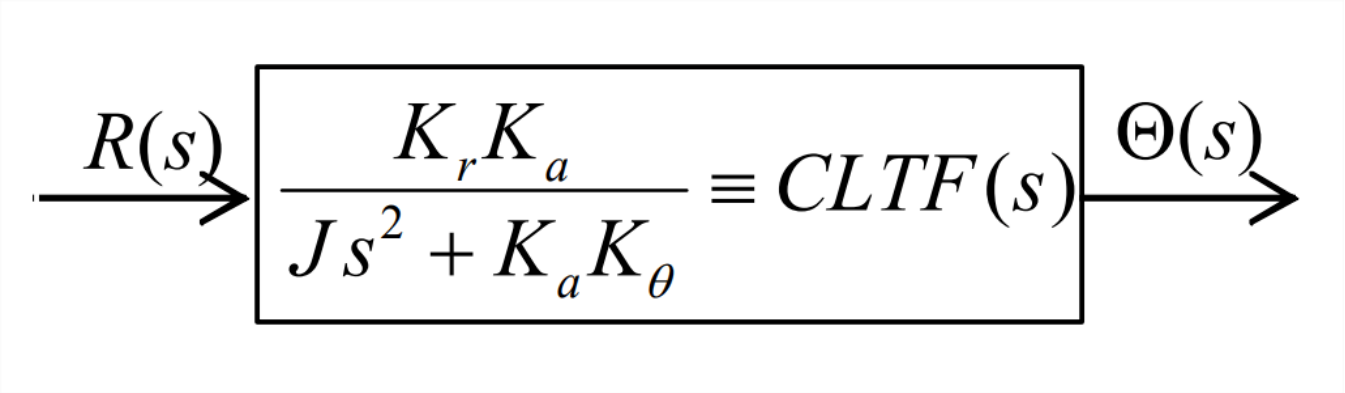


Figure 14.3.6: Step 3. (Copyright; author via source)

For this relatively simple system, the SISO version of ODE of motion Equation 14.3.3 is $J\ddot{\theta} + K_a K_\theta \theta = K_a K_r r(t)$. It is very easy to derive the same $\mathrm{CLTF}(s)$ directly from this ODE, but we have derived $\mathrm{CLTF}(s)$ here using block-diagram algebra in order to establish the approach that is appropriate for more complicated systems.

We shall re-visit this position-feedback control system in Section 14.5, with the objectives of evaluating its control performance and then developing an addition to the systems that will improve the performance.

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14.4: Transfer Function of a Single Closed Loop

Consider Figure 14.4.1, which is a more general version, in standard notation, of the closed loop in Figure 14.3.4. General functions $G(s)$ and $H(s)$ are, respectively, the forward-branch and feedback-branch transfer functions. $E(s)$ is the actuating error signal, and $B(s)$ is the feedback signal. We seek the closed-loop transfer-function, the ratio $\mathrm{Out}(s)/\mathrm{In}(s)$.

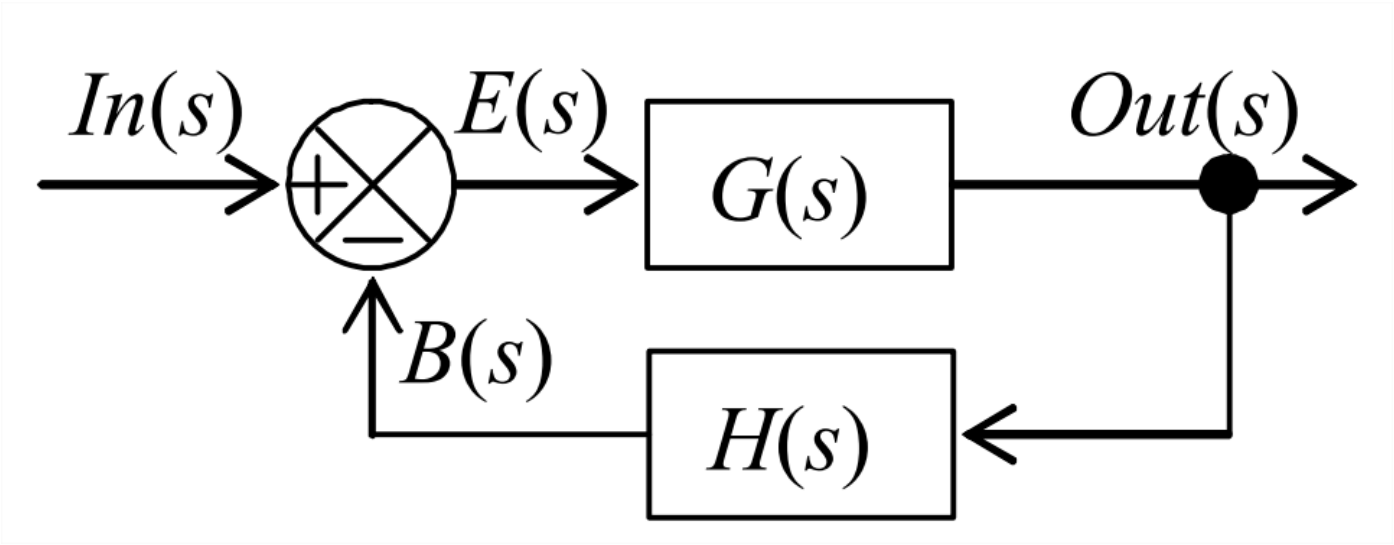


Figure 14.4.1: Laplace block diagram of general closed loop. (Copyright; author via source)

All of the quantities labeled on Figure 14.4.1 are denoted as functions of Laplace variable s, but **it is important to recognize that there are two fundamentally different types of functions**:

1. the "signals" $\mathrm{In}(s)$, $E(s)$, $B(s)$, and $\mathrm{Out}(s)$, which actually are Laplace transforms of time-dependent variables such as motion and voltage; and
2. the transfer functions $G(s)$ and $H(s)$, which represent in the Laplace domain the characteristics of systems and objects such as inertia and circuits. To emphasize the difference in the following derivation of $\mathrm{Out}(s)/\mathrm{In}(s)$, let us omit the functional notation "(s)" from the signals, at least in the intermediate steps. The steps follow naturally from Figure 14.4.1:

$$
\begin{aligned}
& E = In - B \quad \text{and} \quad B = Out \times H(s), \\
\Rightarrow \quad & Out = E \times G(s) = [\mathrm{In} - B] \times G(s) = [\mathrm{In} - Out \times H(s)] \times G(s) \\
& = \mathrm{In} \times G(s) - Out \times H(s) \times G(s) \\
\Rightarrow \quad & Out \times [1 + G(s)H(s)] = \mathrm{In} \times G(s) \\
& \Rightarrow \frac{Out(s)}{\mathrm{In}(s)} = \frac{G(s)}{1 + G(s)H(s)}
\end{aligned}
\tag{14.4.1}
$$

Equation 14.4.1 is an important, general, labor-saving tool for analysis of loops in any control system, not just the systems discussed in this chapter. To illustrate its application, let us use it to derive in detail the loop transfer function written in Step 2 of the block-diagram algebra of Section 14.3. From Step 1 of that process (or from Figure 14.3.4), we identify: $G(s) = K_a/Js^2$ and $H(s) = K_\theta$. Substituting these into Equation 14.4.1 gives $\frac{\mathrm{Out}(s)}{\mathrm{In}(s)} = \frac{K_a/Js^2}{1+(K_a/Js^2)K_\theta} = \frac{K_a}{Js^2+K_aK_\theta}$, which is the loop transfer function shown in the block diagram as the result of Step 2.

A transfer function in Equation 14.4.1 often has the form of a ratio of polynomials in s, such as $G(s)$ in the previous paragraph, so it is useful to derive a version of Equation 14.4.1 in terms of the polynomials. First, we define numerator and denominator polynomials that make up $G(s)$ and $H(s)$:

$$G(s) \equiv \frac{N_G(s)}{D_G(s)} \text{ and } H(s) \equiv \frac{N_H(s)}{D_H(s)} \tag{14.4.2}$$

Next, we carry out the algebra of Equation 14.4.1, dropping the functional notation "(s)" from the transfer functions and polynomials in the interest of notational conciseness:

$$\frac{Out(s)}{\text{In}(s)} = \frac{G}{1+GH} = \frac{N_G/D_G}{1+(N_G/D_G)\times(N_H/D_H)} = \frac{N_G D_H}{D_G D_H + N_G N_H} \tag{14.4.3}$$

For the example of the previous paragraph, the simple polynomials are $N_G = K_a$, $D_G = J_S^2$, $N_H = K_\theta$, and $D_H = 1$; substituting these terms into Equation 14.4.3 obviously leads to the, $\text{Out}(s)/\text{In}(s)$ result derived in the previous paragraph, but with a bit less algebra because the algebra has been completed in Equation 14.4.3. Equation 14.4.3 expresses the closed-loop transfer function as a ratio of polynomials, and it applies in general, not just to the problems of this chapter.

Finally, we will use later an even more specialized form of Equations 14.4.1 and 14.4.3 **for the case of *unity feedback*,** $H(s) = 1 = 1/1$:

$$\frac{\text{Out}(s)}{\text{In}(s)} = \frac{G}{1+G} = \frac{N_G}{D_G + N_G} \tag{14.4.4}$$

Equations 14.4.1, 14.4.3, and 14.4.4 are very useful for the analysis of feedback control. In the future, whenever you encounter a simple loop with the form of Figure 14.4.1 in the Laplace block diagram of a system, you may (and usually **should**) apply whichever of these equations is most appropriate to derive the loop transfer function, without repeating the algebra that goes into the derivation of these equations.

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14.5: Closed-Loop Control of Rotor Position (2)

Let us re-visit the position-feedback control system of Section 14.3 in order, first, to evaluate its control performance and, next, to develop an addition to the system that will improve the performance.

It is easy to evaluate the control performance just by considering the SISO version of ODE of motion Equation 14.3.4, which is

$$J\ddot{\theta} + K_a K_\theta \theta = K_a K_r r(t) \tag{14.5.1}$$

As was noted earlier, the **pseudo-static** response of Equation 14.5.1 is $\theta_{ps}(t) = [K_r/K_\theta]\, r(t)$; this would be good controlled response, but it is **not** necessarily the **total** response. Therefore, let us consider in more detail the total transient response. Equation 14.5.1 is an **undamped** 2nd order ODE of the type analyzed extensively in Chapter 7. By dividing Equation 14.5.1 through by J and then defining appropriate parameters, we can adapt this ODE to the standard form Equation 7.1.5, $\ddot{\theta} + \omega_n^2 \theta = \omega_n^2 u(t)$, in which the natural frequency is $\omega_n = \sqrt{K_a K_\theta / J}$, and the standard input quantity is $u(t) = [K_r/K_\theta]\, r(t)$. We can also adapt any appropriate response solutions already derived in Chapters 7 and 8 to this position-feedback control system. For example, if the operator setting is the step function $r(t) = R_H H(t)$, then $u(t) = [K_r/K_\theta]\, R_H H(t) \equiv U H(t)$, so the standard solution Equation 7.3.8 leads to step response $\theta(t) = (K_r/K_\theta)\, R_H\, (1 - \cos \omega_n t)$. This step response is plotted on the figure at right, with the period defined as $T_n = 2\pi/\omega_n$. Due to the absence of damping, the response oscillates forever **about** the desired constant pseudo-static response, $\theta_{ps} = (K_r/K_\theta)\, R_H$. The response never settles **at** θ_{ps}, so **the control system fails to deliver the desired performance**.

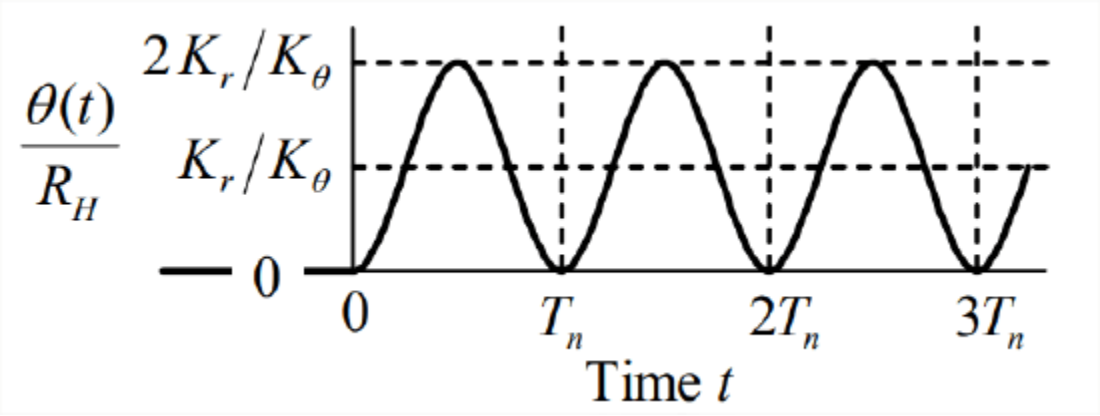


Figure 14.5.1 (Copyright; author via source)

We must conclude from this example that the absence of damping renders the position-feedback control system of Section 14.3 (Figure 14.3.1) unsuitable for most, if not all engineering applications of control. However, there is a relatively simple design feature that can be added to the control system, a feature that will provide damping to the system and will greatly improve the control performance.

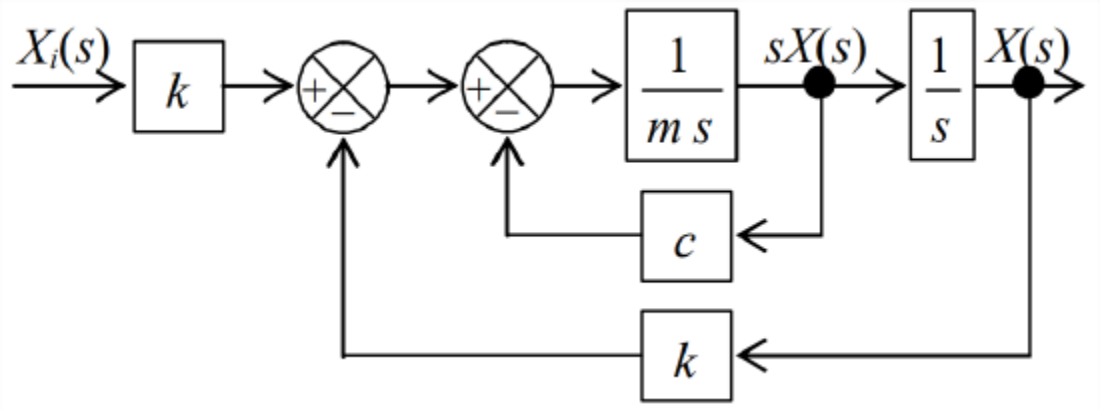


Figure 14.5.2: Laplace block diagram with feedback branches for a base-excited m-c-k system. (Copyright; author via source)

In order to describe the motivation for this new feature of the control design, let us revisit the base-excited m-c-k system of Sections 13.2 and 13.3. The principal elements of Figure 13.2.3, that system's Laplace block diagram, are reproduced on Figure $14.5.2$. The inner feedback loop makes Figure $14.5.2$ more complicated than any control system block diagram that we have encountered previously in this chapter. Consider, however, the case of zero damping, $c = 0$, for which the system is an undamped, base-excited mass-spring (m-k) system, and Figure $14.5.2$ loses the inner feedback loop, taking on the simpler appearance of Figure $14.5.3$. Now, let us compare the Laplace block diagram of the base-excited (m-k) system with that of the rotor position-feedback control system, Figure $14.5.4$, which is repeated here. The Laplace block diagrams of the two systems have nearly identical forms, with, for example, constants K_r and K_a/J in Figure $14.5.4$ being directly analogous, respectively, to constants k and $1/m$ in Figure $14.5.3$. Only the possibility on Figure $14.5.4$ that input block constant K_r and feedback block constant K_θ can be different keeps Figure $14.5.4$ from being completely analogous to Figure $14.5.3$. This is a minor difference that has no substantive effect in the following discussion.

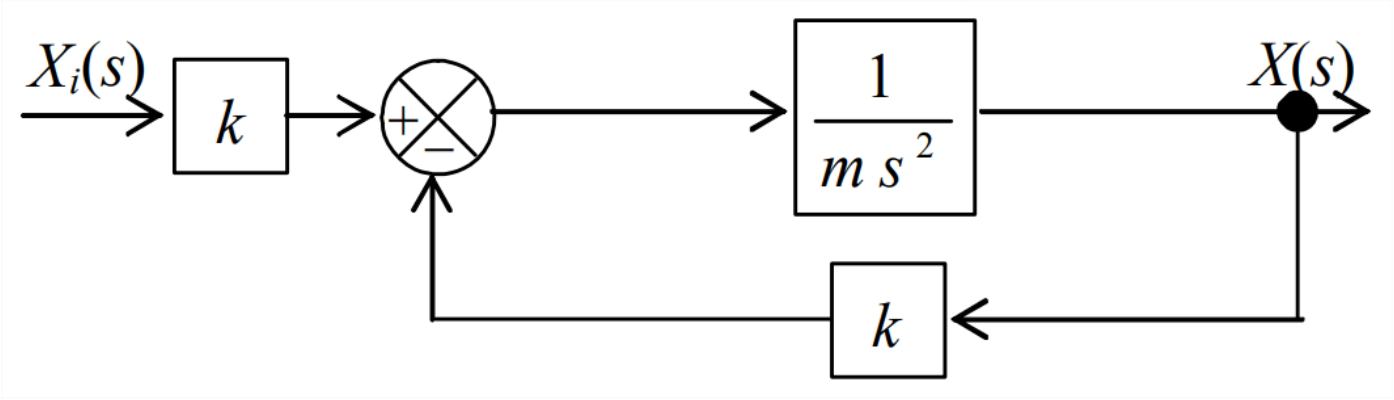


Figure $14.5.3$: Laplace block diagram with feedback branch for a base-excited m-k system. (Copyright; author via source)

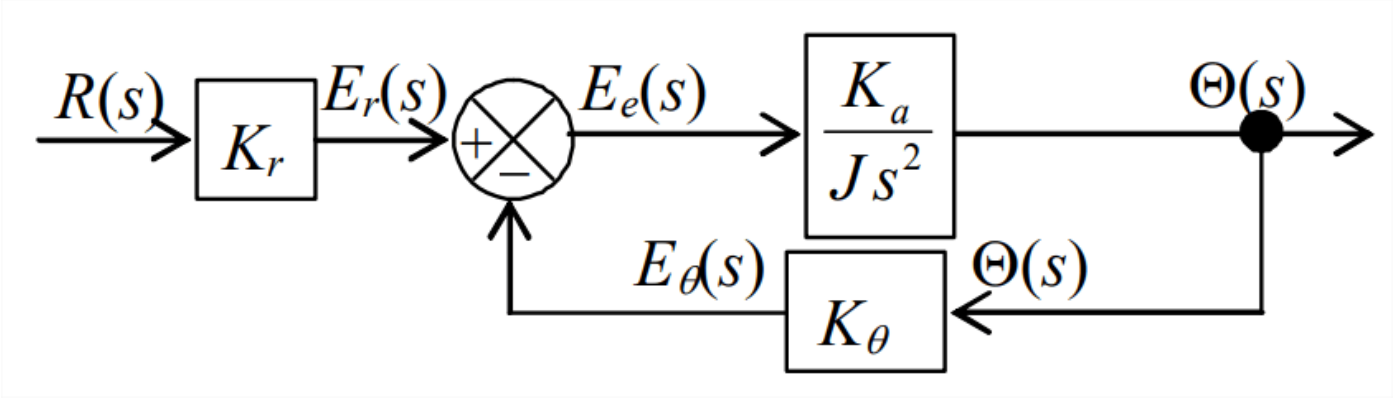


Figure $14.5.4$: Laplace block diagram of SISO position-feedback control system. (Copyright; author via source)

Let us exploit the similarities between the base-excited m-c-k system and the rotor-position control system in order to guide us toward a method for improving the latter. First, we refer back to the solution in Section 13.3 for response of the base-excited m-c-k system to step input $x_i(t) = X_H H(t)$. The step response is presented graphically on Figure $14.5.5$, which is repeated on the next page for your convenience.

Recall the following definitions for the m-c-k system: $\omega_n = \sqrt{k/m}$, $T_n = 2\pi/\omega_n$, and $\zeta = c/(2\sqrt{mk})$. Note on Figure $14.5.5$ that the addition of positive damping ($c > 0 \Rightarrow \zeta > 0$) to the undamped system produces output $x(t)$ that approximates the input; therefore, the positively damped m-c-k system is, indeed, a suitable control system, whereas the undamped m-k system is not.

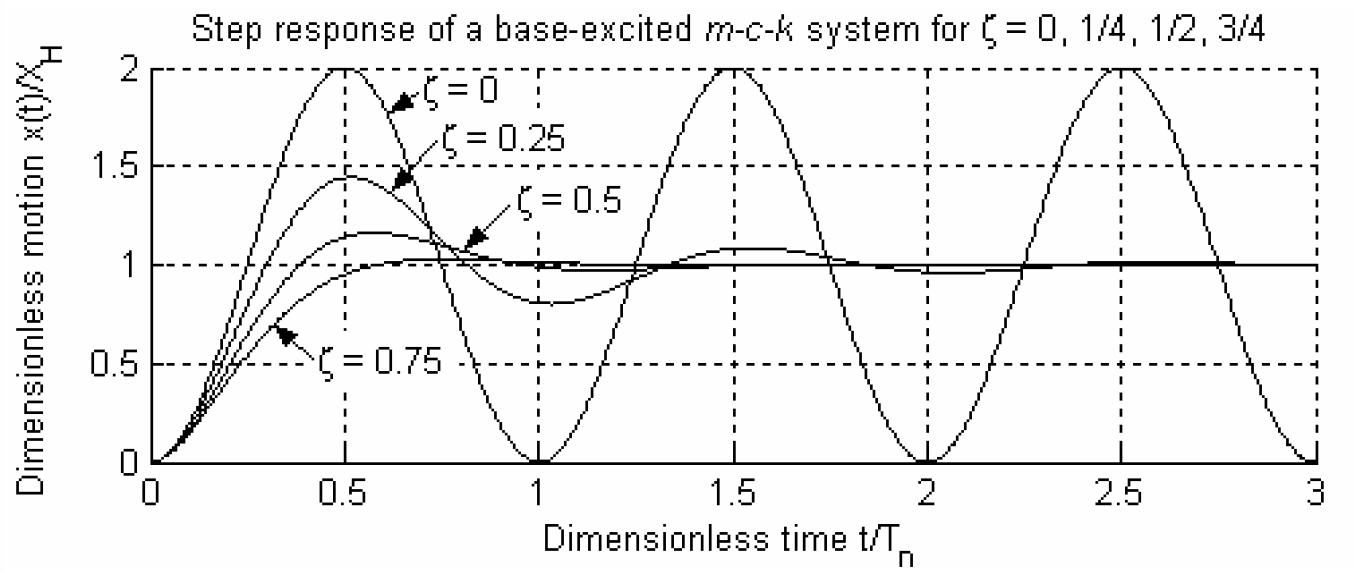


Figure 14.5.5: Base-excitation control of mass position for an m-c-k system. (Copyright; author via source)

Can we modify the undamped rotor-position control system in such a way that it has the same type of positive damping, and therefore suitable control performance, as the m-c-k system? Block diagram Figure 14.5.2 for the positively damped m-c-k system suggests that we might improve the rotor-position control system by adding an inner feedback loop to its block diagram, Figure 14.5.4, a loop that is similar to the inner loop of Figure 14.5.2. But are there practical physical operations and devices corresponding to the purely mathematical addition of this type of loop to Laplace block diagram Figure 14.5.4? To answer this question, let us observe that the Laplace feedback signal of the inner loop on Figure 14.5.2 is $sX(s)$, which is the Laplace transform of velocity $\dot{x}(t)$. Therefore, the inner loop on Figure 14.5.2 represents mathematically the physical operations of detecting the output **velocity** and multiplying that velocity by negative constant $-c$ to create a damping force that **opposes** motion of mass m. We can perform the analogous physical operation for the rotor-position control system by measuring output rotational velocity $p(t) \equiv \dot{\theta}(t)$ with an appropriate sensor (such as a rate gyroscope) and feeding that signal negatively back to the input of the actuator. Figure 14.5.6 depicts the modified rotor-position control system, with **both** this rate feedback (to provide positive damping) **and** the original position feedback (to provide positive stiffness).

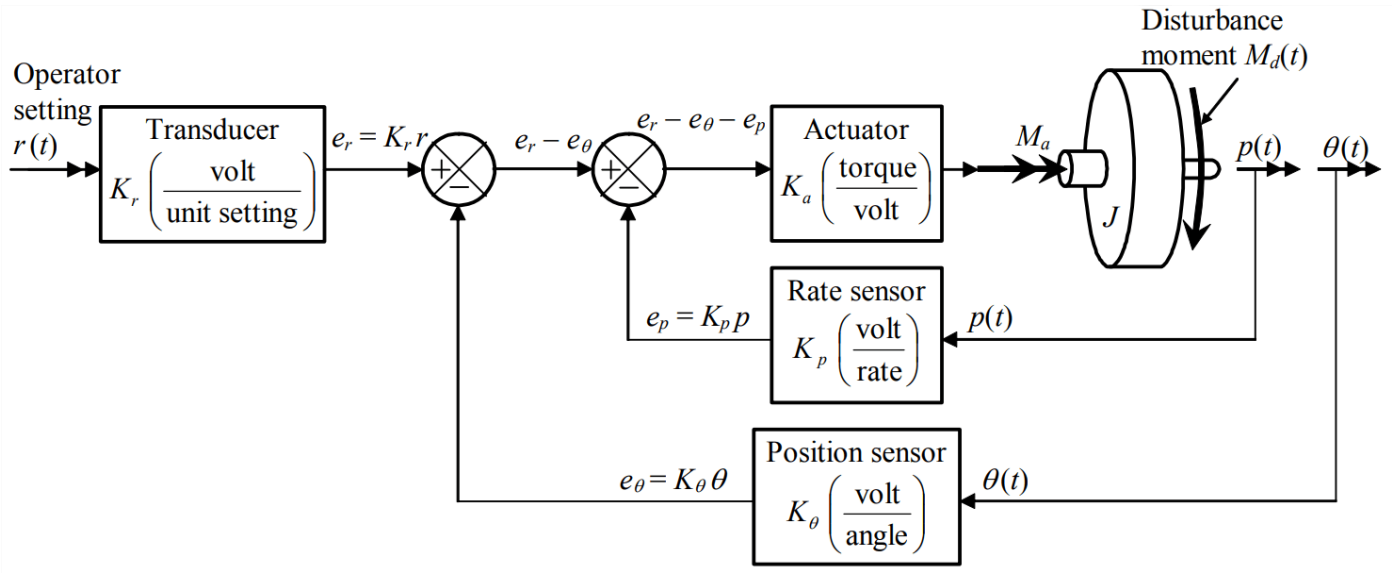


Figure 14.5.6: Functional diagram of rotor-position control system with both position feedback and rate feedback. (Copyright; author via source)

Figure 14.5.7 is the Laplace block diagram corresponding to functional diagram Figure 14.5.6. For measuring and feeding back the rotational velocity, the appropriate form of plant ODE of motion Equation 14.1.1 is $J\dot{p} = M(t)$, so the appropriate plant transfer function in the new inner loop is $L[p(t)]/L[M(t)] \equiv P(s)/\mathrm{M}(s) = 1/(Js)$. Also, the definition of rotational velocity leads to the

forward-branch integration transfer function, $\Theta(s)/P(s) = 1/s$. Comparing Figure $14.5.7$ with Figure $14.5.2$ for the base-excited m-c-k system, we see that the two Laplace block diagrams have nearly identical forms, except for the presence of the disturbance moment on Figure $14.5.7$. Thus, and to summarize, we have modified the rotor-position control system to add positive damping, using the base-excited m-c-k system as a model.

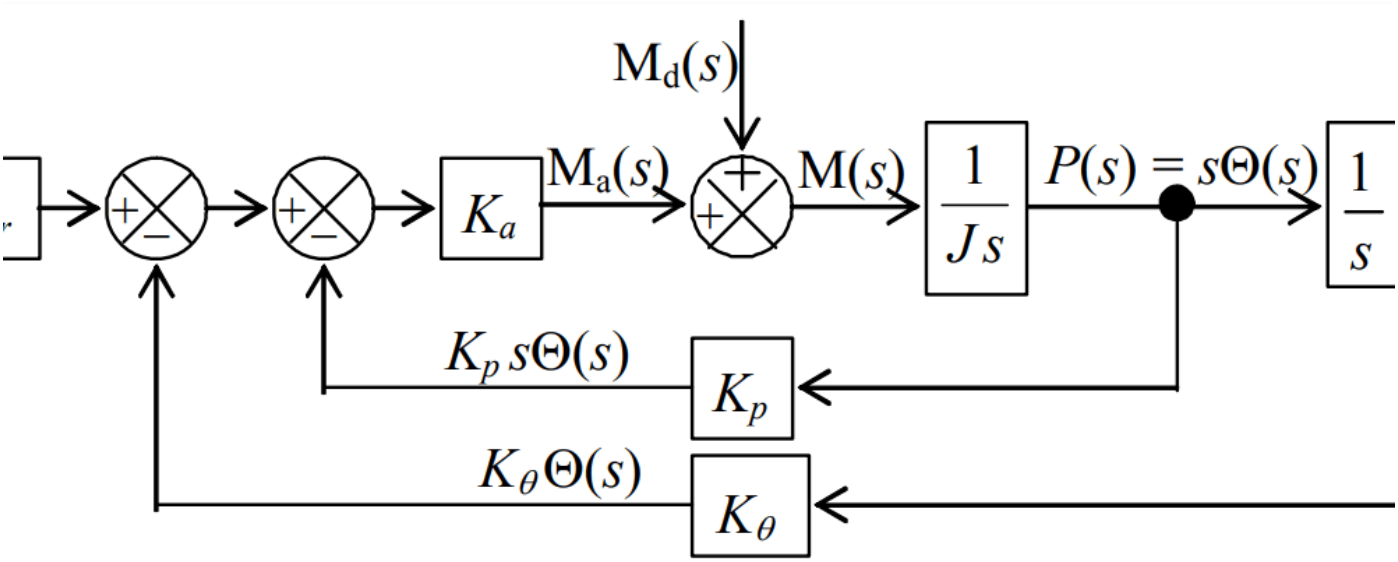


Figure $14.5.7$: Laplace block diagram of rotor-position control system with both position feedback and rate feedback. (Copyright; author via source)

The position-feedback and rate-feedback types of control illustrated by Figures $14.5.6$ and $14.5.7$ are relatively simple *output operations*: this control involves no more than basic arithmetic operations on the sensed output signals, without any modification of the error signal in the forward branch. Chapter 15 introduces control by *input error operations*, which involves more general mathematical operations upon a single error signal.

Next, let us analyze in detail the SISO version of the modified rotor-position control system. Thus, we set $M_d(t) = 0$ and $\mathrm{M_d}(s) = 0$, and we re-draw Figure $14.5.7$ as SISO block diagram Figure $14.5.8$. Our objectives are: first, to derive from Figure $14.5.8$ the closed-loop transfer function, $\mathrm{CLTF}(s) \equiv \Theta(s)/R(s)$; second, to infer from $\mathrm{CLTF}(s)$ the characteristics of the SISO system.

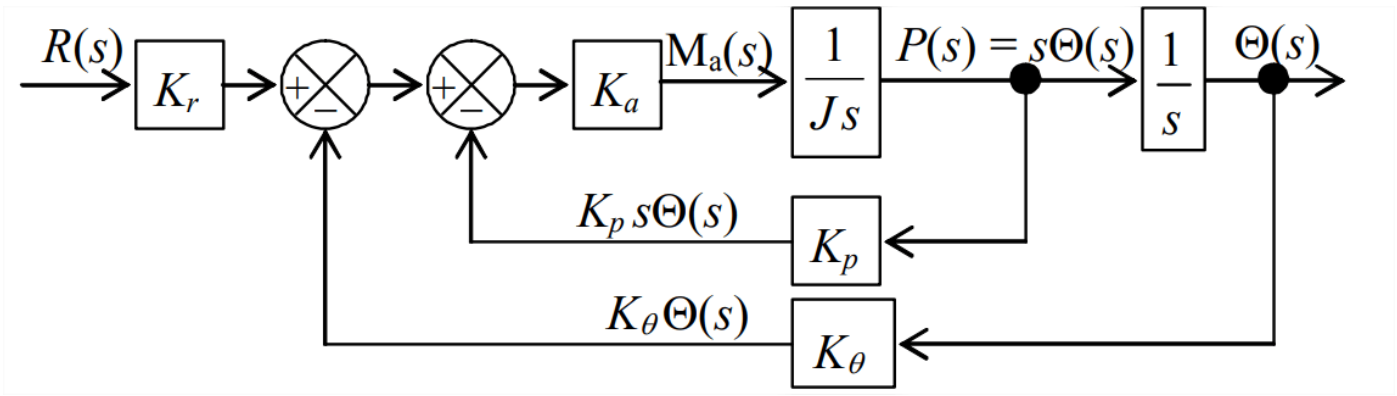


Figure $14.5.8$: Laplace block diagram of SISO rotor-position control system with both position feedback and rate feedback. (Copyright; author via source)

We could resolve block diagram Figure $14.5.8$ by the “brute-force” method of applying Equation 14.4.7 first to the inner loop, then to the outer loop. However, we can achieve the same result more simply by using some sensible block-diagram algebra. The approach in each step is to modify a block and/or branch so as to simplify the diagram without changing the most important output signals. The obvious first step, is to combine the two blocks in the forward branch of the inner loop on Figure $14.5.8$ into transfer function $K_a/(Js)$.

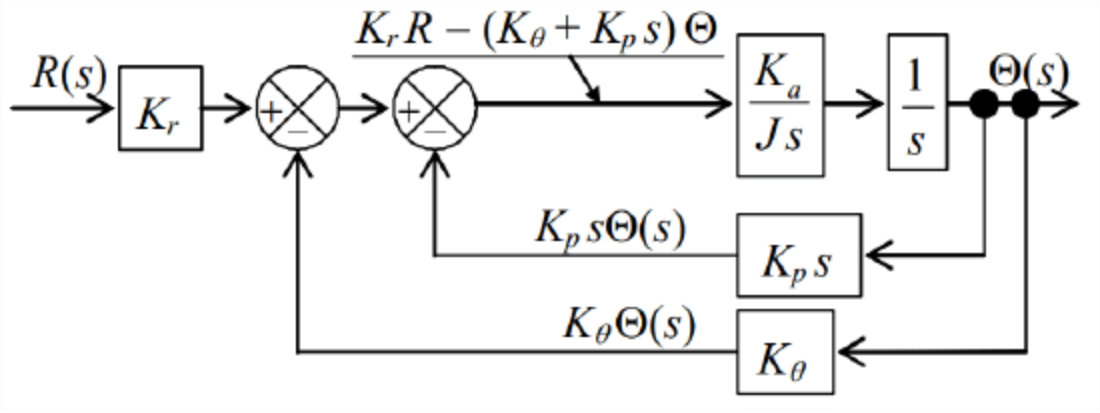


Figure 14.5.9 (Copyright; author via source)

Next, we **move the branch point** of the inner loop to the right of the integration block, and modify accordingly the feedback transfer function of the inner loop, as shown below.

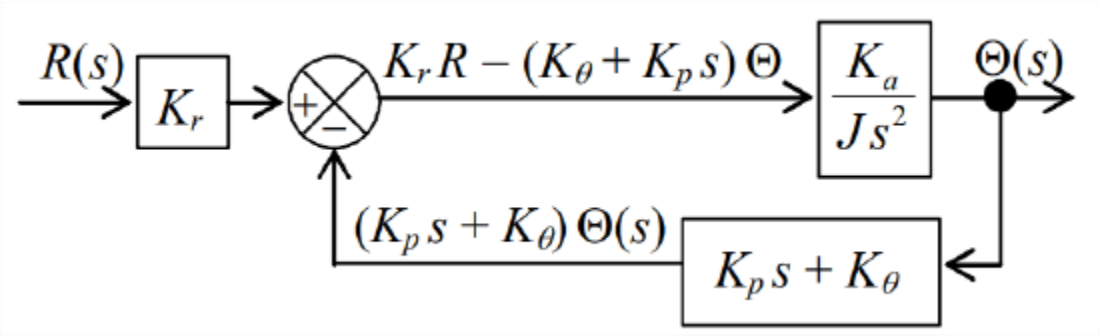


Figure 14.5.10 (Copyright; author via source)

Next, we multiply the two blocks of the forward branch inside the loops into transfer function $K_a/\left(Js^2\right)$. Now, both feedback branches essentially originate at the same branch point and terminate at the same summing junction, so we combine them into a single feedback branch with an appropriate transfer function, as shown at right. This block diagram has only a single loop, so we can apply Equation 14.4.7 with the following transfer-function polynomials: $N_G = K_a$, $D_G = Js^2$, $N_H = K_p s + K_\theta$, and $D_H = 1$. The result is:

$$\frac{N_G D_H}{D_G D_H + N_G N_H} = \frac{K_a \times 1}{Js^2 \times 1 + K_a \times (K_p s + K_\theta)} = \frac{K_a}{Js^2 + K_a K_p s + K_a K_\theta}$$

Finally, multiplying this loop transfer function by the input transducer sensitivity gives

$$\mathrm{CLTF}(s) \equiv \frac{\Theta(s)}{R(s)} = \frac{K_r K_a}{Js^2 + K_a K_p s + K_a K_\theta} \tag{14.5.2}$$

Let us cast system transfer function Equation 14.5.2 into a form that displays more clearly the fundamental characteristics of the closed-loop system. We again use as a guide the base-excited m-c-k system of Section 13.2. The undamped natural frequency of that system is $\omega_n = \sqrt{k/m}$, and the viscous damping ratio is $\zeta = c/(2\sqrt{mk})$. Using these definitions, we can express transfer function Equation 13.2.3 in terms of these fundamental system characteristics rather than the physical parameters m, c, and k:

$$TF(s) \equiv \frac{X(s)}{X_i(s)} = \frac{k}{ms^2 + cs + k} = \frac{k/m}{s^2 + (c/m)s + k/m} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{14.5.3}$$

Manipulating Equation 14.5.2 in a similar manner, with an additional step to account for the possibility that $K_r \neq K_\theta$, gives:

$$\mathrm{CLTF}(s) \equiv \frac{\Theta(s)}{R(s)} = \frac{K_r}{K_\theta} \times \frac{K_a K_\theta / J}{s^2 + (K_a K_p / J)\,s + K_a K_\theta / J} = \frac{K_r}{K_\theta} \times \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{14.5.4}$$

In Equation 14.5.4 the undamped natural frequency and viscous damping ratio of the rotor-position control system are

$$\omega_n = \sqrt{\frac{K_a K_\theta}{J}} \quad \text{and} \quad \zeta = \frac{K_a K_p}{2J\omega_n} \tag{14.5.5}$$

Equation 14.5.4 differs in form from Equation 14.5.3 only in the presence of pseudo-static multiplier K_r/K_θ, which is defined in Equation 14.3.5.

The system with transfer function Equation 14.5.3 behaves as a control system should: in this case, as a positively damped, 2nd order system. For example, if the parameters in Equation 14.5.4 are selected such that ζ is in the range 0.3 to 0.5, and if the operator setting is step function $r(t) = R_H H(t)$, then the inverse Laplace transform developed in homework Problem 9.12 gives the type of step response shown on the figure at right. The output cannot exactly follow this infinitely fast input, but it rises quickly and settles quickly to the desired pseudo-static response. This example illustrates a general characteristic of control systems: it is generally impossible to achieve absolute accuracy instantly (in this case, the pseudo-static response), but a good control system can achieve practically acceptable accuracy, with a practically acceptable time lag between the input and the desired output.

It was observed earlier that position feedback through the sensor with gain K_θ has the effect of attaching between the operator setting and the rotor inertia an artificial restoring spring with stiffness constant $K_a K_\theta$. We see now, additionally, that rate feedback through the sensor with gain K_p has the effect of imposing upon the absolute motion of the rotor an **artificial** viscous damper with damping constant $K_a K_p$.

Finally, let us note that the favorable and **stable** performance of this rotor-position control system is strongly dependent upon the **positivity** of constants $K_a K_\theta$ and $K_a K_p$. In practice, it is possible (in fact, too easy) to make these constants **negative**, usually by accident or negligence, which could cause the system to be **unstable**. Chapters 16 and 17 examine in more detail the important issue of system stability.

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14.6: Comments Regarding Control Theory

In the words of Brogan, 1974, page 1, “Control theory can be divided into two major categories, classical and modern.” Chapters 14, 15, 16, and 17 of this book constitute an introduction to the former, namely, classical control theory for linear, time-invariant (LTI) systems. Whereas classical control theory focuses on relatively simple single-input-single-output (SISO) systems, modern control theory, which is based upon the *state-space* mathematical representation of systems, is very well suited for design and analysis of control for both simple systems and much more complicated multiple-input-multiple-output (MIMO) systems.[1]

The state-space representation of systems is a generalization of the standard 1st order ODE Equation 1.2.1, $\dot{x} - ax = bu(t)$, which has only the single input $u(t)$ and the single dependent variable $x(t)$. In the generalization to an LTI-MIMO system of, say, r inputs and n dependent variables, scalar $x(t)$ becomes the $n \times 1$ state vector (column matrix) $\mathbf{x}(t)$ of dependent variables, scalar $u(t)$ becomes the $r \times 1$ vector of inputs $\mathbf{u}(t)$, single constant a becomes the $n \times n$ matrix $\mathbf{A}$ of system constants, single constant b becomes the $n \times r$ matrix $\mathbf{B}$ of input constants, and single ODE $\dot{x} - ax = bu(t)$ becomes the **system** of n coupled 1st order ODEs written in matrix form, $\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u}(t)$. All of the 2nd order systems analyzed in this book can be expressed in state-space form with $\mathbf{x}(t)$ being \(2 \times 1) state vectors; for example, the two scalar Equations 1.9.3 and 1.9.4 can clearly be written as the following single matrix ODE:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f_x(t)$$

Further, the 2-DOF systems of Chapters 11 and 12 can be expressed in state-space form with $\mathbf{x}(t)$ being 4×1 state vectors.

Chapters 14, 15, 16, and 17 of this book should provide adequate preparation for study of the fundamental aspects of modern control theory. Three appropriate textbooks, among the many available, are Brogan, 1974, Franklin, *et al.*, 1991, and Ogata, 2001.

[1]Franklin, et al., 1991, page 361, wrote that the adjective “modern” is misleading “... since the state-space method of description for differential equations is over 100 years old and was introduced to control design in the late 1950s, ...” Those authors prefer the designations “transform methods” rather than “classical control” and “state-space methods” rather than “modern control”

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14.7: Chapter 14 Homework

1. In Section 14.3 for the rotor-position control system with position feedback, the transfer function relating output rotor position to input operator setting, with zero disturbance, is derived by a series of block-diagram algebraic operations to resolve (simplify) Figure 14.3.3. In practice, for assessment of control system quality, it is often required also to determine the transfer function that relates output rotor position to the **disturbance**, with zero operator setting. The appropriate block diagram for that task in this case comes from Figure 14.3.2, with $R(s) = 0$, as shown at left. Derive **from this block diagram** the transfer function $\frac{\Theta(s)}{\mathrm{M_d}(s)} = \frac{1}{Js^2 + K_e K_\theta}$. This task is not difficult: just write $\mathrm{M_a}(s)$ in terms of $\Theta(s)$, then $\mathrm{M}(s) = \mathrm{M_a}(s) + \mathrm{M_d}(s)$, and $\mathrm{M}(s) \times 1/Js^2 = \Theta(s)$, which steps should lead you to the required result. This result can also be derived directly from the ODE of motion Equation 14.3.4, but the assignment here is to derive it from the block diagram, to give you practice with block-diagram algebra.

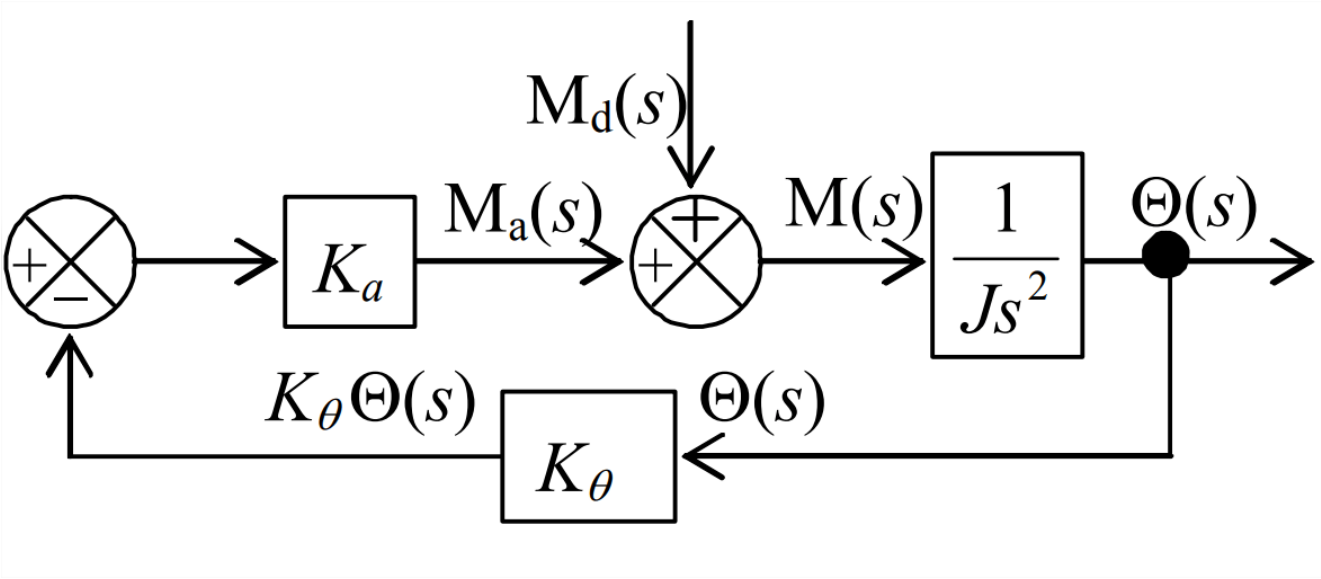


Figure 14.7.1 (Copyright; author via source)

2. In some systems, information is fed **forward** as well as backward. Each of the Laplace block diagrams Figure 14.7.2 and 14.7.3 shown below includes both a *feedforward* branch and a *unity-feedback* branch. Transfer functions $F(s)$ and $G(s)$ represent arbitrary plant, filter, sensor, *etc*. system elements. (We could also include a transfer-function block in the unity-feedback branch, but its value would be simply 1, so it is standard practice to omit that obvious block.) For the part(s) assigned, derive the algebraic equation for the system transfer function, $X(s)/R(s)$, in terms of $F(s)$ and $G(s)$.

1.

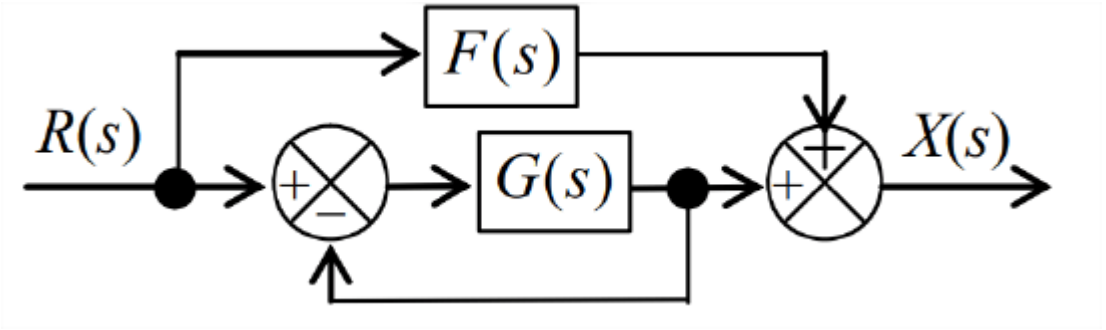


Figure 14.7.2 (Copyright; author via source)

2.

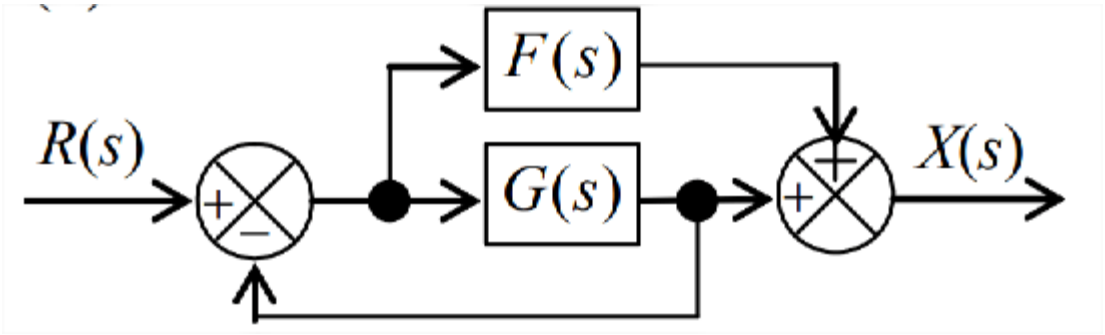


Figure 14.7.3 (Copyright; author via source)

3. The idealized series damper-spring high-pass mechanical filter shown in Figure 3.7.4 has the governing ODE with right-hand-side (RHS) dynamics, $\dot{x}_o + (1/\tau_1)\, x_o = \dot{x}_i(t)$, Equation 3.7.6. The transfer function from this ODE is clearly $X_o(s)/X_i(s) = s/(s + 1/\tau_1)$, where $L[x_i(t)] \equiv X_i(s)$ and $L[x_o(t)] \equiv X_o(s)$. With the definition of a new independent variable $x_d(t)$ and the auxiliary equation $x_o(t) = x_d(t) + x_i(t)$, the ODE is converted into an alternate form without RHS dynamics, $\dot{x}_d + (1/\tau_1)\, x_d = -(1/\tau_1)\, x_i(t)$, Equation 3.7.8. Construct a Laplace block diagram using the alternate ODE and the auxiliary equation; this diagram should have input $X_i(s)$ on the left and output $X_o(s)$ on the right, **and** it should include a feedforward branch. Use block-diagram algebra on this diagram to derive again the system transfer function $X_o(s)/X_i(s) = s/(s + 1/\tau_1)$.

3. Consider a system represented by the somewhat general Laplace block diagram at the right. Constant K_r is an input sensor gain, and transfer functions $G_1(s)$, $G_2(s)$, $H_1(s)$, and $H_2(s)$ represent arbitrary plant, filter, sensor, *etc*. system elements. Use block-diagram algebra to show in detail that the closed-loop transfer function is

$$\mathrm{CLTF}(s) \equiv \frac{X(s)}{R(s)} = \frac{K_r G_1 G_2}{1 + G_1 H_1 + G_1 G_2 H_2}$$

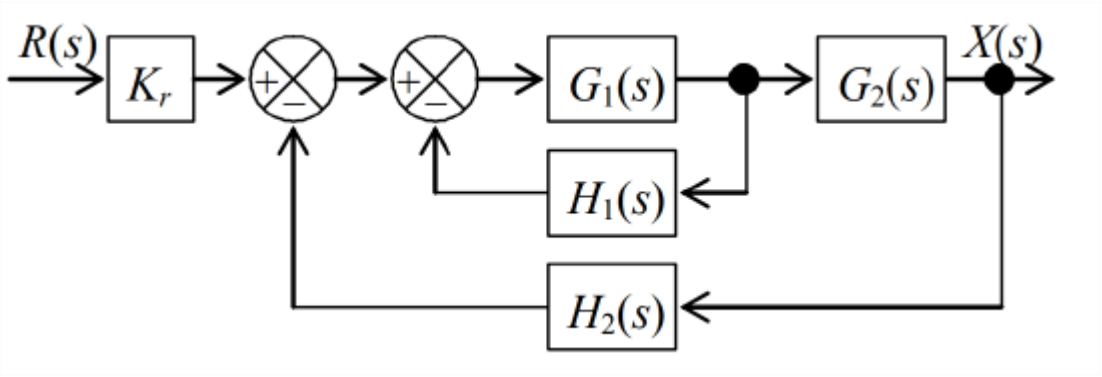


Figure 14.7.4 (Copyright; author via source)

4. Consider a system represented by the block diagram below. The feedback branch is a *unity-feedback* path; we could include a transfer-function block in that branch, but its value would be simply 1, so it is standard practice to omit that obvious block.
   1. Show that the closed-loop transfer function is

$$CLTF(s) = \frac{X(s)}{R(s)} = \frac{10(s+1)}{s^2 + 10s + 10}$$

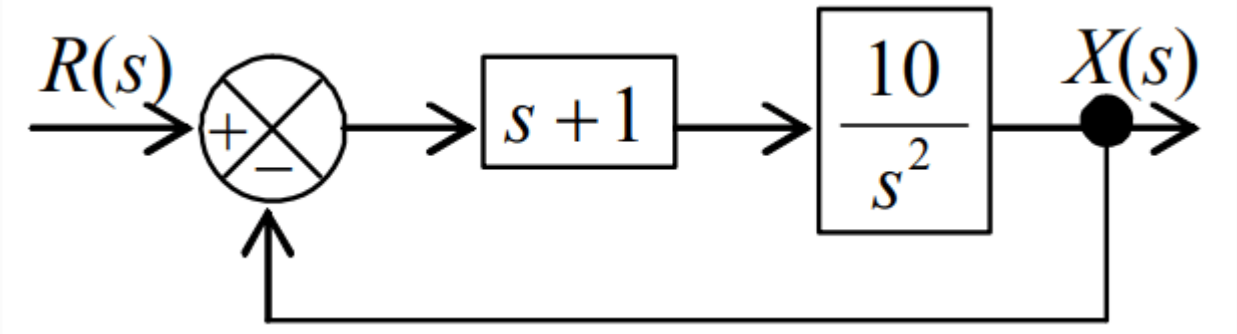


Figure 14.7.5 (Copyright; author via source)

2. Evaluate the denominator quadratic equation to calculate the viscous damping ratio ζ, which should indicate that the closed-loop system is overdamped.
3. Factor the denominator quadratic into the form $(s-p_1)(s-p_2)$, find the values of *poles* p_1 and p_2. (*partial answer*: $p_1 = -1.127$ s^{-1})
4. Suppose that the ICs are zero and that the input is a step function, $r(t) = 5H(t)$. Determine the complete algebraic equation for output $x(t)$. First express $X(s)$ in a partial-fraction expansion, then find the inverse Laplace transform. Use your $x(t)$ equation to find the final value, $\lim_{t\to\infty} x(t)$, after all dynamic motion has damped out. (*partial answer*: one of the terms in the equation for $x(t)$ should be $-5.728e^{-8.873t}$.)

5. Consider position control of a reaction wheel (from Section 3.3) by position feedback plus rate feedback, for which the system Laplace block diagram is:

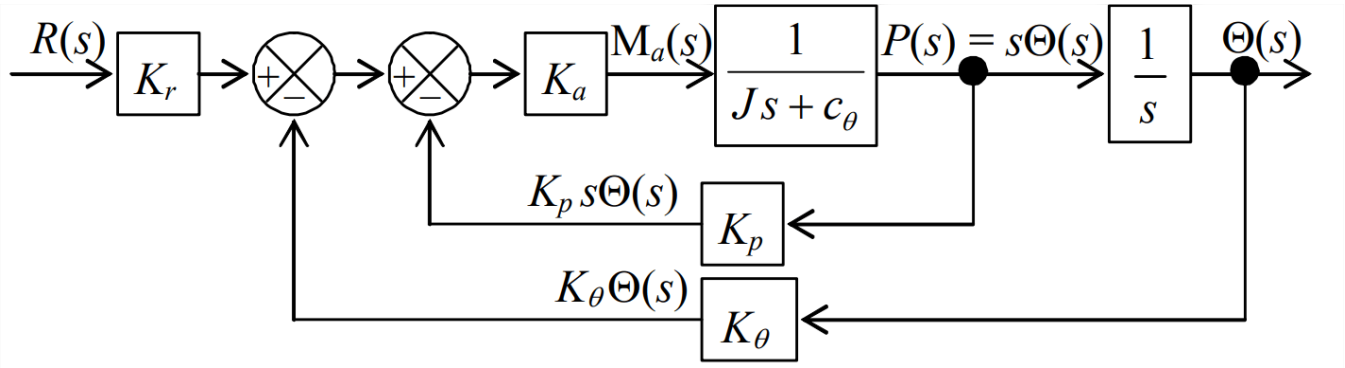


Figure 14.7.6 (Copyright; author via source)

This block diagram is identical to Figure 14.5.8, except that the plant transfer function from Equation 3.3.2 is $1/(Js + c_\theta)$, not just $1/Js$, due to the presence of bearing friction with assumed viscous damping constant c_θ. The following data were measured for a small laboratory reaction-wheel assembly:

$$J = 2.56 \times 10^{-3} \text{lb -inch-sec}^{\,2}, c_\theta = 0.020 \frac{\text{ounce inch}}{\text{rad/s}}, K_a = 0.950 \frac{\text{ounce inch}}{\text{V}}$$

For the case(s) assigned, the control system transducer and sensor sensitivities are as shown in columns 2, 3, and 4 of the following table:

Case	K_r (V/rad)	K_θ (V/rad)	K_p (Vs/rad)	t_r (s)	$\bar{x}_p$
1	15	10	0.36	0.1280	0.3858
2	1.5	1.5	0.27	0.4507	0.1117
3	14	21	0.35	0.08163	0.5355
4	3.5	3.5	0.24	0.2255	0.3260

1. Calculate from the values in table columns 2, 3, and 4 the following quantities for this closed-loop system: pseudo-static multiplier K_r/K_θ, undamped natural frequency ω_n (rad/s), viscous damping ratio ζ, and damped natural frequency ω_d (rad/s).
2. Suppose that the input is a step function, $r(t) = R_H H(t)$, with magnitude $R_H = 30$ degrees. For the subsequent step response, calculate the rise time t_r, the peak time t_p, the maximum overshoot ratio $\bar{x}_p$, and the settling time t_s (see Section 9.8). Also, calculate the **final** static value of wheel position, $\lim_{t\to\infty} \theta(t)$ in degrees, after all dynamic motion has damped out. *Partial answers are given in columns 5 and 6 of the table above to help you know if you are, or are not on the right track, but you still must show all derivations and calculations leading to these answers.*
3. For the same step input as in part 14.5.2, use MATLAB to calculate and plot the step response $\theta(t)$ in degrees from $t = 0$ until at least the settling time $t = t_s$.

6. In Section 14.5 for the rotor-position control system with position feedback and rate feedback, the transfer function relating output rotor position $\Theta(s)$ to input operator setting $R(s)$, **with zero disturbance**, is derived by a series of block-diagram algebraic operations to resolve (simplify) Figure 14.5.8. In practice, for assessment of control system quality, it is often required also to determine the transfer function that relates output rotor position to the **disturbance**, with zero operator setting. The appropriate Laplace block diagram for that task in this case comes from Figure 14.5.7, with $R(s) = 0$, as shown on the figure at right. Derive from this block diagram the algebraic equation for transfer function $\frac{\Theta(s)}{M_d(s)}$.

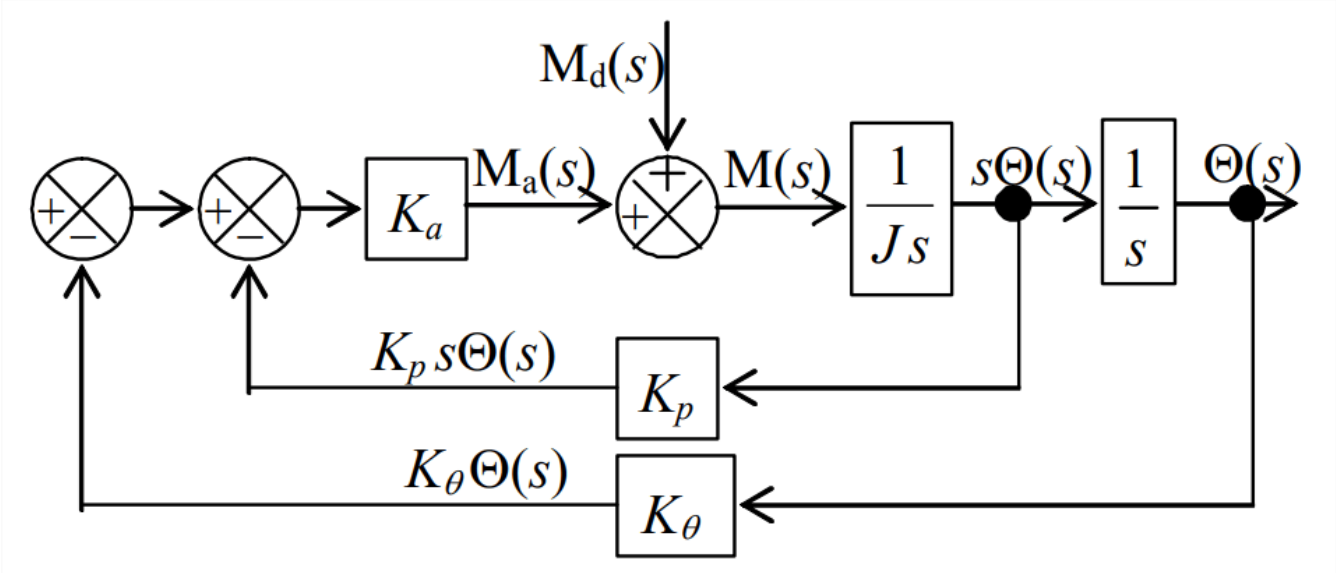


Figure 14.7.7 (Copyright; author via source)

7. Consider a system represented by the Laplace block diagram at right. The outer feedback branch is a *unity feedback* path; we could include a transfer-function block in that branch, but its value would be simply 1, so it is standard practice to omit that obvious block.

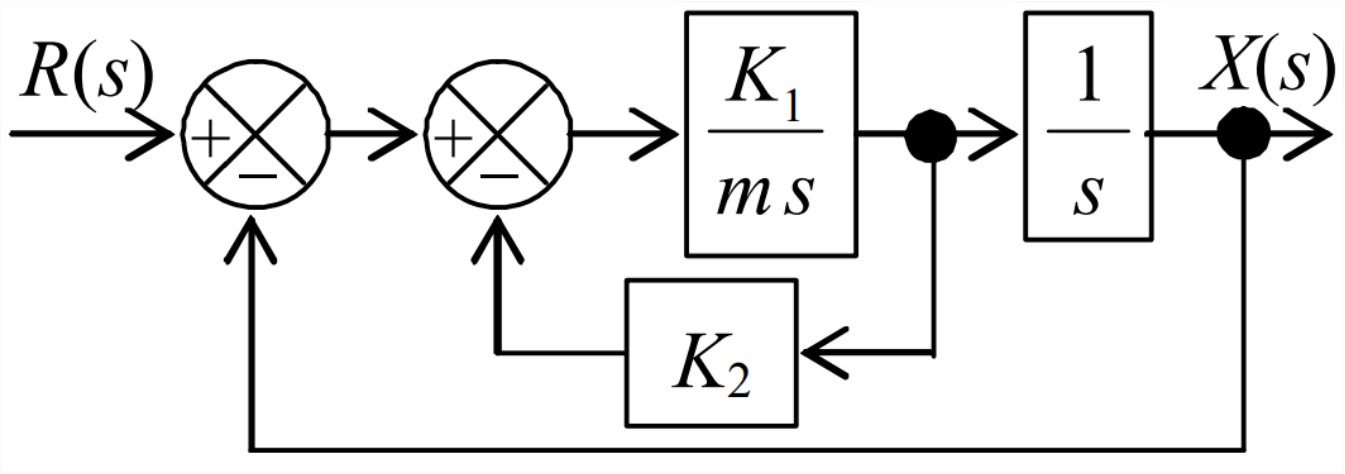


Figure 14.7.8 (Copyright; author via source)

1. Use the result of Problem 14.3 or any other method of your choice to show that the closed-loop transfer function is

$$\frac{X(s)}{R(s)} = \frac{K_1}{ms^2 + K_1K_2s + K_1}$$

2. Suppose that the ICs are zero and that the input to this system is a step function, $r(t) = R_H H(t)$. For the case(s) assigned, the value of the mass-like parameter m (in consistent units) is given in column 2 of the table below. Calculate the values of constants K_1 and K_2 (in consistent units) that will produce the values of maximum overshoot ratio $\bar{x}_p$ and peak time given in columns 3 and 4 of the table (see Section 9.8). Note that this is a design exercise: you are calculating values of system parameters that are intended to produce specified control objectives. Complete the problem by calculating also the rise time t_r and the settling time t_s. *Partial answers are given in columns 5 and 6 of the table below to help you know if you are, or are not on the right track, but you still must show all derivations and calculations leading to these answers.*
3. Let the step magnitude of part 14.7.2 be $R_H = 0.753$ (in consistent units). Calculate the **final** value, $\lim_{t\to\infty} x(t)$, after all dynamic motion has damped out.

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15: INPUT-ERROR OPERATIONS

15.1: Initial Definitions and PID Control
15.2: Examples of P and PI Control
15.3: Derivation of the Final-Value Theorem
15.4: Example of PD Control
15.5: Chapter 15 Homework

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15.1: Initial Definitions and PID Control

Figure 15.1.1 is a general functional block diagram that represents many engineering control systems. The term *input-error operation* can be defined in the context of this diagram: a single output quantity is fed back and differenced from the operator-setting input quantity, and the error signal is manipulated mathematically by a *controller*, also called *control processor* or *filter*, in such a way as to produce effective control. The controller usually consists of analog electronic circuitry and/or a digital processor.

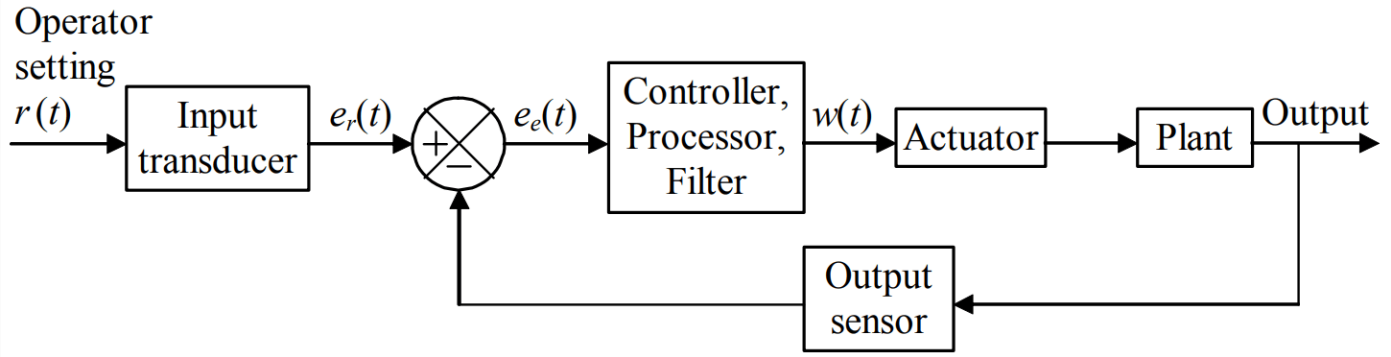


Figure 15.1.1: Functional block diagram of general control system using input-error operations. (Copyright; author via source)

Proportional-integral-derivative (PID) control is a class of input-error operations used widely in industry. Let $e_e(t)$ be the error voltage signal that is input to the controller and $w(t)$ be the output voltage from the controller, as shown on Figure 15.1.1. The mathematical operations performed by an **ideal** PID controller are described by the equation

$$w(t) = P\left[e_e(t) + b_i \frac{1}{\tau_i}\int_{t=0}^{\tau=t} e_e(\tau)d\tau + b_d\tau_d\frac{de_e}{dt}(t)\right] \tag{15.1.1}$$

The physical **constants** in Equation 15.1.1 are *proportional gain* P, *integral time constant* τ_i, and *derivative time constant* τ_d. Each of the binary constants b_i and b_d has dimensionless value 1 or 0, depending upon whether or not integral and/or derivative actions are included in the control. Determining what these constants should be for a particular application is a major part of the control design process. To find the transfer function of the ideal PID controller, we define the Laplace transforms $L\left[e_e(t)\right] \equiv E_e(s)$ and $L[w(t)] \equiv W(s)$. Taking the Laplace transform of Equation 15.1.1 gives the **ideal**-PID-controller transfer function:

$$\frac{W(s)}{E_e(s)} = P\left[1 + b_i\frac{1}{\tau_i s} + b_d\tau_d s\right] \tag{15.1.2}$$

We shall consider in examples three subsets of PID control that are in common usage: for proportional (P) control, only the first term inside the brackets of Equations 15.1.1 and 15.1.2 is used, so $b_i = b_d = 0$; for proportional-integral (PI) control, only the first and second terms are used, so $b_i = 1$ and $b_d = 0$; and for proportional-derivative (PD) control, only the first and third terms are used, so $b_i = 0$ and $b_d = 1$.

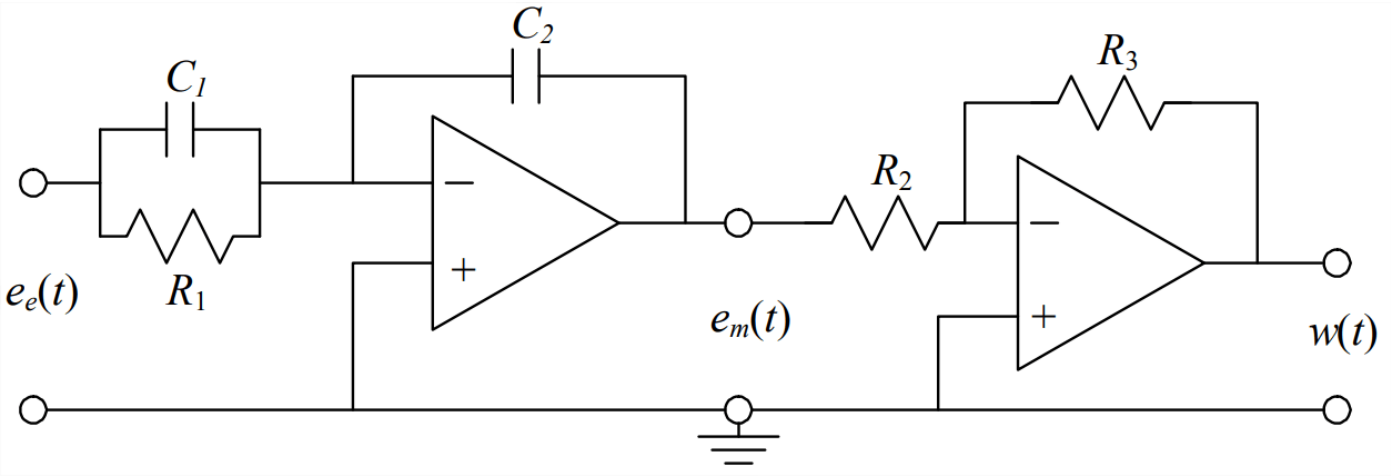


Figure 15.1.2: Electronic analog PI controller. (Copyright; author via source)

To make the concept of PI control, for example, more concrete, Figure 15.1.2 depicts a specific analog electrical circuit that, in principle, acts as an ideal PI controller. (In practice, a great deal of electronic refinement and conditioning is required to produce a circuit that behaves even close to ideally.) By applying the methods of circuit analysis described in Chapter 5 (homework Problem 5.11), we can show that the PI ($b_i = 1$ and $b_d = 0$) constants in Equations 15.1.1 and 15.1.2, expressed in terms of the circuit parameters of Figure 15.1.2, are:

$$P = \frac{R_3 C_1}{R_2 C_2}, \quad \tau_i = R_1 C_1 \tag{15.1.3}$$

The detailed design of PID controllers involves a great deal of engineering art as well as engineering science, and it has been developed extensively; see, *e.g.*, Ogata, 2001, Chapter 10. The present chapter is just an introduction to PID control, so we will not delve deeply into design details; instead, we shall explore some of the more general characteristics of P, PI, and PD types of control in the context of relatively simple examples.

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15.2: Examples of P and PI Control

Figure 15.2.1 on the next page depicts a mechanical plant that is appropriate for the purpose of illustrating P control and PI control. A lightweight rotor is immersed in a viscous liquid, so that the damping moment imposed upon the rotor is $-c_\theta\dot{\theta}$. The rotor is attached to the mechanical operator-setting device (a control wheel, for example) through a highly flexible torsion spring that has rotational stiffness constant k_θ. The rotor has sufficiently low rotational inertia J, and viscous damping constantt c_θ is sufficiently large, that we assume inertial moment $-J\ddot{\theta}$ to be negligible in comparison with the damping and structural moments.

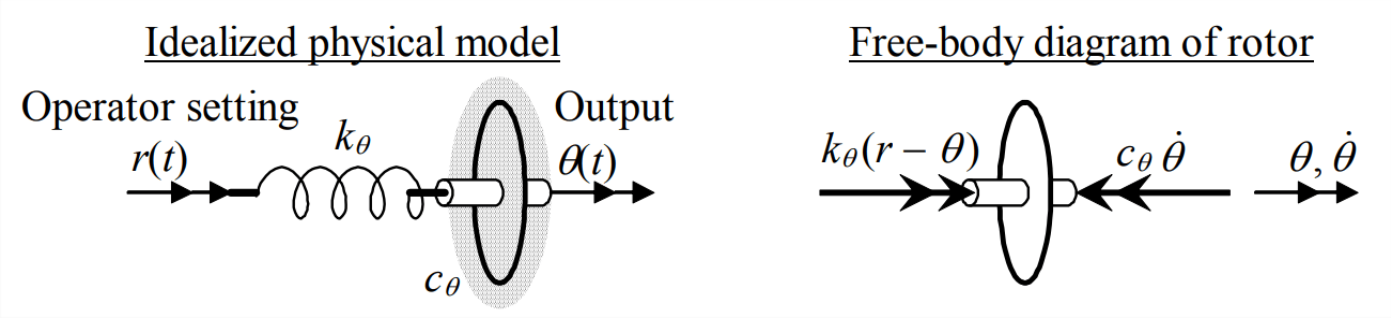


Figure 15.2.1: Lightweight rotor immersed in a viscous liquid, with connection to the input through a soft torsion spring. (Copyright; author via source)

The equation of motion comes from the FBD on Figure 15.2.1:

$$k_\theta(r-\theta)-c_\theta\dot{\theta}=J\ddot{\theta}\approx 0\Rightarrow\frac{c_\theta}{k_\theta}\dot{\theta}+\theta=r\Rightarrow\tau_1\dot{\theta}+\theta=r(t),\ \text{where}\ \tau_1=\frac{c_\theta}{k_\theta} \tag{15.2.1}$$

Taking the Laplace transform of Equation 15.2.1 gives the basic plant transfer function:

$$\text{PTF}(s)=\frac{L[\theta(t)]}{L[r(t)]}\equiv\frac{\Theta(s)}{R(s)}=\frac{1}{\tau_1 s+1} \tag{15.2.2}$$

Clearly, this idealized open-loop mechanical plant by itself is a simple 1st order system with time constant $\tau_1=c_\theta/k_\theta$; furthermore, this system is the rotational equivalent of the translational series damper-spring low-pass filter described by Figure 3.7.4 and Equation 3.7.5.

The input to the plant is operator-setting angle $r(t)$, and the output is rotor position $\theta(t)$. From ODE Equation 15.2.1, we see that the pseudo-static output is exactly equal to the input, $\theta_{ps}(t)=r(t)$. So let us specify that **our control objective for this system is to make the *actual* output $\theta(t)$ be as close as possible to input $r(t)$**. To develop a physical feel for the behavior of the open-loop plant in Figure 15.2.1, let us suppose that the operator quickly moves the control wheel to one side so that input $r(t)$ is close to a step function. The torsion spring will immediately be wound up more tightly, and will impose a moment upon the rotor. However, as we can visualize intuitively, the viscous restraint upon the rotor will prevent it from moving as quickly as the input, and the rotor angle will lag the input. As time goes on and the torsion spring gradually unwinds, the rotor angle will eventually settle to the input angle.

Let us suppose that the open-loop response is too slow for our purposes, and that we want to speed-up the response by using sensors, an actuator, feedback, and P or PI control in the system depicted on the next page, in functional form on Figure 15.2.2 , and as a corresponding Laplace block diagram on Figure 15.2.3. The transfer function on Figure 15.2.3 for the P or PI controller comes from Equation 15.1.2, and the transfer function for the rotor-spring-liquid plant comes from Equation 15.2.2. The input transducer and the output position sensor shown on Figures 15.2.2 and 15.2.3 are familiar from Chapter 14 (on Figure 14.3.1, for example). But the “position actuator” is something new. Whereas the output of most motors is torque, the output of this position actuator is specified to be rotational position. To make an actuator behave in this manner requires a feedback control system within the actuator itself; this is a characteristic of control systems that we now **pause** to examine.

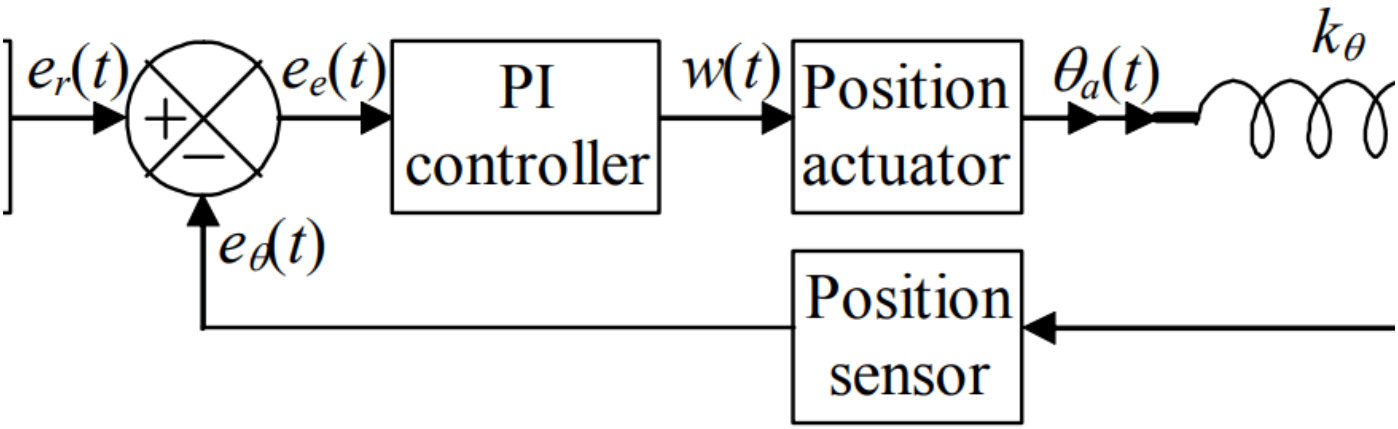


Figure 15.2.2: Functional diagram of a system for controlling the position of a spring-connected rotor in a viscous liquid, based on proportional-integral control. (Copyright; author via source)

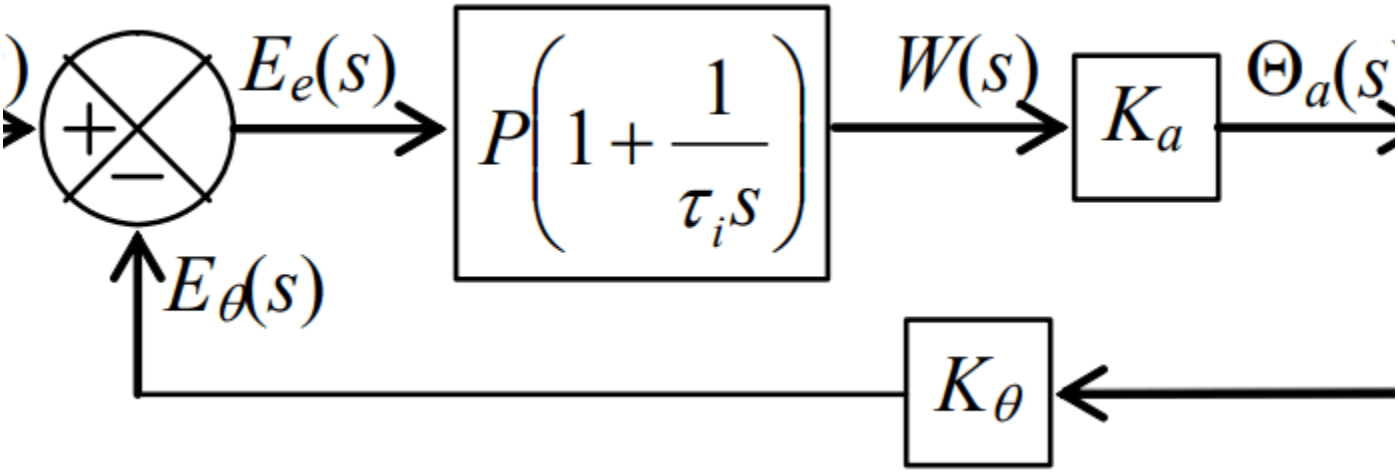


Figure 15.2.3: Laplace block diagram of a system for controlling the position of a spring-connected rotor in a viscous liquid, based on proportional-integral. (Copyright; author via source)

In fact, most actuators and sensors of all types are themselves complicated dynamic systems, often with internal feedback circuits. But this begs the following question: if there are control systems within actuators and/or sensors, then why are not the details of these systems included in the device or sub-system transfer functions shown on block diagrams such as Figure 15.2.3?

A general response to this question is based upon the relative "speeds" of different systems. To illustrate this concept of system speed, we imagine that two standard 2nd order systems, A and B, have equivalent values of subcritical viscous damping ratio ζ, but that System A's undamped natural frequency is 100 times that of System B, $\omega_{nA}/\omega_{nB} = 100$. Consequently, System A's settling time for step inputs, $t_{sA} = 4/(\zeta\omega_{nA})$ [Equation 9.8.11], is only 0.01 that of System B; in this case, we regard "the dynamics of System A" as being "100 times faster" than "the dynamics of System B." Engineers usually select for a control system sensors[1] and, when possible, actuators that are much faster than the dynamics of the plant under control; in such cases, we usually **approximate** the behavior of a sensor or actuator with a simple linear sensitivity constant, in equations such as $e_r(t) = K_r r(t)$, $e_\theta(t) = K_\theta\theta(t)$, and $\theta_a(t) = K_a w(t)$, all of which are represented on Figure 15.2.3. In these cases, we are making the **approximation** that the dynamics of the sensors and actuators are essentially infinitely fast and are independent of, or "uncoupled from" the dynamics of the plant.

Be aware, though, that the full dynamics of devices used for control, especially actuators, **cannot always** be considered uncoupled from the dynamics of the plant. For one example, the dynamics of a reaction-mass actuator (homework Problem 10.15) in a vibration-control system are usually inextricably coupled with those of the structural plant, as in homework Problem 12.6. For

another example, homework Problem 15.7 is an exercise of incorporating the dynamics of a position actuator into a control-system model; in this case, the process involves even more than just inserting the actuator's transfer function in series within the block diagram of the overall system. In yet another example, Dettmer, 1995, Section 1.3.4 describes a control system for which it was necessary to account for the dynamic characteristics of both sensors and actuators.

Returning now to the rotor-position control system of Figures 15.2.2 and 15.2.3, our objective is to make the output follow the input, $\theta(t) \approx r(t)$, so it is appropriate to define a revised (relative to Figure 15.2.1) open-loop configuration. If the feedback loop in Fig. 15- 5 were broken, and if the PI controller were not present (P = 1, b_i = 0), then the transfer function for this open-loop system clearly would be $K_r \times K_a \times 1/(\tau_1 s + 1)$. We want this to be the same as that of the bare (devoid of control devices) plant, Equation 15.2.2, so let us suppose that we have the ability to set the sensitivities of the input transducer and the position actuator so that $K_r \times K_a = 1$. Also, it will greatly simplify the algebra in the following developments if we suppose further that we can use identical sensors on the input angle and the output angle in the feedback path, so that $K_r = K_\theta$. To summarize, we shall apply the following restrictions on the device sensitivities:

$$K_r \times K_a = 1 \quad \text{and} \quad \frac{K_\theta}{K_r} = 1 \tag{15.2.3}$$

To compare open-loop and closed-loop system performances, let us consider response (from zero initial conditions) to step input $r(t) = R_H H(t) \Rightarrow R(s) = R_H/s$. Then, **for the open-loop system**, using transfer function Equation 15.2.2 and partial-fraction expansion gives

$$\Theta(s) = R_H \frac{1}{s} \frac{1}{\tau_1 s + 1} = R_H \left(\frac{1}{s} - \frac{1}{s + 1/\tau_1} \right) \tag{15.2.4}$$

Taking the inverse Laplace transform gives the **step response of the open-loop system**:

$$\theta(t) = R_H \left(1 - e^{-t/\tau_1} \right), \text{ for } t \geq 0 \tag{15.2.5}$$

Equation 15.2.5 is simple 1st order step response of the type shown on Figure 3.4.2. At the end of this section, there is a comparative numerical evaluation of this open-loop response and step responses for P and PI types of closed-loop control. For now, though, let us just observe from Equation 15.2.5 that the asymptotic or "final" value of response for times much greater than time constant τ_1 equals the magnitude of the step input, as required:

$$\lim_{t \to \infty} \theta(t) = R_H \tag{15.2.6}$$

Next, let us analyze **closed-loop proportional (P) control**, for which $b_i = 0$ on Figure 15.2.3. Thus, for the closed loop on Figure 15.2.3, we have for Equation 14.4.6 the polynomials $N_G = P \times K_a$, $D_G = \tau_1 s + 1$, $N_H = K_\theta$, and $D_H = 1$, which we use to find the following $CLTF(s)$:

$$\frac{\Theta(s)}{R(s)} = K_r \times \frac{N_G D_H}{D_G D_H + N_G N_H} = \frac{K_r P K_a}{\tau_1 s + 1 + P K_a K_\theta} = K_r K_a \frac{P}{\tau_1 s + 1 + K_r K_a \frac{K_\theta}{K_r} P}$$

With simplifying assumptions Equation 15.2.3, this becomes

$$CLTF(s) = \frac{\Theta(s)}{R(s)} = \frac{P}{\tau_1 s + 1 + P} = \frac{P}{1+P} \frac{1}{\left(\frac{\tau_1}{1+P}\right) s + 1} \equiv \frac{P}{1+P} \frac{1}{\tau_{1P} s + 1} \tag{15.2.7}$$

in which the **new** time constant for the proportionally controlled system is

$$\tau_{1P} \equiv \frac{\tau_1}{1+P} \tag{15.2.8}$$

For step input $r(t) = R_H H(t) \Rightarrow R(s) = R_H/s$, Equation 15.2.7 gives the transform

$$\Theta(s) = \frac{R_H}{s} \times CLTF(s) = R_H \frac{P}{1+P} \times \frac{1}{s} \frac{1}{\tau_{1P} s + 1} \tag{15.2.9}$$

Hence, we find the **step response of the proportionally controlled system** just by comparing Equation 15.2.9 with Equation 15.2.4 and then modifying Equation 15.2.5 appropriately:

$$\theta(t) = R_H \frac{P}{1+P}\left(1 - e^{-t/\tau_{1P}}\right), \text{ for } t \geq 0 \tag{15.2.10}$$

The asymptotic or "final" value of proportionally controlled step response for times much greater than time constant τ_{1P} is:

$$\lim_{t\to\infty} \theta(t) = R_H \frac{P}{1+P} \tag{15.2.11}$$

This situation provides the opportunity to introduce the *final-value theorem* and to illustrate its application; this is a useful mathematical tool that is derived in the next section. This theorem allows us to find the finite asymptotic value of a time function $f(t)$, assuming that such a value exists, based only on the Laplace transform $L[f(t)] = F(s)$, without requiring that the equation for $f(t)$ be available:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)] \tag{15.2.12}$$

For step response of the proportionally controlled system, the final-value theorem and Equation 15.2.9 give

$$\lim_{t\to\infty} \theta(t) = \lim_{s\to 0} s \times \Theta(s) = \lim_{s\to 0}\left(s \times R_H \frac{P}{1+P}\frac{1}{s}\frac{1}{\tau_{1P}s+1}\right) = \lim_{s\to 0}\left(R_H \frac{P}{1+P}\frac{1}{\tau_{1P}s+1}\right) = R_H\frac{P}{1+P}$$

This is another, easier method for finding result Equation 15.2.11. The final-value theorem was not needed to derive the correct asymptotic solution for this simple case, but it will be a useful, labor-saving tool for more complicated cases in the future.

Comparing Equation 15.2.11 with the required asymptotic value $\lim_{t\to\infty}\theta(t) = R_H$, Equation 15.2.6, we see that, because $P/(1+P) \neq 1$, the proportionally controlled system **fails** to satisfy the requirement exactly. Let us examine possible values of proportional gain P. Equation 15.2.8 requires that $P > -1$ so that the system is stable, $\tau_{1P} > 0$. Equation 15.2.10 requires that $P > 0$ so that the response has the same sign as the input. Moreover, P should be a positive number as large as possible, within practical hardware limitations, because

1. the greater the value of P, the smaller is time constant τ_{1P}, making the system faster than the open-loop system, and
2. the greater the value of P, the closer is the asymptotic value of the output to the value of the input, from Equation 15.2.11.

The value $P = 4$ is a commonly used proportional gain. But even for a large, positive value of P, the steady-state response will be at least a little smaller than the input. The *steady-state output error*, also known as *offset*, is defined as

$$\Delta\theta_{\text{offeet}} \equiv R_H - R_H\frac{P}{1+P} = \frac{R_H}{1+P} \tag{15.2.13}$$

A large value, $P > 9$, is required to reduce the offset to less than 10% of the desired steady-state output.

There is a simple explanation for the offset produced by proportional (P) control, which also will suggest a remedy. Block diagrams Figure 15.2.2 and Figure 15.2.3 show that the final value of θ_a must be non-zero, equal to the **final** value of output θ. In other words, the torsion spring must be completely relaxed (unloaded) at the end of the control process, which is obvious physically. Furthermore, by tracing backwards on the forward branches of the loops on Figures 15.2.2 and 15.2.3, we find the other relevant **final** values: $w = \theta_a/K_a = \theta/K_a$, and, with $b_i = 0$, $e_e = w/P = \theta_a/(K_aP) = \theta/(K_aP) \neq 0$. Ideally, the final error signal $e_e = e_r - e_\theta$ should be zero, corresponding to the output being exactly the desired final value; however, the preceding argument shows that the final error for P control alone is **inevitably** non-zero. This argument also suggests that we seek some way to make the final error equal zero, $e_e = 0$, but still to preserve non-zero input into the position actuator, $w \neq 0$, so that $\theta_a \neq 0$, and so that the final output equals the desired value, $\theta = r$. Integral (I) action can produce this effect: if we add a term to w that is proportional to the **integral** of the error signal, $\int_{\tau=0}^{\tau=t} e_e(\tau)d\tau$, then we can have a final non-zero control signal, $w \neq 0$, even though the final error itself is zero, $e_e = 0$.

Therefore, let us next find the closed-loop transfer function for **closed-loop proportional-integral** (PI) control ($b_i = 1$). For the closed loop on Figure 15.2.3, the forward-branch transfer function is $G = P\left(1 + \frac{1}{\tau_i s}\right) \times K_a \times \frac{1}{\tau_1 s+1} = \frac{PK_a(\tau_i s+1)}{\tau_i s(\tau_1 s+1)}$, so $N_G = PK_a(\tau_i s+1)$ and $D_G = \tau_i s(\tau_1 s+1)$. For the feedback branch, $N_H = K_\theta$ and $D_H = 1$. Therefore, Equation 14.4.6 gives the following $CLTF(s)$):

$$\frac{\Theta(s)}{R(s)} = K_r \times \frac{N_G D_H}{D_G D_H + N_G N_H} = \frac{K_r P K_a (\tau_i s + 1)}{\tau_i s (\tau_1 s + 1) + P K_a (\tau_i s + 1) K_\theta}$$
$$= K_r K_a \frac{P (\tau_i s + 1)}{\tau_i s (\tau_1 s + 1) + K_r K_a \frac{K_\theta}{K_r} P (\tau_i s + 1)}$$

With simplifying assumptions Equation 15.2.3, this becomes

$$\frac{\Theta(s)}{R(s)} = \frac{P (\tau_i s + 1)}{\tau_i s (\tau_1 s + 1) + P (\tau_i s + 1)} = \frac{P (\tau_i s + 1)}{\tau_i \tau_1 s^2 + (1 + P) \tau_i s + P} \tag{15.2.14}$$

Before analyzing Equation 15.2.14 in more detail, let us first determine if, as predicted above, PI control eliminates the P-control offset of Equation 15.2.13. For step input $r(t) = R_H H(t) \Rightarrow R(s) = R_H / s$, Equation 15.2.14 gives the transform

$$\Theta(s) = \frac{R_H}{s} \frac{P (\tau_i s + 1)}{\tau_i s (\tau_1 s + 1) + P (\tau_i s + 1)} \tag{15.2.15}$$

Applying final-value theorem Equation 15.2.12 to Equation 15.2.15 gives

$$\lim_{t \to \infty} \theta(t) = \lim_{s \to 0} s \times \Theta(s) = \lim_{s \to 0} \left(\frac{R_H P (\tau_i s + 1)}{\tau_i s (\tau_1 s + 1) + P (\tau_i s + 1)} \right) = \lim_{s \to 0} \left(R_H \frac{P}{P} \right) = R_H \tag{15.2.16}$$

Therefore, integral action does indeed eliminate the error that is inevitable from purely proportional control.

Returning to transfer function Equation 15.2.14, we see that integral action has increased the order of the system to 2nd order from the 1st order of the plant. Accordingly, we can re-write the transfer function in terms of parameters appropriate for a 2nd order system:

$$\frac{\Theta(s)}{R(s)} = \frac{\frac{P}{\tau_i \tau_1} (\tau_i s + 1)}{s^2 + \frac{1+P}{\tau_1} s + \frac{P}{\tau_i \tau_1}} \equiv \frac{\omega_n^2 (\tau_i s + 1)}{s^2 + 2 \zeta \omega_n s + \omega_n^2} \tag{15.2.17}$$

in which the undamped natural frequency and the viscous damping ratio are, respectively,

$$\omega_n = \sqrt{\frac{P}{\tau_i \tau_1}} \text{ and } \zeta = \frac{1}{2 \omega_n} \frac{1 + P}{\tau_1} = \frac{1}{2} \sqrt{\frac{\tau_i}{\tau_1}} \frac{1 + P}{\sqrt{P}} \tag{15.2.18}$$

Note that this system is **not** a **standard** 2nd order system, as defined in Equation 9.2.2. The term $\omega_n^2 \tau_i s$ in the numerator of Equation 15.2.17 makes this system non-standard; that term corresponds to right-hand-side dynamics in the ODE describing the system, which is $\ddot{\theta} + 2 \zeta \omega_n \dot{\theta} + \omega_n^2 \theta = \omega_n^2 (\tau_i \dot{r} + r)$. The non-standard character means, among other things, that the step-response specifications derived in Section 9.8 do **not** apply exactly for this system.

For comparison with the previous cases of open-loop and P control, let us solve for response of the PI-controlled system to step input $r(t) = R_H H(t) \Rightarrow R(s) = R_H / s$. Equation 15.2.17 gives the transform

$$\Theta(s) = \frac{R_H}{s} \frac{\omega_n^2 (\tau_i s + 1)}{s^2 + 2 \zeta \omega_n s + \omega_n^2} = R_H \left[\frac{\omega_n^2 \tau_i}{\omega_d} \frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} + \frac{1}{s} \frac{\omega_n^2}{(s + \zeta \omega_n)^2 + \omega_d^2} \right] \tag{15.2.19}$$

The second form of Equation 15.2.19 is written subject to the condition $|\zeta| < 1$, so that $\omega_d^2 \equiv \omega_n^2 (1 - \zeta^2) > 0$; since $\zeta > 0$ from Equation 15.2.18, this means that we assume the system to be underdamped, at least initially. With Equation 15.2.19 in the second form, we can apply the following inverse Laplace transforms. First, Equations 9.3.5, 9.3.8, and 9.3.9 give:

$$L^{-1} \left[\frac{\omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} \right] = e^{-\zeta \omega_n t} \sin \omega_d t \tag{15.2.20}$$

Next, Equations 9.3.5 and 9.6.5 give (see homework Problem 9.12):

$$L^{-1} \left[\frac{1}{s} \frac{\omega_n^2}{(s + \zeta \omega_n)^2 + \omega_d^2} \right] = 1 - e^{-\zeta \omega_n t} \left(\cos \omega_d t + \frac{\zeta \omega_n}{\omega_d} \sin \omega_d t \right) \tag{15.2.21}$$

Substituting Equation 15.2.20 and Equation 15.2.21 into the inverse transform of Equation 15.2.19, then combining terms appropriately, gives the **step response of the PI-controlled *under*damped system**:

$$\theta(t) = R_H \left[1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta\omega_n - \omega_n^2 \tau_i}{\omega_d} \sin \omega_d t \right) \right], \text{ for } 0 \leq \zeta < 1 \text{ and } 0 \leq t \tag{15.2.22}$$

In the numerical example to follow, we shall encounter cases of both underdamping and overdamping; therefore, it is appropriate to apply conversion Equation 9.10.2 to underdamped response equation Equation 15.2.22 in order to write the following equation for **step response of the PI-controlled *over*damped system**:

$$\theta(t) = R_H \left[1 - e^{-\zeta\omega_n t} \left(\cosh \mu_d t + \frac{\zeta\omega_n - \omega_n^2 \tau_i}{\mu_d} \sinh \mu_d t \right) \right], \text{ for } 1 < \zeta \text{ and } 0 \leq t \tag{15.2.23}$$

in which $\mu_d = \omega_d \sqrt{\zeta^2 - 1}$.

Now that we have derived the equations for step response, let us evaluate a numerical case with physically plausible parameters; in particular, let us calculate and compare the step responses: Equation 15.2.5 for the open-loop system; Equation 15.2.10 for the P-controlled system; and Equation 15.2.22 or Equation 15.2.23 for the PI-controlled system. We let the time constant of the open-loop system have the value $c_\theta / k_\theta \equiv \tau_1 = 1$ s, and we use the typical value $P = 4$ for the proportional gain. For the PI-controlled system, we consider three different integral time constants, $\tau_i =$ 1/3, 2/3, and 1 s. The following is a MATLAB M-file that calculates the step responses and plots them on a single graph:

```
%MATLABdemo151.m

%1st order system: spring-connected rotor in a viscous fluid

%Step responses: open-loop, P-controlled, and PI-controlled

t=0:.01:4;

tau1=1;%time constant of open-loop system = 1 sec

thOL=1-exp(-t/tau1);%open-loop step response ratio

P=4;tau1P=tau1/(1+P);

thP=P/(1+P)*(1-exp(-t/tau1P));%step response ratio, P control

plot(t,thOL,t,thP),hold

taui=[1/3 2/3 1];%step response ratios, three cases of PI control

for ni=1:3,

    wn=sqrt(P/(taui(ni)*tau1));zeta=(1+P)/(2*wn*tau1);

    sig=zeta*wn;case_taui_zeta=[ni taui(ni) zeta]

    if zeta < 1

        wd=wn*sqrt(1-zeta^2);

        thPI(ni,:)=1-exp(-sig*t).*(cos(wd*t)+(sig-wn^2*taui(ni))/wd*sin(wd*t));

    else

        md=wn*sqrt(zeta^2-1);

        thPI(ni,:)=1-exp(-sig*t).*(cosh(md*t)+(sig-wn^2*taui(ni))/md*sinh(md*t));

    end
```

```
    plot(t,thPI(ni,:),'k')
end
grid,xlabel('Time t (sec)'),ylabel('Step response ratio \theta(t)/R_H')
title('Position control of a spring-connected rotor in a viscous fluid')
```

The MATLAB command and resulting output are printed below. The three cases shown are for PI control with integral time constant ("taui") τ_i = 1/3, 2/3, and 1 sec; note the values of viscous damping ratio ("zeta") ζ: the system is underdamped for τ_i = 1/3 s, but overdamped for τ_i = 2/3 and 1 s.

```
>> MATLABdemo151
Current plot held
case_taui_zeta =
    1.0000 0.3333 0.7217
case_taui_zeta =
    2.0000 0.6667 1.0206
case_taui_zeta =
    3.0000 1.0000 1.2500
```

Figure 15.2.4 is the graphical output from the MATLAB M-file.

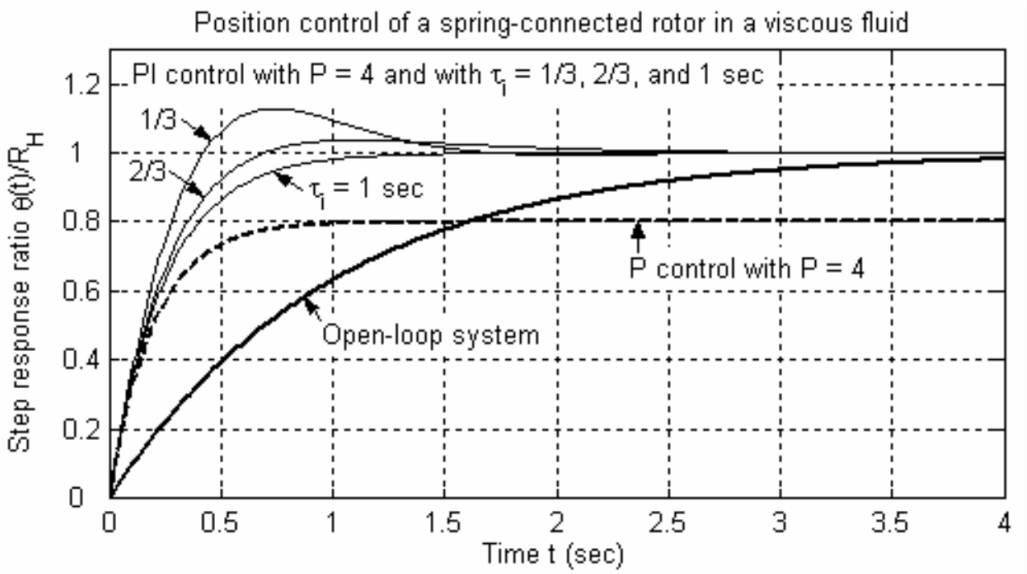


Figure 15.2.4: Step responses for no control (open-loop), proportional (P) control, and proportional-integral (PI) control. (Copyright; author via source)

Figure 15.2.4 illustrates clearly the results derived previously regarding the time constants for open-loop and P control, and the final values of step response for open-loop, P control and PI control. The new and interesting results on Figure 15.2.4 relate to step response of the system under PI control. For all three values of τ_i, the speed of response is much faster than that of the open-loop system, and comparable to that for P control. Moreover, there is a design trade-off for this 2nd order system between rise time and overshoot. Recall from Section 9.8 that rise time is the time required for the response first to reach the desired value, and overshoot is the maximum amount by which the response exceeds the desired value. For most cases, the control designer would like to make rise time as fast as possible, and to minimize overshoot. However, Figure 15.2.4 shows that we cannot simultaneously

do both in this case: rise time is fastest for small τ_i, but overshoot is minimized or eliminated with larger τ_i. Therefore, the control designer must compromise and select a value of τ_i that produces practically **acceptable** values of both rise time and overshoot, even though neither response parameter would be the best possible.

It is interesting to observe also from Figure $15.2.4$ that, for this non-standard 2nd order system, there is overshoot for τ_i = 2/3 s $\Rightarrow \zeta$ = 1.02; in other words, there is overshoot even though the system is overdamped. This is different from the step response of a positively damped standard 2nd order system (Section 9.8), for which overshoot can occur only if $0 \leq \zeta < 1$.

[1]Examples that appear in this book of such sensors are the accelerometer [homework Problem 10.12.2], the rate gyroscope (Example 9.2.2 in Section 9.2, and homework Problem 9.18), and the rate-integrating gyroscope (homework Problem 9.19).

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15.3: Derivation of the Final-Value Theorem

Consider a continuous physical function $f(t)$, with continuous derivative df/dt, and with Laplace transform $L[f(t)] = F(s)$. The final-value theorem expresses the final, steady-state value of $f(t)$ in terms of $F(s)$ as:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)] \tag{15.3.1}$$

This theorem is useful for finding the final value because it is almost always easier to derive the Laplace transform and evaluate the limit on the right-hand side, than to derive the equation for $f(t)$ and evaluate the limit on the left-hand side. Final-value theorem Equation 15.3.1 is valid **provided** that $\lim_{t\to\infty} f(t)$ exists (*i.e.*, is a finite, constant value). But we must apply Equation 15.3.1 with care, because the theorem itself fails to distinguish between functions for which the limit exists and functions which have no limit. Indeed, the theorem can predict falsely that an unstable system has a limit when, in fact, there is none, *i.e.*, that $\lim_{t\to\infty} f(t) \to \pm\infty$.

Derivation of the final-value theorem is based upon definition Equation 2.2.5 of a Laplace transform and the Laplace transform Equation 2.2.9 of a derivative:

$$L\left[\frac{d}{dt}f(t)\right] = \int_{t=0}^{t=\infty} e^{-st}\left[\frac{d}{dt}f(t)\right]dt = sF(s) - f(0)$$

Taking the limit of all terms as $s \to 0$ gives

$$\lim_{s\to 0} L\left[\frac{d}{dt}f(t)\right] = \int_{t=0}^{t=\infty} 1\times\left[\frac{d}{dt}f(t)\right]dt = \lim_{s\to 0}[sF(s)] - f(0)$$

Now the integral is evaluated easily:

$$\int_{t=0}^{t=\infty}\left[\frac{d}{dt}f(t)\right]dt \equiv f(\infty) - f(0) = \lim_{s\to 0}[sF(s)] - f(0)$$

$$\Rightarrow \quad f(\infty) \equiv \lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$$

This completes the derivation of final-value theorem Equation 15.3.1, which was applied earlier to re-derive result Equation 15.2.12 and in Equation 15.2.18.

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15.4: Example of PD Control

Next, we consider the effect of derivative control, the third type of operation on the error signal indicated in Equations 15.1.1 and 15.1.2. The derivative term (with $b_d = 1$) in the actuator signal $w(t)$ is $P\tau_d \times de_e/dt$. This term is proportional to the **rate of change** of the error signal, so it anticipates, in a sense, any increase in the error, and it acts to oppose that increase. Derivative action is used in combination with proportional action or proportional-integral action, but not by itself. Whereas proportional action has the effect of a restoring linear spring, derivative action has the effect of a viscous damper. Therefore, derivative control by itself would fail to provide action that forces system output toward the desired value.

A reality check is appropriate at this point. The derivative output $x_d(t) \equiv \dot{e}_e$ of an **ideal** PID controller in Equation 15.1.1 is physically unrealistic, because it is not possible to measure exactly, in real time, the derivative $\dot{e}_e$ of the error signal. Exact real-time measurement of a derivative at an instant would require information about e e& future values of the signal, but **future** values are unknown. Another manifestation of the unrealism of the derivative term is the associated transfer function, $X_d/E_e = s/1$, which is written here as a fraction on the right-hand side so as to emphasize that the polynomial order (in s) of the numerator is higher than that of the denominator. Thus, this transfer function is *acausal*, meaning that the current output of the ideal differentiator must be dependent upon future, as well as past and present, values of the error signal (see Bélanger, 1995, p. 440).

Since the exact differentiator of Equation 15.1.1 cannot be realized physically, an actual PD controller often includes an **approximate** differentiator in the form of a 1st order *high-pass filter* described by the ODE $\varepsilon\tau_d\dot{x}_d + x_d = \dot{e}_e$; in this ODE, τ_d is the derivative time constant, and $\mathcal{E}$ is a small positive number typically selected to be in the range 0.1 to 0.3 (Ogata, 2001, pages 700 and 727). The transfer function of this approximate differentiator is $X_d(s)/E_e(s) = s/(\varepsilon\tau_d s + 1)$, which is *causal*, therefore physically realizable. For error signals that are slowly varying in time relative to the high-pass-filter break frequency $1/(\varepsilon\tau_d)$ rad/s, this approximate differentiator is very accurate, but for faster error signals the device's output fails to approximate the derivative of the error signal (see homework Problem 15.4). In practice, therefore, the selection of parameters τ_d and $\mathcal{E}$ is based at least partly on the speed of signals that the PD controller is expected to process. The operation of an actual PD controller using the approximate differentiator is $w(t) = P[e_e(t) + \tau_d x_d(t)]$; you can easily show that the transfer function of the actual PD controller is

$$\frac{W(s)}{E_e(s)} = P\left(1 + \tau_d \frac{X_d(s)}{E_e(s)}\right) = P\left(1 + \frac{\tau_d s}{\varepsilon\tau_d s + 1}\right) = P\left(\frac{(1+\varepsilon)\tau_d s + 1}{\varepsilon\tau_d s + 1}\right).$$

It is standard in introductory discussions to describe PD control in the context of the ideal (though physically unrealizable) differentiator $x_d \equiv \dot{e}_e$, rather than the more realistic approximate differentiator $\mathcal{E}\tau_d\dot{x}_d + x_d = \dot{e}_e$. The approximate differentiator leads to a higher-order system and much more complicated algebra, from which it is difficult to infer basic physical characteristics of PD control. Therefore, in the rest of this section, and in the continued examination of this same example in homework Problem 15.1 and in Section 16.2, we shall follow the standard introductory procedure and use the ideal PD-controller transfer function $P(1 + \tau_d s)$ from Equation 15.1.2. Most of the theoretical results produced by this approach are good approximations to what would be realized from a properly designed real PD controller, at least for sufficiently slowly varying inputs and outputs.

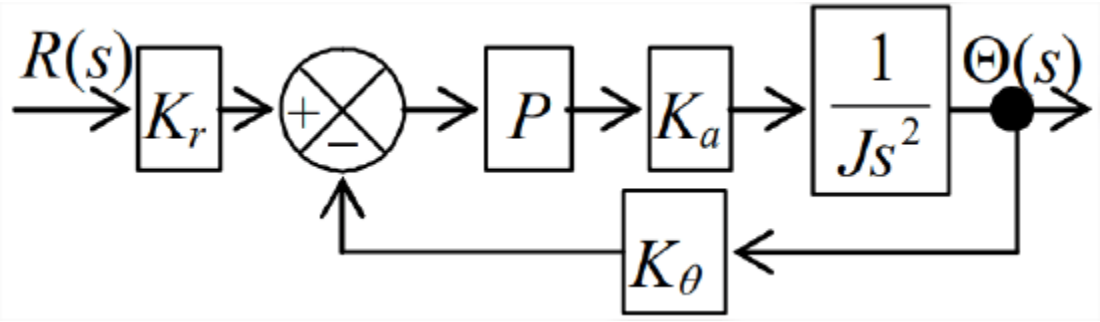


Figure 15.4.1: Proportional control of rotor position. (Copyright; author via source)

To illustrate proportional-derivative (PD) control, we consider again the problem of controlling position $\theta(t)$ of a rotor with significant inertia J, as discussed in Chapter 14 [Figures 14.1.1, 14.2.1, and 14.3.1, and Equations 14.1.1 and 14.1.2]. Recall that the input to the system is operator setting $r(t)$. Let us neglect disturbance moments, $M_d(t) = 0$. Figure 15.4.1 is the Laplace block diagram for the P-controlled system, from which the loop transfer functions are $G(s) = PK_a/(Js^2)$ and $H(s) = K_\theta$. Hence, we use

Equation 14.4.6 to find

$$CLTF(s) = \frac{\Theta(s)}{R(s)} = K_r \times \frac{N_G D_H}{D_G D_H + N_G N_H} = K_r \times \frac{PK_a}{Js^2 + PK_aK_\theta} \equiv \frac{K_r}{K_\theta} \frac{\omega_n^2}{s^2 + \omega_n^2}$$

Undamped natural frequency in this $\mathrm{CLTF}(s)$ is defined by $\omega_n^2 = PK_aK_\theta/J$. Suppose that the input is a step function with magnitude $R_H : r(t) = R_H H(t)$. For this input, the Laplace transform of response is $\Theta(s) = \frac{R_H}{s} \frac{K_r}{K_\theta} \frac{\omega_n^2}{s^2+\omega_n^2}$. Application of inverse transform Equation 15.2.23, with $\zeta = 0$, gives the step response $\theta(t) = (R_H K_r/K_\theta)(1 - \cos\omega_n t)$. This is undamped motion that oscillates forever about the desired final value, $\theta = R_H K_r/K_\theta$ [see Equation 7.3.8 and Figure 7.3.2], so it is unacceptable for a position control system. In this case, P control alone has the effect of a restoring linear spring, but it provides no damping.

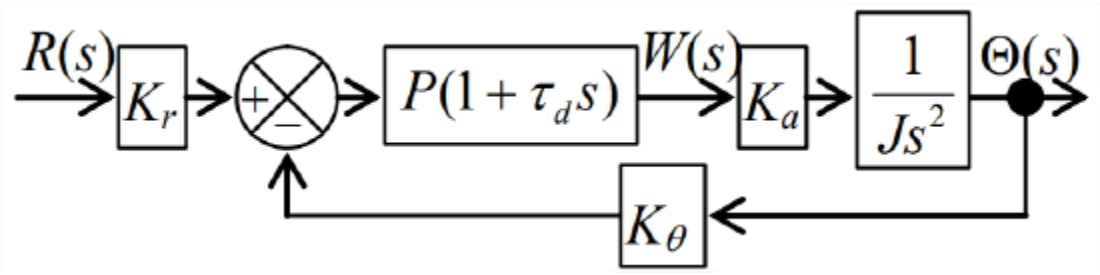


Figure 15.4.2: Ideal proportional-derivative (PD) control of rotor position. (Copyright; author via source)

In order to improve the control performance, we add derivative action, upgrading from P control to ideal PD control. From Equation 15.1.2 with $b_i = 0$ and $b_d = 1$, the ideal PD-controller transfer function is $P(1 + \tau_d s)$, so the Laplace block diagram for the closed-loop system becomes that shown in Figure 15.4.2. In this case, the loop forward-branch transfer function is $G(s) = P(1 + \tau_d s) K_a/(Js^2)$, so the closed-loop transfer function of the system is

$$\frac{\Theta(s)}{R(s)} = K_r \times \frac{N_G D_H}{D_G D_H + N_G N_H} = K_r \times \frac{P(1 + \tau_d s) K_a}{Js^2 + P(1 + \tau_d s) K_a K_\theta} = \frac{K_r PK_a (1 + \tau_d s)}{Js^2 + PK_aK_\theta\tau_d s + PK_aK_\theta}$$

As we did in Equations 15.2.19 and 15.2.20, which apply to PI control (but of a different plant), we can now re-write this PD closed-loop transfer function in terms of parameters appropriate for a damped 2nd order system:

$$\frac{\Theta(s)}{R(s)} = \frac{K_r}{K_\theta} \frac{(PK_aK_\theta/J)(\tau_d s + 1)}{s^2 + (PK_aK_\theta/J)\tau_d s + (PK_aK_\theta/J)} \equiv \frac{K_r}{K_\theta} \frac{\omega_n^2 (\tau_d s + 1)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{15.4.1}$$

In Equation 15.4.1 the undamped natural frequency and the viscous damping ratio are, respectively:

$$\omega_n = \sqrt{\frac{PK_aK_\theta}{J}} \text{ and } \zeta = \frac{1}{2\omega_n} \frac{PK_aK_\theta}{J} \tau_d = \frac{1}{2} \omega_n \tau_d \tag{15.4.2}$$

The obvious effect of the derivative action is to add damping to the 2nd order system, and this damping improves the control performance (see homework Problem 15.1). Note also that the term $\omega_n^2 \tau_d S$ in the numerator of Equation 15.4.1 makes this system non-standard relative to definition Equation 9.2.2 of the standard 2nd order ODE. The non-standard character means, among other things, that the step-response specifications derived in Section 9.8 do **not** apply exactly for this system.

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15.5: Chapter 15 Homework

1. Consider closed-loop transfer function Equation 15.4.3, with definitions Equation 15.4.4, for position control of a rotor by an ideal PD controller.
   1. Suppose that we know rotational inertia J, actuator sensitivity K_a, and **equal** input and feedback transducer sensitivities, $K_r = K_\theta$, and that we want to specify (*i.e.*, "design") the undamped natural frequency ω_n and the viscous damping ratio ζ for this controlled system. Hence, we need to determine appropriate values for proportional gain P and derivative time constant τ_d, which we would set in the PD controller. For a particular system, we have $J = 2.56\text{e}-3$ lb-s^2-inch (the value for a small aluminum wheel about four inches in diameter, Figure 3.3.1), and the product $K_r K_a = 0.0350$ lb-inch/rad; let us specify $f_n = \omega_n/2\pi = 1.00$ Hz (so that step-response rise time will be around ¼ s) and $\zeta = +0.1$ (too low for most practical control systems, but it suits the overall purposes of this problem). Calculate the required values of P and τ_d (s).
   2. Use the MATLAB `residue` operation (see homework Problem 2.15) to expand the transfer function [with the numerical values of part 15.1.1] into a partial-fraction expansion. The poles are critical to the stability of any system, which is the subject of Chapter 16. For this system, you should find a pair of poles that are complex conjugates, and their real part should be **negative**—this means that the system is **stable**.
   3. Let the input be a step, $r(t) = R_H H(t)$. First, apply the final-value theorem to find $\lim_{t\to\infty}\theta(t)$. Next, use inverse transforms Equations 15.2.22 and 15.2.23 [or just appropriately adapt step response Equation 15.2.24 to this case] to show that the step response is

      $$\theta(t) = R_H\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right], \text{ for } |\zeta| < 1 \text{ and } \omega_d = \omega_n\sqrt{1-\zeta^2}$$

      Finally, let the input step magnitude be $R_H = 0.1$ radian, with the numerical values of part 15.1.1, and use MATLAB to plot the output $\theta(t)$ over the time interval 0-2.5 s, about 2½ cycles.
   4. To simulate what would happen if you made a mistake and got the wrong sign for τ_d (and therefore for ζ), let $\zeta = -0.1$, and repeat the operations of parts 15.1.2 and 15.1.3. If you can plot $\theta(t)$ for both $\zeta = +0.1$ and $\zeta = -0.1$ on the same graph, it will enhance comparison of the two cases. For $\zeta = -0.1$, you should find the real part of the complex conjugate poles to be **positive**; the associated time history $\theta(t)$ should indicate system ***in*stability**. Observe for this unstable system that the final-value theorem incorrectly predicts a finite steady-state value for this step response.
2. Consider ideal PID control of the position of a rotor that has significant inertia J. Let us also account for a disturbance moment $M_d(t)$ acting upon the rotor. The Laplace block diagram for this **3rd order** system is drawn above. In the following exercises, assume that this system is stable.

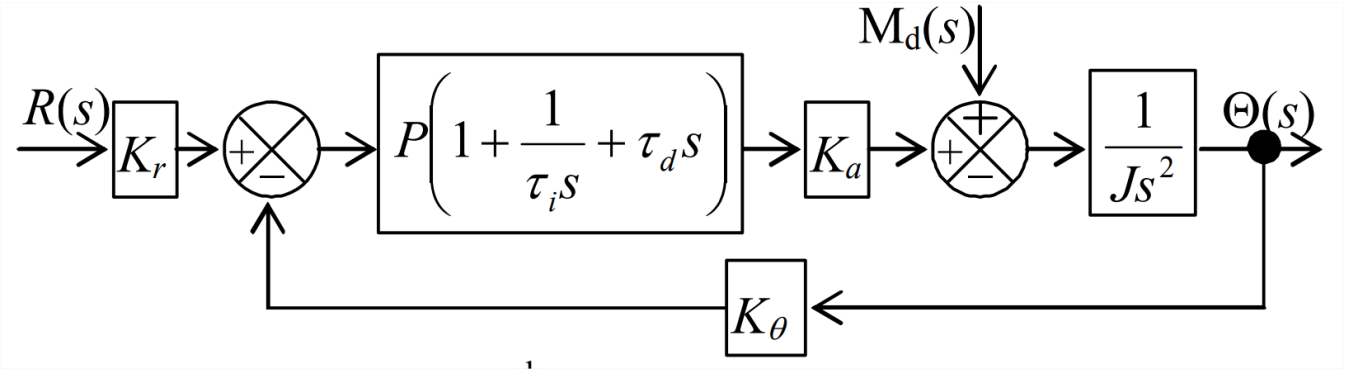


Figure 15.5.1 (Copyright; author via source)

   1. For zero disturbance, $\text{M}_\text{d}(s) = 0$, derive the algebraic equation for closed-loop transfer function$\Theta(s)/R(s)$. For step input, $r(t) = R_H H(t)$, apply the final-value theorem to find $\lim_{t\to\infty}\theta(t)$.
   2. For zero input, $R(s) = 0$, derive the algebraic equation for closed-loop transfer function $\Theta(s)/\text{M}_\text{d}(s)$. For step disturbance, $M_d(t) = M_H H(t)$, apply the final-value theorem to find $\lim_{t\to\infty}\theta(t)$. A control system should suppress permanent output due to a step disturbance.
3. Consider PI control of the **speed** (not position) of a rotor that has significant inertia J. The sensor on the output is a tachometer with gain constant K_p, so the electrical signal that is fed back is $K_p\dot{\theta}$. (This configuration illustrates a general rule-of-thumb in control-system design: sense and feed back the quantity that you want to control.) This is a simplified model, for example, of an automobile's cruise-control system. Let us also allow for a disturbance moment $M_d(t)$ acting upon the rotor. The Laplace block diagram for this system is at left.

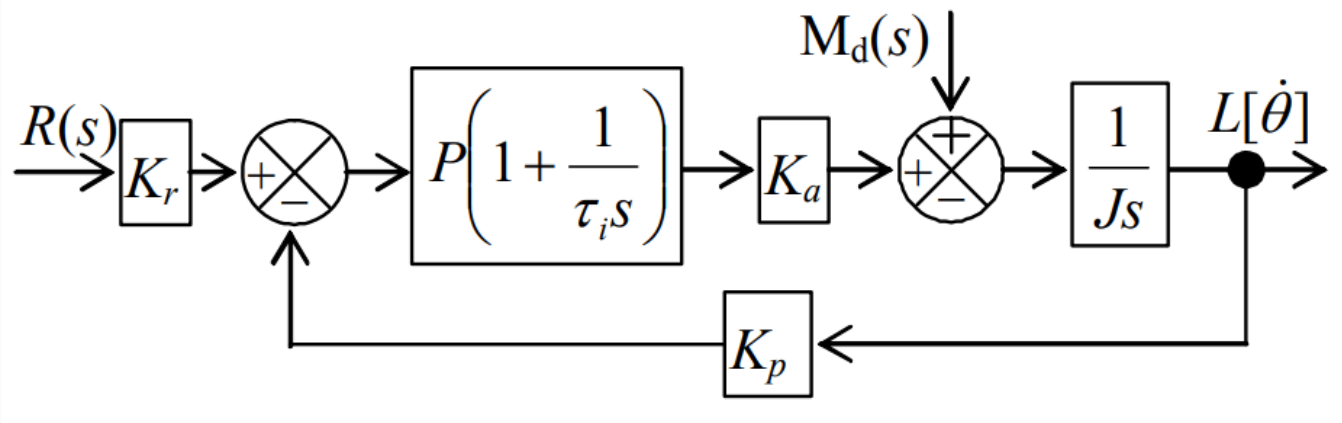


Figure 15.5.2 (Copyright; author via source)

1. For zero disturbance, $\mathrm{M_d}(s) = 0$, derive the algebraic equation in terms of all parameters on the Laplace block diagram for the closed-loop transfer function $L[\dot{\theta}]/R(s)$ that relates output rotational velocity $\dot{\theta}(t)$ to input command $r(t)$. For step input, $r(t) = R_H \times H(t)$, apply the final-value theorem to find the algebraic equation for $\lim_{t\to\infty} \dot{\theta}(t)$, assuming that this asymptotic value exists. Explain (perhaps using an example from your own experience, such as a cruise-control system) the physical significance of the ratio K_r/K_p of the input-transducer sensitivity to the tachometer sensitivity.
2. Derive algebraic equations for undamped natural frequency ω_n and viscous damping ratio ζ of this controlled system, in terms of inertia J, actuator sensitivity K_a, tachometer sensitivity K_p, proportional gain P, and integral time constant τ_i.

4. In this problem, you will produce graphically and compare the frequency-response functions of the exact, but physically unrealizable differentiator $x_d \equiv \dot{e}_e$ (which is used in this book for ideal PD control) and an approximate, but physically realizable differentiator that has the character of a high-pass filter.
   1. Section 15.4 gives the ODE of the approximate differentiator as $\varepsilon\tau_d\dot{x}_d + x_d = \dot{e}_e$, with τ_d being the positive derivative time constant and $\mathcal{E}$ being a small positive number in the range 0.1-0.3; show that the complex frequency-response function (FRF) of output signal $x_d(t)$ relative to input signal $e_e(t)$ is $FRF(\omega) = j\omega/\left(1 + j\omega\varepsilon\tau_d\right)$.
   2. For frequency response, the input signal has the steady-state sinusoidal form $e_e(t) = E_e \cos\omega t$ and the output signal has the corresponding form $x_d(t) = X_d(\omega)\cos(\omega t + \phi)$. Use the FRF of part 15.4.1 to show that the equations for FRF magnitude ratio and phase of the approximate differentiator are $\frac{X_d(\omega)}{\omega_b E_e} = \frac{\bar{\omega}}{\sqrt{1+\bar{\omega}^2}}$ and $\phi(\omega) = \frac{\pi}{2} - \tan^{-1}\bar{\omega}$, in which the break frequency is $\omega_b \equiv 1/\left(\varepsilon\tau_d\right) = 2\pi f_b$ and the dimensionless frequency ratio is $\bar{\omega} \equiv \omega/\omega_b = f/f_b$. Use the magnitude-ratio equation to derive equations for the low-frequency and high-frequency asymptotes, and sketch those asymptotes on a log-log graph such as Figure 4.3.3. The approximate differentiator should exhibit the character of a *high-pass filter*, analogous to the system of homework Problem 4.4. Use the asymptotes as an envelope to guide you in sketching the **actual** curve of $X_d(\omega)/\left(\omega_b E_e\right)$, as is done on Figure 4.3.3. Finally, on a semi-log graph below the log-log graph (analogous to Figure 4.3.2), sketch the variation with frequency of phase $\phi(\omega)$.
   3. Now consider the exact differentiator, with equation $x_d \equiv \dot{e}_e$. Repeat all the steps of parts 15.4.1 and 15.4.2 for the exact differentiator, sketching the magnitude-ratio and phase curves on the part 15.4.2 graphs, but in some different color or line style, so that the curves for exact and approximate differentiators are clearly distinguishable. You should be able to infer from these final graphs the ranges of frequency over which the approximate differentiator is reasonably accurate, or inaccurate.

5. Consider the transfer function of the PD-controlled rotor position of Section 15.4, Equation 15.4.3 with **equal** input and feedback transducer sensitivities, $K_r = K_\theta$, in the form $\frac{\Theta(s)}{R(s)} = \frac{K_r P K_a(1+\tau_d s)}{Js^2 + K_r P K_a \tau_d s + K_r P K_a}$. Use this transfer function and the block diagram of Figure 15.4.2 to write the algebraic equation for the transfer function of the PD controller, $W(s)/R(s)$. Now, examine the nature of $W(s)/R(s)$ in the following steps:
   1. Find the order m of the numerator polynomial and the order n of denominator polynomial of $W(s)/R(s)$. Is this transfer function causal or acausal, and what does that mean in theory regarding the physical realism of $W(s)/R(s)$?
   2. Suppose that the input is the infinitely fast step function $r(t) = R_H H(t)$ of homework Problem 15.1.3. Show that the controller-output transform $W(s)$ must have a constant *direct term*, as defined in homework Problem 2.15. What is the component of time response $w(t)$ that corresponds to the direct term of $W(s)$, and is this a physically realizable signal from a real device? Your conclusions from parts 15.5.1 and 15.5.2 should be compatible.
   3. If you conclude in parts 15.5.1 and 15.5.2 that something is physically unrealistic about these theoretical results, then what is there in the theoretical analysis that makes the results defective? Propose a remedy that should produce physically realizable results, and show succinctly, with appropriate algebraic equations and written explanation (but no calculations), why your proposed remedy should work.

6. Consider control of a spacecraft's pitch attitude, $\theta(t)$, for which the ODE of motion, from Equation 14.2.4, is J

\ddot{\theta}=M_{a}(t)+M_{d}(t), where $M_d(t)$ is a disturbance moment. In this problem, you will apply a physically realizable form of PD control, with the control actuator moment defined as $M_a(t) = K_a \times w(t)$, where K_a is the actuator sensitivity. The output signal from the realizable PD controller is $*w(t) = P\,[e_e(t) + \tau_d x_d(t)]$, where $x_d(t)$ is the dependent variable in the ODE $\varepsilon\tau_d \dot{x}_d + x_d = \dot{e}_e$ (see the relevant discussion in Section 15.4). The input error is $e_e(t) = K_r r(t) - K_\theta \theta(t)$, where $r(t)$ is the input operator setting, and K_r and K_θ are the transducer sensitivities at, respectively, the input and output feedback.

1. Draw and label completely the Laplace block diagram of this system, from input $R(s) \equiv L[r(t)]$ to output $\Theta(s) \equiv L[\theta(t)]$, including disturbance moment signal $\mathrm{M_d}(s) = L\,[M_d(t)]$.
2. For $\mathrm{M_d}(s) = 0$, derive from the block diagram of part 15.6.1 the closed-loop transfer function $\mathrm{CLTF}(s) = \Theta(s)/R(s) = \mathrm{Num}(s)/\,\mathrm{Den}(s)$, where $\mathrm{Num}(s)$ and $\mathrm{Den}(s)$ are polynomials in powers of s. *Partial answer:* $\mathrm{Den}(s) = Js^2\,(\varepsilon\tau_d s + 1) + PK_aK_\theta\,[\tau_d(1+\varepsilon)s + 1]$
3. For step input, $r(t) = R_H H(t)$, apply the final-value theorem to find $\lim_{t\to\infty}\theta(t)$, provided that the system is stable.
4. How does $\mathrm{CLTF}(s)$ in part 15.6.2, for a physically realizable form of PD control, differ from the corresponding closed-loop transfer function that applies for an **ideal** but unrealizable PD controller, for which $w(t) = P\,[e_e(t) + \tau_d \dot{e}_d(t)]$? [Do not re-derive this $\mathrm{CLTF}(s)$ from scratch; instead, just make a simple substitution in the $\mathrm{CLTF}(s)$ of part 15.6.2.] Discuss in one or two sentences.

7. Consider the "position actuator" depicted on functional diagram Figure 15.2.2, which represents a system for control of a rotor immersed in a viscous liquid. Suppose that the position actuator is the rotor-position control system of Figure 14.5.6, and that Figure 14.5.7 is the actuator's Laplace block diagram. Your assignment in this problem is to revise the immersed-rotor control system's Laplace block diagram, Figure 15.2.3, to incorporate the position-actuator dynamics, replacing the simplified block of actuator gain K_a with an appropriate version of Figure 14.5.7, and making any other necessary modifications. There are at least three significant changes required, the first two being notational. (1) You should re-label the input of the modified Figure 14.5.7 to $W(s) \equiv L[w(t)]$, replacing $R(s)$, and also re-label the output to $\Theta_a(s) \equiv L\,[\theta_a(t)]$, replacing $\Theta(s)$. (2) In order to avoid notational ambiguity, you should re-label all system components on Figure 14.5.7 having subscripted symbol K, with subscripted symbol Q, *e.g.*, on the revised Figure 14.5.7, $K_r \to Q$, and $K_a \to Q$, *etc.* (3) It is necessary to recognize that the rotational spring with stiffness k_θ shown on Figure 15.2.2 imposes upon the position actuator a moment $k_d\,(\theta - \theta_a)$. The quantities labeled on Figures 14.5.6 and 14.5.7 that can represent moment $k_d\,(\theta - \theta_a)$ are, respectively, $M_d(t)$ and $\mathrm{M_d}(s)$, for example, $\mathrm{M_d}(s) = k_\theta\,[\Theta(s) - \Theta_a(s)]$. (However, the moment $k_\theta\,(\theta - \theta_a)$ is clearly defined and not at all random or unpredictable mathematically, so it is not truly a *disturbance* in the sense defined in Section 14.2, with reference to Figure 14.2.1.) Accounting for items (1)-(3) and any other necessities, draw a Laplace block diagram of the immersed-rotor control system, a diagram that incorporates the position-actuator dynamics from Figure 14.5.7. This block diagram will be somewhat longer than others appearing in this book. It is **not** required that you simplify this block diagram to derive an algebraic expression for the closed-loop transfer function; it is possible to do so using the operations of block-diagram algebra described in Chapters 13 and 14, but the process would be complicated algebraically, and it would produce a very long and messy equation for $\mathrm{CLTF}(s)$. Modern processes using specialized computer software are more appropriate than algebra-by-hand for a task such as this. For example, if you had numerical values for all the system parameters (J, k_θ, Q_r, ..., τ_1, P, *etc.*), then you could easily generate a single equation for $\mathrm{CLTF}(s)$ with MATLAB by using the functions `tf`, and `feedback`, and `series` (or, rather than `series`, simply the operation of transfer function multiplication).

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16: INTRODUCTION TO SYSTEM STABILITY - TIME-RESPONSE CRITERIA

Section 16.1 is a lengthy and detailed, perhaps almost tedious, development of the theory underlying time-response stability criteria. Section 16.2, on the other hand, is a specific example, based upon a control system from Chapter 15, which illustrates explicitly many of the theoretical results of Section 16.1. Rather than reading the two sections sequentially, in series, you, the student, might find the material more interesting and instructive if you read Sections 16.1 and 16.2 together, in parallel.

16.1: General Time-Response Stability Criterion for LTI Systems

16.2: Stable and Snstable PD-Controlled-Rotor Systems

16.3: Routh's Stability Criteria

16.4: Loci of Roots for Second Order Systems

16.5: Loci of Roots for Third Order Systems

16.6: Open-Loop Transfer Functions and Loci of Roots

16.7: Chapter 16 Homework

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16.1: General Time-Response Stability Criterion for LTI Systems

For any linear, time-invariant (LTI), single-input-single-output (SISO) physical system, we denote the input as $u(t)$ [or $r(t)$, the reference operator setting for controlled systems] and the output as $x(t)$. For an nth order system, in general, the input and output are related by an ODE of the form

$$a_1\frac{d^n x}{dt^n}+a_2\frac{d^{n-1}x}{dt^{n-1}}+\ldots+a_n\frac{dx}{dt}+a_{n+1}x=b_1\frac{d^m u}{dt^m}+b_2\frac{d^{m-1}u}{dt^{m-1}}+\ldots+b_m\frac{du}{dt}+b_{m+1}u \tag{16.1.1}$$

Symbols $a_1,\ldots,a_{n+1}$ and $b_1,\ldots,b_{m+1}$ are constants (with the numbering system keyed to MATLAB notation), and $m\le n$. With use of Equation 2.2.10, especially for initial-condition terms, the Laplace transform of Equation 16.1.1 is

$$\left(a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}\right)L[x(t)]-a_1\left[s^{n-1}x(0)+s^{n-2}\frac{dx}{dt}(0)+\ldots+\frac{d^{n-1}x}{dt^{n-1}}(0)\right]-a_2\left[s^{n-2}x(0)+s^{n-3}\frac{dx}{dt}(0)+\ldots+\cdot\right.$$

The initial-condition terms on the left-hand side are algebraically messy; it will suit our purposes to combine together known a_k constants and known initial conditions into constant coefficients denoted as IC_k, and to express the initial-condition terms in the following simpler algebraic form, arranged in order of descending powers of s:

$$-\overbrace{a_1x(0)}^{=IC_1}s^{n-1}-\overbrace{\left[a_1\frac{dx}{dt}(0)+a_2x(0)\right]}^{\equiv IC_2}S^{n-2}-\ldots-\overbrace{\left[a_1\frac{d^{n-1}x}{dt^{n-1}}(0)+a_2\frac{d^{n-2}x}{dt^{n-2}}(0)+\ldots+a_nx(0)\right]}^{\equiv IC_n}\equiv-IC_1s^{n-1}-IC_2s^{n-2}-\ldots-IC_{n-1}s-$$

Therefore, we can express Equation 16.1.2 in the algebraically simpler form,

$$\left(a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}\right)L[x(t)]-IC_1s^{n-1}-IC_2s^{n-2}-\ldots-IC_{n-1}s-IC_n=\left(b_1s^m+b_2s^{m-1}+\ldots+b_ms+b_{m+1}\right)L[$$

The algebraic solution of Equation 16.1.3 for the Laplace transform of the output is:

$$L[x(t)]=\frac{IC_1s^{n-1}+IC_2s^{n-2}+\ldots+IC_{n-1}s+IC_n}{a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}}+\frac{b_1s^m+b_2s^{m-1}+\ldots+b_ms+b_{m+1}}{a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}}L[u(t)] \tag{16.1.4}$$

It is appropriate now to examine the nature of transform solution Equation 16.1.4. Let us refer to the first term on the right-hand side as the "IC-response transform," and to the second term as the "forced-response transform." Observe first that the coefficient of input transform $L[u(t)]$ in the forced-response transform is the system transfer function, from Equation 4.6.3,

$$TF(s)\equiv\frac{L[x(t)]_{IC s=0}}{L[u(t)]}=\frac{b_1s^m+b_2s^{m-1}+\ldots+b_ms+b_{m+1}}{a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}} \tag{16.1.5}$$

If the system is controlled with use of feedback, then Equation 16.1.5 represents the closed-loop transfer function, which we usually denote as $\mathrm{CLTF}(s)$. It is conventional to express $TF(s)$ in the following form, with both numerator and denominator polynomials factored:

$$TF(s)=\frac{b_1s^m+b_2s^{m-1}+\ldots+b_ms+b_{m+1}}{a_1s^n+a_2s^{n-1}+\ldots+a_ns+a_{n+1}}\equiv\frac{\mathrm{Num}(s)}{\mathrm{Den}(s)}=\frac{b_1\left(s-z_1\right)\left(s-z_2\right)\ldots\left(s-z_m\right)}{a_1\left(s-p_1\right)\left(s-p_2\right)\ldots\left(s-p_n\right)} \tag{16.1.6}$$

The m roots z_k of polynomial equation $\mathrm{Num}(z)=0$ are called the *zeros* of the transfer function, since $TF(s)=0$ if s equals any z_k . The n roots p_k of polynomial equation $\mathrm{Den}(p)=0$ are called the *poles* of the transfer function, because $TF(s)\to\infty$ if s equals any p_k. [Imagine the complex space surface $TF(s)$ graphed as a function of coordinates $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$: for s in the vicinity of any p_k, the surface will look something like that of a circus tent near where it is held up by a structural pole; however, whereas the circustent pole has finite length, the mathematical pole is infinitely long.]

Observe next that both right-hand-side terms of Equation 16.1.4 have in their denominators the term $\mathrm{Den}(s)$ defined in Equation 16.1.6. This term is the key to system stability. The nth degree polynomial equation $\mathrm{Den}(p)=0$ is the general[1] *characteristic equation* of the system, and its n roots are the poles p_k, $k=1,2,\ldots,n$. System stability is determined by the signs of the real parts

of the poles. The general result is:

If $\mathrm{Re}\,(p_k) < 0$ for *all* $k = 1, 2, \ldots, n$, then the system is stable. If, on the other hand, there is at least one pole for which $\mathrm{Re}\,(p_k) > 0$, then the system is unstable.

In order to justify this general stability criterion, we shall continue the analysis of Equation 16.1.4 by deriving the inverse transform conceptually; however, it is appropriate first to state the criterion above, without proof, so that you can see the simple yet very important result of the analysis without being burdened by all the details of the derivation.

First, let us review without proof some basic results from the theory of polynomial equations with real coefficients, a subject covered in most algebra textbooks. The roots of such equations are either real or complex. Moreover, complex roots **always** appear in conjugate pairs. For example, if one of the roots of $\mathrm{Den}(p) = 0$ is complex, $p_k = \sigma_k + j\omega_k$, where σ_k and ω_k are real, then one of the other roots is the complex conjugate, $p_{k+1} = \bar{p}_k = \sigma_k - j\omega_k$.

Returning to Equation 16.1.4, let us evaluate first the IC-response transform. The numerator polynomial is of degree less than n, that of the denominator polynomial, so we can usually expand the entire term into simple partial-fraction form [see the discussion surrounding Equations 2.3.5 and 2.3.6]:

$$\frac{IC_1 s^{n-1} + IC_2 s^{n-2} + \ldots + IC_{n-1} s + IC_n}{a_1 s^n + a_2 s^{n-1} + \ldots + a_n s + a_{n+1}} = \frac{IC_1 s^{n-1} + IC_2 s^{n-2} + \ldots + IC_{n-1} s + IC_n}{a_1 (s - p_1)(s - p_2) \ldots (s - p_n)} = \sum_{k=1}^{n} \frac{C_k}{s - p_k} \tag{16.1.7}$$

Right-hand-side constants C_k in Equation 16.1.7 are called *residues*. Some relevant results from the theory of fractions of polynomials with real coefficients are: if pole p_k is real, then the associated residue C_k also is real; if pole p_k is complex, then the associated residue C_k also is complex, **and** the residue associated with the complex conjugate pole p_{k+1} is the conjugate of C_k, $C_{k+1} = \bar{C}_k$. It will be useful in derivations that follow to express a complex residue in polar form, $C_k \equiv |C_k|\, e^{j\angle C_k}$, in which $|C_k|$ is the absolute value (as in the MATLAB function `abs`), and $\angle C_k$ is the angle (as in the MATLAB function `angle`). See Section 2.1 for a review of complex numbers and complex arithmetic.

Taking the inverse transform of Equation 16.1.7 to find initial-condition response, we deal, in general, with two "Types" of terms:

1. terms that have real poles and real residues; and
2. **pairs** of terms that have complex conjugate poles and complex conjugate residues.

The inverse of any Type 1 term is a simple real exponential, from Equation 2.2.6:

$$L^{-1}\left[\frac{C_k}{s - p_k}\right] = C_k e^{p_k t}, C_k \text{ real, and } p_k \text{ real} \tag{16.1.8}$$

In Equation 16.1.8, if $p_k < 0$, then the term decays exponentially as time increases, which is **stable** response. Such a term decays eventually to a static equilibrium state of zero; let us call this *exponential stability*. However, if $p_k > 0$, then the term is a permanently growing exponential as time increases, which is **unstable** response. Finally, if $p_k = 0$, then the term is constant in time; such response is bounded and therefore not unstable, which is good for practical purposes; but it is also not so stable that it eventually decays to zero.

Next, let us consider Type 2 terms, as defined above Equation 16.1.8, and derive the inverse transform of a pair of terms that have complex conjugate poles and complex conjugate residues. In the following sequence of operations, we apply: first, Equation 2.2.6; next, identities derived in homework Problems 2.3.1 and 2.4; and, finally, Euler's equation, Equation 2.1.12:

$$\begin{aligned} L^{-1}\left[\frac{C_k}{s - p_k} + \frac{\bar{C}_k}{s - \bar{p}_k}\right] &= C_k e^{p_k t} + \bar{C}_k e^{\bar{p}_k t} = C_k e^{p_k t} + \overline{C_k e^{p_k t}} = 2 \times \mathrm{Re}\left[C_k e^{p_k t}\right] \\ &= 2 \times \mathrm{Re}\left[\left(|C_k|\, e^{j\angle C_k}\right) e^{(\sigma_k + j\omega_k)t}\right] = 2\,|C_k|\, e^{\sigma_k t} \times \mathrm{Re}\left[e^{j(\omega_k t + \angle C_k)}\right] \\ &= 2\,|C_k|\, e^{\sigma_k t} \cos\left(\omega_k t + \angle C_k\right) \end{aligned} \tag{16.1.9}$$

Equation 16.1.9[2] varies with time as a sinusoid modulated by an exponential envelope. With respect to stability, the critical term is the exponential envelope, $e^{\sigma_k t}$, and its character is determined by the sign of $\sigma_k \equiv \mathrm{Re}\,(p_k)$. If $\mathrm{Re}\,(p_k) < 0$, then $e^{\mathrm{Re}(p_k)t}$ is a decaying exponential, and the total response is a **stable** positively damped oscillation. However, if $\mathrm{Re}\,(p_k) > 0$, then $e^{\mathrm{Re}(p_k)t}$ is a permanently growing exponential, and the total response is an **unstable** ever-expanding oscillation. Finally, if $\mathrm{Re}\,(p_k) = 0$, then

$e^{\mathrm{Re}(p_k)t}$ is constant in time, and the total response is a sinusoid of constant magnitude; such response is bounded and therefore not unstable; but it is also not so stable that it eventually decays to a static equilibrium state, *i.e.*, it is **not** *exponentially stable*.

Now that we have derived the stability criterion for the IC-response terms of Equation 16.1.4, let us turn our attention to the forced-response transform,

$$L\left[x_f(t)\right] \equiv \frac{b_1 s^m + b_2 s^{m-1} + \ldots + b_m s + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + \ldots + a_n s + a_{n+1}} L[u(t)] = \frac{b_1 \left(s - z_1\right)\left(s - z_2\right) \ldots \left(s - z_m\right)}{a_1 \left(s - p_1\right)\left(s - p_2\right) \ldots \left(s - p_n\right)} L[u(t)] \tag{16.1.10}$$

We shall consider explicitly in this derivation only transfer functions for which the following practically common conditions apply: \(m

In order to find the conceptual inverse of Equation 16.1.10, let us consider the nature of transform $L[u(t)]$ for physically common input functions $u(t)$. In this derivation, we shall consider explicitly only the type of input function whose Laplace transform is a fraction of polynomials in the Laplace variable s, with the numerator polynomial being of lower degree than the denominator polynomial. (However, the same stability criterion that we derive for this special type of input function can be derived for any physically realizable input function.) The following are simple examples of this type of input function, which are defined to be non-zero only for $t \geq 0$: ideal step $H(t)$, for which $L[H(t)] = 1/s$; exponential-unit-impulse $e^{-t/\tau}/\tau$, for which $L\left[e^{-t/\tau}/\tau\right] = 1/(\tau s + 1)$; exponential-unit-step $1 - e^{-t/7}$, for which $L\left[1 - e^{-t/\tau}\right] = 1/[s(\tau s + 1)]$ (see homework Problem 8.6 for a discussion of the exponential-unit-step and exponential-unit-impulse functions); ramped exponential pulse $te^{-t/\tau}$, for which $L\left[te^{-t/\tau}\right] = 1/(s + 1/\tau)^2$; and sinusoid $\cos \omega t$, for which $L\left[te^{-t/\tau}\right] = 1/(s + 1/\tau)^2$. Let us denote the Laplace transform of such input functions in a general form:

$$L[u(t)] \equiv \frac{N_u(s)}{D_u(s)} = \frac{d_1 s^{mu} + d_2 s^{mu-1} + \ldots + d_{mu} s + d_{mu+1}}{c_1 s^{nu} + c_2 s^{nu-1} + \ldots + c_{nu} s + c_{nu+1}} = \frac{d_1 \left(s - w_1\right)\left(s - w_2\right) \ldots \left(s - w_{mu}\right)}{c_1 \left(s - q_1\right)\left(s - q_2\right) \ldots \left(s - q_{nu}\right)} \tag{16.1.11}$$

Constants c_i and d_i are known coefficients of powers of s, and constants q_i and w_i are, respectively, poles and zeros of the input transform. Keep in mind that we assume $N_u(s)$ in Equation 16.1.11 to have lower degree than $D_u(s)$, \(m u

Proceeding with the conceptual forced-response solution, we substitute Equation 16.1.11 into Equation 16.1.10:

$$L\left[x_f(t)\right] = \frac{b_1 \left(s - z_1\right)\left(s - z_2\right) \ldots \left(s - z_m\right)}{a_1 \left(s - p_1\right)\left(s - p_2\right) \ldots \left(s - p_n\right)} \times \frac{d_1 \left(s - w_1\right)\left(s - w_2\right) \ldots \left(s - w_{mu}\right)}{c_1 \left(s - q_1\right)\left(s - q_2\right) \ldots \left(s - q_{nu}\right)} \tag{16.1.12}$$

Forced-response transform Equation 16.1.12 is a fraction of polynomials in s, with numerator polynomial degree $m + mu$ being less than denominator polynomial degree $n + nu$. Therefore, we can, in principle, usually expand Equation 16.1.12 into the following form, in which terms that are associated explicitly with the transfer-function poles have simple partial-fraction form, including residues CU_k:

$$L\left[x_f(t)\right] = \sum_{k=1}^{n} \frac{CU_k}{s - p_k} + F\left(s; q_i\right) \tag{16.1.13}$$

The symbol $F\left(s; q_i\right)$ denotes terms associated explicitly with poles q_i of the input transform. Depending upon the specific nature of $L[u(t)]$, these terms might or might not have simple partial-fraction form. In any case, the terms in $F\left(s; q_i\right)$ have the same poles q_i as the input transform, which are always stable poles, so these terms cannot produce instability. Therefore, instability of the forced-response can exist only in the terms $\sum_{k=1}^{n} CU_k/\left(s - p_k\right)$, *i.e.*, the terms containing the poles of the system transfer function.

The forced-response terms $\sum_{k=1}^{n} CU_k/\left(s - p_k\right)$ are identical in form to the IC-response terms $\sum_{k=1}^{n} C_k/\left(s - p_k\right)$ of Equation 16.1.7. Recall that the analysis of stability for ICresponse [Equations 16.1.8 and 16.1.9 and the associated discussion] was based upon that form. Therefore, the analysis of stability for forced-response is identical to that for ICresponse, so we need not repeat it. However, it is worth repeating, for emphasis, the general stability criterion for LTI systems that has been derived (albeit, not generally) here:

For an nth order LTI system, the poles of the transfer function are denoted as p_k, $k = 1, 2, \ldots, n$. If $\mathrm{Re}\left(p_k\right) < 0$ for *all* $k = 1, 2, \ldots, n$, then the system is stable. If, on the other hand, there is at least one pole for which $\mathrm{Re}\left(p_k\right) > 0$, then the system is unstable.

Observe from the derivation that the **stability or instability of an LTI system does not depend upon ICs or forcing—it is**

an intrinsic property of the system. However, by referring to general transform solution Equation 16.1.4, one might make the mathematical argument that if all initial conditions were zero, and if forcing input $u(t)$ also were zero, then the solution would be $x(t) = 0$, and therefore even a system that has an unstable pole, $\mathrm{Re}\,(p_k) > 0$, would be stable if there were no stimulus. In physical reality, though, there is **always** some perturbation or excitation, no matter how small, acting upon a system; and even the tiniest IC or forcing input, which would be negligible for a stable system, will always be enough to provoke the instability of an unstable LTI system.

[1]The characteristic equation of the free-vibration problem described in Chapter 12 is a specific form, applicable only to undamped free vibration, of this general characteristic equation.

[2]When applying Equation 16.1.9, you should assign the title of p_k to the one root (of the complex conjugate pair of roots) for which $\mathrm{Im}\,(p_k) \equiv \omega_k > 0$, in order to define the oscillatory frequency as being positive. Then you should assign the title of C_k to the residue that is associated with the root for which $\mathrm{Im}\,(p_k) > 0$.

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16.2: Stable and Snstable PD-Controlled-Rotor Systems

In this section, we analyze in detail a specific example of the theory developed in Section 16.1. The situation considered is ideal proportional-derivative (PD) control of position $\theta(t)$ of a rotor with significant inertia J, as described in Section 15.4 and homework Problem 15.1. The sensitivity of identical input and output-feedback transducers is denoted $K_r = K_\theta$, moment actuator sensitivity is K_a, and, for the ideal PD controller, proportional gain is P and derivative time constant is τ_d. The transfer function of this system, from Equations 15.4.3 and 15.4.4, is

$$\frac{\Theta(s)}{R(s)} = \frac{\omega_n^2\left(\tau_d s + 1\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{16.2.1}$$

with 2nd order-system standard parameters defined as

$$\omega_n = \sqrt{\frac{K_r K_a P}{J}} \text{ and } \zeta = \frac{1}{2}\omega_n \tau_d \tag{16.2.2}$$

The first useful thing that we can do with transfer function Equation 16.2.1 is infer the system ordinary differential equation (ODE) by inverse Laplace transformation:

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]\Theta(s) = \left[\omega_n^2\left(\tau_d s + 1\right)\right]R(s) \Rightarrow \ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\left(\tau_d\dot{r} + r\right) \tag{16.2.3}$$

Recall that the transfer function is based upon zero ICs for the output. But we want to account for ICs now, so let us take the complete Laplace transform of the ODE, including the possibly non-zero ICs, initial rotation rate $\dot{\theta}(0) \equiv \dot{\theta}_0$ and initial position $\theta(0) \equiv \theta_0$:

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]\Theta(s) - s\theta_0 - \dot{\theta}_0 - 2\zeta\omega_n\theta_0 = \left[\omega_n^2\left(\tau_d s + 1\right)\right]R(s) \tag{16.2.4}$$

The solution of Equation 16.2.4 for the output transform is

$$\Theta(s) = \frac{\theta_0 s + \left(\dot{\theta}_0 + 2\zeta\omega_n\theta_0\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{\omega_n^2\left(\tau_d s + 1\right)}{s^2 + 2\zeta\omega_n s + \omega_n^2}R(s) \tag{16.2.5}$$

Compare specific transform solution Equation 16.2.5 with the general form Equation 16.1.5. Next, let us factor the denominator polynomial of the transfer function. It is convenient for this purpose to assume that $|\zeta| < 1$, and to define the positive, real, damped natural frequency as $\omega_d = \omega_n\sqrt{1-\zeta^2}$. You can easily derive from the quadratic formula (or just substitute in the right-hand side to verify) the factoring:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s-p)(s-\bar{p}), \text{ where the complex pole is } p = -\zeta\omega_n + j\omega_d \tag{16.2.6}$$

Hence, we re-write transform Equation 16.2.5, and express it in terms of IC-response and forced-response components:

$$\Theta(s) = \frac{\theta_0 s + \left(\dot{\theta}_0 + 2\zeta\omega_n\theta_0\right)}{(s-p)(s-\bar{p})} + \frac{\omega_n^2\left(\tau_d s + 1\right)}{(s-p)(s-\bar{p})}R(s) \equiv \Theta_{ic}(s) + \Theta_f(s) \tag{16.2.7}$$

The IC-response term of Equation 16.2.7 is expanded into partial-fraction form:

$$\Theta_{ic}(s) \equiv \frac{\theta_0 s + \left(\dot{\theta}_0 + 2\zeta\omega_n\theta_0\right)}{(s-p)(s-\bar{p})} = \frac{C}{(s-p)} + \frac{\bar{C}}{(s-\bar{p})} \tag{16.2.8}$$

Using the conventional labor-saving partial-fraction-expansion method of Sections 2.2 and 2.3, and $p = -\zeta\omega_n + j\omega_d$ from Equation 16.2.6, and a little algebra, we find the following complex residue of Equation 16.2.8:

$$C = \left[(s-p)\Theta_{ic}(s)\right]_{s=p} = \left[\frac{\theta_0 p + \left(\dot{\theta}_0 + 2\zeta\omega_n\theta_0\right)}{(s-\bar{p})}\right]_{s=p} = \frac{1}{2}\left(\theta_0 - j\frac{\dot{\theta}_0 + \zeta\omega_n\theta_0}{\omega_d}\right) \tag{16.2.9}$$

Hence, substituting Equations 16.2.6 and 16.2.8 into inverse transform Equation 16.1.12 gives the IC-response [with C given by

Equation 16.2.9]:

$$\theta_{ic}(t) = L^{-1}\left[\Theta_{ic}(s)\right] = 2|C|e^{-\zeta\omega_n t}\cos\left(\omega_d t + \angle C\right) \tag{16.2.10}$$

Next, let us derive the forced response of this controlled system to a specific, nontrivial input, the ramped exponential pulse $r(t) = R_p(t/\tau)e^{-t/\tau}$, the maximum value of which is $r_{\max} = r(t=\tau) = R_p e^{-1}$. The Laplace transform of this pulse is $R(s) = (R_p/\tau)/(s+1/\tau)^2$. The transform of the forced response, from Equation 16.2.7, is

$$L\left[\theta_f(t)\right] = \frac{\omega_n^2\left(\tau_d s+1\right)}{(s-p)(s-\bar{p})}R(s) = \frac{\omega_n^2\left(\tau_d s+1\right)}{(s-p)(s-\bar{p})}\frac{R_p/\tau}{(s+1/\tau)^2} = R_p\frac{\omega_n^2}{\tau}\frac{\tau_d s+1}{(s-p)(s-\bar{p})(s+1/\tau)^2} \tag{16.2.11}$$

The fraction of polynomials in Equation 16.2.11 can be expanded in the following form:

$$L\left[\theta_f(t)\right]/R_p\frac{\omega_n^2}{\tau} \equiv F_f(s) = \frac{\tau_d s+1}{(s-p)(s-\bar{p})(s+1/\tau)^2} = \frac{CR}{s-p} + \frac{\overline{CR}}{s-\bar{p}} + \frac{A}{(s+1/\tau)^2} + \frac{B}{s+1/\tau} \tag{16.2.12}$$

CR, A, and B are constants that we need to find. We have not previously encountered a case of repeated poles such as this. The last two right-hand-side terms in Equation 16.2.12 show the form that a partial-fraction expansion must have for repeated poles (Ogata, 1998, p. 33). We find complex residue CR using the conventional method:

$$CR = \left[(s-p)F_f(s)\right]_{s=p} = \frac{\tau_d p+1}{(p-\bar{p})(p+1/\tau)^2} = \frac{\tau_d p+1}{j2\omega_d(p+1/\tau)^2} \tag{16.2.13}$$

in which $p = -\zeta\omega_n + j\omega_d$. To find constants A and B, it is useful to multiply through Equation 16.2.12 by the denominator factor with repeated poles (Ogata, 1998, p. 33):

$$(s+1/\tau)^2F_f(s) = \frac{\tau_d s+1}{(s-p)(s-\bar{p})} = (s+1/\tau)^2\left(\frac{CR}{s-p} + \frac{\overline{CR}}{s-\bar{p}}\right) + A + (s+1/\tau)B \tag{16.2.14}$$

From Equation 16.2.14, we easily find constant A using the conventional method:

$$A = \left[(s+1/\tau)^2F_f(s)\right]_{s=-1/\tau} = \frac{-\tau_d/\tau+1}{(-1/\tau-p)(-1/\tau-\bar{p})} = \frac{1-\tau_d/\tau}{(1/\tau+p)(1/\tau+\bar{p})} \tag{16.2.15}$$

Finding constant B is expedited by first differentiating Equation 16.2.14 with respect to s, so that constant B stands alone in the last right-hand-side term:

$$\begin{aligned}\frac{d}{ds}\left[(s+1/\tau)^2F_f(s)\right] &= \frac{(s-p)(s-\bar{p})\tau_d - \left(\tau_d s+1\right)(2s-p-\bar{p})}{\left[(s-p)(s-\bar{p})\right]^2} \\ &= \frac{d}{ds}\left[(s+1/\tau)^2\left(\frac{CR}{s-p} + \frac{\overline{CR}}{s-\bar{p}}\right)\right] + B \\ &= 2(s+1/\tau)\left(\frac{CR}{s-p} + \frac{\overline{CR}}{s-\bar{p}}\right) + (s+1/\tau)^2\frac{d}{ds}\left(\frac{CR}{s-p} + \frac{\overline{CR}}{s-\bar{p}}\right) + B\end{aligned} \tag{16.2.16}$$

Hence, we find constant B by evaluating Equation 16.2.16 at $s = -1/\tau$:

$$\begin{aligned}B &= \left\{\frac{(s-p)(s-\bar{p})\tau_d - \left(\tau_d s+1\right)(2s-p-\bar{p})}{\left[(s-p)(s-\bar{p})\right]^2}\right\}_{s=-1/z} \\ &= \frac{(1/\tau+p)(1/\tau+\bar{p})\tau_d + (1-\tau_d/\tau)\left(2/\tau+p+\bar{p}\right)}{\left[(1/\tau+p)(1/\tau+\bar{p})\right]^2}\end{aligned} \tag{16.2.17}$$

Even though p is complex, $p = -\zeta\omega_n + j\omega_d$, the constants A in Equation 16.2.15 and B in Equation 16.2.17 turn out to be real. The factors that appear to be complex are: $(1/\tau+p)(1/\tau+\bar{p}) = (1/\tau-\zeta\omega_n+j\omega_d)(1/\tau-\zeta\omega_n-j\omega_d) = (1/\tau-\zeta\omega_n)^2+\omega_d^2$, which is real; and $2/\tau+p+\bar{p} = 2(1/\tau-\zeta\omega_n)$, which also is real. Thus, the final, more clearly real formulas for constants A in Equation 16.2.15 and B in Equation 16.2.17 are

$$A = \frac{1 - \tau_d/\tau}{(1/\tau - \zeta\omega_n)^2 + \omega_d^2}, B = \frac{\left[(1/\tau - \zeta\omega_n)^2 + \omega_d^2\right]\tau_d + 2\,(1 - \tau_d/\tau)\,(1/\tau - \zeta\omega_n)}{\left[(1/\tau - \zeta\omega_n)^2 + \omega_d^2\right]^2} \tag{16.2.18}$$

With complex residue CR from Equation 16.2.13, and real constants A and B from Equation 16.2.18, the forced-response equation is found from the inverse transform of Equation 16.2.12:

$$\theta_f(t) = R_p \frac{\omega_n^2}{\tau} L^{-1}\left[F_f(s)\right] = R_p \frac{\omega_n^2}{\tau}\left[2|CR|e^{-\zeta\omega_n t}\cos\left(\omega_d t + \angle CR\right) + (At + B)e^{-t/\tau}\right] \tag{16.2.19}$$

The terms $(At + B)e^{-t/\tau}$ in Equation 16.2.19 are associated with the poles of the forcing function $r(t) = R_p(t/\tau)e^{-t/\tau}$. Observe that these terms vary in time similarly to $r(t)$, but that they have **no influence** over the positive or negative stability of the system. Only the exponential function $e^{-\zeta\omega_n t}$, which is in the term associated with the system (transfer-function) poles, determines the state of system stability.

The complete output equation is the combination of IC-response Equation 16.2.10 and forced-response Equation 16.2.19:

$$\theta(t) = \theta_{ic}(t) + \theta_f(t) \tag{16.2.20}$$

Let us calculate and plot responses for a particular system. The basic parameters are: rotor inertia $J = 0.00256$ lb-s^2-inch (the inertia of a small aluminum wheel about four inches in diameter); and product of sensitivities $K_r K_a = 0.0350$ lb-inch/radian. Let us specify for the closed-loop system: undamped natural frequency $f_n = \omega_n/(2\pi) = 1.0$ Hz (so that step-response rise time would be around ¼ s); and damping ratios $\zeta = +0.12$ for a case of positive stability, and $\zeta = -0.03$ for a case of instability. The MATLAB Mfile below uses Equation 15.4.4 to calculate the proportional gain $P = \omega_n^2 J/\left(K_r K_a\right) = 2.888$ and the derivative time constants $\tau_d = 2\zeta/\omega_n = 0.0382$ s and –0.00955 s listed in the alphanumeric output for the two different cases.

In order to have non-trivial initial conditions, let us specify initial rotation rate $\dot{\theta}(0) \equiv \dot{\theta}_0 = -0.29$ rad/s and initial position $\theta(0) \equiv \theta_0 = -0.025$ radian. For the ramped-exponential input function $r(t) = R_p(t/\tau)e^{-t/\tau}$, let us specify magnitude $R_p = 0.1 \times e$ radian = 0.2718 radian and time constant $\tau = 1.5$ s, so that $r_{\max} = R_p e^{-1} = 0.1$ radian at time $t = 1.5$ s. (Most of the input values specified above were chosen by trial and error to produce clear and instructive graphical output.)

A MATLAB script M-file for computing and graphing the response follows:

```
%MATLABdemo161.m

%Stability of a PD-controlled rotor

%Excitation with initial conditions and a ramped exponential pulse

J=2.56e-3;KrKa=0.035;fn=1;wn=2*pi*fn;P=wn^2*J/KrKa;

disp('P ='),disp(P)

zeta=[0.12 -0.03];

for nc=1:2;

zt=zeta(nc);wd=wn*sqrt(1-zt^2);p=-zt*wn+j*wd;Td=2*zt/wn;

disp('Case #, zeta, Td, p ='),disp([nc zt Td p])

%Initial-condition response

th0=-0.025;dth0=-0.29;

C=(th0-j*(dth0+zt*wn*th0)/wd)/2;abC=abs(C);anC=angle(C);

t=0:.05:10;

thic=2*abC*exp(-zt*wn*t).*cos(wd*t+anC);
```

```
%Forced response

Rp=0.1*exp(1);tau=1.5;r=Rp/tau*t.*exp(-t/tau);

CR=(Td*p+1)/(j*2*wd*(p+1/tau)^2);abCR=abs(CR);anCR=angle(CR);

c1=1-Td/tau;c2=1/tau-zt*wn;c3=c2^2+wd^2;A=c1/c3;

B=(c3*Td+2*(1-Td/tau)*c2)/c3^2;

thf=Rp*wn^2/tau*(2*abCR*exp(-zt*wn*t).*cos(wd*t+anCR)+(A*t+B).*exp(-t/tau));

subplot(2,1,nc),plot(t,thic+thf,t,r),grid

end
```

The command to execute the M-file, and the resulting alphanumeric output and graphical output (after use of MATLAB's editing features for annotation and graphical enhancement) are:

```
>> MATLABdemo161

P =

    2.8876

Case #, zeta, Td, p =

    1.0000 0.1200 0.0382 -0.7540 + 6.2378i

Case #, zeta, Td, p =

    2.0000 -0.0300 -0.0095 0.1885 + 6.2804i
```

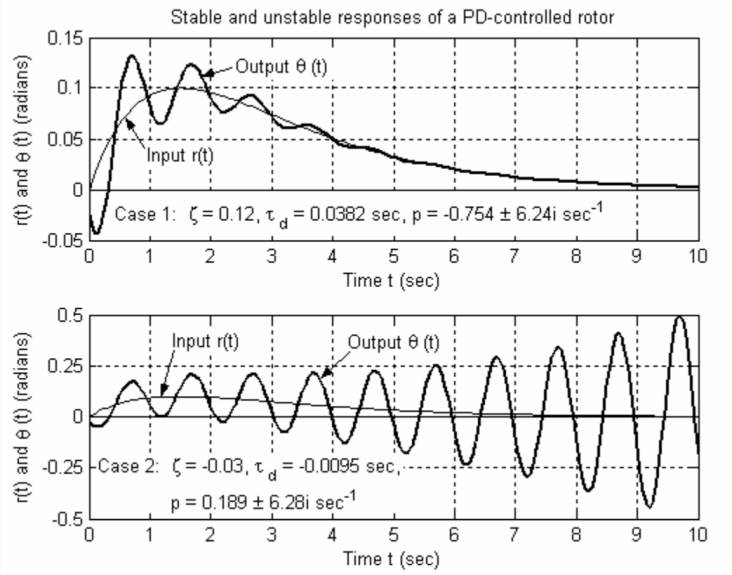


Figure 16.2.1: Ramped-exponential input operator setting $r(t)$ and output rotor positions $\theta(t)$ for an ideal proportional-derivative (PD) controlled rotor, with two different derivative time constants. (Copyright; author via source)

Case 1 of Figure 16.2.1 displays good performance of the PD control system: despite rather adverse initial conditions, the output follows the input nicely, especially after the initial transient response has decayed. In normal engineering practice, derivative time constant τ_d would probably be set considerably higher than in this academic case, which would suppress the transient oscillation even more quickly and produce even better controlled response.

Case 2, on the other hand, exhibits a serious dynamic instability, which would be unacceptable in normal engineering practice. Note the different y-axis scales for Cases 1 and 2. In real life, the oscillations would continue to grow exponentially until some mechanical or electrical part would fail (possibly a protective fuse that is designed to fail under excessive load), or some mechanical governor would limit motion, or some electrical component (*e.g.*, an op-amp) would saturate, or until the control system would somehow be disabled intentionally. This example demonstrates that **feedback control systems can be dangerous**; engineers should design and test any feedback control system with great care in order to make certain that it is positively stable and safe.

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16.3: Routh's Stability Criteria

We denote the transfer function of an nth order LTI system as $TF(s) = \frac{\text{Num}(s)}{\text{Den}(s)}$, in which $\text{Den}(s)$ is an nth degree polynomial in s. As derived in Section 16.1, the system is stable or unstable depending upon the signs of the roots of the characteristic equation,

$$\text{Den}(p) = a_1 p^n + a_2 p^{n-1} + \cdots + a_n p + a_{n+1} = 0 \tag{16.3.1}$$

For positive stability, we must have $\text{Re}\,(p_k) < 0$ for all roots, $k = 1, 2, \ldots, n$. The roots are dependent upon the polynomial coefficients $a_1, a_2, \ldots, a_n, a_{n+1}$, so it would seem that the state of stability could be determined directly from these coefficients, without the necessity of calculating the roots. This is indeed the case, and the conditions that these coefficients must satisfy for positive stability are known as Routh's stability criteria (after Edward John Routh, 1831-1907, English mathematician, physicist, and educator, who developed the systematic framework). These quantitative criteria can be written for a system of any order n, but they become progressively more complicated as n increases. For that reason, we examine in this section only the criteria for 1st, 2nd, and 3rd order systems; Routh's criteria for 4th order systems are presented in homework Problem 16.9(c).

The first requirement for a system of any order, for the system to be positively stable, is that all coefficients of characteristic Equation 16.3.1 be non-zero and have the same sign. This is a necessary condition but, for $n > 2$, not a sufficient condition for stability. Consider the simplest case, any **1st order system**:

$$\text{Den}(p) = a_1 p + a_2 = 0 \Rightarrow p = -\frac{a_2}{a_1} \tag{16.3.2}$$

It is clear that $p < 0$ if a_1 and a_2 are both non-zero and of the same polarity, so no other condition is required for stability. Consider next any **2nd order system**:

$$\text{Den}(p) = a_1 p^2 + a_2 p + a_3 = 0 \tag{16.3.3}$$

The quadratic formula gives the following roots of Equation 16.3.3:

$$p_{1,2} = -\frac{a_2}{2a_1} \pm \sqrt{\left(\frac{a_2}{2a_1}\right)^2 - \frac{a_3}{a_1}} \tag{16.3.4}$$

Careful study of Equation 16.3.4 shows that these roots have negative real parts **only** if a_1, a_2 and a_3 all are non-zero and have the same sign. Therefore, for the 2nd order system, the requirement that all three polynomial coefficients be non-zero and have the same sign is a sufficient as well as necessary condition for stability. Note that $a_2 = 0$ means the system has zero damping; if, in addition, a_1 and a_3 have the same sign, then the free-vibration response of the system is pure sinusoidal oscillation (at the natural frequency) with constant amplitude. In this case, the system is not unstable, since the response is bounded, but it is also not exponentially stable; the response does not decay away to a static equilibrium state.

Finally, let us consider the more challenging **3rd order system**:

$$\text{Den}(p) = a_1 p^3 + a_2 p^2 + a_3 p + a_4 = 0 \tag{16.3.5}$$

For the system to be stable, a_1, a_2, a_3 and a_4 all must be non-zero and have the same sign. The ***other requirement* for positive stability of a 3rd order system** establishes upper and lower bounds on the product $a_1 \times a_4$:

$$a_2 \times a_3 > a_1 \times a_4 > 0 \tag{16.3.6}$$

The derivation of these two Routh criteria for stability of a 3rd order system is relatively simple, but more lengthy than necessary for our study; it is presented clearly by Cannon, 1967, pages 406-409.

Let us evaluate Routh's stability criteria applied to the 3rd order system depicted on Figure 16.3.1. It is similar to the system of Section 14.3 for output-position-feedback control of rotor position, but there are two additions to that system. First, there is bearing friction that produces a passive rotational moment, $-c_\theta \dot{\theta}$, with c_θ being the viscous damping constant. Second, there is, in the feedback branch downstream of the position sensor, an electrical 1st order low-pass filter with time constant τ_L. The filtering

of sensor voltage signals in this fashion is very common in practice. The usual purpose for low-pass filtering is to cut out high-frequency electrical components from the sensor signal. Such unwanted components might arise from the electronics in a "noisy" sensor, from stray electromagnetic fields and/or improperly shielded electrical cables, and even from extraneous mechanical vibrations. So the amplitude-reduction function of a low-pass filter is almost always beneficial to control-system functioning. However, a low-pass filter also changes the **phase** of sensor signals, as well as reducing the amplitude of high-frequency-noise, as shown on Figure 4.3.2.

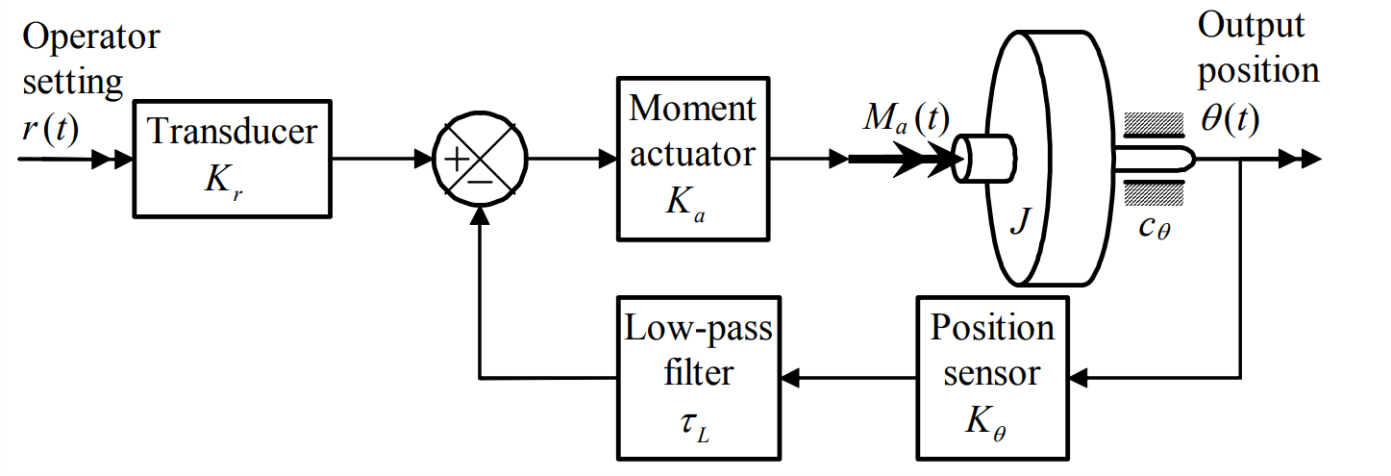


Figure 16.3.1: Functional diagram of a rotor position-feedback control system with a 1st order low-pass filter in the feedback branch. (Copyright; author via source)

Phase shifts can produce adverse effects on control systems, including reduction of control effectiveness and degradation of system stability, even to instability. From Figure 4.3.2, we see that the extreme phase shift introduced by a 1st order low-pass filter is −90°, and we can think of this as "almost changing the sign" of the signal that goes through the filter. According to Routh's criteria, the change of a sign can produce instability. (The effect on stability would be even more dramatic if there were a **2nd** order, rather than 1st order, low-pass filter in the feedback branch, Figure 10.2.1 with $\zeta >\sim 0.5$; the extreme phase shift produced by a 2nd order filter is −180°, which is a complete sign change in the signal that goes through the filter. See homework Problem 16.5.)

The ODE of motion for the damped rotor comes from Equation 3.3.2: $J\dot{p} + c_\theta p = J\ddot{\theta} + c_\theta\dot{\theta} = M_a(t)$. With the notation $L[\theta(t)] \equiv \Theta(s)$, the Laplace transform for zero ICs of the ODE of motion is $(Js^2 + c_\theta s)\,\Theta(s) = L\,[M_a(t)]$, so that the transfer function of the basic damped-rotor plant is

$$\frac{\Theta(s)}{L\,[M_a(t)]} = \frac{1}{Js^2 + c_\theta s} \tag{16.3.7}$$

With plant transfer function Equation 16.3.7 and low-pass-filter transfer function Equation 13.1.3, Figure 16.3.2 is the Laplace block diagram for the system of Figure 16.3.1. The branch transfer functions for the loop in Figure 16.3.1 are $G(s) = \frac{K_a}{Js^2+c_\theta s}$ and $H(s) = \frac{K_\theta}{\tau_L s+1}$. Hence, with use of Equation 14.4.6, we find the following closed-loop transfer function:

$$\frac{\Theta(s)}{R(s)} = K_r \times \frac{N_G D_H}{D_G D_H + N_G N_H} = K_r \times \frac{K_a\,(\tau_L s + 1)}{(Js^2 + c_\theta s)\,(\tau_L s + 1) + K_a K_\theta}$$

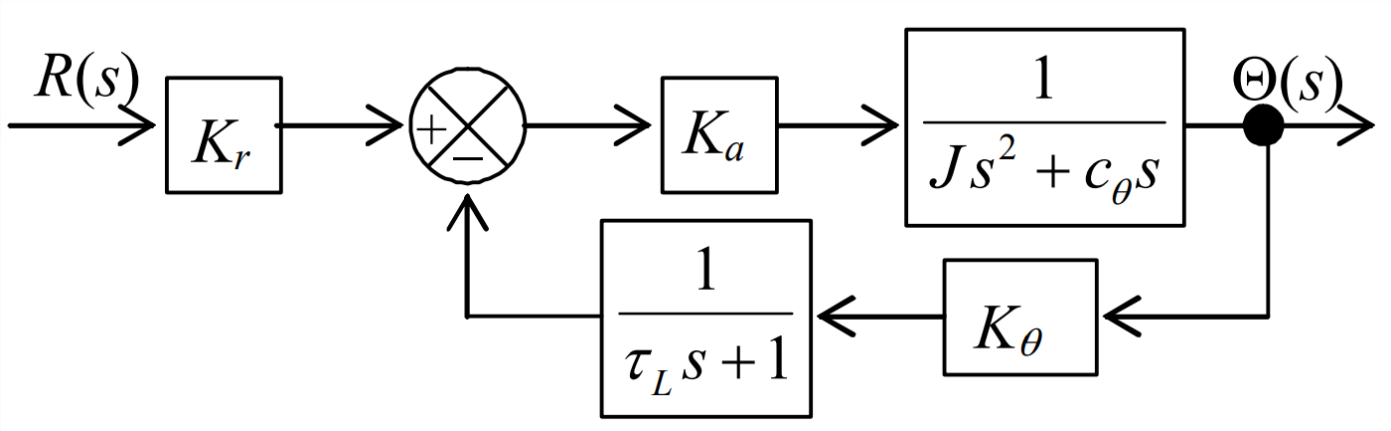


Figure 16.3.2: Laplace block diagram of positionfeedback control system with low-pass filtering. (Copyright; author via source)

Algebraic manipulation of this transfer function casts it into the following more useful forms:

$$\frac{\Theta(s)}{R(s)} = \frac{K_r K_a}{J} \frac{s + 1/\tau_L}{s\left(s + \frac{c_\theta}{J}\right)\left(s + \frac{1}{\tau_L}\right) + \frac{K_a K_\theta}{J\tau_L}} = \frac{K_r K_a}{J} \frac{s + 1/\tau_L}{s^3 + \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)s^2 + \frac{c_\theta}{J}\frac{1}{\tau_L}s + \frac{K_a K_\theta}{J\tau_L}} \tag{16.3.8}$$

The two final algebraic forms of Equation 16.3.8 are intended to expedite both evaluation by Routh's criteria next and by a more general analysis of stability in Section 16.4.

From transfer function Equation 16.3.8 are intended to expedite both evaluation by Routh's criteria next and by a more general analysis of stability in Section , we find the characteristic equation:

$$\text{Den}(p) \equiv a_1 p^3 + a_2 p^2 + a_3 p + a_4 = p^3 + \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)p^2 + \frac{c_\theta}{J}\frac{1}{\tau_L}p + \frac{K_a K_\theta}{J\tau_L} = 0 \tag{16.3.9}$$

Routh's first set of criteria for positive stability requires that all polynomial coefficients be non-zero and of the same sign. Since $a_1 = 1$, the following inequalities are necessary but not sufficient conditions for positive stability:

$$a_2 = \frac{1}{\tau_L} + \frac{c_\theta}{J} > 0, \quad a_3 = \frac{c_\theta}{J}\frac{1}{\tau_L} > 0, \text{ and } \quad a_4 = \frac{K_a K_\theta}{J\tau_L} > 0 \tag{16.3.10}$$

Rotational inertia J and filter time constant τ_L should always be positive physical constants, so Equations 16.3.10 require for positive stability that damping constant c_θ must be positive, and that the product of sensitivities $K_a K_\theta$ must be positive. Routh's final requirement for positive stability of a 3rd order system is Equation 16.3.6:

$$a_2 \times a_3 > a_1 \times a_4 > 0 \Rightarrow \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J}\frac{1}{\tau_L} > \frac{K_a K_\theta}{J\tau_L} > 0 \Rightarrow \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J} > \frac{K_a K_\theta}{J} > 0 \tag{16.3.11}$$

The common factor $1/\tau_L$ was removed to produce the final form of Equation 16.3.11.

Let us suppose that parameters J, c_θ, and τ_L are positive and fixed, but that the positive product of sensitivities $K_a K_\theta$ is a control gain which we have the ability to vary in order to modify the performance and/or stability of the control system. It will be convenient for continued analysis to define a special symbol that denotes the varying fraction in Equation 16.3.11 and to express the stability requirement in terms of that symbol:

$$\frac{K_a K_\theta}{J} \equiv \Lambda \Rightarrow \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J} > \Lambda > 0 \tag{16.3.12}$$

Equation 16.3.12 establishes **upper and lower stability boundaries** for $K_a K_\theta / J \equiv \Lambda$. To illustrate, let us express the stability criterion numerically for some plausible system values. Suppose that the circular break frequency of the low-pass filter is $\omega_b \equiv 1/\tau_L = 300$ rad/s, corresponding to cyclic break frequency $f_b = \omega_b / 2\pi = 47.75$ Hz, a realistic value for a low-pass filter used in the control of a mechanical system. Suppose also that the plant parameters have values such that $c_\theta / J = 100$ s^{-1}. Substituting these constants into Equation 16.3.12 gives the criterion: 40,000 s^{-2} $> \Lambda > 0$ for positive stability.

To test the correctness of Routh's criteria, let us calculate with MATLAB some responses of this system to the initial condition

$\theta(0) \equiv \theta_0 = 0.1$ radian, with all other ICs zero and with zero input, $r(t) = 0$. Let us find the IC response for these three cases:

Case I $\Lambda_{\mathrm{I}} = 0.75 \times \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J} = 30,000$ s^{-2}; this should give stable response.

Case II $\Lambda_{\mathrm{II}} \equiv 1.00 \times \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J} = 40,000$ s^{-2}, the upper boundary of stability; but Λ is supposed to be **less than** 40,000 s^{-2} for stability (not equal to it), so what will be the nature of this response?

Case III $\Lambda_{\mathrm{III}} = 1.25 \times \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right)\frac{c_\theta}{J} = 50,000$ s^{-2}; this should give unstable response.

To solve for IC responses of this system, we adapt Equation 16.1.2 to this problem. A little algebra leads us from Equation 16.1.2 to the following transform of the IC response:

$$\Theta(s) = \frac{\theta_0\left(a_1 s^2 + a_2 s + a_3\right)}{a_1 s^3 + a_2 s^2 + a_3 s + a_4} \tag{16.3.13}$$

The a_k constants in Equation 16.3.13 are defined in Equation 16.3.10, and the numerical parameters are given in the paragraph following Equation 16.3.12. For the numerical values of Cases I, II, and III described above, the partial-fraction expansion of Equation 16.3.13 has the form

$$\Theta(s) = \sum_{k=1}^{3} \frac{C_k}{s - p_k} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + \frac{\overline{C_2}}{s - \overline{p_2}} \tag{16.3.14}$$

in which residue C_1 and pole p_1 are real constants, with p_1 negative for all three cases, and residue C_2 and pole $p_2 \equiv \sigma_2 + j\omega_2$ are complex constants. Therefore, from Equations 16.1.9 and 16.1.12, the time response has the form

$$\theta(t) = C_1 e^{p_1 t} + 2\left|C_2\right| e^{\sigma_2 t}\cos\left(\omega_2 t + \angle C_2\right) \tag{16.3.15}$$

The MATLAB script M-file below computes the constants of transform Equation 16.3.14, using MATLAB's `residue` operation (see homework Problem 2.15), and it computes and plots time response Equation 16.3.15.

```
%MATLABdemo162.m

%Stability of a damped rotor with low-pass-filtered position feedback

%Excitation with initial position

wb=300;covrJ=100;a2=wb+covrJ;a3=wb*covrJ;Lmub=a2*covrJ;

th0=0.1;t=0:0.0004:0.16;format short e,figure,hold

Lm=[0.75 1 1.25]*Lmub;

for nc=1:3;

    a4=Lm(nc)*wb;

    Num=th0*[1 a2 a3];Den=[1 a2 a3 a4];[C,p,k]=residue(Num,Den);

    disp('Case #, Lambda ='),disp([nc Lm(nc)])

    disp('Poles p ='),disp(p.'),disp('Residues C ='),disp(C.')

    sig=real(p(2));wd=imag(p(2));abC=abs(C(2));anC=angle(C(2));

    th=C(1)*exp(p(1)*t)+2*abC*exp(sig*t).*cos(wd*t+anC);

    plot(t,th,'k')

end
```

```
grid,xlabel('Time t (sec)'),ylabel('Position \theta (t) (radian)')

title('Response of a 3^r^d order system with different control gains')
```

The command to execute the M-file, and the resulting alphanumeric output and graphical output (after use of MATLAB's editing features for annotation and graphical enhancement) follow:

```
>> MATLABdemo162

Current plot held

Case #, Lambda =

    1 30000

Poles p =

-3.8302e+002 -8.4887e+000 +1.5305e+002i -8.4887e+000 -1.5305e+002i

Residues C =

1.4354e-002 4.2823e-002 -2.0336e-002i 4.2823e-002 +2.0336e-002i

Case #, Lambda =

    2 40000

Poles p =

-4.0000e+002 -4.3965e-014 +1.7321e+002i -4.3965e-014 -1.7321e+002i

Residues C =

1.5789e-002 4.2105e-002 -1.8232e-002i 4.2105e-002 +1.8232e-002i

Case #, Lambda =

    3 50000

Poles p =

-4.1484e+002 7.4222e+000 +1.9001e+002i 7.4222e+000 -1.9001e+002i

Residues C =

1.6864e-002 4.1568e-002 -1.6786e-002i 4.1568e-002 +1.6786e-002i
```

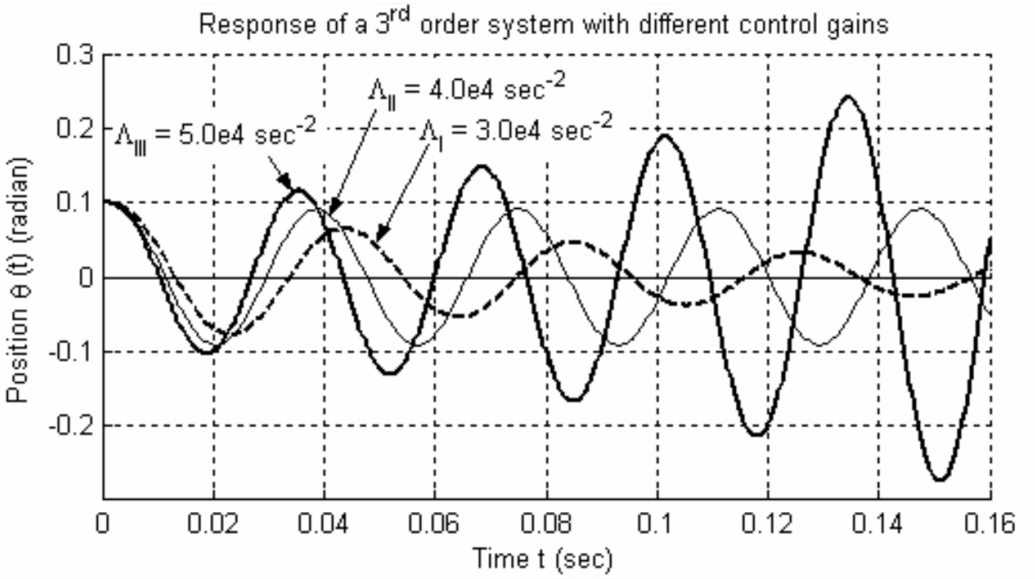


Figure 16.3.3: Initial-condition response of a rotor output-position-feedback control system with a 1st order low-pass filter in the feedback branch. (Copyright; author via source)

For this system with three poles, at least one pole must be real. For Cases I, II, and III evaluated above, this real pole, p_1, turns out to be strongly negative, so it does not contribute to any instability. The second pole, p_2, is complex, and the third pole is the conjugate of the second. (The other possibility for a 3rd order system is that all three poles can be real, but the gains Λ used in this example preclude that.) For gain Λ_{I}, $\mathrm{Re}\,(p_2)$ is negative; consequently, the Case I system is positively stable, as predicted by Routh, Equation 16.3.12. For gain Λ_{III}, $\mathrm{Re}\,(p_2)$ is positive; consequently, the Case III system is unstable, also as predicted by Routh. Gain Λ_{II} is on the upper boundary between positive and negative stability, according to Routh, and the alphanumeric output above shows that $\mathrm{Re}\,(p_2) = 0$. (The tiny number calculated, –4.4e–14, is non-zero due to roundoff error; comparison of this number with the values of the other poles shows that it is effectively zero.) We see from Figure 16.3.3 that the time response associated with Case II is a pure, undamped sinusoid; the response is bounded, therefore not unstable, but it is not exponentially stable, *i.e.*, it does not decay to zero.

The control system of Figures 16.3.1 and 16.3.2 has two major dynamic sub-systems: the damped rotor with transfer function $1/\left[s\left(Js + c_\theta\right)\right]$; and the low-pass filter with transfer function $1/\left(\tau_L s + 1\right)$. Each of these *passive* sub-systems by itself dissipates energy, *i.e.*, acts as an *energy sink*, and is positively stable. But the complete system is clearly capable of being unstable. The complete system includes an *energy source*, the moment actuator, and therefore is said to be *active*. Any system that includes feedback and an energy source is potentially capable of being unstable.

Observe that Routh's criteria essentially tell us only that a system is stable or unstable, the nature of stability in an *absolute* sense. They do not tell us the **type** of response (monotonic-exponential or oscillatory) or the **degree** of stability (essentially, the magnitudes of the real parts of transfer-function poles), which is also known as *relative stability*. The method introduced in the next section finds the relative stability.

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16.4: Loci of Roots for Second Order Systems

Let us seek both the **type** of response and the **degree** of stability, by directly solving for the roots of the characteristic equation $\mathrm{Den}(p) = 0$ [*i.e.*, the poles of $TF(s)$ or $CLTF(s)$, whichever is appropriate]. Further, we will observe how these roots vary as some particular system parameter is changed. The graphical representation of this variation in the complex p-plane is known commonly as $locusofroots$; however, systems of 2nd and higher orders always have multiple roots, so it is more correct to use the Latin plural: *loci* of roots. Stability analysis using loci of roots is more general than the basic Routh's criteria, and more informative about the character of a system and the nature of its stability.

We begin by studying the roots of any 2nd order system. This will be instructive and will also establish the basis for interpretation of loci-of-roots graphs for LTI systems of higher order. The characteristic equation of any 2nd order system can be expressed as:

$$\mathrm{Den}(p) = p^2 + 2\zeta\omega_n p + \omega_n^2 = 0 \tag{16.4.1}$$

Some examples of 2nd order system transfer functions that lead to Equation 16.4.1 are Equations 10.2.3, 10.4.4, 13.2.3, 14.5.4, 15.2.19, and 15.4.3. We are accustomed to studying stable systems for which ζ and ω_n^2 are positive values. However, these parameters are not necessarily always positive in systems; they can be negative, especially for feedback control systems and for *self-excited* systems. An example of a self-excited system is a flexible wing in an airstream, as discussed in Examples 11.3.2 and 11.3.3 of Section 11.3.

First, let us study the roots of Equation 16.4.1 when ζ varies, in both magnitude and sign, while ω_n is fixed and positive, $\omega_n > 0$. Solving first for $|\zeta| \geq 1$, using the quadratic formula, gives the two real-valued poles:

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{16.4.2}$$

For $|\zeta| \leq 1$, the poles are the complex conjugate pair:

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \equiv -\zeta\omega_n \pm j\omega_d \tag{16.4.3}$$

Figure 16.4.1 shows the loci of roots Equations 16.4.2 and 16.4.3 as ζ varies from more negative than –1 to more positive than +1. Note that the x-axis is the real part of the roots, $\mathrm{Re}(p)$, and the y-axis is the imaginary part, $\mathrm{Im}(p)$; these two axes encompass the complex p-plane. For selected values of ζ, the roots are indicated by $\times$'s and the ζ values are labeled. The directions of the loci as ζ increases are indicated by arrowheads.

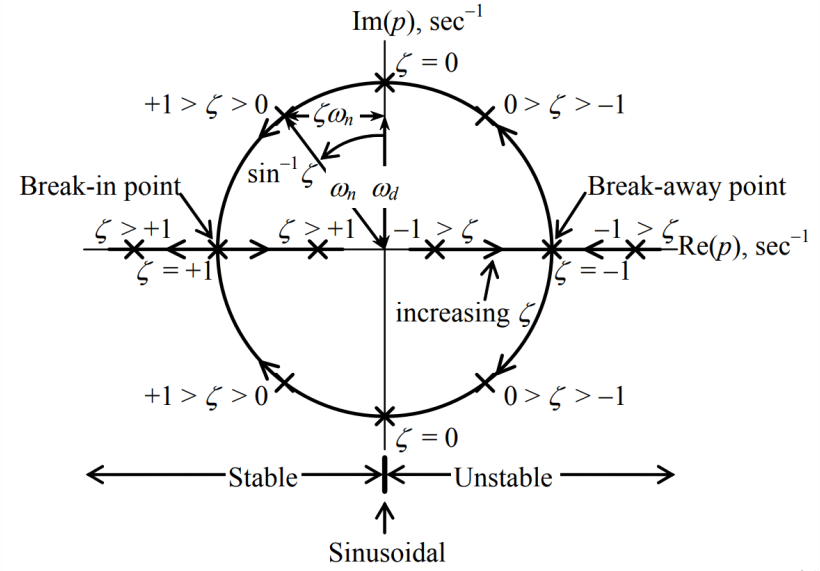


Figure 16.4.1: Loci of roots of $p^2 + 2\zeta\omega_n p + \omega_n^2 = 0$ as ζ varies from less than –1 to greater than +1, with $\omega_n > 0$ held constant. (Copyright; author via source)

For $-1 > \zeta$, the two real roots, Equation 16.4.2, are in the right ("east") half of the p-plane, on the $\mathrm{Re}(p)$ axis; the type of time response associated with these roots is monotonically growing exponential, an instability. At $\zeta = -1$, these two real roots come together with value $+\omega_n$; this is called a "break-away" point because, for the slightest increase in ζ above –1, the roots separate into complex conjugates, Equation 16.4.3, one root headed directly "northward" and the other directly "southward." The type of response associated with complex conjugate roots that are in the right half-plane off the $\mathrm{Re}(p)$ axis is oscillation modulated by a growing exponential envelope, an instability. For $+1 > \zeta > -1$, the loci of roots form a perfect circle of radius ω_n and centered on the origin: $x^2 + y^2 = (-\zeta\omega_n)^2 + (\pm\omega_d)^2 = \omega_n^2$. The roots for $\zeta = 0$ are imaginary, $p_{1,2} = \pm j\omega_n$, for which the time response has pure, undamped sinusoidal form. For $+1 > \zeta > 0$, the complex conjugate roots are in the left ("west") half-plane off the $\mathrm{Re}(p)$ axis; the type of time response associated with these complex conjugate roots is oscillation modulated by a decaying exponential envelope, which is stable response. As increasing ζ approaches the value +1, the two complex roots come together and become real, with value $-\omega_n$, at the "break-in" point. Finally, as ζ increases above +1, one real root heads farther westward on the $\mathrm{Re}(p)$ axis, and the other creeps eastward toward the origin; the type of time response associated with these real roots in the left half-plane is monotonically decaying exponential, which is stable response.

Keep in mind that both monotonic-exponential response and oscillatory response are bounded by the exponential envelope $e^{\mathrm{Re}(p)t}$. If $\mathrm{Re}(p) < 0$, in the left half-plane, then we define, as in Equation 9.4.3, the 2nd order *time constant*, $\tau_2 \equiv -1/\mathrm{Re}(p) > 0$, so that the decaying exponential envelope is e^{-t/τ_2}; τ_2 is the time for the **stable** exponential envelope to **decrease** by the factor e^{-1}. The **degree of positive stability** associated with a particular stable root increases as $\mathrm{Re}(p)$ becomes progressively more negative and τ_2 becomes progressively smaller. In other words, the farther westward from the origin that a pole is located, the greater is the degree of positive stability associated with that pole. On the other hand, if $\mathrm{Re}(p) > 0$, in the right half-plane, then $+1/\mathrm{Re}(p)$ is the time for the **unstable** exponential envelope to **increase** by the factor e^{+1}. The **degree of instability** associated with a particular unstable root increases progressively as $\mathrm{Re}(p)$ becomes progressively more positive; in other words, the instability worsens as the unstable pole moves farther eastward from the origin. Although these observations are made in the context of Figure 16.4.1, they are equally applicable to the loci of roots of any LTI system having any order.

Note on Figure 16.4.1 the construction of a right triangle consisting of horizontal leg $\zeta\omega_n$, vertical leg ω_d, and hypotenuse ω_n, which extends from the origin to a complex pole p. This right triangle is based on the definition $\omega_d^2 = \omega_n^2\,(1 - \zeta^2) = \omega_n^2 - (\zeta\omega_n)^2$, which is valid for $|\zeta| < 1$. Thus, the frequency of vibration (in rad/s) associated with this particular complex pole is the vertical leg ω_d along the $\mathrm{Im}(p)$ axis. Further, the horizontal leg $\zeta\omega_n$ (if in the left half-plane) determines the time constant, $\tau_2 = 1/\zeta\omega_n$ (in s). Finally, the value of damping ratio ζ is the sine of the angle labeled on Figure 16.4.1 between the $\mathrm{Im}(p)$ axis and hypotenuse ω_n; for a stable pole p in the left half-plane, as drawn on Figure 16.4.1, ζ is positive; for an unstable pole p in the right half-plane, ζ would be negative. Although this derivation is based on 16.4.1, these methods derived for determining the frequency ω_d, time constant τ_2, and damping ratio ζ associated with a particular complex pole are applicable to the loci of roots of any LTI system.

Figure 16.4.1 shows the loci of roots of a 2nd order system as generalized **damping**, represented by ζ, varies. Next, let us consider the loci of roots for a 2nd order system as generalized **stiffness** varies. Rather than using the symbols ζ and ω_n, it is better for this task to revert back to the notation of Section 9.1 representing a generalized mass-damperspring system. Accordingly, we have Equation 7.1.3, $\omega_n^2 = k/m$, and from Equation 9.1.6, $\omega_n^2 = k/m$. In these definitions, we specify that k represents a generalized stiffness, which will vary, and that c and m represent, respectively, generalized damping and generalized inertia, which are considered positive constants and will remain fixed while k varies. For convenience, let us use the following simplified notation:

$$\omega_n^2 = k/m \equiv K \quad \text{and} \quad \zeta\omega_n = c/(2m) \equiv C \tag{16.4.4}$$

Therefore, for the case $c/2m \geq k/m$, we re-write the real roots Equation 16.4.2 as

$$p_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -C \pm \sqrt{C^2 - K} \tag{16.4.5}$$

For $k/m \geq c/2m$, the complex conjugate roots Equation 16.4.3 are

$$p_{1,2} = -\zeta\omega_n \pm j\sqrt{\omega_n^2 - (\zeta\omega_n)^2} = -C \pm j\sqrt{K - C^2} \tag{16.4.6}$$

Figure 16.4.2 is an incomplete graph of loci of roots Equations 16.4.5 and ??? as K varies with $C > 0$ fixed. (Homework Problem 16.3 is an exercise for completing the task.) Partial annotation shows the locations of the two real poles for $K = 0$. Note, in particular, that for $K < 0$ one real locus is entirely in the right half-plane. The type of time response associated with this locus is monotonically growing exponential, an instability. (Of course, Routh also shows that the system is unstable for $K < 0$ and $C > 0$, since then coefficients of the characteristic equation have different signs.) The passive mechanical springs with which we are familiar always have positive stiffness, so it might appear that this type of instability is purely academic, without any practical significance. Such is **not** the case. Recall that, in the present discussion, k (and therefore K also) represents generalized or

total stiffness, not just mechanical stiffness. An elastically restrained airfoil within an air stream is one practical engineering system for which the total stiffness is not necessarily positive. As shown in the discussion accompanying Equation 11.3.8, the total mechanical plus aerodynamic stiffness of this system can become zero or even negative at high airspeeds; the resulting unbounded pitching of the airfoil, called *aeroelastic divergence*, is an important example of the type of instability represented in Figure $16.4.2$ for $K < 0$.

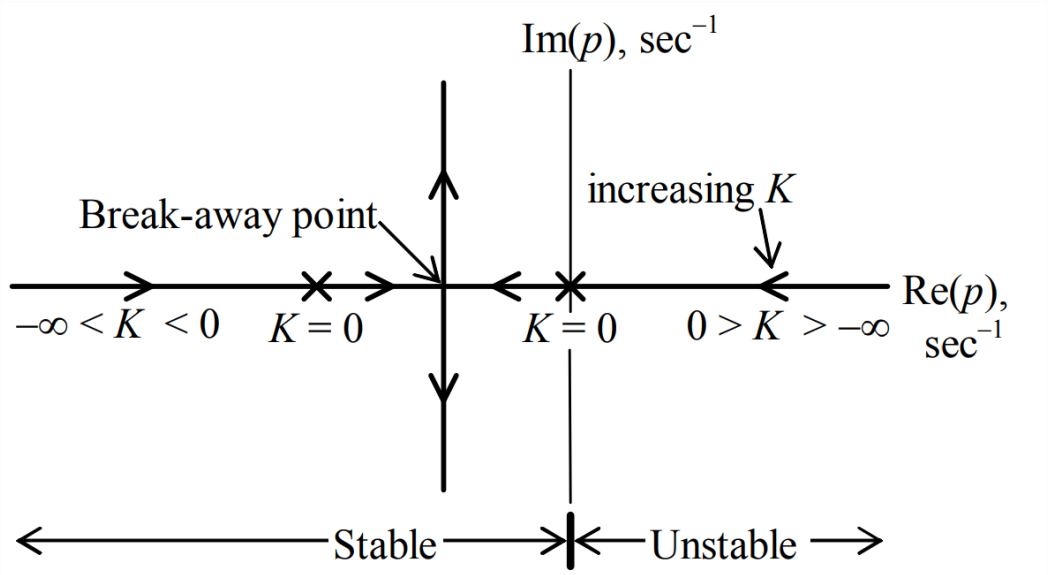


Figure $16.4.2$: Loci roots of $p^2 + 2Cp + K = 0$ as K varies from $-\infty$ to greater than C^2, with $C > 0$ held constant. (Copyright; author via source)

Developing loci-of-roots graphs Figure $16.4.1$ and $16.4.2$ for 2nd order systems is relatively simple, but useful as an instructive introduction to this type of stability analysis. In the next section, we will consider the more demanding task of determining loci of roots for systems of higher order.

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16.5: Loci of Roots for Third Order Systems

In Section 16.4, we used the quadratic formula to evaluate easily the loci of roots for 2nd order systems. It is more typical in practice, however, that engineering systems have higher orders than 2nd order, so that determining loci of roots requires repeatedly solving polynomial equations of 3rd and higher degrees, which can be a challenging task. This section introduces by example the methods of solution for higher-order systems that are commonly used in practice.

For this example, we consider again the system described in Section 16.3 (Figures 16.3.1, 16.3.2, and 16.3.3), a rotor position-feedback control system with a 1st order low-pass filter in the feedback branch. Closed-loop transfer function Equation 16.3.9 gives the 3rd degree characteristic equation in two different algebraic forms, one of which is

$$\text{Den}(p) = a_1 p^3 + a_2 p^2 + a_3 p + a_4 = p^3 + \left(\frac{1}{\tau_L} + \frac{c_\theta}{J}\right) p^2 + \frac{c_\theta}{J}\frac{1}{\tau_L} p + \frac{K_a K_\theta}{J \tau_L} = 0 \tag{16.5.1}$$

In Section 16.3, we held constant rotor inertia J, viscous damping constant c_θ of the bearings, and time constant τ_L of the 1st order low-pass filter, all positive values; in particular, we chose for numerical evaluation the values $c_\theta / J = 100$ s^{-1}, and, for the filter break frequency, $\omega_b \equiv 1/\tau_L = 300$ rad/s. We allowed the value of product $K_a K_\theta$ of control system sensitivities to vary. Using Routh's criteria, we determined the lower and upper stability boundaries for the parameter $K_a K_\theta / J \equiv \Lambda$: the system is positively stable provided that $0 < \Lambda < 40,000$ s^{-2}.

Now, let us conduct a more detailed evaluation of stability by calculating and plotting the loci of roots as Λ varies, using the same numerical values of the other constants as in Section 16.3. We solve numerically for the roots of polynomial Equation 16.5.1 using MATLAB's `root` operation: for each value of Λ, the input is the array of coefficients $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$, and the output is the 3×1 matrix of roots. We will calculate and plot the loci of roots twice: first, for a set of Λ values that we choose (guess) somewhat arbitrarily to encompass the entire possible range of practical interest; second, for another set of Λ values that are refined, based upon the results of the first pass, to focus on the specific roots of greatest practical significance.

For the first pass, we vary Λ from –30,000 s^{-2} to +70,000 s^{-2} in increments of 10,000 s^{-2}. The following is a MATLAB script M-file that computes the roots of polynomial Equation 16.5.1 repeatedly over these values of Λ, then prints the input and output values in a table, then plots the loci of roots:

```
%MATLABdemo163.m

%Stability of a damped rotor with low-pass-filtered position feedback

%Loci of roots

wb=300;covrJ=100;a2=wb+covrJ;a3=wb*covrJ;

Lm=[-3 -2 -1 0 1 2 3 4 5 6 7]*1.0e4;Np=length(Lm);

for i=1:Np;

    a4=Lm(i)*wb;

    p(i,1:3)=roots([1 a2 a3 a4]).';

end

nc=1:Np;anout=[nc' Lm' p];

disp('Roots #, Lambda, p'),disp(anout)

plot(p,'kx'),grid,xlabel('Real part of pole (sec^-^1)')
```

```
ylabel('Imag part of pole (sec^-^1)')

title('Loci of roots of a 3^r^d order system as \Lambda varies')
```

The command to execute the M-file, and the resulting tabular output follows:

```
>> MATLABdemo163

Roots #, Lambda, p

    1.0e+004 *

0.0001 -3.0000 -0.0253 + 0.0141i -0.0253 - 0.0141i 0.0107

0.0002 -2.0000 -0.0242 + 0.0111i -0.0242 - 0.0111i 0.0085

0.0003 -1.0000 -0.0227 + 0.0056i -0.0227 - 0.0056i 0.0055

0.0004 0 0 -0.0300 -0.0100

0.0005 1.0000 -0.0337 -0.0031 + 0.0089i -0.0031 - 0.0089i

0.0006 2.0000 -0.0363 -0.0019 + 0.0127i -0.0019 - 0.0127i

0.0007 3.0000 -0.0383 -0.0008 + 0.0153i -0.0008 - 0.0153i

0.0008 4.0000 -0.0400 -0.0000 + 0.0173i -0.0000 - 0.0173i

0.0009 5.0000 -0.0415 0.0007 + 0.0190i 0.0007 - 0.0190i

0.0010 6.0000 -0.0428 0.0014 + 0.0205i 0.0014 - 0.0205i

0.0011 7.0000 -0.0440 0.0020 + 0.0217i 0.0020 - 0.0217i
```

The associated loci-of-roots plot (after use of MATLAB's editing features for annotation and graphical enhancement) is Figure 16.5.1.

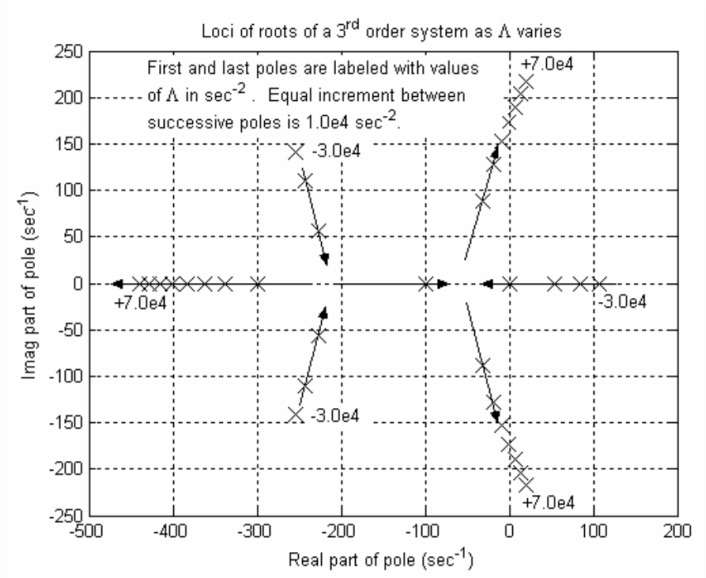


Figure 16.5.1: First-pass loci of roots of Equation 16.5.1 for varying control parameter $\Lambda = K_a K_\theta / J$. (Copyright; author via source)

We infer from Figure 16.5.1 the following relevant information. For $\Lambda < 0$, the system has an unstable pole associated with monotonic exponentially growing time response; we already knew from Routh that the system is unstable for $\Lambda < 0$, but Routh did

not reveal the type of response. For $\Lambda = 0$, the poles are all real and have the values 0, –100, and –300 s^{-1}. This is also obvious in the alternate algebraic form of the 3rd degree characteristic equation that comes from closed-loop transfer function Equation 16.3.9:

$$\text{Den}(p) = p\,(p + c_\theta/J)\,(p + \omega_b) + \omega_b\Lambda = 0 \tag{16.5.2}$$

From Equation 16.5.1 with $\Lambda = 0$, the poles are clearly $p_1 = 0$, $p_2 = -c_\theta/J = -100$ s^{-1}, and $p_3 = -\omega_b \equiv -1/\tau_L = -300$ s^{-1}. The arrows on Figure 16.5.1 indicate the directions of loci as Λ increases. The arrows show that there is a break-in point (as defined on Figure 16.4.1) for Λ somewhere in the range $-10,000s^{-2} < \Lambda < 0$, and a break-away point for Λ somewhere in the range $0 < \Lambda < +10,000$ s^{-2}. For future reference, let us denote the break-away value of Λ as Λ_{ba}, which is not yet determined. We also observe from Figure 16.5.1 and the tabulated roots that the upper boundary of stability (with purely complex roots, $p_{2,3} = \pm j173$ rad/s) is $\Lambda = 40,000$ s^{-2}, and that the time response is oscillatory for values of Λ in the vicinity of the upper boundary; we knew from Routh the value of the upper boundary, but not the type of response. Moreover, since the real pole for $\Lambda > \Lambda_{ba}$ is associated with heavily damped exponential decay, the complex loci for $\Lambda > \Lambda_{ba}$ of Figure 16.5.1 suggest that the time response is primarily oscillatory for $\Lambda > \Lambda_{ba}$. See Figure 16.3.3 for examples of this primarily oscillatory response. For $\Lambda > \Lambda_{ba}$, this system has a heavily damped exponential *mode of response* and a lightly damped oscillatory mode, and the latter, with poles closest to the imaginary axis, is the *dominant* mode.

Now let us refine the loci-of-roots calculation, using a new set of Λ values that would be of greatest practical interest. This set probably would consist of values for which the system is stable, namely $0 < \Lambda < 40,000$ s^{-2}. The value Λ_{ba} is significant: it is the boundary between a system (with $0 < \Lambda < \Lambda_{ba}$) for which all transient response is decaying exponential and a system (with $\Lambda_{ba} < \Lambda$) for which the dominant transient response is oscillatory. We certainly could choose the refined set of Λ values by numerical trial and error, letting MATLAB do most of the work. However, this situation provides the opportunity for introduction of a theoretical and more systematic approach, which we shall consider next. This approach is general, but our limited objective in this introduction will be to determine **theoretically** only the upper boundary of stability, $\Lambda \equiv \Lambda_{ub}$, and the break-away value Λ_{ba}.

This approach is based upon form Equation 16.5.2 of the characteristic equation, which we re-write as

$$p\,(p + c_\theta/J)\,(p + \omega_b) = -\omega_b\Lambda \tag{16.5.3}$$

It is important that the sub-system poles are separated into factors on the left-hand side of Equation 16.5.3. Each factor is a complex number, in general, and it can be expressed in complex polar form, as described in Section 2.1. It is helpful to regard each factor as a two-dimensional complex vector with its tail at the sub-system pole, so that we write

$$p = p - 0 \equiv r_1 e^{j\theta_1}, p + c_\theta/J = p - (-c_\theta/J) \equiv r_2 e^{j\theta_2}, p + \omega_b = p - (-\omega_b) \equiv r_3 e^{j\theta_3} \tag{16.5.4}$$

The real right-hand side of Equation 16.5.3 also is written in polar form: $-\omega_b\Lambda = \omega_b\Lambda \exp[j(\pm\pi, \pm 3\pi, ...)]$. Therefore, the entire characteristic equation is expressed in the following polar form, which any root must satisfy:

$$r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} \times r_3 e^{j\theta_3} = r_1 r_2 r_3 e^{j(\theta_1+\theta_2+\theta_3)} = \omega_b\Lambda \exp[j(\pm\pi, \pm 3\pi, ...)] \tag{16.5.5}$$

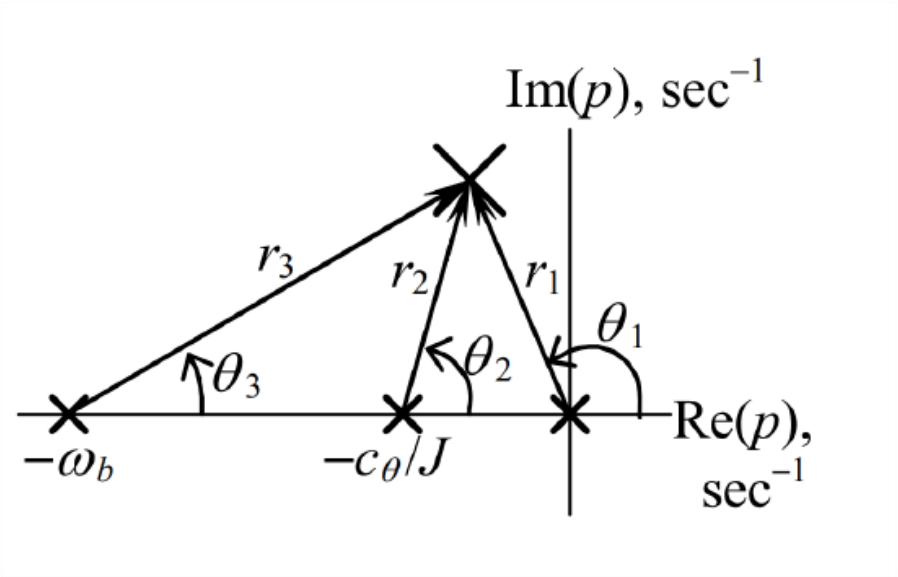


Figure 16.5.2 (Copyright; author via source)

The drawing above is a graphical representation of both the polar vectors of Equation 16.5.4 and characteristic equation Equation 16.5.5. The small $\times$'s are the sub-system poles, at which the tails of the complex vectors are located. The large $\mathbf{X}$ is a pole of the complete system, *i.e.*, a root of Equation 16.5.2, and a value that satisfies Equation 16.5.5. Complex equation Equation 16.5.5 includes two real equations, both of which must be satisfied by **any** system pole. The first equation is the equality of angles on the two sides of Equation 16.5.5:

$$\theta_1 + \theta_2 + \theta_3 = \pm\pi, \pm 3\pi, \ldots \tag{16.5.6}$$

The second equation is the equality of magnitudes:

$$r_1 r_2 r_3 = \omega_b \Lambda \Rightarrow \Lambda = \frac{r_1 r_2 r_3}{\omega_b} \tag{16.5.7}$$

For our first application of this approach, let us use Equation 16.5.6 to find the pole $j\omega_d$ at the upper stability boundary Λ_{ub}, and then Equation 16.5.7 to find the value of Λ_{ub} itself. The pole is shown on the drawing at right. We have by definition for this stability boundary: $r_1 = \omega_d$ and $\theta_1 = \pi/2$. Therefore, Equation 16.5.6 gives $\theta_2 + \theta_3 = \pi/2$. [We rule out the other possibilities $-\pi, \pm 3\pi, \ldots$ in Equation 16.5.6 just by inspecting the figure.] Some trigonometry is required now. Angle α between the arrowheads of r_1 and r_2 is labeled on the drawing. Angles α and θ_3 are the complementary acute angles of a right triangle, so we have also $\alpha + \theta_3 = \pi/2$. Therefore, $\alpha = \theta_2$. Hence, we find ω_d by equating the tangents of the two similar right triangles:

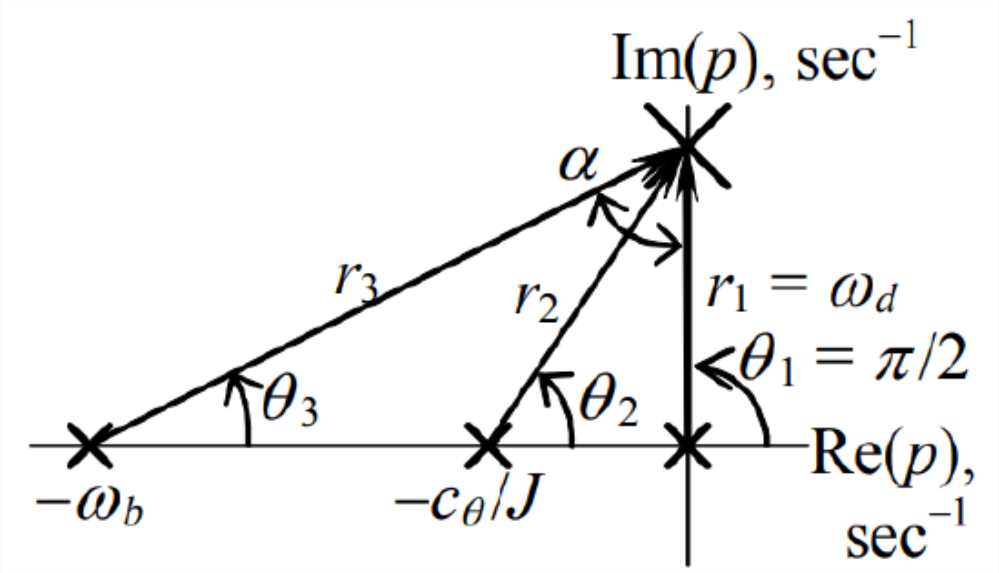


Figure 16.5.3 (Copyright; author via source)

$$\tan\theta_2 = \frac{\omega_d}{c_\theta/J} = \tan\alpha = \frac{\omega_b}{\omega_d} \Rightarrow \omega_d = \sqrt{\omega_b \frac{c_\theta}{J}} = \sqrt{300\text{s}^{-1} \times 100\text{s}^{-1}} = 173.2\frac{\text{rad}}{\text{s}}$$

Having $\omega_d = r_1$, we can use the theorem of Pythagoras to find $r_2 = \sqrt{\omega_d^2 + (c_\theta/J)^2} = \sqrt{\omega_b\,(c_\theta/J) + (c_\theta/J)^2}$ and $r_3 = \sqrt{\omega_d^2 + \omega_b^2}$ $= \sqrt{\omega_b\,(c_\theta/J) + \omega_b^2}$. Finally, we find Λ_{ub} by applying Equation 16.5.7, which, after a little algebra, gives

Extra close brace or missing open brace

This result for Λ_{ub} is, of course, identical to the upper boundary found by Routh's criteria in Equation 16.3.13.

For a more subtle application of this theoretical approach, let us: first, use Equation 16.5.6 to find the value of the two identical poles at the break-away point, which is evident on Figure 16.5.1 (on the real axis, somewhere between the origin and $-c_\theta/J = -100$ s^{-1}); and then, use Equation 16.5.6 to find the associated value of Λ_{ba}. It is obvious that Equation 16.5.6 is satisfied identically **right at** the break-away point, where $\theta_1 = \pi$ and $\theta_2 = \theta_3 = 0$, so this does not provide any useful information. But Equation 16.5.6 must apply also for all other points on the loci of roots, in particular now, for points that are **close to** the break-away point; this fact will lead to a useful result. Suppose that Λ is just slightly greater than Λ_{ba} so that one of the complex poles is a small distance $\mathcal{E}$ above the break-away point on the real axis, as shown on the drawing below. As $\mathcal{E}$ shrinks but remains > 0, the acute angles between the sub-system radii and the real axis become very small, so we can use in Equation 16.5.6 the approximation $\theta \approx \sin\theta$. Hence, Equation 16.5.6 gives

$$\theta_1 + \theta_2 + \theta_3 = \left(\pi - \frac{\varepsilon}{r_1}\right) + \frac{\varepsilon}{r_2} + \frac{\varepsilon}{r_3} = \pi \Rightarrow \varepsilon\left(-\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = 0$$

This equation must be valid even if $\mathcal{E}$ is very small but non-zero, so we conclude that

$$-\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 0 = \frac{-r_2 r_3 + r_1 r_3 + r_1 r_2}{r_1 r_2 r_3} \Rightarrow -r_2 r_3 + r_1 r_3 + r_1 r_2 = 0$$

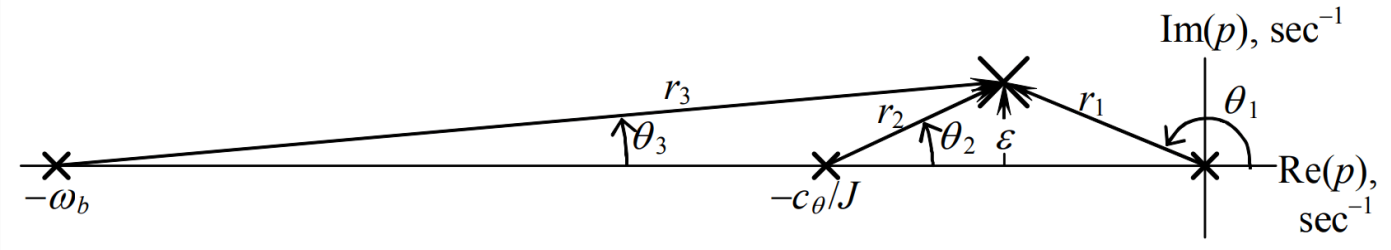


Figure 16.5.4 (Copyright; author via source)

In the limit as the pole goes to the break-away point on the real axis, $\varepsilon = 0$, the radii are related as $r_1 + r_2 = c_\theta / J$ and $r_1 + r_3 = \omega_b$. Substituting into the equation above to eliminate r_2 and r_3 gives $-r_2 r_3 + r_1 (r_3 + r_2) = -(c_\theta / J - r_1)(\omega_b - r_1) + r_1 (\omega_b - r_1 + c_\theta / J - r_1) = 0$, and this leads, after some algebra, to a quadratic equation in unknown r_1:

$$3r_1^2 - 2\left(\omega_b + \frac{c_\theta}{J}\right) r_1 + \omega_b \frac{c_\theta}{J} = 0$$

Solving for $\backslash r_1$ with the quadratic formula gives

$$r_1 = \frac{1}{3}\left[\left(\omega_b + \frac{c_\theta}{J}\right) \pm \sqrt{\left(\omega_b + \frac{c_\theta}{J}\right)^2 - 3\omega_b \frac{c_\theta}{J}}\right]$$

Substituting into this equation $\omega_b = 300$ s^{-1} and $c_\theta / J = 100$ s-1 gives the roots 221.53 and 45.14 s^{-1}, with the latter root corresponding to the break-away pole that we seek between the origin and –100 s^{-1} on the real axis, *i.e.*, $r_1 = 45.14$ s^{-1}. Then Equation 16.5.7 gives the associated value of the varying control parameter:

$$\Lambda_{ba} = \frac{r_1 r_2 r_3}{\omega_b} = \frac{r_1 (c_\theta / J - r_1)(\omega_b - r_1)}{\omega_b} = 2,103.77 \text{s}^{-2}$$

The approach used above to determine theoretically the stability-upper-bound and break-away values of the varying control parameter Λ, based upon Equations 16.5.3-16.5.7, is a small portion of an extensive set of procedures for completely determining loci of roots, which is known as the *root-locus method*. Walter R. Evans (1920-1999, American electrical engineer) developed and published this elegant method around 1950. Using the root-locus method, an experienced engineer can develop quickly, **without the need for a digital computer**, complete loci of roots, even for systems of somewhat higher order. For about a quarter-century after this method became widely known, the slide rule was still an engineer's primary computational tool, and the root-locus method profoundly influenced the design and analysis of control systems during that era. Now that digital computers and highly specialized software are readily available, Evans' root-locus method is no longer so essential if you need only to **analyze** a given system. However, if you want to **design** control systems logically and efficiently, then the concepts underlying Evans' method, represented in this example by Equations 16.5.3-16.5.7, will be extremely valuable. Details of Evans' method are available in many references, one of the clearest and most complete being Cannon, 1967, Sections 11.8 and 11.9 and Chapter 21.

Let us return now to the task of choosing a refined set of Λ values and using MATLAB to calculate and graph loci of roots for this refined set. We specify that this set consists of values for which the system is stable, $0 < \Lambda < 40,000$ s^{-2}, with particular emphasis on the range of values associated with the dominant oscillatory mode of response, $\Lambda_{ba} < \Lambda < 40,000$ s^{-2}. Therefore, the value $\Lambda_{ba} = 2,103.77$ s^{-2} calculated above is useful, and is included in the refined set of Λ values. The other values are chosen by trial and error to produce moderately uniform spacing between the complex poles in the p-plane; the complete set, in ascending order is $\Lambda =$ 0, 1,000, 1,800, Λ_{ba}, 2,300, 3,500, 6,250, 10,000, 20,000, and 40,000 s^{-2}. Except for this different set of Λ values, the MATLAB script M-file is the same as that which produced Figure 16.5.1. The command to execute the M-file, and the resulting tabular output follows:

```
>> MATLABdemo164

Roots #, Lambda, p

    1.0e+004 *

0.0001 0 0 -0.0300 -0.0100

0.0002 0.1000 -0.0305 -0.0083 -0.0012

0.0003 0.1800 -0.0308 -0.0064 -0.0027

0.0004 0.2104 -0.0310 -0.0045 + 0.0000i -0.0045 - 0.0000i

0.0005 0.2300 -0.0311 -0.0045 + 0.0015i -0.0045 - 0.0015i

0.0006 0.3500 -0.0315 -0.0042 + 0.0039i -0.0042 - 0.0039i

0.0007 0.6250 -0.0326 -0.0037 + 0.0066i -0.0037 - 0.0066i
```

```
0.0008 1.0000 -0.0337 -0.0031 + 0.0089i -0.0031 - 0.0089i

0.0009 2.0000 -0.0363 -0.0019 + 0.0127i -0.0019 - 0.0127i

0.0010 3.0000 -0.0383 -0.0008 + 0.0153i -0.0008 - 0.0153i

0.0011 4.0000 -0.0400 -0.0000 + 0.0173i -0.0000 - 0.0173i
```

The associated loci-of-roots plot, Figure 16.5.5, is on the next page. MATLAB's editing features for annotation and graphical enhancement were used on Figure 16.5.5. In particular, the axes were set to be "equal," which means in this case that the x axis [*i.e.*, $\mathrm{Re}(p)$] and the y axis [*i.e.*, $\mathrm{Im}(p)$] have exactly the same linear scale.

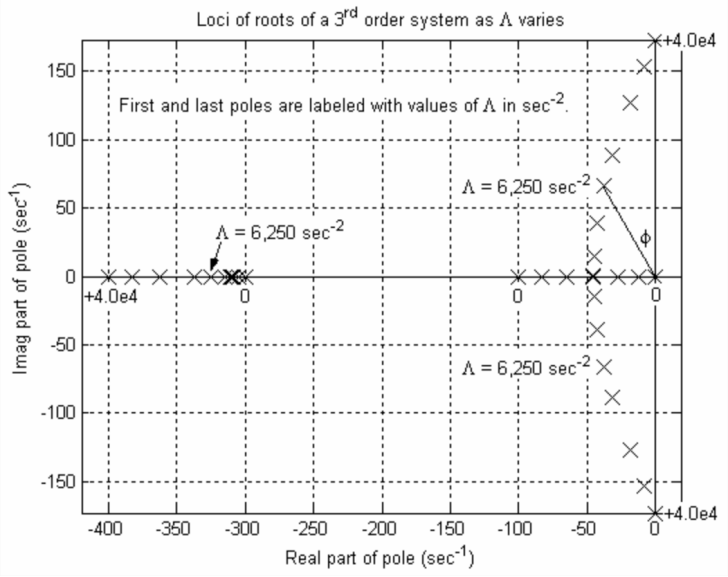


Figure 16.5.5: Refined loci of roots of Equation 16.5.1 for varying control parameter Λ = KaKθ / J. (Copyright; author via source)

The roots on Figure 16.5.5 for one example value, $\Lambda = 6,250$ s^{-2}, are labeled in order to illustrate calculation of basic 1st order and 2nd order parameters for a higher-order system. The real root is $p_1 = -326$ s^{-1} (taken approximately from Figure 16.5.5, but more precisely from the table of roots on the previous page). The associated 1st order time constant is $\tau_1 = -1/p_1 = 3.07\mathrm{e}-3$ s = 3.07 ms. The imaginary roots for $\Lambda = 6,250$ s–2 are $p_{2,3} = -37 \pm j66$ s^{-1}, and the angle labeled on Figure 16.5.5 is $\phi \approx 30°$ (for its relevance, see Figure 16.5.5). The associated 2nd order response parameters are: time constant $\tau_2 = -1/\,\mathrm{Re}\,(p_{2,3}) = 27$ ms (much longer than τ_1); frequency $\omega_d = \mathrm{Im}\,(p_2) = 66$ rad/s; and damping ratio $\zeta = \sin\phi \approx 0.5$ We can use MATLAB's poles to calculate more precisely the damping ratio associated with a stable complex pole, $p_k = \sigma_k + j\omega_k$ (the notation of Section 16.1), by recognizing that $\sigma_k = \mathrm{Re}\,(p_k) \equiv -(\zeta\omega_n)_k$ and $\omega_k = \mathrm{Im}\,(p_k) \equiv (\omega_d)_k$, and that (dropping the k subscripts temporarily) $\omega_d^2 = \omega_n^2 - (\zeta\omega_n)^2$:

$$\zeta^2 = \frac{(\zeta\omega_n)^2}{\omega_n^2} = \frac{(\zeta\omega_n)^2}{(\zeta\omega_n)^2 + \omega_d^2} \equiv \frac{\sigma^2}{\sigma^2 + \omega^2} = \frac{1}{1 + (\omega/\sigma)^2}$$

Therefore (restoring the k subscripts), the damping ratio is

$$\zeta_k = \frac{1}{\sqrt{1 + (\omega_k/\sigma_k)^2}} \tag{16.5.8}$$

Applying Equation 16.5.8 for $\Lambda = 6,250$ s^{-2} gives $\zeta_2 = 1/\sqrt{1 + (66/37)^2} = 0.49$.

The calculations in the previous paragraph suggest the following question: why does this 3rd order system have one real pole that corresponds to monotonic exponential 1st order response, and a pair of complex conjugate poles that correspond to damped oscillatory 2nd order response? In fact, this is a simple example of an important general property of linear time-invariant systems. Every LTI system has *modes of response* that are associated with its poles, the roots of the system's characteristic equation. We

show this by re-writing characteristic equation Equation 16.3.1 in the general factored form:

$$\mathrm{Den}(p) = a_1 p^n + a_2 p^{n-1} + \cdots + a_n p + a_{n+1} = a_1 \prod_{k=1}^{n} (p - p_k) = 0 \tag{16.5.9}$$

where the $\prod_{k=1}^{n}$ symbol denotes multiplication of n terms. Since all coefficients a_k are real, this equation can always be factored into 1st order terms with real poles and 2nd order terms with complex conjugate poles for which $|\zeta_j| < 1$:

$$\frac{1}{a_1}\mathrm{Den}(p) = \prod_{k=1}^{n} (p - p_k) = \ldots (p - p_i) \ldots \left(p^2 + 2\zeta_j \omega_j p + \omega_j^2\right) \ldots \tag{16.5.10}$$

It might be conceptually clearest to think of the modes in terms of response to initial conditions, without the complication of forcing inputs. The mode of IC response associated with any real pole [p_i in Equation 16.5.10] is monotonic exponential, exactly that of a basic 1st order system. The mode of IC response associated with any pair of complex conjugate poles [$-\zeta_j \omega_j \pm j\omega_j \sqrt{1 - \zeta_j^2}$ in Equation 16.5.10] is oscillation modulated by an exponential envelope, exactly that of a basic 2nd order system for which $|\zeta| < 1$. The **total** time response of an LTI system to initial conditions and forcing inputs is the superposition of the responses of all of its individual 1st order and 2nd order modes. If the response is stable, then the greatest contributors to this total response typically are a few, at most, dominant modes, those whose poles are closest to the imaginary axis.

The *modes of vibration* introduced in Section 7.6 and Chapter 12 represent a special case of the more general modes of response discussed in the previous paragraph. An undamped structural system of the type considered in Chapter 12 always has even order (*i.e.*, $n = 2, 4, 6, \ldots$), so that its characteristic equation has an even number of roots. These roots appear in purely imaginary conjugate pairs, each pair for a particular mode of vibration; the absolute value of any root is the natural frequency of that particular mode.

Before proceeding, we should observe a general feature of loci-of-roots diagrams. The poles are either real or, if complex, appear in conjugate pairs, as is expressed by Equation 16.5.10. Therefore, loci-of-roots diagrams are always symmetric about the $\mathrm{Re}(p)$ axis. This symmetry is evident in Figures 16.4.1, 16.5.1, and 16.5.5.

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16.6: Open-Loop Transfer Functions and Loci of Roots

Let us review and consider again Figure 14.4.1, the general Laplace block diagram for an SISO closed-loop system with feedback. In Section 14.4, we used Figure 14.4.1 to derive the closed-loop transfer function, $\mathrm{Out}(s)/\mathrm{In}(s)$:

$$CLTF(s) = \frac{G}{1+GH} = \frac{N_G D_H}{D_G D_H + N_G N_H} \tag{16.6.1}$$

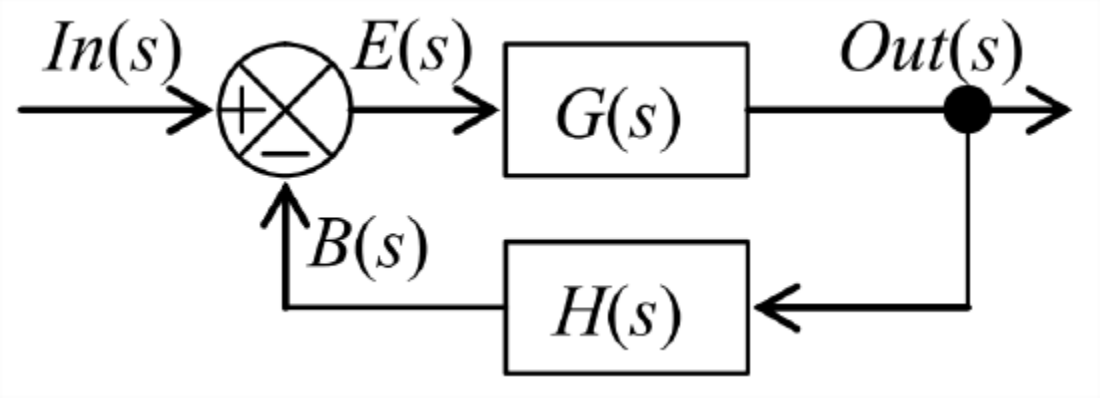


Figure 16.6.1: Laplace block diagram of general closed loop. (Copyright; author via source)

It is common practice also to refer to the ratio $B(s)/E(s)$ as the *open-loop transfer function*, which is expressed more completely as

$$\frac{B(s)}{E(s)} \equiv OLTF(s) = G(s)H(s) = \frac{N_G(s)N_H(s)}{D_G(s)D_H(s)} \equiv C \times \frac{\prod_{k=1}^{m}(s-Z_k)}{\prod_{k=1}^{n}(s-P_k)} \tag{16.6.2}$$

In Equation 16.6.2, C is some physical constant, the finite zeros Z_k of $\mathrm{OLTF}(s)$ are the roots of $N_G(s)N_H(s)=0$, and the poles P_k of $\mathrm{OLTF}(s)$ are the roots of $D_G(s)D_H(s)=0$. [For $n>m$, there are also $n-m$ zeros of $\mathrm{OLTF}(s)$ as $s\to\infty$.]

The nth degree characteristic polynomial of the closed-loop system is Equation 16.5.16, $a_1\prod_{k=1}^{n}(p-p_k)=0$. Note that the denominator of $\mathrm{CLTF}(s)$, Equation 16.6.1, equals the denominator plus the numerator of $\mathrm{OLTF}(s)$, Equation 16.6.2. Therefore, after we divide out coefficient a_1, the characteristic equation for closed-loop system poles p can generally be put into one of the following forms:

$$D_G(p)D_H(p)+N_G(p)N_H(p)=0 \Rightarrow \begin{cases} \text{(a)}\ \prod_{k=1}^{n}(p-P_k)+c\Lambda=0 \\ \text{(b)}\ \prod_{k=1}^{n}(p-P_k)+c\Lambda\prod_{k=1}^{m}(p-Z_k)=0 \end{cases} \tag{16.6.3}$$

In Equations 16.6.3, c is some known physical constant, and Λ is the varying control parameter. The system evaluated by root-locus methods in Section 16.5 includes open-loop poles P_k (called there the "sub-system poles"), but not any finite open-loop zeros Z_k, so that system's characteristic equation, Equation 16.5.2, is of form Equation 16.6.3.a. On the other hand, the systems of homework Problems 16.8, 16.9, and 16.10 have both open-loop poles and open-loop zeros, so their characteristic equations have form Equation 16.6.3.b.

Some simple but general features of loci of closed-loop roots can be inferred from Equations 16.6.3. Both equations show clearly that when $\Lambda=0$, the closed-loop poles are exactly the same as the open-loop poles. At the other extreme, as Λ becomes very large, $\Lambda\to\infty$, m of the loci $\to$ the finite open-loop zeros Z_k. This is obvious from Equation 16.6.3.b, because the term including the open-loop poles becomes progressively less significant in relative size as the magnitude of Λ increases.

As $\Lambda\to\infty$ in Equation 16.6.3.a, in order for an equation of magnitude equality such as Equation 16.5.7, $r_1r_2r_3=\omega_b\Lambda$, to remain satisfied, all n of the loci of that equation must go to some ∞ locations in the p-plane. Similarly, as $\Lambda\to\infty$ in Equation 16.6.3.b, although m of the loci terminate at the finite open-loop zeros Z_k, the remaining $n-m$ loci must go to some ∞ locations in the p-plane. For both Equations 16.6.3 as $\Lambda\to\infty$, the values of the finite open-loop zeros Z_k and of the open-loop poles P_k have a secondary influence, so the **directions** θ_A in the p-plane of the $n-m$ straight-line asymptotes of the $\Lambda\to\infty$ loci are the solutions

for angles of $p^{n-m} + c\Lambda = 0$, which leads to:

$$\left(re^{j\theta_A}\right)^{n-m} = c\Lambda \exp[j(\pm\pi, \pm 3\pi, ...)] \Rightarrow (n-m)\theta_A = \pm\pi, \pm 3\pi, ...$$

$$\theta_A = \frac{1}{n-m}(\pm\pi, \pm 3\pi, ...) \tag{16.6.4}$$

However, Equation 16.6.4 neglects the values of Z_k and P_k, so it implies incorrectly that the asymptotes "radiate" (or "emanate") from the origin of the p-plane; in fact, it can be proved (Cannon, 1967, pages 658-659) that all $n-m$ asymptotes emanate from a point on the real axis defined by the equation:

$$p_A = \frac{1}{n-m}\left(\sum_{k=1}^{n} \mathrm{Re}\,[P_k] - \sum_{k=1}^{m} \mathrm{Re}\,[Z_k]\right) \tag{16.6.5}$$

It is useful also to know the directions in which loci "depart" from the open-loop poles as Λ increases from zero. Suppose that we seek the direction (angle) of departure $(\theta_j)_{dep}$ of the locus of closed-loop roots from open-loop pole P_j. Let us use the following notation: $(\theta P)_{kj}$ is the angle from open-loop pole P_k to open-loop pole P_j; and $(\theta Z)_{kj}$ is the angle from open-loop **zero** Z_k to open-loop pole P_j. Provided that all of the open-loop poles and zeros are single (not repeated or multiple) roots, then the required angle is calculated from the equation (Cannon, 1967, pages 660-661):

$$(\theta_j)_{dep} = \pi - \sum_{\substack{k=1 \\ k\neq j}}^{n} (\theta P)_{kj} + \sum_{k=1}^{m} (\theta Z)_{kj} \tag{16.6.6}$$

We use Section 16.5's numerical example to illustrate the information about loci of roots that is provided by open-loop poles; that example is a 3rd order system for which $n = 3$ and $m = 0$. From Equation 16.5.2 with $\Lambda = 0$, the open-loop poles are $p_1 = 0$ s^{-1}, $p_2 = -100$ s^{-1}, and $p_3 = -300$ s^{-1}. These open-loop poles are marked by $\times$'s on Figure 16.6.2. There are no open-loop zeros, so as $\Lambda \to \infty$, the three loci must go to some ∞ locations in the p-plane; the directions of the lines asymptotic to these loci are given by Equation 16.6.4 as $\theta_A = \pm\pi/3$ and $\theta = \pi$. The point on the real axis from which all three asymptotes emanate is given by Equation 16.6.5 as $p_A = (0 - 100 - 300)/3 = -133\frac{1}{3}$ s^{-1}. The asymptotes are the straight dashed lines on Figure 16.6.2. Equation 16.6.6 gives the angles of departure from open-loop poles as $(\theta_1)_{dep} = \pi - (0 + 0) = \pi$, $(\theta_2)_{dep} = \pi - (\pi + 0) = 0$, and $(\theta_3)_{dep} = \pi - (\pi + \pi) = -\pi$. These angles of departure are represented on Figure 16.6.2 by short, bold arrows. Although the directions of departure for this particular system are either due east or due west, this is not always the case; for systems in general, a direction of departure can be any angle.

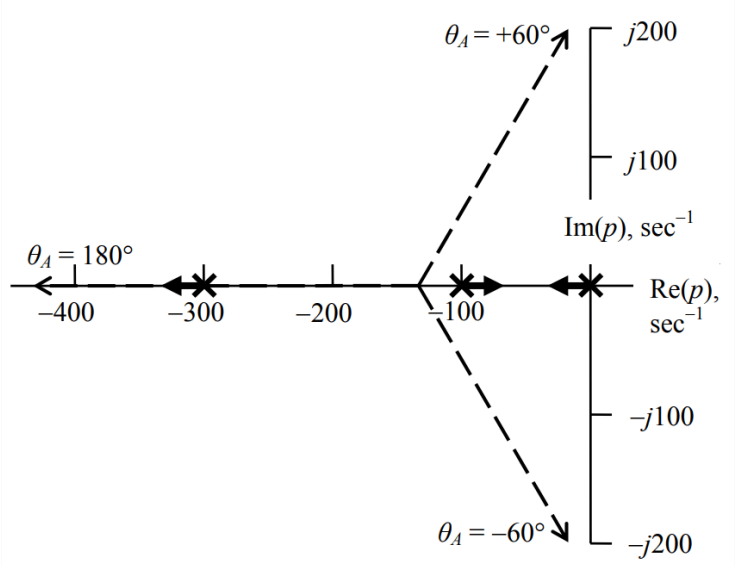


Figure 16.6.2: Example of information about loci of roots that is provided by the open-loop poles. (Copyright; author via source)

We can make educated guesses as to the appearance of the completed diagram of root loci, even from the incomplete diagram Figure 16.6.2. First, it seems likely that the locus originating (*i.e.*, for $\Lambda = 0$) at open-loop pole $p_3 = -300$ s^{-1} just proceeds westerly

along the asymptote $\theta_A = \pi$, without straying away from the $\mathrm{Re}(p)$ axis. That being the case, then the loci originating from the open-loop poles $p_1 = 0$ s^{-1} and $p_2 = -100$ s^{-1} must initially approach each other, then come together somewhere between p_1 and p_2, and then break away from the $\mathrm{Re}(p)$ axis and become the complex conjugate pair of curves to which the asymptotes $\theta_A = \pm\pi/3$ are tangent as $\Lambda \to \infty$. Certainly, the incomplete diagram Figure $16.6.2$ is qualitatively similar to the exact diagram Figure 16.5.5, and Figure $16.6.2$ even includes some of the same significant quantitative information.

When you use software to determine loci of roots, you should also make the easy calculations presented in this section based upon open-loop poles and zeros. If the software calculations match yours, then this check will validate (partially, at least) that the software is correct and that you are using it correctly.

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16.7: Chapter 16 Homework

1. Examine whichever of the following characteristic equations are assigned, using Routh's criteria to determine if all of their roots have negative real parts so that the associated systems are positively stable.
   1. $p-3=0$
   2. $2p+11=0$
   3. $p^2+2p+8=0$
   4. $2p^2-9p+29=0$
   5. $2p^3+4p^2+4p+12=0$
   6. $p^3+6p^2+12p+8=0$
   7. $p^4-p^3+7p^2+p-8=0$
2. Consider the conceptual Laplace closed-loop-system block diagram below, with two possibly variable system parameters, K and Q. Write the equation for the closed-loop transfer function. Next, expand the denominator (characteristic) polynomial and apply Routh's stability criteria to write all of the algebraic inequality relationships involving K and Q (in general) that must be satisfied in order for the system to be positively stable. (Do not worry about the units of K and Q, just assume that they are consistent.) Calculate the **range** of K for which the system is positively stable for whichever of the following cases is assigned: (a) $Q=-0.5$ and $+0.5$; (b) $Q=-1.0$ and $+1.0$; (c) $Q=-1.5$ and $+1.5$.

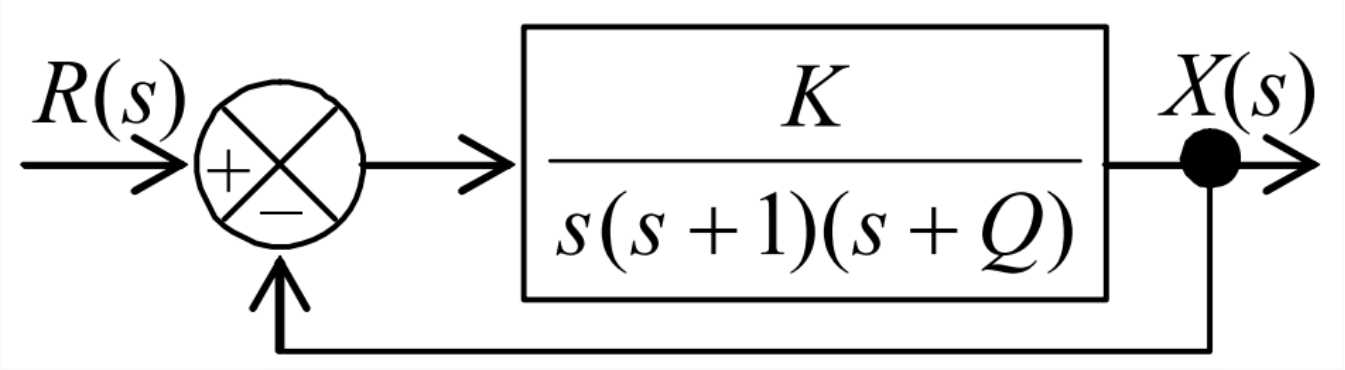


Figure 16.7.1 (Copyright; author via source)

3. Consider the characteristic equation $p^2+2Cp+K=0$. With mass m being positive-definite, this is the characteristic equation of the standard mass-damper-spring system described by equation of motion $m\ddot{x}+c\dot{x}+kx=f_x(t)$, but with the notation $C\equiv c/(2m)$ and $K\equiv k/m$. The roots of this 2nd degree polynomial equation are the poles of the system transfer function. Suppose that damping constant $C=6$ sec^{-1}, and that you want to evaluate the system stability characteristics for different values of stiffness constant K sec^{-2}. Consider first **non-negative values of** K. For what positive value of $K\equiv K_c$ is the damping **critical**, and what values do the polynomial roots have for $K\equiv K_c$? (Definition: if damping is positive but less than critical, then the roots are complex, corresponding to initial-condition response that is oscillation modulated by a decaying exponential envelope; if, on the other hand, damping is equal to or greater than critical, then the roots are negative real, corresponding to monotonic exponentially decaying initial-condition response.) What values are the roots for $K=0$ and for $K=2K_c$? Show each of these roots by marking large $\times$'s at the correct positions on a complex p-plane graph [x-axis is $\mathrm{Re}(p)$, y-axis is $\mathrm{Im}(p)$, both in sec^{-1}]; label the axes and the linear scale of the axes, the **same scale** on both axes, as on the MATLAB graph Figure 16.8. Explain concisely what quality of the roots indicates that the system is stable for all values of $K>0$. Now calculate the values of the roots for a negative stiffness, say $K=-3K_c$, and mark those roots on your p-plane graph. From the roots for a negative stiffness, what can you state conclusively about the system stability for all negative values of K ? Indicate clearly on your p-plane graph the loci of roots as K increases from $-3K_c$ to 0, then from 0 to $+K_c$, then from K_c to $2K_c$, and finally for $K_c>2K_c$. Is there a break-away point on your loci of roots; if so, for what value of K?
4. The following characteristic equations are expressed in completely or partially factored forms. For whichever equations are assigned, find the order n of the associated system, solve for the roots and mark their locations with large $\times$'s on a sketch of the complex p-plane, and state whether the system is stable or unstable; give the reason(s) for your conclusion.
   1. $(p-4)(p+3)(p+4)(p+5)=0$
   2. $p(p+2)\left(p^2-6p+34\right)=0$
   3. $(p+1)(p+3-j5)(p+3+j5)=0$
   4. $(p+3)\left(p^2+2p+2\right)\left(p^2+4p+20\right)=0$
5. The Laplace block diagram at right represents a system similar to that of Figures 16.3.1 and 16.3.2: it is a damped rotor

with position feedback that is low-pass-filtered. But the low-pass filter in this case is 2nd order, whereas the filter of Figure 16.3.2 is 1st order. The filter transfer function on the block diagram is from Equation 10.2.3: low-pass break frequency ω_b is the undamped natural frequency, and γ is the filter viscous damping ratio. We specify the value $\gamma = 1/\sqrt{2}$; as shown on the frequency-response magnitude-ratio graph of Figure 10.2.1, for this γ, the magnitude attenuation at $\omega = \omega_b$ is $1/\sqrt{2}$ (same as for a 1st order filter), and the high frequency roll-off is two decades for each decade increase of frequency (twice as steep as the roll-off of a 1st order filter). (See also the discussion of 2nd order low-pass filters at the end of Section 10.2.)

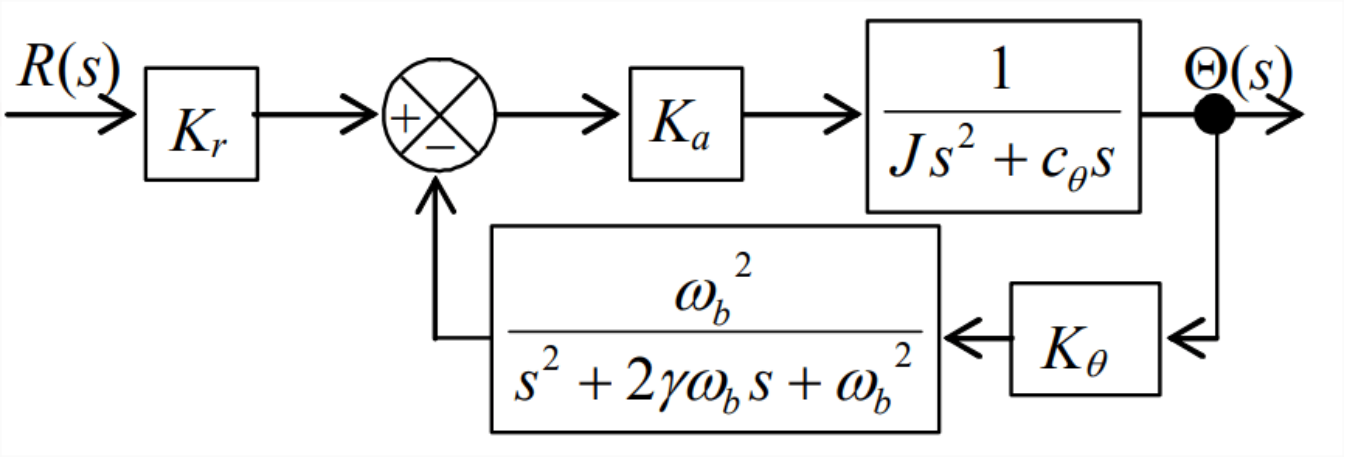


Figure 16.7.2 (Copyright; author via source)

1. Derive from the block diagram the algebraic equation for the closed-loop transfer function, $\mathrm{CLTF}(s) \equiv \Theta(s)/R(s)$ (in terms of the symbols shown, without numbers at this stage). From your $\mathrm{CLTF}(s)$, write the characteristic equation, $\mathrm{Den}(p) = 0$.
2. Use the same values as for the calculations of Chapter 16: $\omega_b = 300$ rad/s and $c_\theta/J = 100$ s^{-1}, as well as $\gamma = 1/\sqrt{2}$. Also as in Chapter 16, denote the varying control-system parameter as $\Lambda \equiv K_a K_\theta / J$. Use MATLAB's `root` operation to calculate, print out, and plot the loci of roots of this system over the entire range of Λ values for which the system is positively stable. Your graph and numerical output should show with at least 2-3 significant-figure precision any break-away points and the upper and lower boundaries of stability. Find by trial and error the approximate value of Λ for which the dominant oscillatory mode of response has the **system** damping ratio $\zeta \approx 1/\sqrt{2}$, and also calculate the time constant τ_2 and the frequency ω_d for this value of Λ.

6. The Laplace block diagram here represents a system to control the angular position of a satellite in one plane. The unfamiliar input-error block in the forward branch of the loop is the transfer function of a *compensator*, which in this case is designed to place a pair of the system's closed-loop poles at $2\pi(-1 \pm j1)$ s^{-1} in the p-plane, for one value of control parameter $\Lambda \equiv K_c K_a K_\theta / J$. For these poles, the frequency is $f_d = \omega_d / 2\pi = 1$ Hz, and the viscous damping ratio is $\zeta = 1/\sqrt{2}$.

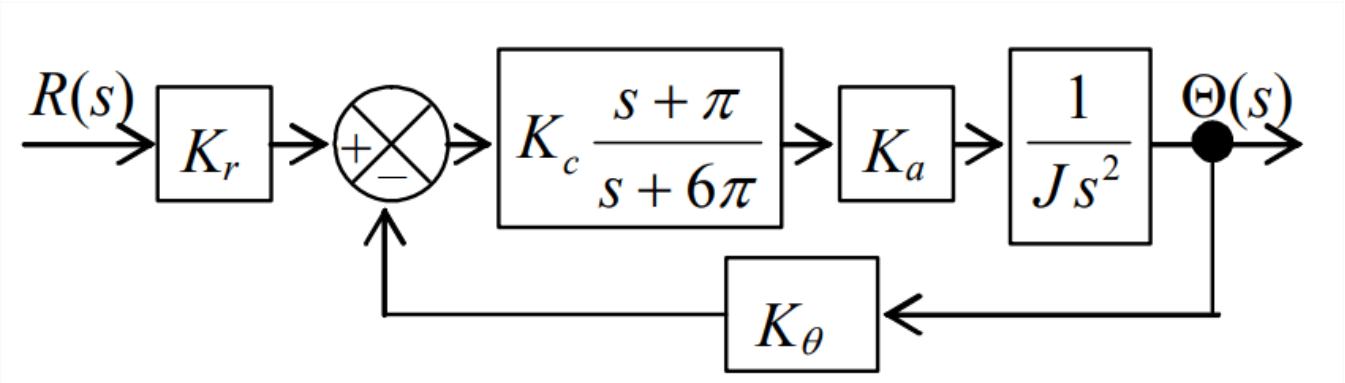


Figure 16.7.3 (Copyright; author via source)

1. Derive from the block diagram the algebraic equation for the closed-loop transfer function, $\Theta(s)/R(s)$, then write the characteristic equation, $\mathrm{Den}(p) = 0$.
2. Use MATLAB's `root` operation to calculate, print out, and plot the loci of roots of this system over the parameter range $0 \le \Lambda \le 200$ s^{-2}.
3. Use Evans' root-locus method to determine the numerical value of Λ that produces poles $2\pi(-1 \pm j1)$ s^{-1}.

7. (Adapted from Franklin, *et al.*, 1991, Problem 5.25 on page 347) The Laplace block diagram of the speed-control system of a magnetic-tape drive is shown in the figure below. The operator setting is input voltage signal $e_{in}(t)$, the Laplace transform of which is $E_{in}(s)$. Each major sub-system of the control system functions as a 1st order system. The sub-system consisting of a power amplifier and a torque actuator has variable sensitivity K_a (N-m/V) and time constant $\tau_a = 1.0$ s. The tape drive has rotational inertia $J = 4.0$ N-m per rad/s^2 and lubricated-shaft viscous damping constant $c_\theta = 1.0$ N-m per rad/s; the output of the tape drive in this application is rotational speed $p(t)$ in rad/s [with Laplace transform $P(s)$], **not** rotational position $\theta(t)$. The rotational-speed sensor in the feedback branch has sensitivity K_p = 1.250 V per rad/s; this sensor is sufficiently slow relative to the other sub-systems that we must account for its time constant, $\tau_p = 0.5$ s.
    1. Derive the closed-loop transfer function $P(s)/E_{in}(s)$, and then show that the characteristic equation of this system can be expressed as
    $$(p + c_\theta/J)\left(p + \tau_a^{-1}\right)\left(p + \tau_p^{-1}\right) + \Lambda = 0, \text{ or } r_1 e^{j\theta_1} \times r_2 e^{j\theta_2} \times r_3 e^{j\theta_3} = \Lambda \exp[j(\pm\pi, \pm 3\pi, ...)]$$
    where the variable control parameter is defined as $\Lambda = K_a K_p / (J\tau_a\tau_p)$, with units of s^{-3}.
    2. Determine (by hand, not MATLAB) all of the open-loop poles, all of the asymptotes (as $\Lambda \to \infty$) of the closed-loop poles, and the directions of departure of the root loci from the open-loop poles as Λ increases from zero. Sketch on graph paper a p-plane, and indicate on it the information that you determined for this system. See Figure 16.6.2 for an example of the type of diagram that yours should be. Make educated guesses and sketch on your diagram the forms you think the root loci should have.
    3. From part 16.7.1, any point on a locus of roots must satisfy the angle equality: $\theta_1 + \theta_2 + \theta_3 = \pm\pi, \pm 3\pi, \ldots$. Use this equality to write a transcendental equation that can be solved for the closed-loop pole $j\omega_d$ at the upper stability boundary Λ_{ub} of the complex oscillatory loci; your equation will include trigonometric functions for the θ_i angles that involve ω_d and the numerical values of the open-loop poles. It is not necessary for you to solve the transcendental equation, but you may show just by substitution that $\omega_d = 1.6583124$ s^{-1} $= \sqrt{11}/2$ s^{-1} is the solution. Finally, again using the result of part 16.7.1, solve for the value $\Lambda_{ub} = r_1 \times r_2 \times r_3$; your equation will include terms for the r_i magnitudes that involve ω_d and the numerical values of the open-loop poles. [Note: you could also solve this part by applying Routh's criteria, Equations 16.3.5 and 16.3.6, to calculate Λ_{ub}, then substituting Λ_{ub} back into the characteristic equation to solve for $p = j\omega_d$.]
8. The open-loop transfer function of a certain control system is
$$G(s)H(s) \equiv OLTF(s) = \Lambda \frac{s^2 + 4s + 104}{(s+1)\left(s^2 + 2s + 26\right)}$$
where Λ (with units s^{-1}) is a variable parameter with possible range $0 \le \Lambda < +\infty$.
    1. Determine (by hand, not MATLAB) all of the **open-loop** poles and open-loop zeros of this system, any asymptotes (as $\Lambda \to \infty$) of the closed-loop poles, and the directions of departure of the root loci from the open-loop poles as λ increases from zero. Sketch on graph paper a p-plane, and indicate on it the information that you determine for this system. See Figure 16.6.2 for an example of the type of diagram that yours should be. However, whereas Figure 16.6.2 does not display any open-loop zeros, your diagram should; it is traditional to mark an open-loop zero by an O, just as an open-loop pole is marked by an $\times$. Make educated guesses and sketch on your diagram the forms you think the root loci should have.
    2. Show that the characteristic equation of the closed-loop system is
    $$p^3 + (3 + \Lambda)p^2 + 4(7 + \Lambda)p + 26(1 + 4\Lambda) = 0$$
    Apply Routh's criteria, Equations 16.3.5 and 16.3.6, to calculate bounds and ranges of Λ for stability; in particular, solve a quadratic equation in unknown Λ to show that two of the bounds are $\Lambda_{1,2} = 8 \pm \sqrt{49.5}$ s^{-1} = 0.964376 s^{-1} and 15.0356 s^{-1}. In order to interpret the significance of these bounds relative to stability, calculate the quadratic equation for values of Λ slightly above and below these two bounds. Next, substitute the values Λ_1 and Λ_2 back into the characteristic equation as $p = j\omega_{d1}$ and $p = j\omega_{d2}$ to solve for the oscillatory frequencies ω_{d1} and ω_{d2} at Λ_1 and (\Lambda_2\), respectively—there will be both a real and an imaginary part of the equation that you can solve; calculate the $\omega_{d1,2}$ from the easier part, then just substitute that value of $\omega_{d1,2}$ into the harder part to show that it also is satisfied. What is the significance of $\omega_{d1,2}$ and $\Lambda_{1,2}$ on your diagram in part 16.8.1?
9. The characteristic equation of a certain LTI-SISO system can be expressed as
$$1 + G(p)H(p) = 1 + \Lambda \frac{p+1}{(p-1)(p+3)\left(p^2 + 2p + 2\right)} = 0$$
where Λ is a variable gain parameter with possible range $0 \le \Lambda < +\infty$.
    1. Determine (by hand, not MATLAB) all of the **open-loop** poles and the open-loop zero of this system, all of the asymptotes (as $\Lambda \to \infty$) of the closed-loop poles, and the directions of departure of the root loci from the open-loop poles as Λ increases from zero. Sketch on graph paper a p-plane, and indicate on it the information that you determine for this system. See Figure 16.6.2 for an example of the type of diagram that yours should be. However, whereas Figure 16.6.2

does not display any open-loop zeros, your diagram should; it is traditional to mark an open-loop zero by an $\circ$, just as an open-loop pole is marked by an $\times$. Make educated guesses and sketch on your diagram the forms you think the root loci should have.

2. Your information in part 16.9.1 should suggest that one locus of roots is on the $\mathrm{Re}(p)$ axis, starting from an open-loop pole at $p = +1$ (for $\Lambda = 0$), and terminating on an open-loop zero at $p = -1$ (for $\Lambda \to \infty$). If that is true, then this particular locus must, for some value of Λ between 0 and inf, cross the origin of the p-plane from $p > 0$ (unstable) to $p < 0$ (stable). Your assignment is to express the characteristic equation in polar form [in the manner of Equations 16.5.4 - 16.5.7], use that form to prove the segment $-1 \leq p \leq +1$ of the $\mathrm{Re}(p)$ axis is or is not a locus, and, if it is, find the value of Λ at $p = 0$, the boundary between instability and stability.
3. Your solution from part 16.9.2 should indicate: there is a locus on the segment $-1 \leq p \leq +1$ of the $\mathrm{Re}(p)$ axis; and that locus has stable roots for Λ greater than its value at $p = 0$. On the other hand, your information from part 16.9.1 should show there is a pair of complex conjugate open-loop poles in the stable western half-plane; it also should suggest that the loci departing from those complex open-loop poles (for $\Lambda = 0$) then proceed in an easterly direction as Λ increases from zero, then, for some positive value of Λ, cross over the $\mathrm{Im}(p)$ axis into the unstable eastern half-plane. So, as Λ increases from zero, one real locus of closed-loop roots goes from being unstable to stable, but a pair of complex conjugate loci go from being stable to unstable. Therefore, we must ask if there is **any** range of Λ for which all of the loci are in the western half-plane and the closed-loop system is stable? Perhaps the best way to answer this question is to use Routh's stability criteria. Consider a general 4th order characteristic equation: $a_1 p^4 + a_2 p^3 + a_3 p^2 + a_4 p + a_5 = 0$. The associated system is stable if and only if all of the following conditions are satisfied (Ogata, 2001, pages 275-277):

   * all $a_i > 0, i = 1, 2, 3, 4, 5$;

   * $R_2 \equiv a_2 \times a_3 - a_1 \times a_4 > 0$;

   * $R_3 \equiv R_2 \times a_4 - a_2^2 \times a_5 > 0$.

   Use Routh's stability criteria to determine the range of Λ for which the closed-loop system is stable, if there is such a range.

10. (adapted from Nelson, 1989, Problem 7.7 on pages 222-223) The 1903 Wright Flyer by itself was inherently unstable in longitudinal motion (coupled pitch and vertical translation). However, Wilbur and Orville Wright stabilized, just barely, the system of airplane + pilot by providing sufficient control authority with a canard and by developing flying skill through extensive experimentation and practice with piloted gliders. A mathematical model[1] of the system was formulated, the Laplace block diagram for which is shown below. The input to the system is commanded pitch attitude $\theta_c(t)$, and the output is actual pitch attitude $\theta(t)$. Incremental canard position is $\delta(t)$, for which the Laplace transform is $\Delta(s)$. The canard-position-to-pitch-attitude transfer function incorporates both rigid-body dynamics and aerodynamics; the estimated model of this transfer function is

$$\frac{\Theta(s)}{\Delta(s)} = \frac{11.0(s+0.5)(s+3.0)}{(s^2+5.9s-11.9)\,(s^2+0.72s+1.44)} = \frac{11.0(s+0.5)(s+3.0)}{(s-1.589)(s+7.489)\,(s^2+0.72s+1.44)}$$

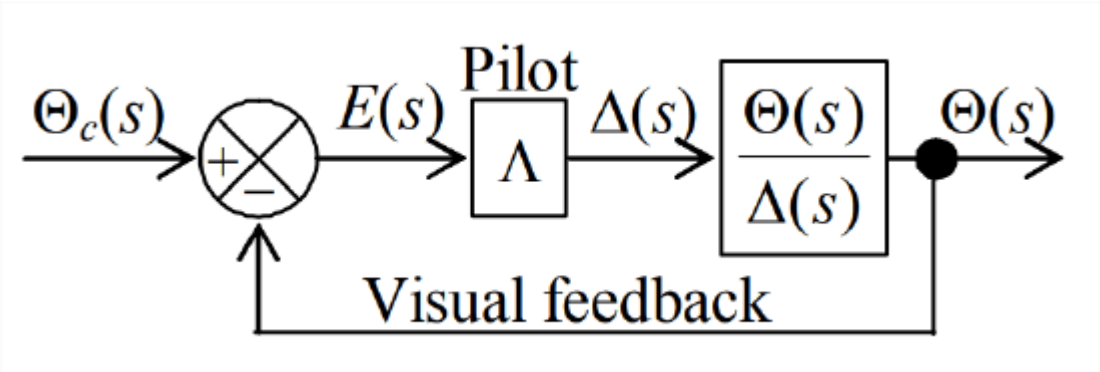


Figure 16.7.4 (Copyright; author via source)

1. Determine (by hand, not MATLAB) all of the open-loop poles and **open-loop** zeros of this system, all of the asymptotes (as pilot gain $\Lambda \to \infty$) of the closed-loop poles, and the directions of departure of the root loci from the open-loop poles as Λ increases from zero. Sketch on graph paper a p-plane, and indicate on it the information that you determined for this system. See Figure 16.6.2 for an example of the type of diagram that yours should be. However, whereas Figure 16.6.2 does not display any open-loop zeros, your diagram should; it is traditional to mark an open-loop zero by an $\circ$, just

as an open-loop pole is marked by an $\times$.

2. Show in detail that the characteristic polynomial equation in the form $a_1p^4 + a_2p^3 + a_3p^2 + a_4p + a_5 = 0$, with all a_i coefficients either numerical or combinations of numbers and pilot gain Λ, is given by:

$$p^4 + 6.62p^3 + (11\Lambda - 6.212)p^2 + (38.5\Lambda - 0.072)p + (16.5\Lambda - 17.136) = 0$$

3. If we wish at this point to program into software the calculation of loci of roots, we know only to consider $0 < \Lambda < \infty$; but, without more information, it is difficult to know what specific range or even order of magnitude of Λ to enter initially in the calculations. However, Routh's stability criteria can provide that necessary additional information. Use Routh's criteria [homework Problem 16.9.3] and the characteristic-equation coefficients to evaluate the absolute stability of the Wright Flyer system, thereby determining quantitatively a candidate range of Λ for use in calculation of loci of roots.[2] (*Answer*: for stability, $\Lambda > 1.3096$)
4. Use MATLAB's `root` operation to calculate, print out, and plot the loci of roots of this system over a range of Λ values for which the system is stable. From Equation 9.5.5, viscous damping ratio $\zeta = 0.11$ of the oscillatory mode produces 50% amplitude decay during each cycle; show from your calculations that $\Lambda = 3.24$ is the lower pilot gain for which $\zeta = 0.11$, to reasonable engineering accuracy. From your results for $\Lambda = 3.24$, do you expect an oscillatory mode to be dominant in time response, or an exponential mode, or perhaps a combination of the two?
5. It is of interest to examine in more detail the case of pilot gain $\Lambda = 3.24$, beginning with the **closed-loop** transfer function $CLTF(s)$. Note that we essentially already have the denominator of $CLTF(s)$: in the characteristic equation of part 16.10.2, just replace $p \rightarrow s$ and set $\Lambda = 3.24$. Show that the closed-loop transfer function, in the form of Equation 16.1.6, is:

$$CLTF(s) = \frac{\Theta(s)}{\Theta_c(s)} = \frac{35.64\,(s^2 + 3.5s + 1.5)}{s^4 + 6.62s^3 + 29.428s^2 + 124.668s + 36.324}$$

Now apply MATLAB's `residue` operation (see homework Problem 2.15) to calculate the residues and poles of $CLTF(s)$, and write out the partial-fraction expansion form of $CLTF(s)$. Note that the poles found in this step should match those of part 16.10.4 for $\Lambda = 3.24$.

6. Finally, consider a forced-response solution in time, with zero initial conditions. Suppose the pilot commands a short pulse to the pitch-control lever; represent this pulse approximately with a Dirac delta function (Section 8.4): $\theta_c(t) = I_\theta \delta(t)$, for which the Laplace transform is $\Theta_c(s) = I_\theta$. Let the impulse magnitude be $I_\theta = 2$ degree-s. Use the partial-fraction expansion form of $CLTF(s)$ from part 16.10.5 to derive the equation for the pitch response $\theta(t)$ to this control excitation, and then calculate and plot this transient response over a 15-s interval.

11. We know from Chapter 16 that the process of calculating and plotting loci of roots is useful in analysis of control-system stability; the objective of this problem is to illustrate another important application of root-loci: analysis of the stability of a system for which the potentially destabilizing feedback is natural and intrinsic, not human-designed. Systems with this characteristic are often described as *self-excited* (see, *e.g.*, Den Hartog, 1956, Chapter 7). The particular phenomenon we consider here, aeroelastic dynamic response of a typical-section airfoil, including unstable flutter, originated in aeronautical engineering but is relevant also to many other applications of force-generating objects that are immersed within moving fluids (*e.g.*, hydrofoils, wind-turbine blades, and propellers). The typical-section airfoil is described in Section 11.3, Examples 11.3.2 and 11.3.3, and we consider here the form depicted below, an uncambered, symmetric thin airfoil supported by springs in a wind tunnel, and immersed within an incompressible airstream of density ρ and freestream velocity V.

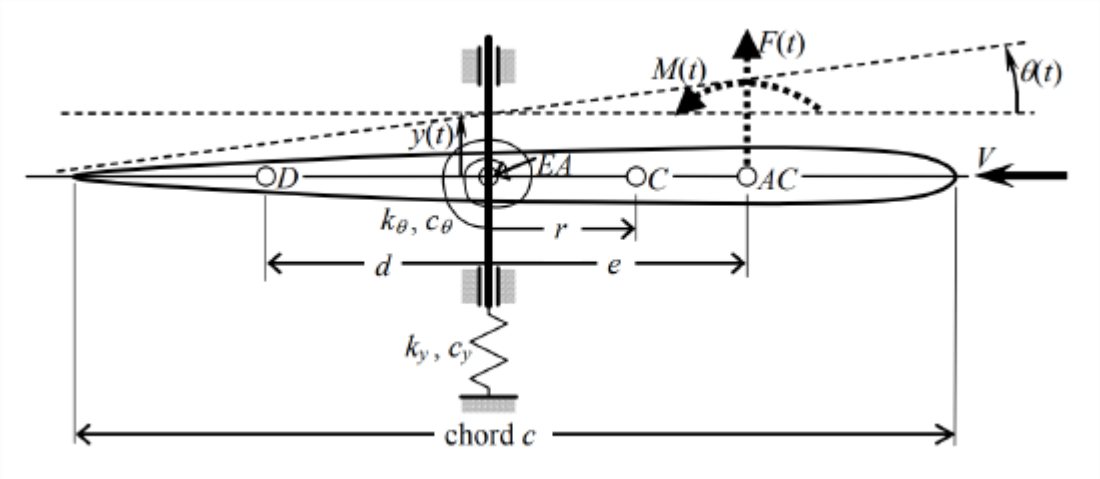


Figure 16.7.5 (Copyright; author via source)

Structural dynamic equations of motion with vertical translation of mass center C, $y_{AC}(t)$, and pitch $\theta(t)$ as degrees of freedom (DOFs) are derived and expressed in matrix form in Equation 11.3.16. However, a more appropriate vertical translation DOF for this problem is $y_{EA}(t) \equiv y(t)$ of the elastic axis EA, as labeled on the figure above; the equations of motion for DOFs $y(t)$ and $\theta(t)$ are derived in homework Problem 11.4 and are expressed below, with the addition of a damping matrix that represents approximately the energy dissipation within the springs (using the viscous-damping mathematical model described in Appendix B, Section 19.5).

$$\begin{bmatrix} m & mr \\ mr & J_{EA} \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c_y & 0 \\ 0 & c_\theta \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_y & 0 \\ 0 & k_\theta \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} F(t) \\ M(t) + F(t)e \end{bmatrix} \tag{16.7.1}$$

We assume that aerodynamic resultant lift force and moment (positive nose-up) are due both to the motions of the typical section itself (self-excitation), and to sources such as gusts that are independent of the motion, which we denote as lift $F_G(t)$ and moment $MG(t)$. There is no exact and readily applicable theory of unsteady, motion-dependent aerodynamics; therefore, we use here one of the many approximate theories, the *quasisteady* aerodynamics of Fung, 1955, pages 191-193. In these equations of unsteady lift and moment, we replace Fung's lift-curve slope for an airfoil of infinite span, $C_{L\alpha} = 2\pi$ per radian, with a reduced value of $C_{L\alpha}$ that applies for an airfoil of finite span. This is appropriate for the specific numerical parameters considered below in part 16.11.2, which are chosen to represent an actual experimental research investigation of typical-section aeroelasticity. Since dynamic pressure is $\bar{q} = 1/2\rho V^2$, it is appropriate to define in Fung's equations the constant $K \equiv 1/2\rho S(2\pi) \rightarrow K \equiv 1/2\rho S C_{L\alpha}$. Thus, the motion-dependent and motion-independent aerodynamic resultant lift and nose-up moment are written as:

$$F(t) = \bar{q}SC_{L\alpha}\left(\theta - \frac{\dot{y} - d\dot{\theta}}{V}\right) + F_G(t) = K\left(V^2\theta - V\dot{y} + dV\dot{\theta}\right) + F_G(t) \tag{16.7.2}$$

$$M(t) = -\bar{q}Sc\frac{\pi c}{8}\frac{\dot{\theta}}{V} + M_G(t) = -K\frac{\pi/8}{C_{L\alpha}}c^2V\dot{\theta} + M_G(t) \tag{16.7.3}$$

1. Substitute Equations 16.7.2 and 16.7.3 into Equation 16.7.1, then transpose the motion-dependent terms from the right-hand side to the left-hand side so that the damping and stiffness matrices in Equation 16.7.1 become more general; for example, the structural-plus-aerodynamic stiffness matrix becomes $\begin{bmatrix} k_y & -KV^2 \\ 0 & k_\theta - eKV^2 \end{bmatrix}$. Now you have a set of equations that, in principle, can be solved for DOFs $y(t)$ and $\theta(t)$ by Laplace transformation. Assume null initial conditions, $y(0) = 0$ and $\theta(0) = 0$, and assume that $F_G(t)$ and $M_G(t)$ are Laplace transformable functions, then express the Laplace-transformed equations of motion in the following form, writing explicit quadratic (in s) polynomials for the four $M_{ij}(s;V)$ elements of the left-hand side coefficient matrix:

$$\begin{bmatrix} M_{11}(s;V) & M_{12}(s;V) \\ M_{21}(s;V) & M_{22}(s;V) \end{bmatrix} \begin{bmatrix} Y(s) \\ \Theta(s) \end{bmatrix} \equiv [\mathbf{M}(s;V)] \begin{bmatrix} Y(s) \\ \Theta(s) \end{bmatrix} = \begin{bmatrix} L\left[F_G(t)\right] \\ L\left[M_G(t)\right] + eL\left[F_G(t)\right] \end{bmatrix} \tag{16.7.4}$$

   The matrix solution of Equation ??? can be written as

$$\begin{bmatrix} Y(s) \\ \Theta(s) \end{bmatrix} = [\mathbf{TF}(s;V)] \begin{bmatrix} L\left[F_G(t)\right] \\ L\left[M_G(t)\right] + eL\left[F_G(t)\right] \end{bmatrix} \tag{16.7.5}$$

   where the 2×2 coefficient matrix $[\mathrm{TF}(s;V)] = [\mathbf{M}(s;V)]^{-1}$ is the multiple-input-multiple-output (MIMO) *transfer-function* ***matrix*** relating two outputs, $Y(s)$ and $\Theta(s)$, to two inputs, $L\left[F_G(t)\right]$ and $\left(L\left[M_G(t)\right] + eL\left[F_G(t)\right]\right)$; however, it is clear that each element of matrix $[\mathbf{TF}(s;V)]$ is itself a single-input-single-output (SISO) transfer function. Use Equations 12.2.16-12.2.18 to write an equation for $[\mathbf{TF}(s;V)]$ in terms of the four $M_{ij}(s;V)$ elements. Note, in particular, that the denominator of each of the SISO transfer functions, $TF_{ij}(s;V)$, is the determinant $\det[\mathbf{M}(s;V)] = M_{11}(s;V) \times M_{22}(s;V) - M_{21}(s;V) \times M_{12}(s;V)$. By the logic of Section 16.1, therefore, the quartic (in p) polynomial equation, $\det[\mathbf{M}(p;V)] = 0$, is the general characteristic equation of the system, and its four roots $p_i(V)$ at each freestream airspeed V determine both the character of unforced time response (*e.g.*, exponential, oscillatory, *etc.*) and the system stability or instability at that particular airspeed.
2. In this part, you will use the equations of part 16.11.1 to calculate and plot the loci of roots, as wind-tunnel freestream airspeed V varies, using numerical parameters that were selected to represent the research project of a Brazilian group, DeMarqui, *et al.*, 2005 and 2006. The parameters of the typical-section airfoil and the wind-tunnel atmosphere are:

   Mass m = 18.0 kg

   Rotational inertia about mass center C: J_C = 0.219 kg-m^2; so $J_{EA} = J_C + mr^2$

Offset of mass center C forward of elastic axis EA: r = 5 mm = 0.005 m

Stiffness constant of vertical-translation spring: k_y = 1,110 N/m

Stiffness constant of pitch-rotation spring: k_θ = 49.8 N-m/radian

Viscous-damping constant of vertical-translation spring: $c_\theta = 0.00318$ s $\times k_y$

Viscous-damping constant of pitch-rotation spring: $c_\theta = 0.00318$ s $\times k_\theta$

Planform: chord c = 0.45 m; *span* = 0.80 m; so, area $S = c \times$ *span* = 0.36 m^2

Offset of $\frac{1}{4}$-chord aerodynamic center AC forward of elastic axis EA : $d = c/4$

Offset of $\frac{3}{4}$-chord dynamic-pitch point D aft of elastic axis EA : $d = c/4$

Standard air density at 800-m average altitude of Sao Paulo, Brazil: $\rho = 1.1337$ kg/m^3

Lift-curve slope for this low-aspect-ratio airfoil: $C_{L\alpha} = 2.64$ per radian

The most significant single result that you seek is the wind-tunnel freestream airspeed V_F at which the typical section experiences aeroelastic flutter. The *flutter airspeed* V_F is that value of V at which the dominant time response of the typical section is a pure sinusoid; for lower airspeeds, \(0 \leq VV_{F}\), the dominant response is unstable exponentially expanding oscillation. Begin the numerical analysis by using Equation 11.3.10 to calculate the *aeroelastic divergence airspeed* V_D in m/s; because usually $V_D > V_F$, this value of V_D will suggest to you the range of airspeeds, \(0 \leq Vconv and `root` operations to calculate, print out, and plot the loci of roots of characteristic equation $M_{11}(p;V) \times M_{22}(p;V) - M_{21}(p;V) \times M_{12}(p;V) = 0$. You might find useful the form of MATLAB script M-file `MATLABdemo164.m` that produced Figures 16.5.1 and 16.5.5 in Section 16.5. In this problem, freestream velocity V plays the same role that control gain parameter Λ plays in `MATLABdemo164.m`.

[1]Culick, F.E.C.: "Building a 1903 Wright 'Flyer'—by Committee," AIAA Paper 88-0094, 1988

[2]This exercise requires several calculations. Rather than calculating manually, you might prefer to use symbolic software. If you use MATLAB's Symbolic Math Toolbox (see homework Problem 1.6), be aware that this symbolic software performs calculations exactly, expressing numbers as fractions of integers if at all possible. But the integers in these fractions can become quite large and unwieldy. You can convert these fractions into decimal numbers by using "**v**ariable **p**recision **a**rithmetic." For example, if a symbolic term is expressed as `v = 4115/2263*w`, you can convert the fractional coefficient into a five-digit decimal with the operations `>> digits(5);v5=vpa(v)`, the result of which is `v5 = 1.8184*w`.

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17: INTRODUCTION TO SYSTEM STABILITY- FREQUENCY-RESPONSE CRITERIA

Frequency-response concepts have been discussed extensively in many previous chapters, most notably in Chapters 4 and 10 relative to methods of calculation and the physical characteristics of frequency response in stable systems. In this chapter, we consider methods using frequency response for assessing the stability of systems.

To examine the stability of a closed-loop control system, we examine in particular the frequency response of the associated open-loop system, for which the transfer function $OLTF(s) = G(s) \times H(s)$ was described in Section 16.6. The open-loop system itself is usually manageable, not unstable, even though its closed-loop version might be unstable. Therefore, the frequency-response function $\mathrm{OLFRF}(\omega) \equiv \mathrm{OLTF}(j\omega) = G(j\omega) \times H(j\omega)$ can typically be measured on actual hardware, even if there is no good mathematical model for the system. The reason for measuring and/or computing $OLFRF(\omega)$ is that **it provides metrics of the absolute and relative stability of the closed-loop system**. In other words, to determine the stability of the closed-loop system, it might not be necessary to close the loop with real hardware, possibly risking damage and/or injury.

Recall that root-locus analysis is based on the characteristic equation $1 + G(s) \times H(s) = 0$, or $G(s) \times H(s) = -1$; obviously, the right-hand-side number –1 is important to the closed-loop poles and to closed-loop system stability. Similarly, the number –1 is also highly significant when we evaluate stability using open-loop frequency response $G(j\omega) \times H(j\omega)$.

17.1: Gain Margins, Phase Margins, and Bode Diagrams

17.2: Nyquist Plots

17.3: The Practical Effects of an Open-Loop Transfer-Function Pole at s = 0 + j0

17.4: The Nyquist Stability Criterion

17.5: Chapter 17 Homework

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17.1: Gain Margins, Phase Margins, and Bode Diagrams

In order to describe and illustrate the most basic form of frequency-response stability analysis, we consider again a familiar system from Chapter 16: the rotor position-feedback control system with a 1st order low-pass filter in the feedback branch, for which the functional diagram is Figure 16.3.1 and the block diagram is Figure 16.3.2. We have already studied this system to determine absolute stability by Routh's criteria in Section 16.3, and relative stability by calculating loci of roots and by applying the root-locus method in Sections 16.5 and 16.6. Knowing the stability characteristics in advance will guide us in interpreting the frequency-response analysis.

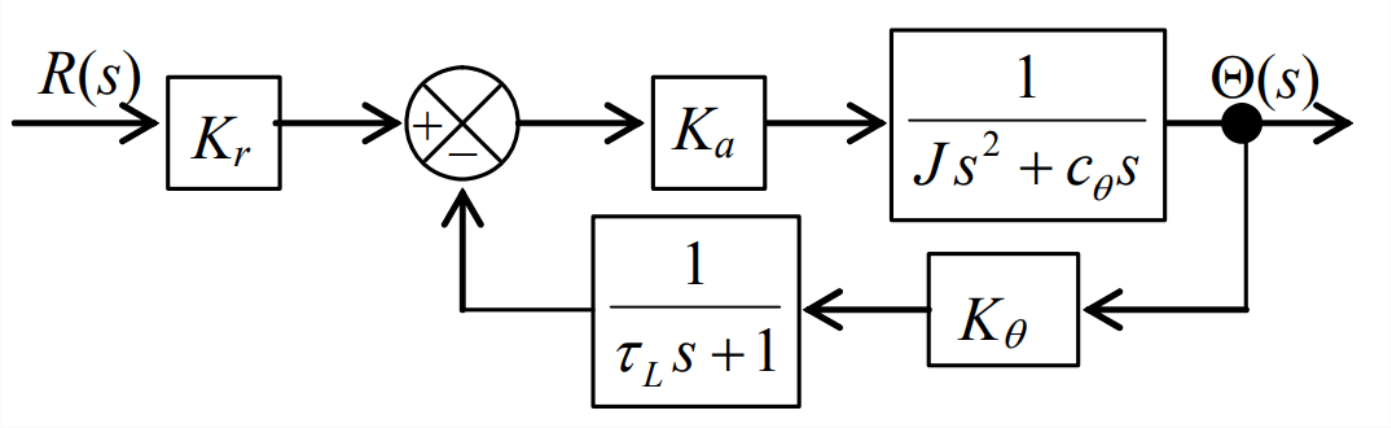


Figure 17.1.1 (Copyright; author via source)

The branch transfer functions for the loop in Figure 16.3.2 are

$$G(s) = \frac{K_a}{Js^2 + c_\theta s} \quad \text{and} \quad H(s) = \frac{K_\theta}{\tau_L s + 1} \tag{17.1.1}$$

The closed-loop transfer function is $CLTF(s) = G(s)/[1 + G(s)H(s)]$; the denominator of $\mathrm{CLTF}(s)$ in this form for the system of Figure 16.3.2 is

$$1 + G(s)H(s) = 1 + \frac{K_a}{Js^2 + c_\theta s} \times \frac{K_\theta}{\tau_L s + 1} = 1 + \frac{K_a K_\theta}{J} \frac{1/\tau_L}{s\,(s + c_\theta/J)\,(s + 1/\tau_L)} \equiv 1 + \Lambda \frac{\omega_b}{s\,(s + c_\theta/J)\,(s + \omega_b)} \equiv 1 + O\,\mathrm{LTF}(s)$$

where $\Lambda \equiv K_a K_\theta / J$ is the control parameter, which we now call *gain*. We evaluate the stability of the closed-loop system as Λ ranges from 0 to $+\inf$. The system constants from Chapter 16 are $\omega_b \equiv 1/\tau_L = 300$ rad/s and c_θ/J = 100 s^{-1}. In Section 16.5, we solved the characteristic equation of the closed-loop system, $1 + G(s)H(s) = 0$.[1] We know from the loci of roots calculated for this system that, with gain Λ positive and increasing, there is a particular value of Λ for which the oscillatory loci of roots intersect the $\mathrm{Im}[s]$ axis; for this value of Λ, the oscillatory time response is purely sinusoidal and periodic, without any exponential envelope. Let us define that intersection as the point of oscillatory ***n**eutral **s**tability* and denote the value of Λ there as Λ_{ns} (the same as Λ_{ub} in Section 16.5). Accordingly, let us denote the oscillatory frequency at that point as ω_{ns} (the same as ω_d in Section 16.5). We determined in Chapter 16 that $\Lambda_{ns} = (c_\theta/J) \times (c_\theta/J + \omega_b) = 40,000$ s^{-2} and $\omega_{ns} = \sqrt{(c_\theta/J)\,\omega_b} = 173.2$ rad/s, and that the closed-loop system is exponentially stable for $0 < \Lambda < \Lambda_{ns}$, but unstable for $\Lambda > \Lambda_{ns}$.

The loci of roots are defined by the characteristic equation:

$$1 + G(s)H(s) \equiv 1 + O\,\mathrm{LTF}(s) = 0 \tag{17.1.3}$$

Therefore, at the point of neutral stability, with $\Lambda = \Lambda_{ns}$ and $s = j\omega_{ns}$, we have

$$1 + O\,\mathrm{LTF}\,(j\omega_{ns}) = 0 \Rightarrow \mathrm{OLTF}\,(j\omega_{ns}) = -1 = 1 \times \exp[j(\pm\pi, \pm 3\pi, ...)]$$

$$\mathrm{OLTF}\,(j\omega_{ns}) \equiv |O\,\mathrm{LTF}\,(j\omega_{ns})| \exp\,[j \angle O\,\mathrm{LTF}\,(j\omega_{ns})] = 1 \times \exp[j(\pm\pi, \pm 3\pi, ...)]$$

Equating magnitude ratios and phase angles in the last equation gives for $\Lambda = \Lambda_{ns}$:

$$|OLTF\,(j\omega_{ns})| = 1 \quad \text{and} \quad \angle OLTF\,(j\omega_{ns}) = (\pm\pi, \pm 3\pi, ...) \tag{17.1.4}$$

Equations 17.1.4 correspond physically, for $\omega = \omega_{ns}$, to the frequency-response test depicted in Figure 17.1.2 (a modification of Figure 16.3.1) below. For steady-state sinusoidal electrical-signal excitation $e_{in}(t) = E_{in} \sin \omega_{ns} t$ of the open-loop system at the neutral-stability frequency of the closed-loop system, $\omega = \omega_{ns}$, the magnitude ratio of the open-loop frequency response is exactly 1 and the output electrical signal is exactly out of phase with the input signal, meaning $e_{out}(t) = -E_{in} \sin \omega_{ns} t$.

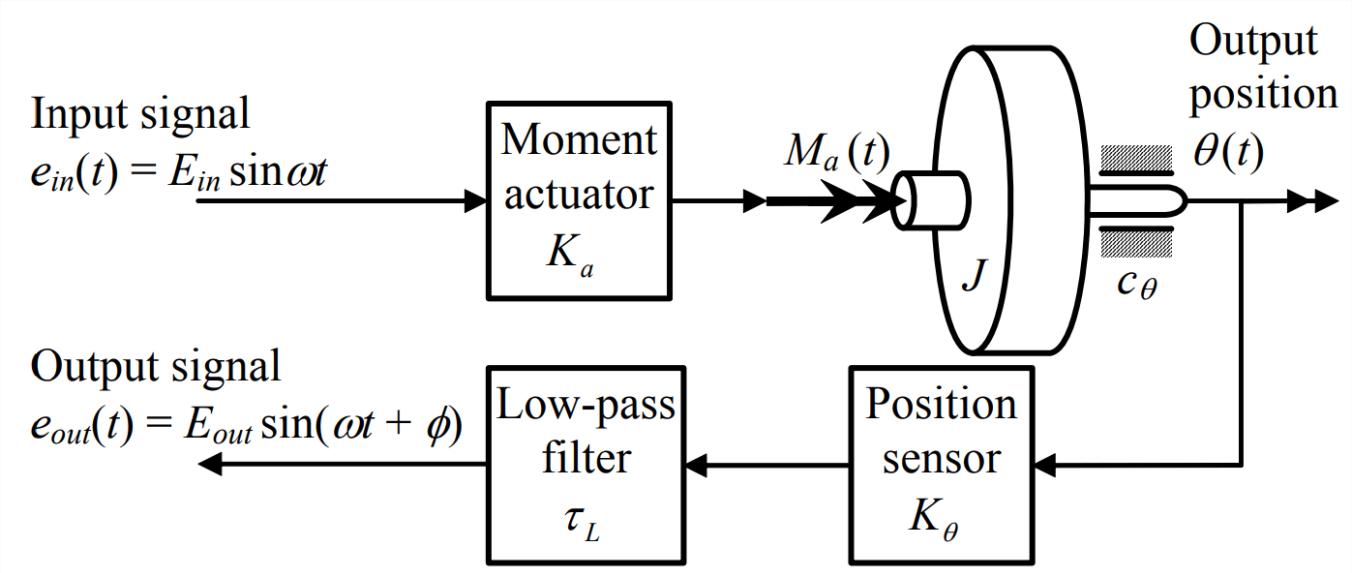


Figure 17.1.2: Functional diagram of frequency-response testing of the open-loop portion of Figures 16.3.1 and 16.3.2. (Copyright; author via source)

The observation that, for $\Lambda = \Lambda_{ns}$, output $e_{\text{out}}(t) = -E_{in} \sin \omega_{ns} t$ in response to input $e_{in}(t) = E_{in} \sin \omega_{ns} t$ provides motivation for conducting steady-state-sinusoidal tests on the open-loop system over a **band** of frequencies surrounding ω_{ns}, not just ω_{ns}, and for other values of Λ, not just Λ_{ns}, in order determine if this type of testing provides information regarding the stability of the closed-loop system. We can simulate such a test by evaluating numerically the open-loop frequency-response function, from Equation 17.1.2:

$$OLFRF(\omega) \equiv OLTF(j\omega) = \Lambda \frac{\omega_b}{j\omega\,(j\omega + c_\theta/J)\,(j\omega + \omega_b)} \tag{17.1.5}$$

To begin, in order to establish a reference set of results for the neutral-stability value of gain, $\Lambda = \Lambda_{ns}$, we calculate and plot from Equation 17.1.5 frequency-response components using the following MATLAB code:

```
>> wb=300;coj=100;wns=sqrt(wb*coj);

>> wbar=logspace(-1,1,100);w=wbar*wns;

>> lm=4e4;olfrf4e4=lm*wb./(j*w.*(j*w+coj).*(j*w+wb));

>> subplot(2,1,1),loglog(wbar,abs(olfrf4e4)),grid

>> faz=atan(imag(olfrf4e4)./real(olfrf4e4))-pi*ones(1,length(w));

>> subplot(2,1,2),semilogx(wbar,faz*180/pi),grid
```

Figure 17.1.3 includes the graphical results from these MATLAB operations. For now, we focus on just the plots of the polar components, magnitude ratio and phase angle, for $\Lambda = \Lambda_{ns} = 40,000$ s^{-2}; the additional annotations on the phase plot for "PM" at other values of Λ will be explained in due course. If we were testing the real hardware of Figure 17.1.2, rather than just numerically simulating a test, then the magnitude ratio would be the measured quantity $E_{\text{out}}/E_{\text{in}}$ as frequency ω is varied, and the phase angle ϕ would be calculated from measurements by the method presented in Section 4.4.

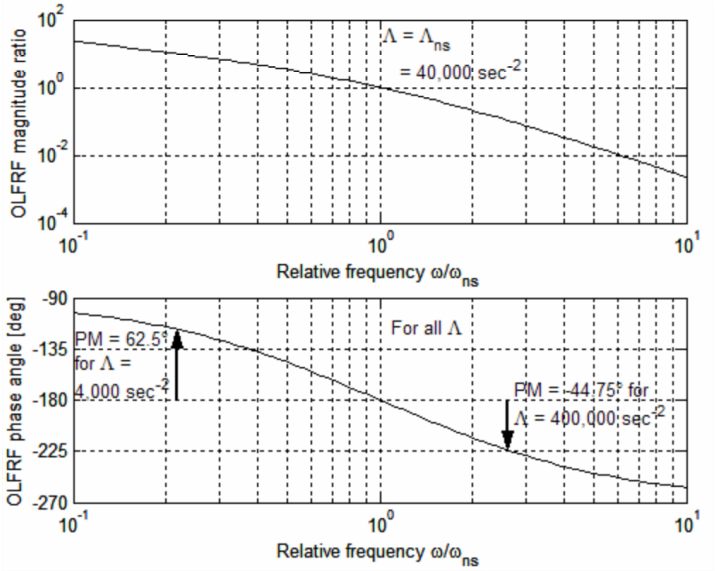


Figure 17.1.3: Modified Bode diagram, Equation 17.1.5, frequency response for $\Lambda = \Lambda_{ns} = 40,000$ s^{-2} of the open-loop system in Figure 17.1.2 (Copyright; author via source)

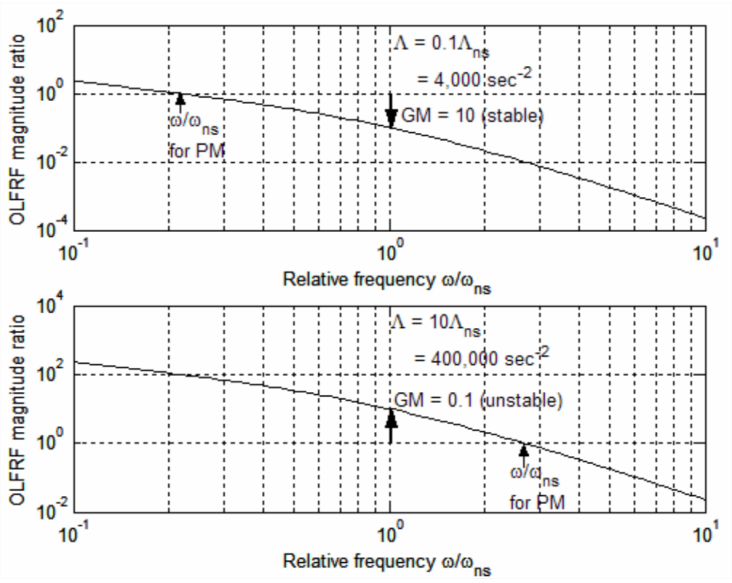


Figure 17.1.4: Magnitude ratios, Equation 17.1.5 for $\Lambda = 4,000$ and $400,000$ s^{-2}. (Copyright; author via source)

The upper plot of Figure 17.1.3 shows that, for $\Lambda = \Lambda_{ns}$, magnitude ratio $|OLFRF|$ varies smoothly from greater than 10 at $\omega = 0.1\omega_{ns}$, through the previously determined value of 1 at $\omega = \omega_{ns}$, down to less than 0.01 at $\omega = 10\omega_{ns}$. The lower plot shows that phase angle $\angle OLFRF$ is a lag ranging from more than –90° at $\omega = 0.1\omega_{ns}$, through the previously determined phase of –180° at $\omega = \omega_{ns}$, down to a lag approaching –270° at $\omega = 10\omega_{ns}$. We have from Equation 17.1.4 for $\Lambda = \Lambda_{ns}$, $\angle OLFRF\,(\omega_{ns}) = (\pm\pi, \pm 3\pi, ...)$, which is ambiguous regarding the multiple and sign of π that is appropriate; the phase plot of Figure 17.1.3 removes the ambiguity, at least for this system: clearly, $\angle OLFRF\,(\omega_{ns}) = -\pi$ rad.[2]

With Figure 17.1.3 for $\Lambda = \Lambda_{ns}$ as the reference open-loop frequency response representing the neutrally stable closed-loop system, we can now compute simulated open-loop frequency responses for other values of Λ, specifically, those for which the closed-loop system is oscillatory and either exponentially stable or unstable. With the following MATLAB code, we calculate and plot on Figure 17.1.4 (on the previous page) the **magnitude-ratio** (only) frequency-response graphs for the gain $\Lambda = 0.1\Lambda_{ns} = 4,000$ s^{-2}, for which we know the closed-loop system is stable, and for the gain $\Lambda = 10\Lambda_{ns} = 400,000$ s^{-2}, for which we know the closed-loop system is unstable:

```
>> wb=300;coj=100;wd=sqrt(wb*coj);
```

```
>> wbar=logspace(-1,1,100);w=wbar*wd;

>> lm=4e3;olfrf4e3=lm*wb./(j*w.*(j*w+coj).*(j*w+wb));

>> lm=4e5;olfrf4e5=lm*wb./(j*w.*(j*w+coj).*(j*w+wb));

>> subplot(2,1,1),loglog(wbar,abs(olfrf4e3)),grid

>> subplot(2,1,2),loglog(wbar,abs(olfrf4e5)),grid
```

If we were to calculate the **phase angles** for **any** value of Λ, we would find them to be identical to those on the phase-angle graph for $\Lambda = \Lambda_{ns}$ of Figure 17.1.3, which is the reason for not repeating the phase-angle calculations. (The theory behind this feature of openloop phase angle, and also the variation of open-loop magnitude for different values of Λ, will be presented at the end of this section.)

By comparing the open-loop magnitude-ratio graphs of Figures 17.1.3 and 17.1.4, we observe the magnitudes for the closed-loop stable case $\Lambda = 0.1\Lambda_{ns}$ to be less than those for neutral stability, $\Lambda = \Lambda_{ns}$, and the magnitudes for the unstable case $\Lambda = 10\Lambda_{ns}$ to be greater. We conclude from this that open-loop magnitudes less than or greater than those for neutral stability are an indication of closed-loop **absolute** stability. If we were to calculate the open-loop magnitude ratios for other values between $0.1\Lambda_{ns}$ and $10\Lambda_{ns}$, we would find that the variation is monotonic and regular; this suggests that open-loop magnitude can also provide a metric for closed-loop **relative** stability, *i.e.*, the **degree** of stability or instability. Indeed, there is a commonly accepted metric called *gain margin* (GM); if we denote the gain margin for a particular value of Λ as $\mathrm{GM}(\Lambda)$, then it is defined as the following dimensionless ratio of open-loop magnitudes:

$$\mathrm{GM}(\Lambda) \equiv \frac{|OLFRF\,(\omega_{ns})|_{\Lambda_{ns}}\ |\ \text{ at }\ \angle OLFRF\,(\omega_{ns})|_{\Lambda_{ns}} = -180^{\circ}}{|OLFRF(\omega)|_{\Lambda}\ |\ \text{ at }\ \angle OLFRF(\omega)|_{\Lambda} = -180^{\circ}} = \frac{1}{|OLFRF(\omega)|_{\Lambda}\ |\ \text{ at }\ \angle OLFRF(\omega)|_{\Lambda} = -180^{\circ}} \tag{17.1.6}$$

In the second form of Equation 17.1.6, which is the working definition, we applied the unit magnitude value derived in Equation 17.1.4. The condition $\angle OLFRF(\omega)|_{\Lambda} = -180^{\circ}$ is called, descriptively, a *phase crossover*.

Applying Equation 17.1.6 to Figures 17.1.3 and 17.1.4, we calculate these gain margins: $\mathrm{GM}\,(0.1\Lambda_{ns}) = 10$ for the closed-loop stable system; and $\mathrm{GM}\,(10\Lambda_{ns}) = 0.1$ for the unstable system. Clearly, $\mathrm{GM} > 1$ implies closed-loop stability and $0 < \mathrm{GM} < 1$ implies instability. Expressing GM in terms of its logarithm to the base 10 (denoted "log") is also of value, since in this form GM is positive for closed-loop stability, zero for neutral stability, and negative for instability. It is traditional in control-system engineering to express GM in *decibels* (dB), in which unit the value of a positive number N is $20 \times \log N$ dB. Accordingly, $\mathrm{GM}\,(0.1\Lambda_{ns}) = +20$ dB for the closed-loop stable case of Figure 17.1.4, and $\mathrm{GM}\,(10\Lambda_{ns}) = -20$ dB for the unstable case.

Another open-loop frequency-response metric for closed-loop degree of stability is the *phase margin* (PM). The phase margin for a given value Λ of the control parameter is defined as the lag angle at the frequency for which the magnitude ratio is unity **relative to** the comparable lag angle for the neutral-stability case, $\Lambda = \Lambda_{ns}$:

$$\mathrm{PM}(\Lambda) = (\angle OLFRF(\omega)|_{\Lambda}\ \text{ at }\ |OLFRF(\omega)|_{\Lambda}\ | = 1) - \left(\angle OLFRF\,(\omega_{ns})|_{\Lambda_{ns}}\ \text{ at }\ |OLFRF\,(\omega_{ns})|_{\Lambda_{ns}}\ | = 1\right)$$

From Equation 17.1.4 and the phase plot of Figure 17.1.3, the reference lag angle is just –180°, so

$$\mathrm{PM}(\Lambda) = 180^{\circ} + (\angle OLFRF(\omega)|_{\Lambda}\ \text{ at }\ |OLFRF(\omega)|_{\Lambda}\ | = 1) \tag{17.1.7}$$

The condition $|OLFRF(\omega)|_{\Lambda}\ | = 1$ is called a *gain crossover*. Reading approximately the appropriate phases from the phase plot of Figure 17.1.3, we calculate these phase margins: $\mathrm{PM}\,(0.1\Lambda_{ns}) \approx 180^{\circ} - 120^{\circ} = +60^{\circ}$ for the closed-loop stable system, and $\mathrm{PM}\,(10\Lambda_{ns}) \approx 180^{\circ} - 225^{\circ} = -45^{\circ}$ for the unstable system. (See homework Problem 17.2(a) for calculation of the more precise values of PM annotated on Figure 17.1.3.) Clearly, $\mathrm{PM} > 0^{\circ}$ implies closed-loop stability and $\mathrm{PM} < 0^{\circ}$ implies instability.

Brogan, 1974, page 25, described succinctly the practical interpretation of these stability margins for an initially stable closed-loop system: "*Gain margin* GM is a measure of additional gain a system can tolerate with no change in phase, while remaining stable. *Phase margin* PM is the additional phase shift that can be tolerated, with no gain change, while remaining stable."

We have defined and illustrated gain and phase margins for stable and unstable feedback control using the physical system of Figures 16.3.1, 16.3.2, and 17.1.2. The behavior of this physical system is fairly "common" in the sense that it is stable for gain $\Lambda = 0$ and for low positive values of Λ, but it is unstable for **all** values of Λ greater than a particular neutral-stability value, Λ_{ns}. For any such common physical system, the open-loop frequency-response metrics of gain margin and phase margin described in

this section are valid measures of absolute and relative stability. However, for “uncommon” physical systems, gain margins and phase margins of open-loop frequency response can be ambiguous and difficult to interpret; for example, such an uncommon system might have $\mathrm{PM} = +78^\circ$ but $\mathrm{GM} = -10$ dB, yet still be closed-loop stable. For such uncommon systems, a more general and involved, but unambiguous, approach to frequency-response analysis is based on the Nyquist stability criterion, which is the subject of Section 17.3.

Now that we know how to calculate open-loop frequency responses in the format of Figures $17.1.3$ and $17.1.4$ using standard MATLAB commands, it is appropriate that we not have to do so for every problem, but, instead, that we can apply the built-in MATLAB functions `margin` and `bode` to produce more easily the same results. Before calling one of these functions, it is necessary first to define with the `tf` command a MATLAB lineartime-invariant (LTI) transfer-function model. The following, for the gain $\Lambda = 10\Lambda_{ns}$ of the same system as that of Figures $17.1.3$ and $17.1.4$, shows one of the many ways to do this, along with the MATLAB responses:

```
>> wb=300;coj=100;

>> s=tf('s')

Transfer function:

s

>> lm=4e5;oltf4e5=lm*wb/(s*(s+coj)*(s+wb))

Transfer function:

    1.2e008

---------------------- s^3 + 400 s^2 + 30000 s

>> margin(oltf4e5)
```

This `margin` command produced the *Bode diagram* or *plot* of Figure $17.1.5$ (next page), which is annotated with the GM and PM for this case, $\Lambda = 10\Lambda_{ns}$, that of the unstable closed-loop system. (The `bode` function itself also produces the Bode diagram, but without calculating and printing the GM and PM.) Note that it was not necessary to specify the frequency band—MATLAB chose an appropriate band. By comparing Figure $17.1.5$ with Figures $17.1.3$ and $17.1.4$, you should observe some differences in format, which is why Figure $17.1.3$ is labeled a “modified” Bode diagram; but there are no differences in the essential content. A traditional Bode diagram such as Figure $17.1.5$ plots the frequency-response magnitude ratio (in decibels) versus frequency (in rad/s or Hz), which is presented on a logarithmic scale in order to accommodate a very wide band.

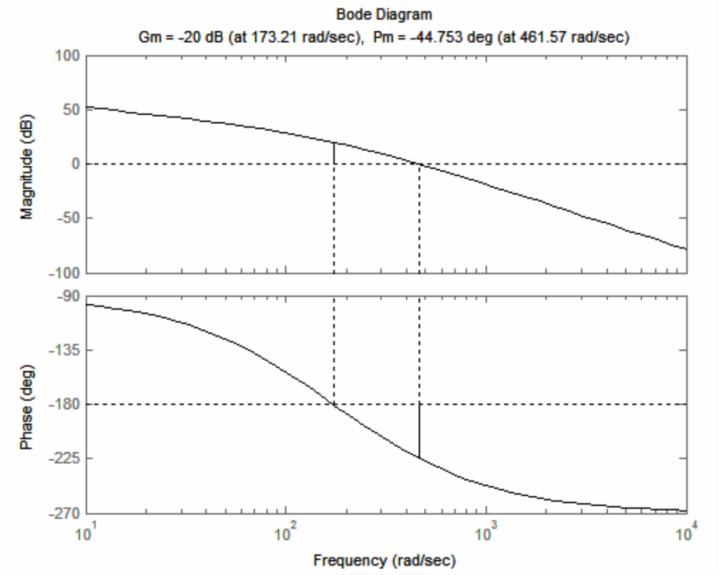


Figure 17.1.5: MATLAB's traditional Bode diagram, Equation 17.1.5, frequency response for $\Lambda = 10\Lambda_{ns} = 400,000$ s^{-2} of the open-loop system in Figure 17.1.2. (Copyright; author via source)

To conclude this section, we consider for systems in general how changes in gain Λ influence open-loop frequency responses and their graphical representations on Bode diagrams (and Nyquist diagrams in Section 17.4). We infer from Equations 16.6.2 and 16.6.3 that an open-loop transfer function can generally be expressed in the form

$$\mathrm{OLTF}(s) = \Lambda \times TF_1(s) \tag{17.1.8}$$

where the *baseline* transfer function $TF_1(s)$ is obviously defined as $OLTF(s)|_{\Lambda=1}$. The associated general frequency-response function is

$$OLFRF(\omega) = \Lambda \times TF_1(j\omega) = \Lambda \times |TF_1(j\omega)| \times \exp\left[j\angle TF_1(j\omega)\right] \tag{17.1.9}$$

Equation 17.1.9 leads to the following general forms for open-loop frequency-response magnitude ratio (MR) and phase angle ϕ:

$$MR(\omega) = |OLFRF(\omega)| = \Lambda \times |TF_1(j\omega)| \equiv \Lambda \times MR_1(\omega) \tag{17.1.10}$$

$$\phi(\omega) = \angle TF_1(j\omega) \tag{17.1.11}$$

Equation 17.1.11 shows that the variation of phase angle with frequency, $\phi(\omega)$, is the same for all values of gain Λ, as was stated earlier in the context of Figures 17.1.3 and 17.1.4.

In the final form of Equation 17.1.10, we define the baseline magnitude ratio (*i.e.*, for unity gain, $\Lambda = 1$) as $MR_1(\omega) \equiv |TF_1(j\omega)|$. It is also informative to express Equation 17.1.10 in a logarithmic form, either in logarithm to base 10 as on Figures 17.1.3 and 17.1.4 , or in decibels as on a traditional Bode plot such as Figure 17.1.5:

$$\log(MR(\omega)) = \log(\Lambda) + \log\left(MR_1(\omega)\right) \tag{17.1.12}$$

$$20 \times \log(MR(\omega)) \equiv [MR(\omega)](\mathrm{dB}) = [\Lambda](\mathrm{dB}) + [MR_1(\omega)]\,(\mathrm{dB}) \tag{17.1.13}$$

We infer from Equations 17.1.10, 17.1.12, and 17.1.13 that the graphical **shape** of the functional variation with frequency of magnitude ratio $MR(\omega)$ does not change with changes in gain Λ. For modified Bode plots such as Figures 17.1.3 and 17.1.4, given the baseline curve $MR_1(\omega)$, we can find from Equation 17.1.12 the curve for any other gain $\Lambda > 1$ just by adding to the baseline curve the value $\log(\Lambda)$, which moves the entire curve up by the distance $\log(\Lambda)$ along the magnitude-ratio axis. The process of finding curves for different gains Λ from Equation 17.1.13 is easy for a traditional Bode plot such as Figure 17.1.5, on which magnitude is plotted in decibels with a linear scale. For example, we can find from Figure 17.1.5 the magnitude curve for $\Lambda = 40,000$ s^{-2} simply by moving downward by 20 dB, *i.e.*, by $20 \times [\log(40,000) - \log(400,000)]$, the magnitude curve for $\Lambda = 400,000$ s^{-2}.

[1]We use the Laplace complex variable s in this chapter because it is traditional for treatment of frequency-response theory to use

s, instead of complex variable p (for $s = p$, a solution of the characteristic equation, *i.e.*, a *pole* of the closed-loop transfer function) that we used in Sections 16.3 through 16.6.

[2]The line of MATLAB code that calculates phase requires the angles to be between $-\pi/2$ and $-3\pi/2$ rad, so the argument here appears to be circular. However, if we were to use the more general MATLAB function `angle` in place of that line of code, we would find, in the range of $0.1 \leq \omega/\omega_{ns} < 1.0$, exactly the same lagging phases as are on the phase plot of Figure $17.1.3$, which does validate the assertion that the appropriate value is $-\pi$ rad. We did not use `angle`, because it is apparently based upon MATLAB's four-quadrant inverse tangent function`atan2`, which returns angle values only between $-\pi$ and $+\pi$ rad. Therefore, on the graph of phase versus ω/ω_{ns}, `angle` would produce an artificial and confusing discontinuous jump in phase from –180° to +180° at $\omega/\omega_{ns} = 1.0$ and, in the range of $1.0 \leq \omega/\omega_{ns} \leq 10$, phase would drop from +180° to a little more than +90°.

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17.2: Nyquist Plots

We now depart temporarily from the analysis of stability in order to describe a type of frequency-response graphical representation that has not appeared previously in this book. Such a graph is commonly called, in the language of control-system engineering, a *Nyquist diagram* or *plot*.

We begin the description of Nyquist plots by re-visiting a familiar system from several previous chapters: the standard, positively damped 2nd order system. The ODE of this system for output $x(t)$ in response to input $u(t)$ is $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=\omega_n^2 u(t)$ [Equation 9.2.2], and the frequency-response function is

$$FRF(\omega)=\frac{\omega_n^2}{\omega_n^2-\omega^2+j2\zeta\omega_n\omega} \tag{17.2.1}$$

The previous graphical representation of Equation 17.2.1 in this book is Figure 10.2.1, a modified Bode diagram displaying the variation with frequency ω of the **polar** components, magnitude ratio $|FRF(\omega)|$ and phase angle $\phi=\angle FRF(\omega)$. A Nyquist plot, in contrast, displays directly the mathematical **rectangular** components, in the format $\mathrm{Im}[FRF(\omega)]$ versus $\mathrm{Re}[FRF(\omega)]$, with frequency ω being the implicit independent variable, for which there is no graduated scale without extra annotation or a three-dimensional third axis perpendicular to the complex $FRF(\omega)$-plane. For example, the following MATLAB commands produce Figure 17.2.1, a Nyquist plot of Equation 17.2.1 for undamped natural frequency $\omega_n=2\pi$ rad/s and damping ratio $\zeta=0.2$:

```
>> wn=2*pi;zt=0.2;

>> w=wn*logspace(-2,2,400);

>> frf=wn^2./(wn^2-w.^2+j*2*zt*wn*w);

>> plot(real(frf),imag(frf)),grid
```

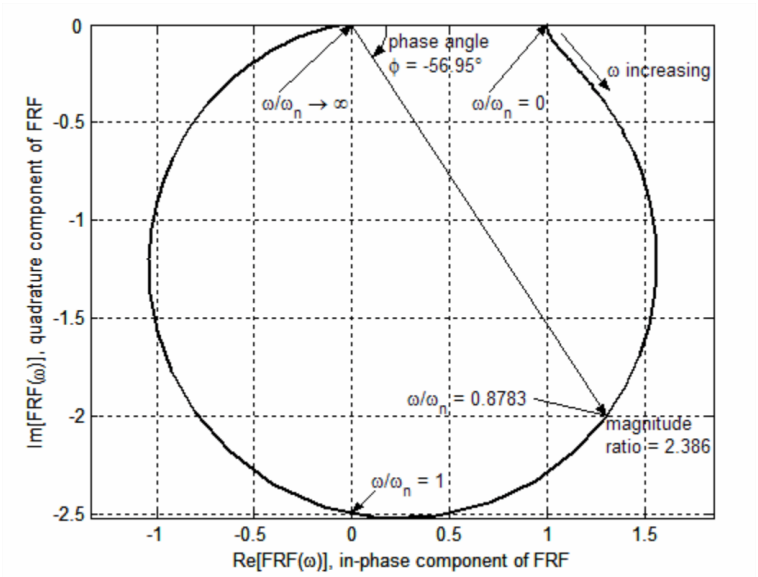


Figure 17.2.1: Nyquist plot of $FRF(\omega)=\omega_n^2/\left(\omega_n^2-\omega^2+j2\zeta\omega_n\omega\right)$ for $\omega_n=2\pi$ rad/s and $\zeta=0.2$. (Copyright; author via source)

Observe on Figure 17.2.1 that the abscissa and ordinate axes are labeled both with the mathematical descriptions, $\mathrm{Re}[FRF(\omega)]$ and \(\operatorname{Im}[F R F(\omega)]), and with the corresponding **physical** descriptions of the response components, respectively: *in-phase* (0°, also known as *coincident*) and *quadrature* (leading by 90°). If you were measuring frequency response experimentally on a damped 2nd order system you might generate the physical information of Figure 17.2.1 in the following manner: first, measure the polar components of response, magnitude ratio $|FRF(\omega)|$ and phase angle $\phi(\omega)=\angle FRF(\omega)$, using the procedure described in Section 17.1 relative to Figure 17.1.3, the modified Bode diagram; then, use Equation 2.1.7 for converting complex numbers from polar form into rectangular form to calculate $FRF_{\text{in-phase}}(\omega)=|FRF(\omega)|\times\cos\phi(\omega)$ and $FRF_{\text{quadrature}}(\omega)$

$= |FRF(\omega)| \times \sin\phi(\omega)$. An example of this conversion is illustrated on Figure 17.2.1 for $\omega = 0.8783\omega_n$: $1.301 = 2.386 \times \cos(-56.95^\circ)$, and $-2.000 = 2.386 \times \sin(-56.95^\circ)$.

The concept of physical in-phase and quadrature frequency-response components might seem strange. Moreover, their values can be either positive or negative (as Figure 17.2.1 shows for the in-phase component), and that might be confusing: if a pure "in-phase" response is negative, then the response is actually 180° **out** of phase with the excitation; and if a pure "quadrature" response is negative, then the response actually lags the excitation by 90°. In order to provide more clarity, even at the risk of being redundant, we present the following table, which lists multiple-of-90° values of phase angle ϕ that correspond to particular values or ranges of the in-phase and quadrature components.

In-phase Component	Quadrature component	Phase angle ϕ
> 0	$= 0$	0° and $\mp 360^\circ$
$= 0$	< 0	-90° and $+270^\circ$
< 0	$= 0$	$\mp 180^\circ$
$= 0$	> 0	-270° and $+90^\circ$

It might be instructive also to compare the Nyquist plot of Figure 17.2.1 with the corresponding polar components for $\zeta = 0.2$ on the modified Bode plots of Figure 10.2.1. On both figures for $\omega \approx 0$, the response is essentially the static value, $FRF(0) = +1 + j0$. As frequency ω increases from $0+$ to ω_n, the magnitude ratio varies (mostly increasing) from +1 to $1/(2\zeta) = 2.5$, while the phase angle, a lag, varies from 0° to −90°. At $\omega = \omega_n$, the response is pure negative quadrature, $FRF(\omega_n) = 0 - j2.5$. As ω increases above ω_n, the magnitude ratio decreases monotonically while the phase lags further, with the magnitude ratio approaching zero and the phase approaching −180° as $\omega \to \infty$.

Frequency response of the standard, positively damped 2nd order system is well behaved, especially for $\omega = 0$, *i.e.*, a non-zero input that is constant in time, for which the system has an unquestionably stable non-zero constant static output. Let us consider now Nyquist plotting for a contrasting system, an open-loop system that lacks a stable static response, the system of Section 17.1 (also Sections 16.3, 16.5, and 16.6). The clearest indication of a possible difficulty is Equation 17.1.7 for $\mathrm{OLFRF}(\omega)$, which indicates an infinite response for $\omega = 0$. The mathematical source of this difficulty is the pole of $O\mathrm{LTF}(s)$ at the s-plane origin, which is apparent in Equation 17.1.2, Figure 16.6.2, and elsewhere. The graphical display of frequency response magnitude becoming very large as $\omega \to 0$ is produced by the following MATLAB commands, which calculate frequency response and produce a Nyquist plot of the same numerical solution as that on Figure 17.1.3, for the neutral-stability case $\Lambda = \Lambda_{ns} = 40,000$ s^{-2}:

```
>> wb=300;coj=100;wns=sqrt(wb*coj);

>> wbar=logspace(-1,1,200);w=wbar*wns;

>> lm=4e4;olfrf4e4=lm*wb./(j*w.*(j*w+coj).*(j*w+wb));

>> plot(real(olfrf4e4),imag(olfrf4e4)),grid
```

Figure 17.2.2 is an annotated portion of the MATLAB graph that displays the noteworthy features of the Nyquist plot. This plot shows that for frequency ω becoming progressively smaller, the negative quadrature component of response becomes progressively larger.

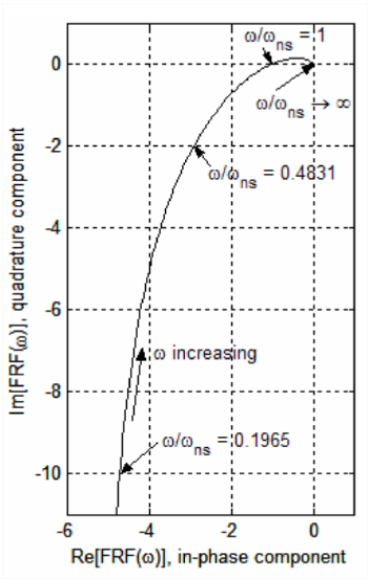


Figure 17.2.2: Nyquist plot, Equation 17.1.7, frequency response for $\Lambda = \Lambda_{ns} = 40,000$ s^{-2} of the open-loop system in Figure 17.1.2, with parameters $\omega_b = 300$ s^{-1}, $c_\theta/J = 100$ s^{-1}, and $\omega_{ns} = \sqrt{\omega_b \times c_\theta/J} = 173.2$ rad/s. (Copyright; author via source)

In fact, we can calculate from Equation 17.1.7 the asymptotic nature of the variation of open-loop frequency response as $\omega \to 0$, by applying the method of complex rectangular division described in Section 2.1:

$$OLFRF(\omega) = \frac{\Lambda\omega_b}{j\omega\left(j\omega + c_\theta/J\right)\left(j\omega + \omega_b\right)} = \frac{40,000 \times 300}{(j\omega)^3 - (100+300)\omega^2 + (100 \times 300)j\omega} = \frac{1.2 \times 10^7 \times \left[-400\omega^2 - j\left(3 \times 10^4\omega - \omega^3\right)\right]}{9 \times 10^8\omega^2 + 1 \times 10^5\omega^4 + \omega^6}$$

Therefore, as $\omega \to 0$,

$$\text{OLFRF}(\omega) \to \frac{1.2 \times 10^7 \times \left[-400\omega^2 - j\left(3 \times 10^4\omega\right)\right]}{9 \times 10^8\omega^2} = -51/3 - j\frac{400}{\omega} \tag{17.2.2}$$

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17.3: The Practical Effects of an Open-Loop Transfer-Function Pole at s = 0 + j0

The development above demonstrates an important characteristic of an open-loop system with a transfer-function pole at the s-plane origin: frequency response of such a system increases in magnitude progressively as excitation frequency decreases, becoming, in theory, infinitely large for zero frequency. What is the significance of this type of response relative to experimental testing? The question is appropriate since, as asserted in the introduction to this chapter, frequency-response of the open-loop portion of a practical system (actual hardware) can indicate the stability or instability of the corresponding closed-loop system. Figure 17.2.2 and Equation 17.2.3 show clearly that the frequency-response magnitude for very low frequencies of excitation could indeed be impractically large. This might lead us to assume that, in order to avoid excessive output magnitudes, we would need only to excite the open-loop system at frequencies above some practical lower limit. The discussion in this section will show that this would be a potentially harmful assumption.

What is the nature of response associated with a system's transfer-function pole at the origin of the s-plane? We know that a pole on the $\mathrm{Re}(s)$ axis in the right-half s-plane means the system's time response (to an impulse, for example) is essentially monotonically growing exponential, an instability. (See the discussion of Section 16.4.) Conversely, a pole on the $\mathrm{Re}(s)$ axis in the left-half s-plane means the response is essentially monotonically decaying exponential, a stable response. Can we infer, then, that a pole on the $\mathrm{Re}(s)$ axis at the origin, between the right- and left-half planes, means time response is neither unstable nor stable, but somehow neutral?

Let us try to answer that question, at least for the open-loop system of current interest that motivated this discussion, by deriving the unit-impulse response, $h(t)$. From Equation 8.9.5, a system's transfer function is also the Laplace transform of its unit-impulse response; accordingly, Equation 17.1.2 gives $L[h(t)] = \Lambda\omega_b / [s\,(s + c_\theta/J)\,(s + \omega_b)]$. We could find the complete equation for $h(t)$, but it is easier and equally relevant for this discussion to apply the final-value theorem [Equation 15.3.1] to find the steady-state response [which we can show follows the decay to zero of stable terms whose time functions are $e^{-(c_\theta/J)t}$ and $e^{-\omega_b t}$]: $\lim_{t\to\infty} h(t) = \lim_{s\to 0} s \times L[h(t)] = \Lambda J/c_\theta$. This constant-in-time response to an impulse excitation is unusual, since impulse response of a stable system normally decays to zero.

Let us probe further and find the response to a step input, which is a better gauge of stability than impulse response. Equation 8.9.10 shows that the unit-impulse response of any LTI-SISO system is the time derivative of the unit-step response, $h(t) = dx_H/dt$. Therefore, the unit-step response as $t \to \infty$ of the open-loop system of current interest must be $\lim_{t\to\infty} x_H(t) = C + (\Lambda J/c_\theta)\,t$, C being a constant. This linear-function-of-time response to a constant input can increase monotonically, without bound. This is not an exponential monotonic instability or exponential oscillatory instability of the type we have observed previously; nevertheless, for all practical purposes, it is a genuine form of instability.

We can be even more specific relative to frequency response. With reference to the functional diagram that depicts testing of the open-loop system, Figure 17.1.2, suppose that the input signal, in addition to the sinusoidal excitation, has a slight constant voltage offset, E_{off}. (Even with high-pass filtering, it is often nearly impossible to eliminate all such offsets from real circuitry used in experimental testing.) Therefore, with $H(t)$ denoting the Heaviside unit-step function, the complete input signal is:

$$e_{in}(t) = E_{in}\sin\omega t + E_{off}H(t) \tag{17.3.1}$$

With use of $OLTF(s)$ of Equation 17.1.2 and inverse Laplace transformation, we can find an algebraic equation for the response in time to input Equation 17.3.1, assuming zero initial conditions. The full equation is quite lengthy. It includes transient-response terms whose time functions are $e^{-(c_\theta/J)t}$ and $e^{-\omega_b t}$, both of which decay quickly to insignificance. If we omit those transient-response terms, then we can express the "steady-state" response in the form (homework Problem 17.?):

$$e_{\mathrm{out}}(t) = E_{\mathrm{in}}\left(MR(\omega)\times\sin(\omega t + \phi(\omega)) + \frac{\Lambda}{\omega\,(c_\theta/J)}\right) + E_{off}\frac{\Lambda J}{c_\theta}\left[t - \frac{(c_\theta/J + \omega_b)}{(c_\theta/J)\,\omega_b}\right] \tag{17.3.2}$$

Open-loop frequency-response magnitude ratio $\mathrm{MR}(\omega)$ and phase angle $\phi(\omega)$ are the quantities defined generally by Equations 17.1.3 and 17.1.4, respectively, and represented for this specific system, from Equation 17.1.7, by the equations:

$$MR(\omega) = \left|\Lambda\frac{\omega_b}{j\omega\,(j\omega + c_\theta/J)\,(j\omega + \omega_b)}\right| \quad \text{and} \quad \phi(\omega) = \angle\left(\frac{\omega_b}{j\omega\,(j\omega + c_\theta/J)\,(j\omega + \omega_b)}\right)$$

Ideally in experimental sinusoidal testing, there would be only steady-state sinusoidal response. Much less ideally, however, there are in Equation 17.3.2 three types of time response: the sinusoidal term, two constant-in-time terms, and a term that increases linearly with time. The constant-in-time terms would be a nuisance, but they could be tolerable, provided they were not so large as to overcome mechanical or electrical limits. However, the unstable linear "drift", $E_{off}\,(\Lambda J/c_\theta)\,t$, would present a serious problem in the testing process. Consider, for example, the numerical solution of Equation 17.3.2 for the open-loop system with the ideal frequency response of Figures 17.1.3 and 17.2.2, and for excitation at the neutral-stability frequency, $\omega = \omega_{ns}$. For this case, Figures 17.1.3 and 17.2.2 show that $MR\,(\omega_{ns}) = 1$ and $\phi\,(\omega_{ns}) = -180^\circ$. Completing the calculations in Equation 17.3.2 with the remaining numerical values gives:

$$e_{out}(t) = E_{in}\left[(-\sin\omega_{ns}t + 2.309) + \frac{E_{off}}{E_{in}}(400t - 51/3)\right] \tag{17.3.3}$$

Equation 17.3.3 shows, for example, that if the offset voltage is even a mere 1% of the input sinusoidal magnitude, $E_{off}/E_{in} = 0.01$, then, after only one second of system response, the linear drift increases to four times the sinusoidal-response magnitude; and the linear drift just continues rising inexorably thereafter. Practical sinusoidal testing of this system would be possible only if the offset voltage were almost completely eliminated, in order to prevent, or at least minimize, this unstable drift.

The type of response represented in Equation 17.3.2 and illustrated numerically in Equation 17.3.3 would pose a formidable challenge for any type of standard experimental testing on any system whose mathematical model has a pole at the origin of the s -plane. This mathematical model represents a general physical deficiency of such a system: the absence of a passive mechanism that tends to restore the system toward a static equilibrium state after the system has been perturbed dynamically. For example, the damped-rotor plant of our system of current interest, Figures 16.3.2 and 17.1.2, is the prime contributor to the unstable drift; this plant lacks a restoring rotational spring, and so its transfer function, Equation 16.3.7, $PTF(s) = 1/\left[s\,(J_S + c_\theta\right]$, has a pole at the origin.

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17.4: The Nyquist Stability Criterion

Section 17.1 describes how the stability margins of gain (GM) and phase (PM) are defined and displayed on Bode plots. It is informative and it will turn out to be even more general to extract the same stability margins from Nyquist plots of frequency response. To begin this study, we will repeat the Nyquist plot of Figure 17.2.2, the closed-loop neutral-stability case, for which $\Lambda = \Lambda_{ns} = 40,000$ s^{-2} and $\omega_{ns} = 100\sqrt{3}$ rad/s, but over a narrower band of excitation frequencies, $100 \leq \omega \leq 1,000$ rad/s, or $1/\sqrt{3} \leq \omega/\omega_{ns} \leq 10/\sqrt{3}$; the intent here is to restrict our attention primarily to frequency response for which the phase lag exceeds about 150°, *i.e.*, for which the frequency-response curve in the $OLFRF$-plane is somewhat close to the negative real axis. Moreover, we will add to the same graph the Nyquist plots of frequency response for a case of positive closed-loop stability with $\Lambda = 1/2\Lambda_{ns} = 20,000$ s^{-2}, and for a case of closed-loop instability with $\Lambda = 2\Lambda_{ns} = 80,000$ s^{-2}. The MATLAB commands follow that calculate [from Equations 17.1.7 and 17.1.12] and plot these cases of open-loop frequency-response function, and the resulting Nyquist diagram (after additional editing):

```
>> wb=300;coj=100;w=logspace(2,3,200);

>> olfrf01=wb./(j*w.*(j*w+coj).*(j*w+wb));

>> olfrf20k=20e3*olfrf01;olfrf40k=40e3*olfrf01;olfrf80k=80e3*olfrf01;

>> plot(real(olfrf80k),imag(olfrf80k),real(olfrf40k),imag(olfrf40k),…

real(olfrf20k),imag(olfrf20k)),grid
```

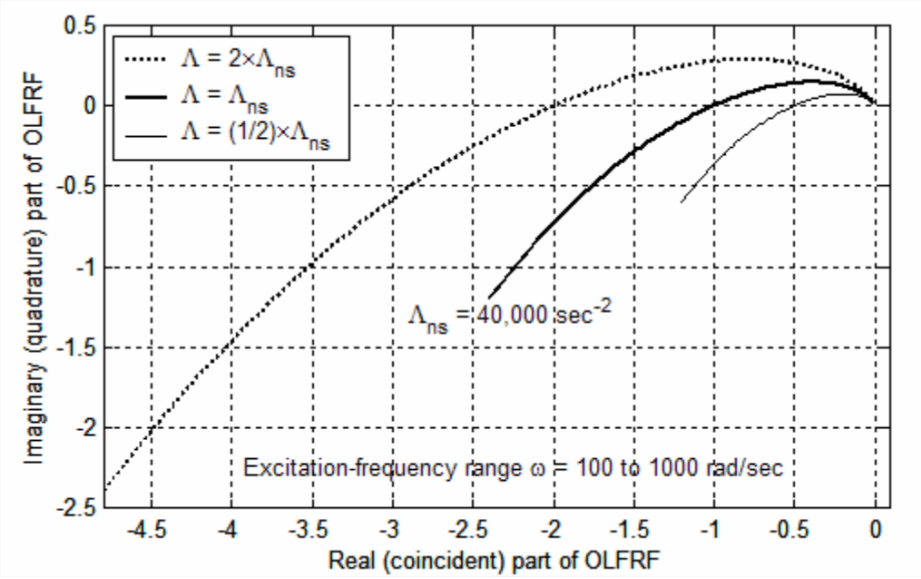


Figure 17.4.1: Nyquist plots of Equation 17.1.7, open-loop frequency-response functions ($OLFRF$) for $\Lambda = (1/2, 1, \text{ and } 2) \times \Lambda_{ns}$ for the system in Figure 17.1.2, with parameters $\omega_b = 300$ s^{-1}, $c_\theta/J = 100$ s^{-1}, and $\omega_{ns} = \sqrt{\omega_b \times c_\theta/J} = 173.2$ rad/s. (Copyright; author via source)

Gain margin and phase margin are present and measurable on Nyquist plots such as those of Figure 17.4.1. Gain margin (GM) is defined by Equation 17.1.8, from which we find

$$\frac{1}{GM(\Lambda)} = |OLFRF(\omega)|_{|} \text{ at } \angle OLFRF(\omega)|_{\Lambda} = -180^\circ \tag{17.4.1}$$

Equation 17.4.1 is illustrated on Figure 17.4.2 for both closed-loop stable and unstable cases. Note that a closed-loop-stable case has $0 < 1/\text{GM}_\text{S} < 1$ so that $\text{GM}_\text{S} > 1$, and a closed-loop-unstable case has $1/\text{GM}_\text{U} > 1$ so that $0 < \text{GM}_\text{U} < 1$.

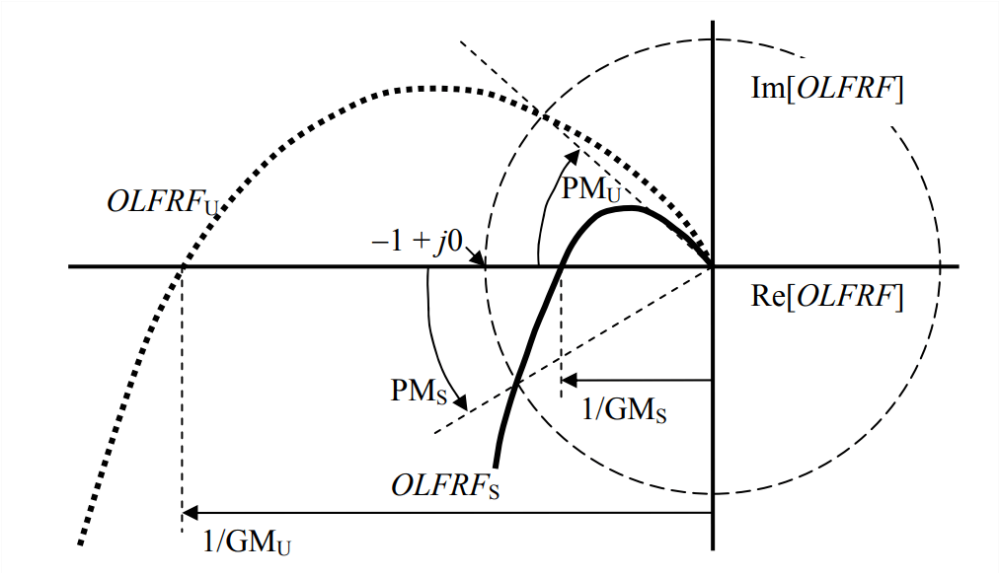


Figure 17.4.2: Schematic Nyquist plots of open-loop frequency-response functions ($OLFRF$) for closed-loop-stable and –unstable cases. (Copyright; author via source)

We can measure phase margin directly by drawing on the Nyquist diagram a circle with radius of 1 unit and centered on the origin of the complex $OLFRF$-plane, so that it passes through the important point $-1+j0$. Phase margin is defined by

$$\mathrm{PM}(\Lambda) = 180^\circ + (\angle OLFRF(\omega)|_\Lambda \text{ at } |OLFRF(\omega)|_\Lambda| = 1) \tag{17.4.2}$$

Phase margins are indicated graphically on Figure 17.4.2. The positive $\mathrm{PM_S}$ for a closed-loop-stable case is the counterclockwise angle from the negative $\mathrm{Re}[OLFRF]$ axis to the intersection of the unit circle with the $OLFRF_S$ curve; conversely, the negative PM_U for a closed-loop-unstable case is the clockwise angle from the negative $\mathrm{Re}[OLFRF]$ axis to the intersection of the unit circle with the $OLFRF_U$ curve.

Note on Figure 17.4.2 that the phase-crossover **point** (phase angle $\phi = -180^\circ$) and the gain-crossover **point** (magnitude ratio $MR = 1$) of an FRF are clearly evident on a Nyquist plot, perhaps even more naturally than on a Bode diagram. On the other hand, a Bode diagram displays the phase-crossover and gain-crossover **frequencies**, which are not explicit on a traditional Nyquist plot.

With a little imagination, we infer from the Nyquist plots of Figure 17.4.1 that the open-loop system represented in that figure has $\mathrm{GM} > 0$ and $\mathrm{PM} > 0$ for $0 < \Lambda < \Lambda_{\mathrm{ns}}$, and that $\mathrm{GM} > 0$ and $\mathrm{PM} > 0$ for all values of gain Λ greater than Λ_{ns}; accordingly, the associated closed-loop system is stable for $0 < \Lambda < \Lambda_{\mathrm{ns}}$, and unstable for all values of gain Λ greater than Λ_{ns}. Thus, this physical system (of Figures 16.3.1, 16.3.2, and 17.1.2) is considered a “common” system, for which gain margin and phase margin provide clear and unambiguous metrics of stability. Let us consider next an “uncommon” system, for which the determination of stability or instability requires a more detailed examination of the stability margins. Suppose that the open-loop transfer function of a system is[1]

$$G(s) \times H(s) \equiv OLTF(s) = \Lambda \frac{s^2 + 4s + 104}{(s+1)\,(s^2 + 2s + 26)} = \Lambda \frac{s^2 + 4s + 104}{s^3 + 3s^2 + 28s + 26} \tag{17.4.3}$$

You should be able to show that the zeros of this transfer function in the complex s-plane are at $(-2 \pm j10)$, and the poles are at ($-1 + j0$) and $(-1 \pm j5)$.

In order to establish the reference for stability and instability of the closed-loop system corresponding to Equation 17.4.3, we determine the loci of roots from the characteristic equation, $1 + GH = 0$, or

$$s^3 + 3s^2 + 28s + 26 + \Lambda\left(s^2 + 4s + 104\right) = s^3 + (3+\Lambda)s^2 + 4(7+\Lambda)s + 26(1+4\Lambda) = 0 \tag{17.4.4}$$

The following MATLAB commands, adapted from the code that produced Figure 16.5.1, calculate and plot the loci of roots:

```
Lm=[0 .2 .4 .7 1 1.5 2.5 3.7 4.75 6.5 9 12.5 15 18.5 25 35 50 70 125 250];
```

```
Np=length(Lm);

for i=1:Np;

    a2=3+Lm(i);a3=4*(7+Lm(i));a4=26*(1+4*Lm(i));

    p(i,1:3)=roots([1 a2 a3 a4]).';

end

plot(p,'kx'),grid,xlabel('Real part of pole (sec^-^1)')

ylabel('Imaginary part of pole (sec^-^1)')
```

The significant roots of Equation ??? are shown on Figure 17.4.3: the complete locus of oscillatory roots with positive imaginary parts is shown; only the beginning of the locus of real (exponentially stable) roots is shown, since those roots become progressively more negative as gain Λ increases from the initial small values. The oscillatory roots on Figure 17.4.3 show that the closed-loop system is stable for $\Lambda = 0$ up to $\Lambda \approx 1$, it is unstable for $\Lambda \approx 1$ up to $\Lambda \approx 15$, and it becomes stable again for Λ greater than ≈ 15. We regard this closed-loop system as being “uncommon” or unusual because it is stable for small and large values of gain Λ, but unstable for a range of intermediate values.

If we were to test experimentally the open-loop part of this system in order to determine the stability of the closed-loop system, what would the open-loop frequency responses be for different values of gain Λ? To simulate that testing, we have from Equation 17.4.3, the following equation for the frequency-response function:

$$OLFRF(\omega) \equiv OLTF(j\omega) = \Lambda \frac{104 - \omega^2 + 4 \times j\omega}{(1 + j\omega)\,(26 - \omega^2 + 2 \times j\omega)} \tag{17.4.5}$$

Let us begin this study by computing $\mathrm{OLFRF}(\omega)$ and displaying it on Nyquist plots for a low value of gain, $\Lambda = 0.7$ (for which the closed-loop system is stable), and for the value corresponding to the transition from stability to instability on Figure 17.4.3 , which we denote as $\Lambda_{ns1} \approx 1$. The value of Λ_{ns1} is not exactly 1, as Figure 17.4.3 might suggest; see homework Problem 17.2(b) for calculation of the more precise value $\Lambda_{ns1} = 0.96438$. The following MATLAB commands calculate and plot the two frequency responses and **also**, for determining phase margins as shown on Figure 17.4.2, an arc of the unit circle centered on the origin of the complex $OLFRF(\omega)$-plane.

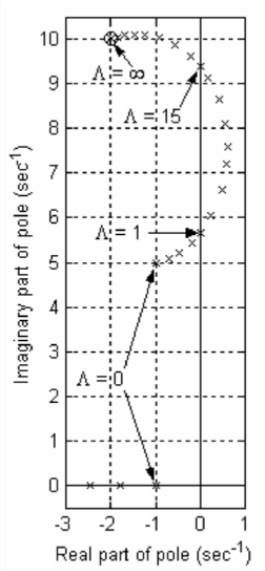


Figure 17.4.3: Loci of roots of Equation ???. (Copyright; author via source)

```
>> w=3.4*logspace(0,2,500);

>> olfrf01=(104-w.^2+4*j*w)./((1+j*w).*(26- w.^2+2*j*w));
```

```
>> olfrf007=0.7*olfrf01;

>> plot(real(olfrf007),imag(olfrf007)),grid

>> cirangrad=0.8*pi:0.01:1.2*pi;

>> hold,plot(cos(cirangrad),sin(cirangrad))

Current plot held

>> olfrfns1=0.96438*olfrf01;

>> plot(real(olfrfns1),imag(olfrfns1))
```

The portions of both Nyquist plots (for $\Lambda = 0.7$ and $\Lambda = \Lambda_{ns1}$) that are closest to the negative $\mathrm{Re}[OLFRF]$ axis are shown on Figure $17.4.4$ (next page). Observe on Figure $17.4.4$ the small loops beneath the negative $\mathrm{Re}[OLFRF]$ axis as driving frequency becomes very high: the frequency responses approach zero from below the origin of the complex $OLFRF$-plane. This is distinctly different from the Nyquist plots of a more "common" open-loop system on Figure $17.4.1$, which approach the origin from above as frequency becomes very high. Another aspect of the difference between the plots on the two figures is particularly significant: whereas the plots on Figure $17.4.1$ cross the negative $\mathrm{Re}[OLFRF]$ axis only once as driving frequency ω increases, those on Figure $17.4.4$ have **two** phase crossovers, *i.e.*, the phase angle is –180° for **two** different values of ω. Since on Figure $17.4.4$ there are two different frequencies at which $\angle OLFRF(\omega)|_{\Lambda} = -180^{\circ}$, the definition of gain margin in Equations 17.1.8 and $17.4.1$ is ambiguous: at which, if either, of the phase crossovers is it appropriate to read the quantity $1/\mathrm{GM}$, as shown on $17.4.2$? Which, if either, of the values calculated from that reading, $\mathrm{GM} = (1/\mathrm{GM})^{-1}$ is a legitimate metric of closed-loop stability? $17.4.4$ includes the Nyquist plots for both $\Lambda = 0.7$ and $\Lambda = \Lambda_{ns1}$, the latter of which by definition crosses the negative $\mathrm{Re}[OLFRF]$ axis at the point $-1 + j0$, not far to the left of where the $\Lambda = 0.7$ plot crosses at about $-0.73 + j0$; therefore, it might be that the appropriate value of gain margin for $\Lambda = 0.7$ is found from $1/\mathrm{GM}_{0.7} \approx 0.73$, so that $\mathrm{GM}_{0.7} \approx 1.37 = 2.7$ dB, a small gain margin indicating that the closed-loop system is just weakly stable. If, on the other hand, we were to calculate gain margin using the other phase crossing, at about $-0.04 + j0$, then that would lead to the exaggerated $\mathrm{GM} \approx 25 = 28$ dB, which is obviously a defective metric of stability. Note that the phase margin for $\Lambda = 0.7$, found as shown on Figure $17.4.2$, is quite clear on Figure $17.4.4$ and not at all ambiguous like the gain margin: $\mathrm{PM}_{0.7} \approx +20^{\circ}$; this value also indicates a stable, but weakly so, closed-loop system.

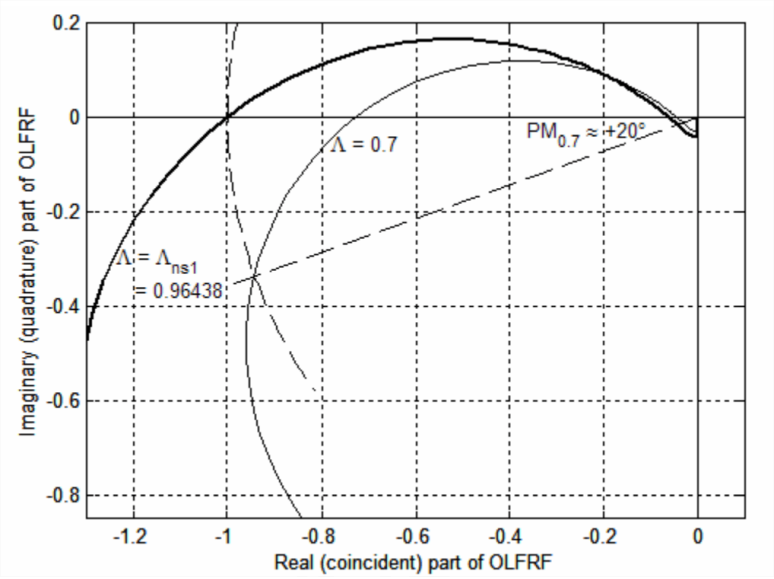


Figure $17.4.4$: Nyquist plots, Equation $17.4.5$, open-loop frequency-response functions ($OLFRF$) for $\Lambda = 0.7$ and $\Lambda = \Lambda_{ns1} = 0.96438$ in the system of Equation $17.4.3$. (Copyright; author via source)

Let us continue this study by computing $OLFRF(\omega)$ and displaying it as a Nyquist plot for an intermediate value of gain, $\Lambda = 4.75$, for which Figure $17.4.3$ shows the closed-loop system is unstable. The following MATLAB commands calculate [from Equations 17.1.12 and $17.4.5$] and plot the frequency response and an arc of the unit circle centered at the origin of the complex $OLFRF(\omega)$ -plane.

```
w=3.4*logspace(0,2,500);

olfrf01=(104-w.^2+4*j*w)./((1+j*w).*( 26-w.^2+2*j*w));

>> olfrf0475=4.75*olfrf01;

>> plot(real(olfrf0475),imag(olfrf0475)),grid

>> cirangrad=0.8*pi:0.01:1.2*pi;

>> hold,plot(cos(cirangrad),sin(cirangrad))
```

The portion of the Nyquist plot for gain $\Lambda = 4.75$ that is closest to the negative $\mathrm{Re}[OLFRF]$ axis is shown on Figure 17.4.5. The $\Lambda = \Lambda_{ns1}$ plot of Figure 17.4.4 is expanded radially outward on Figure 17.4.5 by the factor of $4.75/0.96438 = 4.9254$, so the loop for high frequencies beneath the negative $\mathrm{Re}[OLFRF]$ axis is more prominent than on Figure 17.4.4. The frequency-response curve leading into that loop crosses the $\mathrm{Re}[OLFRF]$ axis at about $-0.315 + j0$; if we were to use this phase crossover to calculate gain margin, then we would find $\mathrm{GM} \approx 1/0.315 = 3.175 = 10.0$ dB. Moreover, if we apply for this system with $\Lambda = 4.75$ the MATLAB `margin` command to generate a Bode diagram in the same form as Figure 17.1.5, then MATLAB annotates that diagram with the values $\mathrm{GM} = 10.007$ dB and $\mathrm{PM} = -23.721^\circ$ (the same as PM4.75 shown approximately on Figure 17.4.5). We know from Figure 17.4.3 that this case of $\Lambda = 4.75$ is closed-loop unstable. However, the positive gain margin 10 dB suggests positive stability. The negative phase margin indicates, to the contrary, instability. Clearly, the calculation $\mathrm{GM} \approx 1/0.315$ is a defective metric of stability. The other phase crossover, at $-4.9254 + j0$ (beyond the range of Figure 17.4.5), might be the appropriate point for calculation of gain margin, since it at least indicates instability, $\mathrm{GM}_{4.75} = 1/4.9254 = 0.20303 = -13.85$ dB.

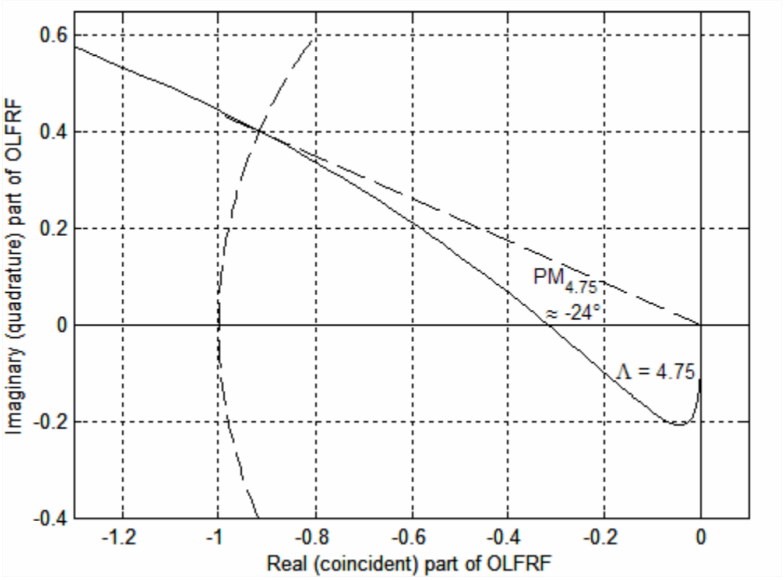


Figure 17.4.5: Nyquist plot, Equation 17.4.5, open-loop frequency-response function ($OLFRF$) for (\Lambda=4.75\) in the system of Equation 17.4.3. (Copyright; author via source)

Let us complete this study by computing $\mathrm{OLFRF}(\omega)$ and displaying it on Nyquist plots for the value corresponding to the transition from instability back to stability on Figure 17.4.3, which we denote as $\Lambda_{ns2} \approx 15$, and for a slightly higher value, $\Lambda = 18.5$, for which the closed-loop system is stable. The value of Λ_{ns2} is not exactly 15, as Figure 17.4.3 might suggest; see homework Problem 17.2(b) for calculation of the more precise value $\Lambda_{ns2} = 15.0356$. The portions of both Nyquist plots (for Λ_{ns2} and $\Lambda = 18.5$) that are closest to the negative $\mathrm{Re}[OLFRF]$ axis are shown on Figure 17.4.6, which was produced by the MATLAB commands that produced Figure 17.4.4, except with gains 18.5 and Λ_{ns2} replacing, respectively, gains 10.7 and Λ_{ns1}.

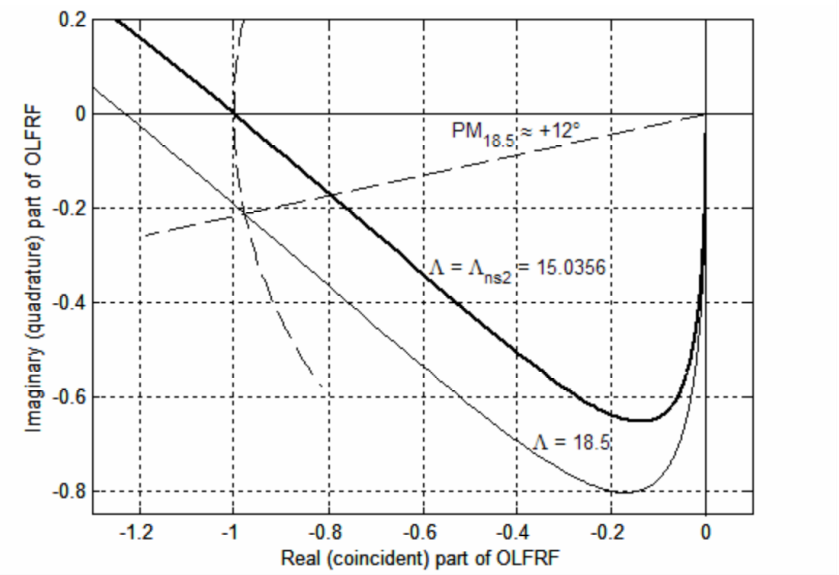


Figure 17.4.6: Nyquist plots, Equation 17.4.5, open-loop frequency-response functions ($OLFRF$) for $\Lambda = \Lambda_{ns2} = 15.0356$ and $\Lambda = 18.5$ in the system of Equation 17.4.3. (Copyright; author via source)

For gain $\Lambda = 18.5$, there are two phase crossovers: one evident on Figure 17.4.6 at $-18.5/15.0356 + j0 = -1.230 + j0$, and the other way beyond the range of Figure 17.4.6 at $-18.5/0.96438 + j0 = -19.18 + j0$. We know from Figure 17.4.3 that the closed-loop system with $\Lambda = 18.5$ is stable, albeit weakly. However, the gain margin calculated from either of the two phase crossovers suggests instability, showing that both are deceptively defective metrics of stability. On the other hand, the phase margin shown on Figure 17.4.6, $\mathrm{PM}_{18.5} \approx +12^{\circ}$, correctly indicates weak stability.

We draw the following conclusions from the discussions above of Figures 17.4.3 through 17.4.6, relative to an "uncommon" system with an open-loop transfer function such as Equation 17.4.3:

1. gain margin as defined on Figure 17.4.5 can be an ambiguous, unreliable, and even deceptive metric of closed-loop stability;
2. phase margin as defined on Figure 17.4.5, on the other hand, is usually an unambiguous and reliable metric, with $\mathrm{PM} > 0$ indicating closed-loop stability, and $\mathrm{PM} < 0$ indicating closed-loop instability.

Conclusion 2. regarding phase margin is a form of the *Nyquist stability criterion*, a form that is pertinent to systems such as that of Equation 17.4.3; it is not the most general form of the criterion, but it suffices for the scope of this introductory textbook.

Proofs of the general Nyquist stability criterion are based on the theory of complex functions of a complex variable; many textbooks on control theory present such proofs, one of the clearest being that of Franklin, *et al.*, 1991, pages 261-280. This reference shows that the form of stability criterion described above [Conclusion 2.] is not sufficiently general to handle all cases that might arise. For example, the unusual case of an open-loop system that has unstable poles requires the general Nyquist stability criterion. However, the actual hardware of such an open-loop system could not be subjected to frequency-response experimental testing due to its unstable character, so a control-system engineer would find it necessary to analyze a mathematical model of the system. It is likely that the most reliable theoretical analysis of such a model for closed-loop stability would be by calculation of closed-loop loci of roots, not by calculation of open-loop frequency response. Another unusual case that would require the general Nyquist stability criterion is an open-loop system with more than one gain crossover, *i.e.*, a system whose frequency response curve intersects more than once the unit circle shown on Figure 17.4.2, thus rendering ambiguous the definition of phase margin.

We conclude this chapter on frequency-response stability criteria by observing that margins of gain and phase are used also as engineering design goals. For example, Brogan, 1974, page 25, wrote "Experience has shown that acceptable transient response will usually require stability margins on the order of $\mathrm{PM} > 30^{\circ}$, $\mathrm{GM} > 6$ dB." Franklin, *et al.*, 1991, page 285, wrote "Many engineers think directly in terms of PM in judging whether a control system is adequately stabilized. In using PM this way, a phase margin of 30° is often judged to be the lowest acceptable PM, with values above 30° desirable."

[1]This transfer function was concocted for the purpose of demonstration. It does not represent any specific real physical system, but it has characteristics that are representative of some real systems. Gain Λ has physical units of s^{-1}, but we will not bother to show units in the following discussion.

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17.5: Chapter 17 Homework

1. The subject of this problem is frequency response of an RC band-pass filter circuit. The appropriate transfer function is Equation 13.1.5:

$$TF_B(s) = \frac{L[e_o]}{L[e_i]} = \frac{\tau_H s}{(\tau_H s + 1)(\tau_L s + 1)} = \frac{\omega_L s}{(s + \omega_H)(s + \omega_L)}$$

   in which $\omega_L \equiv 1/\tau_L$ is the low-pass break frequency and $\omega_H \equiv 1/\tau_H$ is the high-pass break frequency, and for band-pass functioning it is usually specified that $\omega_H << \omega_L$.
   1. This transfer function has (in the s-plane) a zero at the origin and poles at $-\omega_H$ and $-\omega_L$. Consider the case for frequency response: $s = j\omega$, $\omega \geq 0$. Sketch a graph of the s-plane that shows the zero and the poles, and sketch on the graph three complex vectors with tails at the zero and the poles, and heads at an arbitrary point $s = j\omega$ on the positive imaginary axis. Use this graphical construction to derive the frequency-response function in the polar form $FRF(\omega) = MR(\omega)e^{j\phi(\omega)}$, in which you derive explicit formulas for magnitude ratio $MR(\omega)$ and phase angle $\phi(\omega)$. [NOTE: This graphical method of deriving (or calculating) a frequency-response function is somewhat similar to (but simpler than) Evans' general root-locus method described in Section 16.5, with the "locus" in this case being the positive imaginary axis—see, in particular, the unnumbered figure that directly follows Equation 16.5.7.] *Partial answer*: $\phi(\omega) = \pi/2 - \tan^{-1}(\omega/\omega_H) - \tan^{-1}(\omega/\omega_L)$ radians
   2. Let $\omega_H = 5$ rad/s and $\omega_L = 500$ rad/s. Adapt the MATLAB code that produced Figure 17.1.3 to calculate and print a modified Bode diagram for this particular band-pass filter. Your diagram should show that signals at frequencies between about 10 and 200 rad/s are passed through this filter with very little amplitude reduction and relatively little phase change, but that signals at frequencies below about 0.5 rad/s and above about 5,000 rad/s are effectively "filtered out", removed.
2. In these exercises, you will use MATLAB function M-files and the MATLAB command `fzero` to calculate some results that are stated but not explicitly derived in Chapter 17.
   1. Calculate for at least one of the gains $\Lambda = 4,000$ s-2 and $400,000$ s^{-2} the more precise value of PM that is annotated on Figure 17.1.3. The procedure is based on the definition of $\mathrm{PM}(\Lambda)$ in Equation 17.1.10, which shows that you need to find the frequency ω_{PM} at which $|OLFRF(\omega)|_\Lambda| = 1$, *i.e.*, the gain-crossover frequency. Accordingly, write a function M-file that uses Equation 17.1.7 to compute the quantity $|OLFRF(\omega)|_\Lambda| - 1$. Then calculate ω_{PM} by calling that function M-file with the `fzero` command, in which you also include an estimate for ω_{PM} that you read from Figure 17.1.3. Finally, use Equation 17.1.7 again to calculate $\angle OLFRF(\omega)|_\Lambda$ for $\omega = \omega_{\mathrm{PM}}$, and substitute that result into Equation 17.1.10 to obtain the more precise value of $\mathrm{PM}(\Lambda)$.
   2. Calculate for at least one of the neutral-stability points on Figure 17.4.3 the more precise value of gain (Λ_{ns1} and/or Λ_{ns2}) that is stated in the text of Section 17.4. The procedure is based on setting to zero the real part of the appropriate root of the characteristic equation, Equation 17.4.4. Accordingly, write a function M-file that first determines with the `root` command the three roots of Equation 17.4.4, then uses MATLAB flow control features (perhaps a `for` loop containing an `if` loop, or some alternative of your choice) to select the one root of the three that has a positive imaginary part, and finally computes the real part of this root. Then calculate the required Λ_{ns} by calling that function M-file with the `fzero` command, in which you also include an estimate for the Λ_{ns}, which you read from Figure 17.4.3. [See homework Problem 16.8.2 for a different method of solution.]
3. In this problem, re-visit the system of homework Problem 16.5: it is a damped rotor with position feedback that is 2nd order low-pass-filtered. Use the same component values: $\omega_b = 300$ rad/s, $c_\theta/J = 100$ s^{-1}, and $\gamma = 1/\sqrt{2}$. Also as before, denote the varying control-system gain parameter as $\Lambda \equiv K_a K_\theta / J$.

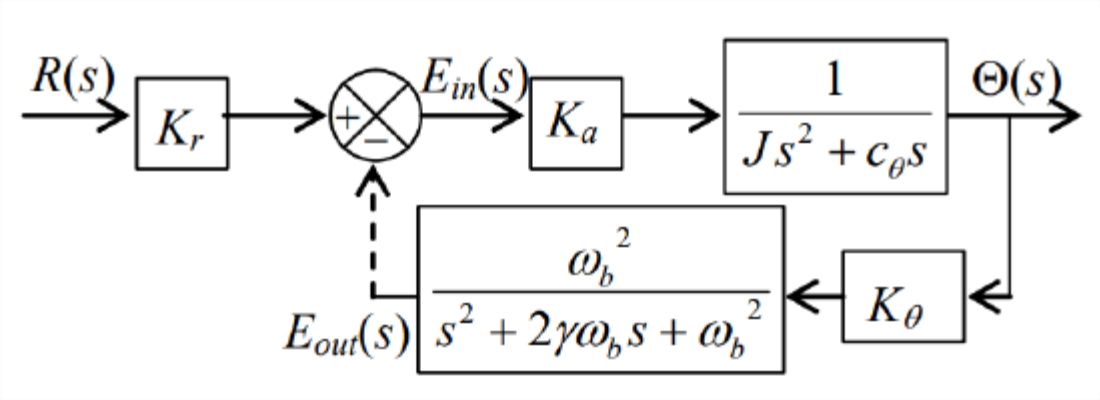


Figure 17.5.1 (Copyright; author via source)

1. Show that the open-loop transfer function is

$$OLTF(s) \equiv \frac{E_{out}(s)}{E_{in}(s)} = \Lambda \frac{\omega_b^2}{s\,(s + c_\theta/J)\,(s^2 + 2\gamma\omega_b s + \omega_b^2)}$$

2. Homework Problem 16.5.2 showed the following by loci-of-roots analysis for the closed-loop system: Your task is to evaluate stability again for $\Lambda = 3,400$ s^{-2} and for $\Lambda = \Lambda_{ns}$, but now using frequency-response criteria. Define the excitation-frequency range of interest to be the band $10 \leq \omega \leq 1000$ rad/s. First, use appropriate MATLAB commands to calculate the baseline open-loop frequency-response function, *i.e.*, $\mathrm{OLFRF}(\omega)$ for $\Lambda = 1$ s^{-2}. Now, use MATLAB to calculate and plot on some type (your choice) of Bode diagram the curves of magnitude ratio and phase angle over the defined frequency band, for both $\Lambda = 3,400$ s^{-2} and $\Lambda = \Lambda_{ns}$. Estimate from your Bode diagram the gain margin GM (in both decimal form and dB) and the phase margin PM (in degrees) for both values of Λ.
    1. $\Lambda \equiv \Lambda_{ns} = 22,000$ s^{-2} is the neutral-stability gain, the upper limit of stability; and
    2. for $\Lambda = 3,400$ s^{-2}, the system is stable, with the dominant mode being oscillatory and having damping ratio $\zeta \approx 1/\sqrt{2}$.
3. Use MATLAB's `margin` function to validate the correctness of your estimated stability margins in part 17.3.2.

4. Suppose that you have an LTI rotary motor with **unknown** transfer function $G(s)$. The input is a voltage signal $e_{in}(t)$, and the output is shaft speed $\dot{\theta}(t) \equiv p(t)$. You want to control the shaft speed by sensing it with a tachometer, which has variable gain Λ, and feeding the tachometer voltage signal $e_p(t)$ back to the input in the standard form shown on the Laplace block diagram.

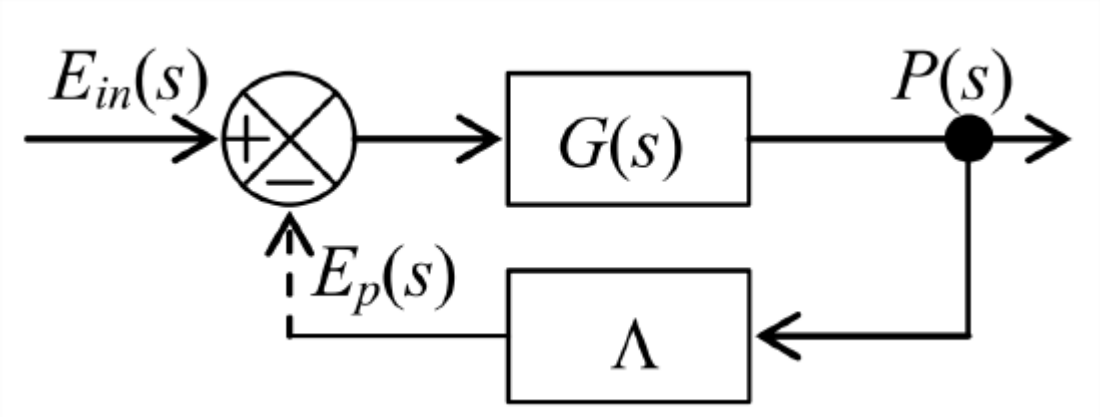


Figure 17.5.2 (Copyright; author via source)

In order to evaluate the stability of the closed-loop system, you conduct a stepped-sine-sweep frequency-response test on the stable **open-loop** system, which consists of the motor and the tachometer in series, with the gain set at $\Lambda = 0.4$ volt per rad/s. Specifically, you drive the open-loop system with sinusoidal input voltage $e_{in}(t) = V_{in} \sin \omega t$ with the frequency increasing in small increments through the range $0.5 \leq \omega \leq 50$ rad/s, you measure at each sine-dwell frequency the steady-state sinusoidal output voltage $e_p(t) = V_p(\omega) \sin[\omega t + \phi(\omega)]$, and then you plot the results for $V_p(\omega)/V_{in}$ and \phi(\omega), leading to the following diagram:

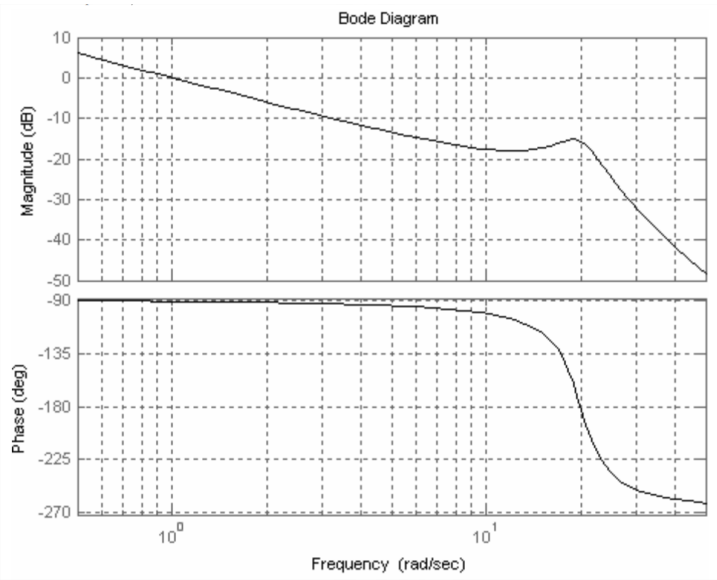


Figure 17.5.3 (Copyright; author via source)

1. Would the closed-loop system be stable with the tachometer gain for which the above data apply, $\Lambda = 0.4$ volt/(rad/s)? If so, calculate from the diagram above, with as much accuracy as the plots permit, the phase margin PM and gain margin GM of stability. Indicate the graphical representations of your PM and GM on a sketch or photocopy of the diagram.
2. Would the closed-loop system be driven unstable by any positive values of tachometer gain, $\Lambda > 0$? If so, estimate those values. Explain your reasoning clearly.

5. Figure 17.2.1 is in the traditional two-dimensional format of a Nyquist plot, *i.e.*, $\mathrm{Im}[FRF(\omega)]$ versus $\mathrm{Re}[FRF(\omega)]$, on which excitation frequency ω is the implicit independent variable. But this format might be considered unsatisfactory for many purposes because it lacks a graduated scale for ω. An enhanced-format Nyquist plot, a **three**-dimensional plot, can show such a graduated scale of frequency in addition to the scales of the rectangular components, $\mathrm{Re}[FRF(\omega)]$ and $\mathrm{Im}[FRF(\omega)]$. In order to explore that format, run in MATLAB the following alternative version of the commands that produced Figure 17.2.1, which applies, in particular, the `plot3` command to plot three-dimensionally:

```
>> wn=2*pi;zt=0.2;

>> w=wn*(0:0.05:2.5);

>> frf=wn^2./(wn^2-w.^2+j*2*zt*wn*w);

>> plot3(real(frf),imag(frf),w/wn),grid

>> axis equal
```

These commands should produce a three-dimensional plot on which the $\mathrm{Re}[FRF(\omega)]$ axis (the x-axis) extends from ≈ -1 to $\approx +1.5$, the $\mathrm{Im}[FRF(\omega)]$ axis (the y-axis) extends from 0 to –2.5, and the ω/ω_n axis (the z-axis) extends from 0 to 2.5. Make a print of the initial plot, which probably will be from the "Default 3-D View" (azimuth = -37.5°, elevation = -30°; Orthographic, not Perspective). The Reference 3-D View, azimuth = 0° and elevation = 0°, is from $y = -\infty$ looking in exactly the $+y$ direction, so it is a two-dimensional projection of the x-z plane. Azimuth angle is positive rotation (in the sense of the right-hand rule) of the viewpoint about the z axis, relative to the Reference 3-D View; thus, for example, the view azimuth = 90° and elevation = 0° is from $x = +\infty$ looking in exactly the $-x$ direction, a two-dimensional projection of the y-z plane.

Consider two consecutive excitation frequencies in a frequency-response experiment, denoted as ω_i and ω_j, with $\omega_j > \omega_i$ [*e.g.*, two consecutive values in the series represented by the MATLAB command `w=wn*(0:0.05:2.5)`], and denote $\Delta\omega_{ij} = \omega_j - \omega_i$; denote the **change** from ω_i and ω_j of the in-phase component $\mathrm{Re}[FRF(\omega)]$ as Δx_{ij}, and that of the quadrature component $\mathrm{Im}[FRF(\omega)]$ as Δy_{ij}; then the line segment between those two points in the $\mathrm{Im}[FRF(\omega)]$-versus-$\mathrm{Re}[FRF(\omega)]$ plane (*e.g.*, Figure 17.2.1, the x-y plane) is $\Delta s_{ij} = \sqrt{\Delta x_{ij}^2 + \Delta y_{ij}^2}$. Engineers who conduct stepped-sine-sweep frequency-response vibration tests on resonant systems (such as that of this problem) consider the **rate of change** $\Delta s_{ij}/\Delta\omega_{ij}$ to be a physically significant quantity that leads to experimental determination of a system's natural frequency, ω_n. Describe how your three-dimensional Nyquist plots (from whatever viewpoints are required) demonstrate that $\Delta s_{ij}/\Delta\omega_{ij}$ is maximum at excitation

frequencies near ω_n, but is below the maximum at frequencies both above and below ω_n.

6. In this problem, re-visit the system of homework Problem 16.7: the Laplace block diagram of the speed-control system of a magnetic-tape drive is shown in the figure on the next page. The operator setting is input voltage signal $e_{in}(t)$, the Laplace transform of which is $E_{in}(s)$. Each major sub-system of the control system functions as a 1st order system. The sub-system consisting of a power amplifier and a torque actuator has variable sensitivity K_a (N-m/V) and time constant $\tau_a = 1.0$ s. The tape drive has rotational inertia $J = 4.0$ N-m per rad/s^2 and lubricated-shaft viscous damping constant $c_\theta = 1.0$ N-m per rad/s; the output of the tape drive in this application is rotational speed $p(t)$ in rad/s [with Laplace transform $P(s)$], **not** rotational position $\theta(t)$. The rotational-speed sensor in the feedback branch has sensitivity $K_p = 1.250$ V per rad/s; this sensor is sufficiently slow relative to the other sub-systems that we must account for its time constant, $\tau_p = 0.50$ s.

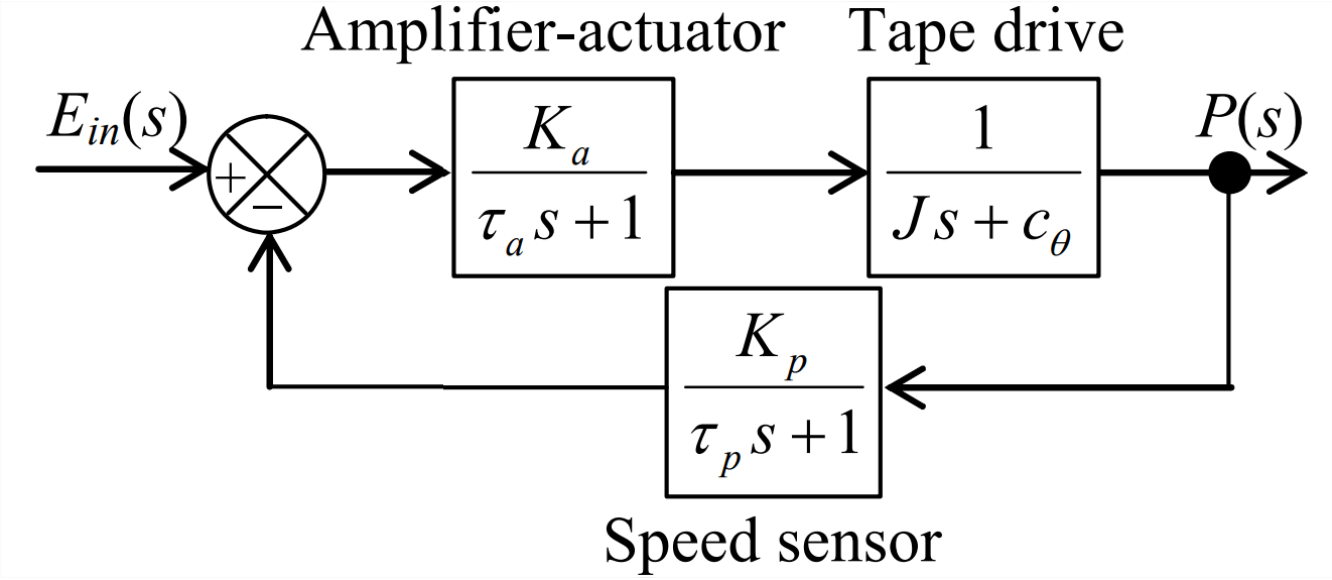


Figure 17.5.4 (Copyright; author via source)

1. Defining the control gain parameter as $\Lambda = K_a K_p / (J \tau_a \tau_p)$, with units of s^{-3}, show that the closed-loop transfer function is $\frac{P(s)}{E_{in}(s)} = \frac{K_a}{J\tau_a} \frac{s+\tau_p^{-1}}{(s+c_\theta/J)(s+\tau_a^{-1})(s+\tau_p^{-1})+\Lambda}$. Consider response to a unit-step input, $e_{in}(t) = (1.0\text{V})H(t)$. Use the final-value theorem (assuming system stability) to find an equation in terms of the system parameters for the steady-state response, after all transients have decayed, $\lim_{t\to\infty} p(t)$. Select the value of sensitivity K_a that makes the overall steady-state sensitivity of the control system be 0.750 rad/s per volt, then calculate gain Λ corresponding to that value of K_a. (*partial solution*: $\Lambda = 7.50$ s^{-3})
2. Show that the open-loop frequency-response function can be written as $OLFRF(\omega) = \Lambda \times \frac{1}{(j\omega+c_\theta/J)(j\omega+\tau_a^{-1})(j\omega+\tau_p^{-1})}$. Calculate for the given parameter values the phase-crossover frequency ω_{ns}, *i.e.*, the frequency at which $\angle OLFRF(\omega) = -\pi$ radians $= -180°$; this task will require the numerical solution of a transcendental equation, which you might want to accomplish by applying the MATLAB function `fzero` in the manner described in homework Problem 17.2. Next, calculate for the baseline case, $\Lambda = 1$ s^{-3}, the magnitude ratio at phase crossover, $MR_1(\omega_{ns})$, and the corresponding gain margin, GM_1. (*partial solution*: GM_1 = 8.4375)
3. Now consider the gain determined in part (a), $\Lambda = 7.50$ s^{-3}. Calculate the magnitude ratio at phase crossover, $MR_{7.50}(\omega_{ns})$, and the corresponding gain margin, $GM_{7.50}$. Calculate the phase margin, $PM_{7.50}$, using the equation for $OLFRF(\omega)$ of part (b)—to do this, you will need first to find the gain-crossover frequency ω_1 [$MR_{7.50}(\omega_1) = 1$], which you might want to accomplish by applying again the MATLAB function `fzero` in the manner described in homework Problem 17.2; then you will need to calculate phase angle $\angle OLFRF(\omega_1)$ to use in the definition of phase margin.
4. To validate your work in the previous parts, produce a diagram showing the Nyquist plots for both values of gain, $\Lambda = 1$ and 7.50 s^{-3}; you might wish just to adapt for this case the MATLAB code that produced Figure 17.4.4. What is your assessment, based on the gain and phase margins, of the control system with gain $\Lambda = 7.50$ s^{-3}?

7. Consider the feedback control system depicted in the Laplace block diagram at right. Assume that units of all constants are consistent, and do not bother to label units.

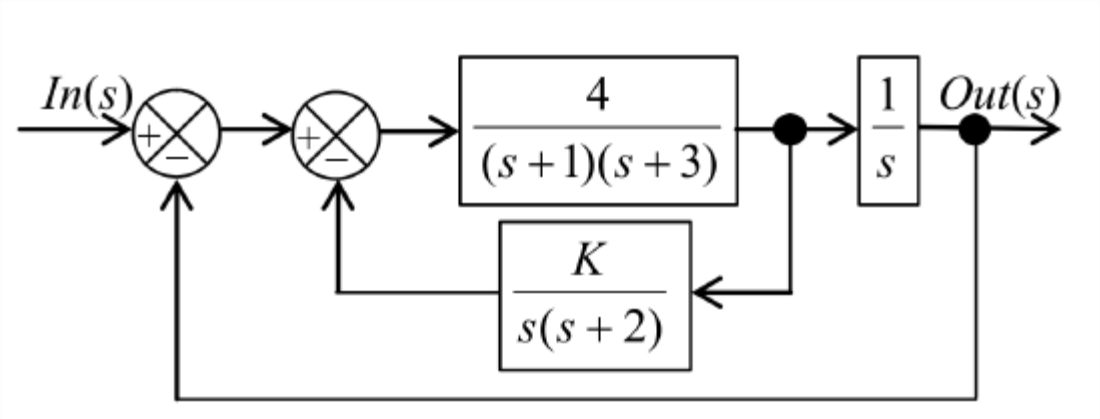


Figure 17.5.5 (Copyright; author via source)

1. Combine the two feedback loops with block-diagram algebra to show that the forward-branch and feedback-branch transfer functions can be written, respectively, as: $G(s) = \frac{4}{s(s+1)(s+3)}$ and $H(s) = \frac{s+(2+K)}{s+2}$. Apply Routh's criteria, as described in homework Problem 16.9.3, to determine the range of the sensitivity constant K over which the control system is stable.
2. Use any method of your choosing to determine the value of K for which the phase margin PM is approximately $35°$. Note that the characteristic equation of this system does **not** have the form of either of Equations 16.6.3: this K does not play the role of Λ in Equations 16.6.3. Therefore, many of the equations developed in Chapter 17, such as Equations 17.1.11-17.1.16, are not applicable in this problem.
3. To validate your work in parts (a) and (b), produce a diagram showing the Nyquist plots for two values of K: both that for neutral stability (the upper-bound K for system stability), and that for $\mathrm{PM} \approx 35°$; you might wish just to adapt for this case the MATLAB code that produced Figure 17.4.4. Explain the features of the Nyquist plots that confirm the correctness of your results in parts (a) and (b), using annotations on the diagram and/or written discussion.

8. Your task in this problem is to derive, with the assistance of MATLAB's symbolic capabilities, the "steady-state" response Equation 17.3.2, which includes the unstable linear drift due to the open-loop transfer function's pole at $s = 0 + j0$. Because the equations that MATLAB produces are long and complicated, it will be advantageous to separate the voltage input function Equation 17.3.1 into the following two parts:

$$e_{in_1}(t) = E_{in} \sin \omega t \quad \text{and} \quad e_{in2}(t) = E_{off} H(t)$$

For each of these parts individually you will seek the voltage output solution, respectively, $e_{\mathrm{out1}}(t)$ and $e_{\mathrm{out2}}(t)$, then you will add the two to obtain the total response.

1. The Laplace transform of $e_{\mathrm{out1}}(t)$, with the use of Equation 17.1.2 and Equation 2.4.7, is

$$L\left[e_{\mathrm{out}\ 1}(t)\right] = L\left[e_{in1}(t)\right] \times OLTF(s) = E_{\mathrm{in}} \frac{\omega}{s^2 + \omega^2} \times \Lambda \frac{\omega_b}{s\left(s + c_\theta/J\right)\left(s + \omega_b\right)}$$

You might be able to find the inverse transform of this equation in some reference, but it is likely that a more efficient process is to define the equation symbolically in MATLAB and then to use the `ilaplace` command to find the inverse transform. The less appealing aspect of this approach is that MATLAB's initial solution will probably be long and difficult to interpret. To convert the initial solution equation into a more transparent form, use the `simple` and `pretty` commands. You seek the "steady-state" response, so discard (from the equation displayed after use of `pretty`) the transient terms involving the time functions $e^{-(c_\theta/J)t}$ and $e^{-\omega_b t}$. At this point, you should be able to express the solution in the form:

$$\frac{e_{out1}(t)}{E_{in}} = \frac{\Lambda}{\omega\left(c_\theta/J\right)} + \Lambda\omega_b \frac{-\left(c_\theta/J + \omega_b\right)\sin\omega t + \left[\omega - \left(c_\theta/J\right)\omega_b/\omega\right]\cos\omega t}{\left[\omega^2 + \left(c_\theta/J\right)^2\right]\left(\omega^2 + \omega_b^2\right)}$$

The second right-hand-side term above is clearly a frequency-response term; but it is not yet expressed in the form from which equations for magnitude ratio and phase angle can be written, so you need to convert it into that form. Use the general trigonometric identity $\sin\theta \times \cos\phi + \cos\theta \times \sin\phi = \sin(\theta + \phi)$, and follow the procedure described in Equations 4.3.2-4.3.5, to show that $C\sin\omega t + S\cos\omega t = \sqrt{C^2 + S^2}\sin(\omega t + \phi)$, where $\phi = \tan^{-1}(S/C)$. Use this result to write explicit equations, in terms of c_θ/J and ω_b and ω, for magnitude ratio $MR(\omega)$ and phase angle $\phi(\omega)$. In the remainder of this problem, do not write out again those somewhat lengthy equations; instead, just use the symbols $MR(\omega)$ and $\phi(\omega)$.
2. Use Equation 17.1.2 again and MATLAB's `ilaplace` command in order to solve for output voltage $e_{\mathrm{out}\ 2}(t)$ in response to $e_{in2}(t) = E_{off} H(t)$.
3. Combine the response equations of parts (a) and (b) to write the complete solution $e_{\mathrm{out}}(t) = e_{\mathrm{out1}}(t) + e_{\mathrm{out2}}(t)$, the

"steady-state" response Equation 17.3.2:

$$e_{out}(t) = E_{in}\left(MR(\omega) \times \sin(\omega t + \phi(\omega)) + \frac{\Lambda}{\omega\left(c_\theta/J\right)}\right) + E_{off}\frac{\Lambda J}{c_\theta}\left[t - \frac{\left(c_\theta/J + \omega_b\right)}{\left(c_\theta/J\right)\omega_b}\right]$$

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18: APPENDIX A- TABLE AND DERIVATIONS OF LAPLACE TRANSFORM PAIRS

18.1: A.1- Table of Laplace Transform Pairs
18.2: A.2- Laplace Transform of a Ratio of Two Polynomials
18.3: A.3- Derivation of the Laplace Transform of a Definite Integral
18.4: A.4- Applications of the Laplace Transform of a Definite Integral
18.5: A.5- Derivation of the Laplace Transform of the Convolution Integral

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18.1: A.1- Table of Laplace Transform Pairs

General conventions:

1. time t is a real number, $t \geq 0$;
2. Laplace variable s is a complex number with dimension of time^{-1};
3. n and k are positive, real integers;
4. p and σ are finite constants, with dimension of time^{-1};
5. t_s is a real, finite constant, with dimension of time;
6. ω is a positive, real, finite constant, with dimension of time^{-1}.

Function of time t, inverse Laplace transform	Function of Laplace variable s, forward Laplace transform	Source and notes
$f(t) = L^{-1}[F(s)]$	$F(s) = L[f(t)]$	General notation, Section 2.2
$f(t)$	$\int_{t=0}^{t=\infty} e^{-st} f(t) dt$, or $\int_{t=0^-}^{t=\infty} e^{-st} f(t) dt$	Definitions: Equation 2.2.5, or Equation 8.4.5 if $f(t)$ involves $\delta(t)$
$\dot{f}(t)$	$sF(s) - f(0^-)$	Equations 2.2.9 and 8.6.3; $t = 0^-$ for ICs precedes all inputs
$\dot{f}(t)$	$s^2F(s) - sf(0^-) - \dot{f}(0^-)$	Equation 2.2.11, Problem 2.4
$\frac{d^n}{dt^n} f(t)$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dot{f}(0^-) - \cdots - f(0^-)$	Equations 2.2.10 and 8.6.4 Ogata, 1998, pp. 25- 26
$\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau) d\tau$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{\tau=-\infty}^{\tau=0} f(\tau) d\tau$	Equation 2.4.9, derivation in Section 18.3; note that \(-\infty
$\int_{\tau=0}^{\tau=t\geq 0} f(\tau) d\tau$	$\frac{1}{s}F(s)$	Equation 2.4.10, derivation in Section 18.3
$CI(t) = \int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} f_1(t-\tau) f_2(\tau) d\tau$	$F_1(s) \times F_2(s)$	Equation 6.1.4, convolution integral, derived in Section 18.5
$e^{\sigma t} f(t)$	$F(s-\sigma)$	Equation 9.3.6; σ is any constant
e^{pt}	$\frac{1}{s-p}$	Equation 2.2.6; p is any constant
$\frac{1}{p}(e^{pt} - 1)$	$\frac{1}{s}\frac{1}{s-p}$	Eq. (A-15); p is any constant
$f(t) = L^{-1}[F(s)]$	$F(s) = L[f(t)]$	General notation, Section 2.2
$f(t)$	$\int_{t=0}^{t=\infty} e^{-st} f(t) dt$, or $\int_{t=0^-}^{t=\infty} e^{-st} f(t) dt$	Definitions: Equation 2.2.5, or Equation 8.4.5 if $f(t)$ involves $\delta(t)$
$\delta(t) \equiv \delta(t-0)$	1	Equation 8.4.6; Dirac delta function of Equation 8.4.1
$H(t-t_s)$	$\frac{e^{-st_s}}{s}$	Equation 2.4.4; general unit-step defined in Equation 2.4.2
$H(t)$	$\frac{1}{s}$	Equation 2.4.5; Heaviside unit-step function defined in Equation 2.4.1
$f(t-t_s)H(t-t_s)$	$e^{-st_s}F(s)$	Ogata, 1998, p. 18; translated function of time
t	$\frac{1}{s^2}$	Equation (A-17)
$te^{\sigma t}$	$\frac{1}{(s-\sigma)^2}$	Equations (A-17) and 9.3.6; σ is any constant
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	Equation 2.4.7; ω is a positive real constant
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	Equation 2.4.8; ω is a positive real constant

Function of time t, inverse Laplace transform	Function of Laplace variable s, forward Laplace transform	Source and notes
$\frac{1}{\omega^2}(1-\cos\omega t)$	$\frac{1}{s(s^2+\omega^2)}$	Equation (A-16); ω is a positive real constant
$\frac{\omega_n^2}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$	$\frac{\omega_n^2}{(s+\zeta\omega_n)^2+\omega_d^2}$	Problem 9.12; $\lvert\zeta\rvert<1$, $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)>0$
$1-e^{-\zeta\omega_n t}\times$ $\left(\cos\omega_d t+\frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)$	$\frac{\omega_n^2}{s\left[(s+\zeta\omega_n)^2+\omega_d^2\right]}$	Problem 9.12; $\lvert\zeta\rvert<1$, $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)>0$
$e^{-\zeta\omega_n t}\times$ $(\omega_d\cos\omega_d t-\zeta\omega_n\sin\omega_d t)$	$\frac{s\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$	Problem 9.15; $\lvert\zeta\rvert<1$, $\omega_d^2=\omega_n^2\left(1-\zeta^2\right)>0$
$2\lvert C\rvert e^{\sigma t}\cos(\omega t+\angle C)$	$\frac{C}{s-p}+\frac{\bar{C}}{s-\bar{p}}$	Equation 16.1.12; C is complex, $p=\sigma+j\omega$, σ and ω are real, $\omega>0$
$\sum_{k=1}^{n}\frac{Num(p_k)}{\mathrm{Den}'(p_k)}e^{p_k t}$	$\frac{\mathrm{Num}(s)}{\mathrm{Den}(s)}$	Eq. (A-7), Section 18.2 for definitions, derivation, and restrictions
$f(t)=L^{-1}[F(s)]$	$F(s)=L[f(t)]$	General notation, Section 2.2
$f(t)$	$\int_{t=0}^{t=\infty}e^{-st}f(t)dt$, or $\int_{t=0^-}^{t=\infty}e^{-st}f(t)dt$	Definitions: Equation 2.2.5, or Equation 8.4.5 if $f(t)$ involves $\delta(t)$
$\lim_{t\to 0^+\text{from } t>0}f(t)\equiv f\left(0^+\right)$	$\lim_{s\to\infty}[sF(s)]$	Initial-value theorem: Equation 8.6.1 and Equation 8.6.6 in Section 8.6
$\lim_{t\to\infty}f(t)$	$\lim_{s\to 0}[sF(s)]$	Final-value theorem: Equation 15.2.13 and Section 15.3, valid **only if** $\lim_{t\to\infty}f(t)$ is a finite, constant value

It is worthy of note that MATLAB's symbolic software, which is introduced in homework Problem 1.6, can sometimes be helpful for finding forward and inverse Laplace transforms by applying, respectively, the `laplace` and `ilaplace` commands. The following are two relatively simple examples that do not appear explicitly in the table of transform pairs. In these examples, MATLAB finds the forward transform $L[\sinh(at)]=a/\left(s^2-a^2\right)$, and the inverse transform $L^{-1}\left[(s+a)/\left[(s+a)^2+\omega^2\right]\right]=e^{-at}\cos\omega t$.

```
>> syms s t a w

>> ft=sinh(a*t)

ft =

sinh(a*t)

>> Lft=laplace(ft,t,s)

Lft =

a/(s^2-a^2)

>> pretty(Lft)

        a
  -------
   2   2
  s  - a

>> Fs=(s+a)/((s+a)^2+w^2)

Fs =
```

```
(s+a)/((s+a)^2+w^2)

>> fFs=ilaplace(Fs,s,t)

fFs =

exp(-a*t)*cos(w*t)
```

For functions of time t or functions of Laplace variable s that are more complicated than those in the examples above, MATLAB might produce solutions that are correct, but are expressed in an unfamiliar form, or in a long and unwieldy form that must be simplified by human touch in order to become useful.

[1]Much more extensive tables of Laplace transform pairs are available in many references, for example, Cannon, 1967, Appendix J.

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18.2: A.2- Laplace Transform of a Ratio of Two Polynomials

Suppose that we have a Laplace transform as the ratio of two polynomials, from Equation 2.2.2:

$$F_n(s) \equiv \frac{\text{Num}(s)}{\text{Den}(s)} = \frac{b_1 s^m + b_2 s^{m-1} + ... + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + ... + a_{n+1}} = \frac{b_1 (s - z_1)(s - z_2) \cdots (s - z_m)}{a_1 (s - p_1)(s - p_2) \cdots (s - p_n)} \tag{18.2.1}$$

The results derived in this section are based upon three assumptions:

1. the roots p_k of $\text{Den}(s)$, which are the poles of $F_n(s)$, are not repeated (such roots are called *simple* poles);
2. the degree of $\text{Den}(s)$ exceeds that of Num(s) , \(0 \leq m
3. none of the zeros of Equation 18.2.1 equals any of the poles. Under these circumstances, we can expand transform Equation 18.2.1 into partial fractions, from Equation 2.3.3:

$$F_n(s) = \sum_{k=1}^{n} \frac{C_k}{s - p_k} \tag{18.2.2}$$

In Equation 18.2.2 the residues are given by Equation 2.3.6 as

$$C_k = \left[(s - p_k) F_n(s)\right]_{s=p_k} = \left[(s - p_k) \frac{\text{Num}(s)}{\text{Den}(s)}\right]_{s=p_k}, k = 1, 2, ..., n \tag{18.2.3}$$

Let us examine what might be considered the “total denominator” of Equation 18.2.3:

$$D_k = \lim_{s \to p_k} \left[\frac{\text{Den}(s)}{(s - p_k)}\right] \tag{18.2.4}$$

Observe from $\text{Den}(s)$ in Equation 18.2.1 that in D_k Equation 18.2.4 has the indeterminate form 0/0. Since we assume that all zeros of $F_n(s)$ are different from the poles, $\text{Num}\,(p_k)$ in Equation 18.2.3 is non-zero and finite. Therefore, D_k must also be non-zero and finite, and we can use l’Hopital’s rule to cast Equation 18.2.4 into a different form:

$$D_k = \lim_{s \to p_k} \left[\frac{\text{Den}(s)}{(s - p_k)}\right] = \lim_{s \to p_k} \left[\frac{\frac{d}{ds}\text{Den}(s)}{\frac{d}{ds}(s - p_k)}\right] \equiv \left[\frac{d}{ds}\text{Den}(s)\right]_{s=p_k} \tag{18.2.5}$$

Thus (Hildebrand, 1962, p. 548), residue Equation 18.2.3 can be expressed alternatively as

$$C_k = \left[(s - p_k) \frac{\text{Num}(s)}{\text{Den}(s)}\right]_{s=p_k} = \left[\frac{\text{Num}(s)}{\frac{d}{ds}\text{Den}(s)}\right]_{s=p_k} \equiv \frac{\text{Num}\,(p_k)}{\text{Den}'\,(p_k)} \tag{18.2.6}$$

Finally (Meirovitch, 1967, p. 532), by substituting Equation 18.2.6 back into Equation 18.2.2 and then taking the inverse Laplace transform of each term in the summation, we find

$$f(t) = L^{-1}\left[F_n(s)\right] = L^{-1}\left[\frac{\text{Num}(s)}{\text{Den}(s)}\right] = \sum_{k=1}^{n} \frac{\text{Num}\,(p_k)}{\text{Den}'\,(p_k)} e^{p_k t}, t \geq 0 \tag{18.2.7}$$

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18.3: A.3- Derivation of the Laplace Transform of a Definite Integral

Suppose that a function $f(t)$ has Laplace transform $F(s) = L[f(t)]$, and that we need the transform of the definite integral $\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau$. Note the lower limit of $\tau = -\infty$; we usually consider $f(t)$ only for $t \geq 0$, but occasionally the integral of $f(t)$ over previous time, $t < 0$, is also needed.

$$L\left[\begin{array}{l}\tau = t \geq 0 \\ \iint_{-\infty} f(\tau)d\tau\end{array}\right] = \int_{t=0}^{t=\infty} \overbrace{\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right]}^{u} \overbrace{e^{-st}dt}^{dv} \tag{18.3.1}$$

Integrating by parts gives

$$L\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right] = \left\{\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right]\left(\frac{e^{-st}}{-s}\right)\right\}\Bigg|_{t=0}^{t=\infty} - \left(\frac{1}{-s}\right)\int_{t=0}^{t=\infty} e^{-st}\frac{d}{dt}\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right]dt \tag{18.3.2}$$

The derivative of the definite integral in the second right-hand-side term of Equation 18.3.2 is a special case of Leibnitz's rule (Hildebrand, 1962, p. 360):

$$\frac{d}{dt}\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right] = f(t) \tag{18.3.3}$$

With the simple result Equation 18.3.3, and with evaluation of the limits of the first right-hand-side term, Equation 18.3.2 becomes

$$L\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right] = \frac{1}{s}\int_{\tau=-\infty}^{\tau=0} f(\tau)d\tau + \frac{1}{s}\int_{t=0}^{t=\infty} e^{-st}f(t)dt \tag{18.3.4}$$

Thus, the final form of the required general transform is

$$L\left[\int_{\tau=-\infty}^{\tau=t\geq 0} f(\tau)d\tau\right] = \frac{1}{s}F(s) + \frac{1}{s}\int_{\tau=-\infty}^{\tau=0} f(\tau)d\tau \tag{18.3.5}$$

For most applications, we have $f(t) = 0$ for $t < 0$, for which the simpler transform is:

$$L\left[\int_{\tau=0}^{\tau=t\geq 0} f(\tau)d\tau\right] = \frac{1}{s}F(s) \tag{18.3.6}$$

If we regard the integral of $f(t)$ as being the first "negative" derivative (antiderivative), then we see that transform Equation 18.3.5 is logically consistent with transform Equation 2.2.9 for a "positive" derivative, with respect to both power of s and the initial value term.

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18.4: A.4- Applications of the Laplace Transform of a Definite Integral

A relatively painless method for deriving certain inverse transforms is based upon the inverse transform of Equation 18.3.6,

$$L^{-1}\left[\frac{1}{s}F(s)\right]=\int_{\tau=0}^{\tau=t\geq 0} f(\tau)d\tau \tag{18.4.1}$$

It is necessary to apply carefully the limits of the definite integral, as is illustrated in the following three examples.

$$L^{-1}\left[\frac{1}{s}\frac{1}{s-p}\right]=\int_{\tau=0}^{\tau=t\geq 0} e^{p\tau}d\tau=\frac{1}{p}\int_{\tau=0}^{\tau=t\geq 0} d\left(e^{p\tau}\right)=\frac{1}{p}\left(e^{pt}-1\right), t\geq 0 \tag{18.4.2}$$

$$L^{-1}\left[\frac{1}{s\left(s^2+\omega^2\right)}\right]=\frac{1}{\omega}\int_{\tau=0}^{\tau=t}\sin\omega\tau d\tau=-\frac{1}{\omega^2}\int_{\tau=0}^{\tau=t} d(\cos\omega\tau)=\frac{1}{\omega^2}(1-\cos\omega t), t\geq 0 \tag{18.4.3}$$

$$L^{-1}\left[\frac{1}{s}\frac{1}{s}\right]=\int_{\tau=0}^{\tau=t\geq 0} H(\tau)d\tau=\int_{t=0}^{\tau=t\geq 0} d\tau=t, t\geq 0 \tag{18.4.4}$$

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18.5: A.5- Derivation of the Laplace Transform of the Convolution Integral

We consider two physically realistic functions of time, $f_1(t)$ and $f_2(t)$, that are zero for all time $t < 0$ and non-zero only for $t \geq 0$. The *convolution integral* is defined to be a definite integral involving $f_1(t)$ and $f_2(t)$ in either of the following forms:

$$CI(t) = \int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=0}^{\tau=t} f_1(t-\tau) f_2(\tau) d\tau \tag{18.5.1}$$

In these definite integrals, τ is the dummy variable of integration, and time t appears both in the upper limit of the integral and in argument $(t-\tau)$ of the integrand.

The Laplace transform $L[CI(t)]$ is called the *convolution transform*. Let us assume that the Laplace transforms of functions $f_1(t)$ and $f_2(t)$ exist: $F_1(s) = L[f_1(t)]$ and $F_2(s) = L[f_2(t)]$. The derivation to follow will show that the product of these two transforms equals the convolution transform:

$$F_1(s) \times F_2(s) = L[CI(t)] = L\left[\int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau\right] \tag{18.5.2}$$

Equation 18.5.2 is certainly not an intuitively obvious result. The short, formal derivation (Meirovitch, 1967, p. 534 and Ogata, 1998, pp. 43-44) involves interchanging of orders of integration. The first step is to revise the upper limit in the convolution integral since, by definition, $f_2(t-\tau) = 0$ for $t - \tau < 0$, that is, for $\tau > t$:

$$CI(t) = \int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=0}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau$$

Next, use the standard definition of a Laplace transform:

$$L[CI(t)] = L\left[\int_{\tau=0}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau\right] = \int_{t=0}^{t=\infty} e^{-st}\left[\int_{\tau=0}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau\right] dt$$

Now interchange the orders of integration between t and τ, an operation permitted by the assumed convergence of the integrals, and re-arrange the terms within the integrands:

$$L[CI(t)] = \int_{\tau=0}^{\tau=\infty} f_1(\tau)\left[\int_{t=0}^{t=\infty} e^{-st} f_2(t-\tau) dt\right] d\tau$$

In the inner integral, change the integration variable to $\lambda = t - \tau$, so that $t = \tau + \lambda$ and $dt = d\lambda$, since τ is regarded as a constant within this integration:

$$L[CI(t)] = \int_{\tau=0}^{\tau=\infty} f_1(\tau) d\tau \left[\int_{\lambda=-\tau}^{\lambda=\infty} e^{-s(\tau+\lambda)} f_2(\lambda) d\lambda\right]$$

Now, since $f_2(t) = 0$ for $\lambda < 0$, in the second integral we set the lower limit to zero. Also, again re-arrange terms within the integrands, to find

$$L[CI(t)] = \int_{\tau=0}^{\tau=\infty} e^{-s\tau} f_1(\tau) d\tau \left[\int_{\lambda=0}^{\lambda=\infty} e^{-s\lambda} f_2(\lambda) d\lambda\right] \equiv F_1(s) \times F_2(s)$$

This completes the derivation of the convolution transform, Equation 18.5.2.

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19: APPENDIX B- NOTES ON WORK, ENERGY, AND POWER IN MECHANICAL SYSTEMS AND ELECTRICAL CIRCUITS

19.1: B.1- Definitions of Work and Power
19.2: B.2- Mechanical Work, Energy, and Power
19.3: B.3- Work, Energy, and Power in Electrical Circuits
19.4: B.4- Analogies Between an m-c-k Mechanical System and an LRC Electrical B-8 Circuit
19.5: B.5- Hysteresis and Dissipation of Mechanical Energy by Damping

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19.1: B.1- Definitions of Work and Power

Work is defined for **translational motion** as the product of a force times a distance through which the force moves. For translation in one dimension, denoted here as x, the differential work of force $f_x(x)$ moving through distance dx at position x is

$$dW = f_x(x) \times dx \tag{19.1.1}$$

So the work by a spatially varying force moving from position x_1 to position x_2 is

$$W = \int_{x=x_1}^{x=x_2} dW = \int_{x=x_1}^{x=x_2} f_x(x)dx \tag{19.1.2}$$

with units of lb-ft, lb-inch, or N-m $\equiv$ J (for joule).

The velocity of one-dimensional motion is $v_x = dx/dt \Rightarrow dx = v_x dt$. Therefore, alternative expressions for work are

$$dW = f_x \times v_x dt \Rightarrow W = \int_{t=t_1}^{t=t_2} f_x v_x dt \tag{19.1.3}$$

This leads to the definition of *power*, the time rate of work:

$$P \equiv \frac{dW}{dt} = f_x \times v_x \tag{19.1.4}$$

with units of lb-ft/s, lb-inch/s, or N-m/s = J/s $\equiv$ W (for watt).

For **rotation** in one dimension, denoted here as θ, the differential work of moment M moving through angle $d\theta$ at position θ is $dW = M \times d\theta$, and the differential work of moment M moving with velocity $\dot{\theta}$ during interval dt is $dW = M \times \dot{\theta}dt$. Therefore, the work of a possibly varying moment between states 1 and 2 can be expressed in either of the two forms:

$$W = \int_{\theta=\theta_1}^{\theta=\theta_2} Md\theta = \int_{t=t_1}^{t=t_2} M\dot{\theta}dt \tag{19.1.5}$$

with units of lb-ft, lb-inch, or J. The associated definition of power is

$$P = M \times \dot{\theta} \tag{19.1.6}$$

with units of lb-ft/s, hp (for horsepower, 1 hp ≡ 550 lb-ft/s), lb-inch/s, or W.

At least some of the work that is done on a real system might be stored in some form of recoverable energy, or work done might be completely dissipated and lost. For all real engineering systems, at least part of the input work that is intentionally done on a system is lost irretrievably, not used for the intended purposes.

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19.2: B.2- Mechanical Work, Energy, and Power

If a force accelerates in **translation** a body with mass m, $f_x = ma_x = m\dot{v}_x$, then the work done by f_x is conserved as *kinetic energy* of the body, from Equation 19.1.3:

$$W \equiv E_K = \int_{t=t_1}^{t=t_2} f_x v_x dt = \int_{t=t_1}^{t=t_2} m\dot{v}_x v_x dt = \frac{1}{2}m\int_{t=t_1}^{t=t_2} \frac{d}{dt}\left(v_x^2\right)dt = \frac{1}{2}m\left[v_x^2\left(t_2\right) - v_x^2\left(t_1\right)\right] \tag{19.2.1}$$

If a force stretches or compresses a standard ideal linear translational spring (assumed here to have negligible mass), $f_x(x) = kx$, then the work done by f_x is stored as *strain energy* (a form of *potential* energy) within the spring, from Equation 19.1.2:

$$W \equiv E_S = \int_{x=x_1}^{x=x_2} f_x(x)dx = \int_{x=x_1}^{x=x_2} kxdx = \frac{1}{2}k\left(x_2^2 - x_1^2\right) \tag{19.2.2}$$

At any instant, therefore, the total mechanical energy present in an ideal mass-spring system, relative to an initially stationary and unstrained state, is

$$E_{Me} = E_K + E_S = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 \tag{19.2.3}$$

In an ideal *conservative* mass-spring system, without any agent of energy augmentation or energy dissipation, E_{Me} remains constant in time, oscillating between the kinetic energy of the mass and the strain energy within the spring.

An ideal translational viscous damper dissipates mechanical energy by exerting a force in opposition to velocity: $f_x = -cv_x = -c\dot{x}$. Thus, from Equation 19.1.4, the rate of energy dissipation by the damper is

$$P_c = f_x v_x = -cv_x^2 \tag{19.2.4}$$

Suppose that a mass-damper-spring system is initially stationary and unstrained, and that an independent, externally applied force $f_x(t)$ is imposed upon the mass. This force is a *source* of power, $P_f = f_x(t)v_x$. The damper, on the other hand, is a *sink* of mechanical energy. Therefore, total mechanical energy E_{Me} varies in time:

$$\frac{d}{dt}E_{Me} = \frac{d}{dt}\left(\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2\right) = P_c + P_f \quad \Rightarrow \quad mv_x\dot{v}_x + kx\dot{x} = -cv_x^2 + f_x(t)v_x \tag{19.2.5}$$

Canceling $v_x = \dot{x}$ out of Equation 19.2.5 and re-arranging terms leads to the general ODE of motion for a mass-damper-spring system[1]:

$$m\ddot{x} + c\dot{x} + kx = f_x(t) m\ddot{x} + c\dot{x} + kx = f_x(t) \tag{19.2.6}$$

The following relations for **rotational** motion can be derived from the basic definitions in the manner used above for translational motion. If a moment M accelerates a body with rotational inertia J about the axis of rotation, then the work done by M is conserved as *kinetic energy* of the body:

$$W \equiv E_K = \int_{t=t_1}^{t=t_2} M\dot{\theta}dt = \frac{1}{2}J\left[\dot{\theta}^2\left(t_2\right) - \dot{\theta}^2\left(t_1\right)\right] \tag{19.2.7}$$

If a moment M twists an ideal linear torsion (rotational) spring (assumed here to have negligible rotational inertia) with spring constant k_θ, then the work done by M is stored as *strain energy* within the spring:

$$W \equiv E_S = \int_{\theta=\theta_1}^{\theta=\theta_2} Md\theta = \frac{1}{2}k_\theta\left(\theta_2^2 - \theta_1^2\right) \tag{19.2.8}$$

A rotational viscous damper with constant c_θ dissipates mechanical energy by exerting a moment in opposition to velocity: $M = -c_\theta\dot{\theta}$. Thus, the rate of energy dissipation by the damper is

$$P_c = M\dot{\theta} = -c_\theta\dot{\theta}^2 \tag{19.2.9}$$

For our final example of mechanical energy, consider translation in the y direction of a mass m that is within a field of constant *gravitational field strength* g (with SI units newton/kilogram). The force required to sustain the mass without acceleration (either stationary or at constant velocity) against gravity is

$$f_y = mg \tag{19.2.10}$$

Therefore, the work required to raise the mass without acceleration against gravity is stored conservatively as *gravitational potential energy*:

$$W \equiv E_G = \int_{y=y_1}^{y=y_2} f_y dy = mg\left(y_2 - y_1\right) \tag{19.2.11}$$

For reference in the next section, we define also the *gravitational potential difference*, $g\left(y_2 - y_1\right)$. The following is an application of Equation 19.2.11. Suppose that we shoot a projectile of mass m straight up against Earth's gravity from surface elevation y_1, with initial velocity v_1 sufficiently low that g remains essentially constant over the entire trajectory. Let us assume that atmospheric drag is viscous with damping constant c.[2] Drag force $-cv_y$ dissipates energy, so the total mechanical energy E_{Me} varies in time:

$$\frac{d}{dt}E_{Me} = \frac{d}{dt}\left(\frac{1}{2}m\left(v_y^2 - v_1^2\right) + mg\left(y - y_1\right)\right) = P_c \Rightarrow mv_y\dot{v}_y + mg\dot{y} = -cv_y^2 \tag{19.2.12}$$

Canceling out $v_y = \dot{y}$ from Equation 19.2.12 and re-arranging terms leads to the 1st order ODE describing projectile velocity:

$$m\dot{v}_y + cv_y = -mg \tag{19.2.13}$$

1The direct power-balance method giving Equation 19.2.5 and leading to ODE of motion Equation 19.2.6 is also used in this appendix to derive governing ODEs (B-19) and (B-34). Each of these applications is for a one-degreeof-freedom (1-DOF) system, *i.e.*, a system that has only one time-dependent variable. (See Chapters 11 and 12 for more detailed definitions of degrees of freedom and examples of multiple-DOF systems.) Unfortunately, this direct approach fails for systems with more than one DOF, as is observed and illustrated by Cannon, 1967, p. 166. For deriving the governing ODEs of multiple-DOF systems, a more general energy method was developed by Joseph Louis Lagrange (French-Italian mathematician and mechanician, 1736-1813). *Lagrange's equations* are derived and illustrated in detail by most textbooks on classical mechanics and structural dynamics, *e.g.*, Bisplinghoff, *et al.*, 1955; Cannon, 1967; Craig, 1981; Greenwood, 1965; and Meirovitch, 1967 and 2001.

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19.3: B.3- Work, Energy, and Power in Electrical Circuits

Electrical fields exist in spaces around and between charged particles or charged objects (Halliday and Resnick, 1960, Chapter 27). The *electrical field strength* $\mathcal{E}$ (with SI units newton/coulomb) at a point in space is analogous in many respects to gravitational field strength g, as in Eq. (B-16). Suppose that an electrical field is oriented in the y direction and is constant. Experimental measurements show that the force required to sustain a particle of charge q_p without acceleration against the electrical field is

$$f_y = q_p \mathcal{E} \tag{19.3.1}$$

(This linear relationship is the basis for definition of $\mathcal{E}$.) Thus, charge q_p within an electrical field of strength $\mathcal{E}$ is analogous to mass m within a gravitational field of strength g. We see from this analogy that the work required to move the charge without acceleration against the electrical field is stored conservatively as *electrical potential energy*:

$$W \equiv E_E = \int_{y=y_1}^{y=y_2} f_y dy = \int_{y=y_1}^{y=y_2} q_p \mathcal{E} dy = q_p \mathcal{E} \left(y_2 - y_1 \right) \tag{19.3.2}$$

It is customary and appropriate in electrical theory and applications to define (in analogy with the gravitational potential difference) the *electrical potential difference* in **volts** between two points in space as (Halliday and Resnick, 1960, Chapter 29)

$$e_2 - e_1 \equiv \mathcal{E} \left(y_2 - y_1 \right), \text{ or differentially, } de \equiv \mathcal{E} dy \tag{19.3.3}$$

With $de \equiv \mathcal{E} dy$ in Equation 19.3.2, the work required to move charge q_p, work equal to the electrical potential energy, is expressed in terms of electrical potential difference as

$$W \equiv E_E = \int_{e=e_1}^{e=e_2} q_p de = q_p \left(e_2 - e_1 \right) \tag{19.3.4}$$

We infer from Equation 19.3.4 the SI unit equivalence: 1 volt = 1 joule/coulomb. Note the sign conventions in Equation 19.3.4: work and potential energy are positive if a positive charge, $q_p > 0$, is moved to a higher electric potential, $e_2 > e_1$.

One useful application of Equation 19.3.4 for electrical potential energy is to a capacitor within a circuit. Halliday and Resnick, 1960, p. 651, explain the physical process:

> ... work must be done to separate two equal and opposite charges. This energy is stored in the system and can be recovered if the charges are allowed to come together again. Similarly, a charged capacitor has stored in it an electrical potential energy U equal to the work W required to charge it. This energy can be recovered if the capacitor is allowed to discharge. We can visualize the work of charging by imagining that an external agent pulls electrons from the positive plate and pushes them onto the negative plate, thus bringing about the charge separation; normally the work of charging is done by a battery, at the expense of its store of chemical energy.

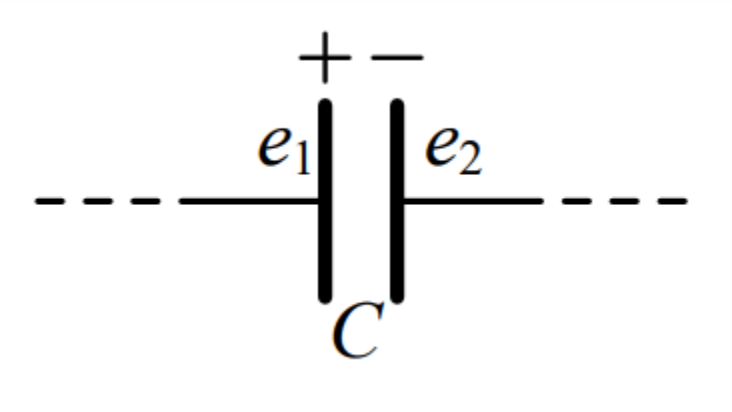


Figure 19.3.1 (Copyright; author via source)

To derive an equation for capacitive potential energy, let us suppose that some quantity of positive charge q_+ has already been transferred from the negative plate of an ideal capacitor to the positive plate; the voltage difference between the two plates due to q_+ is expressed in terms of the capacitance C, from Equation 5.2.3, as $e_{1+} - e_{2+} = (1/C)q_+$, in which e_{1+} is a higher potential than e_{2+}. From Equation 19.3.4, the differential work required to transfer an additional differential quantity of positive charge dq_+ from the negative plate to the positive plate is given by

$$dW = dq_+ \left(e_{1+} - e_{2+}\right) = \frac{1}{C} q_+ dq_+ \tag{19.3.5}$$

We integrate Equation 19.3.5 to find the total work required to fully charge the ideal capacitor from zero to a quantity q, which also is the total electrical potential energy stored in the capacitor:

$$W = E_E = \int_{q_+=0}^{q_+=q} dW = \frac{1}{C} \int_{q_+=0}^{q_+=q} q_+ dq_+ = \frac{1}{2} \frac{1}{C} q^2 \tag{19.3.6}$$

Whereas the energy of a capacitor, Equation 19.3.6, is stored in an electrical field, that of an inductor is stored in the magnetic field within the inductor's coils; this magnetic field exists as a consequence of the electrical current flowing in the wire (Halliday and Resnick, 1960, Chapter 34). Even though it is magnetic field energy, we can still apply Equation 19.3.4 for electrical work to determine an equation for the energy within an inductor (Halliday and Resnick, 1960, Section 36-4). From Equation 19.3.4, the differential work required to move a differential quantity of positive charge through an ideal inductor is

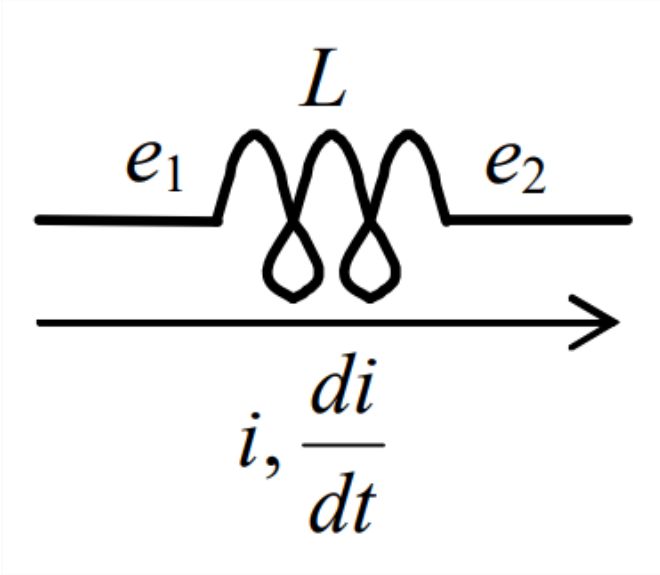


Figure 19.3.2 (Copyright; author via source)

$$dW = dq_+ \left(e_{1+} - e_{2+}\right) \tag{19.3.7}$$

in which e_{1+} is a higher potential than e_{2+}. This is a conservative process, so the electrical work, which might be done by a battery or some other power source, must be stored as magnetic field energy, E_M. The self-induced electrical potential difference across the inductor is given by Equation 5.2.9, $e_{1+} - e_{2+} = L\left(di_+/dt\right)$. Also, current and charge are related by $i_+ = dq_+/dt$, so that $dq_+ = i_+ dt$. Therefore, we re-write Equation 19.3.7 as

$$dW = dE_M = i_+ dt \times L \frac{di_+}{dt} = L i_+ di_+ \tag{19.3.8}$$

Finally, we integrate Equation 19.3.8 to find the total magnetic potential energy stored in the ideal inductor for a quantity of current i:

$$W = E_M = L \int_{i_+=0}^{i_+=i} i_+ di_+ = \frac{1}{2} L i^2 \tag{19.3.9}$$

Next, we apply Equation 19.3.4 for electrical work once again in order to evaluate power sources and power sinks in electrical circuits (Halliday and Resnick, 1960, Section 31-5). Consider a general component within a circuit, a "black box." This might be a

passive component such as a resistor, a source of power such as a battery or signal generator, or a device such as a motor that converts electrical energy into another form of energy. For this derivation, let us consider the upstream terminal of the black box to be at a higher potential than the downstream terminal: $e_1 > e_2$. Suppose that a positive differential charge dq moves through the black box from higher potential to lower potential; Equation 19.3.4 shows that there is a reduction of electrical energy: $dW = dE_E = -dq\,(e_1 - e_2) = -idt\,(e_1 - e_2)$. Electrical power also is reduced:

$$P_E = \frac{dW}{dt} = -i\,(e_1 - e_2) \tag{19.3.10}$$

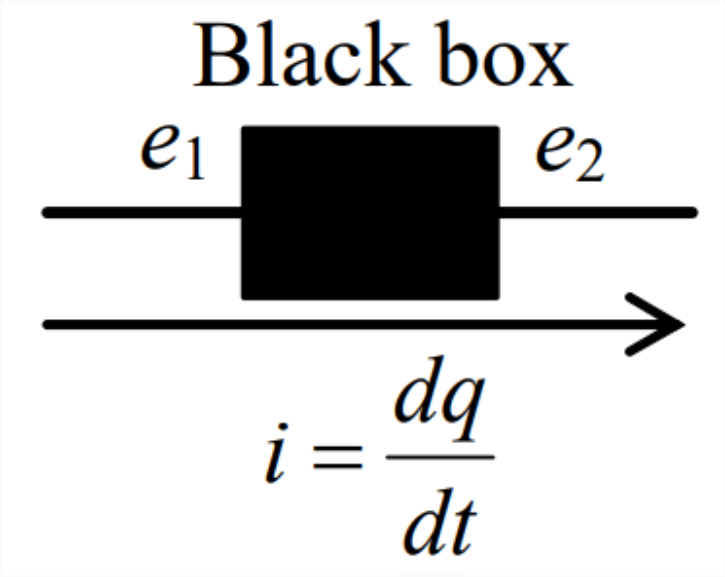


Figure 19.3.3 (Copyright; author via source)

Equation 19.3.10 is an important formula that can be applied usefully for many different black boxes. We shall consider here just two of them. First, suppose that the black box is a resistor with resistance R, for which Equation 5.2.1 gives $e_1 - e_2 = iR$; in this case, the downstream terminal is at lower voltage than the upstream terminal, so Equation 19.3.10 becomes the following famous formula for the electrical power that is dissipated in heating the resistor:

$$P_R = -i^2 R \tag{19.3.11}$$

Next, suppose that the black box is a source of input voltage $e_i(t)$ (battery or signal generator) for a circuit; in this case, the downstream voltage $e_2 = e_i(t)$ exceeds the upstream voltage (for positive current i), so Equation 19.3.10 gives the formula for electrical power introduced into the circuit by the voltage source:

$$P_e = i\,[e_i(t) - e_1] \tag{19.3.12}$$

For most, if not all of the relatively simple circuits considered in this book, the upstream terminal of the voltage source is connected to the ground (zero) potential, so that $e_1 \equiv 0$ and the input power is $P_e = ie_i(t)$.

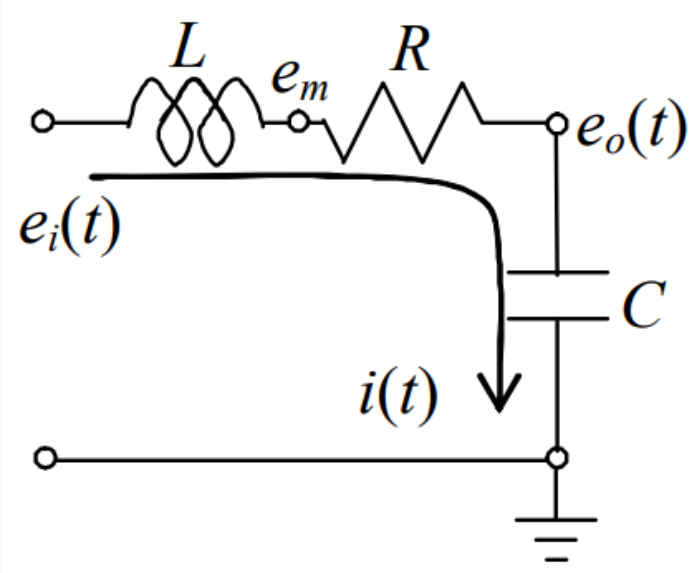


Figure 19.3.4 (Copyright; author via source)

An interesting application of the equations derived in em this section is to the powered LRC circuit drawn above, from the example in Section 9.2. The total stored electromagnetic energy resides in the electrical field of the ideal capacitor and the magnetic field of the ideal inductor:

$$E_{EM} = E_E + E_M = \frac{1}{2}\frac{1}{C}q^2 + \frac{1}{2}Li^2 \tag{19.3.13}$$

The signal generator is a source of electrical power, and the resistor is a sink of electrical power, so the total electromagnetic energy varies in time:

$$\frac{d}{dt}E_{EM} = \frac{d}{dt}\left(\frac{1}{2}\frac{1}{C}q^2 + \frac{1}{2}Li^2\right) = P_R + P_e \Rightarrow Li\frac{di}{dt} + \frac{1}{C}q\frac{dq}{dt} = -i^2R + ie_i(t) \tag{19.3.14}$$

Canceling $i = \dot{q}$ out of Equation 19.3.14 and re-arranging terms leads to the general ODE for charge $q(t)$ on the capacitor plates in the LRC circuit:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = e_i(t) \tag{19.3.15}$$

Since $q = Ce_o$, Equation 19.3.15 is essentially the same as that derived from Kirchhoff's voltage law in Section 9.2 for output voltage $e_o(t)$: $\ddot{e}_o + (R/L)\dot{e}_o + (1/LC)e_o = (1/LC)e_i(t)$.

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19.4: B.4- Analogies Between an m-c-k Mechanical System and an LRC Electrical B-8 Circuit

By comparing ODEs Equation 19.2.6 and Equation 19.3.15 and the many other mechanical and electromagnetic equations developed in this appendix, we can identify some clearly analogous variable quantities and constants between a forced m-c-k mechanical system and a powered LRC electrical circuit (Halliday and Resnick, 1960, Section 38-2). The most obvious analogies are listed in the following table.

Forced m-c-k mechanical system	**Powered LRC electrical circuit**
mass m	inductance L
viscous damping constant c	resistance R
stiffness constant k	inverse capacitance $1/C$
position of the mass $x(t)$	charge on the capacitor $q(t)$
velocity of the mass $v_x(t) = \dot{x}(t)$	current through the circuit $i(t) = \dot{q}(t)$
excitation force $f_x(t)$	input voltage $e_i(t)$
kinetic energy $E_K = \frac{1}{2}mv_x^2$	magnetic energy $E_M = \frac{1}{2}Li^2$
strain energy $E_S = \frac{1}{2}kx^2$	electrical energy $E_E = \frac{1}{2}(1/C)q^2$
power dissipation $P_c = -cv_x^2$	power dissipation $P_R = -Ri^2$
power input $P_f = f_x(t)v_x$	power input $P_e = e_i(t)i$

The mathematical similarity of an m-c-k system to an LRC circuit is one of the simpler analogies between physical systems and circuits, and there are many other analogous electrical circuits of greater complexity (Fifer, 1961, Vol. III, Chapter 19; MacNeal, 1962). In fact, during the 1940s and 1950s, methods of *direct analog computation* were developed extensively for analysis of many physical "prototype" systems with use of electrical circuits. In the introductory chapter of a 1962 book entitled *Electric Circuit Analogies for Elastic Structures*, Richard H. MacNeal wrote (pages 6-7): "The ... direct analog computer is ... a device in which each electrical component (or group of components) is equivalent to a physical component of the prototype system." MacNeal stated further: "This technique has been frequently used for the solution of problems concerning mechanics and elasticity, the conduction of heat, fluid flow, and electromagnetic waves." However, during and after the mid-1960s, direct analog computation was completely eclipsed by the rapidly advancing capabilities of digital computers and software written for digital processing. It is an ironic twist of history that MacNeal himself became an innovative pioneer in the development and implementation of the finiteelement method for high-precision digital computation of structural behavior.

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