EGR 1010: INTRODUCTION TO ENGINEERING FOR ENGINEERS AND SCIENTISTS



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EGR1010 - Introduction to Engineering for Engineers and Scientists

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1: PREFACE

Coordinated engineering curriculum and this textbook course

This is the second text in a new, coordinated engineering curriculum, integrating Introduction to Engineering (which includes the Introduction to Computer Programming), Chemistry, Physics, and Calculus. These courses are commonly taken by first and second year engineering students, however, Chemistry, Physics, and Calculus are not generally aimed for the engineering student in a coordinate fashion. While this coordinated engineering curriculum was designed for Prince George's Community College (PGCC), the authors believe it is generally relevant for any college or university (with modifications unique to their region and educational culture).

The first step to develop in this coordinated engineering curriculum was the PGCC's version of the General Chemistry for Engineering. The book was completed in 2016. The course and the book was well received by the engineering students who felt it more matched their interests. The preface to that book explains how it differs from a General Chemistry course, but basically it accelerates topics already known by freshman engineers and shifts the central theme from biology to materials which is more inline with engineering thinking.

This textbook course is an intensive two semester course introducing engineering and computer programming for engineers and scientists. Topics include brief introductions to most of the relevant courses for engineers which would include chemistry, physics, and calculus.

The traditional approach to introduction to engineering is to teach engineering through history, exemplify some engineers from different disciplines, lecture about relevant equations, and spend the majority of the time on a large engineering project. While that is effective for students who have already gone through a STEM-type program in high school it is not as effective for students who are new to science and technology as you would find in a community college. The approach here does not fully abandon the traditional approach but it shifts emphasis to learning "what is engineering". Exemplifying some engineers from different disciplines is removed as that is not relevant to actual engineering, engineering history is reduced to topics that emphasis "what is engineering" from a historical point of view, and the engineering project is divided into two different projects with one emphasizing real world activities and the other emphasizing laboratory of design and analysis.

In the course of working on this project it was clear that computers was an integral part of the course which is not distinct from the introduction to engineering. Therefore the introduction to computer programming was merged into this textbook course.

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Finally using the LibreTexts platform lessens the substantial cost of a textbook, which exceeds $(US) 250 today.

The authors acknowledge the help and support of Prof. Delmar Larson at UC Davis who created and runs the LibreTexts project.

A number of figures herein were created by the authors using xfig and LibreOffice's Draw program (LibreOffice is NOT connected to LibreTexts though the philosophy of being free with a license is essential the same). Most graphs and programs are from the Octave programming language though some are from Python and Maxima.

Broader impacts of research and teaching are an important part of an engineer's and scientist's life and are made possible by the freedom to explore new things granted by universities and research agencies. Joshua Halpern (Howard University), Scott Sinex, and Scott Johnson (both PGCC) gratefully acknowledge the support of this LibreTexts book by NASA Award 80NSSC18M0126 and the 2018 Department of Education Open Textbook Pilot Project award to LibreTexts.

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1.3: Guide for how to use this book-course

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Prince George's Community College

Introduction to Engineering for Engineers and Scientists

(In Two Semesters)

EGR1010 and EGR1140

Scott Johnson, Scott Sinex
Prince George's Community College

Joshua Halpern
Howard University

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So who wrote the course? Or a book depending on your point of view? There is no underlying text to this particular work, it is written in the standard way of using multiple references, creative commons resources, public domain resources, and widely known engineering techniques, ideas, and procedures.

Scott Johnson, Josh Halpern and Scott Sinex

During the great Pandemic, 2020-2021 (which keep on going into 2022)

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1.3: Guide for how to use this book-course

This course is an integrated course that is meant to go from start to finish in order. However if you wish to not follow the order then the table of contents is given here. Chapters are marked with the course they would go with in a two semester course, however each course has some items that should be introduced in both courses.

The first chapters are just introduction chapters and are fairly short. They include

- Front matter (auto-generated)
  - A Title Page
  - Information Page
  - Table of Contents
- Preface (this discusses the idea of a coordinated engineering curriculum which this is part of)
  - Title Page by hand
  - Creative Commons Comment
  - Table of Contents by hand in order to point out EGR1140 and EGR1010 topics if user wishes to split this
- Description of topics
- What we intend to learn

The "chapters" above are not necessarily part of the course proper.

The course(s):

EGR 1140 is the first engineering course and should be taken the first semester of freshman year (assuming the student is calculus ready). If a student has taken a course equivalent to EGR 1140 then EGR 1010 would be the first engineering course.

EGR 1140 is taken before EGR 1010 despite the number code.

Some students can take these courses simultaneously but only if they have some experience in computers and engineering beforehand.

EGR 1140 Computer Programming for Engineers and Scientists (1st semester)

This course while a computer programming course (not a computer science course which delves into more computer theory) is also an introduction to engineering course in that it discusses engineering with regard to computers. For this reason we combine this into the introduction to engineering course. While this course does discuss a little computer science theory it primarily discusses programming in order to produce program for engineering or scientific projects.

EGR 1010 Introduction to Engineering for Engineers and Scientists (2nd semester)

This course is the true introduction to engineering course, however, it presumes the ability to program. Therefore the computer programming course would be a prerequisite unless a student has previously taken a similar course. Nevertheless there are some aspects of introduction to engineering in the computer programming course, hence the chapter labels below.

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- 4: What is engineering? Who are engineers? (EGR 1010 highly recommended for EGR 1140 as well)
- 5: What is a computer? (EGR 1140 & EGR 1010)
- 6: Understanding (how to investigate on your own) (EGR 1140 & EGR 1010)
- 7: Operating Systems with Brief History (EGR 1140)
- 8: Brief History of Popular Programs (EGR 1140)

- 9: Programming in any language (EGR 1140)
- 10: Parachute Person (EGR 1140 and repeated in EGR 1010)
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2: Description of topics and computer requirements

Description of Course

This course as presented here is a two semester course consisting of a "computer" course and an introduction to theory and practice of engineering course combined into one course. The reason for combining the course here is that some introduction to engineering material should be presented before any material is presented in engineering (like what is an engineer) but the computer course material should be presented before the hardcore portion of introduction to engineering (like laboratory for instance). An instructor can do this in the order they prefer but herein we give what is believed to be a sensible way to go about this.

While this description separates the two courses in distinct semester units as would be expected in traditional program, mixing and matching of this units is a better way of presenting the material.

The first semester of this course would consist of a high-level introduction to computer tools and computer programming for the engineer and scientist. The goal is to develop within the student sufficient knowledge to perform analysis using common engineering and science programming languages. Topics will include algorithm analysis and solution, program structures, data structures, modular design, and overviews of the computer hardware, various computer tools available to solve real world problems, and object-oriented structure. In addition the course will include an introduction to test and control system programming. A variety of languages will be introduced such as MATLAB, Fortran, and C++ with primary emphasis on one of these languages. The results will be to ensure that from the primary language a student can easily master the other languages. Along with the aforementioned languages a number of engineering specific languages such as LABView, Spice, and VHDL will be introduced to the student.

The second semester of this course introduces the student to the theory and practice of engineering. The semester has four main parts. Each part will cover important aspects of engineering giving the student a full picture of the career they are about to embark upon. The first part will help the student understand what an engineer is and what type of work they would be expected to perform in society. Included in this will be discussions of ethics and group dynamics (which will be an optional at the end of the course chapter). The second part will deal with higher level engineering concepts. This will be developed in an application area such as a research laboratory giving students exposure to professional practices common in all engineering disciplines. A number of professional papers will be reviewed leading to a creation of a hypothetical laboratory emphasizing the interaction common to all engineering disciplines. The third part will cover fundamental aspects of engineering including drawing, modeling, problem solving, design, and laboratory experimentation. Basic computer skills will be developed using MATLAB, Fortran, C, or similar high level computer language. Finally a team project will constitute the fourth part. The teams will be expected to develop a product using a number of engineering and software skills.

Team work along with communication skills (oral, written, and graphical) are exercised throughout the course.

The level of this course assumes the student is starting ***calculus*** and ***college level English***. If the first semester of the course is skipped it is assumed the student has experience in computer programming as an engineer or scientist.

Computer programming languages needed

Initially this textbook was written for the computer programming language to run in the textbook itself, but changes were necessitated such that that is not longer true. There are many programs in this book that will require you to have a good set of programming languages. All the programming languages used here are free and available for the three major operating systems (Linux/Unix, OS X, and Windows). To run the programs in here you will need to install on your computer GNU Octave (or MATLAB or Scilab) and Python. Some package installation will be required as well for both languages. In addition for your continued education it would be desirable to install R, Julia, gfortran (Fortran), and g++ (C++). What computer you use, of course, is up to you but if you are financially tight it is suggested you get a cheap Windows computer and convert it to Linux (Ubuntu) (note: these cheap computer's batteries generally don't last long, but it should get you through your first four years of school).

Copy and paste the programs in the book in the appropriate programming language in order to run it on your computer.

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3: What we intend to learn here

Computer agenda

First week: Computer Hardware; Software; Operating Systems (UNIX and others)

Second week: General computer tools; How to use the computer (editing, compiling, etc.); Input and Output; Proper Documentation

Third week: Data Structures and Operators (vector, matrix)

Third and Fourth week: Computer operators

Fourth through sixth week: Computer functions: Traditional and Boolean

Sixth and seventh week: Program control

Eight week: Advance program control (that is beyond the basic control)

Ninth week: Advance Data Structures

Tenth week: Advance Programming Concepts

Eleventh week: Using computers for Engineering, Science, and Analysis

Twelfth week: Overview of advance techniques (analysis, etc.)

Twelfth to fifteenth week: Engineering/Science program design project (this is success-based)

Engineering and Science agenda

Part I:

First week: What is an Engineer:

> What is the difference between an engineer, scientist, and engineer technologist, plus an overview of the many engineering disciplines, and ethics and mechanics of group dynamics

The history case studies of the second week are to give a sense of past engineering innovations and its relationship to modern engineering as well as to give a sense of how engineering actually works as opposite to the layman view of how it works.

Second week: Historical case studies in Engineering (Egypt's pyramids, First flight, and trains)

Third week: A cutting-edge case study on Nanotechnology

Part II:

The major case study in engineering is to include examination of high-level subjects using professional

papers (with instructor guidance). Student is expected to do all research, prepare for discussion each day, and participate. The intend of this exercise is to build something to highlight different engineering disciplines. This particular case study will highlight Civil Engineering, Mechanical Engineering, Optical Engineering, Materials Engineering, Electrical Engineering, Computer Engineering and System Engineering.

Fourth through seventh week: Major case study in engineering:

What it takes to build a laboratory, how to build a laboratory, and putting the laboratory all together

Part III:

Fundamentals of Engineering: The core mathematics, science, and "equations" of engineering.

Eight week to eleventh week:

Units and a mathematics review, statics, dynamics, solid materials, thermodynamics, optics, electrical circuits and quick review of computer programming

Part IV:

For laboratory, student teams would be expect to pick their own subject (with approval) and proceed on their own with teacher guidance. The main objective here is learn how real engineers work. This is about the process not about success, real labs take months to years not weeks. It is very likely only prototypes will be developed by the end.

Twelfth to fifteenth week: Laboratory Project

Creating a project plan under assigned criteria and getting approval, executing a project plan, taking laboratory notes in a laboratory book (composition book) and writing a report

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4: What is engineering? Who are engineers?

General note: the questions or statements with the ¬ symbol means that you can click on them to revel more information or the answer to the question (don't immediately click on them...try answering the question)

Hi, we have a question for you LibreBot,

What is an engineer?

Discuss what an engineer is in class or with friends. Note there is not just one answer as with anything in the "real world" definitions can be complex with multiple subtleties. That being said there is one specific word that is a must to get at the core of an engineer.

- **¬ A person who applies science and mathematics to produce a product?**
- **¬ A person who fixes things?**
- **¬ A person who creates products?**
- **¬ A person who is called an engineer?**
- **¬ A person who works in a team using science to develop products?**
- **¬ A person who through experience and education develops ways to use materials and nature to benefit mankind?**
- **¬ A person who designs using experience and science?**
- **¬ A person who is an artist but with exacting precision?**

So now it is up to you to come up with a final definition. *Why not just go to a book answer?* Well because most answers are insufficient to fully describe engineering and some are just unintentionally wrong. To really get a true answer you need to look at all the aspects of engineering and see it in your own mind. Go for it.

The rest of this chapter tries to define engineering by describing an engineer. It is not a definition, but it certainly is clearer then the previous discussion and will hopefully lead you to find a definition that is satisfactory.

What is the relationship between engineering and similar professions?

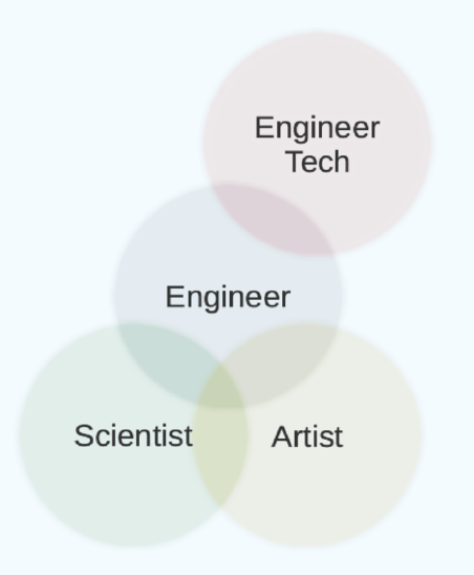


Figure 4.1: This Euler Diagram shows the relationship between similar fields (engineer, engineer tech, and scientist) that people tend to confuse.(Scott Johnson, this work, CC-BY-SA-NC)

Academically how might we distinguish an engineer from say an engineering technician or a scientist? Because of varying academic standards at different colleges and university some variation might occur in the following descriptions, but in general it is what is expected from most academic programs.

- **¬ Engineering academic needs**
- **¬ Engineering technologist academic needs**
- **¬ Scientists academic needs**

Does a person really require a degree to be an engineer or scientist? Technically no, you could become an engineer by experience, though this was more true in the past than the present day. Unless you are independent it is highly unlikely anyone will hire you without the academic requirements. *Question - Would you hire someone claiming to be an engineer without any academic education?* Try and look at this realistically where you might have a significant number of resumes per position you are offering. *How could someone who only has experience get a degree?* No answer here, you should just think about it.

How many credits does someone need to be an engineer? Ok, let's get specific on the number of credits that would be required for the first two years of an engineering degree (in order to fairly compare it to an engineering technologist degree). This particular exercise does have a particular basis towards Prince George's Community College and University of Maryland College Park, though should be generally accurate for anywhere.

- **¬ Engineering credits required**
- **¬ Engineering technologists credits required**
- **¬ Scientist credits required**
- **¬ So what does this all mean?**

In the non-academic world scientists, engineers, and engineering technologists work together very closely so the differences can become hard to distinguish. Normally a scientist does the theoretical work, the engineer does the design, and the engineering technologist does the actual work. However these lines get blurred in the "real" world where each of these three professions might be doing parts of each others jobs. Yes, scientists do solder, but definitely not as well as an engineering technologist.

Engineers can be found to be a form of translator between scientists and engineering technologists and that is one part of their job that does not get blurred in the "real" world.

Assignment: *Write a small essay on what an engineer is to you with particular focus to what discipline you wish to pursue. The essay is to be written like you would for a paper in English class. That is an introduction, a body, and a conclusion. Statements*

made must be supported in the body of the text.

One interesting fact, and sometimes frustrating from an academic point of view, is that engineers come in many different types. Generally we refer to these as engineering disciplines very similar to scientific disciplines. You will find in most schools that science is separated into independent majors like biology, chemistry, physics, etc., however engineering is not. This give the impression to the layperson that engineering is one homogenous group. While engineering does have similar core basis like physics and chemistry (which science does as well), it has many many separate disciplines.

What are some of the engineering disciplines?

- **¬ Types of Engineers**

Beyond these disciplines of engineering are technical divisions within the disciplines which can added up to quite a lot of different courses. What is exciting about this is that your interest in engineering can be combined with your interests in something else (like race cars or music).

What about the money? It is important to remember that job satisfaction is more important than salary. Getting a high paying job that you don't like never turns out well. That being said the average salary of an engineer is about 50% higher than the average salary of an engineering technologist. But you should note that there can be a wide variety of salaries depending on your knowledge base and your experience so it is best to do what you like well then to do what you don't like poorly. As for scientist, nobody becomes a scientist for money it is too underpaid for the work you have to do.

What about stability in the sense that I can be in the field my entire life if I choose? In this case it is well established that engineering is the most stable job. Engineers learn a more general theory of engineering that is easily adaptable to changing practices where as an engineering technologist learns specific practices and devices that can change over time. This can lead to an engineering technologist becoming obsolete without retraining. Fortunately there are many community colleges that offer this retraining, but it could mean starting a career essentially all over again at a starting salary level. As for scientists, it depends on the mood of the world during Luddite times it is not so good, during techophile times it is good. Interesting the times don't generally affect engineers or engineering technologists.

So an engineer is everything and must know everything? Mostly correct. An engineer should be well educated in science, mathematics, and practically everything. But obviously nobody can know everything, so again mostly correct. ;)

That is engineering in a nutshell...

The next step is to learn about computers and computer programming. We will come back to the integral engineering picture later but many tasks that will be done coming up will require a knowledge of computing.

Student questions:

- **¬ I want to be an engineer that doesn't have to program?**
- **¬ Would I have had to learn computers in engineering a century ago?**
- **¬ Would I have had to learn computers in engineering four decades ago?**

```
name = input("What discipline of engineering are you interested in? ","s");
printf("%s is a great discipline, but are you sure?\n",name);
```

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5: What is a computer?

To ask what a computer is we must first define what type of computer. A computer is someone or something that does computations, usually assumed to be quite complicated calculations. In the movie ***Hidden Figures*** you probably noticed they referred to Katherine Johnson, Dorothy Vaughan, and Mary Jackson as "computers." That was not just something for the movie, that is indeed what people who did complicated calculations were called in real life. You might also have notice that one of the characters, Dorothy Vaughan, taught herself FORTRAN (a very smart move) so that she could transfer over to working with *Electronic Computers*.

When we refer to computers today we are referring to *Electronic Computers*. Electronic computers are an electronic machine that does not think but carries out our instructions (commands). They are generally faster then human computers, but they don't think like human computers (this does not change given the advent of newer more sophisticated AI) and they still need a thinking person to provide them with instructions. Anyone can do this, but for complicated calculations like you might find at NASA you would have an engineer, computer scientist, scientist, or someone studying to enter one of these fields preparing these instructions. From here on any reference to computers will mean electronic computers.

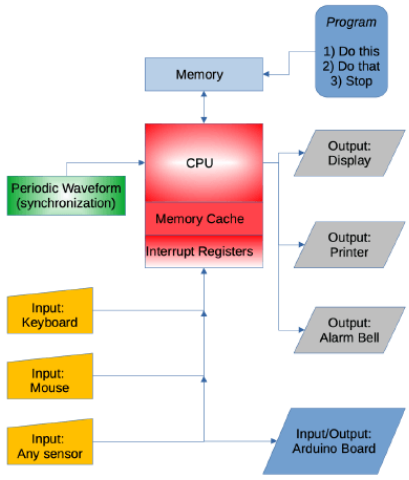
In the previous chapter we discussed the differences between engineers and scientists, but we only lightly touched on computer scientists. Academically there are different types of computer science degrees. For engineering students the distinction is important as other computer science tracks are not sufficient.

- **¬ BS in computer science (engineering program)**
- **¬ BA in computer science**
- **¬ BS in computer science (not through engineering program)**

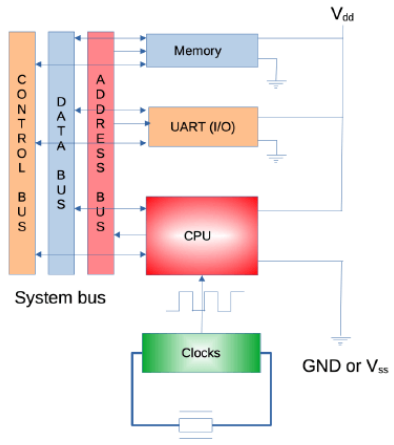
This course will assume an engineering perspective (BS) and will therefore assume the you are in a science/engineering track (or interested in that track). This will mean the programs discussed and ask for here will assume a high level of mathematical preparation, however this course does not delve into computer science theory which can be even more complicated (though quite interesting). So hard but not as hard as it could be...:)

What is a computer? From a technical point of view a computer is an electronic machine just like our brain is an electronic organic machine. ***OR*** the computer is a lump of materials that is just a large man-made ore deposit that can do nothing without some sort of instructions. This is similar to our brain which is a lump of flesh unless instructions are put into it (*of course this is a simplistic analogy useful for building up an understanding of what a computer is but missing nuance*). That is, the hardware itself really cannot be a full-fledge computer without some sort of instructional system. Therefore we really need to view the computer as a whole system which includes *hardware* (the machine or the brain) and *software* (instructions). But first let us look at the hardware.

Computer hardware is the physical aspects of the computer which is center around the Central Processing Unit (CPU) and memory.



Cartoon describing the system design of a computer. The memory would contain both data and program in same location. Listed items for the input and output are examples and could included many different items.



Same cartoon as above but more hardware oriented...

Now let us look at the software.

Computer Software is the virtual aspects of the computer. While it is not physical it plays a crucial part in the whole computer system. Software has flexibility that hardware doesn't have in that you can change your instructions to maybe speed up a calculation but you can't change say the cache of the CPU (hardware which is a fixed size) to speed them up. In our analogy, software is just the electronics computers version of instructions that is store in memory and execute when requested by the user. All virtual aspects of a computer is a form of computer software or a "program". The operating system or systems which work directly with the hardware and the programs that users run to get their jobs done are all forms of software. Operating systems are discussed latter in the course. Here we discuss just the programming languages and user programs. Most engineers will only need to deal with this aspect of computer software.

Software like human languages has many different "languages." Lists of almost all the known programming languages (even extinct ones) can be found on Wikipedia or for a bit of fun on http://99-bottes-of-beer.net (note this web site has disappear for unknown reasons but a backup is available at https://99-bottles-of-beer.spielmannspiel.com) which has programs in different languages that display the famous "long trip" song. You will see there are over 200 programming languages, however, only about twenty of them are the basis for the bulk of programs. In addition, most of the programming languages have many different versions. As an example, Fortran has about 12 versions (not including variants) that go from Fortran I to Fortran 2023, which can seem as different as old English is to modern English. Fortunately like human language there are certain ideas that encompass all languages. ***It is these ideas that are more relevant than the languages themselves.***

Like many other aspects in engineering, software starts from basic building blocks which are then combined to form more complex building boxes which are then combined to form even more complex building boxes and so on. The basic hardware building block for the computer is the transistor which by its design can only have an off and on state (0 or 1 - the beginnings of "software"). Hence software at its basic level is binary. There are some well agreed upon "levels" of software which will be listed here.

- **¬ Level 1**
- **¬ Level 2**
- **¬ Level 3**
- **¬ Level 4**
- **¬ Level 5**

The levels of the programs leads us to discuss the types of programming language, in particular the two most common types: Compiled languages and interpreted languages. Each has their pros and cons and a good programmer usually has at least one favorite compiled language (Fortran, eh?) and one favorite interpreted language (Octave, eh?).

- **¬ Compiled Languages**
- **¬ Interpreted languages**

While computer programming suggests the user makes "programs" for computers, the field has some different names for what could be construed as programs (like, for example, "functions"). The words we will "define" will be program, scripts, functions, and some other words that computer savvy people might use.

- **¬ Programs**
- **¬ Subroutine or Function**
- **¬ Script**
- **¬ Functions**

This course will focus on "GNU Octave"/MATLAB/Scilab but also explore Fortran, C++, Python, IDL/PV-Wave, LISP, and Spice. *Why study many different languages?* Because the main objective is to study the theory which will allow an engineer to master programming in almost any of the over 200 programming languages in the "real" world. When going out to a new job or academic/research center it is impossible to predict what programming languages will be primarily used so it is important to create as large a toolbox as you can before you go off to the "real" world. *Why is the most popular language Java not in this list?* Because this is an engineering/science course and Java is not as relevant in this world, but the study of the theory should allow you to learn Java on your own if you wish.

There are a vast array of ideas you can program into a computer using the languages described above. The program language themselves give a high level of flexibility. A range of programs can be written from very simple programs to fairly difficult programs. While a simple program can be programmed by just about anyone, ones that require difficult concepts need skilled practitioners in the field. This course assumes an engineering perspective (BS) and will therefore assume that you are in a science/engineering track (or interested in that track). This will mean the programs discussed and ask for here will assume a high level of mathematical understanding and the ability to create a complex program.

Programming large complicated systems can be very difficult for a typical programmer even a practitioner of the field. In these cases teams of people program the programs with (interestingly) programs that help them work together without over-writing each others work. As an engineer/scientist you might find yourself as part of a team writing these programs or you might find yourself using the final product. Let's discuss more about the final product.

Very large programs require a team of people to develop and usually the user just gets the final product. These type of programs are referred to as *canned* programs in that they really are not modifiable by the user but are very useful. An example of a layperson

canned program is Apple's Pages or Microsoft's Word. There are also some canned programs that have programming languages in them (so they are not fully "canned"). There are many useful canned programs for use by engineers and scientists.

Some useful programs (with the targeted discipline in braces) in engineering and science include Autocad {CE}, Creo Elements {ME}, MultiSim {EE}, ProSim {ChemE}, COMSOL Multiphysics {Physics, any Engineering disciple}, Molecular Workbench {Chemistry, Biology, any Engineering disciple, mostly educational/free}, RockWorks {Geology/Mining Engineering}, Zemax's OpticalStudio {Optical Engineering}, CMISS {BioEngineering, and maybe ME and CE since this is a finite element package}, and many others. In general you can find less capable but still useful freeware or open-source versions of the aforementioned software packages. This is just a small sample of engineering and science programs, but you can see already that many disciplines have programs associated with them.

Can anyone use these engineering/science type programs? Not really. Engineering and Scientific software requires an understanding of underlying numerical techniques used in the programs as you are often asked to make a choice between different numerical methods. Some have there own underlying language (like Molecular Workbench). The understanding of the theory of programming language and the underlying numerical techniques is required to be able to use these software packages. This is engineering.

Canned software is pretty much as is and anything new or not standard would not be found in said software. To avoid the limitations of canned software the engineer can program their problems in one of a variety of programming languages. In this course only a set of programming languages will be mentioned and discussed to different degrees.

So let us outline some important points:

- There are over 5000 programming languages (in the most broadest viewpoint), just like there are over 5000 spoken languages (in the most broadest viewpoint).
- Excluding programming languages that were not meant for electronic computers (like loom paper punch cards) and assembler languages, the oldest languages include FORTRAN (Speedcoding), LISP, and COBOL.
  - FORTRAN, LISP, and COBOL are still active languages
  - FORTRAN now referred to as Fortran had it's latest version issued recently as Fortran 2023.
- Some languages are "independent," some languages are unique to a software package (not independent), some "independent" languages are attached to certain software packages, some languages have a non-standard character set (APL as the most unusual - like ancient Egyptian hieroglyphics), etc.
- Computer languages in general differ in purpose. A sample of some prominent languages are presented here.
  - Fortran is for scientists (and *engineers*)
    - Equation based
    - Very backward compatible
    - Tries to resemble English words and is one of the most readable.
    - Newest versions of Fortran have interoperability with C build-in (before that there were programs that did that) - this benefits both languages
  - Octave/MATLAB/Scilab and LabVIEW is for *engineers* (and scientists)
    - Octave, MATLAB, and Scilab have similar syntax and commands.
    - These programs all have numerous packages that extend their command set.
    - LabVIEW is graphical with glyphs similar to how some written languages use pictograms.
  - COBOL is for business (although it influenced many non-business languages)
  - LISP is for artificial intelligence and is sometimes used for robotic engineering
  - C and C++ is designed for computer science (operating systems, embedded systems, etc.)
  - R is an open source statistical analysis language (like S, SPSS, SAS, and STATA), any field (economics, business, machine learning, ornithology, etc.) that uses a lot of statistics would use this or a similar language
  - Julia is a new language that is mostly a scientific language but with the addition of some useful R language abilities. In some sense it is a merge of Fortran, Octave/MATLAB, and R but also has some LISP and C++ features. It is reminiscence of PL/I from the deep past but much better.
  - IDL/PVWave is for science in particular Astronomy and Earth Science
    - IDL is used more on the East Coast of the United States where as PVWave is used more on the West Coast.
    - PVWave evolved from an older version of IDL.
  - Python is a general purpose language (heavily influenced by C and Haskell)
    - While a great language it has a disturbing habit of not maintaining compatibility
    - Python 2 (which is end-of-life) is not compatible with Python 3, a source of many complaints
    - Reaching into science through its packages: numpy, scipy, matlibplot, etc.
  - Java is an enterprise language (business and entertainment; heavily influenced by C++)

    - Developed by Sun Microsystems (now Oracle) for interactive television
    - The most popular language but not really for science or engineering
    - Sun licensed OpenJDK to GNU General Public License however Oracle still has a license for their Java components (IceTea which is GNU GPL has similar components)
    - Javascript is not Java though it is similar (yet distinct) and obviously influenced by it
        - Javascript is an essential language for web pages
- Learning a number of languages ensures that any "new" language will be easily self taught, hence the theory behind this book/course.

To review a computer is hardware (the physical, like the brain) and software (the non-physical, like a set of instructions).

So this is what a computer is...now we will go into some history and into more depth...but first a quick sideline to discuss how you should do research on a topic (whether for writing an essay or for trying to figure out something on your own)...

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6: Understanding (how to investigate on your own)

For engineering classes you will write essays on a number of subjects. For introduction to engineering the subjects are to be one page essays on a topic that relates to your desired discipline (or ones your thinking about) within engineering. Essays are due once every one or two weeks. The essays are to be about a particular subject, not about the discipline (so about an aspect of transistors, not about electrical engineering).

Why write essays? For two reasons, one is the simple reason of just practicing your English especially the skills needed for technical writing. The other reason is for you to write something that will help you see what your discipline is all about. For general engineering introduction classes this is the best way for you to learn (that is "teachers only open the door it is up to you to enter them." - Karate class). When doing specific discipline engineering introduction classes found in extremely well-funded schools this might not be necessary, but it doesn't hurt and you still need practice with technical writing in all schools.

Before we start writing essays though we need to set some groundwork on how to do your investigations. Not only will this benefit you for writing your technical essay but it also will give you a guideline on how to study on your own or even do classes on your own (if for some reason you have difficulty getting into a school).

Below we give a brief overview of how to reliably do your own investigations of topics. Unfortunately "fake" news/science/information (Yellow journalism - 100 years ago...this is NOT new) can be found anywhere. That being said their are multiple good sources on information (though they can be occasionally wrong, so always verify).

This is a selection from Astronomy class (substitute Engineering for Science):

Science	Pseudo-Science
Scientific Journal, peer-reviewed	Literature that is aimed at general public
Reproducible results, precise description of experiments, many references of previous work	Results cannot be reproduced consistently, vague results, vague description of experiments, no or very few references OR references that are not relevant but are added to fake legitimacy
Refers to establish theory	Ignores established theory with no basis to do so; no physical phenomena or process to back it up with
Results are the results even if they don't seem to match our beliefs	Appeals to belief rather than evidence
Starts as a hypothesis • "Proved" with multiple experiments • Becomes a theory with a preponderance of evidence (interesting this is a little bit like the law; guilty beyond a reasonable doubt - not 100% no doubt, but reasonable doubt)	Lacks evidence or spurious evidence (not peer reviewed evidence)
Not trying to sell something	Usually selling something: either a product, ***belief, or philosophy***

So a note on theory: In science, a theory is not like the theory in mathematics where you prove a theory with axioms to produce a theorem. It would be near impossible to prove a science theory with axioms to produce a science theorem. Theories in science are more than the name suggests. They are not hypotheses. The theories of science are proven with a preponderance of evidence[1], they are not just ideas that pop from ones head. Unfortunately pseudo-scientists like to use the word "theory" to suggest science is not rigorous, but are just ideas that pop into a scientists head. This is not true. To be clear, *theories in science are, as far as present science allows, **proven** through a preponderance of evidence.*

From the following table you should be able to glean some information on how to tell good information from bad information, but let us get specific with references you are likely to use.

¬ How do we trust the information we see on the web or in the "media"? That is how do we know what is accurate or not?
¬ What is the best way to ensure your knowledge is real or good?
¬ So we all know (because my teacher says so) that Wikipedia (or other web pages) is not to be trusted, right?
¬ So we all know (because my teacher says so) that the New York Times (or other newspapers or magazines) is to be trusted, right?
¬ So we all know that books are to be trusted, right?

Where else can we get information from especially on the internet. Should we get information from the following sources?

¬ X (Twitter)?
¬ Instagram?
¬ Facebook?
¬ Any other social network?

A quick note about search engines. Some search engines (not all) obscure the actual source of the information. Always find the actual source of information, never trust search engines to do the work for you. A good thing to remember in these cases is **trust but verify.** Lately search engines have become more then search engines with added value sections like say a "news" or "science" section. They do not actual do the news or science they get it from others. Find the actual source!

Equally this applies to artificial intelligence (**AI**) applications as well. AI apps work almost identical to search engines except they are better at giving succinct answers that sound authoritative (which often convinces people that wrong answers are actually right). Find the actual source for the information and confirm it is correct! AIs are trained just like children and if a child is told that ants are 8 feet in length they will believe it and so will AI. Human beings are intelligent apps - they are not artificial and hence they are better at doing the work correctly (albeit not as fast).

Now that we have a method of understanding how to investigate ideas, lets look at some web sites you might go to do some investigation that are not Wikipedia. Most of these magazines should be in your library (where paper books are).

Magazines of interest (vetted/engineering):

- Popular Science -- More engineering than science
- IEEE Spectrum -- Electrical Engineering magazine
- Make magazine
- Chemical & Engineering News
- Amateur Radio (currently defunct; though not totally out as of yet)
- Popular Mechanics
- Servo Magazine -- Robotics magazine
- Nuts & Volts -- Electronic magazine
- Optical Engineering -- Optical Engineer magazine
- Nature Biotechnology
- Nature Materials
- Nature Nanotechnology -- Magazine on nanotechnology - first issue free online
- Nature Photonics -- Magazine on optical technology
- Computing in Science and Engineering

Magazines of interest (vetted/science):

- Science News -- Layperson magazine on science and engineering: This is for anyone who is reasonably educated and is published biweekly (used to be weekly, but with the advent of a website it was felt weekly was too much). Started in 1922 as Science News Letters. This is a small operation by news standards but is a better source of science then most newspapers.
- Scientific American -- Science oriented, but occasionally has engineering articles
- Astronomy
- Sky and Telescope
- Astronomy Now
- Discover
- National Geographic
- Nature Physics -- *Physics* is one of the basic sciences along with *Chemistry* (while this particular Nature is rather new, Nature itself started in 1869)
- Nature Chemistry -- *Chemistry* is one of the basic sciences along with *Physics*
- Nature Geoscience -- Might be of interest to petroleum engineers as well as scientists
- Royal Society Open Science -- contains science and engineering (while this Royal Society publication is rather new, the Royal Society's "Philosophical Transactions of the Royal Society" was started in 1665 and was interestingly similar to the original Science News mentioned above)
- Johns Hopkins APL Technical Digest -- Technical Journal for APL

Web sites that are better vetted then most:

- Scholarpedia -- Like Wikipedia (with much less) but peer reviewed
- ScienceWorld -- Eric Weisstein's Scienceworld (sponsored by Wolfram Research - the "Mathematica" dudes)
- MathWorld -- Eric Weisstein's Mathworld
- How's it Made -- A TV show that has sometimes has very interesting stories on engineering processes...this has enough detail that it can be used as a reference and this show has run for 19 years (there are others in this class of TV show that are good as well like *How Do They Do That?, Extreme Engineering, Impossible Engineering, Modern Marvels, How Did They Build That?, Strip the City, etc.*)

Next let us discuss how you should write a paper. This begins with the classic English class refrain: Introduction, Body, Conclusion. This applies to engineering and science papers as well, the only difference is that their should be more seriousness to how the papers are written. No bells and whistles just the facts in your essays.

- Introduction
  - Give an overview of what you are going to present
  - Make in "short and sweet"
  - For a one or two page essay do not do broad subjects, do specific subjects
- Body
  - This is the "meat" of the paper
  - Defend your statements
- Conclusion
  - Summarize what you have presented
  - Don't present substantially new material that needs to be defended

Standard English writing ideas still apply

- No clichés
  - It is raining cats and dogs...so where are the cats and dogs all over the roads???
  - Only time will tell...so what will it tell us???
  - Once we *read between the lines*...so where are the words that you see between the lines???...show it.
  - In this day and age,...you mean now??? Why do you even need to say this???
- No trite statements or hackneyed statements
- No run on sentences
- Paragraphs should normally have at least three sentences unless there is some special reason

- No run on paragraphs...10 sentences or more in a paragraph likely can be split up
- Make sure you have verb agreement and verb/noun agreement
- Don't use words you don't understand, just say it like you would normally say it...fanciness not needed
- If the sentence is awkward, rewrite it to be simple...simple is always better (example from famous playwriter: "To be or not to be")
- Do not use quotes for something you don't understand; either understand it and write it in your own words or don't include it in your essay
- Better yet don't use quotes at all
- Since you all have word processors - CHECK your spelling (if something weird is going on make sure you are set to American English if you are in America and writing in English) - if you are lucky you might even be able to CHECK your grammar.

So lets see some essays!

---

[1]Like in criminal law when you are found guilty beyond a reasonable doubt (usually through a preponderance of evidence or a preponderance of circumstantial evidence). Note you are not found guild beyond any doubt. We all have doubt. Even in science we are never glued completely to an idea if evidence is suggesting something else. Though it is hard to dislodge a scientific theory unless there is a preponderance of evidence.

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7: Operating Systems with Brief History

On operating system (OS) is a set of programs that that runs the various services a computer has to offer including running other programs. The main operating systems today are UNIX (MacOS is a UNIX variant), Linux, and Windows. This chapter attempts to highlight some history of operating systems with a very brief introduction to operating systems. It is not intended for operating system theory which can get pretty involved especially with todays layered operating systems.

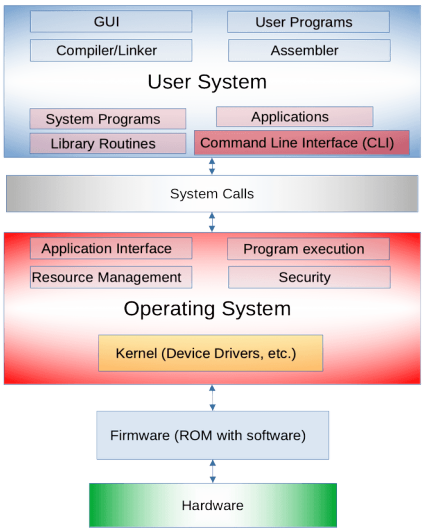
There are literally hundreds of operating systems so this should not be construed as a complete history. A complete history of operating systems would be a 1000 page tome. Not going there, but you can if you wish as this information is readily available on the internet (though be careful as there are bias sources - see the "understanding" chapter).

The three major operating systems take up about a third of the market each.

- Windows (general OS, primarily business and gaming system)
- Unix (many different variants including MacOS and iOS)
- Linux (many different variants including Android)
    - Popular Linux platforms include Ubuntu for it's user friendly installation and Mint
    - Most variants now come from three sources, Red Hat (split into Enterprise and Fedora), SUSE, and Debian (one of the oldest distribution and includes Ubuntu and Mint as variants)
    - All of the top supercomputers use some form of Linux presently (originally Unix was the king of supercomputers)
    - A good number of scientists and engineers use Linux especially in fields that are not as well support by the commercial software industry
    - Linux is free (MacOS is free as well, but it only runs on Macs which can be expensive for students)
    - Most Unix operating systems have some sort of licensing issue, so be careful of "free" Unix operating systems (they probably aren't actually free)

For ***this class*** Linux is recommended as from an academic point of view; it is inexpensive compared to the other operating systems (just need the hardware and some time investment). It can provide all the useful features you might need like word processor, presenter, etc. (through free open-source LibreOffice or Google Docs Editors) and numerous freeware or open sources science and engineering software programs some of which only run on Unix and Linux. Ubuntu is recommended for ease of installation by beginners. For experienced Linux users any Linux is recommended as there is no overriding problem with any major version of Linux.

Operating systems originally were developed from assembler languages but now are usually build using C or C++. While hardware does the actual I/O, memory, shifts, counting, multiplexing, etc the operating software directs the action. Usually engineers do not need to get too much involved with this aspect of computers (unless you are an engineer who designs computers). Like regular programs the operating system is saved in memory and "read" by some very basic hardware operating system (**firmware** or software on hardware that is "burned" on a PROM) which executes the instructions that in essence supersede it in the form of a larger operating system that does more functions (including security and conflict management). This is a layered approach where hardware is controlled by firmware which is controlled by the OS which is used by other programs to run with secure access to lower level hardware commands.



This cartoon describes the interaction between the operating system and the hardware. Both the hardware and the firmware are actually physical hardware items. Firmware is used here as it is more general then BIOS which is a type of firmware. Firmware is "burned" (it isn't actually burned) on to either a ROM (read only memory), PROM, or EPROM which is hardware. All the rest of the cartoon is software but with different levels of access. A number of terms in this figure will be defined later in the course.

Now the history of operating systems starting from the beginning of electronic computers.

- **First computers have no operating system**
- **Computers with beginning operating systems**
- **Workstations start with new types of operating systems**
- **Personal Computers start with new types of operating systems**
- **Mobile computer world's OS**

For more information and an in-depth understanding of operating systems the student should take a traditional computer science course on operating systems. Most engineers do not need to go into this depth of understanding. Installing a Linux operating system and learning how to install programs and configure the computer should be sufficient for most engineers.

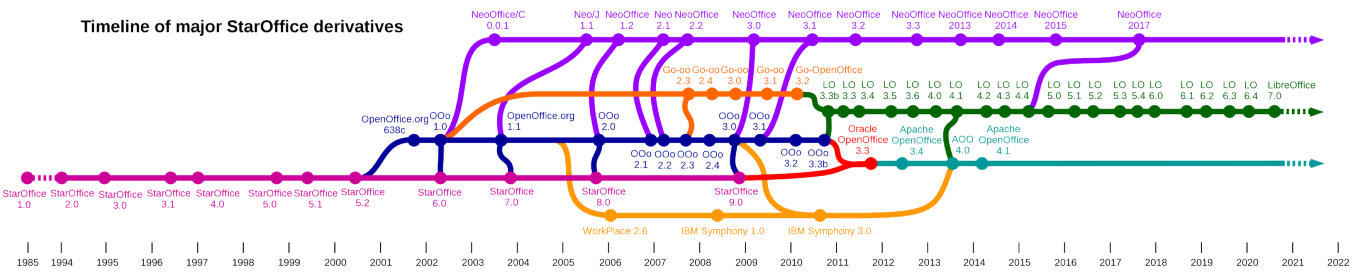
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8: Brief History of Popular Programs

This is a list of very popular programs with history associated with them. This history is meant as a vague timeline and does not have exact dates. There are numerous websites with exact dates that the student may look at if they wish. The goal here is to give a feel of how software developed and to emphasis the unique role Universities had to play in the development. The role of Universities is sometimes downplayed in society which leads to a skewed view of how ideas and products are developed in liberal[1] societies.

- ¬ **Spreadsheets**
- ¬ **Word Processors**
- ¬ **Presenters**
- ¬ **Turtle graphics programming language**
- ¬ **Desktop publishing**
- ¬ **Computer Aided Design (CAD) software**
- ¬ **Simulation software**
- ¬ **SPICE (Simulation Program for Integrated Circuit Emphasis)**

Free software or open-source software has become a major part of the software landscape in recent years. LibreOffice (not related to Libretexts or LibreCad) is one of the most successful of those ventures. LibreOffice derived from StarOffice as is shown in the figure below.



This is the history of LibreOffice which is derived from StarOffice, the software that historically allowed UNIX users to work with MS Word users. David Gerard, Донор, Shunesburg69 / CC0

[1]Here liberal is referring to liberal versus illiberal society which is a different concept then liberal versus conservative political theory.

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9: Programming in any language

Programming

To review, the purpose of programming on an electronic computer is to solve problems in a timely manner that are a difficult to solve by hand. Here we will review five languages as our examples: Octave (MatLab/Scilab), IDL[1] (PV-Wave or GDL), Python, Fortran, and C++, with a heavy emphasis on Octave (the other languages are included to show similarities between these languages, as most languages are very similar).

While you could, as examples, use Octave, Python, or IDL to just add, subtract, divide, or multiply numbers as a sort of calculator that is not the best use of programming languages. A calculator would be better for that sort of work. What makes programming languages powerful is variables, not dissimilar to variables in regular mathematics, especially "variables" that are structures which are reminiscence of matrices or matrices themselves (see parachute person).

We will use Octave for the examples herein and other chapters to show what is being discussed in the text. Please run them and feel free to modify them and run them again (NB: your modifying the programs does not effect them for others...just you. So feel free to play.)

- Here we do simple math in Octave which makes Octave a great calculator, but does not nearly demonstrate its true power.

```
2+2
2*2
(2+2i)*(2+2i)
sind(180)
```

- Here we do simple math in Octave but add a variable component. This increase the power but still doesn't get to the full power of a programming language.

```
x = 0;
x = input("Enter degrees in radians ");
printf("The sine of %f is %f\n",x,sin(x))
```

In this last example we use a variable which we will discuss in the next mini-section.

Numbers, variables, arrays, and other structures

In general we would like our computer programs to provide the user with a method to do either **complicated** calculation, complicated string matching and pattern matching, or a combination of both of these. Everything a computer does we can do by hand but with less ease (though some might argue after struggling with a program that that is not true, but as your skill grows you will find that indeed it is true). In order to do these complicated calculations there must be a way to have and use variables. Variables in a computer languages work just like they do in mathematics (and other fields)...a way of expressing an unknown or many unknowns.

Variables for calculations in computer languages are either declared as or given a specific type of number. A number (a scalar) in mathematics can be a counting number, an integer, a real number, a complex number, etc. In computers we have similar designations though the syntax differs depending on the programming language. The very basic ***data types*** are shown for a set of chosen languages below.

Basic data types	Fortran (mostly required)	C++ (required)	Python (doesn't exist)	Octave/Matlab (doesn't exist)	IDL (optional)	Your math class
Integers	integer	int	Python does this for you	Octave does this for you	int	integers

Basic data types	Fortran (mostly required)	C++ (required)	Python (doesn't exist)	Octave/Matlab (doesn't exist)	IDL (optional)	Your math class
Real	real	float, double	Python does this for you	Octave makes all numbers	float, double	real numbers
Complex	complex		Python does this for you (using j as the imaginary number)	Octave does this for you (using either i or j as imaginary number)	(Need to use a function)	complex number (real and imaginary)
String	character (len=#)	char var_name(len)	Python does this for you	Octave does this for you	string	words

There are many other data types that you will learn as you become a more sophisticated programmer, but for now these are a good start. It is likely that for engineering and science programs a programmer will need to use matrices (or structures) with the numbers having a specific type like in the above table. For equations most programming languages allow you to write it just as you would in mathematics, however just solving one equation still is more like a calculator rather than a programming language. For more utility a programming language would allow you to loop, do conditional branches (if A do B or else do C), and input and output data. These type of statements are the program control statements. But first let us introduce operators and functions (which we have already used above but let's complete the discussion).

Computer Operators

Programming in any language requires operators (like + and -) to do useful functions. has three main control statements.

- Just like in mathematics there are operands and operators
- For most languages computer operators follow the same precedence as mathematics operators
- Below we will briefly look at three of the most common operators

¬ Arithmetic operators
¬ Relational operators
¬ Logic operators

Computer Functions

Programming in any language has functions like print or write.

¬ Functions
¬ Subroutines
¬ Script

Control Statements

Programming in any language has three main control statements.

¬ Loops
¬ Conditionals
¬ Input and output

Below is a script example of each of the major control statements[2]

- This is an example of a loop.

```
for i = 1: 10
        printf('This is a loop in Octave (or Scilab or Matlab)...counting numbers %i\n',i)
endfor
```

- This is the example of a conditional statement, however it does have input and output as well.

```
yn = input("Is engineering fun? (Y/N)","%s");
if (yn == "Y" || yn == "y")
        printf("Yes engineering is fun!!!");
else
        printf("Hmmmm...");
endif
```

- This is the example for input and output, however it does have a conditional statement as well.

```
snumber = input("The input statement allows us to input numbers or letters.  How many suns
if (snumber == 1)
        printf("That is correct.\n");
else
    printf("An astronomy course is recommended for you.\n");
endif
name = input("My name is ","%s");
printf("Hello %s, as you can see the printf statement is a form of output.\n",name);
```

So we have used Octave (as our example program) as a fancy calculator (which has its uses) but we have not used the true power of Octave. The true power of a programming language is to write a program (or function).

- This is an example of a function that has input and output, a conditional, and some loops. This is a different example of input and output where it uses a file (you are more likely to use files in "real world" work). Normally this program is saved in a file with an extension of *name_of_program.m* (or in this case *readwritefile.m*) and when you run it in octave you just run readwritefile(), but in this Libretext format we have the file a special area so you can see it, edit it (yes you can edit this - it won't be permanent) and run it.

```
%
% Reading and writing files ("advanced" input and output)
%
function nul = readwritefile()
%
%
num1 = [1,2,3];
num2 = [];
sznum1 = size(num1);
%
% Note there are many commands to write a file in octave including
% fprintf (which we use here), fputs, fwrite, dlmwrite, etc...
% this does not include octaves own method of
% saving matricies (for a much faster access) of just "save"...the better way to go for
% mathematical work...
%
fidw = fopen("testfile","w");
for i = 1:sznum1(2)
    fprintf(fidw,"%d\n",num1(i))
    disp("Wrote in testfile"), disp(num1(i))
end %for
fclose(fidw);
disp("File written and closed")
disp("Now we will read it and print the result")
%
% Note there are many commands to read a file in octave including
% fgets (which we use here), fread, fscanf, dlmread, importdata, etc....
% this does not include ocaves own method of
```

```
% reading its matricies (for much faster access) of just "load"
%
fidr = fopen("testfile","r");
test = fgets(fidr);
do
    rnumber = str2num(test);
    num2 = [num2,rnumber];
    test = fgets(fidr);
until(test == -1)
for i = 1:sznum1(2)
    disp(num2(i))
end %for
disp("This is the end - JM")
return
end

readwritefile()
```

There are multiple languages to write programs in but herein we will just detail five languages. In order to do this we will show examples of the same routine written in different languages, starting with Octave (because it can run on in this format and it is an excellent language for engineers).

Example of different languages

In this section we will compare five languages (Octave, GDL, Python, Fortran, and C++) using the same routine for calculating a root of a number $ans = \sqrt[n]{A}$. First the function as a simple Octave routine (this is not for comparison) as it should be written, but is not as good for an example.

```
% Function to figure out roots: A^(1/n)  x0 is initial guess;
% tol is the tolerance that we are willing to accept
% Example:  octave:2> root = nroot(20,3,2.2,1e-6)
%           root =  2.7144
%
function root=nroot(A,n,x0,tol)
x1=x0;
x2 = 1/n*[(n-1)*x0+A/(x0)^(n-1)];
while (abs(x2-x1) >= tol)
   x1=x2;
   x2 = 1/n*[(n-1)*x1+A/(x1)^(n-1)];
end %while
root=x2;
return  % root
end

root=nroot(20,3,2.2,1e-6)
```

Here we have a function that has all the inputs in the function itself, rather then asking the user to enter them one by one. In the next example we will ask questions rather than putting the inputs in the function. Doing this makes the program more interactive but it does not make it very reusable, so this is mostly for an academic exercise. If you were to use a function use the one above.

```
% Function to figure out roots: A^(1/n)
% Need a tolerance for how close you want the answer to be
% And an initial guess
%
% This is a demo program to show control structures in action while solving
% a math problem (the nth root of A). While most programing languages
% can do that without a program being written this is a method to examplify
% writing simple program.
%
function nul=control_root()
base = input("What is your base? ");
degree = input("What is the degree for the root you want? ");
```

```
tol = input("What tolerance do you wish to work towards? ");
guess = input("You will need an initial guess, what would it be? ");
answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
while (abs(answr-guess) >= tol)
   guess = answr;
   answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
end %while
printf("The answer is %g\n",answr);
return
end

control_root()
```

Note that the comments section (indicated by the %) are useful in Octave as that is where the help comes from.

```
octave:1> help control_root
'control_root' is a function from the file [redacted]/howtos/exampleprograms/
control_root.m

 Function to figure out roots: A^(1/n)
 Need a tolerance for how close you want the answer to be
 And an initial guess

 This is a demo program to show control structures in action while solving
 a math problem (the nth root of A). While most programing languages
 can do that without a program being written this is a method to examplify
 writing simple program.
```

To compare the programming languages to each other we will first present the program and how it runs, then compare the programs to either Octave or Fortran.

Octave

Welcome | control_root.pro | control_root.py | control_root.m

```
1  % Function to figure out roots: A^(1/n)
2  % Need a tolerance for how close you want the answer to be
3  % And an initial guess
4  %
5  % This is a demo program to show control structures in action while solving
6  % a math problem (the nth root of A). While most programing languages
7  % can do that without a program being written this is a method to examplify
8  % writing simple program.
9  %
10 function nul=control_root()
11 base = input("What is your base? ");
12 degree = input("What is the degree for the root you want? ");
13 tol = input("What tolerance do you wish to work towards? ");
14 guess = input("You will need an initial guess, what would it be? ");
15 answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
16 while (abs(answr-guess) >= tol)
17     guess = answr;
18     answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
19 end %while
20 printf("The answer is %g\n",answr);
21 return
22 end
23
24
```

```
GNU Octave, version 6.2.0
Copyright (C) 2021 The Octave Project Developers.
This is free software; see the source code for copying
There is ABSOLUTELY NO WARRANTY; not even for MERCHANTA
FITNESS FOR A PARTICULAR PURPOSE.  For details, type 'w

Octave was configured for "x86_64-apple-darwin20.3.0".

Additional information about Octave is available at htt

Please contribute if you find this software useful.
For more information, visit https://www.octave.org/get-

Read https://www.octave.org/bugs.html to learn how to s
For information about changes from previous versions, t

octave:1> control_root()
What is your base? 20
What is the degree for the root you want? 3
What tolerance do you wish to work towards? 1e-6
You will need an initial guess, what would it be? 2.2
The answer is 2.71442
octave:2> root=nroot(20,3,2.2,1e-6)
root = 2.7144
octave:3>
```

The program above is written in Octave, we have input and output, and a conditional and loop all in one (*while* statement).	This is the run of the program on the left. First Octa executed and then the program called control_root i Octave program.

Welcome | control_root.pro | control_root.py | control_root.m

```
% Function to figure out roots: A^(1/n)
% Need a tolerance for how close you want the answer to be
% And an initial guess
%
% This is a demo program to show control structures in action while solving
% a math problem (the nth root of A). While most programing languages
% can do that without a program being written this is a method to examplify
% writing simple program.
%
function nul=control_root()
base = input("What is your base? ");
degree = input("What is the degree for the root you want? ");
tol = input("What tolerance do you wish to work towards? ");
guess = input("You will need an initial guess, what would it be? ");
answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
while (abs(answr-guess) >= tol)
  guess = answr;
  answr = 1/degree*((degree-1)*guess+base/guess^(degree-1));
end %while
printf("The answer is %g\n",answr);
return
end
```

Comparing the Octave program here to the Fortran program on the right shows some differences but a number of similarities. Octave does not have data declarations and combines the *print* and *read* statements (into *input*). But the equation is the same (except for the power symbol) and there is a similar *while* statement.

control_root.pro | control_root.py | control_root.m | control_root.f

```
      PROGRAM CONTROL_ROOT
c
c Function to figure out roots: A^(1/n)
c Need a tolerance for how close you want the answer to be
c And an initial guess
c
c This is a demo program to show control structures in action while solving
c a math problem (the nth root of A). While most programing languages
c can do that without a program being written this is a method to examplify
c writing simple program.
c This is written in Fortran ---- use gfortran if you have install it
c
c

      double precision :: degree, base
      double precision :: tol
      double precision :: guess
      double precision :: answr
      print *, 'What is your base? '
      read *, base
      print *, 'What is the degree for the root you want? '
      read *, degree
      print *, 'What tolerance do you wish to work towards? '
      read *, tol
      print *, 'You will need an intial guess, what would it be? '
      read *, guess
      answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      do while (abs(answr-guess) >= tol)
        guess = answr;
        answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      enddo
      print *, 'The answer is '
      print *, answr
      stop
      end
```

Comparing Fortran to the Octave program on the le some differences, but is essentially the same. The d declarations being the biggest difference.

GDL (IDL)

Welcome | control_root.pro

```
; Function to figure out roots: A^(1/n)
; Need a tolerance for how close you want the answer to be
; And an initial guess
;
; This is a demo program to show control structures in action while solving
; a math problem (the nth root of A). While most programing languages
; can do that without a program being written this is a method to examplify
; writing simple program.
;
pro control_root
print,"What is your base? "
read,base
print,"What is the degree for the root you want? "
read,degree
print,"What tolerance do you wish to work towards? "
read,tol
print,"You will need an initial guess, what would it be? "
read,guess
answr = 1/degree*((degree-1)*guess+base/guess^(degree-1))
while (abs(answr-guess) GE tol) do begin
  guess = answr
  answr = 1/degree*((degree-1)*guess+base/guess^(degree-1))
endwhile
print,"The answer is ",answr
return
end
```

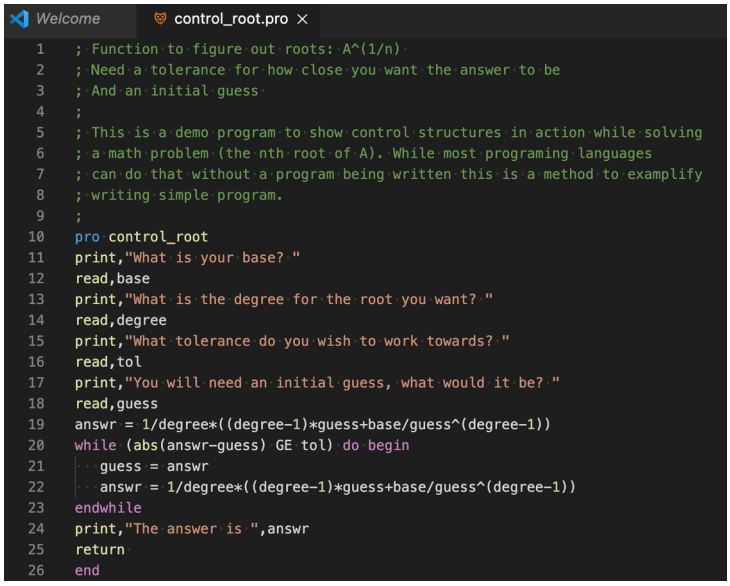
The program above is written in GDL, we have input and output, and a conditional and loop all in one (*while* statement)

```
  GDL - GNU Data Language, Version 0.9.9

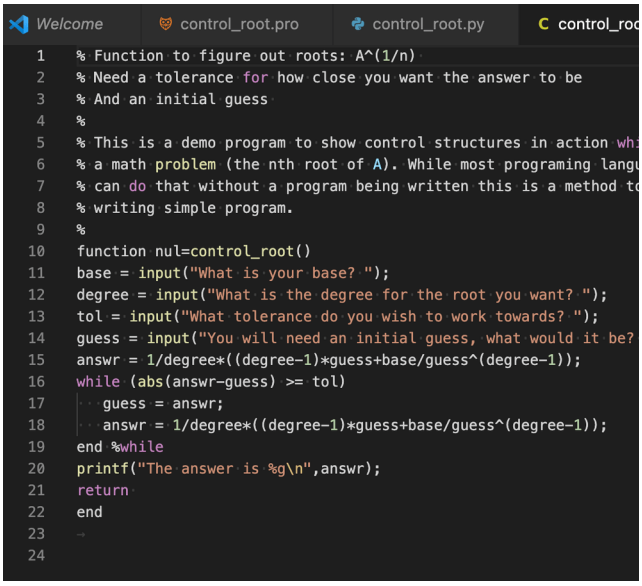
- For basic information type HELP,/INFO
- Default library routine search path used (GDL_PA
:
  +/opt/local/share/gnudatalanguage/lib
- No startup file read (GDL_STARTUP/IDL_STARTUP en
- Please report bugs, feature or help requests and
  https://github.com/gnudatalanguage/gdl

GDL> control_root
% Compiled module: CONTROL_ROOT.
What is your base?
: 20
What is the degree for the root you want?
: 3
What tolerance do you wish to work towards?
: 1e-6
You will need an initial guess, what would it be?
: 2.2
The answer is       2.71442
GDL>
```

This is the run on the program on the left. First GDL itself ha program called control_root is executed in the GDL program. that can run IDL programs. IDL is a common program at NA Washington DC area. PV-Wave similar to IDL but is primarily

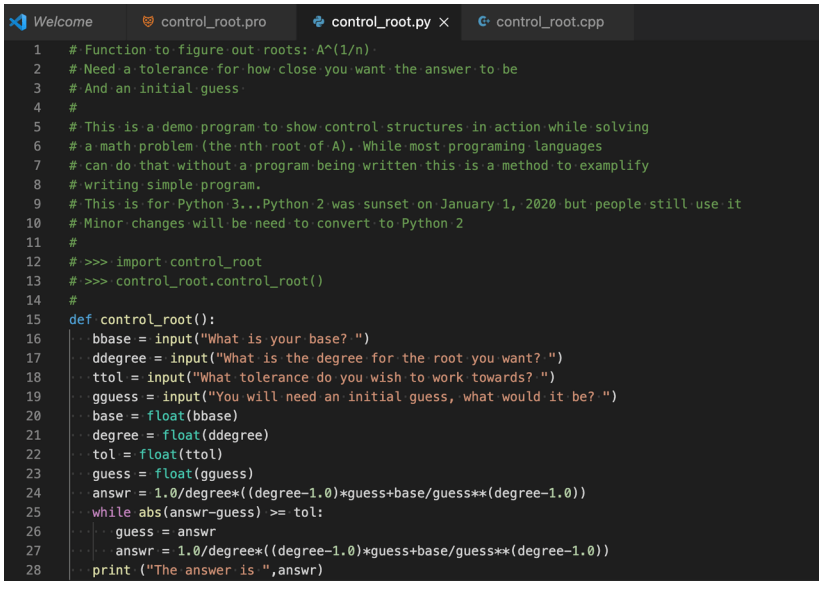


Comparing the GDL program here to an Octave program on the right shows some differences but a number of similarities. GDL is very much like Octave with similar function and motivations. Note there are no data declarations in either programming language because they are interpreted languages which in general declare on the fly. GDL does have some features that are reminiscent of Fortran (like the *print* and *read*).

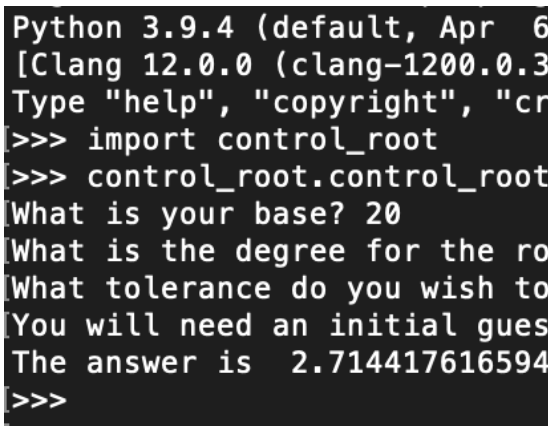


Comparing Octave to the GDL program on the left shows sor but many more similarities.

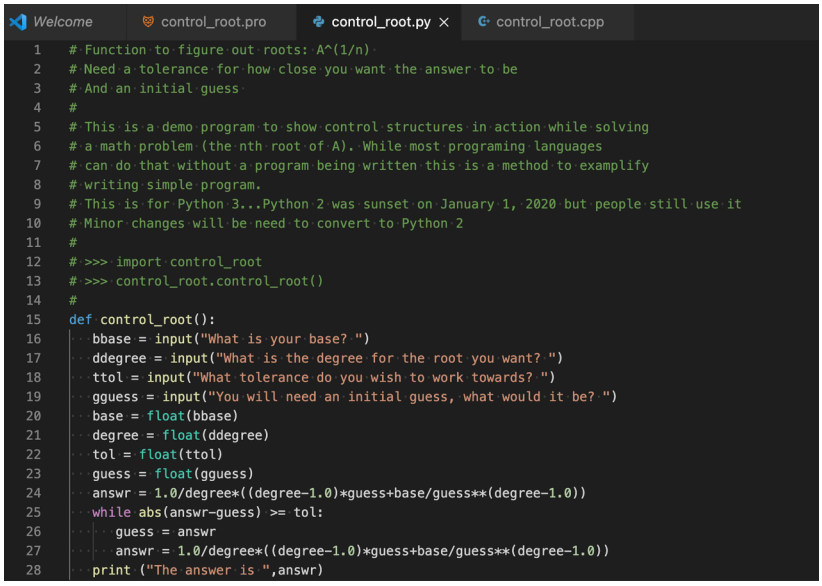
Python



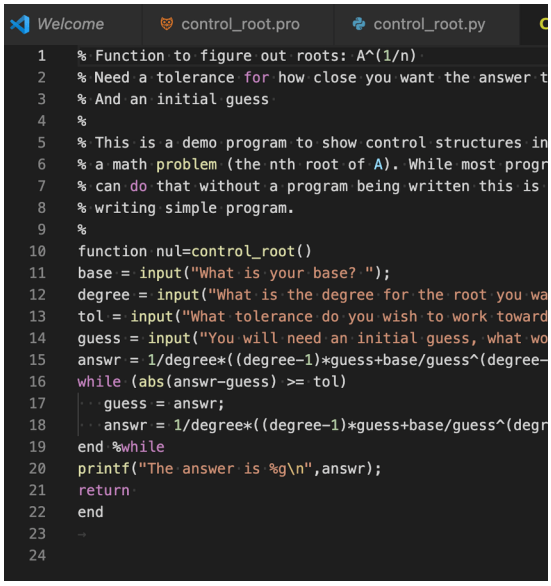
The program above is written in Python (Python 3 in this case), we have input and output, and a conditional and loop all in one (*while* statement). Because in Python *input* returns a string, the function *float* is used to convert it to a number. Most languages have commands to convert strings to numbers and numbers to strings. Please note how a program is written as most Python tutorials do not spend much time on the *def* command.



This is the run of the program on the left. First Pytho imported and then run. Python is a bit more complic

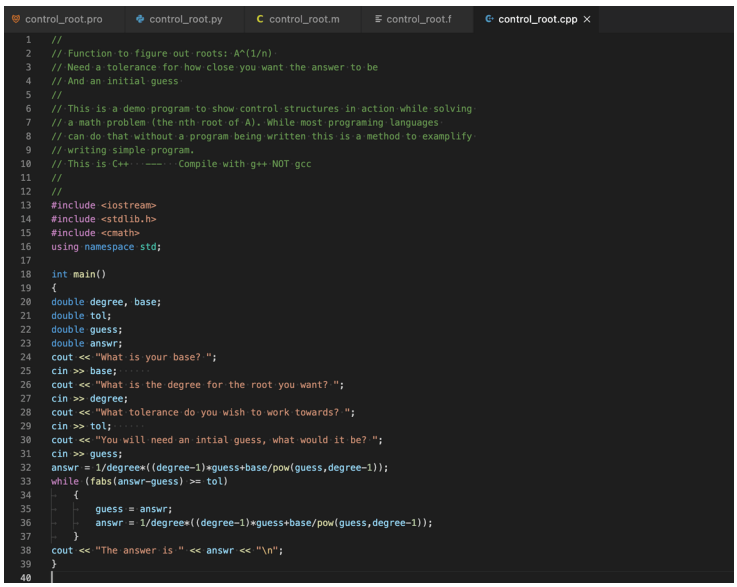


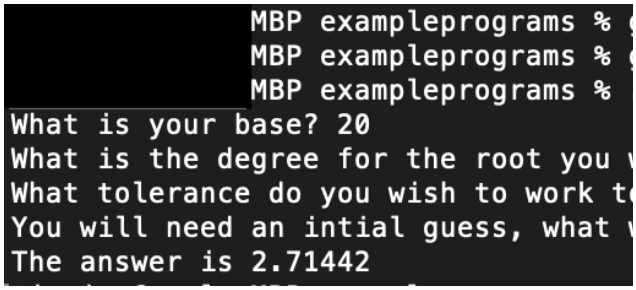
Comparing the Python program here to an Octave program on right shows a lot more similarities then differences. One big difference is the indenting of the lines which are required for Python. Python does not have *endwhile*, or ends of any kind, so the way Python tells where the end of a loop is is by the indentation.



Comparing Octave to the Python program on the lef similarities then differences. Here *input* is used in a of these programming languages. However, *input* in returns a number and for Python it is always a string commands). Octave's *input* can be made to produce additional parameter. Python is more natural for stri is more natural in the numerical world - hence the p MATLAB/Scilabs among engineers.

C++





The program above is written in C++ (not C which is different but very similar since C++ is a direct descendant of C), we have input and output, and a conditional and loop all in one (*while* statement). Note that C++ requires include statements to be able to do a lot of its functions like *fabs*, *cin*, and *cout*. You might also notice that this has data declarations like Fortran.

This is the compiling, linking, and running of the program or for C, not for C++. The executable now can be shared with a compiler. (Note the file ends in .cpp which is the convention ending carefully).

control_root.pro | control_root.py | control_root.m | control_root.f | control_root.cpp

```cpp
//
// Function to figure out roots: A^(1/n)
// Need a tolerance for how close you want the answer to be
// And an initial guess
//
// This is a demo program to show control structures in action while solving
// a math problem (the nth root of A). While most programing languages
// can do that without a program being written this is a method to examplify
// writing simple program.
// This is C++ ---- Compile with g++ NOT gcc
//
//
#include <iostream>
#include <stdlib.h>
#include <cmath>
using namespace std;

int main()
{
double degree, base;
double tol;
double guess;
double answr;
cout << "What is your base? ";
cin >> base;
cout << "What is the degree for the root you want? ";
cin >> degree;
cout << "What tolerance do you wish to work towards? ";
cin >> tol;
cout << "You will need an intial guess, what would it be? ";
cin >> guess;
answr = 1/degree*((degree-1)*guess+base/pow(guess,degree-1));
while (fabs(answr-guess) >= tol)
    {
        guess = answr;
        answr = 1/degree*((degree-1)*guess+base/pow(guess,degree-1));
    }
cout << "The answer is " << answr << "\n";
}
```

Comparing the C++ program here to the Fortran program on the right shows more similarities then they did to the interpretive language. C++ has data declarations like Fortran and does not have a combined *print* and *read* statement like Fortran.

control_root.pro | control_root.py | control_root.m | control_root.f | control_root.cpp

```fortran
      PROGRAM CONTROL_ROOT
c
c Function to figure out roots: A^(1/n)
c Need a tolerance for how close you want the answer to be
c And an initial guess
c
c This is a demo program to show control structures in action while solving
c a math problem (the nth root of A). While most programing languages
c can do that without a program being written this is a method to examplify
c writing simple program.
c This is written in Fortran ---- use gfortran if you have install it
c
c

      double precision :: degree, base
      double precision :: tol
      double precision :: guess
      double precision :: answr
      print *, 'What is your base? '
      read *, base
      print *, 'What is the degree for the root you want? '
      read *, degree
      print *, 'What tolerance do you wish to work towards? '
      read *, tol
      print *, 'You will need an intial guess, what would it be? '
      read *, guess
      answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      do while (abs(answr-guess) >= tol)
          guess = answr;
          answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      enddo
      print *, 'The answer is '
      print *, answr
      stop
      end
```

Comparing Fortran to the C++ program on the left has some differences, but is essentially the same.

Fortran

Fortran has already been compared to C++ and Octave so we will not do that here. We will just show the program and the run.

control_root.pro | control_root.py | control_root.m | control_root.f | control_root.cpp

```fortran
      PROGRAM CONTROL_ROOT
c
c Function to figure out roots: A^(1/n)
c Need a tolerance for how close you want the answer to be
c And an initial guess
c
c This is a demo program to show control structures in action while solving
c a math problem (the nth root of A). While most programing languages
c can do that without a program being written this is a method to examplify
c writing simple program.
c This is written in Fortran ---- use gfortran if you have install it
c
c

      double precision :: degree, base
      double precision :: tol
      double precision :: guess
      double precision :: answr
      print *, 'What is your base? '
      read *, base
      print *, 'What is the degree for the root you want? '
      read *, degree
      print *, 'What tolerance do you wish to work towards? '
      read *, tol
      print *, 'You will need an intial guess, what would it be? '
      read *, guess
      answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      do while (abs(answr-guess) >= tol)
          guess = answr;
          answr = 1/degree*((degree-1)*guess+base/guess**(degree-1))
      enddo
      print *, 'The answer is '
      print *, answr
      stop
      end
```

```
[redacted]-MBP exampleprograms % gfortr
[redacted]-MBP exampleprograms % gfortr
[redacted]-MBP exampleprograms % ./cont
 What is your base?
20
 What is the degree for the root you want?
3
 What tolerance do you wish to work toward
1e-6
 You will need an intial guess, what would
2.2
 The answer is
   2.7144176165949063
```

The program above is written in Fortran (which is the same as used in the comparisons), we have input and output, and a conditional and loop all in one (*while* statement). Note that Fortran does not include *include* statements like C and C++. A number of Fortran's packages are presumed in linking. This makes Fortran easier for scientists and engineers and one of many reasons Fortran never seems to go away. Modern Fortran (2008 and beyond) does have a command that is similar to the C/C++ *include* called *use* which is associated with a *module*.	This is the compiling, linking, and running of the program on the left. shared with anyone who has a similar operating system without the u

Comparison of programming languages (morale of previous section)

- Most programming languages are very similar in looks
  - Syntax is the major difference with each programming language having its own unique syntax
  - Program flow is essentially the same in all the five languages introduced above
    - This is true for most languages
    - LISP and LabVIEW are notable exceptions
- The main difference above beyond the syntax is the difference between the interpreted languages and the compiled languages
  - Interpreted languages like MATLAB/Octave/Scilab, Python, and IDL/GDL require another program to run before you can execute your functions
    - That is you cannot execute a function you have made in this languages without running the MATLAB/Octave/Scilab, Python, or IDL/GDL main program
    - Some of these programs do have a way of making a compilable executable but then this makes them a compiled language...
  - Compiled languages like C++ and Fortran do not require another program to execute on any similar machine (with the same OS)
    - Of course you need to compile it first so would need gcc or gfortran (or a similar compiler) before you send your executable to someone else
    - The advantage of the compiled executable is you can send your program to people who do not have a compiler

End of chapter statement

The conclusion to this section is that you can learn to program in any language as long as you have the basic ideas of programming. Hence programming in any language is in your grasp. *So you might ask why should I learn how to program in any programming language when company XYZ says they only use Python?* Company XYZ might not keep you on forever so what happens when you go to company ABC and they use Julia? Knowing how to program is like learning how to fish. Using the modern proverb originating from Anne Isabella Thackeray Ritchie →If you give a person a fish, the person eats for a day; If you teach a person to fish, the person eats for a lifetime. Using the same philosophy, we can conclude that it is better to learn ***programming*** not learn a specific programming language. At least one author of this text has seen literally hundreds of new programming languages appear since University (of course, Fortran has been there and is still there...so if you want one language, go for that).

In the following chapter we will develop a program from a simple physics problem in order to demonstrate the process of writing a program for engineering (or science). It is limited in scope so as to gently introduce the concept however it will take effort on the students part...

---

[1]Interactive Data Language (IDL) is primarily a local language for PG Community College because of its proximity to NASA/GSFC, other colleges or universities might want to substitute a different regional language. The other languages are global and relatively popular.

[2]While scripts are good for educational purposes and quick calculations, they are not full fledged program which is the final goal. See parachute person for an example of a full fledged (albeit simple) program

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10: Parachute Person

Parachute person is an example of a problem relevant to engineers and scientists that can be made fairly simple for beginning programmers. This problem can be analytically solved using differential equations, however, there are many problems that **cannot** be solved analytically, but **can** be solved on a computer using an approximation method (a simple numerical method which in this case makes an approximation of a differential equation).

So why write a program if it can be solved analytically? Because this program is simpler than an intractable problem (that you would more likely write a program for) and thus will be easier to grasp for a beginning programmer. *Will real world problems be as simple?* No. This example can be expanded to intractable real world problems, but a good understanding of numerical methods is necessary to truly do this sort of work (therefore it is wise for a student to take a numerical methods course or a signals and systems course that includes numerical methods).

First we discuss the physics and show how to design a computer program from a "word" problem. Then we show the working program and the results we can get.

So lets start with the problem.

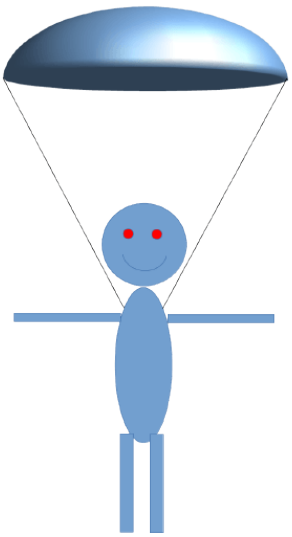
A person has jumped out of a plane with a parachute, what is the velocity as the person reaches the ground?

As with all programming an initial program is made and then new versions enhance the initial program. Assumptions can systematically be removed or refined for new versions, but for now we will do the initial program. Because this is such a general problem the instructor has obviously :) given us liberty to make some *reasonable* assumptions. So let us list some assumptions.

- The ground is sufficiently far enough that the parachutist doesn't hit the ground before reaching terminal velocity
- The parachutist opens the parachute as soon as he jumps out of the plane (this of course would be one of the first assumptions to be revisited)
- The drag constant will include drag coefficient, parachute area size, design of parachute, and density of the air (more assumptions that would eventually be revisited)
- Weight of parachute is negligible
- Wind will be assumed to be zero in all directions (this assumption might be difficult to revisit, but give it a shot when you are more expert at programming)

So this is a physics problem at its heart. Since the parachutist is moving we recognize this as a dynamic problem.

- First step in a dynamic physics problem is to set up your force diagram.
- Before you draw your force diagram, draw your coordinate directions (which way does "x" go....which way does "y" go)
- Use the parachutist below to draw your forces.



¬ Parachute dude's force diagram (with coordinate system)
¬ Forces defined
¬ Force equation
¬ Force equation redefined...this is where it moves from physics to computer science
¬ Vector array, say what?

- Here is one version of parachute person
- The students should try to write this in a number of other languages (Fortran, C++, IDL, and Python)
- The student should try to improve upon this program (a number of real world concepts have been sacrificed here for simplicity)

```
%
% Function parachute person
% Syntax: [time,velocity] = paraperson(0,50,0,57.2,200,9.8,0.1)
% Author: Scott D. Johnson as an example to understanding MATLAB/octave
%
% Inputs:
%
% ti is initial time
% tf is final time
% vi is initial velocity
% c is coefficient of drag of the person and parachute combined (this is an equation in it
% m is mass of person and parachute
% g is gravity
% tsteps is the size of the time increments
%
% Outputs:
%
% t is a time vector
% v is a velocity vector (terminal velocity)
%
% Note:  This is an extremely simplified physics of skydiving problem...why?  So the stude
%            the more complex version.
%
%-------------------------------------------------------------------------------------
%
function [t,v]=paraperson(ti,tf,vi,c,m,g,tsteps)
numsteps = (tf-ti)/tsteps;
i=1;
v(i)=vi;
t(i)=ti;
```

```
for i = 1:numsteps-1
   v(i+1)=v(i) + [g-c*v(i)^2/m]*tsteps;
   t(i+1)=t(i) + tsteps;
end %for
return
end%
%
% This is the run of the function
%
[time,vel]=paraperson(0,50,0,57.2,200,9.8,0.1);
%
% Plotting the data
%
plot(time,vel,';Parachute Person;');
xlabel('Time in seconds');
ylabel('Velocity in m/s');
title('Velocity of person with a parachute jumping out of a plane')
axis([0,20,0,8])
```

Parachute person as presented here becomes more complex as we remove the simplified assumptions. Final solutions (as in the program, not the "answer") would involve fluid dynamics and would likely be a system of equations (usually expressed in matrix form).

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11: HISTORICAL CASE STUDIES IN ENGINEERING

In order to understand engineering it is helpful to follow some historical case studies. These are NOT complete studies, that is for historians. This is a study specifically oriented to a moral of the story approach when it comes to *engineering*. It is for the student to fill in details.

This course will look at three case studies out of literally thousands of case studies that could be written. The cases studies presented here should not be viewed as the end of a students look at engineering in a historical context rather as a beginning. To further those ends after the studies presented here the student might wish to look at the books by engineer and author Henry Petroski. He has written a number of books that looks into the history of engineering and design. A list of his books can be instructive on what is engineering is about. His books are very interesting and fun to read, but you do not have to limit yourself to those books; you can do your own research and see engineering design from your own point of view. A quick look at the list of books below clearly indicates that he is a civil engineer. Even so there are lessons for all engineers in books from different engineering disciplines.

This is a selection of his books, it is not complete, but essentially captures the overall themes of all his books:

- The Pencil: The history of design and circumstances
  - This book is actually about the pencil - the whole book!
  - How long? 448 pages long!
  - This is engineering, you need to design it to not break, etc.
  - Materials matter
- The Toothpick: Technology and Culture
  - This book is actually about the toothpick - the whole book!
  - *How long?* 464 pages long!
  - *Fascinating?* Yes!
  - *Are there trade secrets in the toothpick industry?* Yes, really!
  - Toothpicks have to be strong but flexible and NOT damage the teeth
    - This is engineering, you need to design for this
    - Materials matter
- The Evolution of Useful Things: How Everyday Artifacts - From Forks and Pins to Paper Clips and Zippers - Came to be as they are
- Engineers of Dreams: Great Bridge Builders and the Spanning of America
- Remaking the World: Adventures in Engineering
- Pushing the Limits: New Adventures in Engineering
  - This has a chapter on Washington DC's Wilson bridge (the original span which is now demolished)
- To Engineer is Human: The Role of Failure in Successful Design
  - Disasters teach us how to build better
  - Most disasters are not intentional but a missing fact about design
  - Engineers need to take lessons from this

What is interesting in this selection of his books is the idea that design (engineering) is part of the seemingly **simplest of things**: the pencil, the toothpick, the paper clip, and more.

Engineering is about the great **and small** designs, it is about success **and failure**, it is about science **and art**, it is about exact mathematics and **approximations**,...

The histories in the following sections talk about the great designs, mainly to dispel misconceptions, rather than the small design but the student is certainly encouraged to read about the small designs.

11.1: First Flight

11.2: The pyramids of Egypt

11.3: Trains, trains, trains

11.4: What are the lessons learned?

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11.1: First Flight

History of Engineering: Case History of Flight

A case history of flight for the purpose of teaching an Engineering lesson
by Dr. Scott Johnson

Invention of the Aeroplane

The invention of the airplane was accomplished by the accumulation of knowledge, testing, and creative engineering of a ***many*** individuals and teams. To first understand the invention of the airplane, what an airplane is needs to be clearly understood (note this is a general theme in engineering and science - know what you are talking about first). In general we refer to an airplane as a *heavier than air powered flying machine*. Other types of airplanes would be *lighter than air flying machines* such as balloons and dirigibles, *unpowered flying machines* such as gliders, and *minimally powered flying machines* such as the modern demoiselles (single-person very light airplanes powered by human motion).

Lighter than air flying machines depended on balloons for lift. Balloons were first developed by competing groups, the Montgolfier brothers (paper mill owners) and the Paris Academy of Sciences. The Montgolfier brother's balloon used an open flame hot air balloon design while the Paris Academy of Science's lead physicist-engineer Jacques Alexandre Cesar Charles' balloon used hydrogen gas. Interestingly, the Montgolfier's theory of why their balloon flew was mostly wrong, even a little comical given our modern understanding. This theme of successful design yet lack of understanding of why the design is a success repeats itself in engineering.

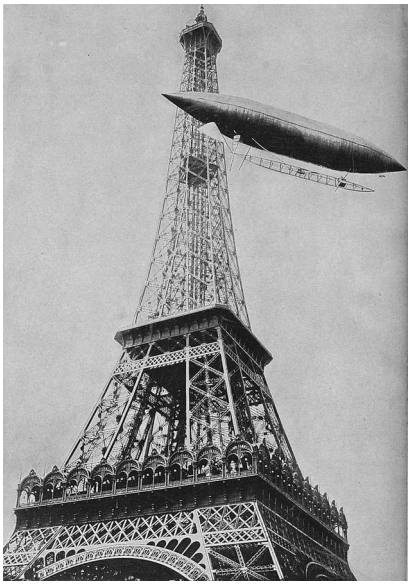
For a small period of time balloons, charlières and montgolfières, as a method of transportation was very popular. However the fashion faded relatively fast given that there was little practical purposes to them. Eventually powered balloons would be developed but control was still an issue. For any reasonable purpose the flying machine must go from one place to another in a controllable manner.

Alberto Santos-Dumont

The Brazilian Alberto Santos-Dumont developed the first true lighter than air powered flying machine giving his cigar-shaped dirigibles a highly effective control system with his rudder and elevator. He rounded the Eiffel tower in dirigible No. 6 on October 19, 1901 after a 30 minute controlled flight (which he won money for and gave the money, which he did not need because he was rich, to the poor and underpaid staff). Santos-Dumont flew the first true-controllable airplane inspiring many others to continue their work in heavier than air flight. Alberto Santos-Dumont is considered the inventor of the airplane in some countries, this has lead to heavy and heated debate. All unnecessary because real invention is a team effort.



Caricature of Santos-Dumont from *Vanity Fair* (British Magazine; defunct), 1901. Public domain from Wikipedia: Alberto Santos-Dumont.



Flying around Eiffel tower in dirigible No. 5 (which crashed latter on in the trip). This is often mistaked for dirigible No. 6 (the successful flight). Public Domain.

Of crucial design in this airplane was the **automobile**-like engine which Santos-Dumont had to redesign to allow for air flight with the any possible inclinations. Engine design was crucial to all successful powered airplane flights whether heavier or lighter than air. Many of these designs could only have been manufactured at the time airplanes started to take the skies. While this is not a history of the automobile, it is clear that the invention (1700s) and continual improvement of the automobile (combustion engine early 1800s) was required before powered flight.

Sir George Cayley and Otto Lilienthal

Heavier than air machines unlike lighter than air machines needed to achieve lift using some form of wing like a bird. Studies of birds were used to design the first wings. Detailed work by Sir George Cayley of England, lead to the first flight of man, possibly one of Cayley's servant, in a heavier than air unpowered flying machine in 1853.

His work lead to the design of Otto Lilienthal's glider that formed the basis of Samuel Pierpont Langley's, the Wright brother's, and the Aerial Experiment Association's airplanes. With the glider's development and aerial engines that were used in lighter than air flying machines, the only problem left to solve was the stability problem that heavier than air machines all seem to have. Cayley had made some progress using a rudder to enhance stability, but for powered machines the rudder was not sufficient.

Samuel Langley

The head of the Smithsonian Institute an astronomer named Samuel Langley, who had experiment with heavier than air flying machines since 1889, developed a rudimentary aileron called a dihedral wing to alleviate the stability problem. While some worked in secret to develop their airplane, Langley was widely published and undoubtedly instrumental in other designers development of the airplane. Langley's model planes (not toys, prototypes) flew very well, but Langley's manned planes never flew and he was ridiculed for his failures with statements indicating that humans would never fly just all the government money spent on flight. Note, however, his work which included confirmation of previous work before him made possible the continued progress towards a sustainable useful heavier than air powered flight.

The Wright brother's developed the warp wing idea to alleviate the stability problem. This worked well but was very difficult to control and their planes were very dangerous to fly. Wilbur Wright seemed to have the uncanny ability to control their planes and flew from a rail the first heavier than air powered flight on December 17, 1903. However this was not like any plane we know today.

It should be noted that in July 1901 the first heavier-than-air helicopter was briefly flown with two passengers on board however there is no evidence it was controllable like Santos-Dumond's dirigible No. 5 and 6 of the same time period. Interestingly, Santos-Dumond had a design of a helicopter, the Wright brothers were inspired by a toy helicopter, and Leonardo da Vinci had his own design of a helicopter along with his "aeroplane." Helicopters are another fascinating story, but are not the topic of this essay, however it might be of interest for the student to write an essay on this topic (i.e. The history of the helicopter).

Bell's AEA

It was up to the Aerial Experiment Association (AEA) whose members consisted of Glenn Curtiss, Alexander Graham Bell, Frederick Baldwin, Thomas E. Selfridge, and Douglas McCurdy to fly the first plane as we understand it today. They solved the stability problem by using ailerons which all modern planes use in some fashion. Glenn Curtiss flew their first successful plane, the June Bug, in 1908 and continued developing innovations that enabled him to built a successful business that would eventually absorb the Wright brother's business. This plane, unlike the Wright brothers' plane, took off using wheels like modern planes. The Wright brothers' plane took off using a rail and catapult system (like Langley which possibly contributed to his failures), though interestingly some French pilots licensed by the Wright brothers did fit wheels to the aircraft. The AEA planes (the most famous being the June Bug and Silver Dart) had superior stability over the Wright brothers' planes and other advantages, such as wheels for take-off and landing and a perfected engine (which would be expected since Curtiss' company originally centered around engine design for his "motorcycle").

With their successful flights and the death of one of their members in a Wright brothers' plane, the AEA disbanded though Glenn Curtiss continued to produce innovative airplanes, using the designs of the AEA, through the first licensed US aircraft manufacturer, the Curtiss Company. In 1910 Glenn Curtiss broke the Wright brothers' distance record by flying from Albany, New York to New York City along the Hudson river, 150 miles, proving to the US the viability of airplanes and establishing his company as well. Eventually in the 1920s the Curtiss company merged with the Wright brothers company forming the Curtiss-Wright Company (note Wright company became a division). The Curtiss (and Curtiss-Wright) company produced some of the finest aircrafts until superseded by other companies that had new innovation such as Douglas (DC-3) and Lockheed (U-2). The large airplanes we fly today are not based off the original airplanes described here, they are another history of flight story.

Unfortunately, lawsuits between the Wright brothers and Glenn Curtiss stifled progress on airplanes in the United States for years until the government demanded an end to the petty bickering due to the war effort at the time (World War I). Airplanes were crucial to the war effort and Europe had made considerable strides in this technology while the United States was embroiled in legal limbo. The lawsuits contented that warped wings and ailerons were the same idea. Each were initially patented separately, but the courts disputed this with no clear resolution.

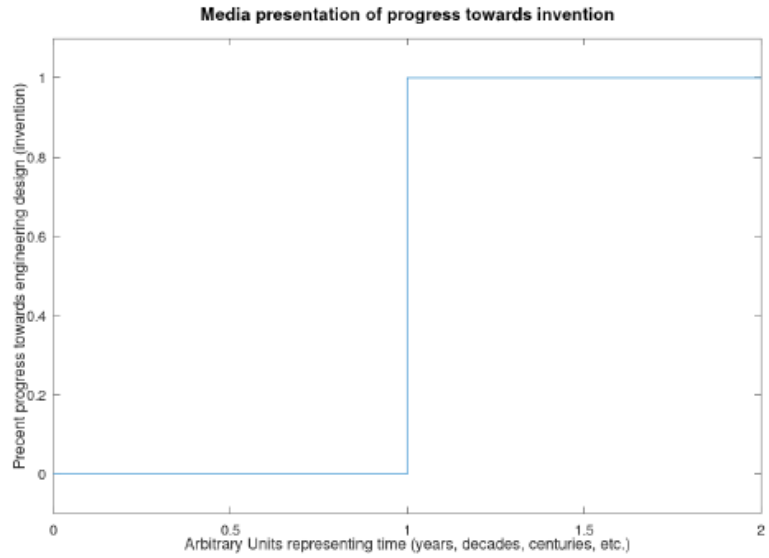
Today it would be hard to make the contention that the two technologies are the same as the ideas are sufficiently different as to be considered separate distinct ideas. However, patent laws were substantially different 100 years ago (even a vague idea could be patented). We might consider it fortunate that the government stepped in to allow progress of the airplane to continue despite the patent laws.

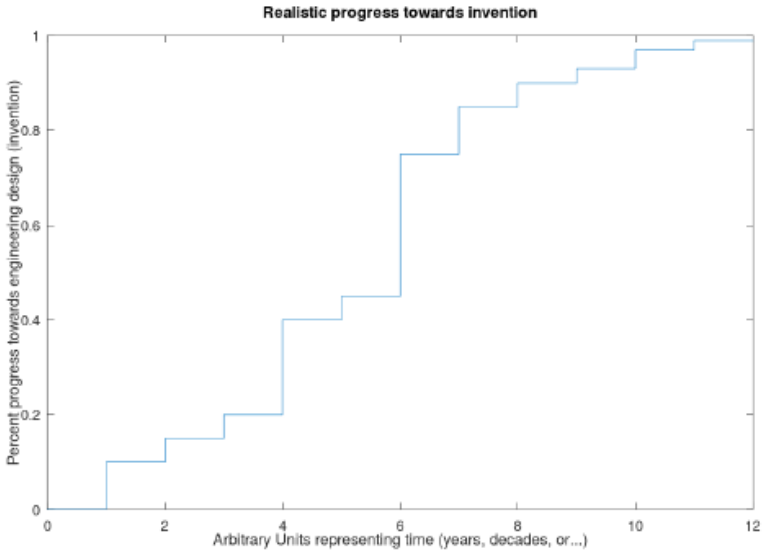
Lessons Learned

What are the lessons learned? Discuss in class or with friends.

- **Wright brothers or Santos-Dumont (Brazil) invented the airplane, right?**
  The invention of the airplane is really an accumulation of the work of many people with both the Wright brothers and Santos-Dumont adding major contributions. The idea of one group or one person inventing something is more motived by national pride rather than an actual assessment. It is much better to tell the true story of design, science, and engineering rather than the "all were fools until this person (or persons) came along." The later approach is more of a politician, media or filmmaker approach in order to have a "hero."

- **Engineering designs (inventions) are accomplished by a few and not the many?**
  No. Engineering designs (inventions) are more a product of many people over multiple generations. This is true for science as well. All of Einstein's theories and equations were build upon works by other scientists including Heisenberg, Bohr, Planck, Lorentz, Poincaré, Michelson, Morley, Riemann, Grossmann, Noether, Minkowski (one of Einstein's math teacher), and more. Check out "Too Big for a Single Mind" (see references below) for the science view. This is not to say that there is no ingenious ideas involved in discovery and invention; it is to say it is a group effort.





¬ Anyone can be an engineer look at the uneducated poor like Montgolfier brothers and the Wright brothers.

Let get this straight, these were NOT poor people from the streets. They were rich or very well-off middle class. (Santos-Dumont can also be classified as rich). The Montgolfier brothers owned a very successful paper company with major inventions in that field (including very very thin paper). The Wright brothers owned a bicycle shop (don't think of today, think of back when bicycles were all over the place and an important mode of transportation) and were well off enough to leave other people in charge while they went gallivanting off playing with their aeroplanes.

As for education, the Wright brothers finished high school and then started a successful businesses that funded their interests in flying. They read the published works of Leonardo De Vinci, Cayley, Lilienthal, Langley, Chanute, and more. It is clear their self motivated studies are equivalent to an engineering degree (it's not impossible to do this, just difficult, and unless you ***fund*** yourself it is unlikely you will be taken seriously - even if you are a genius). As for the Montgolfier brothers, paper making is a high technology engineering field (in the past and in the present) and the Montgolfier brothers had the education to continue to lead this business successfully.

It is clear that none of the inventors/scientists/engineers lacked education (whether formal or self-aquired) or financial resources. **Funding** is crucial in science and engineering and stories of people with lack of funding producing major engineering/science advances are more for political reasons (to justify not funding science and technology) rather than real world reasons. It should be also noted that self-acquired education requires funding as well (for both time off to study and resources like books, etc.).

¬ Innovation is spurred by patents OR innovation is hindered by patents?

This is an interesting questions and there are multiple point of views. The prevailing point of view is that patents spur innovation by protecting a businesses interests but this is a business point of view with a lack of understanding of innovation. *What is the science point of view (well since there are no patents in science I guess that is the wrong question, eh?)? Does science fail to have innovation because it is not patentable? What is the engineering point of view?*

For a counter point of view, let us first note that Santos Dumont did not patent anything he did and expected the world to share his ideas. His most famous invention, the demoiselles, was publicly published in many magazines and was free for anyone to build (even the general public, gasp!). The idea behind patents is that it allows business to recover their initial expense at developing an invention. This idea rest on the dubious proposition that inventions are a single person feat which is only true for small items (like ailerons but not airplanes) and even that is a bit dubious. Glen Curtiss was also known for not patenting most of his inventions having adopted the principle that business should keep on innovating and the only thing that is ever patented is inventions that are from the past, not the future. *Now the answer to the question is up to you.*

¬ The study of birds in order to understand flight was pioneered by the Wright brothers?

Not at all. From Leonardo da Vinci's "*Codex on the Flight of Birds*" to Cayley's study of birds and Otto Lilienthal's "*Birdflight As the Basis of Aviation*" there are many examples of early engineers/scientists studying birds in order to learn how to fly.

Suggested books

1. *Wings of Madness: Alberto Santos-Dumont and the Invention of Flight*, Paul Hoffman, Theia, New York (2003)
2. *Progress in Flying Machines*, Octave Chanute, Dover Publications, New York (1997) -- Originally published in 1894
3. *Unlocking the Sky*, Seth Shulman, Perennial (HarperCollins), New York (2002)
4. *Rocket Man*, David A. Clary, Theia, New York (2003)
5. *How We Invented the Airplane*, Orville Wright, Dover Publications, New York(1988) -- Note that most books and web sites on the Wright brothers' planes are derived from this book, and are largely unoriginal with no independent research - it is best just to read the original and read other works not centered on the Wright brothers (i.e. do your own research). Almost all Wikipedia page suffers from this as well (and extends to discrediting others work in the history of flight). Do real research. Wikipedia is not very good for topics that are politically charged (this includes science and engineering). Non-politically charged subjects are usually done very well by Wikipedia however.
6. *Too Big for a Single Mind: How the Greatest Generation of Physicists Uncovered the Quantum World*, Tobias Hürter, The Experiment LLC, New York (2022) -- this is more about science but follows along the same lines as above -- many not one (or a few) are involved with discovery and invention. This book was reference above.

Some important dates

1. *Unmanned Langley Aerodrome model number 5 flies for about 0.5 miles - May 6, 1896: Samuel Langley and Charles Manly (powerful light weight motor)*
2. *Flight Around Eiffel Tower - October 19, 1901: Santos-Dumont (rudder, elevator, and airplane motor). Dirigible No 6, though technically he rounded the Eiffel Tower on July 13, 1901 with dirigible No. 5 but that one crashed on its way back and is not considered a success.*
3. *"Flying Car" lands at cafe for breakfast - July 23, 1903: Santos-Dumont, Dirigible No. 9.*
4. *Full scale Aerodrome fails to fly - October 7 and December 8, 1903: Samuel Langley and Charles Manly (airplane motor 4 times more powerful than the engine the Wright brothers used just a few days later)*
5. *First flight heavier than air machine: **Wright Flyer** - December 17, 1903: Wright Brothers (rudder, elevator, and warp-wing; while the Wright brothers engine was less powerful then other engines it did have one significant innovation - the first use of aluminum for part of the system)...the documentation on this flight is mostly the Wright Brothers documentation*
6. *First **public** flight of heavier than air machine: **14-bis** (bird of prey) - October 23, 1906: Santos-Dumont* - This was likely less stable then the Wright Brothers (this had *ailerons*, but the wings were problematic though it could be said that most of the early wings of all the airplane inventors were problematic)
7. *First **public** filming of a heavier than air machine - November 12, 1906: Santos-Dumont*
8. *First **public** flight of heavier than air machine in North America - March 12, 1908: Aerial Experiment Association*

9. *First flight of air machine to win first Aeronautical prize in the United States: the **June Bug** - May 21, 1908: Aerial Experiment Association* - This was stable and worked well. (This had ailerons with good wings).
10. *First flight of the modified Langley Aerodrome - 1914: Glenn Curtiss* - There is a debate on the extent of the modifications (whether "modern" material was used or not) with each side clearly being heavily biased (Numerous items of this plane would have worked - especially the engine). Unfortunately a clear history at this point is impossible to determine because of the partisan nature of this debate. This is a case where you will have to figure it out on your own by reading and studying a lot. If you are unwilling to do this, then it is probably best that you stay on the sidelines.

Notes

1. *Working with Samuel Langley was Charles Manly who produced the most powerful airplane light weight airplane engines for the Aerodromes.*
2. *Ailerons were patented (and probably invented by...but not sure) by Matthew Boulton in 1868 from a paper the British scientist wrote in 1864. He clearly recognized this device as a lateral control device as did the patent office that issued him the patent. At the time these were referred to as rudders along with what we now call rudders. The patent details what we call ailerons, not rudders however. The first "modern" aileron was probably designed by Henri Farman and both the Curtiss corporation and the Wright corporation were using his designs by about 1915.*
3. *Wing warping was known before the Wright brothers use of it, but it was not believed to be primarily used for lateral control, but instead a breaking method to allow for turning (which can be done with rudders only). It is believed the Wright brothers first recognized its use as a primary control device when used in conjunction with the rudder and they successfully implemented the idea. It should be noted that the patent office probably should not have given a patent for this device as it was already known, however patents is less of a science and more of a government sanctioned business method of control even today.*

Items not invented by the Wright brothers, but erroneously credit to them by others (the Wright brothers never claimed they invented these items):

1. *Wind tunnel - invented in 1871 by Francis Wenham. The idea was developed from the "whirling arm" invented by Benjamin Robins and used by Sir George Cayley for his "wind tunnel" experiments*
2. *Air-screw propeller - The idea is that the twisted shape of the propeller is due to the Wright brothers, but it is not. Many early airplane pioneers used this propeller shape including Samuel Langley and Alberto Santos Dumont. The idea to make these cambered type propellers which lead to the better designs of Samuel Langley, Alberto Santos Dumont, and the Wright brothers was pioneered by George Cayley; there is no credit of invention on this, just continual improvement.*
3. *Yaw-pitch-roll control method of flying - Not really an invention, so this is baffling why this is called an invention by some. Everyone knew you needed to control yaw, pitch, and roll and all early 'airplanes' had designs to control the yaw, pitch, and roll with most of them being ineffective (including the Wright brother's implementation of wing warping). This is usually an invention credited to the Wright brothers in order to overinflate their accomplishments (which were important, but was continual improvement over the past workers in the field).*
4. *Planform of the airplane - Not really an invention, so this is baffling why this is called an invention by some. There were many different yet effective planforms from the beginning of airplane design. Leonardo da Vinci (maybe even earlier) was a pioneer in this (from studying birds and bats; just like Cayley and Lilienthal). Note: Santos Dumont was known to be influenced by Leonardo da Vinci's ideas and drawings (check them out for yourself).*

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11.2: The pyramids of Egypt

History of Engineering: Building the Pyramids

One of the greatest feats of ancient engineering was the building of the great pyramids. While the pyramid structure is found in many cultures (Mesopotamia's ziggurats, Mesoamerica's pyramid, North America's pyramid-shaped mounds, China's flat-top pyramid, etc.), the great pyramids of Egypt captivate the imagination more than any other. The great pyramids of Egypt are over 145 meters tall dwarfing the Statue of Liberty in height and just a bit smaller than the Washington Monument which is 169 meters tall. Their base length is over 230 meters and they are estimated to have more than two million blocks of stone with the average weight of one of those stone being approximately 2.3 metric tons.

The pyramid structure seems to have originated from mastabas in Nubia. The stepped mastaba (Arabic for a type of bench) appeared around the Third Dynasty (2650-2575 BCE) in ancient Egypt and look very much like a true pyramid. These "pyramids" were surrounded by a number of other structures associated with the pyramids function. Sometime between the Third and Forth Dynasty a number of attempts were made to convert stepped mastabas into true pyramids. The conversion from a stepped mastaba to a true pyramid did have a number of failures, but the Egyptian engineers would work around the problems or in the case of complete failure note the problems and re-design avoiding the reason for failure. The great pyramids were built in the Forth and Fifth Dynasties using idle laborers[1] during the flooding season. These pyramids are testament to the success of their engineering designs. Besides the great pyramids there are over 100 pyramids[2] in Egypt and over 200 pyramids[2] in the Sudan (Nubia). It is highly likely a team of engineers and specialized workers had to be employed full time working on the pyramid construction during these Dynasties.

Building a pyramid required a good grasp of mathematics, geometry, and knowledge of the qualities of the available materials. The design and positioning of the pyramids were very precise. The great pyramids are perfectly situated to have their sides line up to north, south, east, and west. The accuracy is uncanny for the time period. It is likely this was achieved by accurately observing the north star and using a plumb line. Other geometrical techniques were used to assure that the pyramid's edges would align at the apex.



Above is the Saqqara ostracon which is public domain on Wikipedia. A sketch of this ostracon is on page 52 (Figure 53) from *Ancient Egyptian Construction and Architecture*. It is were we see an architect's diagram using skilled mathematical techniques. The figure here shows what would amount to a numerical method of calculating curves (done by calculus now...unless you are using a computer, then you would do numerical methods again). This curve is an incomplete catenary, which suggests the Egyptians had knowledge of the usefulness of this shape (though it is not proof). However, the point to take here is the the Ancient Egyptians had significant mathematical knowledge especially from an engineering point of view.

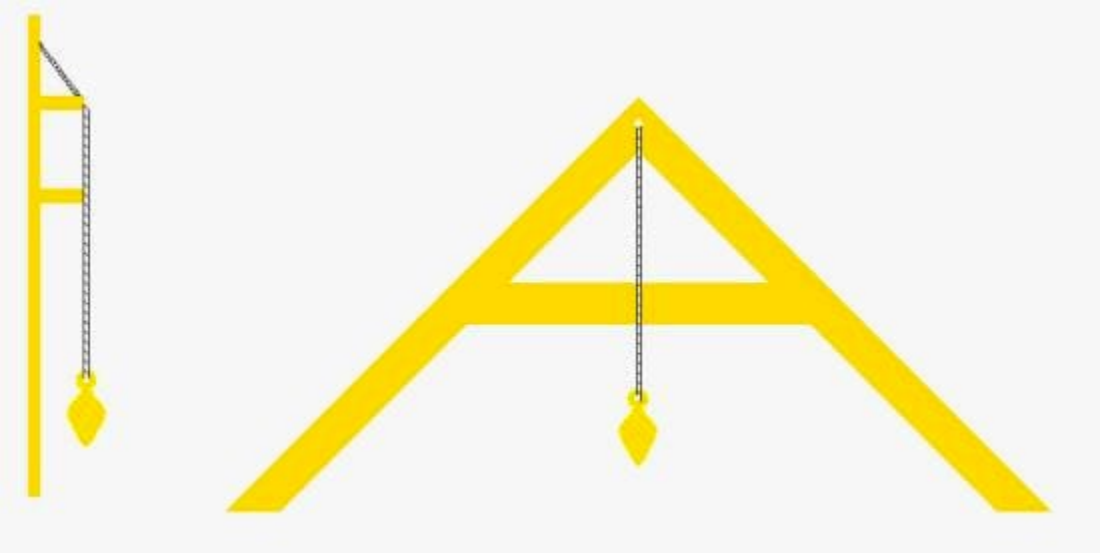
To build a pyramid the land had to be surveyed for a relatively flat area above the flood plain covering an area greater than the size of the proposed pyramid. Taking into consideration the enormous load to be placed on the proposed site, the bedrock had to be surveyed for defects to ensure a stable foundation. This is the type of job a civil engineer would do today.

After a site was designated it would be prepared by leveling off the rock foundation. Building would begin on this prepared site. In Egypt measurements were generally done using body parts. The standard unit of measure was the cubit which was a length from the elbow to the tip of the middle finger. For basic sort of work this measurement was sufficient, but for the great pyramids the royal cubit was used. This measured 52.5 cm and would normally be represented by a rod. The pyramid's base was design to be almost perfectly square using the rod as a measuring tool.



In the above picture you see a cubit rod. This picture is public domain (released by owner) found on Wikipedia. This picture was taken in the Liverpool World Museum in 2009.

In building any structure to have level surfaces is extremely important. Egyptians developed levels using the force of gravity called a plumb level which is an application of the well-known plumb line. Plumb lines are still used today just as in ancient Egypt.



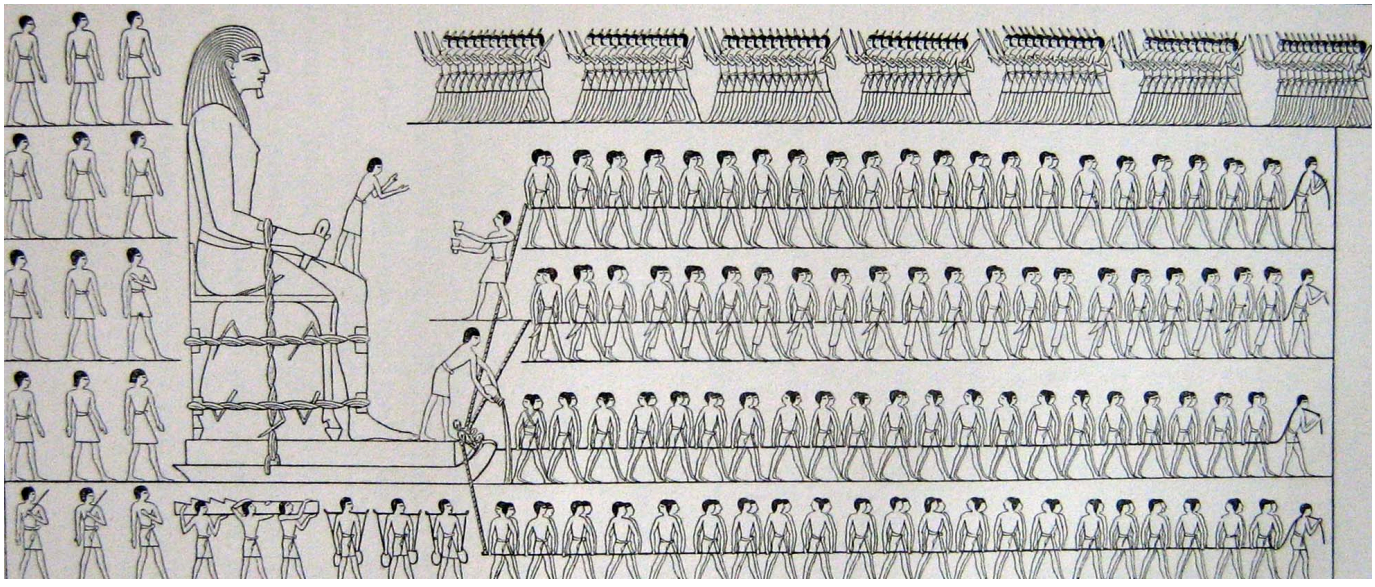
In the above figure you see graphical representations of levels from the tomb of Sennedjem (TT1). The level to the right is a representation of the so-called "plumb level of Sennedjem". The real plumb level of Sennedjem is damaged with one of the legs of the triangle broken off. Also these levels have hieroglyphics adorning them. Searching the web you will find images of these levels, however, it is unclear who owns these images and how they are licensed so they will not appear here. The level to the left would be used to ensure a level vertical surfaces and the plumb level of Sennedjem would be used to ensure a level horizontal surfaces. These levels are in essence no difference from the bubble levels we use today. Searching the web you can see other ancient Egypt tools as well.

Most of the large stone blocks of the great pyramids where of low quality limestone, however the internal chambers and hallways used granite, and the pyramid was encased in a fine limestone. This encasement was polished to give the great pyramids a spectacular white appearance. The majority of the blocks to build the pyramid where quarried right next to the pyramids. Giza was chosen partly because of the good building materials in the area.



In the above figure (Figure 13, page 17) from *Ancient Egyptian Construction and Architecture* we see the clear evidence of blocks being removed from near the pyramid. This book is out of copyright and the images our public domain.

Blocks were moved using sleds with water (or possibly oil) poured underneath. The blocks were taken up the pyramid using ramps of mud brick and rubble. While ancient Egyptians used ropes and levers there is no evidence of the use of pulleys. Even if they had knowledge of pulleys it is unclear if a block and tackle of sufficient strength could be made to move such large blocks. Ropes and levers were used to position in certain instances as well.



In the above sketch we see a heavy object (in this case a statue) being moved by a sled with a worker pouring water (or maybe oil but this seems less likely - *why would they use water or oil?* you answer this; debate) in the front of the sled. This sketch is based off a painting in the tomb of Djehutihotep on Wikipedia by John Gardner Wilkinson and is public domain "PD-US-expired" in the United States. A similar sketch is in *Ancient Egyptian Construction and Architecture* (figure 79, page 85) with some additional discussion.

Mortar was used for settling the stones but not for cement as might be used today. The mortar, made of gypsum (plaster and sand), was very soft like butter. The advantages of gypsum mortar is that it is slow-setting and, while it is setting, it can act as a sort of lubricant to lessen the friction between large blocks. Gypsum mortar is still used for certain construction (in dry environments or internal structures) in modern times. Rockers were used for placement of the blocks while the gypsum mortar sets when careful placement was necessary.

As the building of the pyramid progressed ramps would be placed along side the edges of the pyramid to allow the workers using sleds to drag the large blocks up the ramps. New evidence suggests ramps might have been inside the pyramid as it was built as well. The outside ramps would be removed after their usefulness was at an end. After completion a limestone covering was placed on the pyramid as stated above. These coverings have all but dissolved today.

The building of the great pyramids of Egypt took an accumulation of knowledge, testing, and creative engineering. The Great Pyramid of Giza is one of the largest pyramids in the world and was the tallest structure in the world until the beginning of the renaissance in Europe many thousands of years after their construction. It along with the great pyramids of Mesoamerica are a testament to the ability of engineers to create grand structures and stretch our horizons. In modern times the Egyptian pyramids captivate the imagination of engineers and spur them to reach new heights as far as the moon and beyond.



The Giza pyramids from a Department of Defense photo. Note this is a combination picture showing an airplane (engineering history lesson 1) with pyramids (engineering history lesson 2)



Aerial photograph from 1904 of the pyramids of Giza taken on a balloon by Eduard Spelterini. Public Domain.



The Giza pyramids from a Department of Defense photo at a different angle.

To further investigations, I suggest the following books and websites:

1. *When the Pyramids were Built: Egyptian Art of the Old Kingdom*, Dorothea Arnold, The Metropolitan Museum of Art, New York (1999)
2. *Egyptology: An Introduction to the History, Culture and Art of Ancient Egypt*, James Putnam, Crescent Books, New York (1990)
3. *Ancient Egyptian Construction and Architecture*, Somers Clarke and R. Engelback, Dover Publications, Inc., New York (1990). Original was published in 1930 and is out of copyright. This book has a lot of images that are not available anywhere else, it also has a number of facts that are now understood to be wrong, however from an engineering point of view it has a lot of still valid facts. Also while it says "architecture" in the title and "architect" in the book it is clear they are actually referring to engineers and engineering - a common misconception by laypeople.
4. Egyptian Museum in Cairo (EMC), the main museum for Egyptian artifacts, https://egyptianmuseumcairo.eg
5. The Theban Mapping Project has interesting information on current research, https://thebanmappingproject.com
6. Osirisnet has interesting information on tomb research plus some 3D virtual looks at some of the tombs, https://osirisnet.net
7. *Dr. Zahi Hawass*, One of the most famous modern Egyptologists, https://www.hawasszahi.com
8. *Dr. Kara Cooney*, Young new Egyptologist who is very media savvy, http://www.karacooney.squarespace.com/
9. The Grand Egyptian Museum (Giza Museum), https://www.visit-gem.com/

[1]Slaves were not used in building pyramids in Ancient Egypt. The story of slavery (for building the pyramids) in Egypt might have originated from ancient Greek writings that were inaccurate either by accident or by design. Herodotus ("the father of history") claimed that the pyramids were build by slaves, but he said he was told this and it was many many years after the construction of the pyramids. Some believe that the idea of slavery was "fake news" originated from ancient Greece, Egypt's enemy. However the origin of the story came about it has been clear for many years now that enslaved people were not used for the building of pyramids in Ancient Egypt.

[2]There is some debate on what a pyramid is and isn't so the exact number of pyramids is disputed, but all estimates are higher than the number sited here (hence the "over").

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11.3: Trains, trains, trains

Dynamics of our railway system



The train as we understand it today consists of a locomotive (though not necessarily) and as many as 150 cars to transport either people or freight. Locomotives in a train system have a number of advantages such as replacement so that the train as a whole can still go on if the engine fails. When locomotives become obsolete the rest of the train system can still be used saving money.

Advantages:

- Energy efficient compared to cars/trucks
- Minimal Land Use
- Low air and noise pollution
- High safety record compared to cars/trucks
- Moderate cost of service
- Flexibility: using different locomotives depending on conditions
- Can change out obsolete or broken locomotives without having to replace the whole system
- Special trains can plow snow, can be used in military campaigns (armored with some having gun mounts or carrying missiles), extra cushioned (usually air) train cars for transport of extremely fragile items, NASA railroad (stopped operating for a small period of time when the shuttle program was mothballed) with specialized trains for heavy loads, and...well why don't you all look up more special trains...except...
- Some special trains also have special tracks so that they can go up mountains which are near impossible for trucks and cars at a sustain rate



Cog train at Pikes Peak, CO. A cog train can travel up grades greater than 10%. The Pikes Peak Cog Train's greatest ascent is 25% grade. A new cog train is to be completed sometime in 2021. Looks like there is a new cog train, check the website. The above picture is the old cog train.

Disadvantages:

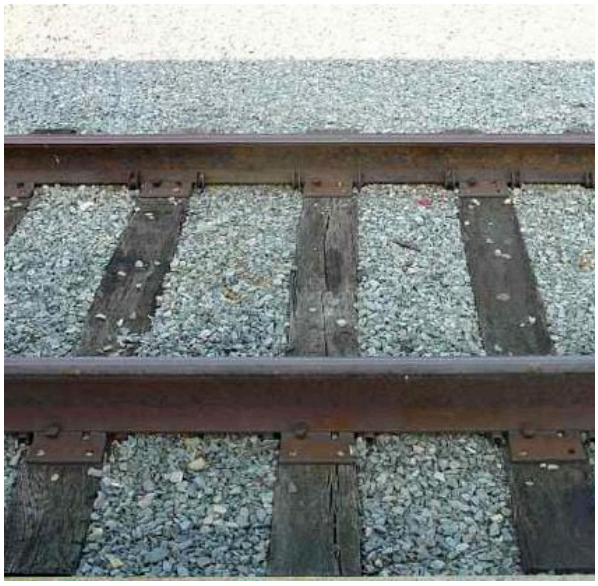
- Not as flexible as an automobile's ability to go where you want (obviously; note you don't need to drive your car on a road)
- Not private (not good for social distancing - spread of disease easier; this is true of airplanes as well) - written during COVID so...
  - However trains would be easier to design for some social distancing than airplanes
  - Windows can be opened on trains (at least slow moving ones)

To get trains to go faster might not be what you think. Most trains with current technology (with slight mods) can go very very fast (>500km/hr). It is *passenger comfort* and other issues that keep the trains from going fast. Here are some things that are done to improve speed.

- Tilting technology: Internally tilt train for comfort
- Increase radius of curvature of track (same as above)
- Deeper ballast
- More sleepers (ties) using concrete (and maybe steel)
- Heavy rail
- Better track alignment (spend the extra money to improve what is alright)
- Large diameter tunnels (for less air pressure)

"Railways" were actually created in ancient times (Greece and Rome) with stone rail roads for transporting large items such as boats to prevent them from leaving the path. These eventually lead to wood rails and then in the 1700s iron rails. However these railways could not be effectively used for trains that have locomotives.

The first railway lines in the United States were laid by Charles Carroll (signer of the Declaration of Independence) in 1828 in Baltimore starting what was known at the Baltimore & Ohio Rail Road (B & O, think Monopoly). The first stone is still on display at the "Birthplace of American Railroading" at the B & O Railroad Museum.

	
Western Maryland Scenic Railroad train in Cumberland, MD.	Classic I-beam structure with classic ties (sleepers) that wood (for the Acela, in the picture on top, the ties are m

The first trains on the B & O railroad (and other railways in the United States) where pulled by horses, not locomotives because of the sharp curves (locomotives of the time could not take sharp curves). Interestingly there were even some initial trains that were powered by sails. Sails are practical on land and on sea, but would have never been practical for balloon flight (going back to history lesson 1) which necessitated the invention of the heavier-than-air airplane. The steam locomotive made all those initial trains become a historical footnote.

Locomotives were first developed between 1802 to 1804 by Richard Trevithick as a steam locomotive. The first non-steam locomotives were developed between 1890 to 1920 though the steam locomotive was dominant until the 1950s. Interestingly the first electric locomotives were also developed between 1890 to 1920. Many companies were involved in development of different locomotives and there never was a major lack of funding to develop locomotives and railroads until the advent of the automobile and trucks. However, recently environmental considerations have again made development of locomotives a well funded activity once again.

The modern locomotive involves many different engineering disciplines. The most obvious, of course, is ***mechanical engineering*** with the engines, bogies, wheels, brakes, vibration control, etc. ***Civil engineering*** (or ***railway civil engineering***) would deal with the railway itself...the track, stations, etc. Less obvious are ***electrical engineering*** and ***computer engineering*** (monitoring systems including drones), ***chemical engineering*** (fuels from petrol to hydrogen and sanitation), and ***system engineering*** to make sure all of these systems work together. While traditionally trains are viewed as a mechanical or civil engineering career this is no longer the case.

Some sites of interest:
Progress Rail -- One of the largest builders of locomotives (EMD is incorporated into this Caterpillar company)
Alstom -- Formally Bombardier Transportation is a train company that is the builder of the Acela Express (with help from some others including TVG). Bombardier was a very large Canadian airplane and train company that now only specializes in airplanes. Alstom is finishing the replacement to the Acela Express, the Avelia Liberty
Association of American Railroads - public policy organization for railroads in the United States
Wabtec corporation -- Parts for many transportation systems but mostly locomotives
Railway City (Wabtec) -- Clickable railcity that students can get a better idea of all the engineering that goes into the railway...COOL! UPDATE: Site deleted; try Wayback Machine (old web address: https://wabtec-city.com/)
MotivePower -- A Wabtec company that builds locomotives (The local MARC system has purchased 26 advanced locomotives [MPXpress] from this company) UPDATE: This company has been totally merged into Wabtec - pages deleted; try Wayback Machine (old web address: https://www.wabtec.com/business-units/motivepower)
Norfolk Southern - one of oldest rail system in United States
Baltimore & Ohio Railroad Museum - B&O railroad museum (B&O oldest common carrier railroad system in the United States)
Trains.com -- Magazine on trains

Trainweb.org -- Hosting service for train systems (lots of small operations)
MARC -- Maryland's local train system - Amtrak runs part of it and Alstom runs the other part
VRE -- Northern Virginia's local train system: cross-over agreement with MARC
MetroRail -- Washington D.C.'s subway rail system
Amtrak -- Rather large railway system in the United States; not the only one though
Amtrak's Acela Express -- Currently the fastest train in the USA (Boston to Washington D.C. in the so-called Northeastern corridor) - to be replaced in 2024
Avelia Liberty -- Soon to be the fastest train in the USA (replaces Acela Express)

References:

1. *Handbook of Railway Vehicle Dynamics,* ed. Simon Iwnicki, CRC Press, New York (2006). This book is not a train enthusiast book but a true engineering book (familiarity with engineering and calculus is required).
2. *EMD Locomotives,* Brain Solomon, Voyageur Press, St. Paul, MN (2006).
3. *Baltimore & Ohio Railroad Museum,* http://www.borail.org
4. *Railway Technologies Laboratory,* https://cvess.me.vt.edu/labs/rtl.html

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11.4: What are the lessons learned?

What are the lessons learned from these historical examples? Discuss in class or with friends. Below are *some* answers (which you should not look at until you have as a class (group) discussed what you think are the lessons learned)...this is not all the possible answers. The student is expected to get all the answers.

¬ Answer A

The top answer to the lessons learned in these historical topics is that engineering (and science) is a **step-by-step process** by many groups and individuals. It is not one massive leap after another; everything is build on another previous result. This is one reason engineers and scientists are always reading journals, books, and other relevant material - they are looking to find something that is close to the answers they are seeking in order to advance one step further. An engineer or scientist can be a success just with a sequence of little improvements and innovations.

¬ Answer B

The other top answer to the lessons learned is that engineering (and science) needs **funding**.

¬ Answer C

The media prefers a good story over a true or accurate story. The general story of invention is complicated and is perceived to be boring by the media who like to have a singular center around one person or a small group of people. The idea is to entertain. Most news media fits into this entertainment media; there are however a small set of the news organizations that spends the extra time to do a complete story usually in print form with low subscriptions (because they don't entertain).

¬ Answer D

References (papers/books) are always required for human progress. We always build on other's works.

¬ Answer E

Resources are required to do any engineering (or science). Money, as previously stated, but also **materials** and **equipment** that usually have a high cost.

¬ Answer F

Engineers/Scientists are **well educated** either through school or by library learning (which usually requires free time which usually is reserved for the rich).

¬ Answer G

Materials science must be developed enough to be able to innovate. Many inventions and innovations had to wait until our materials science was developed sufficiently, for instance, light weight paper; light weight metals; etc. for **air flight**. This is a good reason to support scientific research through funding, even for theoretically work.

In science a recent example of the importance of this is the mRNA vaccine (note RNA is a polymer so this loosely falls under the materials science development) that was used on **COVID-19.** The theoretical work on this type of vaccine started in the 1980s and then in small experimental trails (all in **academic** settings; 1990 at University of Wisconsin being the most exciting step). Recent increases in **funding** (from a mixture of government and private investors BEFORE the pandemic) allowed for a very effective vaccine from two different companies. Note to make this vaccine work **bioengineering** was required to protect against the quick degradation of mRNA. Finally note that to keep the two vaccines stable they needed to be cooled to very cold temperatures. *Who designed those refrigerators that go to very cold temperatures?* Yes, engineers (with scientists and engineering technicians), of course!

¬ Answer H

There is one engineer/scientist who actually was eventually worshiped as of god (long after is death), Imhopet. Imhopet was the ancient Egyptian equivalent of Leonardo Da Vinci and may have been buried in one of the lesser pyramids (unfortunately the location of his burial place was lost not in modern times but in ancient Egyptian times...so it is going to be a hard mystery to solve). This leads to the conclusion that in great civilizations engineers and scientists are **revered** to the point of worship. Maybe you will have a different conclusion. *What do you think?*

¬ Answer I

Given Glenn Curtiss' position (as well as Santos Dumont) that it is better to keep innovating rather than to patent innovations (which he viewed as a waste of time and resources), there is a debate if **patents actually stifle** progress. It does seem that at least with regard to the history of flight there was considerable stifling of progress which due to the emergency situation of World War I cause the government to step in. Curtiss has hundreds of innovations to his name only a few of which he patented mostly due to others who wanted to charge him for what he felt was his innovations (or his groups innovations). As for business, Curtis though that rather then sitting on your laurels you should constantly innovate to stay on top, people who copy will always be behind you. Santos Dumont did not patent at all as he felt innovations should be free to all - even you. He did not rest on his laurels either. *This is a good debate topic. What do you think?*

This concludes this lesson of engineering through a historical perspective. Please continue thinking about these issues through out your academic career and your "real world" career.

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12: Case Study on Nanotechnology

Here we delve into a case study on nanotechnology which is an ancient technology as well as a cutting-edge modern technology. This contradiction is exactly why this is an interesting case study for learning what engineering (and science) is all about.

This section is meant to be accompanied with an inexpensive textbook. Fortunately wikibooks has such a textbook (free): **The Opensource Handbook of Nanoscience and Nanotechnology**

This book is an excellent if a bit incomplete introduction (for an engineer or scientist) to nanotechnology. Some of the topics however might be overly advanced for an introduction to engineering class, so in this section nanotechnology will be reviewed with an assumption that the student will use the textbook above (or another one of their choice) to supplement. This section is not meant to take more than a week in an actually instructive setting.

What is naontechnology?

To begin with let us do another class discussion that asks the question: *What is nanotechnology?* Discuss before looking at some answers.

¬ What is nanotechnology?
¬ What does technology in the nanometer range mean?
¬ So what is nanometer really mean?
¬ Given the scales you have seen what is nanotechnology now?
¬ Nanotechnology, while something humankind manipulates, is natural as well. Can you think of a food technology were we use nanotechnology?
¬ What about a product developed in ancient Egypt (or maybe earlier) that is nanotechnology?
¬ However even before humankind started using nature to assist us in nanotechnology, an amazing piece of nanotechnology already existed billions of years before humankind. This nanotechnology is a technology that we might someday be able to employ to do work for us one day. What is this technology?
¬ Can you think of any other natural nanotechnology?

Carbon allotropes

Because "buckyballs" are the start of the modern revival of nanotechnology (at least from a media point of view) let us go over some of carbon allotropes that are making headlines.

Buckyballs

While nanotechnology is an old technology, a new modern revival of the technology came about with discovery of C_{60} or the Buckminsterfullerene (buckyball) named after Buckminster Fuller because of his penchant for building geodesic domes. *Why geodesic domes?* Because these domes are based off the Platonic solids[3] and C_{60} is a truncated icosahedron (one of the Platonic solids).

C_{60} were produced in 1985 during an experiment to help understand certain carbon molecules that might have been generated in space. *Why do such an experiment?* Because most stars have debris surrounding them with carbon in it and some have very long chains that are of interest to astronomers. Hence the experiment. The actual generation of C_{60} was not intended but serendipity. From an engineering and science point of view, the ***analysis*** after the experiment was the real research because C_{60} was identified through analysis after the experiment that did not aim to produce them or even know of their existence.

The buckyball is now considered a part of the fullerene family. An outline of facts about buckyballs:

- Truncated Icosahedron (like a Telstar football or "a soccer ball circa 1970s")
- 0.7 nm in diameter with a spacing of about 1 nm between adjacent buckyballs
- Inert

- Can be made into a superconductor
- Offshoot studies led to the discovery of the carbon nanotube (next topic)
- Has been detected in burning candles (a modern addition to Faraday's *The Chemical History of a Candle*, yes?)
- Buckyballs have been found in space officially by the satellite Spitzer
  - Stacked buckyballs
  - A huge amount, not miniscule
- The most massive particle to show wave-particle duality (Nature 1999)

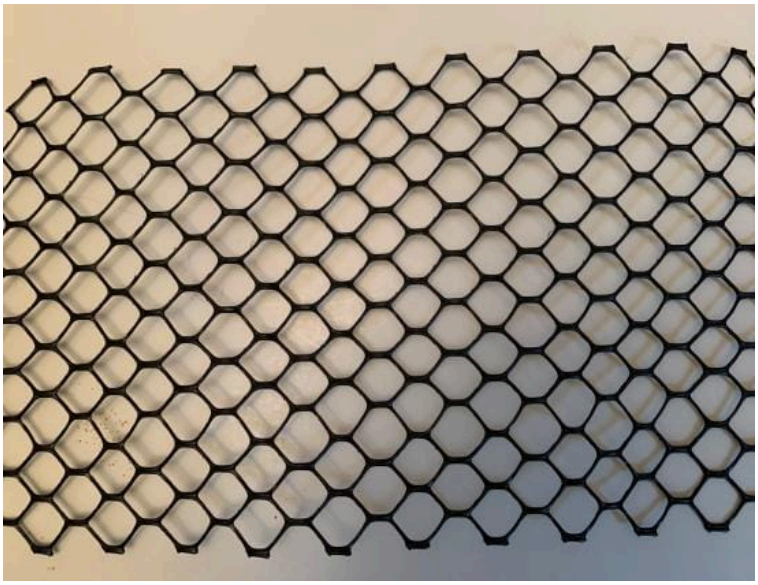
There are many articles about buckyballs and interesting uses of buckyballs (though some are totally false, so be careful! See Understanding). In this brief review though we will move onto the carbon nanotube as there have been actual products developed from this fullerene. That's not to say that buckyballs will never have products produced from them, there time just hasn't come yet.

Carbon Nanotubes

Carbon nanotubes were first observed in 1991 and produced in 1992. Because of this discovery interest in buckyball technology shifted to these nanotubes. Carbon nanotubes are like an individual layer of graphite (which is now called graphene) that is wrapped around to meet end to end. An outline of facts about carbon nanotubes:

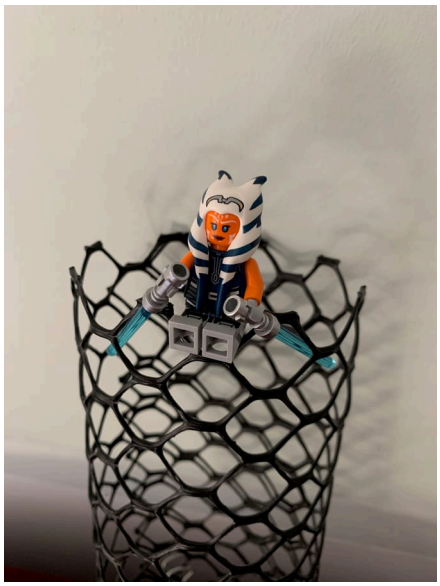
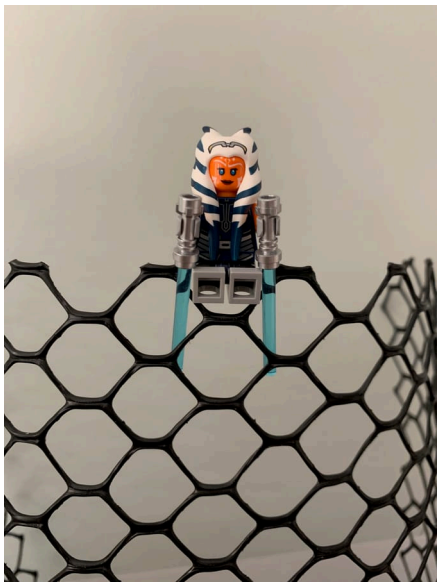
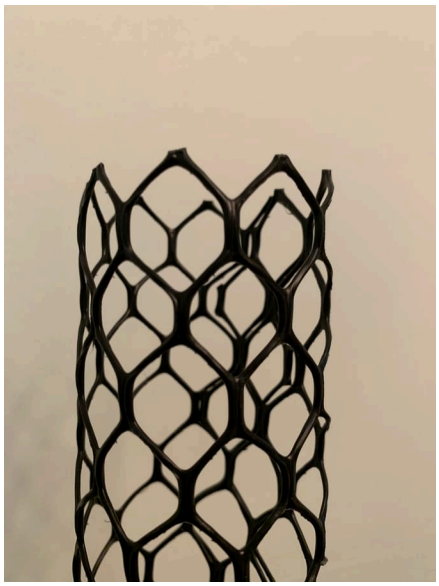
- Extremely strong
- Known as buckytubes at one time
- Found in Damascus sabers, an extremely strong and flexible saber in the 13th century and beyond (*Nature* 444, 286, 2006)
  - Science in making the sabers but serendipity that CNTs were involved (just like bread making, etc.)
  - Modern techniques make better sabers, but at the time they were the best (and their legend lives on)
- Varying diameters from 1 nm to 100 nm and can in theory be as long as you desire, but in practice not so long (yet)
- Good conductor of electricity
- Or can be a semiconductor
- Called (carbon) nanowires when discussing electrical properties (note: this is not the only type of nanowire)
- Single-walled (SWCNT or SWNT) and multi-walled (MWCNT or MWNT)
- Forest of MWCNTs can produce the blackest substance known this is very important for astronomical purposes (telescopes)
  - Buckypaper offers many possible applications, but still is in its infancy
  - GSFC/NASA continues their groundbreaking work on carbon nanotube technology
- CNT has been tested for such diverse ideas such as water filtration, supercapacitors, heat shields, etc.

A great way to look at nanotubes is to get a piece of chicken wire (plastic preferably) and cut out a rectangle (at this point you have graphene) and wrap it around (nanotube). You can do this at home which is way better then a flat screen simulation and definitely inexpensive.

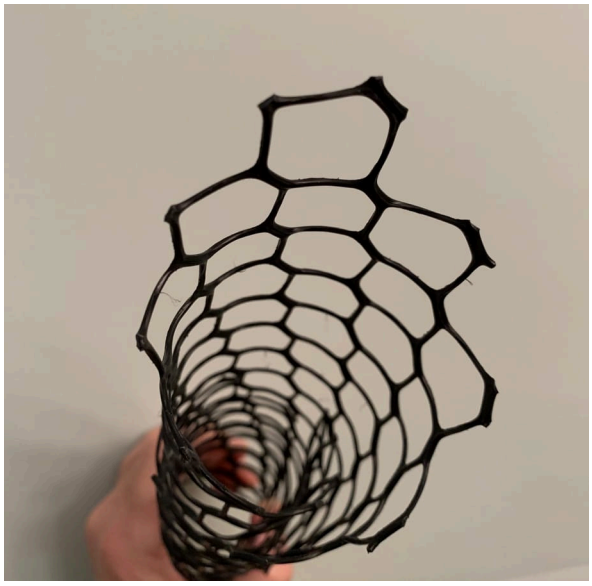
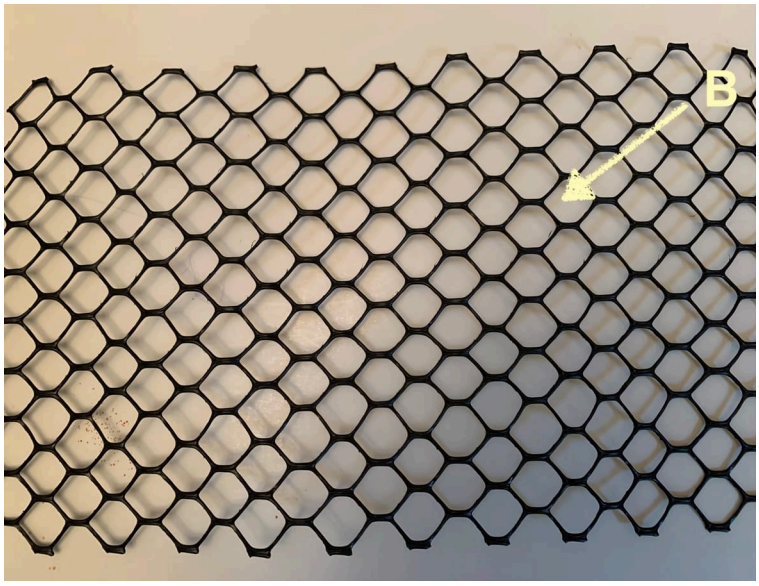


Chicken wire to represent one layer of graphite which is called graphene. Graphene will be discussed later in the chapter.

Different wraps of graphene can produce different properties for carbon nanotubes. That is, depending on how you wrap the nanotube you can have metallic nanotubes or semiconductor nanotubes (or at this point you might want to call it a nanowire). Note that the ends of the wrap which normally don't have a cap in our representations represents the end of the nanotube itself.

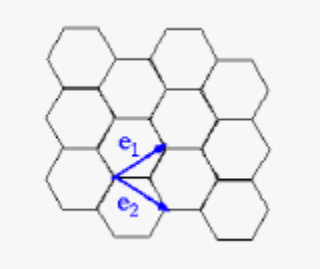
			
Armchair wrap of the "graphene" from the image above. This warp will give you a very good conductor.	Armchair wrap. Here Ahsoka is demonstrating why it is called an armchair. The wrap is not complete here because it obscured the visual aspect of the armchair.	Zig-zag wrap. This wrap produces semiconductor nanowires.	Zi sc th

There are two other possible wraps for the carbon nanotube and that is the chiral wraps. Chiral CNTs are stereoisomers and are mostly semiconductors.

		
Chiral wrap - delta isomer. This in essences is a twisted cylinder. Chiral wraps can produce either conductor or semiconductor nanowires.	Using the "graphene" sheet if we wrap the sheet starting at B in the direction of the arrow (right; clockwise) we will produce a chiral wrap at left.	

For carbon nanotubes we can define a coordinate system that has unit vectors that help us describe the armchair, zig-zag, and

chiral nanotubes.



Using the unit vectors ($\vec{e_1}$ and $\vec{e_2}$) defined in the figure above we can write an equation that describes the various nanotubes as $m\vec{e_1} + n\vec{e_2}$ where m and n are integers and $m + n \geq 2$. Given this equation if m or n equal zero then we have a zig-zag CNT (semiconductor), if m=n we have an armchair CNT (metallic), and otherwise it is chiral CNT. In general chiral CNTs are semiconductors but if $|(m - n)|$ is a multiple of 3 then it is metallic[4].

Fullerene research is just at its infancy and there will be more to discover which will include its share of disappointments, but that is science.

Graphene

So what about that sheet of graphite we discussed above? A single sheet of graphite is called graphene. Through studies of the laminar nature of graphite oxide starting as early as the 1860s where chemist Benjamin Brodie produced thin layers of the crystal which he studied and was able to get atomic weight of graphite. Studies on this structure continued with every thinner layers which had high strength and noteworthy optical properties. In 1947, physicist's P. R. Wallace produced a theoretical framework for graphene in order to understand the electronic properties of graphite. Work continued on thin layers of graphite both experimentally and theoretically with some work possibly being on graphene (there would be no way to distinguish between one and a few layers of graphite). In 1961 chemist Hanns-Peter Boehm reported on very thin layers of graphite flacks and called a single layer of graphite, "graphene." The term would be revived in the late 1990s when disscussing carbon nanotubes. Finally in 2004, physicists' Andre Geim and Konstantin Novoselov isolated and characterized free-standing graphene. And this is when things got interesting...

In the following outline we will list some properties of graphene that can possibly lead to exciting new products or are just very interesting scientifically:

- Graphene properties
  - Single atom thickness (carbon)
  - Graphene is a zero-gap semiconductor
    - Normally a semiconductor has a greater than zero band gap and it is metals you would expect to have no band gap
  - Graphene for a single atom layer has an unusually high opacity from visible to near infrared (and maybe mid infrared)
    - That is the graphene actual absorbs light (over 2%)
    - This feature mean you can actual "see" graphene in certain conditions
    - Graphene's strong interaction with photons maybe useable for nanophotonics
  - Graphene is theoretically an excellent material for spintronics due to carbon coupling and long spin lifetimes (theory)
  - Graphene is mechanically very strong
    - Lightest strongest material with large tensile strength
    - Small spring constant (flexible)
    - Very robust
  - Graphene is brittle (like ceramics and not like metals)
    - But it has a impressive ability to distribute the force of an impact
    - This allows it to bend like metals
  - Graphene has high surface area to mass ratio (almost goes without saying) which could make it good for supercapacitors (instead of the currently favored idea of activated carbon)
  - It is possible to sculpture graphene into a honeycomb shape referred to as 3D graphene
    - Can by used for energy storage, filtration, and other applications

That was just a few of the interesting properties of graphene. But this is not the last word on nanotechology as up and coming

new technology includes the hexogonal Boron Nitrite (h-BN) that has just as many interesting properties as graphene. And we can go even further with combining fullerenes, graphene, and h-BN. Already combining graphene with CNTs has produced interesting research avenues as well as graphene with bismuth nanowires and graphene with h-BN (hexagonal Boron Nitrite).

So let us move on to discussing nanotechnology in more general way to give just a brief overview.

Nanotechnology by discipline

Nanotechnology spans multiple engineering disciplines which we will list briefly below. For electrical engineering the processes of making integrated circuits (ICs) has been in the nanotechnology range for decades, but new techniques are possible with nanotechnology elements.

- Bioengineering
  - Bionanosensors
  - Utilizing natures nanotechnology (like mRNA for vaccines, etc.)
- Chemical Engineering/Food Engineering
  - Nanofoods (nano-manipulation of food to improve taste, texture, etc.)
  - Nanopackaging (using nanomaterials to improve packaging)
- Environmental Engineering
  - Nanomembranes (for filtering)
  - Nanocatalysts (for water remediation)
- Materials Science Engineering (can include Mechanical/Civil/Chemical/Aerospace/etc.)
  - Nanocoating (including CNT coating)
  - Nanosurface protection (including uses of CNT mechanical properties)
  - Quantum dots
- Electrical Engineering
  - Lithography (been at nano-level for a long time)
  - DNA nanoarray
  - Nanowires or nanosemiconductors
  - Nano-optics

The outline above is just a taste of nanotechology and how it effects a number of engineering disciplines.

There are three different areas of research in nanotechnology which usually are the domain of different disciplines.

- "Wet" nanotechnology
  - Liquid environment
  - Usually biological
  - Membranes
  - Filters (CNT) and example of cross-over technology
- Dry nanotechnology
  - Silicon and other inorganic materials
  - Metals, semiconductors
  - Too reactive so they can't operate in wet conditions
- Computational nanotechnology
  - Modeling and simulation of complete nanostructures
    - This should be in addition to actual experimentation and prototyping
    - Be careful of papers that say they have discovered something but it is actual a simulation (you see this a lot)
      - While this is important and could produce some excellent product or insight, it still has to be verified experimentally
      - So don't get excited until the process is complete
  - This is required to fully understand nanotechnology

What is so exciting about Nanotechnology?

The physical rules of the "macro" world are relevant all they way down to the microscopic level, but things change when you pass into the nano realm. Surface effects, chemical effects, optical effects, and physical effects are different in the nanoscale when compared to the macro or micro scale.

- This creates new and interesting products like
  - Stain resistant clothes
  - Sweat absorbing clothes
  - Antimicrobial socks
- This creates new avenues of research
  - New exciting discoveries await
  - However, disappointments await as well
  - This is the nature of research
  - Of course, going back to the history lecture, funding is essential
    - *Is some money going to be wasted?* Yes that is the nature of searching for things. "Failure" is an integral part of engineering and science. We want success but we want to progress as well and that means some failures
    - *Can we predict where our money should go?* Yes and no. Simulations can give us clues, but it is not a perfect solution
    - *Should we only do research that is proven out by a simulation?* No, but we should not ignore the contribution of simulation

Understanding the different effects at the nanolevel requires an understanding of physics. For engineers and scientists this is why physics is essential. Some ideas require a graduate level physics background, but even with a calculus-based physics understanding the ideas behind nanotechnology become clearer. Simulations are going to require graduate school level education.

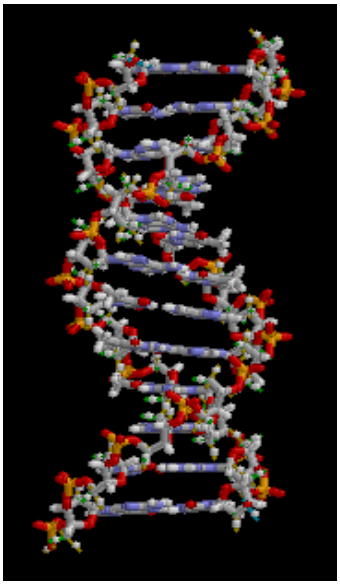
- Some concepts that are important
  - Scaling laws
  - Quantum mechanics
    - Transport phenomena
    - Hartree-Fock (computational physics - approximation method for wave functions)
  - Surface theories
    - Hydrophobic and hydrophilic
    - Diffusion, transport in all dimensions

Practical ways to do Nanotechnology

How do you go about making something in nanotechnology? There are two methods

- Top-down
  - Building nanotechnology using larger elements
  - Primary method in manufacturing at present
  - No atomic-level control
  - Nanophotolithography
    - State of photolithography for a couple of decade
    - Uses deep UV laser light
      - Laser is a larger element producing smaller nano-element
- Bottom-up
  - Build from molecular components
  - Self-assembly
    - Static self-assembly utilizes nature to reach minimum free energy
    - Dynamic self-assembly requires energy to force a solution
    - Molecular recognition
      - That is components assemble themselves based off of a code
      - Need some form of coding
        - *What in nature might be used as a model for this?*
        - *What are some problematic issues with using this method?*

The answer to our coding is DNA which we discussed at the start of this chapter.



DNA animation, the code of life, and maybe the code to build nanotechnology in the future. https://commons.wikimedia.org/wiki/F..._animation.gif, brian0918, Public domain, via Wikimedia Commons

DNA is a coding device that is used in nature, but some have proven it can be used by humans. DNA is nanometer in size. Let us view a TED Talk by Paul Rothemund explaining his creation of DNA faces.

Interactive Content

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

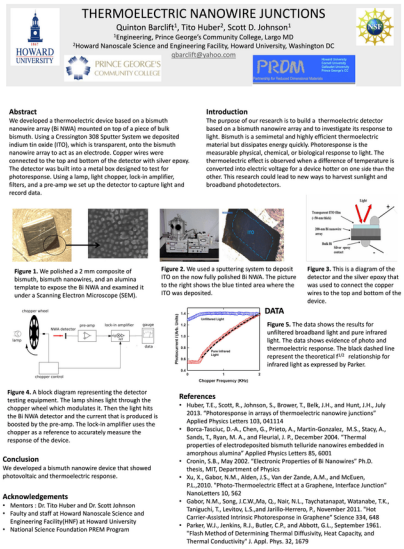
Note that the method described here is not the only method people are researching. You can go to the Rothemund Lab web page (under research) to get links to other researchers in the field.

Nanotechnology Examples

Because nanotechnology is so vast and covers so many disciplines we have picked only a few examples as a way of introduction. There are many many many more applications and examples in the literature. We encourage you to read as many as you can. And maybe one of your essays can be on nanotechnology in your field!

Bismuth Nanowires

Bismuth in has been used in one form or another for thermocouples and thermopiles for more than a century. Bismuth is a semimetal even in nanowire form until about 50 nm when it transitions to a semiconductor form. Most research is done, however, with Bismuth nanowires in the semimetal form as it is difficult to produce good nanowires below 50 nm (though advances continue). Nanowires offer different properties that can aid in the thermocouple/thermopile are of research such as optical properties and reduction of thermal conductivity (bulk semimetal general dissipate energy to quickly due to higher thermal conductivity.



This presentation discusses one application of nanotechnology. As a bonus this presentation also included an example of a detector testing system that is a classic chopping method for getting small signal response which is important in engineering beyond nanotechnology.

Nanotechnology and the environment

- Water purification
  - Humans need clean consumable water for survival
  - Environmental contaminates are a serious problem that reduces the amount of consumable water to unacceptable levels
- *Nanotechnology to the rescue?*
  - Ceramic nanomembranes
    - Ultrafiltration
    - Added reactive component (iron oxide ceramic membranes) add an extra-level of removal of contaminates
    - Aluminum oxide ceramic membranes are another membrane being investigated
  - Iron remediation
    - Iron oxidization causes certain organic molecules (including toxic ones) to break down
    - As the scale of the iron particles are reduced the iron becomes more reactive (greater overall surface area)
      - Therefore nanoscale iron can improve remediation
      - Smaller size allows the iron to go further into the soil (percolation)

Nanotechnology materials

- Nanocrystalline metals
  - All metals are crystalline with the grain size (not particle size but a boundary feature in metals...take Materials for more info)
    - The grain size is an important characterization of metal (regardless if we are taking nanotechnology or not) that defines among other things the yield strength
    - The Hall-Petch equation defines the yield strength with regards to the grain size
      - $\sigma_y = \sigma_0 + \frac{k}{\sqrt{d}}$ where σ_y is the yield strength, σ_0 and k are constants that depend on the particular metal, and d is the average grain size diameter
      - The equation implies that smaller grain sizes give better yield strength
      - Valid for microcrystalline structures and mostly valid for nanocrystalline structures (above 30 nm)
        - Possible negative Hall-Petch effect below 30 nm
        - Questions remain; studies needed
    - Issues are worsening corrosion and creep as the grain size gets smaller
    - Future shows promise however
  - Polymers

        - DNA
        - RNA
    - Ceramics
        - Ceramic nanoparticles
        - Possible bone repair (see next example)

Nanotechnology and bones

A large portion of our bones are nanosize hydroxyapatite which could be repairable using bioactive and resorbable ceramics. The mechanism of this repair would be osteoinduction. This is a very promising research avenue.

Spintronics (or magnetoelectronics)

The idea behind spintronics is to develop electronics that uses the spin of the electron rather than the "movement" of the electrons. The promise of this technology is to make transistors smaller and faster.

- *So why is this nanotechnology? An electron certainly is not in the nanometer range? Is it?*
    - Technically spintronics is not nanotechnology, however, nanotechnology offers the best approach for its practical use
- *How?*
    - By creating ferromagnetic semiconductors that require layers that are only a few nanometers (← there you go)

Nanotechnology Machines

Can there be nanotechnology machines? No, not really, nanomachines are not very practical. But nanoparts for use by microelectromechanical systems (MEMS) is possible. For nanoelectromechanical systems (NEMS) we will outline some possible parts without getting into the details of how to control the motion (some sort of voltage will need to be applied).

- Nanomotor
    - Use multi-walled nanotubes
    - One tube rotates inside the other
    - Kinds of emulates rotational bearings
- Molecular actuator
    - The nanomotor would be controlled by the use of a nanocrystal ram (sort of like a piston)
    - Control by voltage in some fashion
- Nanoelectromechanical relaxation oscillator (RC oscillator; multivibrator)
    - In general electronics this can be used as a clock or for blinking lights on a car
    - This works using liquid metal droplets that exchange mass
    - Utilizes surface tension (which in would be very strong at this scale)
- Nano-Pressure sensor using graphene
    - Graphene has relatively small spring constant and therefore is relatively flexible
    - Graphene is very robust as well

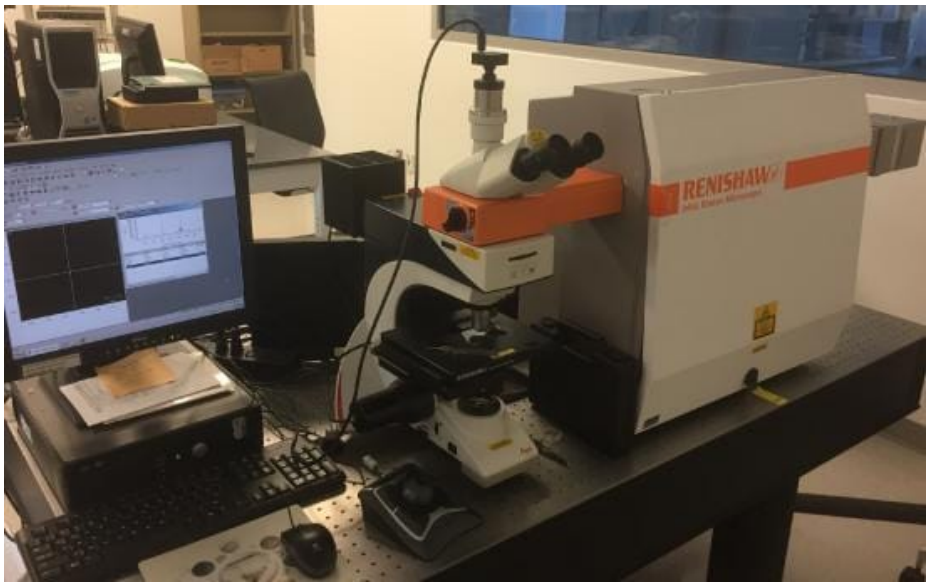
Tools used in nanotechnology

A microscope is an optical device that uses light to magnify the object it is viewing, because visible light has a wavelength between 400 nm to 800 nm. Typically a "microscope" can at best see an object about twice the wavelength of light that is used. This means a normal optical microscope could at best see about 1 μm which is in its name...a **micro**scope. This would be cellular level. It is possible to infer some nanotechnology from a powerful microscope, but it would be better to use something else. Also there are UV microscopes, but still it would be better to use something else. So in this section we will go over the tools for nanotechnology.

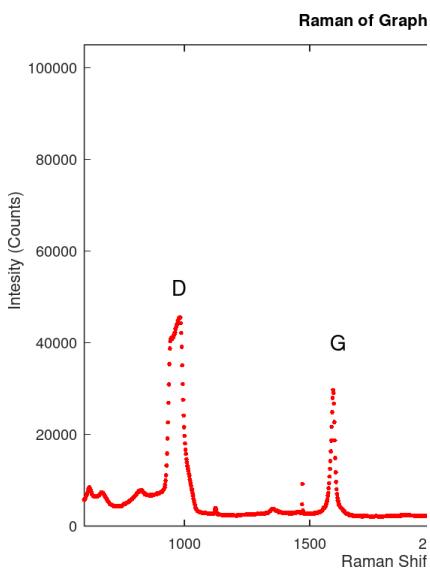
- **Electron microscopes (EM)**
    - Scanning electron microscope (SEM)
        - Focused beam of electrons
        - Electrons' wavelength is much smaller than 1 nm (so this will work for nanotechnology)
        - 5 to 10 nm resolution; some special SEMs can get down to just less than 1 nm

        - Surface scanner
    - Transmission electron microscope (TEM)
        - Electrons penetrate the sample (typically less the 1 μm)
        - Magnets used to manipulate the electrons into the sample
        - 0.2 nm resolution (but field of view is severely reduced in exchange for this better resolution)
    - Analytic electron microscope (AEM)
        - SEM, TEM with equipment like spectrometers
        - 0.1 nm resolution
    - While there are versions that can be used in a liquid environment, these Liquid-phase EMs have limited uses
    - Need to prepare certain samples by sputtering metal (like gold) on them
    - Sample is placed in a vacuum of at least 10^{-4} torr
    - New innovations allow for "desktop" Scanning electron microsopes
- **Scanning Probe Microscopes (SPM)**
    - Scanning tunneling microscope (STM)
        - Used electrical properties from tip to sample
        - 0.1 nm resolution
        - 0.01 nm depth resolution
    - Atomic Force Microscope (AFM)
        - Uses force properties (this is how it distinguishes from STM) using a cantilever
        - Three modes
            - Non-contact mode
                - Detects the Van der Waals forces by oscillating very close to the surface
                - Difficult mode to work because of its being close to the surface which induces troublesome forces
            - Contact mode
                - Most common mode
            - Tapping mode
                - For soft surfaces
        - Different probes[5] that detect different things can be attached to most AFMs
            - There are many different type of probes (maybe 100 or so)
            - List of a few interesting probes (to highlight what can be done)
                - Nanoscale Thermal Analysis probes for thermal maps of the sample
                - Scanning Microwave Impedance Microscopy probe for scanning local electrical properties
                - Magnetic probes for probing magnetic fields above the sample
                - Scanning Capacitance Mode probes for getting a sense of carrier concentrations in semiconductors
                - Deep Trench probe used for the integrated circuit industry
                - Tip Enhanced Raman Spectroscopy probe
    - Field of view is significantly smaller than SEMs
        - Millimeters for Electron Microscopes
        - Micrometers for Scanning Probe Microscopes
    - Slow scan compared to SEM
    - No vacuum requirement
        - Unless you really want to get to the atomic level then you need high vacuum
        - In the case of atomic level however we are not discussing nanotechnology any more though this could be of benefit to nanotechnology in the research sense
    - Liquids are not an issue except for what type of probe you use (some might get damaged in liquid environment)
        - Tapping mode is usually used here
        - Usually use same sort of probes as with solid but designed for liquid (Silver Nitride)
- **Nanomanipulation**
    - Probes for AFMs can be used to do nanomanipulation (nanolithography or nanobuilding)
    - Nanomanpulators are available for SEMs as well
- **Spectrometers**
    - There are many spectrometers (nearly 100 types) with a number of types very useful in the study of nanotechnology
        - Only two types will be outlined here, more are covered in materials class
        - Spectroscopy is the study of how light interacts with materials
        - Spectrometers utilize these studies to determine properties of various objects (stars) or materials (say, CNTs)
            - Basic spectrometers that most people are familiar with determine elements in a system but other spectrometers determine much more
            - Studying spectrometers could actual be a year-long course in itself, fortunately there are numerous web sites

on spectroscopy for most types of spectrometers
- X-ray diffraction spectrometer
  - Determines type of crystal structure along with defects and any other structural information
  - Some methods are non-destructive
- Raman spectrometer
  - Determines many characteristics of materials but one of the most important uses is being able to determine allotropes
    - "Common" spectroscopy in general determines if you have say carbon or not but not what form of carbon
    - Raman spectroscopy can determine the allotrope of carbon you have
      - Allotropes of carbon: buckyball, CNTs, graphite, diamond, graphene, glassy carbon, carbon nanobuds, etc.
      - Basis of this spectroscopy is Stokes Raman scattering (as opposed to say Mie or Rayleigh scattering)
  - This is covered more thoroughly in the materials science course



Raman spectrometer with traditional microscope to position the sample and computer to run the spectrometer and to do analysis. Power and laser is behind the instrument.



Raman spectrum of a graphene sample. T graphene in a Raman spectrum. Other car Raman spectrums.

- **Theory, Modeling, and Simulation (TMS)**
  - New advances have been produced in the lab (real) because of simulation that were originally preformed based off new theories or ideas
  - TMS is an important tool to help direct studies in nanotechnology
    - Main idea is to study nanotechnology to direct how to use it better or to make new discoveries
      - Theories are made into models which are then simulated
    - Another factor is the need to interpret the data from nanotools
      - Need models of measuring tools and the materials to understand interactions
  - Protocol for this "tool"
    - Theory: what do we know about the materials and tools
    - Model: represent the theory in a testable fashion (equations; numerical analysis techniques)
    - Simulate: Test the models
      - Success for known results
        - Use the model to predict some new results
        - Laboratory test for the new results to confirm the model
      - Failure (not bugs but failure of the model)
        - Re-work the model
        - In rare instances look at the theory

Nanotechnology involves almost everything

- Nanomaterials
  - Semiconducting nanostructures
    - Buckyballs
    - Nanotubes
    - Nanowires
    - Nanoparticles (like quantum dots)
- Nano-optics or nanophotonics
  - Light and its interaction at a nanoscale
  - Metamaterials (negative index of refraction among other "non-natural" properties) are the most promising here
- Nanomechanics
- Nanofluidics (study of fluids confined to a nanostructure)
- Nanobiotechnology

Additional websites to satiate your curiosity on nanotechnology

- https://www.nano.gov
- https://www.nature.com/nnano/ - Nature Magazine's Nanotechnology Journal
- https://www.ornl.gov/facility/cnms
- https://nanohub.org - this is for educators and researchers can be very high level
- https://nanocenter.umd.edu
- https://www.olympus-lifescience.com/en/microscope-resource/primer/java/electronmicroscopy/magnify1/ - simulation of an electron microscope
- https://www.renishaw.com/en/raman-spectroscopy--6150 - Renishaw's Raman Spectroscopy page (they have links to a lot of literature on Raman spectroscopy)
- http://mw.concord.org/modeler/ - Molecular Workbench: Simulator program for learning science in a realistic manner
- https://www.sciencenews.org - General science periodical but you can search for Nanotechnology and get interesting articles
- https://www.nanowerk.com - kinda like a warehouse of nanotechnology links (more for learning)
- https://www.graphene-info.com - kinda like a warehouse of graphene articles and links
- https://www.nationalgeographic.org/encyclopedia/nanotechnology/ - National Geographic article on Nanotechnology
- https://science.howstuffworks.com/nanotechnology.htm
- https://www.agilent.com/labs/features/2011_101_nano.html
- https://www.cdc.gov/niosh/programs/nano/default.html - CDC laboratory that investigates the safety of nanotechnology
- https://www.open-raman.org - open source Raman project so you can build your won Raman spectrometer (costs a bit, still)
- https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6982820/ - An article on this history of nanotechnology that might be of interest to some

This is just a sampling of nanotechnology, a more detail look at nanotechnology will be provide in materials science class. This is the last teacher-led case study; now it is the students turn - starting in the next section.

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[1]For a more modern version of the *Powers of Ten* you might want to look at the *Cosmic Eye* version:

YouTube Video

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this YouTube Video online

Another interesting approach is the tool on AAAS' ScienceNetlink that gives more scales then just the power of 10 movie: Scale of Universe 2. Still the original movie from 1977 is still amazingly good and has music from the famous American composer, Elmer Bernstein (*The Ten Commandments, Magnificent Seven*,...).

[2]The tendency is to use grain size here but that actually means something else with regards to metallurgy so instead we will say nanoparticle size. Gold is obviously gold when we look at it, but a 30 nm nanoparticle size of gold is red. As you make larger and large nanoparticles it starts to change from red to a bluish-purple hue. The shape also can cause color change so rather than grinding it like you would in ancient times you would purposely make spheres or prismoids to get different colors (note that the sphere would be different color then prismoid if both were the same size).

[3]The Platonic solids were described by Plato (or, maybe, Pythagoras) and consist of five solids: the cube, tetrahedron, octahedron, icosahedron, and dodecahedron. These solids are very interesting in the field of mathematics and crystallography (and by association materials science).

[4]You can examine this more by using one of Scott Sinex's Material Sciences Excelets (in particular one named "Carbon Nanotube"). This, while designed for Excel, will run on LibreOffice's spreadsheet but does not work on MacOS Numbers.

[5]The example list of probes herein is from Bruker, a company that sells scientific equipment, in particular AFM and STM probes (Bruker probes division).

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13: STUDENT LED CASE STUDY IN ENGINEERING

This is more of a student guided learning chapter. Since this is not a typical textbook section that consists of an instructor lecturing the student, this chapter might at times seem incomplete. It is intended to be incomplete as it is up to the student to complete it with guidance from a live teacher (whether in-person or over the internet). The basic idea is simple: a real engineering project is discussed in detail by the teacher in a manner that is consistent with real world activities as opposed to a more typical lecture (which has its place, just not here).

The case study herein is to design and construct a detector characterization laboratory which will include examination of said design using **professional papers** (examples of pre-peer reviewed papers can be found at: https://arxiv.org). The student is expected to do all the research to prepare for discussion each day as if they were in a meeting to determine the direction of the engineering project. This is a participatory activity. In general there are really no wrong or right answers as long as they are within the scope of the research (if we ask what a chair is and you say it is a coffee cup...then yes that is wrong - but to say a chair is a couch is open to debate which should then occur among your classmates). It is expected that the instructor will be sufficiently skilled[1] in the subject matter to be able to take over the conversation and help the student navigate the subjects that they are not ready to handle as freshman. Lectures might occur when these type of road blocks emerge, but they should be infrequent.

The intend of this excursion is to build something to highlight different engineering disciplines. This particular case study centering on detectors will highlight electrical engineering, civil engineering, mechanical engineering, optical engineering, system engineering, chemical engineering, and materials engineering. There is no reason a different case study (say bridges) could be done, but it should follow the methodology highlighted here.

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[1]Note if the instructor is not sufficiently skilled in detector characterization then we would suggest the instructor modifies this to something he is skilled in, like a, the design of a bridge or water treatment plant, etc.

13.1: Example case study involving detectors characterization

13.2: Information for example case study involving detector characterization

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13.1: Example case study involving detectors characterization

Outline for the creation of a Detector Characterization Laboratory

Scott D. Johnson

The assignment for this section:

THIS IS A LECTURE/PROBLEM POSER, NOT YOUR ESSAY. You may use this as reference for numbers and facts, but DO NOT use this as a style reference. You are to write a ***technical*** essay following all the rules in class. This essay will be extensive (25 or more pages) so DON'T leave it to the last minute. The laboratory building must be done in a CAD, not by hand.

Note: The numbers should be changed for different semesters (try to be challenging). If is possible that the numbers are not doable. That is part of the problem, though you should try to get close to the numbers suggested.

The starting problem poser:

The customer, an astronomer (scientist), has stated she wishes us to use our engineering expertise to pick an appropriate detector (or maybe detectors) for her astronomical purposes, test it for this purpose, and design the camera (instrument) along with components of a satellite to make an effective astronomical tool (imager and maybe spectrometer as well...we need to advise) at a high Earth orbit.

The customer wishes the detector(s) to achieve the following minimum specifications which includes a *dark current* that must be < 0.001 e^-/pixel/sec especially in the visible portion of the spectrum, a *read noise* that must be less than 2 e^-, a spectral range cover the wavelengths between 100 nm (for the Lyman-alpha forest) to 14000 nm, and she would like for a resolution of 0.05 arcseconds in the UV and Vis, 1 arcseconds in the IR with a 1' for field of view all wavelengths. These requirements are just some of the requirements desired; we of course want to study the best device possible and will search for excellent quantum efficiency, dark current, and the like.

To achieve the customer's wishes we will design a laboratory that will test the detector we advice her to purchase. We will also advise the customer on the design of the satellite and camera (instrument) with regards to the best way to use the detector.

Our deliverable will be a report outlining the laboratory design along with our advice on the satellite and camera. Included in this report will be, at minimum, a CAD of the laboratory floor plan and a detail area maps (in particular the seismology of the chosen area).

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13.2: Information for example case study involving detector characterization

Discussion of a Detector Characterization Laboratory

Scott D. Johnson

Our goal is to characterize a number of scientific grade detectors for use in an astronomical setting. The main purpose behind this exercise is to understand a portion of what each engineer in the major disciplines might do in *designing* a laboratory. Our objective will be to design this advanced laboratory. We will need to investigate cutting-edge material to achieve this. For the sake of clarity some subjects will have to be cursorily studied in class, but should be more fully studied outside of class.

A report is to be produced about the laboratory design itself, not about what the various engineers would do (which might be discussed in class but is not to be in a report - not everything talked about in meetings is report worthy). For example we might want to design a very low vibration laboratory and civil engineer might work on how to do that, we will report on the results of the investigation not on what the civil engineer did to get these results.

Why design a laboratory to characterize detectors? Why not let the company give you the characterization? Or why not hire someone? Because science in general has very cutting-edge engineering/science requirements. Regular companies generally do not look at cutting-edge requirements because that is not their business (it is not very profitable in the short term). Companies that make detectors maybe able to give you some characterization information but they are in the business of selling detectors to everyone and cannot afford to do detailed tests for astronomical purposes. *Besides would you buy a product from someone based on their telling you about the product without independent verification? Is that how you buy a car???*

Optics considerations:

Let us say for purpose of an example that we looking to chose something that has about a 20 μm pitch and at least 1024x1024 pixels, with a spectral range of 100 nm to 14000 nm (note this spans from UV to IR...a very large range). This however is not fully possible, so we will need to consider how to achieve this with multiple strategies. We would like to consider all possible ranges of detectors from CCDs to CMOS devices.

We do not have a specific focal length or diameter of the telescope as the astronomer is going to leave that up to us in the first design phase.

If we assume an effective focal length of 5 m[1] and a 20 μm pitch, we get a sampling resolution of about $\text{resolution} = \arctan(\frac{20\times10^{-6}}{5}) \rightarrow 0.8 \text{ arcsec}$ (good for IR, but not for UV/Vis) and a field of view of $\text{FOV} = \arctan(0.235/5) \rightarrow 14'$. In general this does not meet any specification given on the problem poser. So we will need to do better than this. *So what have you figured out so far?*

To assure ourselves that we are not diffraction limited we will need to make sure our diameter of this example telescope is at least large enough to handle the 0.8 arcsec we wish to achieve. For practical purposes (since theory is in a perfect world) we will aim for 0.4 arcsec.

Given that we will calculate what we think would be a good diameter of the telescope. This will also give us an idea our our optical needs. We will need to discuss this with the astronomers to assure ourselves they will have the appropriate optics. The system will be more diffraction limited at the red-end of the spectrum, therefore we will only do this calculation for the 14000 nm wavelength. Using a standard equation[2] to calculate this.

$$0.4 \times \frac{2\pi}{360\times60\times60} = \frac{1.22\times14000\times10^{-9}}{x}$$

x = 8.8 meters

Therefore an effective 5 meter focal length with a 9 meter diameter telescope should work. This diameter is very large. *Can it be done?* Note that the f-number on this would be awful, so we must think about a much larger focal length. The larger the number the better the light we should be able to get in the telescope. This leads to a design issue not expressed in the problem poser. We

would like high resolution but also we would like an f-number say around 30 to 50 (f/50). The following calculation do not allow for this. This will need to be investigated more fully.

So we want to characterize a detector given a possibly large focal length and optics with a f-number of maybe f/50. It is unlikely we will be able to design a lab that will be able to handle optics as large as those prescribed so we will need design f/50 optics with a few "tricks." We will need an **optical engineer** to work with us on this design.

It maybe possible to find a detector that achieves a low dark current as required in the specifications, but it is more likely we will need to cool the detector to achieve a very low dark current. If we do need to cool, we will need to investigate what temperature is necessary to achieve the specified dark current. A cooler of some sort might be required for this project. There are many different type coolers that are used in astronomical missions, some require power some do not. If we choose a cooler that needs power we have to remember that in any satellite-type telescope (like Hubble) there are limited power requirements. This may require a redesign of a cooler for size and power. We will need an **electrical engineer**, a **mechanical engineer**, a **physicist** and possibly a **chemical engineer** to help us on this portion of the laboratory.

While we might find a detector that says it achieves a read noise of ~2 e^-, it is very likely that clocking will need to be modified to achieve this. If this requires slow clocking, this may cause some problems with our dark current accumulation. Our laboratory will need electronics that is designed for flexibility so that we can test different read out speeds (clock speed) along with different temperatures. There are data acquisition systems that might be able to handle this, but it likely will have to be at least partially redesigned. We will need to both and **electrical engineer** and a **computer scientist** on our team to help us with this work.

It is clear that a lot of testing will need to be done and we will need a laboratory to do our specific tests. We could redesign and reuse an existing laboratory. Rather then reusing a laboratory space (which could be done) for this assignment it would be more instructive to actually design it. Since we intend this laboratory to be useful beyond the current work, we would like a well designed scientific-type laboratory. We will need the services of a **civil engineer** to pick the best site and design the best usable building. We will need a **mechanical engineer** to help us design (or purchase) specialized machinery within the building. We will need an electrical engineer to advise on how to keep the electronics as noise free as possible.

Finally *we will act* as a **system engineers** to design the whole system for characterization. We will design for this project, but keep in mind that we wish to utilize what we design after the project is completed. The resources are likely to be pretty expensive and doing a one-time project does not make sense for a laboratory.

Comments and Web Sites:

Detectors

- CCD: e2v (now owned by Teledyne)
- IR Detectors: Teledyne (formally Rockwell)

Vibration considerations

- Microphonics is electric noise due to **vibration**.
- A good page on the pros and cons of **vibrations** systems (but a little biased): (Herzan tutorial)
- BTW, vibration is such an important topic you can find it in courses for **mechanical engineers**, **civil engineers**, and **electrical engineers**. Some schools include vibration in their two year control systems course (or signals and systems;cross-listed into all disciplines in some schools) and some schools have a course dedicated to vibration.

Building Site considerations

- This is a great product to investigate the Earth (Google Earth): Google Earth
- seismic activity MD
- Seismic Activity

Vibration isolation tables

- Optical benches and vibration isolators: http://www.kineticsystems.com
- Good tables for vibration isolation: http://www.minusk.com
- More tables for vibration isolation: Speir Robertson

Telescopes

- Understanding Cassegrain Telescopes: https://www.scientus.org/Reflecting-Telescope-History.html
- Telescope Overview (one type): http://www.rcopticalsystems.com/telescopes/
- Geometric Optics with PhET: PhET simulations (University of Colorado Boulder)
- Aberrations, etc.: http://www.astrosurf.com/luxorion/report-aberrations2.htm

Cryogen System

- Good Dewars for a number of scientific applications: IRLABS

Pulse Tube

- Cryocooler Discussion: cryocooler
- Cyrtogenics Discussion (note there are professional papers here; just a click away): Cyrogenics at NIST
- Lakeshore.com (Janis) - cryostat/cryocoolers

Thermal Electric

- https://www.ii-vi.com/thermoelectrics/

Technical Equipment

- Sells a number of technical products that are useful for a laboratory: Newport Corporation
- Sells a number of technical products that would be useful for a laboratory: Vere Inc.

DAQ

- Data acquisition system: http://www.astro-cam.com/

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[1]This is a rather small focal length for a true massive astronomical project. *Why?* Because this is just a help aid, it is up to you to figure out the real answers.

[2]Rayleigh criterion for diffraction for a circular aperture.

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14: FUNDAMENTALS OF ENGINEERING

First a helpful suggestion

The Schaum's outline series is a series of outlines (more then 200 inexpensive books started in the 1930s) with solved problems that cover many of the subjects in engineering, science, mathematics, and other disciplines that the student might want to invest in at some point in their career.

Now the introduction to this chapter

This chapter consists of the academic core of engineering with a skim of the mathematics, science, and discipline specific ideas. This chapter while seeming purely academic is meant to give engineers a taste of the hard core portion of engineering. While someone might desire to become an engineer, the high-level of mathematics and science might discourage a possible engineer. This chapter gives a taste of what is to come. This chapter is another piece of the puzzle on what an engineer is and will hopefully help a student make a decision on whether they wish to become an engineer. Some topics might be difficult, but don't let that discourage you, it takes time to learn complex ideas (even for geniuses).

14.1: The importance of Units

14.2: Arithmetic

14.2.1: Powers and Roots

14.2.1.1: Algebraic Expressions and Formulas

14.2.1.2: Rules of Exponents and Scientific Notation

14.2.1.3: Roots and Radicals

14.2.1.4: The Binomial Theorem

14.2.2: Partial Fractions

14.2.3: Matrices and how to use them

14.2.3.1: Matrices and Matrix Operations

14.2.3.2: System of equations and Matrices

14.2.4: Combinations and Permutations

14.2.5: Complex Numbers

14.2.6: Sequences and Series

14.2.6.1: Prelude to Sequence and Series

14.2.6.2: Sequences

14.2.6.3: Infinite Series

14.2.7: Power Series

14.2.7.1: Prelude to Power Series

14.2.7.2: Power Series and Functions

14.2.7.3: Taylor and Maclaurin Series

14.2.7.4: Working with Taylor Series

14.2.7.5: Fourier Series and Transform

14.2.7.6: The Laplace Transform

14.3: Geometry
14.3.1: Basic Trigonometry
14.3.2: Trigonometric Functions
14.3.3: Trigonometric Identities

14.4: Analytic Geometry
14.4.1: Circles
14.4.2: The Parabola
14.4.3: The Hyperbola
14.4.4: The Ellipse
14.4.5: Hyperbolic Functions
14.4.6: Coordinate Systems

14.5: Scalars, vectors, and tensors

14.6: Calculus

14.7: Infinitesimal calculus for derivatives
14.7.1: Table of Derivatives
14.7.2: Derivatives of Exponential and Logarithmic Functions
14.7.3: Maxima and Minima
14.7.4: Derivatives and the Shape of a Graph
14.7.5: L'Hôpital's Rule
14.7.6: Newton's Method

14.8: Infinitesimal Calculus for integration
14.8.1: Table of Integrals
14.8.2: Substitution
14.8.3: Applications of Integration
14.8.3.1: Prelude to Applications of Integration
14.8.3.2: Volumes of Revolution - Cylindrical Shells
14.8.3.3: Arc Length of a Curve and Surface Area
14.8.3.4: Physical Applications of Integration
14.8.3.5: Moments and Centers of Mass

14.9: Statistics and Probability

14.10: Differential equations
14.10.1: First-order Differential Equations
14.10.2: Second-order Differential Equations

14.11: Mechanics
14.11.1: Statics
14.11.2: Dynamics
14.11.2.1: Kinematics
14.11.2.2: Kinetics

14.11.3: Hydraulics
14.11.3.1: Hydrostatics
14.11.3.2: Hydrodynamics
14.11.4: Solid Mechanics
14.12: Thermodynamics (Statistical Physics)
14.13: Electrical Circuits
14.14: Signals and Systems (Control systems)
14.15: Optics
14.16: Chemistry

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14.1: The importance of Units

When working with engineering and science it is important to keep an eye on your units. This is a theme that professors are always harping on, but with seemingly little success. This might be because when assignments or tests are time constrained it seems like something you can skip. This is not TRUE. Don't skip your units check. In real life always be very careful with your units, but for a time constrained tests you need to develop a fast method to check units which usually involve various shortcuts (these shortcuts depend on the individual so that will not be discussed here, but you certainly can find many shortcut methods in the various test prep books out there - note these shortcut methods are to only be used for tests, *never when it matters the most,* like when you are building the bridge ***we or your or our family members*** are going to drive my/your/our car on).

First let us have a review of units which is to say let's just take a look at the following NIST site:

https://www.nist.gov/pml/owm/metric-si/si-units

This site covers all the most important aspects of units. In particular note how there are only seven basic units that all other units are build from. *Why these seven?* (no answer - set up a group chat to figure out why).

Let us say a car is moving at 10 ⅔ meters per second (mps), how long does it take for it to go 5 ⍰ meters? Try this problem in class and discuss the answer.

¬ The answer is...

The answer is 87 / 160 seconds (about a half of a second). Sometimes people with perfectly excellent math skills get this problem wrong. One of the most common answers to this question is 160 / 87 seconds (this is wrong!). Why is that? The simple answer, because people don't take the time to check their units. Generally they know what the unit should be but they don't check to see if their procedure will get that unit. Lets do a unit check. Let us first try each possible unit combination :

$$\frac{meters}{seconds} + meters \tag{14.1.1}$$

Ok this clearly can't be done, the units are not alike and there is no way to convert one to the other, so as suspected addition (and by association subtraction) is not the way to solve this. Ok we did the obvious, lets go to less obvious now.

$$\frac{meter}{seconds} \times meters = \frac{meters^2}{seconds} \tag{14.1.2}$$

Ok this also is not going to work. You probably suspected that, but did you check the units? Next division, but here we have two possibilities because division is not commutative (at least not until you make it multiplication).

$$\frac{meters}{seconds} \div \frac{meters}{1} = \frac{\cancel{meters}}{seconds} \times \frac{1}{\cancel{meters}} = \frac{1}{seconds} \tag{14.1.3}$$

Hmmm...looks kinda of right, but it is wrong. Dividing this way is the reason that 160 / 87 (the unit is "assumed" not checked) is one of the most common erroneous answers. Let's check the right way for completeness sake.

$$\frac{meters}{1} \div \frac{meters}{seconds} = \frac{\cancel{meters}}{1} \times \frac{seconds}{\cancel{meters}} = seconds \tag{14.1.4}$$

So now this gets us the right units and hence the right answer. Note this is a rather easy example (and can be done in your head), when problems get tougher (and can't be done in your head) this unit check can be used to help ensure you are getting the correct answer (sometimes it can even help you "remember" your formulas).

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14.2: ARITHMETIC

Arithmetic in the mathematical sense, not the elementary sense. Most of the material in this section is from other Libretexts sources with minor edits.

14.2.1: Powers and Roots

14.2.1.1: Algebraic Expressions and Formulas

14.2.1.2: Rules of Exponents and Scientific Notation

14.2.1.3: Roots and Radicals

14.2.1.4: The Binomial Theorem

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14.2.7.6: The Laplace Transform

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14.2.1: POWERS AND ROOTS

This is a review of basic powers and roots. In this class you should already know this as it is a prerequisite, but it never hurts to review.

14.2.1.1: Algebraic Expressions and Formulas

14.2.1.2: Rules of Exponents and Scientific Notation

14.2.1.3: Roots and Radicals

14.2.1.4: The Binomial Theorem

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14.2.1.1: Algebraic Expressions and Formulas

Learning Objectives

- Identify the parts of an algebraic expression.
- Apply the distributive property.
- Evaluate algebraic expressions.
- Use formulas that model common applications.

Algebraic Expressions and the Distributive Property

In algebra, letters called variables are used to represent numbers. Combinations of variables and numbers along with mathematical operations form **algebraic expressions**, or just **expressions**. The following are some examples of expressions with one variable, x:

Table 14.2.1.1.1

$2x+3$	x^2-9	$\frac{1}{x}+\frac{x}{x+2}$	$3\sqrt{x}+x$

Terms in an algebraic expression are separated by addition operators and **factors** are separated by multiplication operators. The numerical factor of a term is called the **coefficient**. For example, the algebraic expression $x^2y^2+6xy-3$ can be thought of as $x^2y^2+6xy+(-3)$ and has three terms. The first term, x^2y^2, represents the quantity $1x^2y^2=1\cdot x\cdot x\cdot y\cdot y$ where 1 is the coefficient and x and y are the variables. All of the variable factors with their exponents form the **variable part of a term**. If a term is written without a variable factor, then it is called a **constant term**. Consider the components of $x^2y^2+6xy-3$,

Table 14.2.1.1.2

Terms	*Coefficient*	*Variable Part*
x^2y^2	1	x^2y^2
$6xy$	6	xy
-3	-3	

The third term in this expression, -3, is called a constant term because it is written without a variable factor. While a variable represents an unknown quantity and may change, the constant term does not change.

Example 14.2.1.1.1:

List all coefficients and variable parts of each term: $10a^2-5ab-b^2$.

Solution

We want to think of the third term in this example $-b^2$ as $-1b^2$.

Table 14.2.1.1.3

Terms	Coefficient	Variable Part
$10a^2$	10	a^2
$-5ab$	-5	ab
$-b^2$	-1	b^2

Answer: Coefficients: $\{-5,\ -1,\ 10\}$; Variable parts: $\{a^2,\ ab,\ b^2\}$

In our study of algebra, we will encounter a wide variety of algebraic expressions. Typically, expressions use the two most common variables, x and y. However, expressions may use any letter (or symbol) for a variable, even Greek letters, such as alpha (α) and beta (β). Some letters and symbols are reserved for constants, such as $\pi \approx 3.14159$ and $e \approx 2.71828$. Since there is only a limited number of letters, you will also use subscripts, $x_1, x_2, x_3, x_4, \ldots$, to indicate different variables.

The properties of real numbers are important in our study of algebra because a variable is simply a letter that represents a real number. In particular, the **distributive property** states that if given any real numbers a, b and c, then,

$$a\ (b + c) = ab + ac$$

This property is one that we apply often when simplifying algebraic expressions. To demonstrate how it will be used, we simplify $2(5 - 3)$ in two ways, and observe the same correct result.

Table 14.2.1.1.4

Working parenthesis first.	Using the distributive property.
$2(5-3) = 2(2)$ $= 4$	$2(5-3) = 2\cdot 5 - 2\cdot 3$ $= 10 - 6$ $= 4$

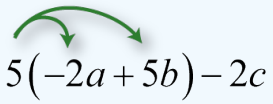
Certainly, if the contents of the parentheses can be simplified we should do that first. On the other hand, when the contents of parentheses cannot be simplified any further, we multiply every term within it by the factor outside of it using the distributive property. Applying the distributive property allows us to multiply and remove the parentheses.

Example 14.2.1.1.2:

Simplify: $5(-2a + 5b) - 2c$.

Solution

Multiply only the terms grouped within the parentheses for which we are applying the distributive property.



$$5(-2a+5b)-2c$$

$$=5\cdot(-2a)+5\cdot 5b-2c$$

$$=-10a+25b-2c$$

Answer: $-10a+25b-2c$

Recall that multiplication is commutative and therefore we can write the distributive property in the following manner, $(b+c)a=ba+ca$.

Example 14.2.1.1.3:

Simplify: $(3x-4y+1)\cdot 3$.

Solution

Multiply all terms within the parenthesis by 3.

$$(3x-4y+1)\cdot 3=3x\cdot 3-4y\cdot 3+1\cdot 3$$

$$=9x-12y+3$$

Answer: $9x-12y+3$

Terms whose variable parts have the same variables with the same exponents are called **like terms**, or **similar terms**. Furthermore, constant terms are considered to be like terms. If an algebraic expression contains like terms, apply the distributive property as follows:

$$5x+7x=(5+7)x=12x$$

In other words, if the variable parts of terms are exactly the same, then we can add or subtract the coefficients to obtain the coefficient of a single term with the same variable part. This process is called **combining like terms**. For example,

$$12x^2y^3+3x^2y^3=15x^2y^3$$

Notice that the variable factors and their exponents do not change. Combining like terms in this manner, so that the expression contains no other similar terms, is called **simplifying the expression**[1]**.** Use this idea to simplify algebraic expressions with multiple like terms.

Example 14.2.1.1.4:

Simplify:

$x^2-10x+8+5x^2-6x-1$.

Solution

Identify the like terms and add the corresponding coefficients.

Answer: $6x^2-16x+7$

Example 14.2.1.1.5:

Simplify: $a^2b^2 - ab - 2\left(2a^2b^2 - 5ab + 1\right)$.

Solution

Distribute -2 and then combine like terms.

Answer: $-3a^2b^2 + 9ab - 2$

Evaluating Algebraic Expressions

An algebraic expression can be thought of as a generalization of particular arithmetic operations. Performing these operations after substituting given values for variables is called **evaluating**. In algebra, a variable represents an unknown value. However, if the problem specifically assigns a value to a variable, then you can replace that letter with the given number and evaluate using the order of operations.

Example 14.2.1.1.9:

Evaluate $\sqrt{b^2 - 4ac}$ where $a = -1, b = -7$, and $c = \frac{1}{4}$.

Solution

Substitute in the appropriate values and then simplify.

$$\begin{aligned} &= \sqrt{49 + 4(\frac{1}{4})} \\ &= \sqrt{49 + 1} \\ &= \sqrt{50} \\ &= \sqrt{25 \cdot 2} \\ &= 5\sqrt{2} \end{aligned}$$

Aligned: $5\sqrt{2}$

Exercise 14.2.1.1.1

Evaluate $\frac{\sqrt{3\pi Vh}}{\pi h}$ where $V = 25\pi$ and $h = 3$.

Answer available at the end of this section.

Using Formulas

The main difference between algebra and arithmetic is the organized use of variables. This idea leads to reusable **formulas**, which are mathematical models using algebraic expressions to describe common applications. For example, the volume of a right circular cone depends on its radius r and height h and is modeled by the formula:

$$V = \frac{1}{3}\pi r^2 h$$

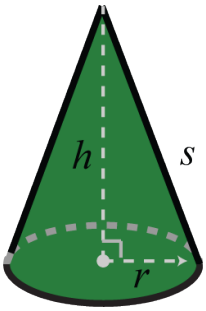


Figure 14.2.1.1.1: Cone with height h, radius r, and side length s.

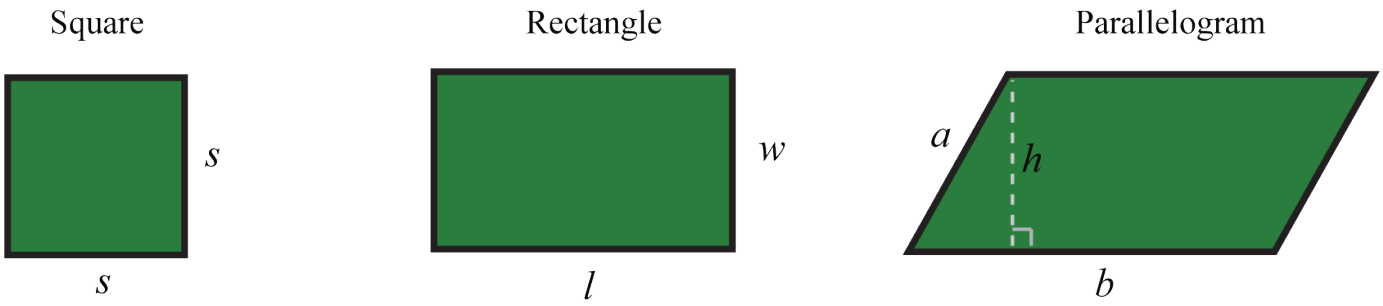
In this equation, variables and constants are used to describe the relationship between volume and the length of the base and height. If the radius of the base measures 3 meters and the height measures 5 meters, then the volume can be calculated using the formula as follows:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(3m)^2(5m) \\ &= \frac{1}{\cancel{3}}\pi\cdot \cancel{9}^{\,3}\cdot 5m^3 \\ &= 15\pi \mathrm{m}^3 \end{aligned}$$

Using $\pi \approx 3.14$, we can approximate the volume: $V \approx 15(3.14) = 47.1$ cubic meters.

Formulas for areas and perimeters of common plane figures

A list of formulas that describe the area and perimeter of common plane figures follows. The letter *P* represents perimeter and is measured in linear units. The letter *A* represents area and is measured in square units.



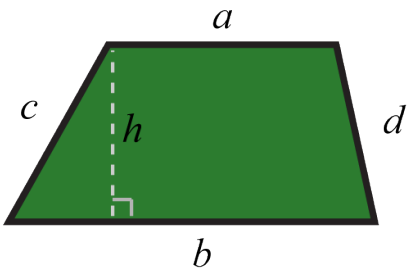
$P = 4s$ $A = s^2$

$P = 2l + 2w$ $A = lw$

$P = 2a + 2b$ $A = bh$

Figure 14.2.1.1.2: Square, rectangle, and parallelogram formulas for perimeter and area.

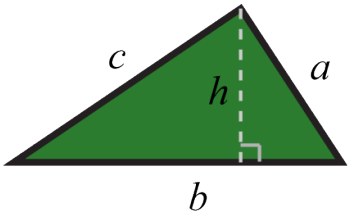
Trapezoid



$$P = a + b + c + d$$

$$A = \frac{1}{2}h(a + b)$$

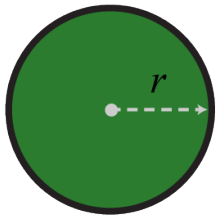
Triangle



$$P = a + b + c$$

$$A = \frac{1}{2}bh$$

Circle



$$C = 2\pi r$$

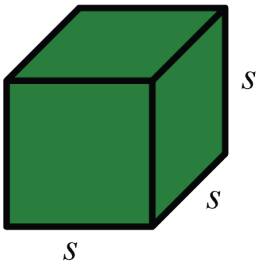
$$A = \pi r^2$$

Figure 14.2.1.1.3: Trapezoid, Triangle, Circle formulas for perimeter and area.

List of formulas for surface area and volume for common figures

A list of formulas that describe the surface area and volume of common figures follows. Here *SA* represents surface area and is measured in square units. The letter *V* represents volume and is measured in cubic units.

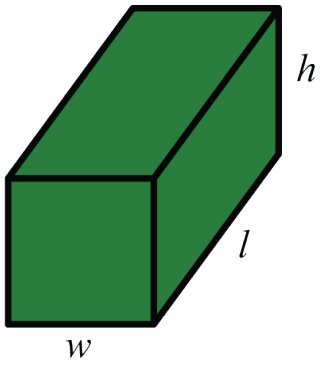
Cube



$$SA = 6s^2$$

$$V = s^3$$

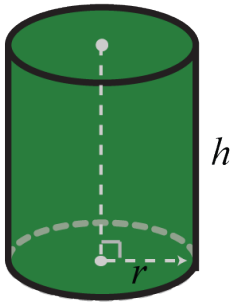
Rectangular Solid



$$SA = 2lw + 2lh + 2wh$$

$$V = lwh$$

Right Circular Cylinder

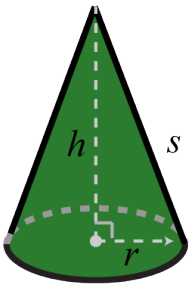


$$SA = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$

Figure 14.2.1.1.4: Cube, rectangular solid, and right circular cylinder formulas for volume and surface area.

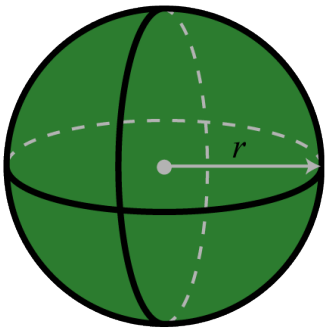
Right Circular Cone



$$SA = \pi r^2 + \pi rs$$

$$V = \frac{1}{3}\pi r^2 h$$

Sphere



$$SA = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$

Figure 14.2.1.1.5: Right circular cone and sphere formulas for volume and surface area.

Example 14.2.1.1.10:

The diameter of a spherical balloon is 10 inches. Determine the volume rounded off to the nearest hundredth.

Solution

The formula for the volume of a sphere is

$$V = \tfrac{4}{3}\pi r^3$$

This formula gives the volume in terms of the radius, r. Therefore, divide the diameter by 2 and then substitute into the formula. Here, $r = \frac{10}{2} = 5$ inches and we have

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(5\text{in})^3 \\ &= \frac{4}{3}\pi \cdot 125\text{in}^3 \\ &= \frac{500\pi}{3}\text{in}^3 \approx 523.60\text{in}^3 \end{aligned}$$

Answer: The volume of the balloon is approximately 523.60 cubic inches.

Uniform motion

Formulas can be found in a multitude of subjects. For example, **uniform motion** is modeled by the formula $D = rt$, which expresses distance D, in terms of the average rate, or speed, r and the time traveled at that rate, t. This formula, $D = rt$, is used often and is read, "*distance equals rate times time*." Note we use here speed, not velocity as we have not yet introduced vectors.

Example 14.2.1.1.11:

Jim's road trip took $2\ \frac{1}{2}$ hours at an average speed of 66 miles per hour. How far did he travel?

Solution

Substitute the appropriate values into the formula and then simplify.

$$\begin{aligned} D &= r \cdot t \\ &= (66\frac{\text{mi}}{\text{hr}}) \cdot (2\frac{1}{2}\text{hr}) \\ &= \frac{66}{1} \cdot \frac{5}{2}\text{mi} \\ &= 33 \cdot 5\text{mi} \\ &= 165\text{mi} \end{aligned}$$

Answer: Jim traveled 165 miles.

Simple interest I is given by the formula $I = prt$, where p represents the principal amount invested at an annual interest rate r for t years.

Example 14.2.1.1.12:

Calculate the simple interest earned on a 2-year investment of $\$1,250$ at an annual interest rate of $3\ \frac{3}{4}$.

Solution

Convert $3\ \frac{3}{4}$ to a decimal number before using it in the formula.

$$r = 3\tfrac{3}{4}\% = 3.75\% = 0.0375$$

Use this and the fact that $p = \$1,250$ and $t = 2$ years to calculate the simple interest.

$$\begin{aligned} I &= prt \\ &= (1,250)(0.0375)(2) \\ &= 93.75 \end{aligned}$$

Answer: The simple interest earned is $\$93.75$.

---

[1]Note that combining like terms is relevant to units as well, following the same rules. For instance, cm + cm^2 are not like terms and can't be added but cm^2 + cm^2 are like terms and can be added together.

Answers

Exercise 14.2.1.1.1

5

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14.2.1.2: Rules of Exponents and Scientific Notation

Learning Objectives

- Review the rules of exponents.
- Review the definition of negative exponents and zero as an exponent.
- Work with numbers using scientific notation.

Review of the Rules of Exponents

In this section, we review the rules of exponents. Recall that if a factor is repeated multiple times, then the product can be written in exponential form x^n. The positive integer exponent n indicates the number of times the base x is repeated as a factor.

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n\,times}$$

Consider the product of x^4 and x^6,

$$x^4 \cdot x^6 = \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}_{10\ factors\ of\ x} = x^{10}$$

Expanding the expression using the definition produces multiple factors of the base which is quite cumbersome, particularly when n is large. For this reason, we have useful rules to help us simplify expressions with exponents. In this example, notice that we could obtain the same result by adding the exponents.

$$x^4 \cdot x^6 = x^{4+6} = x^{10} \quad Product\ rule\ for\ exponents$$

In general, this describes the **product rule for exponents**[1]. In other words, when multiplying two expressions with the same base we add the exponents. Compare this to raising a factor involving an exponent to a power, such as $\left(x^6\right)^4$.

$$\begin{aligned}\left(x^6\right)^4 &= \underbrace{x^6 \cdot x^6 \cdot x^6 \cdot x^6}_{4\ factors\ of\ x^6} \\ &= x^{6+6+6+6} \\ &= x^{24}\end{aligned}$$

Here we have 4 factors of x^6, which is equivalent to multiplying the exponents.

$$\left(x^6\right)^4 = x^{6 \cdot 4} = x^{24} \quad Power\ rule\ for\ exponents$$

This describes the **power rule for exponents**[2]. Now we consider raising grouped products to a power. For example,

$$\begin{aligned}\left(x^2y^3\right)^4 &= x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \cdot x^2y^3 \\ &= x^2 \cdot x^2 \cdot x^2 \cdot x^2 \cdot y^3 \cdot y^3 \cdot y^3 \cdot y^3 \quad Commutative\ property \\ &= x^{2+2+2+2} \cdot y^{3+3+3+3} \\ &= x^8y^{12}\end{aligned}$$

After expanding, we are left with four factors of the product x^2y^3. This is equivalent to raising each of the original grouped factors to the fourth power and applying the power rule.

$$(x^2y^3)^4 = (x^2)^4 (y^3)^4 = x^8y^{12}$$

In general, this describes the use of the power rule for a product as well as the power rule for exponents. In summary, the rules of exponents streamline the process of working with algebraic expressions and will be used extensively as we move through our study of algebra. Given any positive integers m and n where $x, y \neq 0$ we have

Table 14.2.1.2.1

Product rule for exponents:	$x^m \cdot x^n = x^{m+n}$
Quotient rule for exponents:	$\frac{x^m}{x^n} = x^{m-n}$
Power rule for exponents:	$(x^m)^n = x^{m \cdot n}$
Power rule for a product:[3]	$(xy)^n = x^n y^n$
Power rule for a quotient:[4]	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

These rules allow us to efficiently perform operations with exponents.

Example 14.2.1.2.1:

Simplify: $\frac{10^4 \cdot 10^{12}}{10^3}$

Answer available at the end of this section.

In the previous example, notice that we did not multiply the base 10 times itself. When applying the product rule, add the exponents and leave the base unchanged.

Example 14.2.1.2.2:

Simplify: $(x^5 \cdot x^4 \cdot x)^2$

Solution: Recall that the variable x is assumed to have an exponent of one, $x = x^1$.

Answer available at the end of this section.

The base could in fact be any algebraic expression.

Example 14.2.1.2.3:

Simplify: $(x+y)^9(x+y)^{13}$

Solution: Treat the expression $(x+y)$ as the base.

Answer available at the end of this section.

The commutative property of multiplication allows us to use the product rule for exponents to simplify factors of an algebraic expression.

Example 14.2.1.2.4:

Simplify: $-8x^5y \cdot 3x^7y^3$

Answer available at the end of this section.

Division involves the quotient rule for exponents.

Example 14.2.1.2.5:

Simplify: $\frac{33x^7y^5(x-y)^{10}}{11x^6y(x-y)^3}$.

Answer available at the end of this section.

The power rule for a quotient allows us to apply that exponent to the numerator and denominator. This rule requires that the denominator is nonzero and so we will make this assumption for the remainder of the section.

Example 14.2.1.2.6:

Simplify: $\left(\frac{-4a^2b}{c^4}\right)^3$

Answer available at the end of this section.

Using the quotient rule for exponents, we can define what it means to have zero as an exponent. Consider the following calculation:

Twenty-five divided by twenty-five is clearly equal to one, and when the quotient rule for exponents is applied, we see that a zero exponent results. In general, given any nonzero real number x and integer n,

This leads us to the definition of **zero as an exponent**[5],

$x^0 = 1 \qquad x \neq 0$

It is important to note that 0^0 is indeterminate. If the base is negative, then the result is still positive one. In other words, any nonzero base raised to the zero power is defined to be equal to one. In the following examples assume all variables are nonzero.

Example 14.2.1.2.7:

Simplify:

a. $(-2x)^0$
b. $-2x^0$

Answer available at the end of this section.

In general, given any nonzero real number x and integer n, the definition of **negative exponents**[6] is:

$x^{-n} = \frac{1}{x^n} \qquad x \neq 0$

An expression is completely simplified if it does not contain any negative exponents.

Example 14.2.1.2.8:

Simplify: $(-4x^2y)^{-2}$

Solution

Rewrite the entire quantity in the denominator with an exponent of 2 and then simplify further.

Answer available at the end of this section.

Sometimes negative exponents appear in the denominator.

Example 14.2.1.2.9:

Simplify: $\frac{x^{-3}}{y^{-4}}$

Answer available at the end of this section.

The previous example suggests a property of **quotients with negative exponents**[7]. Given any integers m and n where $x \neq 0$ and $y \neq 0$, then

$$\frac{x^{-n}}{y^{-m}} = \frac{\frac{1}{x^n}}{\frac{1}{y^m}} = \frac{1}{x^n} \cdot \frac{y^m}{1} = \frac{y^m}{x^n}$$

This leads us to the property

$$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$$

In other words, negative exponents in the numerator can be written as positive exponents in the denominator and negative exponents in the denominator can be written as positive exponents in the numerator.

Example 14.2.1.2.10:

Simplify: $\frac{-5x^{-3}y^3}{z^{-4}}$

Solution

Take care with the coefficient -5, recognize that this is the base and that the exponent is actually positive one: $-5 = (-5)^1$. Hence, the rules of negative exponents do not apply to this coefficient; leave it in the numerator.

Answer available at the end of this section.

Exercise 14.2.1.2.11

Simplify: $\left(\frac{2x^{-2}y^3}{z}\right)^{-4}$

Answer available at the end of this section.

Scientific Notation

Real numbers expressed using **scientific notation**[8] have the form,

$$a \times 10^n$$

where n is an integer and $1 \leq a < 10$.This form is particularly useful when the numbers are very large or very small. For example the number of atoms in a mole is 602200000000000000000000 $\frac{atoms}{mole}$ or Planck constant $\hbar$ which is 0.000000000000000000000000000000000663 $J \cdot sec$. It is cumbersome to write all the zeros in both of these cases. Scientific notation is an alternative, compact representation of these numbers. For scientific notation, the factor 10^n indicates the power of ten to multiply the coefficient by to convert back to decimal form:

$$9.46 \times 10^{15} = 9.46 \times 1,\overbrace{000,000,000,000,000}^{15\ zeros} = 9,460,000,000,000,000$$

This is equivalent to moving the decimal in the coefficient fifteen places to the right.

A negative exponent indicates that the number is very small:

$$2.5 \times 10^{-11} = 2.5 \times \frac{1}{10^{11}} = \frac{2.5}{\underbrace{100,000,000,000}_{11\ zeros}} = 0.000000000025$$

This is equivalent to moving the decimal in the coefficient eleven places to the left.

We will leave it to you to convert Planck constant and number of atoms per mole into scientific notation (you can check the internet for the correct answer).

Example 14.2.1.2.12:

Write 0.000003045 using scientific notation.

Answer available at the end of this section.

Often we will need to perform operations when using numbers in scientific notation. All the rules of exponents developed so far also apply to numbers in scientific notation.

Example 14.2.1.2.13:

Multiply: $\left(4.36 \times 10^{-5}\right)\left(5.3 \times 10^{12}\right)$

Answer available at the end of this section.

Example 14.2.1.2.14:

Divide: $\left(3.24 \times 10^{8}\right) \div \left(9.0 \times 10^{-3}\right)$

Answer available at the end of this section.

Example 14.2.1.2.15:

The speed of light is approximately 6.7×10^8 miles per hour. Express this speed in miles per second.

Answer available at the end of this section.

Example 14.2.1.2.16:

The Sun moves around the center of the galaxy in a nearly circular orbit. The distance from the center of our galaxy to the Sun is approximately $26,000$ light years. What is the circumference of the orbit of the Sun around the galaxy in meters?

Solution

One light-year measures 9.46×10^{15} meters. Therefore, multiply this by $26,000$ or 2.60×10^4 to find the length of $26,000$ light years in meters.

$$\begin{aligned}\left(9.46 \times 10^{15}\right)\left(2.60 \times 10^4\right) &= 9.46 \cdot 2.60 \times 10^{15} \cdot 10^4 \\ &\approx 24.6 \times 10^{19} \\ &= 2.46 \times 10^1 \cdot 10^{19} \\ &= 2.46 \times 10^{20}\end{aligned}$$

The radius r of this very large circle is approximately 2.46×10^{20} meters. Use the formula $C = 2\pi r$ to calculate the circumference of the orbit.

$$\begin{aligned}C &= 2\pi r \\ &\approx 2(3.14)\left(2.46 \times 10^{20}\right) \\ &= 15.4 \times 10^{20} \\ &= 1.54 \times 10^1 \cdot 10^{20} \\ &= 1.54 \times 10^{21}\end{aligned}$$

Answer available at the end of this section.

Exercise 14.2.1.2.17

Divide: $\left(3.15 \times 10^{-5}\right) \div \left(12 \times 10^{-13}\right)$.

Answer available at the end of this section.

Key Takeaways

- When multiplying two quantities with the same base, add exponents: $x^m \cdot x^n = x^{m+n}$.
- When dividing two quantities with the same base, subtract exponents: $\frac{x^m}{x^n} = x^{m-n}$.
- When raising powers to powers, multiply exponents: $(x^m)^n = x^{m \cdot n}$.
- When a grouped quantity involving multiplication and division is raised to a power, apply that power to all of the factors in the numerator and the denominator: $(xy)^n = x^n y^n$ and $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.
- Any nonzero quantity raised to the 0 power is defined to be equal to 1: $x^0 = 1$.
- Expressions with negative exponents in the numerator can be rewritten as expressions with positive exponents in the denominator: $x^{-n} = \frac{1}{x^n}$.
- Expressions with negative exponents in the denominator can be rewritten as expressions with positive exponents in the numerator: $\frac{1}{x^{-m}} = x^m$.
- Take care to distinguish negative coefficients from negative exponents.
- Scientific notation is particularly useful when working with numbers that are very large or very small.

Exercise 14.2.1.2.3

Simplify. (Assume all variables represent nonzero numbers.)

1. $10^4 \cdot 10^7$
2. $7^3 \cdot 7^2$
3. $\dfrac{10^2 \cdot 10^4}{10^5}$
4. $\dfrac{7^5 \cdot 7^9}{7^2}$
5. $x^3 \cdot x^2$
6. $y^5 \cdot y^3$
7. $\dfrac{a^8 \cdot a^6}{a^5}$
8. $\dfrac{b^4 \cdot b^{10}}{b^8}$
9. $\dfrac{x^{2n} \cdot x^{3n}}{x^n}$
10. $\dfrac{x^n \cdot x^{8n}}{x^{3n}}$
11. $\left(x^5\right)^3$
12. $\left(y^4\right)^3$
13. $\left(x^4y^5\right)^3$
14. $\left(x^7y\right)^5$
15. $\left(x^2y^3z^4\right)^4$
16. $\left(xy^2z^3\right)^2$
17. $\left(-5x^2yz^3\right)^2$
18. $\left(-2xy^3z^4\right)^5$
19. $\left(x^2yz^5\right)^n$
20. $\left(xy^2z^3\right)^{2n}$
21. $\left(x \cdot x^3 \cdot x^2\right)^3$
22. $\left(y^2 \cdot y^5 \cdot y\right)^2$
23. $\dfrac{a^2 \cdot \left(a^4\right)^2}{a^3}$
24. $\dfrac{a \cdot a^3 \cdot a^2}{\left(a^2\right)^3}$
25. $(2x+3)^4(2x+3)^9$
26. $(3y-1)^7(3y-1)^2$
27. $(a+b)^3(a+b)^5$
28. $(x-2y)^7(x-2y)^3$
29. $5x^2y \cdot 3xy^2$
30. $-10x^3y^2 \cdot 2xy$
31. $-6x^2yz^3 \cdot 3xyz^4$
32. $2xyz^2\left(-4x^2y^2z\right)$
33. $3x^ny^{2n} \cdot 5x^2y$
34. $8x^{5n}y^n \cdot 2x^{2n}y$
35. $\dfrac{40x^5y^3z}{4x^2y^2z}$
36. $\dfrac{8x^2y^5z^3}{16x^2yz}$
37. $\dfrac{24a^8b^3(a-5b)^{10}}{8a^5b^3(a-5b)^2}$
38. $\dfrac{175m^9n^5(m+n)^7}{25m^8n(m+n)^3}$
39. $\left(-2x^4y^2z\right)^6$
40. $\left(-3xy^4z^7\right)^5$
41. $\left(\dfrac{-3ab^2}{2c^3}\right)^3$

42. $\left(\frac{-10a^3b}{3c^2}\right)^2$
43. $\left(\frac{-2xy^4}{z^3}\right)^4$
44. $\left(\frac{-7x^9y}{z^4}\right)^3$
45. $\left(\frac{xy^2}{z^3}\right)^n$
46. $\left(\frac{2x^2y^3}{z}\right)^n$
47. $(-5x)^0$
48. $(3x^2y)^0$
49. $-5x^0$
50. $3x^2y^0$
51. $(-2a^2b^0c^3)^5$
52. $(-3a^4b^2c^0)^4$
53. $\frac{(9x^3y^2z^0)^2}{3xy^2}$
54. $\frac{(-5x^0y^5z)^3}{25y^2z^0}$
55. $-2x^{-3}$
56. $(-2x)^{-2}$
57. $a^4 \cdot a^{-5} \cdot a^2$
58. $b^{-8} \cdot b^3 \cdot b^4$
59. $\frac{a^8 \cdot a^{-3}}{a^{-6}}$
60. $\frac{b^{-10} \cdot b^4}{b^{-2}}$
61. $10x^{-3}y^2$
62. $-3x^{-5}y^{-2}$
63. $3x^{-2}y^2z^{-1}$
64. $-5x^{-4}y^{-2}z^2$
65. $\frac{25x^{-3}y^2}{5x^{-1}y^{-3}}$
66. $\frac{-9x^{-1}y^3z^{-5}}{3x^{-2}y^2z^{-1}}$
67. $(-5x^{-3}y^2z)^{-3}$
68. $(-7x^2y^{-5}z^{-2})^{-2}$
69. $\left(\frac{2x^{-3}z}{y^2}\right)^{-5}$
70. $\left(\frac{5x^5z^{-2}}{2y^{-3}}\right)^{-3}$
71. $\left(\frac{12x^3y^2z}{2x^7yz^8}\right)^3$
72. $\left(\frac{150xy^8z^2}{90x^7y^2z}\right)^2$
73. $\left(\frac{-9a^{-3}b^4c^{-2}}{3a^3b^5c^{-7}}\right)^{-4}$
74. $\left(\frac{-15a^7b^5c^{-8}}{3a^{-6}b^2c^3}\right)^{-3}$

Answer available at the end of this section.

Exercise 14.2.1.2.4

The value in dollars of a new mobile phone can be estimated by using the formula $V = 210(2t + 1)^{-1}$, where t is the number of years after purchase.

1. How much was the phone worth new?
2. How much will the phone be worth in 1 year?
3. How much will the phone be worth in 3 years?
4. How much will the phone be worth in 10 years?
5. How much will the phone be worth in 100 years?
6. According to the formula, will the phone ever be worthless? Explain.
7. The height of a particular right circular cone is equal to the square of the radius of the base, $h = r^2$. Find a formula for the volume in terms of r.
8. A sphere has a radius $r = 3x^2$.Find the volume in terms of x.

Answer available at the end of this section.

Exercise 14.2.1.2.5

Convert to a decimal number.

1. 5.2×10^8
2. 6.02×10^9
3. 1.02×10^{-6}
4. 7.44×10^{-5}

Answer available at the end of this section.

Exercise 14.2.1.2.6

Rewrite using scientific notation.

1. $7,050,000$
2. $430,000,000,000$
3. 0.00005001
4. 0.000000231

Answer available at the end of this section.

Exercise 14.2.1.2.7

Perform the operations.

1. $\left(1.2 \times 10^9\right)\left(3 \times 10^5\right)$
2. $\left(4.8 \times 10^{-5}\right)\left(1.6 \times 10^{20}\right)$
3. $\left(9.1 \times 10^{23}\right)\left(3 \times 10^{10}\right)$
4. $\left(5.5 \times 10^{12}\right)\left(7 \times 10^{-25}\right)$
5. $\frac{9.6 \times 10^{16}}{1.2 \times 10^{-4}}$
6. $\frac{4.8 \times 10^{-14}}{2.4 \times 10^{-6}}$
7. $\frac{4 \times 10^{-8}}{8 \times 10^{10}}$
8. $\frac{2.3 \times 10^{23}}{9.2 \times 10^{-3}}$
9. $987,000,000,000,000 \times 23,000,000$
10. $0.00000000024 \times 0.00000004$
11. $0.000000000522 \div 0.0000009$
12. $81,000,000,000 \div 0.0000648$
13. The population density of Earth refers to the number of people per square mile of land area. If the total land area on Earth is 5.751×10^7 square miles and the population in 2007 was estimated to be 6.67×10^9 people, then calculate the population density of Earth at that time.

14. In 2008 the population of New York City was estimated to be 8.364 million people. The total land area is 305 square miles. Calculate the population density of New York City.
15. The mass of Earth is 5.97×10^{24} kilograms and the mass of the Moon is 7.35×10^{22} kilograms. By what factor is the mass of Earth greater than the mass of the Moon?
16. The mass of the Sun is 1.99×10^{30} kilograms and the mass of Earth is 5.97×10^{24} kilograms. By what factor is the mass of the Sun greater than the mass of Earth? Express your answer in scientific notation.
17. The radius of the Sun is 4.322×10^{5} miles and the average distance from Earth to the Moon is 2.392×10^{5} miles. By what factor is the radius of the Sun larger than the average distance from Earth to the Moon?
18. One light year, 9.461×10^{15} meters, is the distance that light travels in a vacuum in one year. If the distance from our Sun to the nearest star, Proxima Centauri, is estimated to be 3.991×10^{16} meters, then calculate the number of years it would take light to travel that distance.
19. It is estimated that there are about 1 million ants per person on the planet. If the world population was estimated to be 6.67 billion people in 2007, then estimate the world ant population at that time.
20. The radius of the earth is 6.3×10^{6} meters and the radius of the sun is 7.0×10^{8} meters. By what factor is the radius of the Sun larger than the radius of the Earth?
21. A gigabyte is 1×10^{9} bytes and a megabyte is 1×10^{6} bytes. If the average song in the MP3 format consumes about 4.5 megabytes of storage, then how many songs will fit on a 4-gigabyte memory card?
22. Water weighs approximately 18 grams per mole. If one mole is about 6×10^{23} molecules, then approximate the weight of each molecule of water.

Answer available at the end of this section.

Exercise 14.2.1.2.8

1. Use numbers to show that $(x+y)^n \neq x^n + y^n$.
2. Why is 0^0 indeterminate?
3. Explain to a beginning algebra student why $2^2 \cdot 2^3 \neq 4^5$.
4. René Descartes (1637) established the usage of exponential form: a^2, a^3, and so on. Before this, how were exponents denoted?

Answer available at the end of this section.

---

[1] $x^m \cdot x^n = x^{m+n}$; the product of two expressions with the same base can be simplified by adding the exponents.

[2] $(x^m)^n = x^{mn}$; a power raised to a power can be simplified by multiplying the exponents.

[3] $(xy)^n = x^n y^n$; if a product is raised to a power, then apply that power to each factor in the product.

[4] $(xy)^n = x^n y^n$; if a quotient is raised to a power, then apply that power to the numerator and the denominator.

[5] $x^0 = 1$; any nonzero base raised to the 0 power is defined to be 1.

[6] $x^{-n} = \frac{1}{x^n}$, given any integer n, where x is nonzero.

[7] $\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$, given any integers m and n, where $x \neq 0$ and $y \neq 0$.

[8] Real numbers expressed the form $a \times 10^n$, where n is an integer and $1 \leq a < 10$.

[9] $\frac{x^m}{x^n} = x^{m-n}$; the quotient of two expressions with the same base can be simplified by subtracting the exponents.

Answers

Example 14.2.1.2.1**:**

10^{13}

Example 14.2.1.2.2**:**

x^{20}

Example 14.2.1.2.3**:**

$(x+y)^{22}$

Example 14.2.1.2.4**:**

$-24x^{12}y^4$

Example 14.2.1.2.5**:**

$3xy^4(x-y)^7$

Example 14.2.1.2.6**:**

$-\frac{64a^6b^3}{c^{12}}$

Example 14.2.1.2.7**:**

a. Any nonzero quantity raised to the zero power is equal to 1.

b. In the example, $-2x^0$, the base is x, not $-2x$.

Example 14.2.1.2.8**:**

$\frac{1}{16x^4y^2}$

Example 14.2.1.2.9**:**

$\frac{y^4}{x^3}$

Example 14.2.1.2.10**:**

$\frac{-5y^3z^4}{x^3}$

Exercise 14.2.1.2.11

$\frac{x^8z^4}{16y^{12}}$

Example 14.2.1.2.12**:**

3.045×10^{-6}

Example 14.2.1.2.13**:**

2.3108×10^{8}

Example 14.2.1.2.14**:**

3.6×10^{10}

Example 14.2.1.2.15**:**

The speed of light is approximately 1.9×10^{5} miles per second.

Example 14.2.1.2.16**:**

The circumference of the Sun's orbit is approximately 1.54×10^{21} meters.

Exercise 14.2.1.2.17

2.625×10^{7}

Exercise 14.2.1.2.3

1. 10^{11}
3. 10
5. x^5
7. a^9
9. x^{4n}
11. x^{15}
13. $x^{12}y^{15}$
15. $x^8y^{12}z^{16}$
17. $25x^4y^2z^6$
19. $x^{2n}y^nz^{5n}$
21. x^{18}
23. a^7
25. $(2x+3)^{13}$
27. $(a+b)^8$
29. $15x^3y^3$
31. $-18x^3y^2z^7$
33. $15x^{n+2}y^{2n+1}$
35. $10x^3y$
37. $3a^3(a-5b)^8$
39. $64x^{24}y^{12}z^6$
41. $-\frac{27a^3b^6}{8c^9}$

43. $\frac{16x^4y^{16}}{z^{12}}$
45. $\frac{x^n y^{2n}}{z^{3n}}$
47. 1
49. -5
51. $-32a^{10}c^{15}$
53. $27x^5y^2$
55. $-\frac{2}{x^3}$
57. a
59. a^{11}
61. $\frac{10y^2}{x^3}$
63. $\frac{3y^2}{x^2z}$
65. $\frac{5y^5}{x^2}$
67. $-\frac{x^9}{125y^6z^3}$
69. $\frac{x^{15}y^{10}}{32z^5}$
71. $\frac{216y^3}{x^{12}z^{21}}$
73. $\frac{a^{24}b^4}{81c^{20}}$

Exercise 14.2.1.2.4

1. $\$210$
3. $\$30$
5. $\$1.04$
7. $V = \frac{1}{3}\pi r^4$

Exercise 14.2.1.2.5

1. $520,000,000$
3. 0.00000102

Exercise 14.2.1.2.6

1. 7.05×10^6
3. 5.001×10^{-5}

Exercise 14.2.1.2.7

1. 3.6×10^{14}
3. 2.73×10^{34}
5. 8×10^{20}
7. 5×10^{-19}
9. 2.2701×10^{22}
11. 5.8×10^{-4}
13. About 116 people per square mile
15. 81.2
17. 1.807
19. $6.67 \times 10^{15} ants$
21. Approximately 889 songs

Exercise 14.2.1.2.8

1. Answer may vary
3. Answer may vary

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14.2.1.3: Roots and Radicals

Square and Cube Roots

Recall that a **square root**[1] of a number is a number that when multiplied by itself yields the original number. For example, 5 is a square root of 25, because $5^2 = 25$. Since $(-5)^2 = 25$, we can say that -5 is a square root of 25 as well. Every positive real number has two square roots, one positive and one negative. For this reason, we use the radical sign $\sqrt{\ }$ to denote the **principal (nonnegative) square root**[2] and a negative sign in front of the radical $-\sqrt{\ }$ to denote the negative square root.

Zero is the only real number with one square root.

$\sqrt{0} = 0$ because $0^2 = 0$

the **radicand**[3], the number inside the radical sign, can be factored as the square of another number, then the square root of e number is apparent. In this case, we have the following property:

$$\sqrt{a^2} = a \quad \text{if} \quad a \geq 0$$

Or more generally,

$$\sqrt{a^2} = |a| \quad \text{if} \quad a \in R$$

The absolute value is important because a may be a negative number and the radical sign denotes the principal square root. For example,

Make use of the absolute value to ensure a positive result.

> Simplify: $\sqrt{(x-2)^2}$.
>
> *Answer available at the end of this section.*

The importance of the use of the absolute value in the previous example is apparent when we evaluate using values that make the radicand negative. For example, when $x = 1$,

Next, consider the square root of a negative number. To determine the square root of -25, you must find a number that when squared results in -25. However, any real number squared always results in a positive number. The square root of a negative number is requires complex numbers where $i = \sqrt{-1}$. In this case the answer is $\pm 5i$ which is an imaginary number. As such, $\sqrt{-25}$ is not a real number. Therefore, the **square root function**[4] given by $f(x) = \sqrt{x}$ is not defined to be a real number if the x -values are negative. The smallest value in the domain is zero. For example, $f(0) = \sqrt{0} = 0$ and $f(4) = \sqrt{4} = 2$. Recall the graph of the square root function.

$f(x)=\sqrt{x}$

x	$f(x)$
0	0
1	1
4	2
9	3

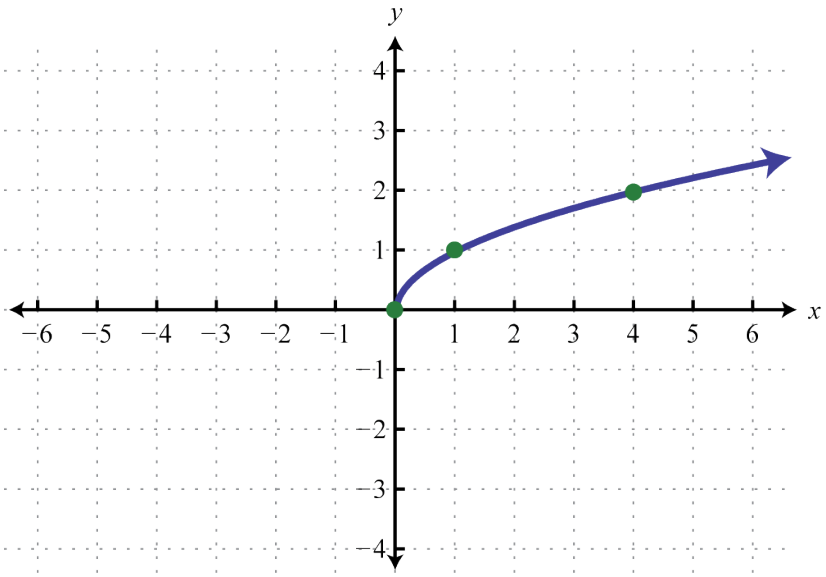


Figure 14.2.1.3.1: Graph of the square root of x using a table to figure out some of the points of the line.

The domain and range both consist of real numbers greater than or equal to zero: $[0,\infty)$. To determine the domain of a function involving a square root we look at the radicand and find the values that produce nonnegative results.

Example 14.2.1.3.3:

Determine the domain of the function defined by $f(x)=\sqrt{2x+3}$.

Solution

Here the radicand is $2x+3$. This expression must be zero or positive. In other words,

$$2x+3\geq 0$$

Solve for x.

Answer available at the end of this section.

A **cube root**[5] of a number is a number that when multiplied by itself three times yields the original number. Furthermore, we denote a cube root using the symbol $\sqrt[3]{}$, where 3 is called the **index**[6]. For example,

$$\sqrt[3]{64}=4,\ \text{because } 4^3=64$$

The product of three equal factors will be positive if the factor is positive and negative if the factor is negative. For this reason, any real number will have only one real cube root. Hence the technicalities associated with the principal root do not apply. For example,

$$\sqrt[3]{-64}=-4,\ \text{because } (-4)^3=-64\backslash)$$

In general, given any real number a, we have the following property:

$\sqrt[3]{a^3}=a \quad \text{if} \quad a\in R$

When simplifying cube roots, look for factors that are perfect cubes.

Example 14.2.1.3.4:

Evaluate.

1. $\sqrt[3]{8}$
2. $\sqrt[3]{0}$
3. $\sqrt[3]{\frac{1}{27}}$
4. $\sqrt[3]{-1}$
5. $\sqrt[3]{-125}$

Solution

1. $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$
2. $\sqrt[3]{0} = \sqrt[3]{0^3} = 0$
3. $\sqrt[3]{\frac{1}{27}} = \sqrt[3]{\left(\frac{1}{3}\right)^3} = \frac{1}{3}$
4. $\sqrt[3]{-1} = \sqrt[3]{(-1)^3} = -1$
5. $\sqrt[3]{-125} = \sqrt[3]{(-5)^3} = -5$

It may be the case that the radicand is not a perfect square or cube. If an integer is not a perfect power of the index, then its root will be irrational. For example, $\sqrt[3]{2}$ is an irrational number that can be approximated on most calculators using the root button $\sqrt[x]{}$ .Depending on the calculator, we typically type in the index prior to pushing the button and then the radicand as follows:

$$3 \quad \sqrt[x]{y} \quad 2 \quad =$$

Therefore, we have

$$\sqrt[3]{2} \approx 1.260, \qquad \text{because} \qquad 1.260\text{^}3 \approx 2$$

Since cube roots can be negative, zero, or positive we do not make use of any absolute values.

Since the cube root of a function could be either negative or positive, we conclude that the domain consists of all real numbers. Sketch the graph by plotting points. Choose some positive and negative values for x, as well as zero, and then calculate the corresponding y-values.

Table 14.2.1.3.1

x	$f(x)$	$f(x) = \sqrt[3]{x}$	*Ordered Pairs*
-8	-2		$(-8, -2)$
-1	-1		$(-1, -1)$
0	0	$f(0) = \sqrt[3]{0} = 0$	$(0, 0)$
1	1	$f(1) = \sqrt[3]{1} = 1$	$(1, 1)$
8	2	$f(8) = \sqrt[3]{8} = 2$	$(8, 2)$

Plot the points and sketch the graph of the cube root function.

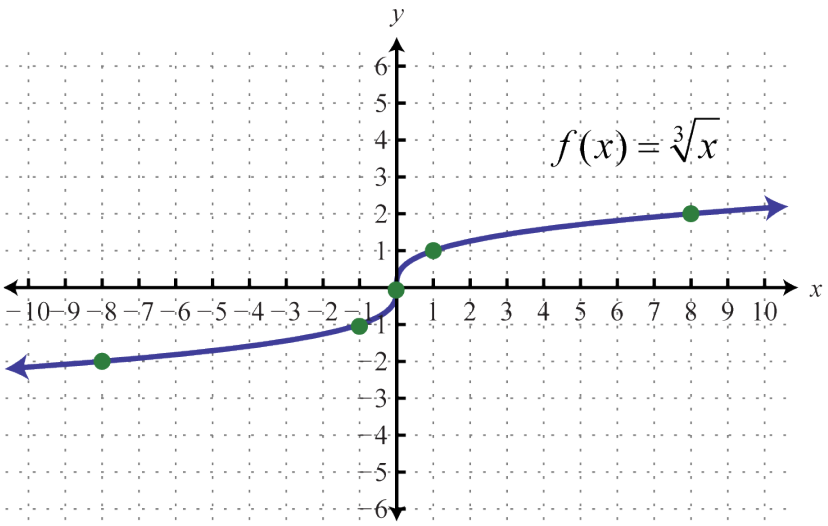


Figure 14.2.1.3.2: Graph of cube root of x

The graph passes the vertical line test and is indeed a function. In addition, the range consists of all real numbers.

Example 14.2.1.3.5:

Given $g(x) = \sqrt[3]{x+1} + 2$, find $g(-9), g(-2), g(-1)$, and $g(0)$. Sketch the graph of g.

Solution

Replace x with the given values.

Table 14.2.1.3.2

x	$g(x)$	$g(x) = \sqrt[3]{x+1} + 2$	*Ordered Pairs*
-9	0		$(-9, 0)$
-2	1		$(-2, 1)$
-1	2		$(-1, 2)$
0	3	$g(0) = \sqrt[3]{0+1} + 2 = \sqrt[3]{1} + 2 = 1 + 2 = 3$	$(0, 3)$

We can also sketch the graph using the following translations:

$$y = \sqrt[3]{x} \qquad \textit{Basic cube root function}$$
$$y = \sqrt[3]{x+1} \qquad \textit{Horizontal shift left 1 unit}$$
$$y = \sqrt[3]{x+1} + 2 \qquad \textit{Vertical shift up 2 units}$$

Answer:

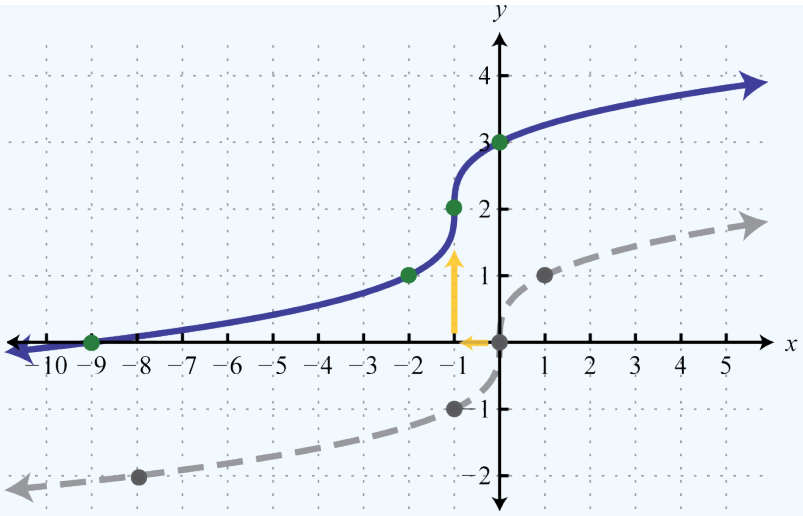


Figure 14.2.1.3.3: Example of shifting the graph of the cube root of x by taking the cube root of (x-1) plus 2, so that it shifts one to the left (negative x) and two up (positive y).

nth Roots

For any integer $n \geq 2$, we define an **nth root**[7] of a positive real number as a number that when raised to the nth power yields the original number. Given any nonnegative real number a, we have the following property:

$$\sqrt[n]{a^n} = a, \quad \text{if} \quad a \geq 0$$

Here n is called the index and a^n is called the radicand. Furthermore, we can refer to the entire expression $\sqrt[n]{A}$ as a **radical**[8]. When the index is an integer greater than or equal to 4, we say "fourth root," "fifth root," and so on. The nth root of any number is apparent if we can write the radicand with an exponent equal to the index.

Example 14.2.1.3.7:

Simplify:

1. $\sqrt[4]{81}$
2. $\sqrt[5]{32}$
3. $\sqrt[7]{1}$
4. $\sqrt[4]{\frac{1}{16}}$

Solution

1. $\sqrt[4]{81} = \sqrt[4]{3^4} = 3$
2. $\sqrt[5]{32} = \sqrt[5]{2^5} = 2$
3. $\sqrt[7]{1} = \sqrt[7]{1^7} = 1$
4. $\sqrt[4]{\frac{1}{16}} = \sqrt[4]{\left(\frac{1}{2}\right)^4} = \frac{1}{2}$

Note

If the index is $n = 2$, then the radical indicates a square root and it is customary to write the radical without the index; $\sqrt[2]{a} = \sqrt{a}$.

Lastly, for any real number a we have,

$$\sqrt[n]{a^n} = |a| \quad When\ n\ is\ even$$
$$\sqrt[n]{a^n} = a \quad When\ n\ is\ odd$$

When n *is odd*, the nth root is *positive or negative* depending on the sign of the radicand.

$$\sqrt[3]{27} = (-3)^3 = -3$$

When n *is even*, the nth root is *positive or not real* depending on the sign of the radicand.

Exercise 14.2.1.3.2

Simplify: $-8\sqrt[5]{-32}$.

Answer available at the end of this section.

Simplifying Radicals

It will not always be the case that the radicand is a perfect power of the given index. If it is not, then we use the **product rule for radicals**[10] and the **quotient rule for radicals**[11] to simplify them. Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,

Table 14.2.1.3.3

Product Rule for Radicals:	$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$
Quotient Rule for Radicals:	$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$

A **radical is simplified**[12] if it does not contain any factors that can be written as perfect powers of the index.

Example 14.2.1.3.10:

Simplify: $\sqrt{150}$.

Solution

Here 150 can be written as $2 \cdot 3 \cdot 5^2$.

$$\begin{aligned} \sqrt{150} &= \sqrt{2 \cdot 3 \cdot 5^2} && Apply\ the\ product\ rule\ for\ radicals. \\ &= \sqrt{2 \cdot 3} \cdot \sqrt{5^2} && Simplify. \\ &= \sqrt{6} \cdot 5 \\ &= 5\sqrt{6} \end{aligned}$$

We can verify our answer on a calculator:

$$\sqrt{150} \approx 12.25 \quad \text{and} \quad 5\sqrt{6} \approx 12.25$$

Also, it is worth noting that

$$12.25^2 \approx 150$$

Answer:

$5\sqrt{6}$

Note

$5\sqrt{6}$ is the exact answer and 12.25 is an approximate answer. We present exact answers unless told otherwise.

Fractional exponents as radicals

A fractional exponent is equivalent to a radical with an index being the reciprocal of the exponent. For instance $x^{\frac{1}{2}}$ is the same as $\sqrt[2]{x}$. This notation is used as a convenience when there are other exponents in your formula. To summarize,

$$x^{\frac{1}{n}} = \sqrt[n]{x} \tag{14.2.1.3.1}$$

All rules of exponents apply as before. For instance $(x^{\frac{1}{2}})^4 = x^{(\frac{1}{2})(4)} = x^2$.

---

[1]A number that when multiplied by itself yields the original number.

[2]The positive square root of a positive real number, denoted with the symbol $\sqrt{\ }$.

[3]The expression A within a radical sign, $\sqrt[n]{A}$.

[4]The function defined by $f(x) = \sqrt{x}$.

[5]A number that when used as a factor with itself three times yields the original number, denoted with the symbol $\sqrt[3]{\ }$.

[6]The positive integer n in the notation $\sqrt[n]{\ }$ that is used to indicate an n[th] root.

[7]A number that when raised to the nth power $(n \geq 2)$ yields the original number.

[8]Used when referring to an expression of the form $\sqrt[n]{A}$.

[9]The positive n[th] root when n is even.

[10]Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,$\sqrt[n]{A \cdot B} = \sqrt[n]{A} \cdot \sqrt[n]{B}$.

[11]Given real numbers $\sqrt[n]{A}$ and $\sqrt[n]{B}$,$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$ where $B \neq 0$.

[12]A radical where the radicand does not consist of any factors that can be written as perfect powers of the index.

Answers

$|x-2|$

Example 14.2.1.3.3**:**

Domain: $[-\frac{3}{2}, \infty)$

Exercise 14.2.1.3.2

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14.2.1.4: The Binomial Theorem

Binomial Theorem

A binomial is a polynomial with exactly two terms. The ***binomial theorem*** gives a formula for expanding $(x+y)^n$ for any positive integer n.

How do we expand a product of polynomials? We pick one term from the first polynomial, multiply by a term chosen from the second polynomial, and then multiply by a term selected from the third polynomial, and so forth. In the special case of $(x+y)^n$, we are selecting either x or y from each of the n binomials $x+y$ to form a product. Some of these products will be identical, hence, we need to collect their coefficients. The expansion of $(x+y)^3$ is demonstrated below.

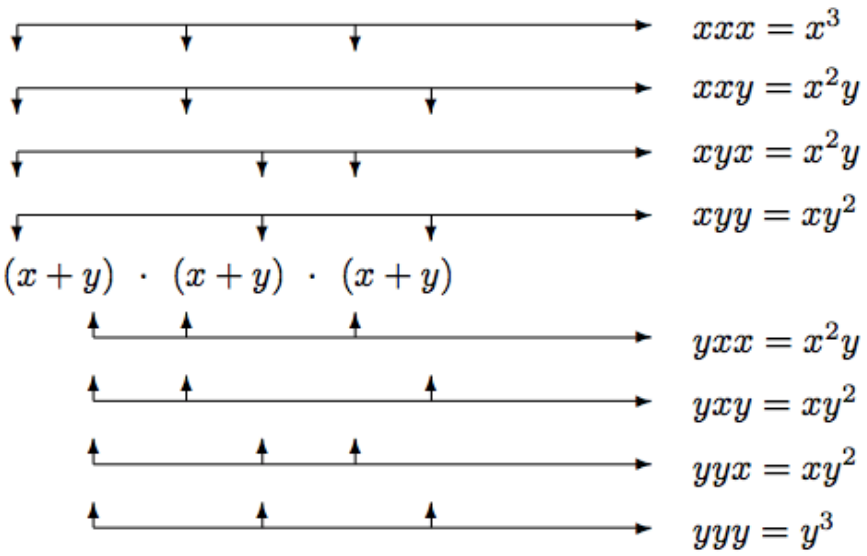


Figure 14.2.1.4.1: The expansion of $(x+y)^3$ is demonstrated here in a sequential sense.

We find

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + x^2y + x^2y + xy^2 + x^2y + xy^2 + xy^2 + y^3\\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

What happens when we expand $(x+y)^n$?

If we select y from k copies of the $(x+y)$s, and x from the other $n-k$ copies, their product will be $x^{n-k}y^k$. Therefore, in the expansion of $(x+y)^n$, a typical term will be of the form $x^{n-k}y^k$, where $0 \le k \le n$. The question is, what is its coefficient in the expansion, after we collect like terms? This coefficient is the number of times the product $x^{n-k}y^k$ appears when we multiply out $(x+y)^n$ in the way described above. It depends on which k copies of the $(x+y)$s we will choose y from. There are $\binom{n}{k}$ choices, hence, the product $x^{n-k}y^k$ appears $\binom{n}{k}$ times. Thus, the coefficient is $\binom{n}{k}$. For this reason, we also call $\binom{n}{k}$ the ***binomial coefficients***.

Theorem 14.2.1.4.1 (Binomial Theorem)

For any positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Because of the symmetry in the formula, we can interchange x and y. In addition, we also have $\binom{n}{k} = \binom{n}{n-k}$. Consequently, the binomial theorem can be written in three other forms:

$$\begin{aligned}(x+y)^n &= \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k,\\ (x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{k} y^{n-k},\\ (x+y)^n &= \sum_{k=0}^{n} \binom{n}{n-k} x^{k} y^{n-k}.\end{aligned}$$

You need not worry which one to use. They are all the same! This is how to remember these four different forms. In each term, the powers of x and y always add up to n. If the power of one of the two variables is k, where $0 \le k \le n$, then the power of the other must be $n-k$, and we need to multiply the coefficient $\binom{n}{k}$, which is the same as $\binom{n}{n-k}$, to their product.

When expanding $(x+y)^n$, it may be helpful if you first lay out all the terms x^n, $x^{n-1}y$, $x^{n-2}y^2$, and so forth. Then you fill in with the binomial coefficients. For instance, to expand $(x+y)^3$, we first list all the terms that we expect fo find:

$$(x+y)^3 = _x^3 + _x^2 y + _xy^2 + _y^3.$$

Next we fill in the binomial coefficients:

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2 y + \binom{3}{2}xy^2 + \binom{3}{3}y^3.$$

Finally, evaluate the binomial coefficients and simplify the result.

$$(x+y)^3 = x^3 + 3x^2 y + 3xy^2 + y^3.$$

In a similar way, we also find $(x-y)^3 = x^3 - 3x^2 y + 3xy^2 - y^3$. Note the similarity between the two expansions.

Example 14.2.1.4.1

Compute $(x+y)^4$.

Solution available at the end of this section.

Example 14.2.1.4.2

Compute $(x-y)^4$.

Solution available at the end of this section.

Example 14.2.1.4.3

Expand $(2x-3y)^5$.

Solution available at the end of this section.

hands-on Exercise 14.2.1.4.1

Use the binomial theorem to expand $(3x-5y)^4$.

Example 14.2.1.4.4

Find the coefficient of x^3 in the expansion of $(1+x)^{102}$.

Solution available at the end of this section.

Example 14.2.1.4.5

What is the coefficient of t^4 in the expansion of $(2+3t)^9$?

Solution available at the end of this section.

Example 14.2.1.4.6

What is the coefficient of t^5 in the expansion of $(3-2t)^7$?

Solution available at the end of this section.

hands-on Exercise 14.2.1.4.2

What is the coefficient of t^5 in $(1+3t)^8$?

hands-on Exercise 14.2.1.4.3

What is the coefficient of t^4 in the expansion of $(2-5t)^9$?

Example 14.2.1.4.7

What is the coefficient of t^6 in the expansion of $(4+5t^2)^8$?

Solution available at the end of this section.

hands-on Exercise 14.2.1.4.4

What is the coefficient of t^9 in the expansion of $(3-2t^3)^8$?

The constant term in an expansion does not contain any variable. It can be interpreted as the term containing x^0.

Example 14.2.1.4.8

Find the constant term in the expansion of $\left(x+\frac{2}{x}\right)^8$.

Solution available at the end of this section.

hands-on Exercise 14.2.1.4.5

Find the constant term in the expansion of the two expressions $\left(x+\frac{3}{x}\right)^9$ and $\left(2x-\frac{3}{x}\right)^{10}$.

Example 14.2.1.4.9

Determine the coefficient of x^7 in the expansion of $(1+x+x^2)(1+x)^{10}$.

Solution available at the end of this section.

hands-on Exercise 14.2.1.4.6

Find the coefficient of x^8 in the expansion of $(1-2x+3x^2)(1+2x)^{12}$.

Pascal's Triangle

To compute the binomial coefficients quickly, one may use the ***Pascal triangle***, in which the nth row ($n \geq 0$) consists of the binomial coefficients $\binom{n}{k}$, where $0 \leq k \leq n$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Constructing the Pascal triangle is easy. We generate the rows one at a time. The extreme ends are always 1. Each of the interior entries is the sum of the two entries right above it in the preceding row. For instance, the next row (for $n=7$) should be

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$

Such computations produce the right binomial coefficients, because of the next result.

Theorem 14.2.1.4.2 (Pascal's Identitity)

For all integers n and k satisfying $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(Analytic Proof) available at the end of this section.

hands-on Exercise 14.2.1.4.7

Determine the 8th and the 9th rows in the Pascal's triangle.

Example 14.2.1.4.10

Use the Pascal's triangle to expand

a. $(C-D)^5$
b. $(2A+5B)^3$
c. $(3C-4B)^4$

Solution available at the end of this section.

Many interesting results can be derived from the binomial theorem.

Example 14.2.1.4.11

Setting $x=y=1$, we obtain a simple (analytic) proof of the familiar identity $2^n=\sum_{k=0}^{n}\binom{n}{k}$.

Example 14.2.1.4.12

Letting $x=1$ and $y=-1$ yields $0=\sum_{k=0}^{n}(-1)^k\binom{n}{k}$. We can rewrite it as

$$\binom{n}{0}+\binom{n}{2}+\cdots=\binom{n}{1}+\binom{n}{3}+\cdots.$$

Combinatorially, this means the number of subsets of even cardinalities equals the number of subsets of odd cardinalities.

Summary and Review

- The binomial theorem can be expressed in four different but equivalent forms.
- The expansion of $(x+y)^n$ starts with x^n, then we decrease the exponent in x by one, meanwhile increase the exponent of y by one, and repeat this until we have y^n.
- The next few terms are therefore $x^{n-1}y$, $x^{n-2}y^2$, etc., which end with y^n.
- In general, the sum of exponents in x and y is always n. Hence, the general term is x^ky^{n-k}, whose coefficient is $\binom{n}{k}$.
- The expansion of $(x+y)^n$ and $(x-y)^n$ look almost identical, except that the signs in $(x-y)^n$ alternate.

Exercise 14.2.1.4.1

Use binomial theorem to expand the following expressions:

a. $(x+y)^5$
b. $(s-t)^6$
c. $(a+3b)^4$

Exercise 14.2.1.4.2

Find the coefficient of

a. $x^{11}y^3$ in $(x+y)^{14}$
b. x^4y^7 in $(2x-y)^{11}$
c. x^4y^3 in $(3x+2y)^7$
d. x^5 in $(1-x+x^2)(1+x)^7$

Exercise 14.2.1.4.4

Find the constant term in the expansion of

a. $\left(x+\frac{1}{x}\right)^4$
b. $\left(3x-\frac{2}{5x^2}\right)^9$
c. $\left(3x^2-\frac{5}{7x^3}\right)^4$
d. $\left(1-x^2+x^3\right)\left(3x^2-\frac{5}{7x^3}\right)^6$

Exercise 14.2.1.4.4

Show that $\sum_{k=0}^{n}\binom{n}{k}2^k=3^n$ for any positive integer n.

Exercise 14.2.1.4.5

Let n be a positive integer. Evaluate $\sum_{k=0}^{n}\binom{n}{k}r^k$ for any real number r.

Exercise 14.2.1.4.6

Find a closed form for the summation $\sum_{k=0}^{n}k\binom{n}{k}$.

Hint available at the end of this section.

Exercise 14.2.1.4.7

The objective of this problem is to derive a formula for $\sum_{k=1}^{n}k^2$.

a. Use induction to show that

$$\sum_{k=1}^{n}\binom{k}{1}=\frac{n(n+1)}{2}$$

for any positive integer n.

b. Use induction to show that

$$\sum_{k=1}^{n}\binom{k}{2}=\frac{n(n+1)(n-1)}{3!}$$

for any positive integer n

c. Find the integers a and b such that

$$k^2=a\binom{k}{2}+b\binom{k}{1}.$$

d. From part (c), we obtain

$$\sum_{k=1}^{n}k^2=a\sum_{k=1}^{n}\binom{k}{2}+b\sum_{k=1}^{n}\binom{k}{1}.$$

Apply the results from parts (a) and (b) to derive a formula for $\sum_{k=1}^{n}k^2$.

Hint (b) available at the end of this section.

Hint (c) available at the end of this section.

Exercise 14.2.1.4.8

The objective of this problem is to derive a formula for $\sum_{k=1}^{n}k^3$.

a. Use induction to show that

$$\sum_{k=1}^{n} \binom{k}{3} = \frac{n(n+1)(n-1)(n-2)}{4!}$$

for any positive integer n.

b. Find the integers a, b, and c such that

$$k^3 = a\binom{k}{3} + b\binom{k}{2} + c\binom{k}{1}.$$

c. Apply the results from parts (a) and (b) to derive a formula for $\sum_{k=1}^{n} k^3$.

Hint (a) available at the end of this section.

Hint (b) available at the end of this section.

Answers

Example Label 'eg:binom-01' multiply defined

Following the steps we outlined above, we find

$$\begin{aligned}(x+y)^4 &= \tbinom{4}{0}x^4 + \tbinom{4}{1}x^3y + \tbinom{4}{2}x^2y^2 + \tbinom{4}{3}xy^3 + \tbinom{4}{4}y^4\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

Since $\binom{n}{0} = \binom{n}{n} = 1$, the expansion always starts with x^n and ends with y^n.

Example Label 'eg:binom-02' multiply defined

We find

$$\begin{aligned}(x-y)^4 &= [x+(-y)]^4\\ &= \tbinom{4}{0}x^4 + \tbinom{4}{1}x^3(-y) + \tbinom{4}{2}x^2(-y)^2 + \tbinom{4}{3}x(-y)^3 + \tbinom{4}{4}(-y)^4\\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4.\end{aligned}$$

Take note of the alternating signs in the expansion. This suggests that we could expand $(A-B)^n$ the exact same way we would with $(A+B)^n$, except that the signs alternate.

We can carry out the expansion by following these steps. First, list all the terms we expect to find

$$(x+y)^4 = _x^4 \quad _x^3y \quad _x^2y^2 \quad _xy^3 \quad _y^4.$$

Next, fill in the signs:

$$(x+y)^4 = _x^4 - _x^3y + _x^2y^2 - _xy^3 + _y^4,$$

and then the binomial coefficients:

$$(x+y)^4 = \binom{4}{0}x^4 - \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 - \binom{4}{3}xy^3 + \binom{4}{4}y^4.$$

Finally, compute the binomial coefficients to finish the expansion.

Example Label 'eg:binom-03' multiply defined

The expansion yields

$$(2x)^5 - \binom{5}{1}(2x)^4(3y) + \binom{5}{2}(2x)^3(3y)^2 - \binom{5}{3}(2x)^2(3y)^3 + \binom{5}{4}(2x)(3y)^4 - (3y)^5.$$

Therefore, $(2x-3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

Example Label 'eg:binom-04' multiply defined

Since

$$(1+x)^{102} = \sum_{k=0}^{102} \binom{102}{k} x^k,$$

the term containing x^3 is $\binom{102}{3}x^3$. Therefore, the coefficient is $\binom{102}{3}$. Depending on which form of the binomial theorem you use, you may end up with the term $\binom{102}{99}x^3$. Numerically, this gives us the same coefficient, because $\binom{102}{99} = \binom{102}{102-99} = \binom{102}{3}$.

Example Label 'eg:binom-05' multiply defined

Since

$$(2+3t)^9 = \sum_{k=0}^{9} \binom{9}{k} 2^{9-k}(3t)^k,$$

we need $k=4$. The coefficient is $\binom{9}{4}2^5 \cdot 3^4 \cdot$.

Example Label 'eg:binom-06' multiply defined

Since $(3-2t)^7 = \sum_{k=0}^{7}\binom{7}{k}3^{7-k}(-2t)^k$, we need $k=5$, and the coefficient is $\binom{7}{5}3^2 \cdot (-2)^5 = -\binom{7}{5}3^2 \cdot 2^5$.

Example Label 'eg:binom-07' multiply defined

The general term in the expansion is $\binom{8}{k}4^{8-k}(5t^2)^k = \binom{8}{k}4^{8-k}\cdot 5^k t^{2k}$. Hence, we need $k=3$, and the coefficient is $\binom{8}{3}4^5\cdot 5^3$.

Example Label 'eg:binom-08' multiply defined

The general term in the expansion is

$$\binom{8}{k}x^{8-k}\left(\frac{2}{x}\right)^k = \binom{8}{k}x^{8-k}\cdot\frac{2^k}{x^k} = \binom{8}{k}2^k x^{8-2k}.$$

We need $8-2k=0$ or $k=4$. Therefore, the coefficient is $\binom{8}{4}2^4$.

Example Label 'eg:binom-09' multiply defined

Expand $(1+x+x^2)(1+x)^{10}$ as follows:

$$\begin{aligned}(1+x+x^2)(1+x)^{10} &= (1+x+x^2)\textstyle\sum_{k=0}^{10}\binom{10}{k}x^k \\ &= \textstyle\sum_{k=0}^{10}\binom{10}{k}x^k + \sum_{k=0}^{10}\binom{10}{k}x^{k+1} + \sum_{k=0}^{10}\binom{10}{k}x^{k+2}.\end{aligned} \tag{14.2.1.4.1}$$

So the coefficient of x^7 is $\binom{10}{7}+\binom{10}{6}+\binom{10}{5}$.

Theorem 14.2.1.4.2 **(Pascal's Identitity)**

It follows from the definition of binomial coefficients that

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,(n-k)!}+\frac{(n-1)!}{k!\,(n-k-1)!} \\ &= \frac{(n-1)!}{(k-1)!\,(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right) \\ &= \frac{(n-1)!}{(k-1)!\,(n-k-1)!}\cdot\frac{n}{k(n-k)} \\ &= \frac{n!}{k!\,(n-k)!}.\end{aligned}$$

This completes the proof.

Let A be an n-element set. Then $\binom{n}{k}$ counts the number of k-element subsets of A. These subsets can be classified according to whether they contain a fixed element, say x. If a subset contains x, then the other $k-1$ elements must be selected from the remaining $n-1$ elements of A. Otherwise, if the subset does not contain x, then all its k elements must be selected from the other $n-1$ elements of A. The numbers of these two kinds of subsets are given by $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$, respectively. The theorem now follows immediately by applying the addition principle.

Example Label 'eg:binom-10' multiply defined

Draw the values of $\binom{n}{k}$ from the Pascal triangle directly. The answers are:

a. $(C-D)^5 = C^5-5C^4D+10C^3D^2-10C^2D^3+5CD^4-D^5$.
b. $(2a+5B)^3 = 8A^3+60A^2B+150AB^2+125B^3$.
c. $(3C-4B)^4 = 81C^4-432C^3B+864C^2B^2-768CB^3+256B^4$.

Exercise Label 'ex:binom-06' multiply defined

Differentiate $(1+x)^n$ with respect to x.

Exercise Label 'ex:binom-07' multiply defined

Note that $\binom{1}{2}=0$.

Exercise Label 'ex:binom-07' multiply defined

Compare coefficients.

Exercise Label 'ex:binom-08' multiply defined

Note that $\binom{1}{2}=0$.

Exercise Label 'ex:binom-08' multiply defined

Compare coefficients.

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14.2.2: Partial Fractions

Learning Objectives

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Non-repeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right)&=\frac{2x+4-x+3}{(x+2)(x-3)}\\&=\frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{14.2.2.1}$$

How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 14.2.2.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = (x+2)(x-1)\left[\frac{A}{(x+2)}\right] + (x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x = A(x-1) + B(x+2)$$

Expand the right side of the equation and collect like terms.

$$3x = Ax - A + Bx + 2B$$
$$3x = (A+B)x - A + 2B$$

Set up a system of equations associating corresponding coefficients.

$$3 = A + B$$
$$0 = -A + 2B$$

Add the two equations and solve for B.

$$3 = A + B$$
$$\underline{0 = -A + 2B}$$
$$3 = 0 + 3B$$
$$1 = B$$

Substitute $B = 1$ into one of the original equations in the system.

$$3 = A + 1$$
$$2 = A$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the A or B term equal 0. If we let $x = 1$, the A term becomes 0 and we can simply solve for B.

$$3x = A(x-1) + B(x+2)$$
$$3(1) = A[(1)-1] + B[(1)+2]$$
$$3 = 0 + 3B$$
$$1 = B$$

Next, either substitute $B = 1$ into the equation and solve for A, or make the B term 0 by substituting $x = -2$ into the equation.

$$3x = A(x-1) + B(x+2)$$
$$3(-2) = A[(-2)-1] + B[(-2)+2]$$
$$-6 = -3A + 0$$
$$\frac{-6}{-3} = A$$
$$2 = A$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

Exercise 14.2.2.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Answer available at the end of this section.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors.

We must remember that we account for repeated factors by writing each factor in increasing powers.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{14.2.2.2}$$

Write the denominator powers in increasing order.

How to: decompose a rational expression with repeated linear factors

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{14.2.2.3}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 14.2.2.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4x}{(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right]x(x-2)^2$$
$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$A + B = -1 \tag{14.2.2.4}$$
$$-4A - 2B + C = 2 \tag{14.2.2.5}$$
$$4A = 4 \tag{14.2.2.6}$$

Solving for A in Equation 14.2.2.6, we have

$$4A = 4$$
$$A = 1$$

Substitute $A = 1$ into Equation 14.2.2.4.

$$B = -2$$

Then, to solve for C, substitute the values for A and B into Equation 14.2.2.5.

$$-4A - 2B + C = 2$$
$$-4(1) - 2(-2) + C = 2$$
$$-4 + 4 + C = 2$$
$$C = 2$$

Thus,

$$\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x} = \frac{1}{x} - \frac{2}{(x-2)} + \frac{2}{(x-2)^2}$$

Exercise 14.2.2.2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x - 11}{(x-1)^2}$$

Answer available at the end of this section.

Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax + B$, $Bx + C$, etc.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{(a_1x^2 + b1_x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)} \tag{14.2.2.7}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

How to: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as $A_1x + B_1$, $A_2x + B_2$, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)} + \frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)} + \cdots + \frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 14.2.2.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$(x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] = \left[\frac{A}{(x+3)} + \frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2)$$

$$8x^2+12x-20 = A(x^2+x+2) + (Bx+C)(x+3)$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx + C$ term equal 0. Let $x = -3$ and substitute it into the equation.

$$\begin{aligned} 8x^2+12x-20 &= A(x^2+x+2) + (Bx+C)(x+3) \\ 8(-3)^2+12(-3)-20 &= A((-3)^2+(-3)+2) + (B(-3)+C)((-3)+3) \\ 16 &= 8A \\ A &= 2 \end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned} 8x^2+12x-20 &= 2(x^2+x+2) + (Bx+C)(x+3) \\ 8x^2+12x-20 &= 2x^2+2x+4+Bx^2+3B+Cx+3C \\ 8x^2+12x-20 &= (2+B)x^2+(2+3B+C)x+(4+3C) \end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2 + B = 8 \tag{14.2.2.8}$$
$$2 + 3B + C = 12 \tag{14.2.2.9}$$
$$4 + 3C = -20 \tag{14.2.2.10}$$

Solve for B using Equation 14.2.2.8

$$2 + B = 8$$
$$B = 6$$

and solve for C using Equation 14.2.2.10.

$$3C = -24$$
$$C = -8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2 + 12x - 20}{(x+3)(x^2+x+2)} = \frac{2}{(x+3)} + \frac{6x-8}{(x^2+x+2)}$$

Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2 + 12x - 20 = Ax^2 + Ax + 2A + Bx^2 + 3B + Cx + 3C$$
$$8x^2 + 12x - 20 = (A+B)x^2 + (A + 3B + C)x + (2A + 3C)$$

So the system of equations would be:

You answer

Exercise 14.2.2.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2 - 6x + 7}{(x-1)(x^2+1)}$$

Answer available at the end of this section.

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n} \qquad (14.2.2.11)$$

Write the denominators in increasing powers.

How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)} = \frac{A}{ax+b} + \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 14.2.2.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution

The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,

$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x) \\ x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1) + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \qquad \text{Expand the right side.} \\ &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \end{aligned}$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1 = (A+B)x^4 + (C)x^3 + (2A+B+D)x^2 + (C+E)x + A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$A + B = 1$$
$$C = 1$$
$$2A + B + D = 1$$
$$C + E = -1$$
$$A = 1$$

We can use substitution from this point. Substitute $A = 1$ into the first equation.

$$1 + B = 1$$
$$B = 0$$

Substitute $A = 1$ and $B = 0$ into the third equation.

$$D = -1$$

Substitute $C = 1$ into the fourth equation.

$$E = -2$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0, C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2 + 1)^2} = \frac{1}{x} + \frac{1}{(x^2 + 1)} - \frac{x + 2}{(x^2 + 1)^2}$$

Exercise 14.2.2.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$$

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors
- Partial Fraction Decomposition With Linear and Quadratic Factors

Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

  $$\frac{A}{a_1 x + b_1} + \frac{B}{a_2 x + b_2}.$$

  Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example 14.2.2.1).
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example 14.2.2.2).

- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x}+\frac{Bx+C}{(ax^2+bx+c)}$ (see Example 14.2.2.3).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

See Example 14.2.2.4.

Contributors and Attributions

- Template:ContribOpenStaxPreCalc

Answers

Exercise 14.2.2.1

$\frac{3}{x-3}-\frac{2}{x-2}$

Exercise 14.2.2.2

$$\frac{6}{x-1}-\frac{5}{(x-1)^2}$$

Exercise 14.2.2.3

$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$

Exercise 14.2.2.4

$$\frac{x-2}{x^2-2x+3}+\frac{2x+1}{(x^2-2x+3)^2}$$

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14.2.3: MATRICES AND HOW TO USE THEM

Matrices are a system to organize numbers to make transforms easier, including solving a system of equations.

14.2.3.1: Matrices and Matrix Operations

14.2.3.2: System of equations and Matrices

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14.2.3.1: Matrices and Matrix Operations

Learning Objectives

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 14.2.3.1.1 shows the needs of both teams.

Table 14.2.3.1.1

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs $\$300$; a ball costs $\$10$; and a jersey costs $\$30$. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named A, B, and C are shown below.

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij},we look for the entry in row i,column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A **square matrix** is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.
- A **row matrix** is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- A **column matrix** is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic **matrix operations**.

Definition: MATRICES

A *matrix* is a rectangular array of numbers that is usually named by a capital letter: A, B, C,and so on. Each entry in a matrix is referred to as a_{ij},such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

Example 14.2.3.1.1: Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A:

a. What are the dimensions of matrix A?
b. What are the entries at a_{31} and a_{22}?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution

a. The dimensions are 3×3 because there are three rows and three columns.
b. Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries. To do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

ADDING AND SUBTRACTING MATRICES

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A + B = C \tag{14.2.3.1.1}$$

such that $a_{ij} + b_{ij} = c_{ij}$

$$A - B = D \tag{14.2.3.1.2}$$

such that $a_{ij} - b_{ij} = d_{ij}$

Matrix addition is *commutative*.

$$A + B = B + A \tag{14.2.3.1.3}$$

It is also *associative*.

$$(A + B) + C = A + (B + C) \tag{14.2.3.1.4}$$

Example 14.2.3.1.2A: Finding the Sum of Matrices

Find the sum of A and B, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Solution

Add corresponding entries.

$$\begin{aligned} A + B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$

Example 14.2.3.1.2B: Adding Matrix A and Matrix B

Find the sum of A and B.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11}, of B. Continue the pattern until all entries have been added.

$$\begin{aligned} A + B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

Example 14.2.3.1.2C: Finding the Difference of Two Matrices

Find the difference of A and B.

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

We subtract the corresponding entries of each matrix.

$$A-B=\begin{bmatrix}-2 & 3\\ 0 & 1\end{bmatrix}-\begin{bmatrix}8 & 1\\ 5 & 4\end{bmatrix}$$
$$=\begin{bmatrix}-2-8 & 3-1\\ 0-5 & 1-4\end{bmatrix}$$
$$=\begin{bmatrix}-10 & 2\\ -5 & -3\end{bmatrix}$$

Example 14.2.3.1.2D: Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B:

a. Find the sum.
b. Find the difference.

$$A=\begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}$$

and

$$B=\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}$$

Solution

a. Add the corresponding entries.

$$A+B=\begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}+\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}$$
$$=\begin{bmatrix}2+6 & -10+10 & -2-2\\ 14+0 & 12-12 & 10-4\\ 4-5 & -2+2 & 2-2\end{bmatrix}$$
$$=\begin{bmatrix}8 & 0 & -4\\ 14 & 0 & 6\\ -1 & 0 & 0\end{bmatrix}$$

b. Subtract the corresponding entries.

$$A-B=\begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}-\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}$$
$$=\begin{bmatrix}2-6 & -10-10 & -2+2\\ 14-0 & 12+12 & 10+4\\ 4+5 & -2-2 & 2+2\end{bmatrix}$$
$$=\begin{bmatrix}-4 & -20 & 0\\ 14 & 24 & 14\\ 9 & -4 & 4\end{bmatrix}$$

Exercise 14.2.3.1.1

Add matrix A and matrix B.

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$$

Answer available at the end of this section.

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15 more equipment is needed in both labs. The school's current inventory is displayed in Table $14.2.3.1.2$.

Table $14.2.3.1.2$

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Converting the data to a matrix, we have

$$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.

$$(0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is

$$\begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.

$$\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

This means

$$C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

SCALAR MULTIPLICATION

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is

$$\begin{aligned} cA &= c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ &= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \end{aligned}$$

Scalar multiplication is distributive. For the matrices A, B,and C with scalars a and b,

$$a(A+B) = aA + aB \tag{14.2.3.1.5}$$

$$(a+b)A = aA + bA \tag{14.2.3.1.6}$$

Example 14.2.3.1.3: Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

Multiply each entry in A by the scalar 3.

$$\begin{aligned} 3A &= 3 \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 8 & 3 \cdot 1 \\ 3 \cdot 5 & 3 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

Exercise 14.2.3.1.2

Given matrix B, find $-2B$ where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

Answer available at the end of this section.

Example 14.2.3.1.4: Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

Solution

First, find $3A$, then $2B$.

$$\begin{aligned} 3A &= \begin{bmatrix} 3 \cdot 1 & 3(-2) & 3 \cdot 0 \\ 3 \cdot 0 & 3(-1) & 3 \cdot 2 \\ 3 \cdot 4 & 3 \cdot 3 & 3(-6) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2B &= \begin{bmatrix} 2(-1) & 2 \cdot 2 & 2 \cdot 1 \\ 2 \cdot 0 & 2(-3) & 2 \cdot 2 \\ 2 \cdot 0 & 2 \cdot 1 & 2(-4) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \end{aligned}$$

Now, add $3A + 2B$.

$$\begin{aligned} 3A + 2B &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} + \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \\ &= \begin{bmatrix} 3-2 & -6+4 & 0+2 \\ 0+0 & -3-6 & 6+4 \\ 12+0 & 9+2 & -18-8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 2 \\ 0 & -9 & 10 \\ 12 & 11 & -26 \end{bmatrix} \end{aligned}$$

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product matrix AB is an $m \times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\underset{2\times 3}{A} \cdot \underbrace{\underset{3\times 3}{B}}_{\text{same}}$$

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of **matrix multiplication** becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3,the product of AB will be a 2×3 matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB,multiply the first row in A by the first column in B,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB,multiply the first row of A by the second column in B,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB,multiply the first row of A by the third column in B,and add.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

For the matrice A, B,and C the following properties hold.

- Matrix multiplication is *associative*:

$$(AB)C = A(BC). \tag{14.2.3.1.7}$$

- Matrix multiplication is *distributive*:

$$C(A + B) = CA + CB \tag{14.2.3.1.8}$$

$$(A + B)C = AC + BC. \tag{14.2.3.1.9}$$

Note that matrix multiplication is not commutative.

Example 14.2.3.1.5A: Multiplying Two Matrices

Multiply matrix A and matrix B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Example 14.2.3.1.5B: Multiplying Two Matrices

Given A and B:

a. Find AB.
b. Find BA.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution

a. As the dimensions of A are 2×3 and the dimensions of B are 3×2,these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1(5)+2(-4)+3(2) & -1(-1)+2(0)+3(3) \\ 4(5)+0(-4)+5(2) & 4(-1)+0(0)+5(3) \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

b. The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.

$$\begin{aligned} BA &= \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5(-1)+-1(4) & 5(2)+-1(0) & 5(3)+-1(5) \\ -4(-1)+0(4) & -4(2)+0(0) & -4(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} \end{aligned}$$

Analysis

Notice that the products AB and BA are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is **not** commutative.

```
#
# Example routine of multiplying matrices
#
A = [-1 2 3;4 0 5]
B = [5 -1;-4 0;2 3]
A*B
B*A
```

Q&A: Is it possible for AB to be defined but not BA?

Yes, consider a matrix A with dimension 3×4 and matrix B with dimension 4×2. For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

Example 14.2.3.1.6: Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have Table 14.2.3.1.3, representing the equipment needs of two soccer teams.

Table 14.2.3.1.3

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

We are also given the prices of the equipment, as shown in Table 14.2.3.1.4.

Table 14.2.3.1.4

Goal	$300
Ball	$10
Jersey	$30

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = [300 \quad 10 \quad 30]$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= [300 \quad 10 \quad 30] \cdot \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= [300(6) + 10(30) + 30(14) \quad 300(10) + 10(24) + 30(20)] \\ &= [2,520 \quad 3,840] \end{aligned}$$

The total cost for equipment for the Wildcats is $\$2,520$, and the total cost for equipment for the Mud Cats is $\$3,840$.

Example 14.2.3.1.7: Using a Calculator to Perform Matrix Operations

Find $AB - C$ given

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}$$

Solution

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[A]$, matrix B above as the matrix variable $[B]$, and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

However for real work it is better if you use one of the modern programming languages where you can write a program. *Why?* Because doing calculations in isolation (a single calculation) is not a real world thing.

```
#
# Use octave to solve these problems, it is easier
#
A = [-15 25 32;41 -7 -28;10 34 -2]
B = [45 21 -37;-24 52 19;6 -48 -31]
C = [-100 -89 -98;25 -56 74;-67 42 -75]
#
# A*B-C
#
AxBmC = A*B-C
```

Media

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See Example $14.2.3.1.1$.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example $14.2.3.1.2$, Example $14.2.3.1.3$, Example $14.2.3.1.4$, and Example $14.2.3.1.5$.
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example $14.2.3.1.6$.
- Scalar multiplication is often required before addition or subtraction can occur. See Example $14.2.3.1.7$.
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B,is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B,and so on. See Example $14.2.3.1.8$ and Example $14.2.3.1.9$.
- Many real-world problems can often be solved using matrices. See Example $14.2.3.1.10$.

Contributors and Attributions

- Template:ContribOpenStaxPreCalc

Answers

Exercise $14.2.3.1.1$

$$A+B=\begin{bmatrix}2 & 6\\ 1 & 0\\ 1 & -3\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 5\\ -4 & 3\end{bmatrix}$$
$$=\begin{bmatrix}2+3 & 6+(-2)\\ 1+1 & 0+5\\ 1+(-4) & -3+3\end{bmatrix}$$
$$=\begin{bmatrix}5 & 4\\ 2 & 5\\ -3 & 0\end{bmatrix}$$

Exercise 14.2.3.1.2

$$-2B=\begin{bmatrix}-8 & -2\\ -6 & -4\end{bmatrix}$$

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- 2.12: Choosing Mnemonic Variable Names by Chuck Severance is licensed CC BY-NC-SA 4.0. Original source: https://www.py4e.com/html3.

14.2.3.2: System of equations and Matrices

System of Equations

To solve a system of equations there are a number of methods that you have already learned in previous classes. Let us look at an example system of equations (two variables; two equations).

$$3x + 4y = 7 \tag{14.2.3.2.1}$$

$$-2x + 2y = 2 \tag{14.2.3.2.2}$$

¬ What are three of the four methods to solve this system of equations that you are aware of presently?

- Substitution method
  - Take one of the equations and rearrange it so you have either x = ... or y = ... and plug it into the other equation
  - This method gets much harder for three or more equations
- Addition method
  - Multiply the equations by a coefficient that will allow you to subtract the equations from each other and zero out one of the variables
  - This method gets harder for three or more equations
- Graphical method
  - This is where you graph the functions and determine the answer from the intersection
  - For three equation you would need a to plot in 3D which is already difficult because we don't really plot 3D (plot 3D on 2D is what we do), for more than three you would have to go into hyper dimensions
  - This method is not practical for multiple equations though it is instructive

The better method for three or more equations would be a matrix method (the fourth method). As an example we will solve the simple of system of equations in octave.

```
A = [3 4;-2 2]
b = [7;2]
x = inv(A)*b
```

So this is pretty easy but lets verify this with a graph.

```
x = [-2:0.01:2];
y1 = (-3/4).*x + (7/4);
y2 = x + 1;
hold off
plot(x,y1)
hold on
plot(x,y2)
xlabel("x")
ylabel("y")
title("Solving equations by graphing")
```

In the last program you can see the lines intersect at the location that the matrix method gave us for an answer. So that was easy. So now lets solve another problem. You can use any of the methods you know. Here we go.

$$3u + 2v - 7x + 4y - z = 9 \tag{14.2.3.2.3}$$

$$-2u - 3v + 2x - y + 2z = 4 \tag{14.2.3.2.4}$$

$$7u + v + 3x + 2y + 3z = 5 \tag{14.2.3.2.5}$$

$$12u + 5v - 4x + y - 3z = -3 \tag{14.2.3.2.6}$$

$$-5u - 7v - 2x - 4y + 3z = -6 \tag{14.2.3.2.7}$$

So the answer is with round-off u=1.3, v=-6.8, x=1.9, y=6.7, and z = -5.5 right? *Is this what you got? Did you solve it in a minute like we did?* Hmmm....let's learn a new better method that allows us to use the wonderful world of mathematical programs.

First lets start write the system of equations using matrix notations. Let us first put all the coefficients of the equations in a matrix we will call A.

$$A = \begin{pmatrix} 3 & 2 & -7 & 4 & -1 \\ -2 & -3 & 2 & -1 & 2 \\ 7 & 1 & 3 & 2 & 3 \\ 12 & 5 & -4 & 1 & -3 \\ -5 & -7 & -2 & -4 & 3 \end{pmatrix} \tag{14.2.3.2.8}$$

For the results we will make a column vector and call it b.

$$b = \begin{pmatrix} 9 \\ 4 \\ 5 \\ -3 \\ -6 \end{pmatrix} \tag{14.2.3.2.9}$$

Then we can write the equations above in matrix format.

$$\mathbf{Ax} = \mathbf{b} \tag{14.2.3.2.10}$$

where

$$x = \begin{pmatrix} u \\ v \\ x \\ y \\ z \end{pmatrix} \tag{14.2.3.2.11}$$

While we were just doing one particular example it turns out we created a general equation that would work for any size array (assuming we could take the inverse of it which is not a given with very large arrays). Now we can solve for $\mathbf{x}$. First let us "define" the inverse of a matrix as $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{I}$ is called the identity matrix and it acts like the one of matrices. That is $\mathbf{IA} = \mathbf{A}$. Now we can continue to solve this equation. Rewriting our general equation.

$$\mathbf{Ax} = \mathbf{b} \tag{14.2.3.2.12}$$

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{bA}^{-1} \tag{14.2.3.2.13}$$

¬ So we multiplied by the inverse of the matrix on each side of the equation and what do we do next?

- We cross out what we did because it is WRONG. Matrix multiplication is not commutative. If you multiply on the left side of the equation you need to multiply on the left side on the other side of the equation. ~~$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{bA}^{-1}$~~

So let us rewrite the last line.

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \tag{14.2.3.2.14}$$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} \tag{14.2.3.2.15}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \tag{14.2.3.2.16}$$

And now we can solve this using a mathematical programming language, in this case Octave.

```
A = [3 2 -7 4 -1;-2 -3 2 -1 2;7 1 3 2 3;12 5 -4 1 -3;-5 -7 -2 -4 3]
b = [9;4;5;-3;-6]
#
# Note * is matrix multiplication; element by element multiplication is .*
```

```
#
x = inv(A)*b
```

Once you know the equation it takes minutes to type it in and solve the problem.

This is just a bit of what matrix multiplication, we have also hinted at transforms in the last section, etc. You can get more detail with numerical methods in the signals and system course.

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14.2.4: Combinations and Permutations

Investigate!

You have a bunch of chips which come in five different colors: red, blue, green, purple and yellow.

1. How many different two-chip stacks can you make if the bottom chip must be red or blue? Explain your answer using both the additive and multiplicative principles.
2. How many different three-chip stacks can you make if the bottom chip must be red or blue and the top chip must be green, purple or yellow? How does this problem relate to the previous one?
3. How many different three-chip stacks are there in which no color is repeated? What about four-chip stacks?
4. Suppose you wanted to take three different colored chips and put them in your pocket. How many different choices do you have? What if you wanted four different colored chips? How do these problems relate to the previous one?

A *permutation* is a (possible) rearrangement of objects. For example, there are 6 permutations of the letters *a, b, c*:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

The six permutations are listed above. Note there are 3 choices for which letter we put first, then 2 choices for which letter comes next, which leaves only 1 choice for the last letter. Thus we can multiply $3 \cdot 2 \cdot 1$ or $n!$ to get the number of permutations of the three letters.

Example 14.2.4.1

How many permutations are there of the letters *a, b, c, d, e, f*?

Answer available at the end of this section.

A piece of notation is helpful here: $n!$, read "n factorial", is the product of all positive integers less than or equal to n (in mathematics 0! is defined to be 1). So the number of permutation of 6 letters, as seen in the previous example is $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Permatation of n elements:

Permutations of n elements

There are $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ or

$$n! = \prod_{1}^{n} \tag{14.2.4.1}$$

permutations of n (distinct) elements.

Counting Bijective Functions

How many functions $f : \{1, 2, \ldots, 8\} \to \{1, 2, \ldots, 8\}$ are *bijective*?

Solution available at the end of this section.

Sometimes we do not want to permute all of the letters/numbers/elements we are given.

Example 14.2.4.3

How many 4 letter "words" can you make from the letters *a* through *f*, with no repeated letters?

Solution available at the end of this section.

k-permutations of n elements

$P(n,k)$ is the number of k-permutations of n elements, the number of ways to arrange k objects chosen from n distinct objects.

$$P(n,k) = \frac{n!}{(n-k)!}.$$

Note that when $n = k$, we have $P(n,n) = \frac{n!}{(n-n)!} = n!$ (remember $0!$ is 1). This makes sense —we already know $n!$ gives the number of permutations of all n objects.

Counting injective functions

How many functions $f : \{1,2,3\} \to \{1,2,3,4,5,6,7,8\}$ are *injective*?

Solution available at the end of this section.

Here is another way to find the number of k-permutations of n elements: first select which k elements will be in the permutation, then count how many ways there are to arrange them. Once you have selected the k objects, we know there are $k!$ ways to arrange (permute) them. But how do you select k objects from the n? You have n objects, and you need to *choose* k of them. You can do that in $\binom{n}{k}$ ways. Then for each choice of those k elements, we can permute *them* in $k!$ ways. Using the multiplicative principle, we get another formula for $P(n,k)$:

$$P(n,k) = \binom{n}{k} \cdot k!.$$

Now since we have a closed formula for $P(n,k)$ already, we can substitute that in:

$$\frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!.$$

If we divide both sides by $k!$ we get a closed formula for $\binom{n}{k}$.

Closed formula for $\binom{n}{k}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We say $P(n,k)$ counts *permutations*, and $\binom{n}{k}$ counts *combinations*. The formulas for each are very similar, there is just an extra $k!$ in the denominator of $\binom{n}{k}$. That extra $k!$ accounts for the fact that $\binom{n}{k}$ does not distinguish between the different orders that the k objects can appear in. We are just selecting (or choosing) the k objects, not arranging them. Perhaps "combination" is a misleading label. We don't mean it like a combination lock (where the order would definitely matter). Perhaps a better metaphor is a combination of flavors — you just need to decide which flavors to combine, not the order in which to combine them.

To further illustrate the connection between combinations and permutations, we close with an example.

Example 14.2.4.5

You decide to have a dinner party. Even though you are incredibly popular and have 14 different friends, you only have enough chairs to invite 6 of them.

1. How many choices do you have for which 6 friends to invite?
2. What if you need to decide not only which friends to invite but also where to seat them along your long table? How many choices do you have then?

Solution available at the end of this section.

So let us review with a table and also add some additional possibilities.

Permutation without considering position (no repeating): *Combination*	Permutation without considering position (repeating): *Multicombination*	Permutation with considering position (no repeating): *Permutation*	Considering position with repeating[1]: *tuples*	Permutations for many sets within a set[2]: *Multiset permutation*
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$	$\left(\binom{n}{k}\right) = \binom{n+k-1}{k} = \frac{n+k-1!}{k!(n-1)!}$	$P(n,k) = \frac{n!}{n-k}!$	n^k	$\frac{n_T!}{n_1!n_2!n_3!\ldots}$

[1]You might find this in an alphabetic cipher system, say.

[2]You might find this in statistical mechanics where there are many sets of molecules at each possible energy level for a total number of molecules, say, n_T.

Answers

Example 14.2.4.1

Rather then list them out we will pick a letter to write down first. There are 6 choices for that letter. For each choice of first letter, there are 5 choices for the second letter (we cannot repeat the first letter; we are rearranging letters and only have one of each), and for each of those, there are 4 choices for the third, 3 choices for the fourth, 2 choices for the fifth and finally only 1 choice for the last letter. So there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ permutations of the 6 letters.

Counting Bijective Functions

Remember what it means for a function to be bijective: each element in the codomain must be the image of exactly one element of the domain. Using two-line notation, we could write one of these bijections as

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 7 & 6 & 2 & 4 \end{pmatrix}$$

What we are really doing is just rearranging the elements of the codomain, so we are creating a permutation of 8 elements. In fact, "permutation" is another term used to describe bijective functions from a finite set to itself.

If you believe this, then you see the answer must be $8! = 8 \cdot 7 \cdot \cdots \cdot 1 = 40320$. You can see this directly as well: for each element of the domain, we must pick a distinct element of the codomain to map to. There are 8 choices for where to send 1, then 7 choices for where to send 2, and so on. We multiply using the multiplicative principle.

Example 14.2.4.3

This is just like the problem of permuting 4 letters, only now we have more choices for each letter. For the first letter, there are 6 choices. For each of those, there are 5 choices for the second letter. Then there are 4 choices for the third letter, and 3 choices for the last letter. The total number of words is $6 \cdot 5 \cdot 4 \cdot 3 = 360$. This is not $6!$ because we never multiplied by 2 and 1. We could start with $6!$ and then cancel the 2 and 1, and thus write $\frac{6!}{2!}$.

In general, we can ask how many permutations exist of k objects choosing those objects from a larger collection of n objects. (In the example above, $k = 4$ and $n = 6$). We write this number $P(n,k)$ and sometimes call it a k-permutation of n elements. From the example above, we see that to compute $P(n,k)$ we must apply the multiplicative principle to k numbers, starting with n and counting backwards. For example

$$P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7$$

Notice again that $P(10,4)$ starts out looking like $10!$, but we stop after 7. We can formally account for this "stopping" by dividing away the part of the factorial we do not want:

$$P(10,4) = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!}$$

Careful: The factorial in the denominator is not $4!$ but rather $(10-4)!$

Counting injective functions

Note that it doesn't make sense to ask for the number of *bijections* here, as there are none (because the co-domain is larger than the domain, there are no surjections). But for a function to be injective, we just can't use an element of the co-domain more than once.

We need to pick an element from the codomain to be the image of 1. There are 8 choices. Then we need to pick one of the remaining 7 elements to be the image of 2. Finally, one of the remaining 6 elements must be the image of 3. So the total number of functions is $8 \cdot 7 \cdot 6 = P(8,3)$.

What this demonstrates in general is that the number of injections $f : A \to B$, where $|A| = k$ and $|B| = n$, is $P(n,k)$.

Example $14.2.4.5$

1. You must simply choose 6 friends from a group of 14. This can be done in $\binom{14}{6}$ ways. We can find this number either by using Pascal's triangle or the closed formula: $\frac{14!}{8! \cdot 6!} = 3003$.
2. Here you must count all the ways you can permute 6 friends chosen from a group of 14. So the answer is $P(14,6)$, which can be calculated as $\frac{14!}{8!} = 2192190$.

Notice that we can think of this counting problem as a question about counting functions: how many injective functions are there from your set of 6 chairs to your set of 14 friends (the functions are injective because you can't have a single chair go to two of your friends).

How are these numbers related? Notice that $P(14,6)$ is much larger than $\binom{14}{6}$. This makes sense. $\binom{14}{6}$ picks 6 friends, but $P(14,6)$ arranges the 6 friends as well as picks them. In fact, we can say exactly how much larger $P(14,6)$ is. In both counting problems we choose 6 out of 14 friends. For the first one, we stop there, at 3003 ways. But for the second counting problem, each of those 3003 choices of 6 friends can be arranged in exactly $6!$ ways. So now we have $3003 \cdot 6!$ choices and that is exactly 2192190.

Alternatively, look at the first problem another way. We want to select 6 out of 14 friends, but we do not care about the order they are selected in. To select 6 out of 14 friends, we might try this:

$$14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9.$$

This is a reasonable guess, since we have 14 choices for the first guest, then 13 for the second, and so on. But the guess is wrong (in fact, that product is exactly $2192190 = P(14,6)$). It distinguishes between the different orders in which we could invite the guests. To correct for this, we could divide by the number of different arrangements of the 6 guests (so that all of these would count as just one outcome). There are precisely $6!$ ways to arrange 6 guests, so the correct answer to the first question is

$$\frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{6!}.$$

Note that another way to write this is

$$\frac{14!}{8! \cdot 6!},$$

which is what we had originally.

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14.2.5: Complex Numbers

Complex Numbers

The complex number $z = x + iy$ is interpreted as follows: the real part is x, the imaginary part is y, and $i = \sqrt{-1}$ (imaginary). DeMoivre's theorem connects complex z with the complex exponential. It states that $cos\ \theta + i\ sin\ \theta = e^{i\theta}$, and so we can visualize any complex number in the two-dimensional plane, where the axes are the real part and the imaginary part. We say that $Re(e^{i\theta}) = cos\ \theta$, and $Im(e^{i\theta}) = sin\ \theta$, to denote the real and imaginary parts of a complex exponential[1]. More generally, $Re(z) = x$ and $Im(z) = y$.

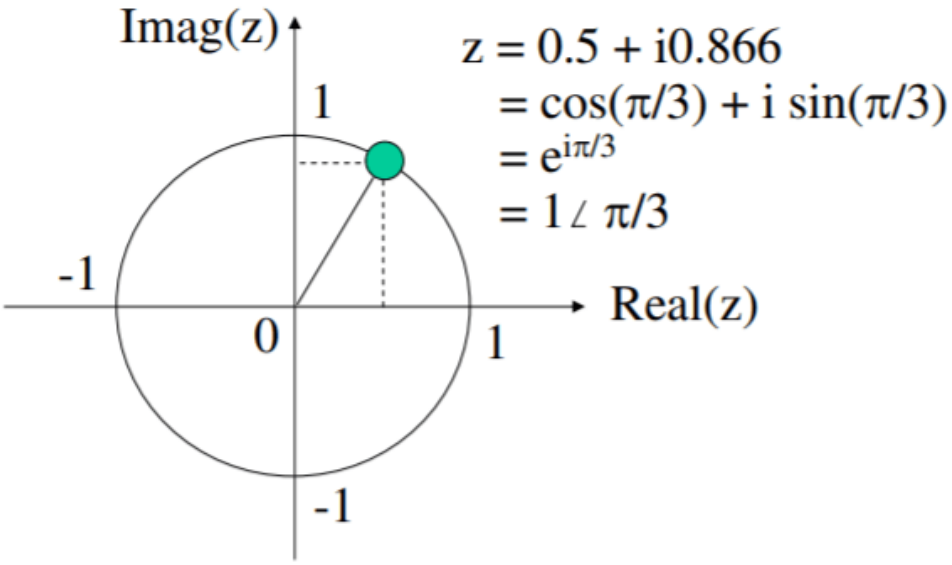


Figure 14.2.5.1: Imaginary number z represented as a graph on the Cartesian plane, with the x-axis representing the real part of z and the y-axis representing the imaginary part of z.

A complex number has a magnitude and an angle: $|z| = \sqrt{x^2 + y^2}$, and[2] arg $(z) = arctan2(y, x)$. We can refer to the $[x, y]$ description of z as Cartesian coordinates, whereas the [magnitude, angle] description is called polar coordinates. This latter is usually written as $z = |z|\angle$ arg(z). Arithmetic rules for two complex numbers z_1 and z_2 are as follows:

$$
\begin{aligned}
z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\
z_1 - z_2 &= (x_1 - x_2) + i(y_1 - y_2) \\
z_1 * z_2 &= |z_1||z_2|\angle \arg(z_1) + \arg(z_2) \\
z_1/z_2 &= \frac{|z_1|}{|z_2|}\angle \arg(z_1) - \arg(z_2)
\end{aligned}
$$

Note that, as given, addition and subtraction are most naturally expressed in Cartesian coordinates, and multiplication and division are cleaner in polar coordinates.

Contributors and Attributions

- Franz S. Hover & Michael S. Triantafyllou
- Some modification and additions of this section were done by Scott D. Johnson, Joshua Halpern, and Scott Sinex. The majority of the work is however Hover's and Triantafyllou's

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[1]Note also that we can now define sine and cosine as $sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ and $cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$. Later you can compare this to the hyperbolic sine and cosine in terms of exponents.

[2]Arctan2 is computerese for a two argument arctangent to ensures a unique arctangent for the ratio $\frac{y}{x}$. This was introduced into FORTRAN early on in its development. Other programming languages adopted this notation and eventual different fields not related to programming adopted the terminology as well. Run the program below to help see what this is all about.

```
y = 5;
x = 6;
printf("atan and atan2 with values y = %f and x = %f\n",y,x)
printf("atan(y/x) = %f\n",atan(y/x))
printf("atan2(y,x) = %f\n",atan2(y,x))
y = -5;
x = -6;
printf("atan and atan2 with values y = %f and x = %f\n",y,x)
printf("atan(y/x) = %f\n",atan(y/x))
printf("atan2(y,x) = %f\n",atan2(y,x))
```

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14.2.6: SEQUENCES AND SERIES

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

14.2.6.1: Prelude to Sequence and Series

14.2.6.2: Sequences

14.2.6.3: Infinite Series

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14.2.6.1: Prelude to Sequence and Series

The Koch snowflake[1] is constructed from an infinite number of non-overlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake.

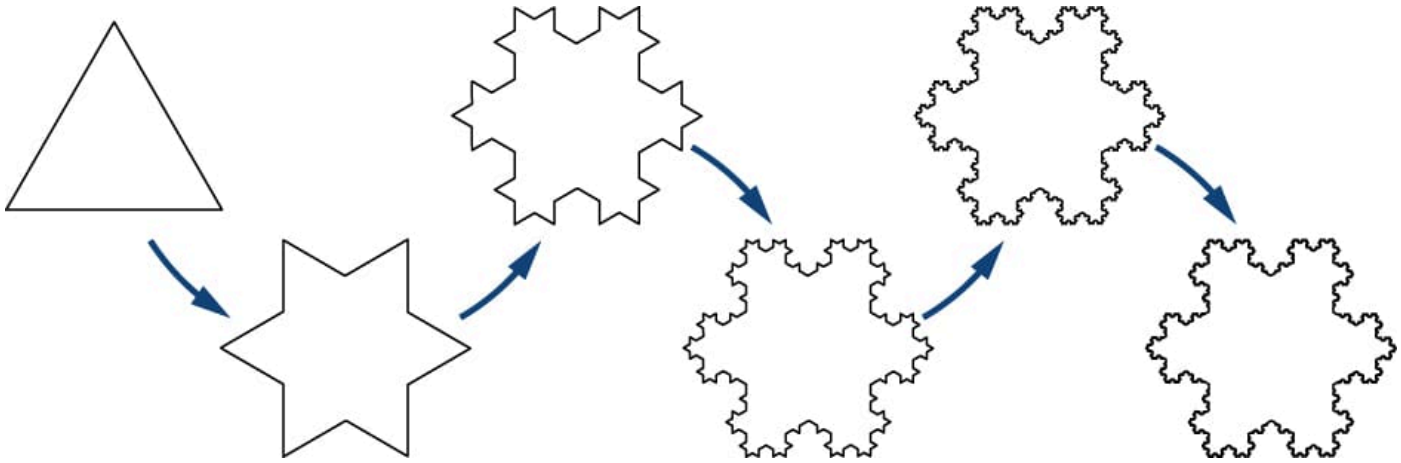


Figure 14.2.6.1.1: The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as "infinite polynomials." We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

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[1]The Koch snowflake is a Koch curve which is a fractal curve. Fractals are seen in nature (snowflake for instance as well as trees, coastlines, broccoli, galaxies, etc.) but are also seen in constructions such as the Khajuraho temples in India (see below). Fractal analysis (many methods which are usually used together for robust comparison) is an analysis that looks for fractal characteristics to extract information from a signal. One application of fractal analysis is cancer detection.

Note that there are different types of fractals. The Koch snowflake is fractal curve that lends itself to the discussion in this section, other fractals are not applicable to this section. Fractals are real mathematics/science but there are a numerous pseudo-science writings that use fractals in questionable ways. Follow the guidelines discussed herein before when reading about fractals.



A couple of the Khajuraho temples which have a fractal structure to them. Wikipedia image (Abinthomas0007, CC BY-SA 4.0,9/30/2017).

Contributors and Attributions

- Template:ContribOpenStaxCalc
- Footnote contributed by Scott D. Johnson, Joshua Halpern, and Scott Sinex

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14.2.6.2: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{14.2.6.2.1}$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$a_n|_{n=1}^{\infty} \tag{14.2.6.2.2}$$

or simply a_n, to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots \tag{14.2.6.2.3}$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2, a_2 = 4,$ and $a_3 = 8.$ You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1, a_2 = 2^2, a_3 = 2^3, a_4 = 2^4 \text{ and } a_5 = 2^5 \tag{14.2.6.2.4}$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n$. Using this notation, we can write this sequence as

$$2^n|_{n=1}^{\infty} \tag{14.2.6.2.5}$$

or

$$2^n \tag{14.2.6.2.6}$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence a_n where $a_1 = 2$ and for all $n \geq 2$, each term an is defined by the **recurrence relation**

$$a_n = 2a_{n-1} \tag{14.2.6.2.7}$$

Definition: infinite sequence

An **infinite sequence** a_n is an ordered list of numbers of the form

$$a_1, a_2, \ldots, a_n, \ldots$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by **explicit formulas,** in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0, a_1, a_2, \ldots \tag{14.2.6.2.8}$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence a_n has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence a_n consists of all points (n, a_n) for all positive integers n. Figure $14.2.6.2.1$ shows the graph of 2^n.

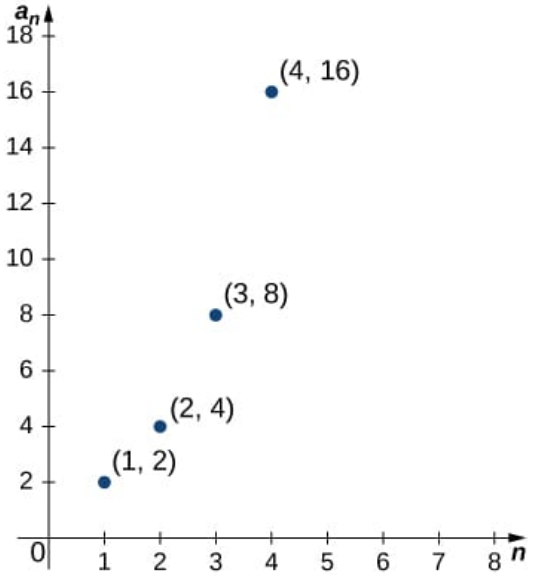


Figure $14.2.6.2.1$: The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3, 7, 11, 15, 19, \ldots \tag{14.2.6.2.9}$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, for\ n \geq 2 \end{cases} \tag{14.2.6.2.10}$$

Note that

$$a_2 = 3 + 4 \tag{14.2.6.2.11}$$

$$a_3 = 3 + 4 + 4 = 3 + 2 \cdot 4 \tag{14.2.6.2.12}$$

$$a_4 = 3 + 4 + 4 + 4 = 3 + 3 \cdot 4 \tag{14.2.6.2.13}$$

Thus the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n-1) = 4n - 1 \tag{14.2.6.2.14}$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$

In a geometric sequence**,** the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \frac{2}{81}, \ldots \tag{14.2.6.2.15}$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$a_1 = 2 \tag{14.2.6.2.16}$$

$$a_n = -\frac{1}{3} \cdot a_{n-1}, for\ n \geq 2 \tag{14.2.6.2.17}$$

Alternatively, since

$$a_2 = -\frac{1}{3} \cdot 2 \tag{14.2.6.2.18}$$

$$a_3 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2 \tag{14.2.6.2.19}$$

$$a_4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2, \tag{14.2.6.2.20}$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1} \tag{14.2.6.2.21}$$

The sequence 2^n that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 14.2.6.2.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$

b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots.$

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n + 1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1} \tag{14.2.6.2.22}$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n + 1$. Therefore, we can describe the sequence by the explicit formula $a_n = \dfrac{3^n}{3n+1.}$

Exercise 14.2.6.2.1

Find an explicit formula for the nth term of the sequence $\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.2.6.2.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2, a_n = -3a_{n-1}$ for $n \geq 2$
b. $a_1 = \left(\frac{1}{2}\right), a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2 2 \\ a_4 &= -3a_3 = (-3)^3 2. \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}$$

b. Write out the first few terms:

$$a_1 = \frac{1}{2}$$

$$a_2 = a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$a_3 = a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$a_4 = a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$$

Exercise 14.2.6.2.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Hint available at the end of this section.

Answer available at the end of this section.

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 14.2.6.2.2):

a. $1 + 3n = 4, 7, 10, 13, \ldots$. The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.
b. $1 - \left(\frac{1}{2}\right)^n = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$ .... The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.
c. $(-1)^n = -1, 1, -1, 1, \ldots$. The terms alternate but do not approach one single value as $n \to \infty$.
d. $\frac{(-1)^n}{n} = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots$. The terms alternate for this sequence as well, but $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$.

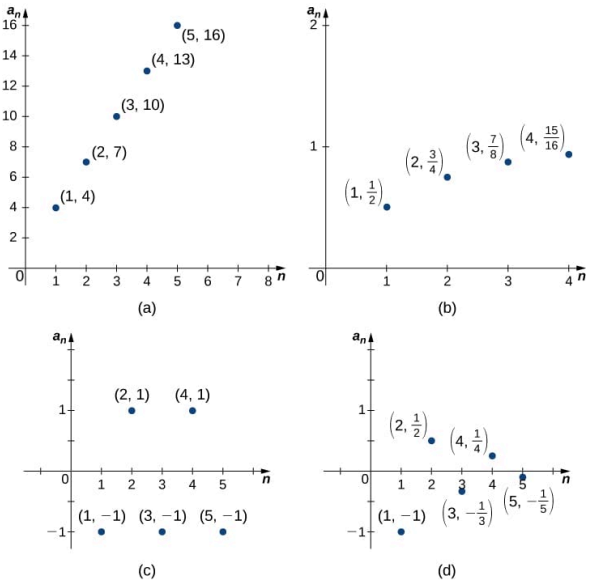


Figure 14.2.6.2.2: (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Limits: Convergent and Divergent sequences

Definition: convergent and divergent sequences

Given a sequence a_n, if the terms an become arbitrarily close to a finite number L as n becomes sufficiently large, we say a_n is a **convergent sequence** and L is the **limit of the sequence.** In this case, we write

$$\lim_{n\to\infty} a_n = L \tag{14.2.6.2.23}$$

If a sequence a_n is not convergent, we say it is a **divergent sequence.**

From Figure $14.2.6.2.2$, we see that the terms in the sequence $1-\left(\frac{1}{2}\right)^n$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $1-\left(\frac{1}{2}\right)^n$ is a convergent sequence and its limit is 1. In contrast, from Figure $14.2.6.2.2$, we see that the terms in the sequence $1+3n$ are not approaching a finite number as n becomes larger. We say that $1+3n$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms “arbitrarily close” and “sufficiently large.” Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the more formal definition of limit for a sequence and show these ideas graphically in Figure $14.2.6.2.2$.

Definition: Convergence

A sequence a_n converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ $|a_n - L| < \varepsilon$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \tag{14.2.6.2.24}$$

In this case, we say the sequence a_n is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence a_n depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1, b_2, \ldots, b_N, a_1, a_2, \ldots, \tag{14.2.6.2.25}$$

this new sequence will converge if a_n converges and diverge if a_n diverges. Further, if the sequence a_n converges to L, this new sequence will also converge to L.

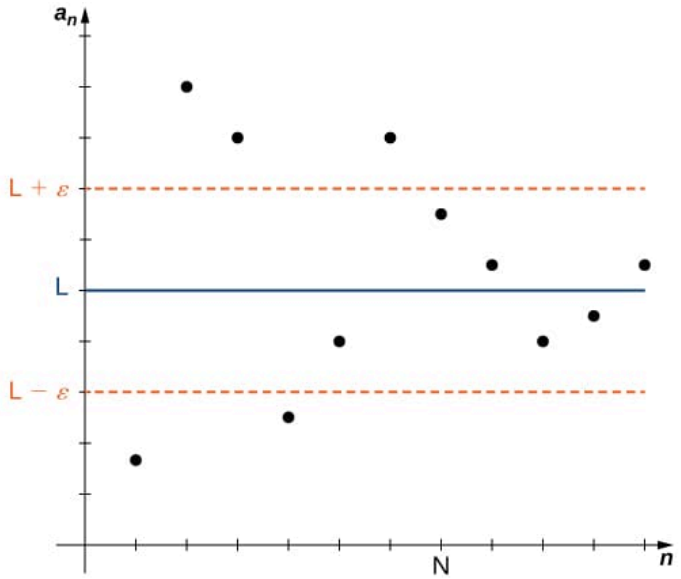


Figure $14.2.6.2.3$: As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $1+3n$ and $(-1)^n$ shown in Figure $14.2.6.2.2$ diverge. However, different sequences can diverge in different ways. The sequence $(-1)^n$ diverges

because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $1 + 3n$ diverges because the terms $1 + 3n \to \infty$ as $n \to \infty$. We say the sequence $1 + 3n$ diverges to infinity and write $\lim_{n\to\infty}(1 + 3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $1 + 3n$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $-5n + 2$ diverges to negative infinity because $-5n + 2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n\to\infty}(-5n + 2) =\to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence a_n and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\frac{1}{n}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\frac{1}{n}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Limit of a Sequence Defined by a Function

Consider a sequence a_n such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L \tag{14.2.6.2.26}$$

then a_n converges and

$$\lim_{n\to\infty} a_n = L \tag{14.2.6.2.27}$$

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $(1/2)^n$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty}(1/2)^x = 0$, we conclude that the sequence $(1/2)^n$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence r^n converges. On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence 1^n is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence r^n diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$r^n \to 0 \text{ if } \backslash(\backslash\text{displaystyle } 0$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1.$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $(2/3)^n + (1/4)^n$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $(2/3)^n$ and $(1/4)^n$. As we describe in the following algebraic limit laws, since $(2/3)^n$ and $1/4)^n$ both converge to 0, the sequence $(2/3)^n + (1/4)^n$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of a_n and b_n.

Algebraic Limit Laws

Given sequences a_n and b_n and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$
ii. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n = cA$
iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$
iv. $\lim_{n\to\infty} (a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n) = A \cdot B$
v. $\lim_{n\to\infty} (\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$, $|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \frac{1}{n^2}$. As shown earlier, $\lim_{n\to\infty} \frac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \tag{14.2.6.2.28}$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 14.2.6.2.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $5 - \frac{3}{n^2}$
b. $\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}$
c. $\frac{2^n}{n^2}$
d. $\left(1 + \frac{4}{n}\right)^n$

Solution

a. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} . \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3\lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\lim_{n\to\infty}\frac{3n^4-7n^2+5}{6-4n^4}=\lim_{n\to\infty}\frac{3-\frac{7}{n^2}+\frac{5}{n^4}}{\frac{6}{n^4}-4}$$

$$=\frac{\lim_{n\to\infty}(3-\frac{7}{n^2}+\frac{5}{n^4})}{\lim_{n\to\infty}(\frac{6}{n^4}-4)}$$

$$=\frac{(\lim_{n\to\infty}(3)-\lim_{n\to\infty}\frac{7}{n^2}+\lim_{n\to\infty}\frac{5}{n^4})}{(\lim_{n\to\infty}\frac{6}{n^4}-\lim_{n\to\infty}(4))}$$

$$=\frac{(\lim_{n\to\infty}(3)-7\cdot\lim_{n\to\infty}\frac{1}{n^2}+5\cdot\lim_{n\to\infty}\frac{1}{n^4})}{(6\cdot\lim_{n\to\infty}\frac{1}{n^4}-\lim_{n\to\infty}(4))}$$

$$=\frac{3-7\cdot 0+5\cdot 0}{6\cdot 0-4}=-\frac{3}{4}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x)=2^x/x^2$ defined on all real numbers $x>0$. Since $2^x\to\infty$ and $x^2\to\infty$ as $x\to\infty$, apply L'Hôpital's rule and write

$$\lim_{x\to\infty}\frac{2^x}{x^2}=\lim_{x\to\infty}\frac{2^x\ln(2)}{2x}$$ Take the derivatives of the numerator and denominator.

$$=\lim_{x\to\infty}\frac{2^x\ln(2)^2}{2}$$ Take the derivatives again.

$$=\infty$$

We conclude that the sequence diverges.

d. Consider the function $f(x)=(1+\frac{4}{x})^x$ defined on all real numbers $x>0$. This function has the indeterminate form 1^∞ as $x\to\infty$ Let

$$y=\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y)=\ln[\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x].$$

Since the function $f(x)=\ln(x)$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y)=\lim_{x\to\infty}[\ln\left(1+\frac{4}{x}\right)^x].$$

Using properties of logarithms, we write

$$\lim_{x\to\infty}[\ln\left(1+\frac{4}{x}\right)^x]=\lim_{x\to\infty}x\ln\left(1+\frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty\cdot 0$, rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form 0/0, we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x\to\infty}\frac{\ln(1+4/x)}{1/x} = \lim_{x\to\infty}\frac{4}{1+4/x} = 4$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left(1+\frac{4}{n}\right)^n$ converges to e^4.

Exercise 14.2.6.2.3

Consider the sequence $(5n^2+1)/e^n$ Determine whether or not the sequence converges. If it converges, find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that if f is a continuous function at a value L, then $f(x)\to f(L)$ as $x\to L$. This idea applies to sequences as well. Suppose a sequence $a_n\to L$, and a function f is continuous at L. Then $f(a_n)\to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5-\frac{3}{n^2}}$. From Example a. we know the sequence $5-\frac{3}{n^2}\to 5$. Since $\sqrt{x}$ is a continuous function at $x=5$,

$$\lim_{n\to\infty}\sqrt{5-\frac{3}{n^2}} = \sqrt{\lim_{n\to\infty}(5-\frac{3}{n^2})} = \sqrt{5} \qquad (14.2.6.2.29)$$

Continuous Functions Defined on Convergent Sequences

Consider a sequence a_n and suppose there exists a real number L such that the sequence a_n converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n\geq N$, and the sequence $f(a_n)$ converges to $f(L)$ (Figure 14.2.6.2.4).

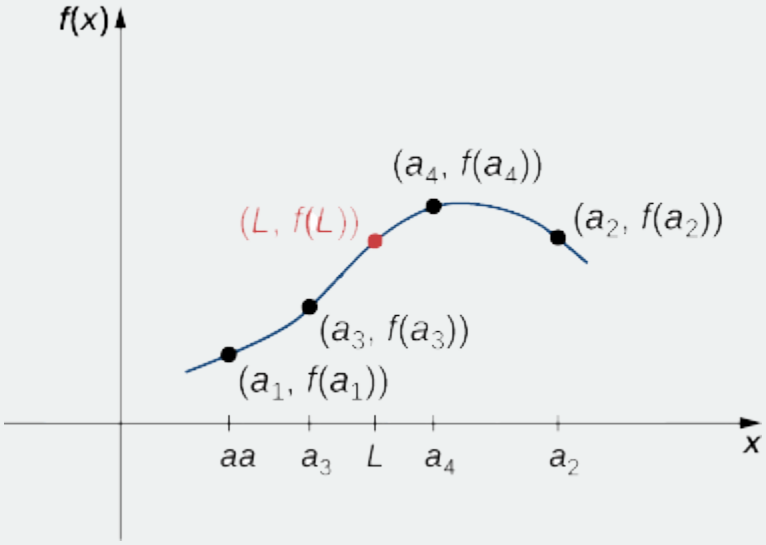


Figure 14.2.6.2.4: Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$

.

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence a_n converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $f(a_n)$ converges to $f(L)$.

Example 14.2.6.2.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $cos(3/n^2)$ converges. If it converges, find its limit.

Solution:

Since the sequence $3/n^2$ converges to 0 and $cos(x)$ is continuous at $x = 0$, we can conclude that the sequence $cos(3/n^2)$ converges and

$$\lim_{n\to\infty} cos\left(\frac{3}{n^2}\right) = cos(0) = 1$$

Exercise 14.2.6.2.4

Determine if the sequence $\sqrt{\frac{2n+1}{3n+5}}$ converges. If it converges, find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Squeeze theorem for Sequences

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits.

Squeeze Theorem for Sequences

Consider sequences $a_n, b_n,$ and c_n. Suppose there exists an integer N such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n, \tag{14.2.6.2.30}$$

then b_n converges and $\lim_{n\to\infty} b_n = L$ (Figure 14.2.6.2.5).

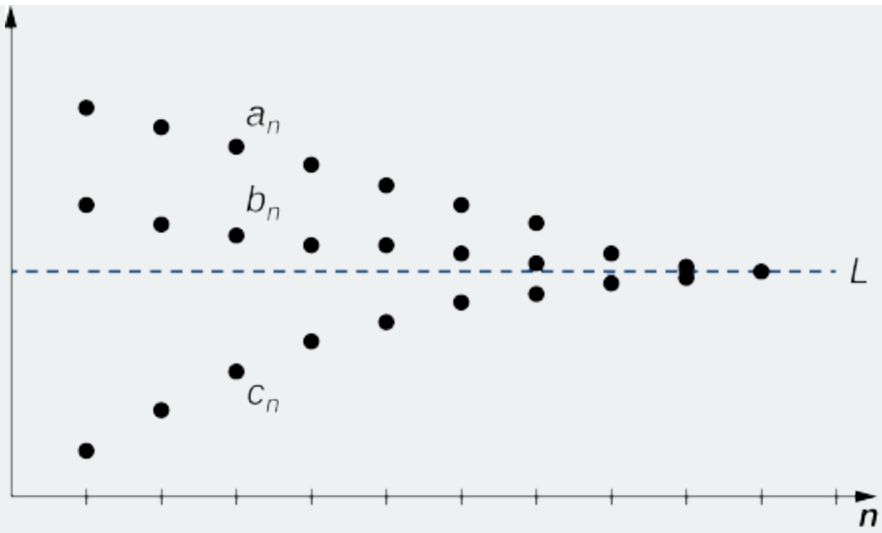


Figure 14.2.6.2.5: Each term b_n satisfies $a_n \le b_n \le c_n$ and the sequences a_n and c_n converge to the same limit, so the sequence b_n must converge to the same limit as well.

Proof

Let $\varepsilon > 0$. Since the sequence a_n converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \ge N_1$. Similarly, since c_n converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \ge N_2$. By assumption, there exists an integer N such that $a_n \le b_n \le c_n$ for all $n \ge N$. Let M be the largest of N_1, N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \ge M$. For all $n \ge M$,

$$-\varepsilon < -|a_n - L| \le a_n - L \le b_n - L \le c_n - L \le |c_n - L| < \varepsilon \qquad (14.2.6.2.31)$$

Therefore, \(\displaystyle –ε

Example 14.2.6.2.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\dfrac{cos(n)}{n^2}$

b. $(-\dfrac{1}{2})^n$

Solution

a. Since $-1 \le cos(n) \le 1$ for all integers n, we have

$$-\frac{1}{n^2} \le \frac{cos(n)}{n^2} \le \frac{1}{n^2}$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $cos(n)/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \overset{\le}{} (-\frac{1}{2})^n \le \frac{1}{2^n}$$

for all positive integers $n, -1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise 14.2.6.2.5

Find $\lim_{n\to\infty} \frac{2n - sin(n)}{n}$

Hint available at the end of this section.

Answer available at the end of this section.

Using the idea from Exampleb. we conclude that $r^n \to 0$ for any real number r such that \(\displaystyle –1

$$r^n \to 0 \text{ if } |r| < 1 \tag{14.2.6.2.32}$$

$$r^n \to 1 \text{ if } r = 1 \tag{14.2.6.2.33}$$

$$r^n \to \infty \text{ if } r > 1 \tag{14.2.6.2.34}$$

$$r^n diverges \text{ if } r \leq -1 \tag{14.2.6.2.35}$$

Bounded Sequences

Monotone Convergence Theorem

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bound Sequences

A sequence a_n is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence a_n is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all positive integers n.

A sequence a_n is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $1/n$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $1/n$ is a bounded sequence. On the other hand, consider the sequence 2^n. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, 2^n is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence a_n is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms an that are arbitrarily large in magnitude as n gets larger. As a result, the sequence a_n cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Convergent Sequences Are Bounded

If a sequence a_n converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $(-1)^n$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence a_n. Suppose the sequence a_n is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that a_n converges. For example, consider the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \tag{14.2.6.2.36}$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots \tag{14.2.6.2.37}$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition

A sequence a_n is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence a_n is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence a_n is a *monotone sequence* for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Definition: Monotone Convergence Theorem

If a_n is a bounded sequence and there exists a positive integer n_0 such that a_n is monotone for all $n \geq n_0$, then a_n converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure $14.2.6.2.6$).

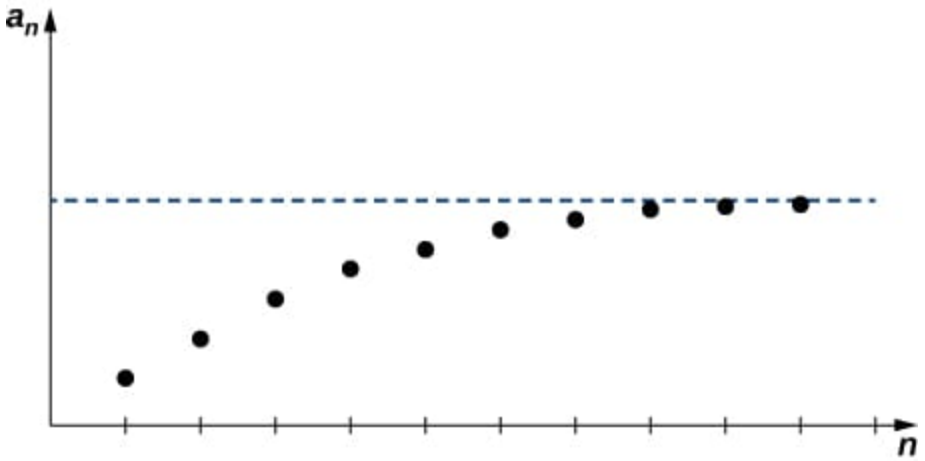


Figure 14.2.6.2.6: Since the sequence a_n is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 14.2.6.2.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\dfrac{4^n}{n!}$

b. a_n defined recursively such that
$a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{1}{2a_n}$ for all $n \geq 2$

Solution

a. Writing out the first few terms, we see that

$$\frac{4^n}{n!} = 4, 8, \frac{32}{3}, \frac{32}{3}, \frac{128}{15}, ...$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \text{ if } n \geq 3$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim\limits_{n\to\infty} a_n$. Now note this important observation. Consider $\lim\limits_{n\to\infty} a_{n+1}$. Since

$$a_{n+1} = a_2, a_3, a_4, ...,$$

the only difference between the sequences a_{n+1} and a_n is that a_{n+1} omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0 \cdot L = 0$$

b. Writing out the first several terms,

$$2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, \ldots$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n}.$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a^2 n + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n.$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2.$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Since a_n is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} (\frac{a_n}{2} + \frac{1}{2a_n}),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L}.$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1.$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Exercise 14.2.6.2.6

Consider the sequence a_n defined recursively such that $a_1 = 1$, $a_n = a_{n-1}/2$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Hint available at the end of this section.

Answer available at the end of this section.

Fibonacci Numbers

Golden ratio or Divine Proportion

Definition: Fibonacci Numbers

The *Fibonacci numbers* are defined recursively by the sequence F_n where $F_0 = 0, F_1 = 1$ and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.

2. Find a closed formula for the Fibonacci sequence by using the following steps.

    a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.

    b. Using the result from part a. as motivation, look for a solution of the equation

    $F_n = F_{n-1} + F_{n-2}$

    of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.

    c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2} \tag{14.2.6.2.38}$$

The number $\phi=(1+\sqrt{5})/2$ is known as the **golden ratio** (Figure 14.2.6.2.7 and Figure 14.2.6.2.8). The golden ratio is also known as the **divine proportion**.



Figure 14.2.6.2.7: The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

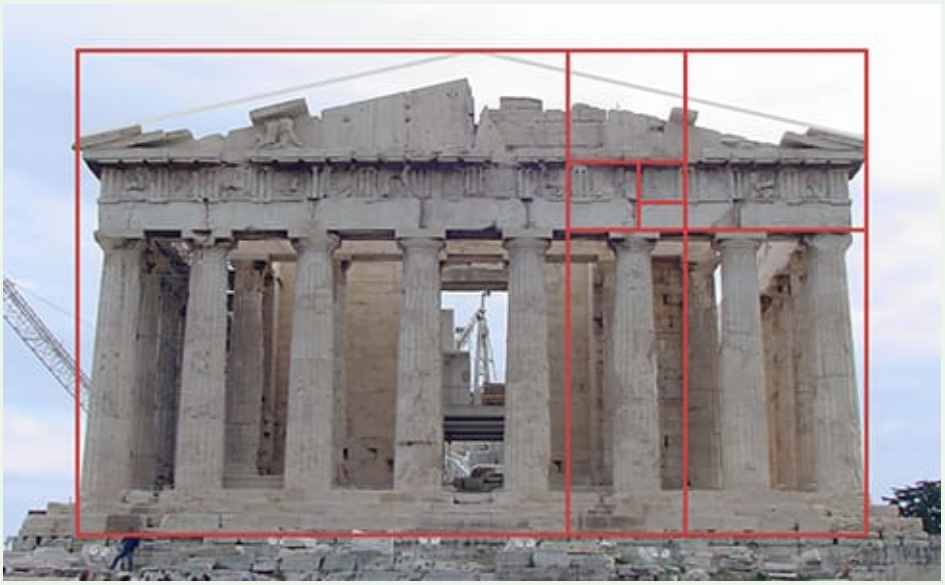


Figure 14.2.6.2.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr).

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n=f(n)$, we use the properties of limits for functions.
- If a_n and b_n are convergent sequences that converge to **A** and **B**, respectively, and **c** is any real number, then the sequence ca_nconverges to c · A, the sequences {an±bn} converge to A±B, the sequence $a_n\cdot b_n$ converges to A · B, and the sequence a_n/b_n converges to A/B, provided B≠0.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence r^n converges if and only if $|r|<1$ or $r=1$.

Glossary

arithmetic sequence
: a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above
: a sequence a_n is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below
: a sequence a_n is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence
: a sequence a_n is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence
: a convergent sequence is a sequence a_n for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence
: a sequence that is not convergent is divergent

divine proportion
: the golden ratio

explicit formula
: a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence
: a sequence a_n in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

golden ratio
: a golden ratio is a ratio of two quantities, say a and b, that are equal to the ratio of the larger quantity and the sum of the quantities. It is said that this ratio defines a unique natural structure and hence it is sometimes called the divine proportion. Interestingly this quantity is used in numerical methods.

index variable
: the subscript used to define the terms in a sequence is called the index

limit of a sequence
: the real number L to which a sequence converges is called the limit of the sequence

monotone sequence
: an increasing or decreasing sequence

recurrence relation
: a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence
: an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term
: the number a_n in the sequence a_n is called the n^{th} term of the sequence

unbounded sequence
: a sequence that is not bounded is called unbounded

Additional resources

This web site has fun information on the Fibonacci sequence: https://www.mathsisfun.com/numbers/fibonacci-sequence.html

A link on the previous web sites leads to some fun information on the Golden ratio: https://www.mathsisfun.com/numbers/nature-golden-ratio-fibonacci.html

The Divine Proportion: A Study in Mathematical Beauty, H. E. Huntley, Dover Books, 1970. This book is not the ***De Divina Proportione*** (1509) written by Luca Pacioli and illustrated by Leonardo da Vinci. The Huntley book is an inexpensive book while still being an excellent book. The Pacioli version is generally less informative to the modern readier, however, the illustrations by Leonardo da Vinci might be worth the expense (hardcopy book) for the 1509 version. There are many facsimile productions of "De Divina Proportione", just do a web search.

This web site discusses the relationship between the Golden ratio and quasi-crystals: https://www.goldennumber.net/quasi-crystals/

Contributors and Attributions

- Template:ContribOpenStaxCalc
- Additional resources section plus some minor edits contributed by Scott D. Johnson, Joshua Halpern, and Scott Sinex

Answers

Exercise 14.2.6.2.1

The denominators form an arithmetic sequence.

Exercise 14.2.6.2.1

$$a_n = \frac{(-1)^{n+1}}{3+2n} \tag{14.2.6.2.39}$$

Exercise 14.2.6.2.2

This is an arithmetic sequence.

Exercise 14.2.6.2.2

$a_n = 6n - 10$

Exercise 14.2.6.2.3

Use L'Hôpital's rule.

Exercise 14.2.6.2.3

The sequence converges, and its limit is 0

Exercise 14.2.6.2.4

Consider the sequence $\frac{2n+1}{3n+5}$

Exercise 14.2.6.2.4

The sequence converges, and its limit is $\sqrt{2/3}$.

Exercise 14.2.6.2.5

Use the fact that $-1 \leq sin(n) \leq 1$

Exercise 14.2.6.2.5

2

Exercise 14.2.6.2.6

Show the sequence is decreasing and bounded below.

Exercise 14.2.6.2.6

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14.2.6.3: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$S_1 = 1$$

$$S_2 = 1 + 0.5 = 1 + \tfrac{1}{2}$$

$$S_3 = 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4}$$

$$S_4 = 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}$$

$$S_5 = 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

At the same time, as $k\to\infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1=1$$

$$S_2=1+\tfrac{1}{2}=\tfrac{3}{2}$$

$$S_3=1+\tfrac{1}{2}+\tfrac{1}{4}=\tfrac{7}{4}$$

$$S_4=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}=\tfrac{15}{8}$$

$$S_5=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}=\tfrac{31}{16}$$

Plotting some of these values in Figure, it appears that the sequence S_k could be approaching 2.

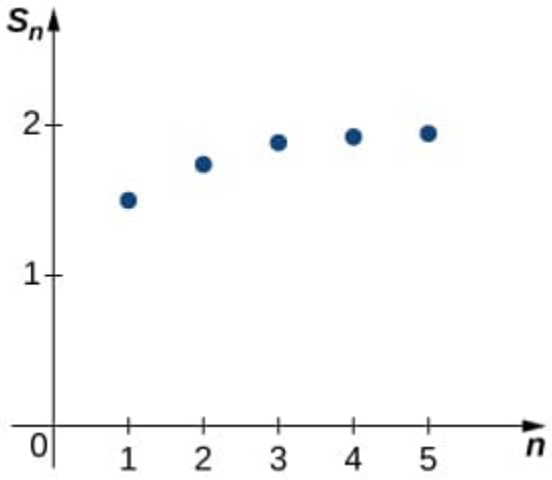


Figure 14.2.6.3.1: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k=2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we

summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots. \tag{14.2.6.3.1}$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \tag{14.2.6.3.2}$$

is called the k^{th} **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S. \tag{14.2.6.3.3}$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}$$

Example 14.2.6.3.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3}$$

$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$$

$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right)$$

$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right)$$

$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right)$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1 = -1$$

$$S_2 = -1 + 1 = 0$$

$$S_3 = -1 + 1 - 1 = -1$$

$$S_4 = -1 + 1 - 1 + 1 = 0$$

From this pattern we can see the sequence of partial sums is

$$S_k = -1, 0, -1, 0, \ldots$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

$$S_5 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}$$

From this pattern, we can see that the k^{th} partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

Since $k/(k+1) \to 1$, we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Exercise 14.2.6.3.1

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint available at the end of this section.

Answer available at the end of this section.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{14.2.6.3.4}$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$ terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right)$$

Using the same idea for S_8, we see that

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right).$$

From this pattern, we see that $S_1 = 1, S_2 = 1 + 1/2, S_4 > 1 + 2(1/2)$, and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty$, we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note 14.2.6.3.1: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n - b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 14.2.6.3.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$

Solution

We showed earlier that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2$$

Since both of those series converge, we can apply the properties of Note 14.2.6.3.1 to evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)} + \left(\frac{1}{2}\right)^{n-2}\right]$$

Using the sum rule, write

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\left(\frac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=3(1)+\left(\frac{1}{2}\right)^{-1}(2)=3+2(2)=7$$

Exercise 14.2.6.3.2

Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}$

Hint available at the end of this section.

Answer available at the end of this section.

Geometric Series

A **geometric series** is any series that we can write in the form

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1} \tag{14.2.6.3.5}$$

Because the ratio of each term in this series to the previous term is *r*, the number *r* is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty}ar^{n-1} \tag{14.2.6.3.6}$$

when $a>0$. Its sequence of partial sums S_k is given by

$$S_k=\sum_{n=1}^{k}ar^{n-1}=a+ar+ar^2+\cdots+ar^{k-1} \tag{14.2.6.3.7}$$

Consider the case when $r=1$. In that case,

$$S_k=a+a(1)+a(1)^2+\cdots+a(1)^{k-1}=ak \tag{14.2.6.3.8}$$

Since $a>0$, we know $ak\to\infty$ as $k\to\infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r=1$. For $r\neq 1$, to find the limit of S_k, multiply the equation by $1-r$. Doing so, we see that

$$(1-r)S_k=a(1-r)(1+r+r^2+r^3+\cdots+r^{k-1})=a[(1+r+r^2+r^3+\cdots+r^{k-1})-(r+r^2+r^3+\cdots+r^k)]=a(1-r^k)\quad($$

All the other terms cancel out.

Therefore,

$$S_k = \frac{a(1-r^k)}{1-r} \text{ for } r \neq 1$$

From our discussion in the previous section, we know that the geometric sequence $r^k \to 0$ if $|r| < 1$ and that r^k diverges if $|r| > 1$ or $r = \pm 1$. Therefore, for $|r| < 1, S_k \to \frac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ if } |r| < 1$$

If $|r| \geq 1, S_k$ diverges, and therefore

$$\sum_{n=1}^{\infty} ar^{n-1} \text{ diverges if } |r| \geq 1$$

Definitions: Diverging and Converging Series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \tag{14.2.6.3.10}$$

If $|r| < 1$, the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1 \tag{14.2.6.3.11}$$

If $|r| \geq 1$, the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form above, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots = \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1-2/3} = \frac{4}{3}$$

Example 14.2.6.3.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to

$$\frac{9}{1-(-3/4)} = \frac{9}{7/4} = \frac{36}{7}$$

b. Writing this series as

$$e^2 \sum_{n=1}^{\infty} (e^2)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$ Therefore, the series diverges.

Exercise 14.2.6.3.3

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 14.2.6.3.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.\overline{26}$ as a fraction of integers.

Solution

Since $3.\overline{26}— = 3.262626\ldots$, first we write

$$\begin{aligned}3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{1000} + \frac{26}{100,000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots\end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}$$

Thus,

$$3.262626\ldots = 3 + \tfrac{26}{99} = \tfrac{323}{99}$$

Exercise 14.2.6.3.4

Write $5.2\overline{7}$ as a fraction of integers.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.2.6.3.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 14.2.6.3.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.**

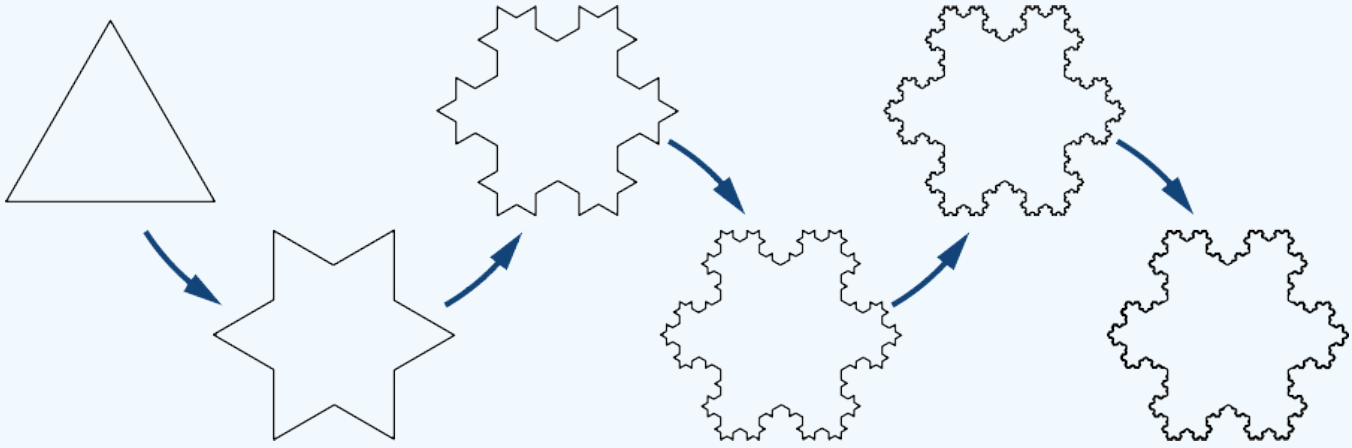


Figure 14.2.6.3.2: The first four figures, F_0, F_1, F_2, and F_3, in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let ln denote the length of each side of F_n . Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3}$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \dfrac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n \tag{14.2.6.3.12}$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n \tag{14.2.6.3.13}$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n \to \infty} L_n = \infty \tag{14.2.6.3.14}$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1} \tag{14.2.6.3.15}$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4} \tag{14.2.6.3.16}$$

Writing out the first few terms A_0, A_1, A_2, we see that

$A_0 = \frac{\sqrt{3}}{4}$

$A_1 = A_0 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{3}{4} \cdot \left(\frac{4}{9}\right) \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right]$

$A_2 = A_1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right)\right] + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2 \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}[1 + \frac{3}{4} \cdot \left(\frac{4}{9}\right) + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^2]$

More generally,

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4}\left(\tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^n\right)\right]$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{1}{3}\left(1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^{n-1}\right)\right]$$

The expression $1+\left(\frac{4}{9}\right)+\left(\frac{4}{9}\right)^2+\cdots+\left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1+\tfrac{4}{9}+\left(\tfrac{4}{9}\right)^2+\cdots+\left(\tfrac{4}{9}\right)^{n-1}=\frac{1-(4/9)^n}{1-(4/9)}$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n=\frac{\sqrt{3}}{4}\left[1+\frac{1}{3}(\frac{1-(4/9)^n}{1-(4/9)})\right]=\frac{\sqrt{3}}{4}\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]$$

Therefore, the area of Koch's snowflake is

$$A=\lim_{n\to\infty}A_n=\frac{2\sqrt{3}}{5}$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y=1/x^2$ and the x-axis on the interval $[1,\infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\,dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$. We discussed this series in Example, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k=\dfrac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right]=\left(1+\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$S_1=1-\tfrac{1}{2}$$

$$S_2=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)=1-\tfrac{1}{3}$$

$$S_3=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)=1-\tfrac{1}{4}$$

In general,

$$S_k=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{k}-\tfrac{1}{k+1}\right)=1-\frac{1}{k+1}$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k=1-1/(k+1)$ and $1/(k+1)\to 0$ as $k\to\infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$S_1 = b_1 - b_2$$

$$S_2 = (b_1 - b_2) + (b_2 - b_3) = b_1 - b_3$$

$$S_3 = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) = b_1 - b_4$$

In general, the k^{th} partial sum of this series is

$$S_k = b_1 - b_{k+1}$$

Since the k^{th} partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to $b_1 - B$, and therefore

$$\sum_{n=1}^{\infty}[b_n - b_{n+1}] = b_1 - B$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 14.2.6.3.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$S_1 = \cos(1) - \cos(\tfrac{1}{2})$$

$$S_2 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) = \cos(1) - \cos(\tfrac{1}{3})$$

$$S_3 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) + (\cos(\tfrac{1}{3}) - \cos(\tfrac{1}{4}))$$

$$= \cos(1) - \cos(\tfrac{1}{4})$$

In general,

$$S_k = \cos(1) - \cos\left(\tfrac{1}{k+1}\right)$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1$$

Exercise 14.2.6.3.5

Determine whether $\sum_{n=1}^{\infty}[e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint available at the end of this section.

Answer available at the end of this section.

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$ In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right) \to \gamma \text{ as } k \to \infty$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right)$ Evaluate T_k for various values of k.

2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.

   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1 + 1/k > 1/(k+1))$

   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)

   c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.

3. Now estimate how far T_k is from γ for a given integer k. Prove that for \(k≥1, 0

   a. Show that $\ln(k+1) - \ln k < 1/k$

   b. Use the result from part a. to show that for any integer k,

   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

   $$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j)$$

   Use the result from part b. combined with this telescoping sum to conclude that

   $$T_k - T_j < \frac{1}{k} - \frac{1}{j}$$

a. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}$$

e. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ and the corresponding sequence of partial sums S_k where $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$, the series converges if and only if the sequence S_k converges.
- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$.
- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges.
- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The k^{th} partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k\to\infty} b_{k+1}$ exists. In that case, $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k\to\infty} (b_{k+1})$.

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1$$

Glossary

convergence of a series
a series converges if the sequence of partial sums for that series converges

divergence of a series
a series diverges if the sequence of partial sums for that series diverges

geometric series
a geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

harmonic series
the harmonic series takes the form $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

infinite series
an infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

partial sum

the k^{th} partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$

telescoping series

a telescoping series is one in which most of the terms cancel in each of the partial sums

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Answers

Exercise 14.2.6.3.1

Look at the sequence of partial sums.

Exercise 14.2.6.3.1

The series diverges because the k^{th} partial sum $S_k > k$.

Exercise 14.2.6.3.2

Rewrite as $\sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1}$

Exercise 14.2.6.3.2

10

Exercise 14.2.6.3.3

$r = -2/5$

Exercise 14.2.6.3.3

$5/7$

Exercise 14.2.6.3.4

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Exercise 14.2.6.3.4

$475/90$

Exercise 14.2.6.3.5

Write out the sequence of partial sums to see which terms cancel.

Exercise 14.2.6.3.5

$e - 1$

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14.2.7: POWER SERIES

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

14.2.7.1: Prelude to Power Series

14.2.7.2: Power Series and Functions

14.2.7.3: Taylor and Maclaurin Series

14.2.7.4: Working with Taylor Series

14.2.7.5: Fourier Series and Transform

14.2.7.6: The Laplace Transform

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14.2.7.1: Prelude to Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n \tag{14.2.7.1.1}$$

is known as a **power series**. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as “infinite polynomials.” In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

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Modifications to compact this for review. Scott Johnson, Scott Sinex, and Joshua Halpern.

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14.2.7.2: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots, \tag{14.2.7.2.1}$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \tag{14.2.7.2.2}$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition 14.2.7.2.1: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots \tag{14.2.7.2.3}$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots \tag{14.2.7.2.4}$$

is a *power series* centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{14.2.7.2.5}$$

and

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \ldots \tag{14.2.7.2.6}$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \ldots \tag{14.2.7.2.7}$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Note 14.2.7.2.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R > 0$ such that the series converges if \(|x–a|R\). At the values x where |x–a|=R, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that \(|x|

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii. of Note, then the series converges for all x such that \(|x–a|0\), and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1,1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1,1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: radius of convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for \(|x–a|R,\) then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure $14.2.7.2.1$).

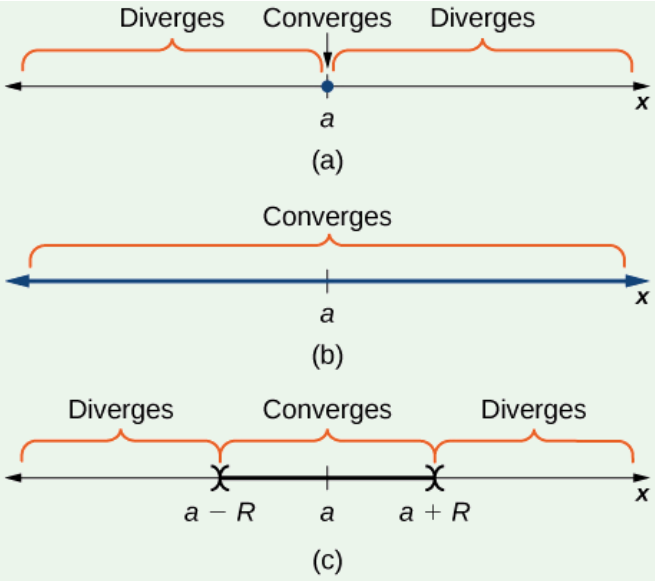


Figure 14.2.7.2.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R=0$, graph (b) shows a radius of convergence at $R=\infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x=a+R$ and $x=a-R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 14.2.7.2.1, we show the three different possibilities illustrated in Figure 14.2.7.2.1.

Example 14.2.7.2.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n!x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0$, we see that

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| \\ &= \lim_{n\to\infty} |(n+1)x| \\ &= |x| \lim_{n\to\infty} (n+1) \\ &= \infty.\end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x \neq 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right| \\ &= \frac{|x-2|}{3}.\end{aligned}$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that \(–31\) if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

Exercise 14.2.7.2.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Hint available at the end of this section.

Answer available at the end of this section.

Representing Functions as Power Series

Being able to represent a function by an “infinite polynomial” is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n. \tag{14.2.7.2.8}$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + ... \tag{14.2.7.2.9}$$

converges if and only if $|r| < 1$. In that case, it converges to $\frac{a}{1-r}$. Therefore, if $|x| < 1$, the series in Example $14.2.7.2.1$ converges to $\frac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x} \quad for \quad |x| < 1. \tag{14.2.7.2.10}$$

As a result, we are able to represent the function $f(x) = \frac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + ... \quad when \quad |x| < 1. \tag{14.2.7.2.11}$$

We now show graphically how this series provides a representation for the function $f(x) = \frac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example $14.2.7.2.2$: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \frac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure you see that as N increases, S_N becomes a better approximation for $f(x) = \frac{1}{1-x}$ for x in the interval $(-1, 1)$.

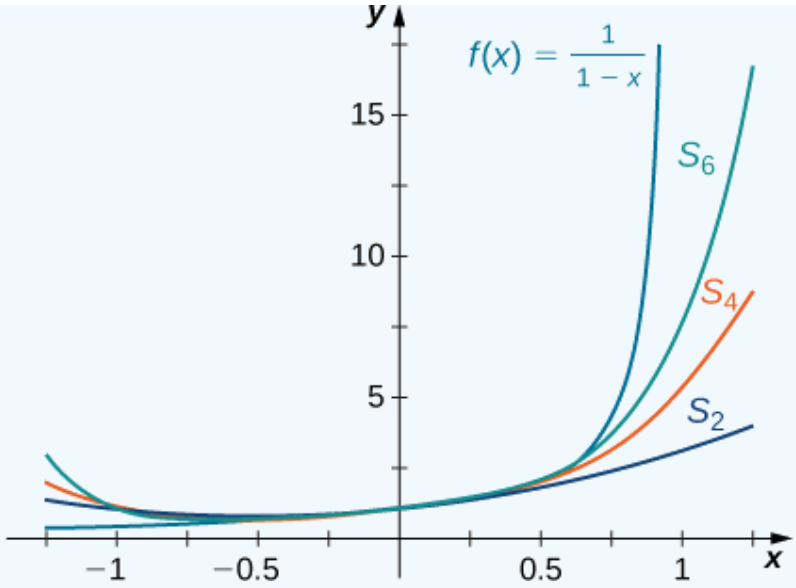


Figure 14.2.7.2.2: The graph shows a function and three approximations of it by partial sums of a power series.

Exercise 14.2.7.2.2

Sketch a graph of $f(x) = \dfrac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint available at the end of this section.

Answer available at the end of this section.

Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 14.2.7.2.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \dfrac{1}{1+x^3}$

b. $f(x) = \dfrac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r| < 1$, $\dfrac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots,$$

we see that, for $|-x^3| < 1$,

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}$$
$$= \sum_{n=0}^{\infty} (-x^3)^n$$
$$= 1 - x^3 + x^6 - x^9 + \ldots .$$

Since this series converges if and only if $|-x^3| < 1$, the interval of convergence is $(-1, 1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots \; for |x| < 1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\frac{x^2}{4-x^2} = \frac{x^2}{4\left(\frac{1-x^2}{4}\right)}$$
$$= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}.$$

Therefore, we have

$$\frac{x^2}{4-x^2} = \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}$$
$$= \frac{\frac{x^2}{4}}{1-\left(\frac{x}{2}\right)^2}$$
$$= \sum_{n=0}^{\infty} \frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.$$

The series converges as long as $|(\frac{x}{2})^2| < 1$ (note that when $|(\frac{x}{2})^2| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\frac{x^2}{4-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}}$$
$$= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \ldots$$

for $|x| < 2$.

Exercise 14.2.7.2.3

Represent the function $f(x) = \dfrac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint available at the end of this section.

Answer available at the end of this section.

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for \(|x–a|R\). In this case, the radius of convergence is R.
- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at** $x = 0$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at** $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

interval of convergence
: the set of real numbers x for which a power series converges

power series
: a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence
: if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for \(|x–a|R\), then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.2.7.2.1

Apply the ratio test to check for absolute convergence.

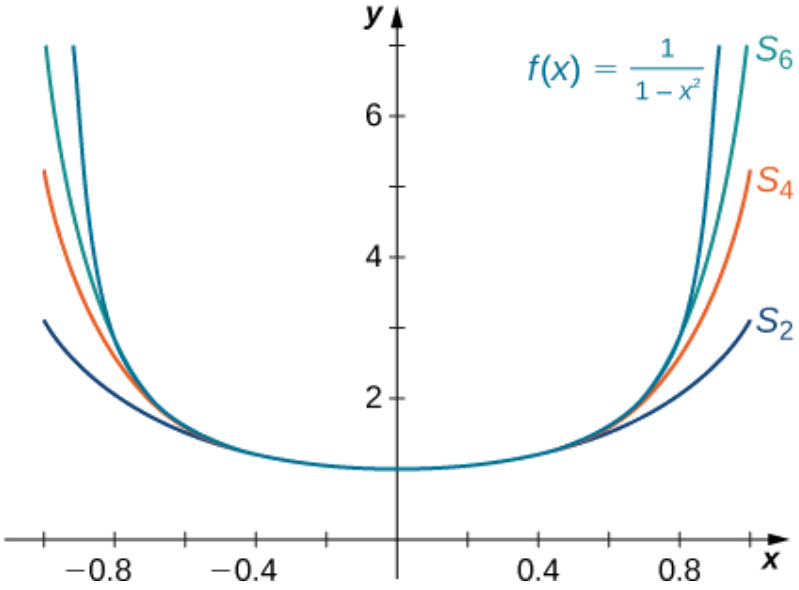
Exercise 14.2.7.2.1

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Exercise 14.2.7.2.2

$S_N(x) = 1 + x^2 + \ldots + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$

Exercise 14.2.7.2.2



Exercise 14.2.7.2.3

Rewrite f in the form $f(x) = \frac{g(x)}{1-h(x)}$ for some functions g and h.

Exercise 14.2.7.2.3

$\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

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14.2.7.3: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions--specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Taylor Series Definition

We will first define the Taylor[1] series (and Maclaurin series) and then go about showing how it is developed in a step by step fashion.

Definition 14.2.7.3.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f centered **at a** is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)|_{x=a}}{n!}(x-a)^n = f(x)|_{x=a} + f'(x)|_{x=a}(x-a) + \frac{f''(x)|_{x=a}}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(x)|_{x=a}}{n!}(x-a)^n + \cdots \quad (14.2.7.3.$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

In short notation when we evaluate at x=a we just put the a in the function ahead of time so the equation looks like

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \quad (14.2.7.3.2)$$

For numerical methods being at a is changed to being "in the vicinity of a".

Informal Proof of Taylor Series

Consider a function f that has a power series representation at $x = a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots. \quad (14.2.7.3.3)$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 14.2.7.3.3 is a representation for f at $x = a$, we certainly want the series to equal $f(a)$ at $x = a$. Evaluating the series at $x = a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \quad (14.2.7.3.4)$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f\,'(a)$ at $x = a$. Differentiating Equation 14.2.7.3.4 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots. \tag{14.2.7.3.5}$$

Therefore, at $x = a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{14.2.7.3.6}$$

Therefore, the derivative of the series equals $f\,'(a)$ if the coefficient $c_1 = f\,'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 14.2.7.3.6 will agree with all the corresponding derivatives of f at $x = a$. The second and third derivatives of Equation 14.2.7.3.5 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{14.2.7.3.7}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots. \tag{14.2.7.3.8}$$

Therefore, at $x = a$, the second and third derivatives

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{14.2.7.3.9}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{14.2.7.3.10}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3\cdot 2}$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{14.2.7.3.11}$$

This power series for f is known as the Taylor series for f at a. If $x = 0$, then this series is known as the Maclaurin series for f.

ıter in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a nction will converge to that function. Here, we state an important result. Recall that power series representations are unique. ıerefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Uniqueness of Taylor Series

Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The n^{th} partial sum of the Taylor series for a function f at a is known as the n^{th}-degree Taylor polynomial. For example, the 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} partial sums of the Taylor series are given by

$$\begin{aligned} p_0(x) &= f(a) \\ p_1(x) &= f(a) + f'(a)(x-a) \\ p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{aligned}$$

respectively. These partial sums are known as the 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} degree Taylor polynomials of f at a, respectively. If $x = a$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 14.2.7.3.2: Maclaurin polynomial

If f has n derivatives at $x = a$, then the n^{th}-degree Taylor polynomial of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{14.2.7.3.12}$$

The n^{th}-degree Taylor polynomial for f at 0 is known as the n^{th}-degree Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 14.2.7.3.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$f(x) = \ln x \; f(1) = 0$$

$$f'(x) = \frac{1}{x} \; f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \; f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \; f'''(1) = 2$$

Therefore,

$$p_0(x) = f(1) = 0,$$
$$p_1(x) = f(1) + f'(1)(x-1) = x - 1,$$
$$p_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$$
$$p_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 14.2.7.3.1.

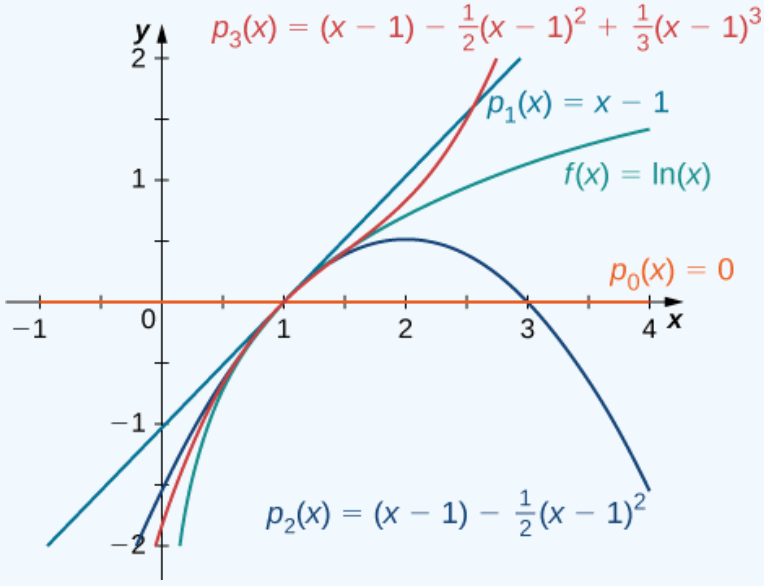


Figure 14.2.7.3.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Exercise 14.2.7.3.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

Hint available at the end of this section.

We now show how to find Maclaurin polynomials for $e^x, \sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 14.2.7.3.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the n^{th} -degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

Since $f(x) = e^x$,we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers n. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$p_0(x) = f(0) = 1,$$

$$p_1(x) = f(0) + f\,'(0)x = 1 + x,$$

$$p_2(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2,$$

$$p_3(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3,$$

$$p_n(x) = f(0) + f\,'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

The function and the first three Maclaurin polynomials are shown in Figure 2.

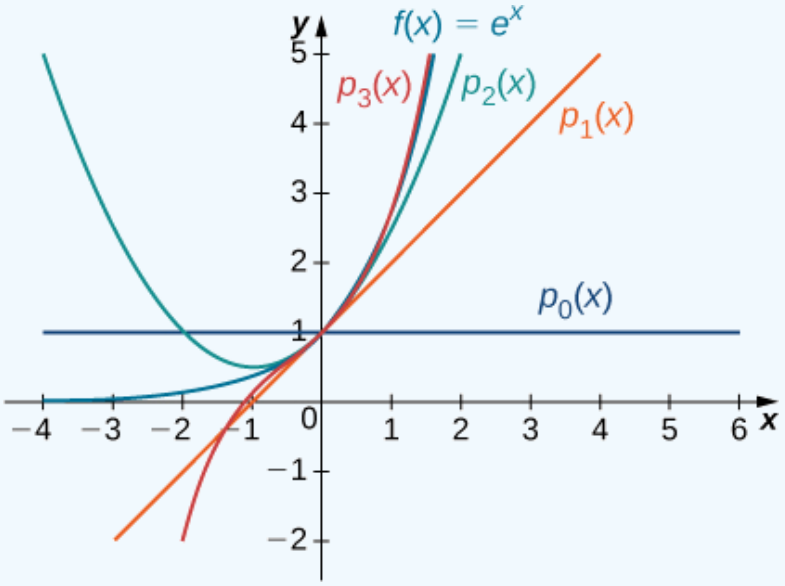


Figure 14.2.7.3.2: *The graph shows the function $y = e^x$ and the Maclaurin polynomials $p_0,\ p_1,\ p_2$ and p_3.*

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \sin x \ f(0) = 0$$

$$f\,'(x) = \cos x \ f\,'(0) = 1$$

$$f''(x) = -\sin x \ f''(0) = 0$$

$$f'''(x) = -\cos x \ f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \ f^{(4)}(0) = 0.$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$p_0(x) = 0,$$

$$p_1(x) = 0 + x = x,$$

$$p_2(x) = 0 + x + 0 = x,$$

$$p_3(x) = 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!},$$

$$p_4(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!},$$

$$p_5(x) = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

and for $m \geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 3.

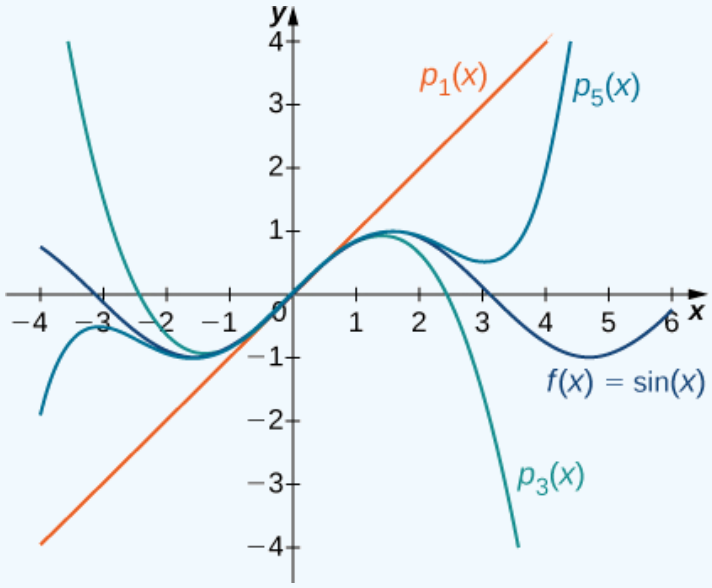


Figure 14.2.7.3.3: *The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.*

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$f(x) = \cos x \; f(0) = 1$$

$$f'(x) = -\sin x \; f'(0) = 0$$

$$f''(x) = -\cos x \; f''(0) = -1$$

$$f'''(x) = \sin x \; f'''(0) = 0$$

$$f^{(4)}(x) = \cos x \; f^{(4)}(0) = 1.$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$p_0(x) = 1,$$

$$p_1(x) = 1 + 0 = 1,$$

$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!},$$

$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},$$

$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Graphs of the function and the Maclaurin polynomials appear in Figure 4.

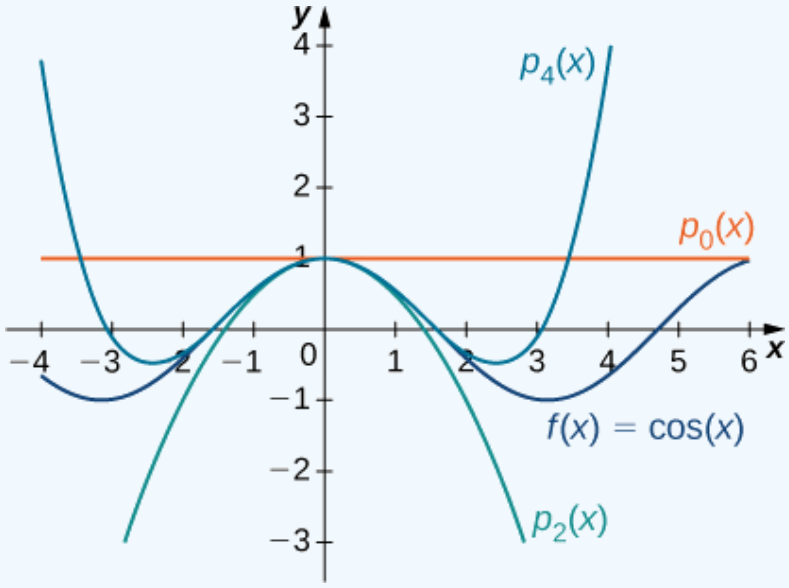


Figure 14.2.7.3.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.*

Exercise 14.2.7.3.2

Find formulas for the Maclaurin polynomials $p_0,\ p_1,\ p_2$ and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the n^{th}-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint available at the end of this section.

Taylor's Theorem with Remainder

Recall that the n^{th}-degree Taylor polynomial for a function f at a is the n^{th} partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x). \tag{14.2.7.3.13}$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the n^{th}-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0^{th} Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f\,'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a). \tag{14.2.7.3.14}$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the n^{th} remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \tag{14.2.7.3.15}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $| f^{(n+1)}(x) |$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \tag{14.2.7.3.16}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its n^{th}-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n + 1$ times on an interval I containing the real number a. Let p_n be the n^{th} -degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x) \tag{14.2.7.3.17}$$

be the n^{th} remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1} \tag{14.2.7.3.18}$$

.

If there exists a real number M such that $| f^{(n+1)}(x) | \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - a|^{n+1} \tag{14.2.7.3.19}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n - R_n(x)\frac{(x - t)^{n+1}}{(x - a)^{n+1}}. \tag{14.2.7.3.20}$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t = a$ and $t = x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g\,'(c) = 0$. We now calculate $g\,'$. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{14.2.7.3.21}$$

Consequently,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)\right.$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \tag{14.2.7.3.23}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g\,'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} \tag{14.2.7.3.24}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0. \tag{14.2.7.3.25}$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{14.2.7.3.26}$$

as desired. From this fact, it follows that if there exists M such that $|\, f^{(n+1)}(x)\,| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \tag{14.2.7.3.27}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 14.2.7.3.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$f(x) = \sqrt[3]{x}, f(8) = 2$$

$$f\,'(x) = \frac{1}{3x^{2/3}},\ f\,'(8) = \frac{1}{12}$$

$$f''(x) = \frac{-2}{9x^{5/3}},\ f''(8) = -\frac{1}{144.}$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$p_1(x) = f(8) + f\,'(8)(x-8)$$

$$= 2 + \frac{1}{12}(x-8)$$

$$p_2(x) = f(8) + f\,'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

The function and the Taylor polynomials are shown in Figure 4.

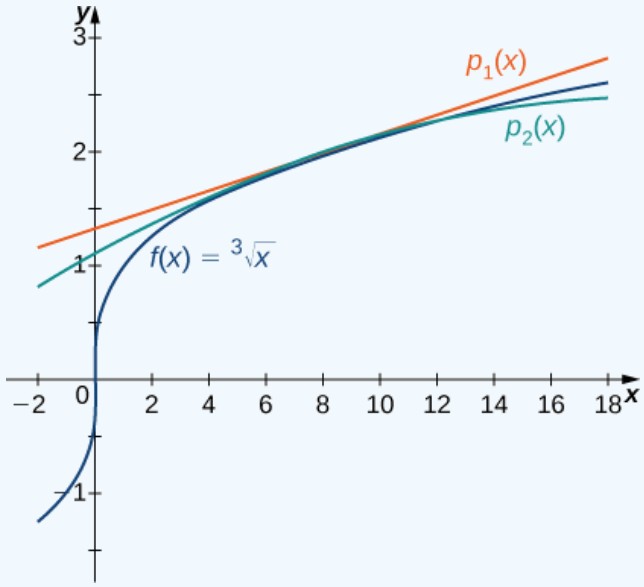


Figure 14.2.7.3.5: *The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x = 8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11-8) = 2.25. \tag{14.2.7.3.28}$$

Using the second Taylor polynomial at $x = 8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875. \tag{14.2.7.3.29}$$

c. By Note, there exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2. \tag{14.2.7.3.30}$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\frac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\frac{1}{144}$, we obtain

$$|R_1(11)| \leq \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \frac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \leq \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

Exercise 14.2.7.3.3:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x = 4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint available at the end of this section.

Example 14.2.7.3.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example b., the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.
b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{14.2.7.3.31}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right) \approx p_5\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \approx 0.173648. \tag{14.2.7.3.32}$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Note, the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7 \tag{14.2.7.3.33}$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $|\, f^{(7)}(x)\,| \leq 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}. \tag{14.2.7.3.34}$$

b.

We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001. \tag{14.2.7.3.35}$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

Exercise 14.2.7.3.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint available at the end of this section.

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 14.2.7.3.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$f(x) = \frac{1}{x}\ f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \quad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \quad f''(1) = 2!$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4} \quad f'''(1) = -3!$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \quad f^{(4)}(1) = 4!.$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|\,(-1)^{n+1}(x-1)n^{+1}\,|}{|(-1)^n(x-1)^n|} = |x-1|.$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for \(0

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Exercise 14.2.7.3.5

Find the Taylor series for $f(x) = \frac{1}{2}$ at $x = 2$ and determine its interval of convergence.

Hint available at the end of this section.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}. \tag{14.2.7.3.36}$$

That is, f can be represented by the geometric series $\sum_{n=0}^{\infty} (1-x)^n$. Since this is a geometric series, it converges to $\frac{1}{x}$ as long as $|1-x| < 1$. Therefore, the Taylor series found in Example does converge to $f(x) = \frac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the n^{th} partial sum is given by the n^{th}-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n \to \infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

We now state this theorem formally.

Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{14.2.7.3.37}$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0 \tag{14.2.7.3.38}$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \tag{14.2.7.3.39}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 14.2.7.3.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Note to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the n^{th}-degree Maclaurin polynomial for e^x found in Example a., we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \ge 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \le b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the n^{th}-degree Maclaurin polynomial for $\sin x$ found in Example b., we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}.$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $| f^{(n+1)}(c) | \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 14.2.7.3.6

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint available at the end of this section.

Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$ for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$ to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the n^{th}-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The n^{th}-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for $e^x, \sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f\,'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial
: a Taylor polynomial centered at 0; the n^{th}-degree Taylor polynomial for f at 0 is the n^{th}-degree Maclaurin polynomial for f

Maclaurin series
: a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials
: the n^{th}-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f\,'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series
: a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the n^{th}-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between x and a; if there exists an interval I containing a and a real number M such that $|\ f^{(n+1)}(x)\ | \leq M$ for all x in I, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

Contributors and Attributions

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[1]Visit the MacTutor History of Mathematics archive to read brief biographies of Brook Taylor and Colin Maclaurin and how they developed the concepts named after them.

Answers

Exercise 14.2.7.3.1

Find the first three derivatives of f and evaluate them at $x = 1$.

$$p_0(x) = 1 \tag{14.2.7.3.40}$$

$$p_1(x) = 1 - 2(x-1) \tag{14.2.7.3.41}$$

$$p_2(x) = 1 - 2(x-1) + 3(x-1)^2 \tag{14.2.7.3.42}$$

$$p_3(x) = 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 \tag{14.2.7.3.43}$$

Exercise 14.2.7.3.2

Evaluate the first four derivatives of f and look for a pattern.

$p_0(x) = 1;\ \ p_1(x) = 1 - x;\ \ p_2(x) = 1 - x + x^2;\ \ p_3(x) = 1 - x + x^2 - x^3;\ \ p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$

Exercise 14.2.7.3.3**:**

Evaluate $f(4), f\,'(4)$, and $f''(4)$.

$p_1(x) = 2 + \dfrac{1}{4}(x-4); p_2(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \leq 0.0625; |R_2(6)| \leq 0.015625$

Exercise 14.2.7.3.4

The fourth Maclaurin polynomial is $p_4(x) = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$.

0.96593

Exercise 14.2.7.3.5

$f^{(n)}(2) = \dfrac{(-1)^n n!}{2^{n+1}}$

$\dfrac{1}{2}\sum_{n=0}^{\infty}(\dfrac{2-x}{2})^n$. The interval of convergence is $(0, 4)$.

Exercise 14.2.7.3.6

Use the Maclaurin polynomials for $\cos x$.

$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \dfrac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

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14.2.7.4: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate nonelementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x) = (1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r = 0, 1, 2, 3, 4, f(x) = (1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1 \\ f(x) &= (1+x)^1 = 1+x \\ f(x) &= (1+x)^2 = 1+2x+x^2 \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3 \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4 \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{14.2.7.4.1}$$

and

$$f(x) = (1+x)^r = \binom{r}{0}1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r = \sum_{n=0}^{r}\binom{r}{n}x^n \tag{14.2.7.4.2}$$

For example, using this formula for $r = 5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \tbinom{5}{0}1+\tbinom{5}{1}x+\tbinom{5}{2}x^2+\tbinom{5}{3}x^3+\tbinom{5}{4}x^4+\tbinom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5 \end{aligned}$$

We now consider the case when the exponent r

is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x) = (1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^r \\ f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} \\ f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} \\ f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} \\ f'''(0) &= r(r-1)(r-2) \\ f(n)(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} \\ f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!} \tag{14.2.7.4.3}$$

We note that if r is a nonnegative integer, then the $(r+1)st$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation for the coefficients agrees with Equation for the coefficients, and the formula for the binomial series agrees with Equation for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n} = \frac{(r-1)(r-2)\cdots(r-n+1)}{n!} \tag{14.2.7.4.4}$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \tag{14.2.7.4.5}$$

We now need to determine the interval of convergence for the binomial series Equation 14.2.7.4.5. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!} \cdot \frac{n}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x| < 1 \tag{14.2.7.4.6}$$

if and only if $|x| < 1$, we conclude that the interval of convergence for the binomial series is $(-1, 1)$. The behavior at the endpoints depends on r. It can be shown that for $r \geq 0$ the series converges at both endpoints; for \(–1

Definition: binomial series

For any real number r, the Maclaurin series for $f(x) = (1+x)^r$ is the binomial series. It converges to f for $|x| < 1$, and we write

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + r\frac{(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \tag{14.2.7.4.7}$$

for $|x| < 1$.

We can use this definition to find the binomial series for $f(x) = \sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 14.2.7.4.1: Finding Binomial Series

a. Find the binomial series for $f(x) = \sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r = \frac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \cdots \\ &= 1 + \frac{1}{2}x - \frac{1}{2!}\frac{1}{2^2}x^2 + \frac{1}{3!}\frac{1\cdot 3}{2^3}x^3 - \cdots + \frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n + \cdots \\ &= 1 + \textstyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n!}\dfrac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3.$$

Therefore,

$$\sqrt{1.5} = \sqrt{1+0.5} \approx 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 \approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5) = \frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x) = -\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $|\, f^{(4)}(x)\,|$ on the interval $(0, 0.5)$ occurs at $x = 0$, we have

$$|R_3(0.5)| \le \frac{15}{4!2^4}(0.5)^4 \approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure.

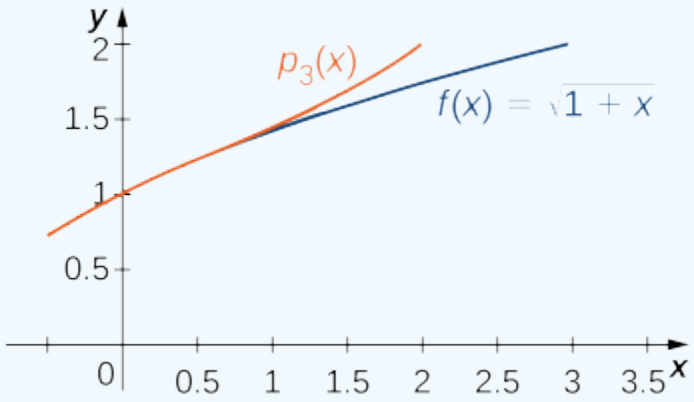


Figure 14.2.7.4.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x) = \sqrt{1+x}$ for x near zero.

Exercise 14.2.7.4.1

Find the binomial series for $f(x) = \dfrac{1}{(1+x)^2}$.

Hint available at the end of this section.

Answer available at the end of this section.

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x) = (1+x)^r$. In Table, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x) = ln(1+x)$ at the endpoint $x = 1$ and the Maclaurin series for $f(x) = tan^{-1}x$ at the endpoints $x = 1$ and $x = -1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence	Note
$f(x) = \dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	\(–1	
$f(x) = e^x$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	\(–∞	
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	\(–∞	small angle approximation[1]
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	\(–∞	small angle approximation[1]
$f(x) = \ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$	\(–1	
$f(x) = tan^{-1}x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	\(–1	
$f(x) = (1+x)^r$	$\sum_{n=0}^{\infty} \binom{r}{n} x^n$	\(–1	

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table, to create Maclaurin series for other functions.

Example 14.2.7.4.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table.

a. $f(x) = \cos\sqrt{x}$
b. $f(x) = \sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges to $cos\sqrt{x}$ for all x in the domain of $cos\sqrt{x}$; that is, for all $x \geq 0$.

b. To find the Maclaurin series for $sinhx$, we use the fact that

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the n[th] term in the Maclaurin series for $sinhx$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $sinhx$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Exercise 14.2.7.4.2

Find the Maclaurin series for $\sin(x^2)$.

Hint available at the end of this section.

Answer available at the end of this section.

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

Example 14.2.7.4.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n.$$

Exercise 14.2.7.4.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint available at the end of this section.

Answer available at the end of this section.

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y \tag{14.2.7.4.8}$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 14.2.7.4.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem

$$y' = y, y(0) = 3 \tag{14.2.7.4.9}$$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$c_0 = c_1,$$

$$c_1 = 2c_2,$$

$$c_2 = 3c_3,$$

$$c_3 = 4c_4,$$

$$\vdots$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$c_1 = c_0 = 3 = \frac{3}{1!},$$

$$c_2 = \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},$$

$$c_3 = \frac{c_2}{3} = \frac{3}{3 \cdot 2} = \frac{3}{3!},$$

$$c_4 = \frac{c_3}{4} = \frac{3}{4 \cdot 3 \cdot 2} = \frac{3}{4!}.$$

Therefore,

$$y = 3[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3\frac{1}{4!}x^4 + \cdots] = 3\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

You might recognize

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

Exercise 14.2.7.4.4

Use power series to solve $y' = 2y, y(0) = 5$.

Hint available at the end of this section.

Answer available at the end of this section.

We now consider an example involving a differential equation that cannot be solve with "simple" methods. This differential equation

$$y'' - xy = 0 \tag{14.2.7.4.10}$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

Example 14.2.7.4.5: Power Series Solution of Airy's Equation

Use power series to solve

$$y'' - xy = 0 \tag{14.2.7.4.11}$$

with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Differentiating this function term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y'' = 2 \cdot 1c_2 + 3 \cdot 2c_3 x + 4 \cdot 3c_4 x^2 + \cdots$$

If y satisfies the equation $y'' = xy$, then

$$2 \cdot 1c_2 + 3 \cdot 2c_3x + 4 \cdot 3c_4x^2 + \cdots = x(c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots)$$

Using [link] on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$2 \cdot 1c_2 = 0$$

$$3 \cdot 2c_3 = c_0$$

$$4 \cdot 3c_4 = c_1$$

$$5 \cdot 4c_5 = c_2$$

$$\vdots$$

More generally, for $n \geq 3$, we have $n \cdot (n-1)c_n = c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2 = 0$. Then

$$c_3 = \frac{c_0}{3 \cdot 2}$$

$$c_4 = \frac{c_1}{4 \cdot 3}$$

For c_5, c_6, c_7, we see that

$$c_5 = \frac{c_2}{5 \cdot 4} = 0$$

$$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}$$

$$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$$

Therefore, the series solution of the differential equation is given by

$$y = c_0 + c_1x + 0 \cdot x^2 + \frac{c_0}{3 \cdot 2}x^3 + \frac{c_1}{4 \cdot 3}x^4 + 0 \cdot x^5 + \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}x^7 + \cdots$$

The initial condition $y(0) = a$ implies $c_0 = a$. Differentiating this series term by term and using the fact that $y'(0) = b$, we conclude that $c_1 = b$.

Therefore, the solution of this initial-value problem is

$$y = a(1 + \frac{x^3}{3 \cdot 2} + \frac{x}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots) + b(x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots)$$

Exercise 14.2.7.4.5

Use power series to solve $y'' + x^2y = 0$ with the initial condition $y(0) = a$ and $y'(0) = b$

Hint available at the end of this section.

Answer available at the end of this section.

Evaluating Non-elementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2}dx$ Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with non-complicated function. For example, the function $f(x) = \sqrt{x^2 - 3x} + e^{x^3} - sin(5x + 4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral.**

Nonelementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2}dx$

Example 14.2.7.4.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2}dx$ as an infinite series.
b. Evaluate $\int_0^1 e^{-x^2}dx$ to within an error of 0.01

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Therefore,

$$\int e^{-x^2}dx = \int (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots)dx = C + x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \cdots$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216} \approx 0.0046296 < 0.01$

Exercise 14.2.7.4.6

Express $\int cos\sqrt{x}dx$ as an infinite series. Evaluate $\int_0^1 cos\sqrt{x}dx$ to within an error of 0.01

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned above, the integral $\int e^{-x^2}dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}dx \tag{14.2.7.4.12}$$

(See Figure.)

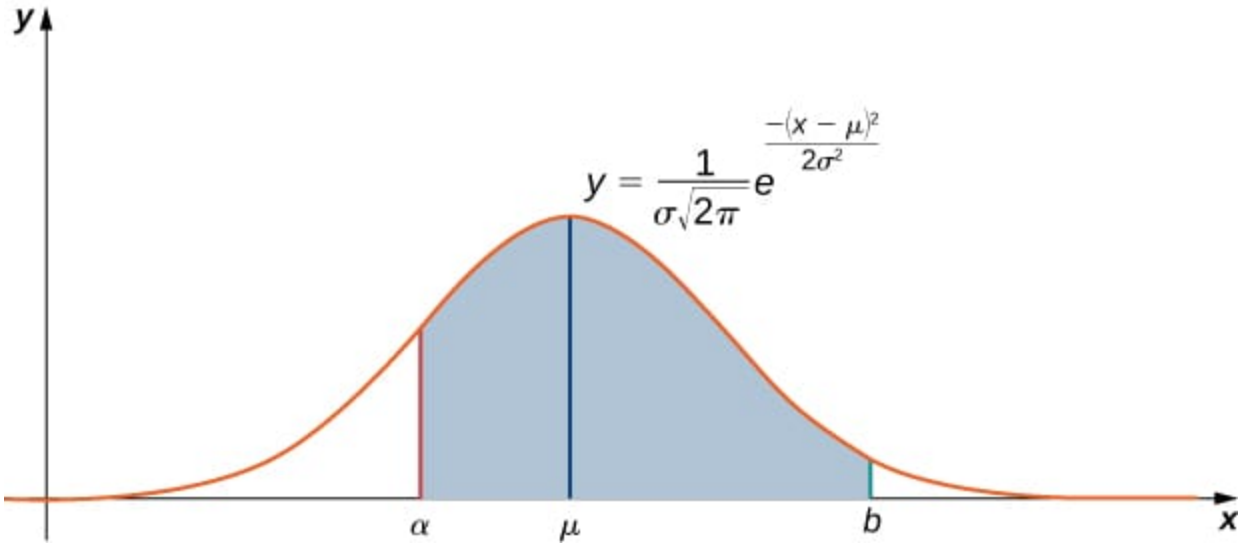


Figure 14.2.7.4.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity *z* is known as the *z* score of a data value. With this simplification, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2} dz \tag{14.2.7.4.13}$$

In Example, we show how we can use this integral in calculating probabilities.

Example 14.2.7.4.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50,$ and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-\frac{x^2}{2})^n}{n!} = 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} \int (1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots) dz = \frac{1}{\sqrt{2\pi}} (C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)2^n \cdot n!} + \cdots)$$

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2} dz = \frac{1}{\sqrt{2\pi}} (2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}}\frac{2^{13}}{13\cdot 2^6\cdot 6!}\approx 0.00546 \tag{14.2.7.4.14}$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95 Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5. The estimate, combined with the bound on the accuracy, falls within this range.

Exercise 14.2.7.4.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint available at the end of this section.

Answer available at the end of this section.

Another application in which a nonelementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}} \tag{14.2.7.4.15}$$

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 14.2.7.4.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}}$$

where g is the acceleration due to gravity and $k=sin(\frac{\theta_{max}}{2})$ (see Figure). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $sin\theta$ is approximated by θ.)

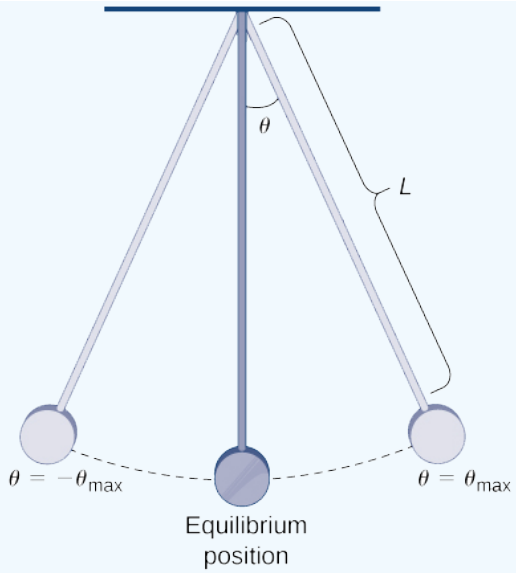


Figure 14.2.7.4.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2 sin^2\theta$ Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}$$

If θ_{max} is small, then $k = sin(\frac{\theta_{max}}{2})$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2} (1 + \frac{1}{2}k^2 sin^2\theta + \frac{1 \cdot 3}{2!2^2} k^4 sin^4\theta + \cdots) d\theta$$

Since $|sinx| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} (\frac{1}{2}k^2 + \frac{1.3}{2!2^2} k^4 + \cdots) d\theta < \frac{\pi}{2} (\frac{1}{2}k^2 + \frac{1 \cdot 3}{2!2^2} k^4 + \cdots)$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2} (1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1 - k^2},$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}(1+\frac{1}{2}k^2 sin^2\theta)d\theta = 2\pi\sqrt{\frac{L}{g}}(1+\frac{k^2}{4})$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by $(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$ for $|x| < 1$

nonelementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

Contributors and Attributions

- Template:ContribOpenStaxCalc

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[1]This series is the basis for the **small angle approximation** used in physics, sometimes without warning you. In this approximation when we have a small angle in our problem we can say that $\cos(\theta) = 1 - \frac{\theta^2}{2}$ and/or $\sin(\theta) = \theta$. The students is advised to watch out for this appearing in problem solutions and proofs where the author neglects to inform you about the use of the small angle approximation.

Answers

Exercise 14.2.7.4.1

Use the definition of binomial series for $r = -2$.

Exercise 14.2.7.4.1

$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$

Exercise 14.2.7.4.2

Use the Maclaurin series for $\sin x$.

Exercise 14.2.7.4.2

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$

Exercise 14.2.7.4.3

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Exercise 14.2.7.4.3

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$

Exercise 14.2.7.4.4

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1, c_1 = 2 \cdot 2c_2, c_2 = 2 \cdot 3c_3, \ldots$. In general, for all $n \geq 0, c_n = 2(n+1)C_{n+1}$.

Exercise 14.2.7.4.4

$y = 5e^{2x}$

Exercise 14.2.7.4.5

The coefficients satisfy $c_0 = a, c_1 = b, c_2 = 0, c_3 = 0$, and for $n \geq 4, n(n-1)c_n = -c_{n-4}$.

Exercise 14.2.7.4.5

$y = a(1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \cdots) + b(x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \cdots)$

Exercise 14.2.7.4.6

Use the series found in Example.

Exercise 14.2.7.4.6

$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01

Exercise 14.2.7.4.7

Evaluate $\int_0^1 e^{-z^2/2} dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$

Exercise 14.2.7.4.7

The estimate is approximately 0.3414

This estimate is accurate to within 0.0000094

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14.2.7.5: Fourier Series and Transform

Fourier Series

The Fourier transform is the underlying principle for frequency-domain description of signals. We begin with the Fourier series. Consider a signal $f(t)$ continuous on the time interval $[0, T]$, which then repeats with period T off to negative and positive infinity. It can be shown that

$$f(t) = A_o + \sum_{k=1}^{\infty} [A_n \cos(n\omega_o t) + B_n \sin(n\omega_o t)] \qquad \text{in which:} \tag{14.2.7.5.1}$$

$$\omega_o = 2\pi/T,$$

$$A_0 = \frac{1}{T} \int_0^T f(t)\, dt,$$

$$A_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega_o t)\, dt,$$

$$B_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega_o t)\, dt.$$

This says that the time-domain signal $f(t)$ has an exact (if you carry all the infinity of terms) representation of a constant plus scaled cosines and sines. As we will see later, the impact of this second, frequency-domain representation is profound, as it allows an entirely new set of tools for manipulation and analysis of signals and systems. A compact form of these expressions for the Fourier series can be written using complex exponentials:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_o t}, \tag{14.2.7.5.2}$$

$$\text{where } C_n = \frac{1}{T} \int_0^T f(t) e^{-in\omega_o t}\ dt.$$

Of course, C_n can be a complex number. In making these inner product calculations, orthogonality of the harmonic functions is useful:

$$\int_0^{2\pi} sin(nt)\ sin(mt)\ dt = 0, \text{ for } n \geq 1,\ m \geq 1,\ n \neq m$$

$$\int_0^{2\pi} cos(nt)\ cos(mt)\ dt = 0, \text{ for } n \geq 1,\ m \geq 1,\ n \neq m$$

$$\int_0^{2\pi} sin(nt)\ cos(mt)\ dt = 0, \text{ for } n \geq 1,\ m \geq 1.$$

Now let's go to a different class of signal, one that is not periodic, but has a finite integral of absolute value. Obviously, such a signal has to approach zero at distances far from the origin. We can write a more elegant transformation:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt, \tag{14.2.7.5.3}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega\tau}\, d\tau. \tag{14.2.7.5.4}$$

Fourier Transform

This is the real Fourier transform: a time-domain signal is transformed into a (complex) frequency-domain version, and it can be transformed back. On working it through, we see that derivatives and integrals look this way through the transform:

$$f(t) \longleftrightarrow F(\omega) \tag{14.2.7.5.5}$$

$$\frac{d^n f(t)}{dt^n} \longleftrightarrow (i\omega)^n F(\omega) \tag{14.2.7.5.6}$$

$$\int_{-\infty}^{t} f(\tau)\, d\tau \longleftrightarrow \frac{1}{i\omega} F(\omega). \tag{14.2.7.5.7}$$

Another very important property of the Fourier transform is Parseval's Relation:

$$\int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega)\, d\omega = \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega, \tag{14.2.7.5.8}$$

where the $*$-superscript indicates the complex conjugate. We will give more properties for the related Laplace transform in a later section. But as is, the Fourier transform is immediately useful for solving linear differential equations with constant coefficients (LTI systems):

$$mx'' + bx' + cx = u(t) \longleftrightarrow [-m\omega^2 + i\omega b + k]X(\omega) = U(\omega), \tag{14.2.7.5.9}$$

$$\text{so that } X(\omega) = \frac{1}{-m\omega^2 + i\omega b + k} U(\omega).$$

Hence, the action of the differential equation to relate $f(t)$ with $x(t)$ is, in the frequency domain, captured by the function

$$H(\omega) = \frac{1}{-m\omega^2 + i\omega b + k} \tag{14.2.7.5.10}$$

Putting two and two together, we then assert that $X(\omega) = H(\omega)U(\omega)$; in the Fourier space, the system response is the product of impulse response function, and the input! To back this up, we show now that convolution in the time-domain is equivalent to multiplication in the frequency domain:

$$\begin{aligned}
X(\omega) &= \mathcal{F}\left[\int_{-\infty}^{\infty} u(\tau)h(t-\tau)\, d\tau\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)h(t-\tau)e^{-i\tau t}\, dt d\tau \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)h(\xi)e^{-i\omega(\xi+\tau)}\, d\xi d\tau \qquad \text{because } e^{-i\omega t} = e^{-i\omega(t-\tau+\tau)} \\
&= \int_{-\infty}^{\infty} e^{-i\omega\xi}h(\xi)\, d\xi \int_{-\infty}^{\infty} e^{-i\omega\tau}u(\tau)\, d\tau \\
&= H(\omega)U(\omega).
\end{aligned}$$

The central role of the impulse response should be reiterated here. It is a complete definition of the system, and for systems of differential equations, it is a specific function of the parameters and of the frequency ω. The Fourier Transform of the impulse response is called the system transfer function, and we often refer to the transfer function as "the system," even though it is actually a (transformed) signal.

By way of summary, we can write

$$y(t) = h(t) * u(t), \tag{14.2.7.5.11}$$

and

$$Y(\omega) = H(\omega)U(\omega). \tag{14.2.7.5.12}$$

Fast Fourier Method (FFT)

This method of Fourier transforms is very good when not using a computer, but converting this as is for computers is very cumbersome. The solution to this is the Fast Fourier Method (FFT) which is really a Discrete Fourier Transform (DFT). The main idea of the FFT is to do a couple of "tricks" to handle sums faster. Gauss had developed just such a method where you divide your signal into many smaller discrete Fourier transforms. This technique is referred to as the Cooley-Tukey algorithm based off the paper that James Cooley and John Tukey developed using Gauss' method. Note the Cooley-Tukey algorithm is designed so that other similar algorithms can be included to avoid problems with the original algorithm.

The FFT is used extensively in engineering and science for analysis and other purposes. Almost all higher level languages have access to a FFT algorithm as will as some other Discrete Fourier Transform (DFT) algorithms. This topic will be discussed in your numerical methods class (signals and systems).

Note that the generalized form of DFT leads to the chirp Z-transform (CZT). This transform is in the z-space as opposed to the frequency space of the Fourier transform and the s-space of the Laplace transform. The algorithms associated with this transform is Bluestein's algorithm which uses FFTs.

Other FFT algorithms include the Rader-Brenner's algorithm, Bruun's algorithm, Winogard's FFT algorithm, etc. While the study of these algorithms are very interesting, from a practical point-of-view it is only the final product that is of interest to engineers and scientists as the FFT is a method that allows us to use Fourier transforms on a computer to maximum effect. FFTs are demonstrated in the data analysis section of this book/course.

Contribution

- Franz S. Hover & Michael S. Triantafyllou
- FFT addition by Scott D. Johnson, Joshua Halpern, and Scott Sinex

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14.2.7.6: The Laplace Transform

The causal version of the Fourier transform is the Laplace transform; the integral over time includes only positive values and hence only deals with causal impulse response functions[1]. In our discussion, the Laplace transform is chiefly used in control system analysis and design.

Definition of Laplace Transform

The Laplace transform projects time-domain signals into a complex frequency-domain equivalent. The signal $y(t)$ has transform $Y(s)$ defined as follows:

$$Y(s) = L(y(t)) = \int_0^\infty y(\tau) e^{-s\tau}\, d\tau, \tag{14.2.7.6.1}$$

where s is a complex variable, properly constrained within a region so that the integral converges. $Y(s)$ is a complex function as a result. Note that the Laplace transform is linear, and so it is distributive: $L(x(t) + y(t)) = L(x(t)) + L(y(t))$. The following tables give a list of some useful transform pairs and other properties, for reference.

Table of Laplace Transforms

Table 14.2.7.6.1: List of some common Laplace transforms. Contributors: Wikipedia.

Function	**Time domain** $f(t) = \mathcal{L}^{-1}\{F(s)\}$	**Laplace s-domain** $F(s)\mathcal{L}\{f(t)\}$	**Region of convergence**
unit impulse	$\delta(t)$	1	all s
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$	$\mathrm{Re}(s) > 0$
unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$	$\mathrm{Re}(s) > 0$
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
nth power (for integer n	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$ $(n > -1)$
qth power (for complex q)	$t^q \cdot u(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$	$\mathrm{Re}(s) > 0$ $\mathrm{Re}(q) > -1$
nth root	$\sqrt[n]{t} \cdot u(t)$	$\frac{1}{s^{\frac{1}{n}+1}}\Gamma\left(\frac{1}{n}+1\right)$	$\mathrm{Re}(s) > 0$
nth power with frequency shift	$t^n e^{-\alpha t} \cdot u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$
delayed nth power with frequency shift	$(t-\tau)^n e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{n! \cdot e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}(s) > -\alpha$
two-sided exponential decay (only for bilateral transform)	$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2 - s^2}$	$-\alpha < \mathrm{Re}(s) < \alpha$

exponential approach	$(1-e^{-\alpha t})\cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}(s)>0$
sine	$\sin(\omega t)\cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s)>0$
cosine	$\cos(\omega t)\cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s)>0$
hyperbolic sine	$\sinh(\alpha t)\cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\mathrm{Re}(s)>\lvert\alpha\rvert$
hyperbolic cosine	$\cosh(\alpha t)\cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$\mathrm{Re}(s)>\lvert\alpha\rvert$
exponentially decaying sine wave	$e^{-\alpha t}\sin(\omega t)\cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}(s)>-\alpha$
exponentially decaying cosine wave	$e^{-\alpha t}\cos(\omega t)\cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$\mathrm{Re}(s)>-\alpha$
natural logarithm	$\ln(t)\cdot u(t)$	$\frac{-\ln(s)-\gamma}{s}$	$\mathrm{Re}(s)>0$
Bessel function of the first kind, of order n	$J_n(\omega t)\cdot u(t)$	$\frac{\left(\sqrt{s^2+\omega^2}-s\right)^n}{\omega^n\sqrt{s^2+\omega^2}}$	$\mathrm{Re}(s)>0$ $(n>-1)$
Error function	$\mathrm{erf}(t)\cdot u(t)$	$\frac{e^{s^2/4}}{s}\left(1-\mathrm{erf}\left(\frac{s}{2}\right)\right)$	$\mathrm{Re}(s)>0$

Properties fo Laplace Transform

Table 14.2.7.6.2: some useful properties of the unilateral Laplace transform, given the functions $f(t)$ and $g(t)$ and their respective Laplace transforms $F(s)$ and $G(s)$. Contributors: Wikipedia.

	Time domain	s domain	Comments
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$	Can be proved using basic rules of integration
Frequency-domain derivative	$tf(t)$	$-F'(s)$	F' is first derivative of F with respect to s
Frequency-domain general derivative	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Most general form, nth derivative of $F(s)$.
Derivative	$f'(t)$	$sF(s)-f(0^-)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts.
Second derivative	$f''(t)$	$s^2F(s)-sf(0^-)$ $-f'(0^-)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$
General derivative	$f^{(n)}(t)$	$s^nF(s)$ $-\sum_{k=1}^{n}s^{n-k}f^{(k-1)}(0^-)$	f is assumed to be n-times differentiable with nth derivative of exponential type. Follows by mathematical induction.
Frequency-domain integration	$\frac{1}{t}f(t)$	$\int_s^\infty F(\sigma)\ d\sigma$	This is deduced using the nature of frequency differentiation and conditional convergence.

Time-domain integration	$\int_0^t f(\tau)\ d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$	$u(t)$ is the Heaviside step function and $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$a > 0,\ u(t)$ is the Heaviside step function.
Time scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	$a > 0$
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$	$F(s) \cdot G(s)$	
Conjugation	$f^*(t)$	$F^*(s^*)$	

Properties 4 and 8 of Table 2.10.2 are of special importance. You will see that in this book/course in the systems section the two example transform blocks are $\frac{1}{s}$ and s which is because of their importance. For control system design, the differentiation of a signal is equivalent to multiplication of its Laplace transform by s; integration of a signal is equivalent to division by s. The other terms that arise will cancel if $y(0) = 0$, or if $y(0)$ is finite.

Convergence

We note first that the value of s affects the convergence of the integral. For instance, if $y(t) = e^t$, then the integral converges only for $Re(s) > 1$, since the integrand is e^{1-s} in this case. Although the integral converges within a well-defined region in the complex plane, the function $Y(s)$ is defined for all s through analytic continuation. This result from complex analysis holds that if two complex functions are equal on some arc (or line) in the complex plane, then they are equivalent everywhere. It should be noted, however, that the Laplace transform is defined only within the region of convergence.

Convolution Theorem

One of the main points of the Laplace transform is the ease of dealing with dynamic systems. As with the Fourier transform, the convolution of two signals in the time domain corresponds with the multiplication of signals in the frequency domain. Consider a system whose impulse response is $g(t)$, being driven by an input signal $x(t)$; the output is $y(t) = g(t) * x(t)$. The *Convolution Theorem* is:

Convolution Theorem:

$$y(t) = \int_0^t g(t-\tau)x(\tau)\,d\tau \leftrightarrow Y(s) = G(s)X(s). \tag{14.2.7.6.2}$$

Here's the proof given by Siebert:

Proof available at the end of this section.

$$
\begin{aligned}
Y(s) &= \int_0^\infty y(t)e{-}st\,dt \\
&= \int_0^\infty \left[\int_0^t g(t-\tau)x(\tau)\,d\tau\right] e^{-st}\,dt \\
&= \int_0^\infty \left[\int_0^\infty g(t-\tau)h(t-\tau)x(\tau)\,d\tau\right] e^{-st}\,dt \\
&= \int_0^\infty x(\tau)\left[\int_0^\infty g(t-\tau)h(t-\tau)e^{-st}\,dt\right] d\tau \\
&= \int_0^\infty x(\tau)G(s)e^{-st}\,d\tau \\
&= G(s)X(s).
\end{aligned}
$$

where $h(t)$ is the unit step function. When $g(t)$ is the impulse response of a dynamic system, then $y(t)$ represents the output of this system when it is driven by the external signal $x(t)$.

Solution of Differential Equations by Laplace Transform

The Convolution Theorem allows one to solve (linear time-invariant) differential equations in the following way:

1. Transform the system impulse response $g(t)$ into $G(s)$, and the input signal $x(t)$ into $X(s)$, using the transform pairs.
2. Perform the multiplication in the Laplace domain to find $Y(s)$.
3. Ignoring the effects of pure time delays, break $Y(s)$ into partial fractions with no powers of s greater than 2 in the denominator.
4. Generate the time-domain response from the simple transform pairs. Apply time delay as necessary.

Control Systems Application of Laplace Transform

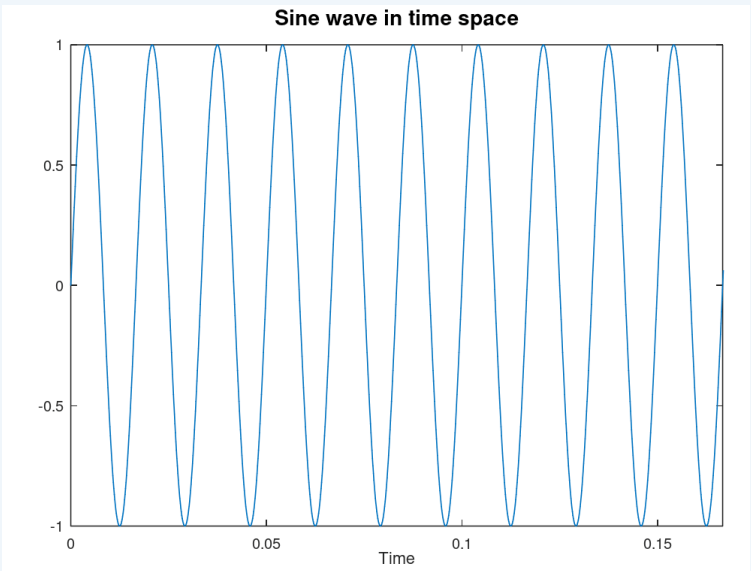
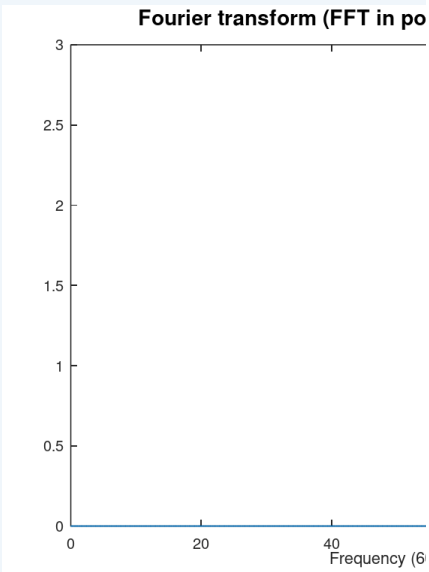
Let us look back at your case study of building a laboratory. In order to mode the various vibrations, forces, etc. that a structure might undergo requires a number of simultaneous differential equations (similar to simultaneous algebraic equations). While you learned methods for solving simultaneous equations (substitution, addition, graphing, or with matrices), these method seemingly do not work for differential equations. However the Laplace transform can come to the rescue.

time space		s-space (Laplace)
System of differential equations for, say **X**(t)	Laplace transform →	System of simultaneous equations, **X**(s)
	$\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$	← Matrix solution of simultaneous equations ↓
Solutions of system of differential equations in time space	← Inverse Laplace transform	Solutions of system of differential equations in s-space

This table does not include the convolution portion of the solving of these equations but that can be addressed, just not herein. This along with non-linear equations is numerical methods (through signals and systems).

In engineering, the transfer function is usually a vector or matrix structure that usually have differential equation that act as operators. For a structure the basis of the differential equation will be a force equation $\mathbf{M}_s\ddot{\mathbf{x}} + \mathbf{C}_s\dot{\mathbf{x}} + \mathbf{K}_s\mathbf{x} = \beta_s f + \ldots$. We will discuss this later in the systems section.

Laplace Transform compared to Fourier Transform

	Time space	Frequency space
Function representation	x(t)	x(ω)
Example graph		
Digital Form	Vector array	Fast Fourier Transf

Contributors and Attributions

- Franz S. Hover & Michael S. Triantafyllou
- Some modification of this section, particularly the last two tables, were done by Scott D. Johnson, Joshua Halpern, and Scott Sinex. The majority of the work is however Hover's and Triantafyllou's

---

[1]Causal systems only depend on present and past. Non-causal systems have a future component which would not be found in the real world, but in data processing we might be able to use future values to process present values (post-processing).

Answers

Convolution Theorem:

$$
\begin{aligned}
y(t) &\longleftrightarrow Y(s) && \\
\delta(t) &\longleftrightarrow 1 && \text{(Impulse)} \\
1(t) &\longleftrightarrow \frac{1}{s} && \text{(Unit Step)} \\
t &\longleftrightarrow \frac{1}{s^2} && \text{(Unit Ramp)} \\
e^{-\alpha t} &\longleftrightarrow \frac{1}{s+\alpha} && \\
\sin \omega t &\longleftrightarrow \frac{\omega}{s^2+\omega^2} && \\
\cos \omega t &\longleftrightarrow \frac{s}{s^2+\omega^2} && \\
e^{-\alpha t} \sin \omega t &\longleftrightarrow \frac{\omega}{(s+\alpha)^2+\omega^2} && \\
e^{-\alpha t} \cos \omega t &\longleftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega^2} && \\
\frac{1}{b-a}(e^{-at}-e^{-bt}) &\longleftrightarrow \frac{1}{(s+a)(s+b)} && \\
\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right] &\longleftrightarrow \frac{1}{s(s+a)(s+b)} && \\
\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t &\longleftrightarrow \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} && \\
1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\, t+\phi\right) &\longleftrightarrow \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} && \\
\left(\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) & && \\
y(t-\tau)1(t-\tau) &\longleftrightarrow Y(s)e^{-s\tau} && \text{(Pure Delay)} \\
\frac{dy(t)}{dt} &\longleftrightarrow sY(s)-y(0) && \text{(Time Derivative)} \\
\int_0^t y(\tau)\, d\tau &\longleftrightarrow \frac{Y(s)}{s}+\frac{\int_{0^-}^{0^+} y(t)\, dt}{s} && \text{(Time Integral)}
\end{aligned}
$$

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14.3: GEOMETRY

This is just a general section on geometry that the engineering student should already be familiar with, but we will review a bit.

14.3.1: Basic Trigonometry

14.3.2: Trigonometric Functions

14.3.3: Trigonometric Identities

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14.3.1: Basic Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for 30°($\frac{\pi}{6}$),45°($\frac{\pi}{4}$),and 60°($\frac{\pi}{3}$).
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We can defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, however, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

We can used a unit circle to define the *trigonometric functions*. In this section, we use definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $14.3.1.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

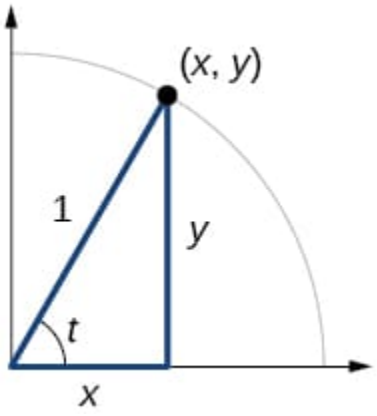


Figure $14.3.1.1$: Graph of quarter circle with radius of 1. Inscribed triangle with an angle of t. Point of (x,y) is at intersection of terminal side of angle and edge of circle.

We know

$$\cos t = \frac{x}{1} = x \tag{14.3.1.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{14.3.1.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $14.3.1.2$.

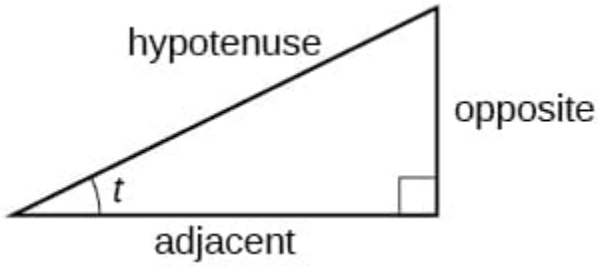


Figure $14.3.1.2$: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{14.3.1.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{14.3.1.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{14.3.1.5}$$

A common mnemonic for remembering these relationships is SohCahToa[1], formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example $14.3.1.1$: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure $14.3.1.3$, find the value of $\cos \alpha$.

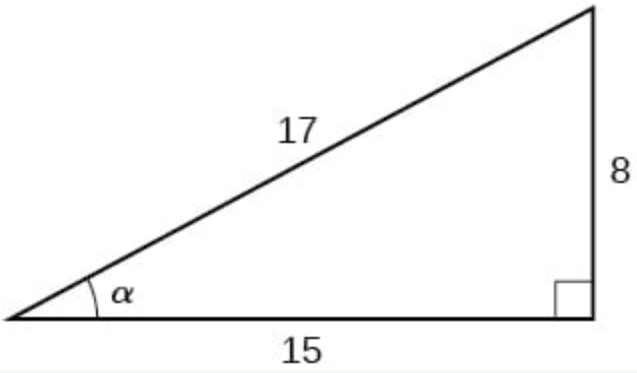


Figure 14.3.1.3: A right triangle with side lengths of 8, 15, and 17. Angle alpha also labeled which is opposite to the side labeled 8.

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 14.3.1.4:

$$\begin{aligned}\cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17}\end{aligned}$$

Exercise 14.3.1.1

Given the triangle shown in Figure 14.3.1.4, find the value of $\sin t$.

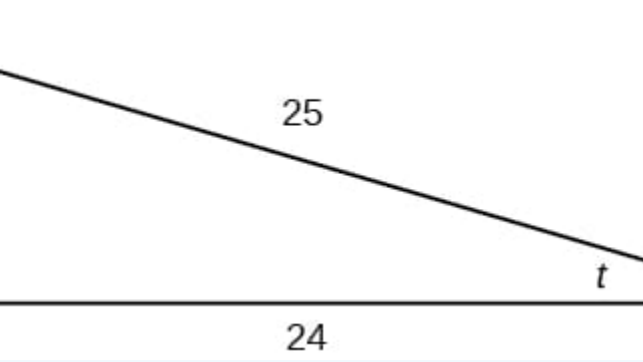


Figure 14.3.1.4: A right triangle with sides of 7, 24, and 25. Also labeled is angle t which is opposite the side labeled 7.

Answer available at the end of this section.

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 14.3.1.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

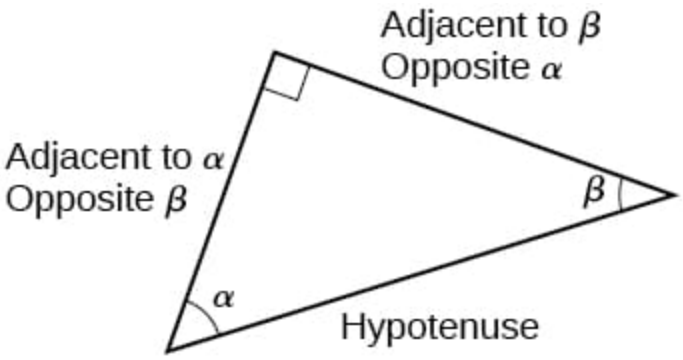


Figure 14.3.1.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 14.3.1.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 14.3.1.6, evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$.

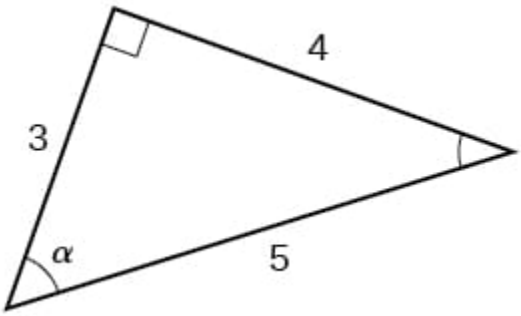


Figure 14.3.1.6: Right triangle with sides of 3, 4, and 5. Angle alpha is also labeled which is opposite the side labeled 4.

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 14.3.1.2

Using the triangle shown in Figure 14.3.1.7, evaluate $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.

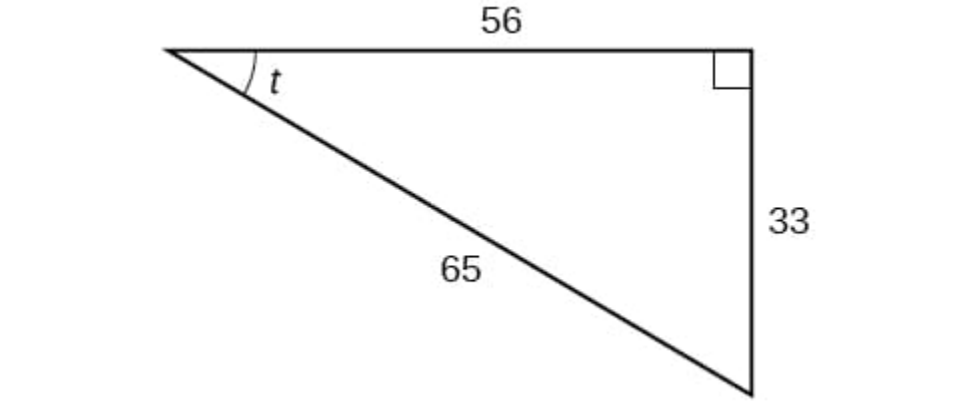


Figure 14.3.1.7: Right triangle with sides 33, 56, and 65. Angle t is also labeled which is opposite to the side labeled 33.

Answer available at the end of this section.

Finding Trigonometric Functions of Special Angles Using Side Lengths

Here we use unit circle relationships to evaluate triangles that contain **special angles**. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30\,°, 60\,°,$ and $45\,°$, however, remember that when dealing with right triangles, we are limited to angles between $0\,°$ and $90\,°$.

Suppose we have a $30\,°, 60\,°, 90\,°$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45\,°, 45\,°, 90\,°$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 14.3.1.8.

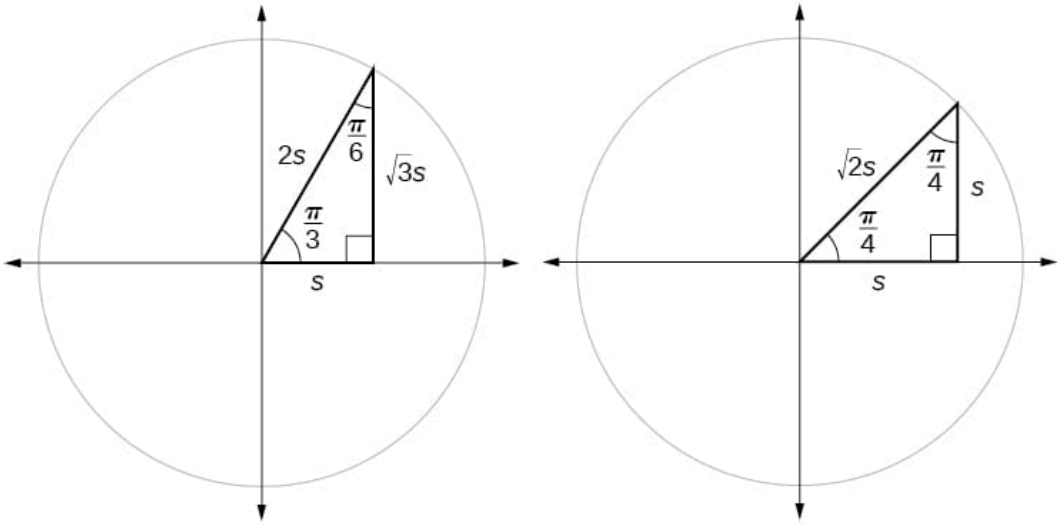


Figure 14.3.1.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 14.3.1.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 14.3.1.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 14.3.1.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer available at the end of this section.

Using Equal Co-function of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure $14.3.1.9$).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

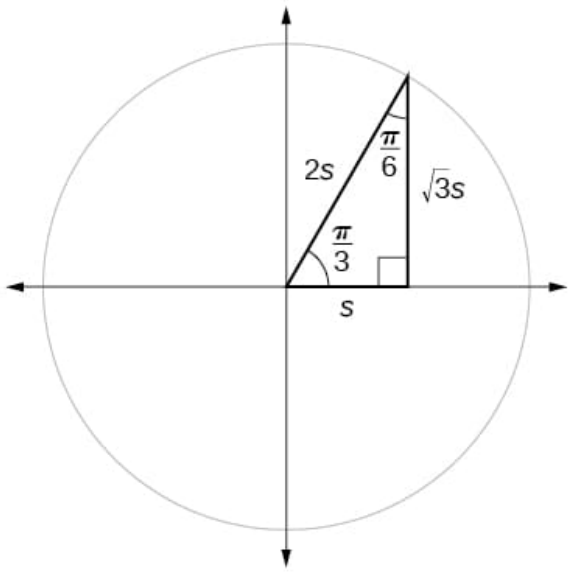
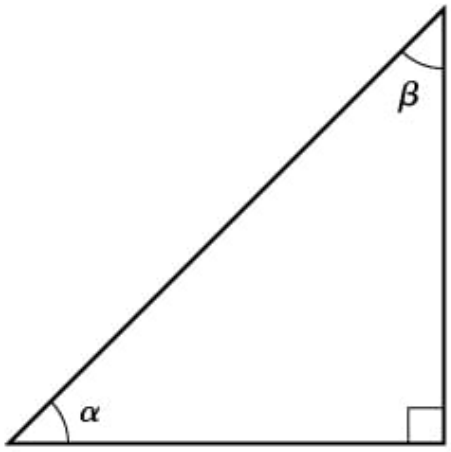


Figure $14.3.1.9$: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure $14.3.1.9$, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, π,and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure $14.3.1.10$.



$$\sin\alpha = \cos\beta$$
$$\sin\beta = \cos\alpha$$

Figure 14.3.1.10: Co-function identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle $t, \cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

CO-FUNCTION IDENTITIES

The **co-function identities** in radians are listed in Table 14.3.1.1.

Table 14.3.1.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$
$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 14.3.1.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 14.3.1.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 14.3.1.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 14.3.1.11.

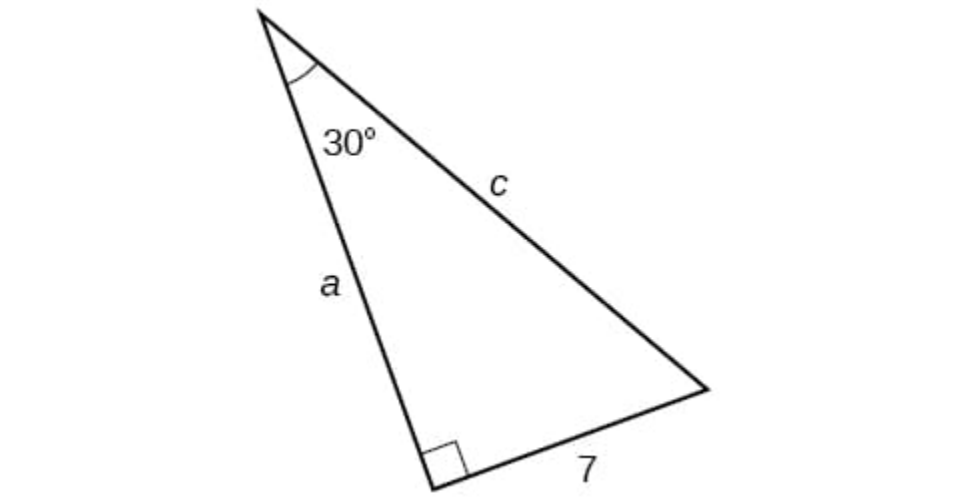


Figure 14.3.1.11: A right triangle with sides a, c, and 7. Angle of 30 degrees is also labeled which is opposite the side labeled 7.

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30°)} \tag{14.3.1.6}$$
$$= 12.1 \tag{14.3.1.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30°)} = 14$$

Exercise 14.3.1.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer available at the end of this section.

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 14.3.1.12.

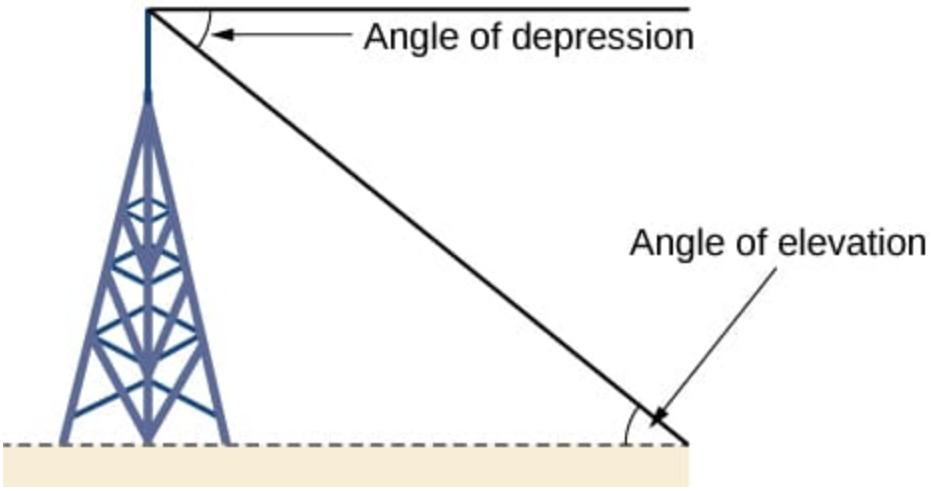


Figure 14.3.1.12: Diagram of a radio tower with line segments extending from the top and base of the tower to a point on the ground some distance away. The two lines and the tower form a right triangle. The angle near the top of the tower is the angle of depression. The angle on the ground at a distance from the tower is the angle of elevation.

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 14.3.1.6: Measuring a Distance Indirectly[2]

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 14.3.1.13. Find the height of the tree.

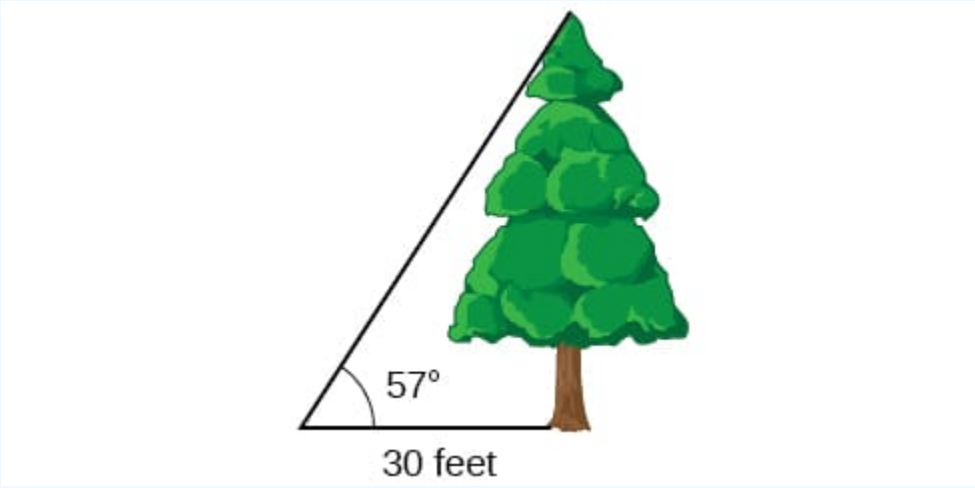


Figure 14.3.1.13: A tree with angle of 57 degrees from vantage point. Vantage point is 30 feet from tree.

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of $57\,°$, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57°) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57°) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{14.3.1.8}$$

The tree is approximately 46 feet tall.

Exercise 14.3.1.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer available at the end of this section.

Key Equations

Co-function Identities

$$\begin{aligned} \cos t &= \sin\left(\frac{\pi}{2} - t\right) \\ \sin t &= \cos\left(\frac{\pi}{2} - t\right) \\ \tan t &= \cot\left(\frac{\pi}{2} - t\right) \\ \cot t &= \tan\left(\frac{\pi}{2} - t\right) \\ \sec t &= \csc\left(\frac{\pi}{2} - t\right) \\ \csc t &= \sec\left(\frac{\pi}{2} - t\right) \end{aligned}$$

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

---

[1]Other ways are "Oscar Had A Heap Of Apples" though this way you need to know sine, cosine, and tangent are in that order, or "Studying Our Homework Can Always Help To Obtain Achievement." Wolfram Math has some more alternatives to SOHCAHTOA which might be preferred by some.

[2]A famous example of measuring indirectly does not require trigonometry but is nevertheless an interesting tale. That tale is about Thales and his technique to measure the pyramid with the shadow of a stick (or himself). There is some question if it really happened or not, though the technique was actually correct regardless.

Answers

Exercise 14.3.1.1

$\frac{7}{25}$

Exercise 14.3.1.2

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Exercise 14.3.1.3

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Exercise 14.3.1.5**:**

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3};$ missing angle is $\frac{\pi}{6}$

Exercise 14.3.1.6**:**

About 52 ft

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- 12.3: Prandtl-Meyer Function has no license indicated.

14.3.2: Trigonometric Functions

Learning Objectives

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ, let s be the length of the corresponding arc on the unit circle (Figure $14.3.2.1$). We say the angle corresponding to the arc of length 1 has radian measure 1.

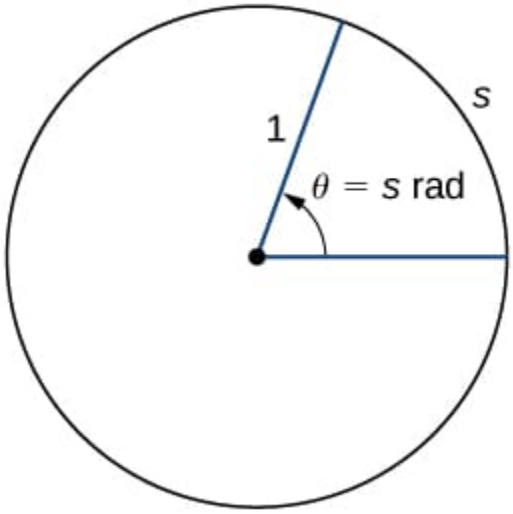


Figure $14.3.2.1$: The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of $360\,^\circ$ corresponds to the circumference of a circle, or an arc of length 2π, we conclude that an angle with a degree measure of $360\,^\circ$ has a radian measure of 2π. Similarly, we see that $180\,^\circ$ is equivalent to π radians. Table $14.3.2.1$ shows the relationship between common degree and radian values.

Table $14.3.2.1$: Common Angles Expressed in Degrees and Radians

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π

Degrees	Radians	Degrees	Radians
90	$\pi/2$		

Converting between Radians and Degrees

a. Express 225 ° using radians.
b. Express $5\pi/3$ rad using degrees.

Solution

Use the fact that $180°$ is equivalent to π radians as a **conversion factor** (Table $14.3.2.1$):

$$1 = \frac{\pi \text{ rad}}{180°} = \frac{180°}{\pi \text{ rad}}.$$

a. $225\,° = 225\,° \cdot \left(\frac{\pi}{180}\right)° = \left(\frac{5\pi}{4}\right) rad$
b. $\frac{5\pi}{3}$ rad = $\frac{5\pi}{3} \cdot \frac{180°}{\pi}$=300°

Exercise $14.3.2.1$

a. Express 210 ° using radians.
b. Express $11\pi/6$ rad using degrees.

Hint available at the end of this section.

Answer available at the end of this section.

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship between the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x-axis and with a terminal side that is the line segment OP. An angle in this position is said to be in standard position (Figure $14.3.2.2$). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y.

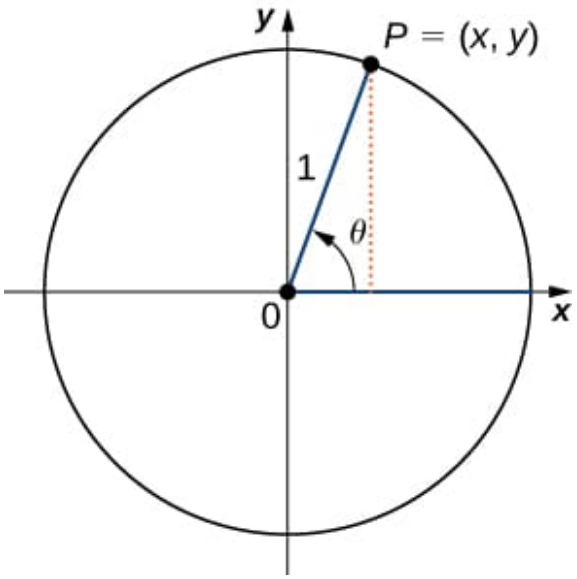


Figure 14.3.2.2: The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y.

Definition: Trigonometric functions

Let $P = (x, y)$ be a point on the unit circle centered at the origin O. Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP. The trigonometric functions are then defined as

$\sin\theta = y$	$\csc\theta = \dfrac{1}{y}$
$\cos\theta = x$	$\sec\theta = \dfrac{1}{x}$
$\tan\theta = \dfrac{y}{x}$	$\cot\theta = \dfrac{x}{y}$

If $x = 0, \sec\theta$ and $\tan\theta$ are undefined. If $y = 0$, then $\cot\theta$ and $\csc\theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ, the coordinates x and y satisfy

$$\cos\theta = \frac{x}{r} \tag{14.3.2.1}$$

$$x = r\cos\theta \tag{14.3.2.2}$$

and

$$\sin\theta = \frac{y}{r} \tag{14.3.2.3}$$

$$y = r\sin\theta. \tag{14.3.2.4}$$

The values of the other trigonometric functions can be expressed in terms of x, y, and r (Figure 14.3.2.3).

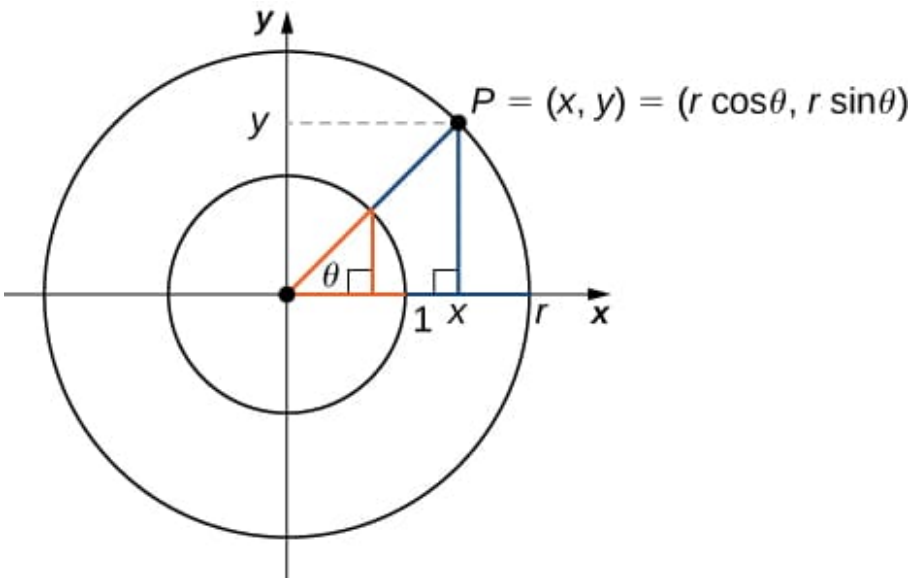


Figure 14.3.2.3: For a point $P = (x, y)$ on a circle of radius r, the coordinates x and y satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Table 14.3.2.2 shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin\theta$ and $\cos\theta$.

Table 14.3.2.2: Values of $\sin\theta$ and $\cos\theta$ at Major Angles θ in the First Quadrant

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Example 14.3.2.2: Evaluating Trigonometric Functions

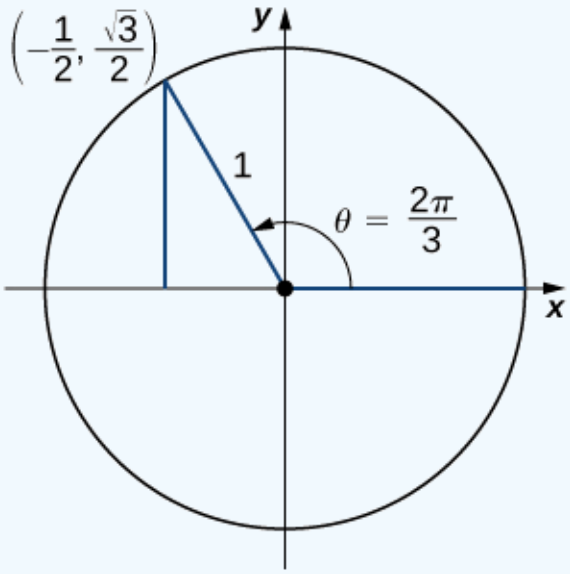
Evaluate each of the following expressions.

a. $\sin\left(\frac{2\pi}{3}\right)$
b. $\cos\left(-\frac{5\pi}{6}\right)$
c. $\tan\left(\frac{15\pi}{4}\right)$

Solution:

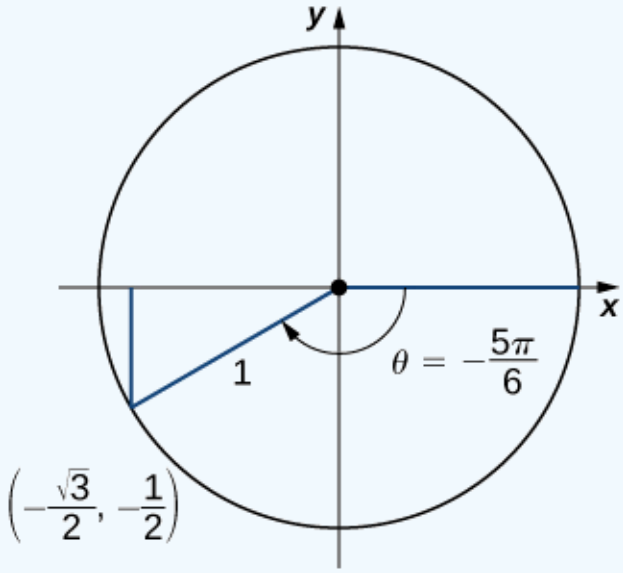
a) On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore,

$$\sin\left(\frac{2\pi}{3}\right) = y = \left(\frac{\sqrt{3}}{2}\right). \tag{14.3.2.5}$$



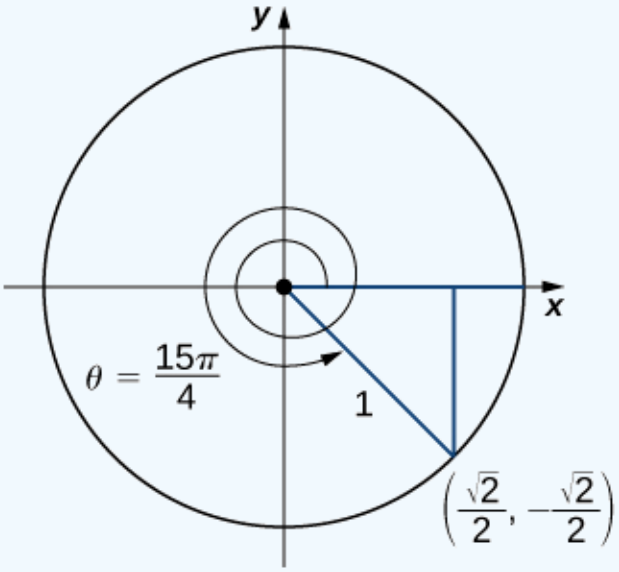
b) An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore,

$$\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}. \tag{14.3.2.6}$$



c) An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, we can conclude that

$$\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1. \tag{14.3.2.7}$$



Exercise 14.3.2.2

Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H, as shown in Figure 14.3.2.4, we see that A, H, and O satisfy the following relationships with θ:

$\sin\theta = \frac{O}{H}$	$\csc\theta = \frac{H}{O}$
$\cos\theta = \frac{A}{H}$	$\sec\theta = \frac{H}{A}$
$\tan\theta = \frac{O}{A}$	$\cot\theta = \frac{A}{O}$

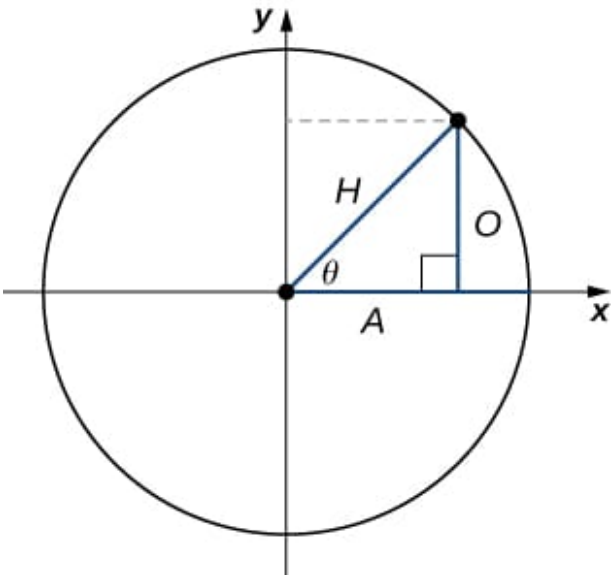


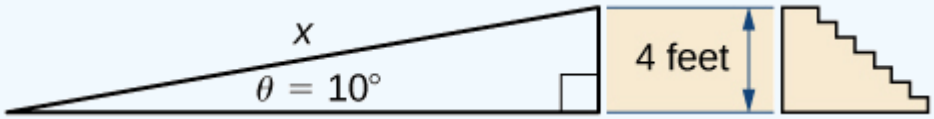
Figure 14.3.2.4: By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ.

Example 14.3.2.3: Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be $10°$, how long does the ramp need to be?

Solution

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10\ °) = 4/x$. Solving this equation for x, we see that $x = 4/\sin(10\ °) \approx 23.035$ ft.



Exercise 14.3.2.3

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be $60°$, how far from the house should she place the base of the ladder?

Hint available at the end of this section.

Answer available at the end of this section.

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Trigonometric Identities

Reciprocal identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{14.3.2.8}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{14.3.2.9}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{14.3.2.10}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{14.3.2.11}$$

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{14.3.2.12}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{14.3.2.13}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{14.3.2.14}$$

Addition and subtraction formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{14.3.2.15}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{14.3.2.16}$$

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{14.3.2.17}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 \tag{14.3.2.18}$$

$$= 1 - 2\sin^2\theta \tag{14.3.2.19}$$

$$= \cos^2\theta - \sin^2\theta \tag{14.3.2.20}$$

Example 14.3.2.4: Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos(2\theta) = \cos\theta$
b. $\sin(2\theta) = \tan\theta$

Solution

a) Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

$$1 + \cos(2\theta) = \cos\theta$$

if and only if

$$1 + 2\cos 2\theta - 1 = \cos\theta,$$

which is true if and only if

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \ldots,$$

and $\cos\theta = 1/2$ when

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \ldots or \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \ldots,$$

we conclude that the set of solutions to this equation is

$$\theta = \frac{\pi}{2} + n\pi,\ \theta = \frac{\pi}{3} + 2n\pi$$

and

$$\theta = -\frac{\pi}{3} + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots.$$

b) Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions. Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0 or \cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \ldots$. After checking for extraneous solutions, the set of solutions to the equation is

$$\theta = n\pi$$

and

$$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$$

with $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 14.3.2.4

Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.3.2.5: Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the Pythagorean identity (Equation 14.3.2.12)

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

$$\tan^2\theta + 1 = \sec^2\theta.$$

Exercise 14.3.2.5

Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Answer available at the end of this section.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle . Since the angle θ and $\theta + 2\pi$ correspond to the same point P, the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f. The sine, cosine, secant, and cosecant functions have a period of 2π. Since the tangent and cotangent functions repeat on an interval of length π, their period is π (Figure 14.3.2.5).

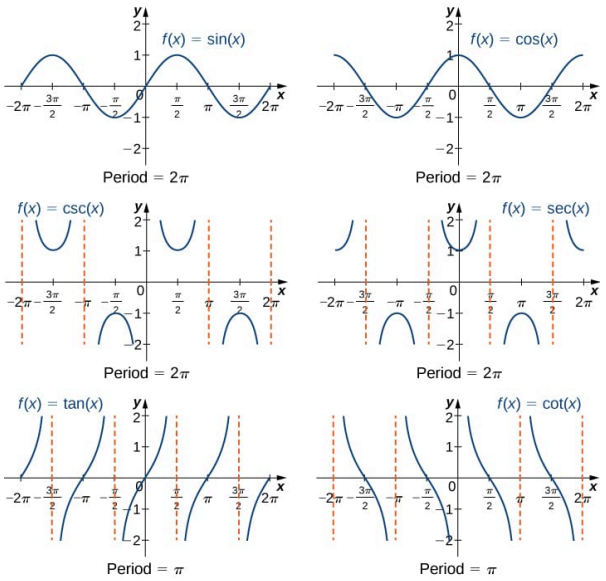


Figure 14.3.2.5: The six trigonometric functions are periodic.

Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

$$f(x) = A\sin(B(x-\alpha)) + C. \tag{14.3.2.21}$$

In Figure 14.3.2.6, the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the "amplitude of f." The constant C causes a vertical shift.

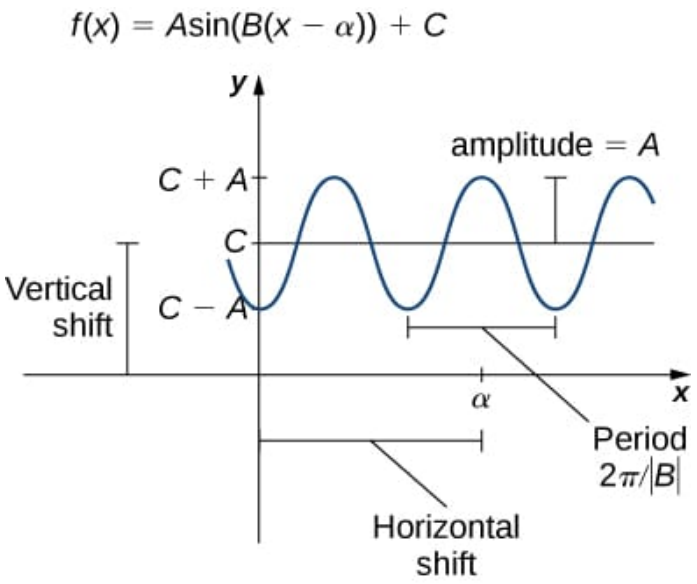


Figure 14.3.2.6: A graph of a general sine function.

Notice in Figure 14.3.2.6 that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write

$$\cos x = \sin(x + \pi/2). \tag{14.3.2.22}$$

Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the

shortest day of the year with 8.3 hours. It can be shown that the function

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(x - 80.5)\right) + 12 \tag{14.3.2.23}$$

is a model for the number of hours of daylight h as a function of day of the year t (Figure 14.3.2.7).

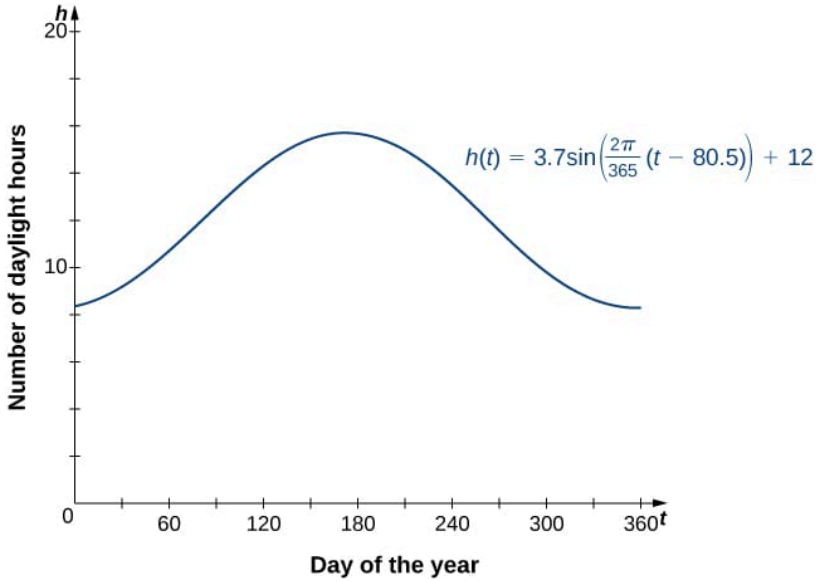


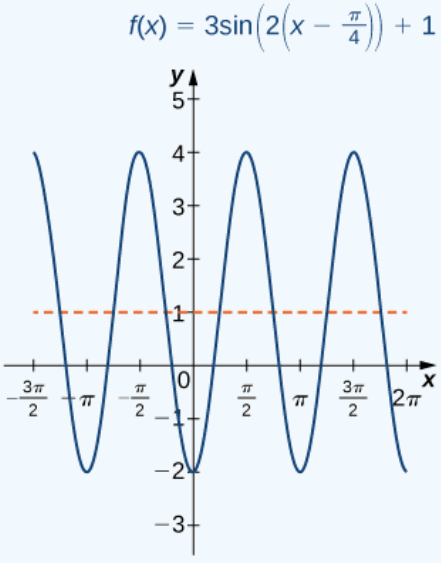
Figure 14.3.2.7: The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example 14.3.2.6: Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3\sin(2(x - \pi 4)) + 1$.

Solution

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π.



Exercise 14.3.2.6

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of $180°$ has a radian measure of π rad.
- For acute angles θ,the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ.
- For a general angle θ, let (x, y) be a point on a circle of radius r corresponding to this angle θ. The trigonometric functions can be written as ratios involving x, y, and r.
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π. The tangent and cotangent functions have period π.

Glossary

periodic function
: a function is periodic if it has a repeating pattern as the values of x move from left to right

radians
: for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions
: functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity
: an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

Contributors and Attributions

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Answers

Exercise 14.3.2.1

π radians is equal to 180°

Exercise 14.3.2.1

a. $7\pi/6$
b. 330°

Exercise 14.3.2.2

Look at angles on the unit circle.

Exercise 14.3.2.2

$$\cos(3\pi/4) = -\sqrt{2}/2$$
$$\sin(-\pi/6) = -1/2$$

Exercise 14.3.2.3

Draw a right triangle with hypotenuse 20.

Exercise 14.3.2.3

10 ft

Exercise 14.3.2.4

Use the double-angle formula for cosine (Equation 14.3.2.17).

Exercise 14.3.2.4

$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi$

for $n = 0, \pm 1, \pm 2, \ldots$.

Exercise 14.3.2.5

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Exercise 14.3.2.6

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Exercise 14.3.2.6

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

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14.3.3: Trigonometric Identities

Pythagorean Identities

$\cos^2 x + \sin^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

Double-Angle Identities

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$

Half-Angle Identities

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$

Angle Sum and Difference Identities

$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta)$

$\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$

$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$

Angle Reflections and Shifts

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos x$

$\cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin x$

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14.4: ANALYTIC GEOMETRY

Analytic geometry is geometry within a coordinate system. While there are many coordinate systems here we will focus on just the three most used in academia: Cartesian (Rectangular), Cylindrical, and Spherical. There are others which we will mention later, but for now let us start with the conic sections.

14.4.1: Circles

14.4.2: The Parabola

14.4.3: The Hyperbola

14.4.4: The Ellipse

14.4.5: Hyperbolic Functions

14.4.6: Coordinate Systems

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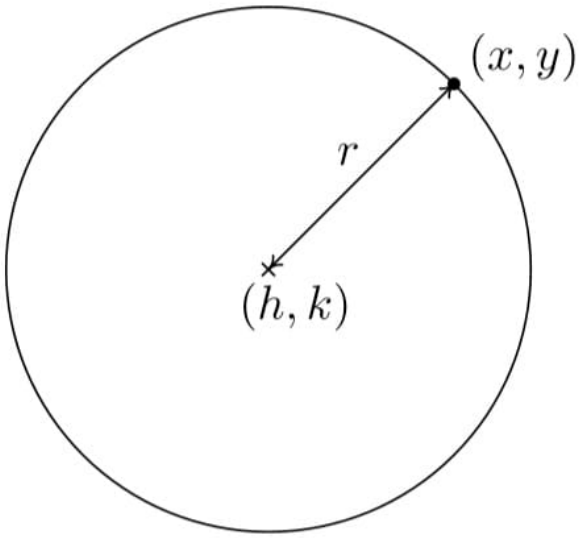
14.4.1: Circles

A circle

A circle can be determined by fixing a point at the center and a positive number, the radius.

Definition 14.4.1.1: Circles

A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula (Equation 14.4.1.1), as

$$r = \sqrt{(x-h)^2 + (y-k)^2} \tag{14.4.1.1}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

The Equation of a Circle

Note 14.4.1.1: The Standard Equation of a Circle

The equation of a circle with center (h, k) and radius $r > 0$ is

$$(x-h)^2 + (y-k)^2 = r^2. \tag{14.4.1.2}$$

or

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)}{r^2} = 1 \tag{14.4.1.3}$$

The second form of the equation is for comparison to hyperbolas and ellipses coming in later sections.

Example 14.4.1.1: The Standard Equation

Write the standard equation of the circle with center (–2,3) and radius 5.

Solution.

Here, $(h,k) = (-2,3)$ and $r = 5$, so we get

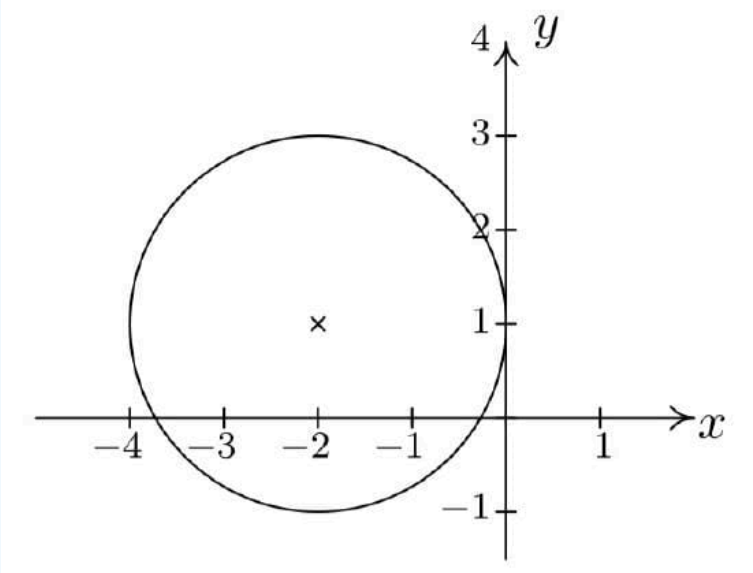
$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

Example 14.4.1.2:

Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Solution

From the standard form of a circle, Equation 14.4.1.2, we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us



If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2 + (y-1)^2 = 4$, we'd be contending with $x^2 + 4x + y^2 - 2y + 1 = 0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 14.4.1.2 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 14.4.1.3:

Complete the square to find the center and radius of $3x^2 - 6x + 3y^2 + 4y - 4 = 0$.

Solution

$$\begin{aligned}
3x^2 - 6x + 3y^2 + 4y - 4 &= 0 && \\
3x^2 - 6x + 3y^2 + 4y &= 4 && \text{add 4 to both sides} \\
3\left(x^2 - 2x\right) + 3\left(y^2 + \frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\
3\left(x^2 - 2x + \underline{1}\right) + 3\left(y^2 + \frac{4}{3}y + \underline{\underline{\frac{4}{9}}}\right) &= 4 + 3\underline{(1)} + 3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x,\ y \qquad (14.4.1.4) \\
3(x-1)^2 + 3\left(y + \frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\
(x-1)^2 + \left(y + \frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3}
\end{aligned}$$

From Equation 14.4.1.2, we identify $x - 1$ as $x - h$, so $h = 1$, and $y + \frac{2}{3}$ as $y - k$, so $k = -\frac{2}{3}$. Hence, the center is $(h, k) = \left(1, -\frac{2}{3}\right)$. Furthermore, we see that $r^2 = \frac{25}{9}$ so the radius is $r = \frac{5}{3}$.

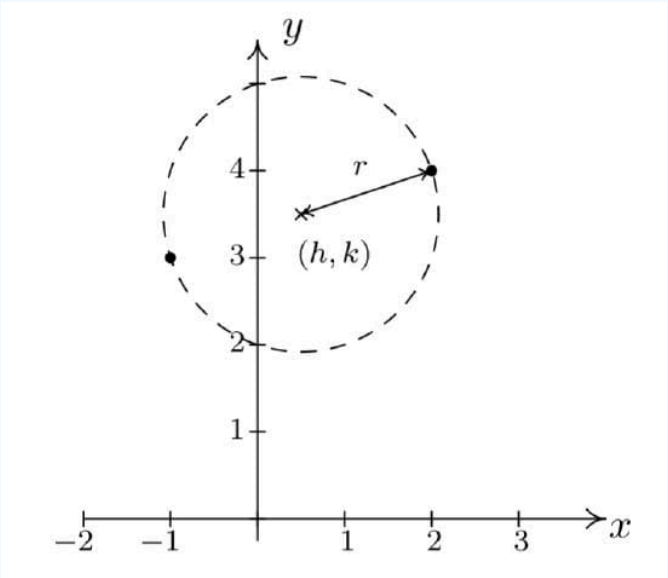
It is possible to obtain equations like $(x-3)^2 + (y+1)^2 = 0$ or $(x-3)^2 + (y+1)^2 = -1$, neither of which describes a circle. The reader is encouraged to think about what, if any, points lie on the graphs of these two equations.

Example 14.4.1.4:

Write the standard equation of the circle which has $(-1, 3)$ and $(2, 4)$ as the endpoints of a diameter.

Solution

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h, k) is the center of the circle. This is easily solved as

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned} \tag{14.4.1.5}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned} \tag{14.4.1.6}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$

The Unit Circle

We close this section with an important circle in mathematics[1]: the **Unit Circle**.

Definition 14.4.1.2: Unit Circle

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

Example 14.4.1.5:

Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution

We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

$$\begin{aligned}x^2+y^2 &= 1\\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1\\ \frac{3}{4}+x^2 &= 1\\ x^2 &= \frac{1}{4}\\ x &= \pm\sqrt{\frac{1}{4}}\\ x &= \pm\frac{1}{2}\end{aligned} \tag{14.4.1.7}$$

Our final answers are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

Contributors and Attributions

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[1]The unit circle is used in trigonometry to define and evaluate trigonometric functions like sine and cosine.

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14.4.2: The Parabola

Learning Objectives

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror (Figure 14.4.2.1), which focuses light rays from the sun to ignite the flame.



Figure 14.4.2.1: The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

Graphing Parabolas with Vertices at the Origin

Conic sections are formed when a plane cuts through a right circular cone as seen below. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a *parabola* (Figure 14.4.2.2).

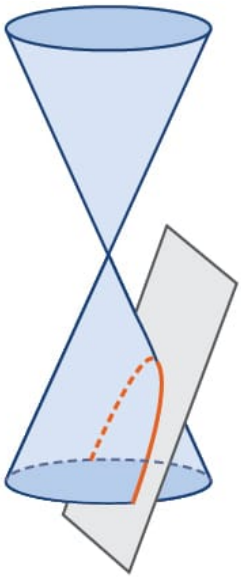


Figure 14.4.2.2: Parabola created by cutting a plane through a right circular cone.

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the *directrix*, and a fixed point (the *focus*) not on the directrix.

Key features of the parabola are given in (Figure 14.4.2.3). Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus. The line segment that passes through the focus and is parallel to the directrix is called the *latus rectum*. The endpoints of the latus rectum lie on the curve. By definition, the distanced d from the focus to any point P on the parabola is equal to the distance from P to the directrix.

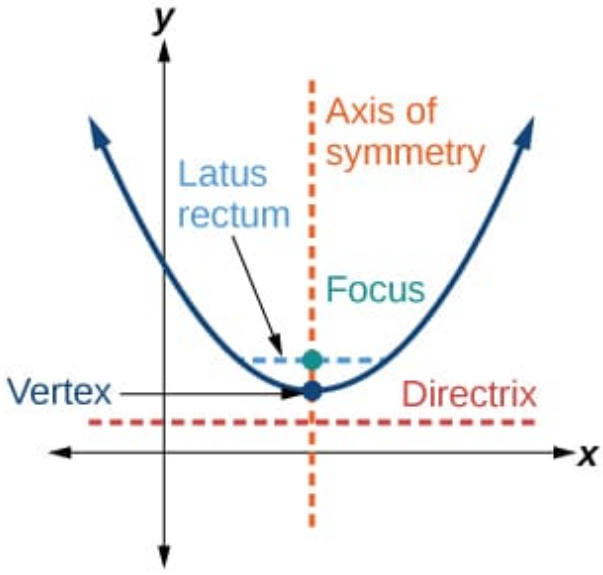


Figure 14.4.2.3: Key features of the parabola

To work with parabolas in the *coordinate plane*, we consider two cases: those with a vertex at the origin and those with a *vertex* at a point other than the origin. We begin with the former.

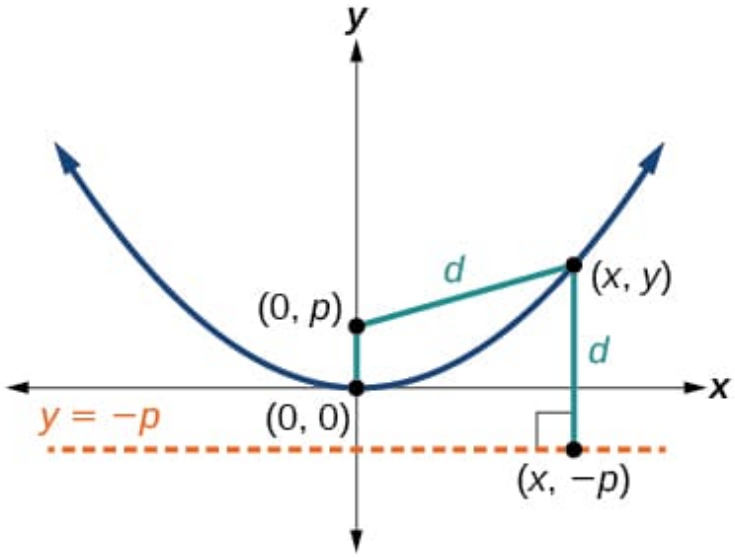


Figure 14.4.2.4: Parabola graphed at the origin in a coordinate plane.

Let (x, y) be a point on the parabola with vertex $(0, 0)$, focus $(0, p)$,and directrix $y = -p$ as shown in Figure 14.4.2.4. The distanced d from point (x, y) to point $(x, -p)$ on the directrix is the difference of the *y*-values: $d = y + p$. The distance from the focus $(0, p)$ to the point (x, y) is also equal to d and can be expressed using the distance formula.

$$\begin{aligned} d &= \sqrt{(x-0)^2 + (y-p)^2} \\ &= \sqrt{x^2 + (y-p)^2} \end{aligned}$$

Set the two expressions for d equal to each other and solve for y to derive the equation of the parabola. We do this because the distance from (x, y) to $(0, p)$ equals the distance from (x, y) to $(x, -p)$.

$$\sqrt{x^2 + (y-p)^2} = y + p \tag{14.4.2.1}$$

We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py \end{aligned}$$

The equations of parabolas with vertex $(0, 0)$ are $y^2 = 4px$ when the *x*-axis is the axis of symmetry and $x^2 = 4py$ when the *y*-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

Standard Forms of Parabolas with Vertex at 0,0

STANDARD FORMS OF PARABOLAS WITH VERTEX $(0, 0)$

Table 14.4.2.1 and Figure 14.4.2.5 summarize the standard features of parabolas with a vertex at the origin.

Table 14.4.2.1

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
x-axis	$y^2 = 4px$	$(p, 0)$	$x = -p$	$(p, \pm 2p)$

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
y-axis	$x^2 = 4py$	$(0, p)$	$y = -p$	$(\pm 2p, p)$

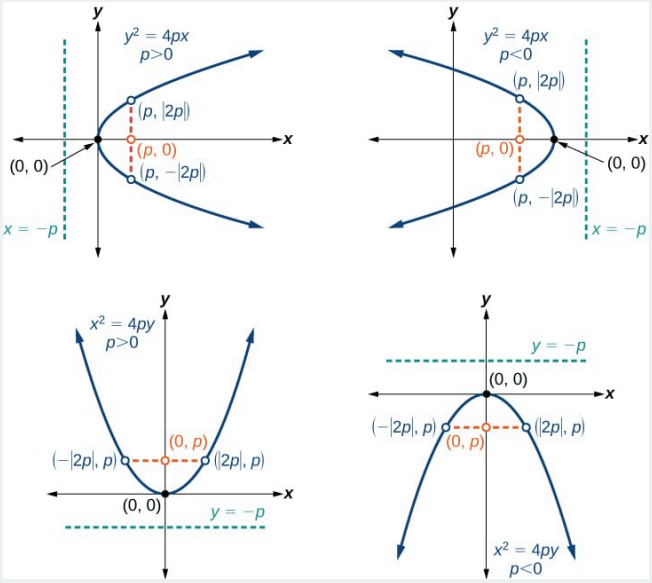


Figure $14.4.2.5$: (a) When $p > 0$ and the axis of symmetry is the x-axis, the parabola opens right. (b) When $p < 0$ and the axis of symmetry is the x-axis, the parabola opens left. (c) When $p > 0$ and the axis of symmetry is the y-axis, the parabola opens up. (d) When $p < 0$ and the axis of symmetry is the y-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and *latus rectum* (Figure $14.4.2.5$). When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola. A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the *latus rectum*, these lines intersect on the axis of symmetry, as shown in Figure $14.4.2.6$.

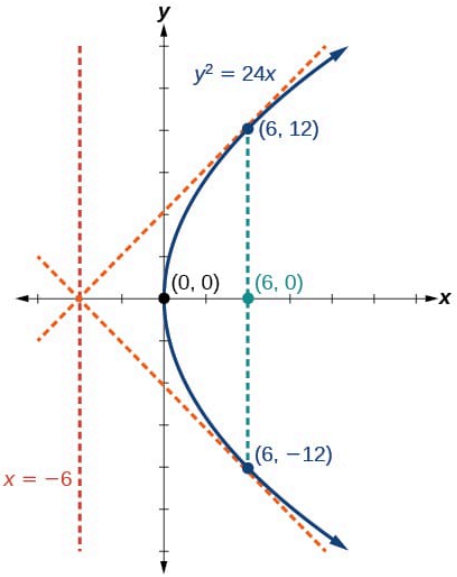


Figure $14.4.2.6$: Example of a tangent line where the line touches the parabola at one point only. The two tangent lines above intersect at the axis of symmetry, the x-axis.

How to: Given a standard form equation for a parabola centered at $(0,0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation: $y^2=4px$ or $x^2=4py$.
2. Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $y^2=4px$, then
     - the axis of symmetry is the x-axis, $y=0$
     - set $4p$ equal to the coefficient of x in the given equation to solve for p. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left.
     - use p to find the coordinates of the focus, $(p,0)$
     - use p to find the equation of the directrix, $x=-p$
     - use p to find the endpoints of the latus rectum, $(p,\pm 2p)$. Alternately, substitute $x=p$ into the original equation.
   - If the equation is in the form $x^2=4py$,then
     - the axis of symmetry is the y-axis, $x=0$
     - set $4p$ equal to the coefficient of y in the given equation to solve for p. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down.
     - use p to find the coordinates of the focus, $(0,p)$
     - use p to find equation of the directrix, $y=-p$
     - use p to find the endpoints of the latus rectum, $(\pm 2p,p)$
3. Plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

x-axis as the Axis of Symmetry

Graph $y^2=24x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Solution

The standard form that applies to the given equation is $y^2=4px$. Thus, the axis of symmetry is the *x*-axis. It follows that:

- $24=4p$, so $p=6$. Since $p>0$, the parabola opens right
- the coordinates of the focus are $(p,0)=(6,0)$
- the equation of the directrix is $x=-p=-6$
- the endpoints of the latus rectum have the same *x*-coordinate at the focus. To find the endpoints, substitute $x=6$ into the original equation: $(6,\pm 12)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure $14.4.2.7$).

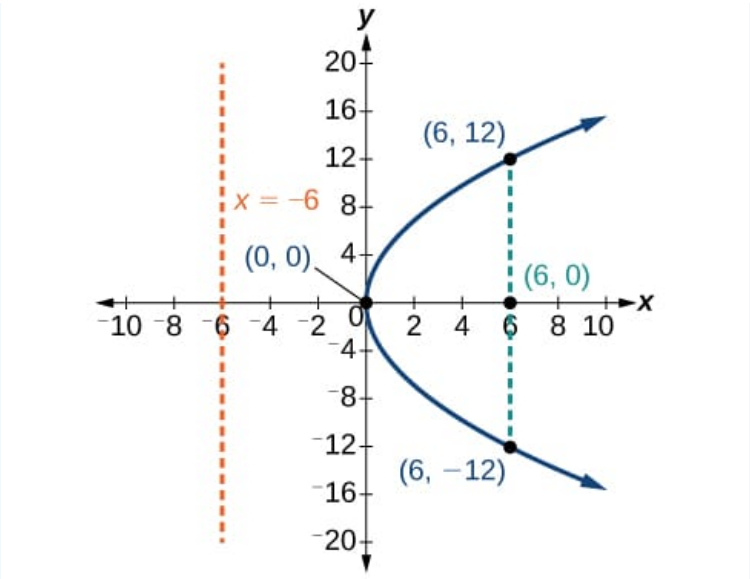


Figure 14.4.2.7: A graph of the solution to the problem presented herein.

Exercise 14.4.2.1

Graph $y^2 = -16x$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Answer available at the end of this section.

y-axis as the Axis of Symmetry

Graph $x^2 = -6y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Solution

The standard form that applies to the given equation is $x^2 = 4py$. Thus, the axis of symmetry is the y-axis. It follows that:

- $-6 = 4p$,so $p = -\frac{3}{2}$. Since $p < 0$, the parabola opens down.
- the coordinates of the focus are $(0, p) = (0, -\frac{3}{2})$
- the equation of the directrix is $y = -p = \frac{3}{2}$
- the endpoints of the latus rectum can be found by substituting $y = \frac{3}{2}$ into the original equation, $(\pm 3, -\frac{3}{2})$

Next we plot the focus, directrix, and *latus rectum*, and draw a smooth curve to form the parabola.

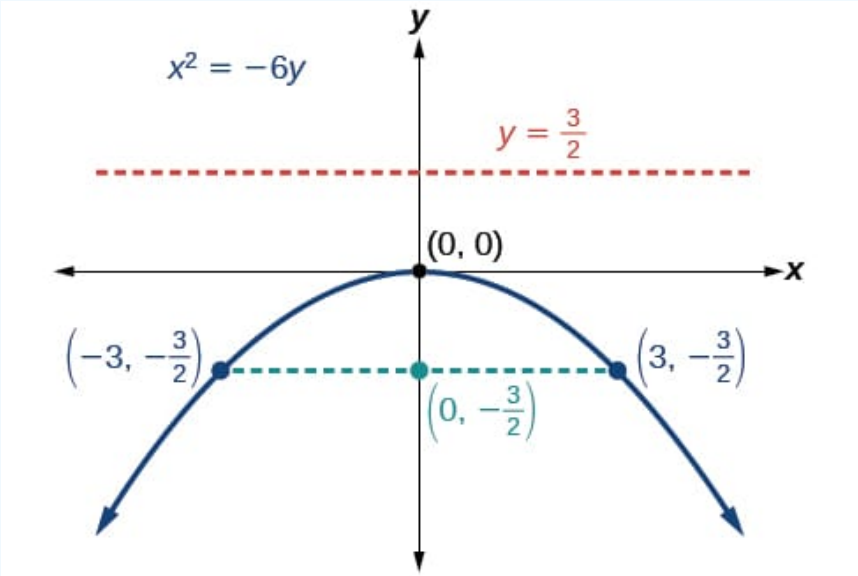


Figure 14.4.2.9: A graph of the solution to the problem presented herein.

Exercise 14.4.2.2

Graph $x^2 = 8y$. Identify and label the focus, directrix, and endpoints of the *latus rectum*.

Answer available at the end of this section.

Writing Equations of Parabolas in Standard Form

In the previous examples, we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

How to: Given its focus and directrix, write the equation for a parabola in standard form

1. Determine whether the axis of symmetry is the x- or y-axis.
   i. If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the x-axis. Use the standard form $y^2 = 4px$.
   ii. If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the y-axis. Use the standard form $x^2 = 4py$.
2. Multiply $4p$.
3. Substitute the value from Step 2 into the equation determined in Step 1.

Example 14.4.2.3: Writing the Equation of a Parabola in Standard Form Given its Focus and Directrix

What is the equation for the parabola with focus $(-\frac{1}{2}, 0)$ and directrix $x = \frac{1}{2}$?

Solution

The focus has the form $(p, 0)$, so the equation will have the form $y^2 = 4px$.

- Multiplying $4p$, we have $4p = 4(-\frac{1}{2}) = -2$.
- Substituting for $4p$, we have $y^2 = 4px = -2x$.=

Therefore, the equation for the parabola is $y^2 = -2x$.

Exercise 14.4.2.3

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y = -\frac{7}{2}$?

Answer available at the end of this section.

Graphing Parabolas with Vertices Not at the Origin

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated h units horizontally and k units vertically, the vertex will be (h, k). This translation results in the standard form of the equation we saw previously with x replaced by $(x - h)$ and y replaced by $(y - k)$.

To graph parabolas with a vertex (h, k) other than the origin, we use the standard form $(y - k)^2 = 4p(x - h)$ for parabolas that have an axis of symmetry parallel to the x-axis, and $(x - h)^2 = 4p(y - k)$ for parabolas that have an axis of symmetry parallel to the y -axis. These standard forms are given below, along with their general graphs and key features.

Standard Forms of Parabolas with Vertex at (h,k)

STANDARD FORMS OF PARABOLAS WITH VERTEX (H, K)

Table 14.4.2.2 and Figure 14.4.2.11 summarize the standard features of parabolas with a vertex at a point (h, k).

Table 14.4.2.2

Axis of Symmetry	Equation	Focus	Directrix	Endpoints of Latus Rectum
$y = k$	$(y - k)^2 = 4p(x - h)$	$(h + p, k)$	$x = h - p$	$(h + p, k \pm 2p)$
$x = h$	$(x - h)^2 = 4p(y - k)$	$(h, k + p)$	$y = k - p$	$(h \pm 2p, k + p)$

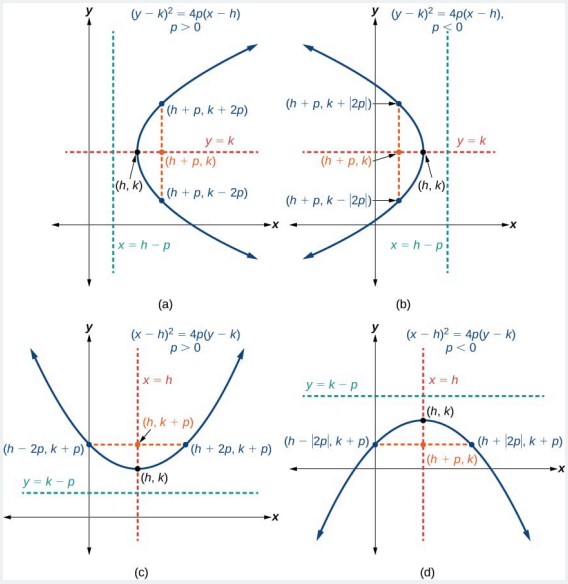


Figure 14.4.2.11: (a) When $p > 0$, the parabola opens right. (b) When $p < 0$, the parabola opens left. (c) When $p > 0$, the parabola opens up. The equation to this becomes more familiar as $y = ax^2 + bx + c$. (d) When $p < 0$, the parabola opens down.

How to: Given a standard form equation for a parabola centered at (h, k), sketch the graph

1. Determine which of the standard forms applies to the given equation: $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the latus rectum.
   - If the equation is in the form $(y-k)^2 = 4p(x-h)$,then:
     - use the given equation to identify h and k for the vertex, (h, k)
     - use the value of k to determine the axis of symmetry, $y = k$
     - set $4p$ equal to the coefficient of $(x-h)$ in the given equation to solve for p. If $p > 0$,the parabola opens right. If $p < 0$, the parabola opens left.
     - use h, k, and p to find the coordinates of the focus, $(h+p, k)$
     - use h andp p to find the equation of the directrix, $x = h-p$
     - use h, k, and p to find the endpoints of the latus rectum, $(h+p, k \pm 2p)$
   - If the equation is in the form $(x-h)^2 = 4p(y-k)$,then:
     - use the given equation to identify h and k for the vertex, (h, k)
     - use the value of h to determine the axis of symmetry, $x = h$
     - set $4p$ equal to the coefficient of $(y-k)$ in the given equation to solve for p. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down.
     - use h, k, and p to find the coordinates of the focus, $(h, k+p)$
     - use k and p to find the equation of the directrix, $y = k-p$
     - use h, k, and p to find the endpoints of the latus rectum, $(h \pm 2p, k+p)$
3. Plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.

Example 14.4.2.4: Graphing a Parabola with Vertex (h, k) and Axis of Symmetry Parallel to the x-axis

Graph $(y-1)^2 = -16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the latus rectum.

Solution

The standard form that applies to the given equation is $(y-k)^2 = 4p(x-h)$. Thus, the axis of symmetry is parallel to the x -axis. It follows that:

- the vertex is $(h,k) = (-3,1)$
- the axis of symmetry is $y = k = 1$
- $-16 = 4p$,so $p = -4$. Since $p < 0$, the parabola opens left.
- the coordinates of the focus are $(h+p,k) = (-3+(-4),1) = (-7,1)$
- the equation of the directrix is $x = h - p = -3-(-4) = 1$
- the endpoints of the latus rectum are $(h+p, k \pm 2p) = (-3+(-4), 1 \pm 2(-4))$, or $(-7,-7)$ and $(-7,9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 14.4.2.10).

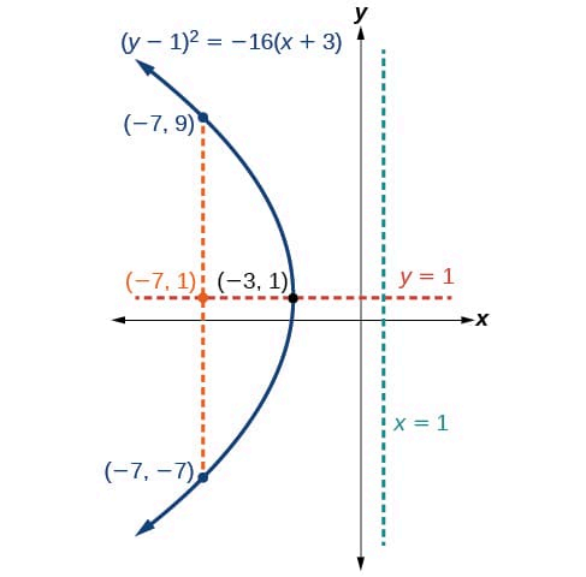


Figure 14.4.2.12: A graph of the solution to the problem presented herein.

Exercise 14.4.2.4

Graph $(y+1)^2 = 4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Answer available at the end of this section.

Example 14.4.2.5: Graphing a Parabola from an Equation Given in General Form

Graph $x^2 - 8x - 28y - 208 = 0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Solution

Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^2 = 4p(y-k)$. Thus, the axis of symmetry is parallel to the y-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable x in order to complete the square.

$$x^2 - 8x - 28y - 208 = 0$$
$$x^2 - 8x = 28y + 208$$
$$x^2 - 8x + 16 = 28y + 208 + 16$$
$$(x-4)^2 = 28y + 224$$
$$(x-4)^2 = 28(y+8)$$
$$(x-4)^2 = 4 \cdot 7 \cdot (y+8)$$

It follows that:

- the vertex is $(h, k) = (4, -8)$
- the axis of symmetry is $x = h = 4$
- since $p = 7$, $p > 0$ and so the parabola opens up
- the coordinates of the focus are $(h, k + p) = (4, -8 + 7) = (4, -1)$
- the equation of the directrix is $y = k - p = -8 - 7 = -15$
- the endpoints of the latus rectum are $(h \pm 2p, k + p) = (4 \pm 2(7), -8 + 7)$, or $(-10, -1)$ and $(18, -1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and latus rectum, and draw a smooth curve to form the parabola (Figure 14.4.2.14).

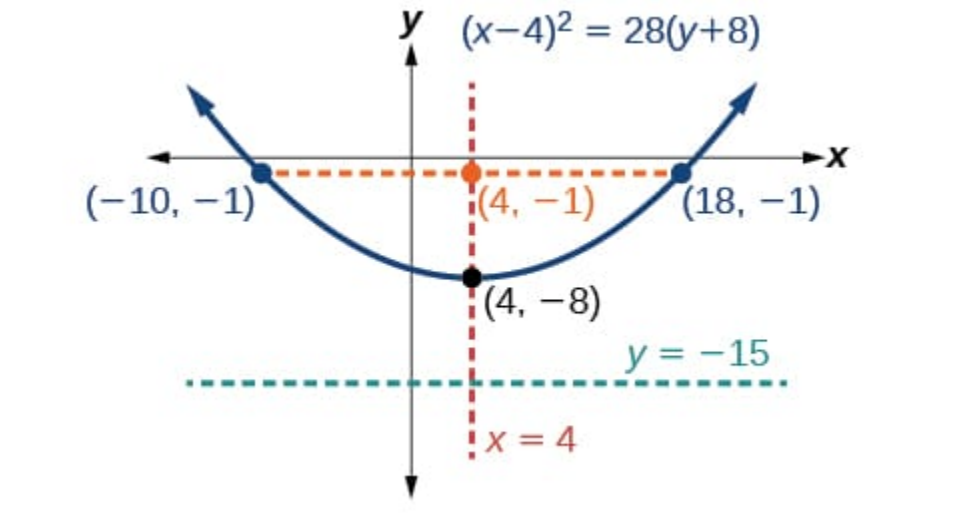


Figure 14.4.2.14: A graph of the solution to the problem presented herein.

Exercise 14.4.2.5

Graph $(x + 2)^2 = -20(y - 3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the *latus rectum*.

Answer available at the end of this section.

Solving Applied Problems Involving Parabolas

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus (Figure 14.4.2.16). This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.

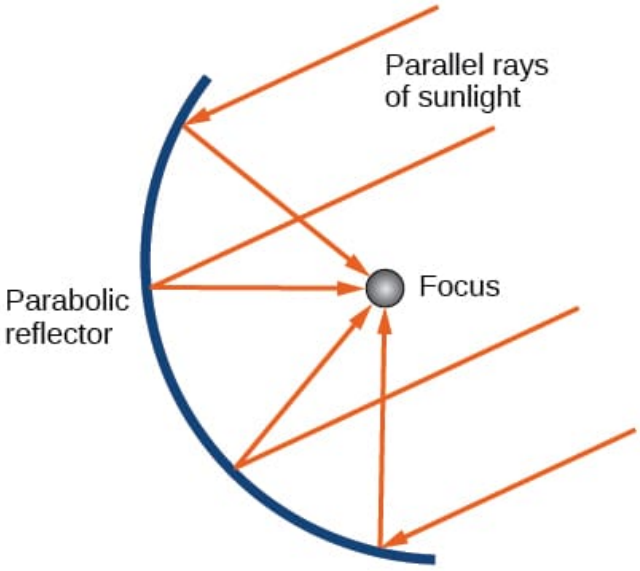


Figure 14.4.2.16: Reflecting property of parabolas

Solar Cookers

Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus, parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

Example 14.4.2.6: Solving Applied Problems Involving Parabolas

A cross-section of a design for a travel-sized solar fire starter is shown in Figure 14.4.2.17. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

a. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
b. Use the equation found in part (a) to find the depth of the fire starter.

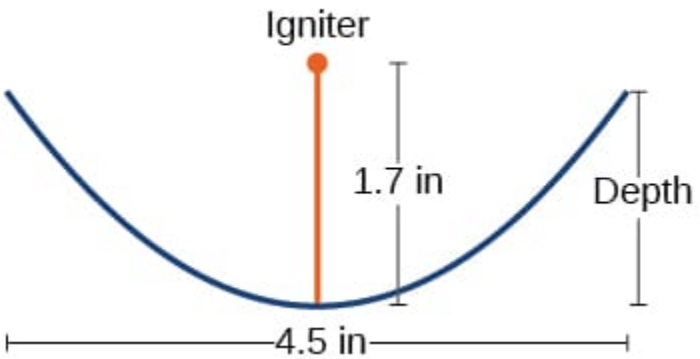


Figure 14.4.2.17 Cross-section of a travel-sized solar fire starter

Solution

a. The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^2 = 4py$,where $p > 0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p = 1.7$.

$$\begin{aligned} x^2 &= 4py && \text{Standard form of upward-facing parabola with vertex } (0,0) \\ x^2 &= 4(1.7)y && \text{Substitute } 1.7 \text{ for } p \\ x^2 &= 6.8y && \text{Multiply.} \end{aligned}$$

b. The dish extends $\frac{4.5}{2} = 2.25$ inches on either side of the origin. We can substitute 2.25 for x in the equation from part (a) to find the depth of the dish.

$$\begin{aligned} x^2 &= 6.8y && \text{Equation found in part } (a) \\ (2.25)^2 &= 6.8y && \text{Substitute } 2.25 \text{ for } x \\ y &\approx 0.74 && \text{Solve for } y \end{aligned}$$

The dish is about 0.74 inches deep.

Exercise 14.4.2.6

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm. The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.

a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has the *x*-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

Answer a available at the end of this section.

Key Equations

Parabola, vertex at origin, axis of symmetry on *x*-axis	$y^2 = 4px$
Parabola, vertex at origin, axis of symmetry on *y*-axis	$x^2 = 4py$
Parabola, vertex at (h, k),axis of symmetry on *x*-axis	$(y-k)^2 = 4p(x-h)$
Parabola, vertex at (h, k),axis of symmetry on *y*-axis	$(x-h)^2 = 4p(y-k)$

Key Concepts

- A parabola is the set of all points (x, y) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex $(0, 0)$ and the *x*-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example $14.4.2.1$.
- The standard form of a parabola with vertex $(0, 0)$ and the *y*-axis as its axis of symmetry can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example $14.4.2.2$.
- When given the focus and directrix of a parabola, we can write its equation in standard form. See Example $14.4.2.3$.
- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the x-axis can be used to graph the parabola. If $p > 0$, the parabola opens right. If $p < 0$, the parabola opens left. See Example $14.4.2.4$.
- The standard form of a parabola with vertex (h, k) and axis of symmetry parallel to the y-axis can be used to graph the parabola. If $p > 0$, the parabola opens up. If $p < 0$, the parabola opens down. See Example $14.4.2.5$.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides. See Example $14.4.2.6$.

Contributors and Attributions

- Template:ContribOpenStaxPreCalc

Answers

Exercise 14.4.2.1

- Focus: $(-4, 0)$
- Directrix: $x = 4$
- Endpoints of the latus rectum: $(-4, \pm 8)$

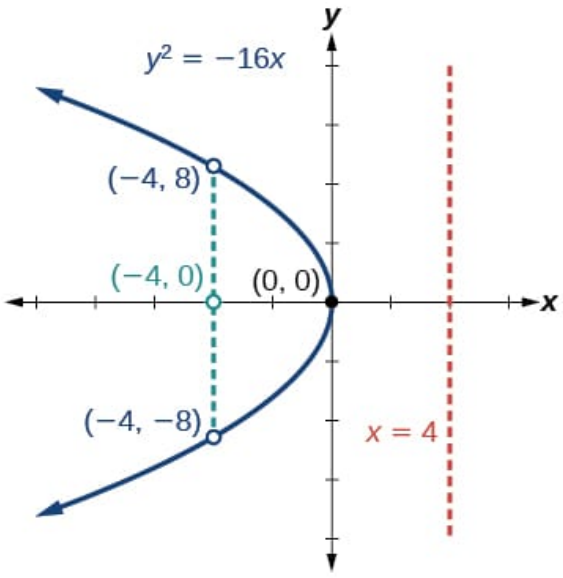


Figure 14.4.2.8: A graph of the solution to the problem presented herein.

Exercise 14.4.2.2

- Focus: $(0, 2)$
- Directrix: $y = -2$
- Endpoints of the latus rectum: $(\pm 4, 2)$.

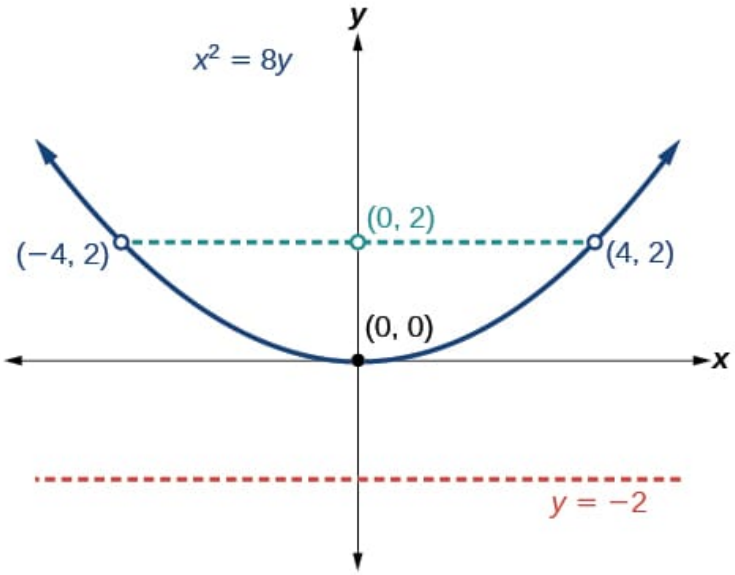


Figure 14.4.2.10: A graph of the solution to the problem presented herein.

Exercise 14.4.2.3

$x^2 = 14y$.

Exercise 14.4.2.4

- Vertex: $(8, -1)$

- Axis of symmetry: $y = -1$
- Focus: $(9, -1)$
- Directrix: $x = 7$
- Endpoints of the *latus rectum*: $(9, -3)$ and $(9, 1)$.

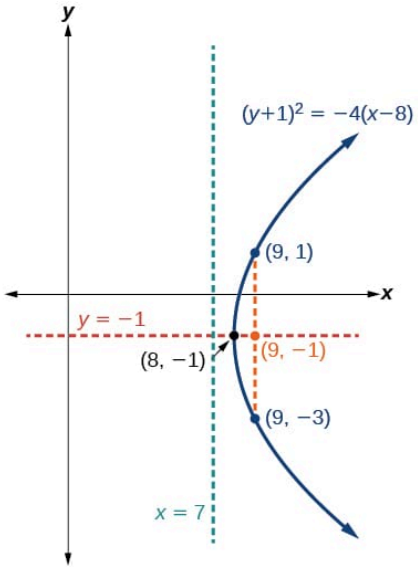


Figure 14.4.2.13: A graph of the solution to the problem presented herein.

Exercise 14.4.2.5

- Vertex: $(-2, 3)$
- Axis of symmetry: $x = -2$
- Focus: $(-2, -2)$
- Directrix: $y = 8$
- Endpoints of the *latus rectum*: $(-12, -2)$ and $(8, -2)$.

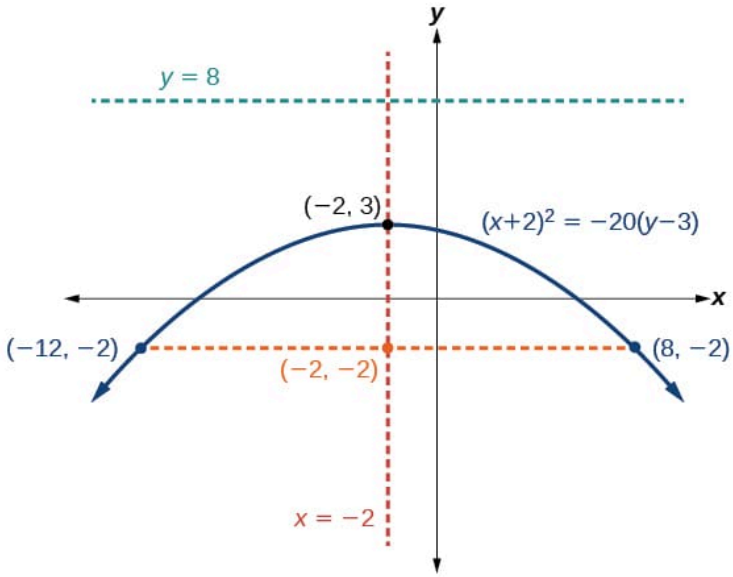


Figure 14.4.2.15: A graph of the solution to the problem presented herein.

Exercise 14.4.2.6

$y^2 = 1280x$

The depth of the cooker is 500 mm

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14.4.3: The Hyperbola

Learning Objectives

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of **conic**. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom (Figure 14.4.3.1).

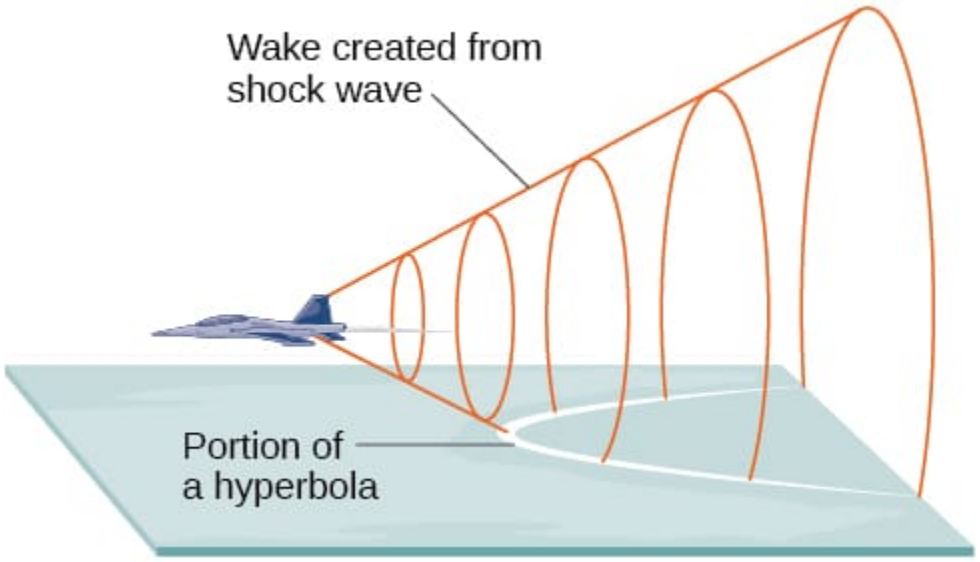


Figure 14.4.3.1: This image shows a jet producing a shock wave that intersecting the ground forming a portion of a conic which is a hyperbola.

Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

Locating the Vertices and Foci of a Hyperbola

In analytic geometry, a **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other (Figure 14.4.3.2).

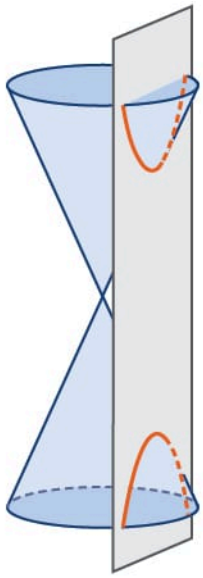


Figure 14.4.3.2: A hyperbola that is formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected.

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points (x, y) in a plane such that the difference of the distances between (x, y) and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

As with the ellipse, every hyperbola has two **axes of symmetry**. The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints. The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle (Figure 14.4.3.3).

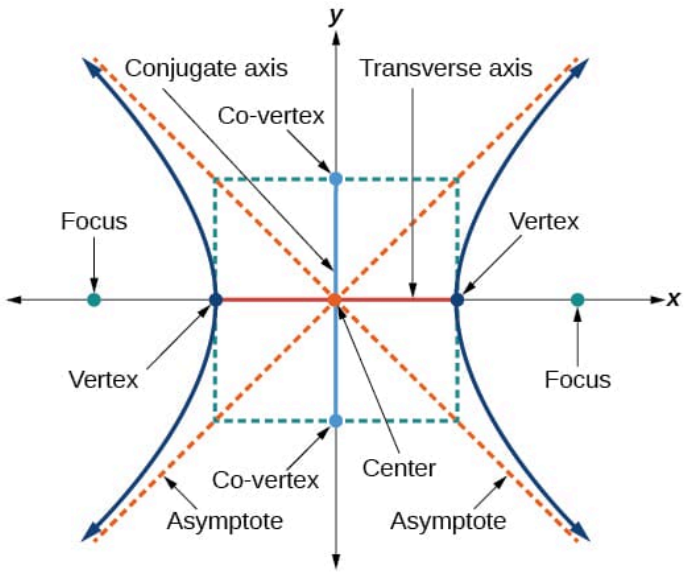


Figure 14.4.3.3: Key features of the hyperbola

In this section, we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the x- and y-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

Deriving the Equation of an Ellipse Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant. See Figure 14.4.3.4.

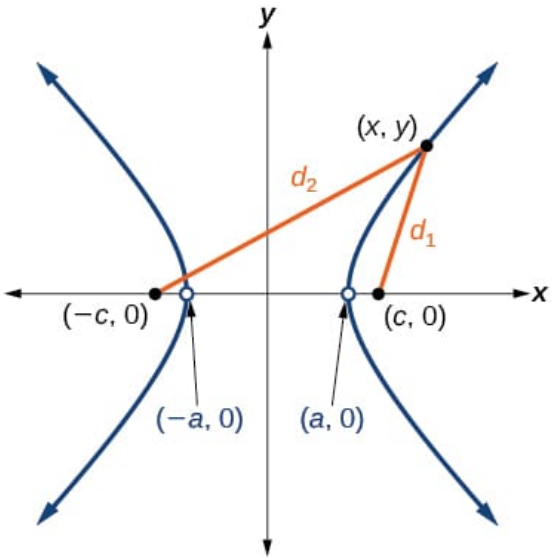


Figure 14.4.3.4: A horizontal hyperbola in the x-y coordinate system centered at (0, 0) with Vertices at (-a, 0) and (a, 0) and Foci at (-c, 0) and (c, 0), with lines of length d1 and d2 connecting a point on the right branch of the hyperbola to the foci.

If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a - (-c) = a + c$. The distance from $(c, 0)$ to $(a, 0)$ is $c - a$. The sum of the distances from the foci to the vertex is

$$(a + c) - (c - a) = 2a$$

If (x, y) is a point on the hyperbola, we can define the following variables:

$$d_2 = \text{the distance from } (-c, 0) \text{ to } (x, y)$$

$$d_1 = \text{the distance from } (c, 0) \text{ to } (x, y)$$

By definition of a hyperbola, $d_2 - d_1$ is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2a$ for the vertex $(a, 0)$. It follows that $d_2 - d_1 = 2a$ for any point on the hyperbola. As with the derivation of the equation of an ellipse, we will begin by applying the **distance formula**. The rest of the derivation is algebraic. Compare this derivation with the one from the previous section for ellipses.

$$\begin{aligned}
d_2 - d_1 &= 2a \\
\sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} &= 2a && \text{Distance Formula} \\
\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a && \text{Simplify expressions.} \\
\sqrt{(x+c)^2 + y^2} &= 2a + \sqrt{(x-c)^2 + y^2} && \text{Move radical to opposite side.} \\
(x+c)^2 + y^2 &= \left(2a + \sqrt{(x-c)^2 + y^2}\right)^2 && \text{Square both sides.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 && \text{Expand the squares.} \\
x^2 + 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2 && \text{Expand remaining square.} \\
2cx &= 4a^2 + 4a\sqrt{(x-c)^2 + y^2} - 2cx && \text{Combine like terms.} \\
4cx - 4a^2 &= 4a\sqrt{(x-c)^2 + y^2} && \text{Isolate the radical.} \\
cx - a^2 &= a\sqrt{(x-c)^2 + y^2} && \text{Divide by 4.} \\
(cx - a^2)^2 &= a^2\left[\sqrt{(x-c)^2 + y^2}\right]^2 && \text{Square both sides.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2(x^2 - 2cx + c^2 + y^2) && \text{Expand the squares.} \\
c^2x^2 - 2a^2cx + a^4 &= a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 && \text{Distribute } a^2 \\
a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 && \text{Combine like terms.} \\
c^2x^2 - a^2x^2 - a^2y^2 &= a^2c^2 - a^4 && \text{Rearrange terms.} \\
x^2(c^2 - a^2) - a^2y^2 &= a^2(c^2 - a^2) && \text{Factor common terms.} \\
x^2b^2 - a^2y^2 &= a^2b^2 && \text{Set } b^2 = c^2 - a^2 \\
\frac{x^2b^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} &= \frac{a^2b^2}{a^2b^2} && \text{Divide both sides by } a^2b^2 \\
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1
\end{aligned}$$

This equation defines a hyperbola centered at the origin with vertices $(\pm a, 0)$ and co-vertices $(0, \pm b)$.

STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(0, 0)$

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(\pm c, 0)$

- the equations of the asymptotes are $y = \pm\frac{b}{a}x$

See Figure $14.4.3.5a$.

The standard form of the equation of a hyperbola with center $(0, 0)$ and transverse axis on the y-axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(\pm b, 0)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}x$

See Figure $14.4.3.5b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

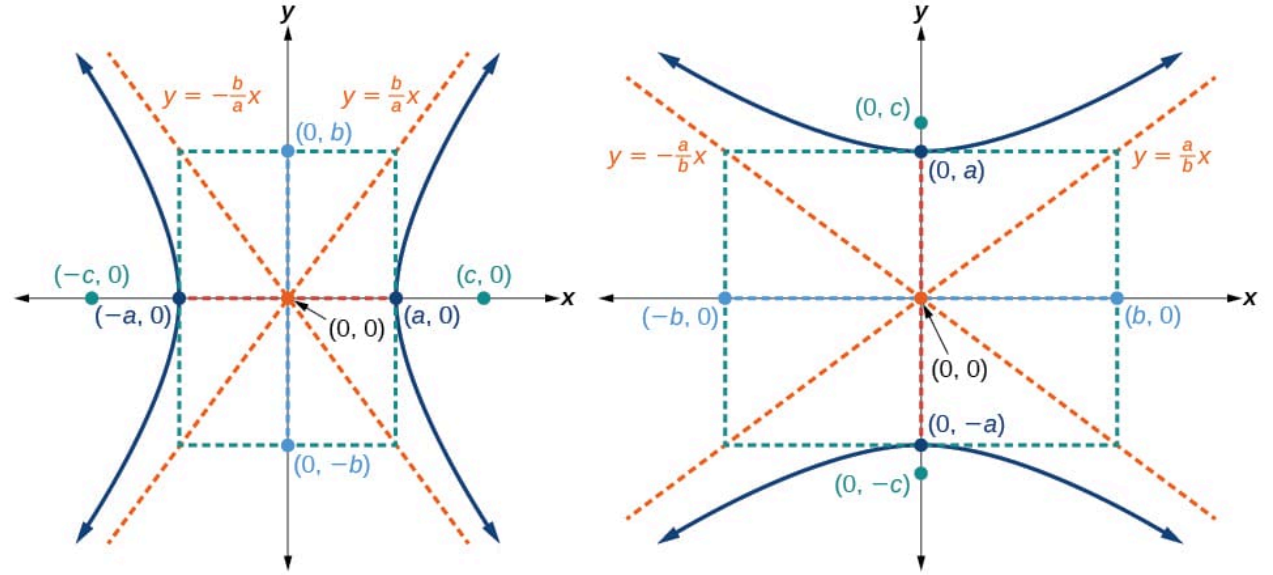


Figure $14.4.3.5$: (a) Horizontal hyperbola with center $(0, 0)$ (b) Vertical hyperbola with center $(0, 0)$

How to: Given the equation of a hyperbola in standard form, locate its vertices and foci

1. Determine whether the transverse axis lies on the x- or y-axis. Notice that a^2 is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
   - If the equation has the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the transverse axis lies on the x-axis. The vertices are located at $(\pm a, 0)$, and the foci are located at $(\pm c, 0)$.

- If the equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, then the transverse axis lies on the y-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.

2. Solve for a using the equation $a = \sqrt{a^2}$.
3. Solve for c using the equation $c = \sqrt{a^2 + b^2}$.

Example 14.4.3.1: Locating a Hyperbola's Vertices and Foci

Identify the vertices and foci of the hyperbola with equation $\frac{y^2}{49} - \frac{x^2}{32} = 1$.

Solution

The equation has the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, so the transverse axis lies on the y-axis. The hyperbola is centered at the origin, so the vertices serve as the *y*-intercepts of the graph. To find the vertices, set $x = 0$, and solve for y.

$$\begin{aligned} 1 &= \frac{y^2}{49} - \frac{x^2}{32} \\ 1 &= \frac{y^2}{49} - \frac{0^2}{32} \\ 1 &= \frac{y^2}{49} \\ y^2 &= 49 \\ y &= \pm\sqrt{49} \\ &= \pm 7 \end{aligned}$$

The foci are located at $(0, \pm c)$. Solving for c,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{49 + 32} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0, 9)$.

Exercise 14.4.3.1

Identify the vertices and foci of the hyperbola with equation $\frac{x^2}{9} - \frac{y^2}{25} = 1$.

Answer available at the end of this section.

Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0, 0)$,we see that the vertices, co-vertices, and foci are related by the equation $c^2 = a^2 + b^2$. Note that this equation can also be rewritten as $b^2 = c^2 - a^2$. This relationship is used to write the

equation for a hyperbola when given the coordinates of its foci and vertices.

How to: Given the vertices and foci of a hyperbola centered at $(0,0)$, write its equation in standard form

1. Determine whether the transverse axis lies on the x- or y-axis.
   - If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$, respectively, then the transverse axis is the x-axis. Use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
   - If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the y-axis. Use the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.
2. Find b^2 using the equation $b^2 = c^2 - a^2$.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

Example 14.4.3.2: Finding the Equation of a Hyperbola Centered at $(0,0)$ Given its Foci and Vertices

What is the standard form equation of the hyperbola that has vertices $(\pm 6, 0)$ and foci $(\pm 2\sqrt{10}, 0)$?

Solution

The vertices and foci are on the x-axis. Thus, the equation for the hyperbola will have the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The vertices are $(\pm 6, 0)$, so $a = 6$ and $a^2 = 36$.

The foci are $(\pm 2\sqrt{10}, 0)$, so $c = 2\sqrt{10}$ and $c^2 = 40$.

Solving for b^2, we have

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= 40 - 36 \quad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 4 \quad \text{Subtract.} \end{aligned}$$

Finally, we substitute $a^2 = 36$ and $b^2 = 4$ into the standard form of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{4} = 1$, as shown in Figure 14.4.3.6.

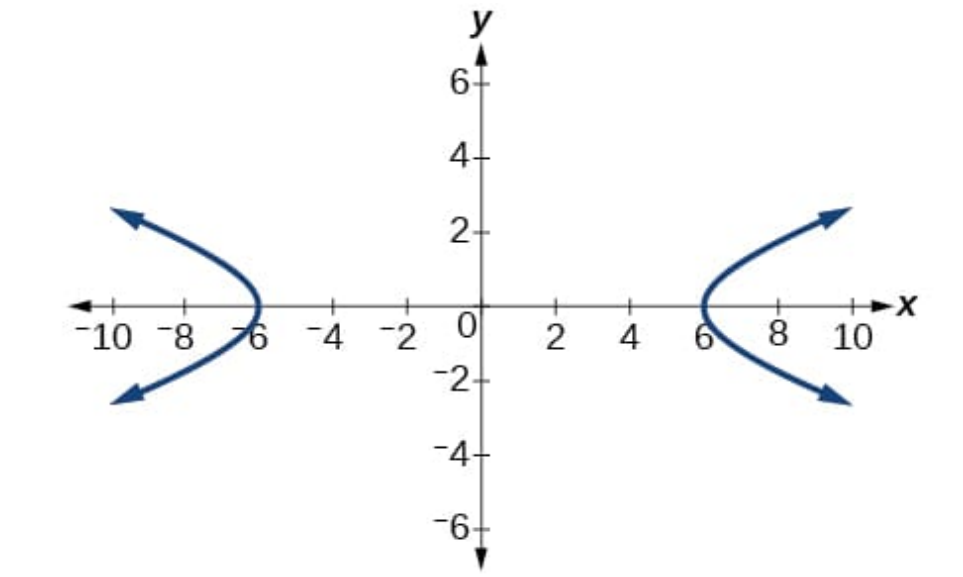


Figure 14.4.3.6: A horizontal hyperbola centered at (0, 0) in the x-y coordinate system with Vertices at (-6, 0) and (6, 0).

Exercise 14.4.3.2

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $(0, \pm 2\sqrt{5})$?

Answer available at the end of this section.

Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated h units horizontally and k units vertically, the center of the hyperbola will be (h, k). This translation results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and y replaced by $(y - k)$.

STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER (H, K)

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{14.4.3.1}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the distance between the foci is $2c$, where $c^2 = a^2 + b^2$
- the coordinates of the foci are $(h \pm c, k)$

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2a$ and its width is $2b$. The slopes of the diagonals are $\pm\frac{b}{a}$,and each diagonal passes through the center (h, k). Using the point-slope formula, it is simple to show that the equations of the asymptotes are $y = \pm\frac{b}{a}(x - h) + k$. See Figure 14.4.3.7*a*.

The standard form of the equation of a hyperbola with center (h, k) and transverse axis parallel to the y-axis is

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1 \tag{14.4.3.2}$$

where

- the length of the transverse axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the distance between the foci is $2c$, where $c^2=a^2+b^2$
- the coordinates of the foci are $(h, k \pm c)$

Using the reasoning above, the equations of the asymptotes are $y=\pm\frac{a}{b}(x-h)+k$. See Figure $14.4.3.7b$.

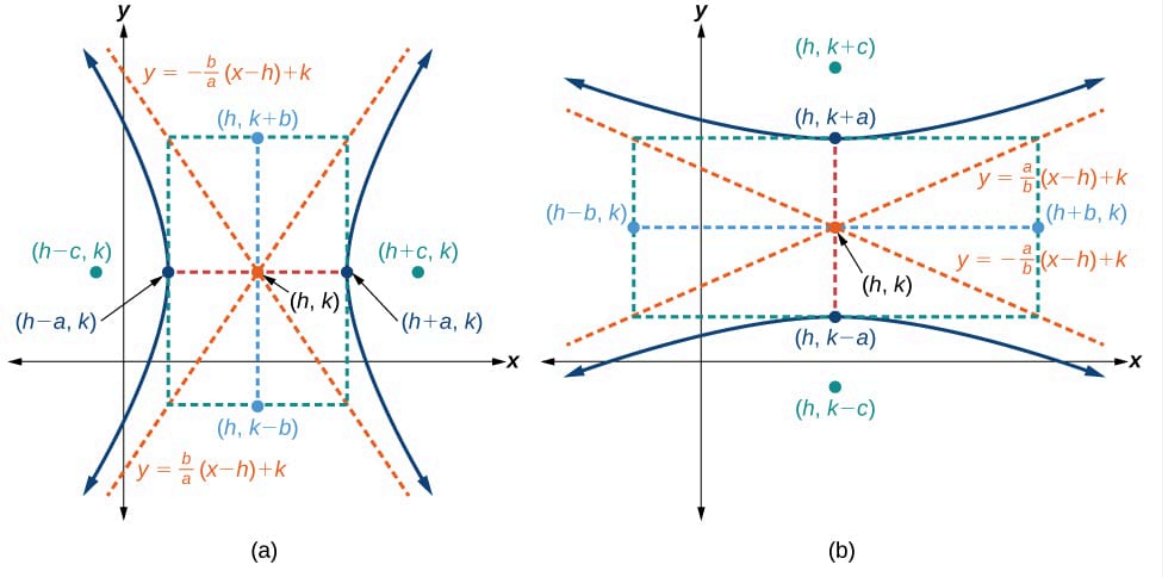


Figure $14.4.3.7$: (a) Horizontal hyperbola with center (h, k) (b) Vertical hyperbola with center (h, k)

Like hyperbolas centered at the origin, hyperbolas centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2=a^2+b^2$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

How to: Given the vertices and foci of a hyperbola centered at (h, k),write its equation in standard form

1. Determine whether the transverse axis is parallel to the x- or y-axis.
   - If the y-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the x-axis. Use the standard form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.
   - If the x-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the y-axis. Use the standard form $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$.
2. Identify the center of the hyperbola, (h, k),using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the transverse axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k found in Step 2 along with the given coordinates for the foci.
5. Solve for b^2 using the equation $b^2=c^2-a^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Example 14.4.3.3: Finding the Equation of a Hyperbola Centered at (h, k) Given its Foci and Vertices

What is the standard form equation of the **hyperbola** that has vertices at $(0, -2)$ and $(6, -2)$ and foci at $(-2, -2)$ and $(8, -2)$?

Solution

The y-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the x-axis. Thus, the equation of the hyperbola will have the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

First, we identify the center, (h, k). The center is halfway between the vertices $(0, -2)$ and $(6, -2)$. Applying the midpoint formula, we have

$$(h, k) = \left(\frac{0+6}{2}, \frac{-2+(-2)}{2}\right) = (3, -2)$$

Next, we find a^2. The length of the transverse axis, $2a$,is bounded by the vertices. So, we can find a^2 by finding the distance between the x-coordinates of the vertices.

Now we need to find c^2. The coordinates of the foci are $(h \pm c, k)$. So $(h-c, k) = (-2, -2)$ and $(h+c, k) = (8, -2)$. We can use the x-coordinate from either of these points to solve for c. Using the point $(8, -2)$, and substituting $h = 3$,

$$\begin{aligned} h + c &= 8 \\ 3 + c &= 8 \\ c &= 5 \\ c^2 &= 25 \end{aligned}$$

Next, solve for b^2 using the equation $b^2 = c^2 - a^2$:

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 25 - 9 \\ &= 16 \end{aligned}$$

Finally, substitute the values found for h, k, a^2,and b^2 into the standard form of the equation.

$$\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$$

Exercise 14.4.3.3

What is the standard form equation of the hyperbola that has vertices $(1, -2)$ and $(1, 8)$ and foci $(1, -10)$ and $(1, 16)$?

Answer available at the end of this section.

Graphing Hyperbolas Centered at the Origin

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for horizontal hyperbolas and the standard form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ for vertical hyperbolas.

How to: Given a standard form equation for a hyperbola centered at $(0,0)$, sketch the graph

1. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
   - If the equation is in the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then
     - the transverse axis is on the x-axis
     - the coordinates of the vertices are $(\pm a,0)$
     - the coordinates of the co-vertices are $(0,\pm b)$
     - the coordinates of the foci are $(\pm c,0)$
     - the equations of the asymptotes are $y=\pm\frac{b}{a}x$
   - If the equation is in the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, then
     - the transverse axis is on the y-axis
     - the coordinates of the vertices are $(0,\pm a)$
     - the coordinates of the co-vertices are $(\pm b,0)$
     - the coordinates of the foci are $(0,\pm c)$
     - the equations of the asymptotes are $y=\pm\frac{a}{b}x$
3. Solve for the coordinates of the foci using the equation $c=\pm\sqrt{a^2+b^2}$.
4. Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.

Example 14.4.3.4: Graphing a Hyperbola Centered at $(0,0)$ Given an Equation in Standard Form

Graph the hyperbola given by the equation $\frac{y^2}{64}-\frac{x^2}{36}=1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Solution

The standard form that applies to the given equation is $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$. Thus, the transverse axis is on the y-axis

The coordinates of the vertices are $(0,\pm a)=(0,\pm\sqrt{64})=(0,\pm 8)$

The coordinates of the co-vertices are $(\pm b,0)=(\pm\sqrt{36},0)=(\pm 6,0)$

The coordinates of the foci are $(0,\pm c)$, where $c=\pm\sqrt{a^2+b^2}$. Solving for c, we have

$$c=\pm\sqrt{a^2+b^2}=\pm\sqrt{64+36}=\pm\sqrt{100}=\pm 10$$

Therefore, the coordinates of the foci are $(0,\pm 10)$

The equations of the asymptotes are $y=\pm\frac{a}{b}x=\pm\frac{8}{6}x=\pm\frac{4}{3}x$

Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola, as shown in Figure 14.4.3.8.

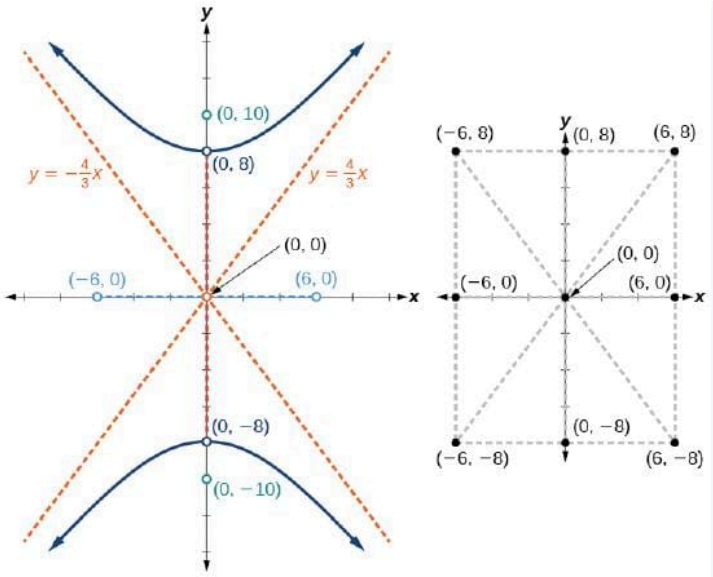


Figure 14.4.3.8: A vertical hyperbola centered at (0, 0) in the x-y coordinate system with Vertices at (0, 8) and (0, -8) and Foci at (0, -10) and (0, 10). Also shown are the slant asymptotes, $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$. The points (-6, 0) (6, 0) and (0, 0) are labeled.

Exercise 14.4.3.4

Graph the hyperbola given by the equation $\frac{x^2}{144} - \frac{y^2}{81} = 1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

Answer available at the end of this section.

Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point (h, k) other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ for horizontal hyperbolas, and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

How to: Given a general form for a hyperbola centered at (h, k), sketch the graph

1. Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
2. Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
    1. If the equation is in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the x-axis
        - the center is (h, k)
        - the coordinates of the vertices are $(h \pm a, k)$
        - the coordinates of the co-vertices are $(h, k \pm b)$
        - the coordinates of the foci are $(h \pm c, k)$
        - the equations of the asymptotes are $y = \pm\frac{b}{a}(x-h) + k$
    2. If the equation is in the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$, then
        - the transverse axis is parallel to the y-axis
        - the center is (h, k)

- the coordinates of the vertices are $(h, k \pm a)$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$
- the equations of the asymptotes are $y = \pm\frac{a}{b}(x-h)+k$

3. Solve for the coordinates of the foci using the equation $c = \pm\sqrt{a^2+b^2}$.
4. Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.

Example 14.4.3.5: Graphing a Hyperbola Centered at (h, k) Given an Equation in General Form

Graph the hyperbola given by the equation $9x^2 - 4y^2 - 36x - 40y - 388 = 0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Solution

Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(9x^2 - 36x) - (4y^2 + 40y) = 388$$

Factor the leading coefficient of each expression.

$$9(x^2 - 4x) - 4(y^2 + 10y) = 388$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$9(x^2 - 4x + 4) - 4(y^2 + 10y + 25) = 388 + 36 - 100$$

Rewrite as perfect squares.

$$9(x-2)^2 - 4(y+5)^2 = 324$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x-2)^2}{36} - \frac{(y+5)^2}{81} = 1$$

The standard form that applies to the given equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, where $a^2 = 36$ and $b^2 = 81$,or $a = 6$ and $b = 9$. Thus, the transverse axis is parallel to the x-axis. It follows that:

the center of the ellipse is $(h, k) = (2, -5)$

the coordinates of the vertices are $(h \pm a, k) = (2 \pm 6, -5)$, or $(-4, -5)$ and $(8, -5)$

the coordinates of the co-vertices are $(h, k \pm b) = (2, -5 \pm 9)$, or $(2, -14)$ and $(2, 4)$

the coordinates of the foci are $(h \pm c, k)$, where $c = \pm\sqrt{a^2+b^2}$. Solving for c,we have

$$c = \pm\sqrt{36+81} = \pm\sqrt{117} = \pm 3\sqrt{13}$$

Therefore, the coordinates of the foci are $(2 - 3\sqrt{13}, -5)$ and $(2 + 3\sqrt{13}, -5)$.

The equations of the asymptotes are $y = \pm\frac{b}{a}(x-h)+k = \pm\frac{3}{2}(x-2)-5$.

Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola, as shown in Figure 14.4.3.10.

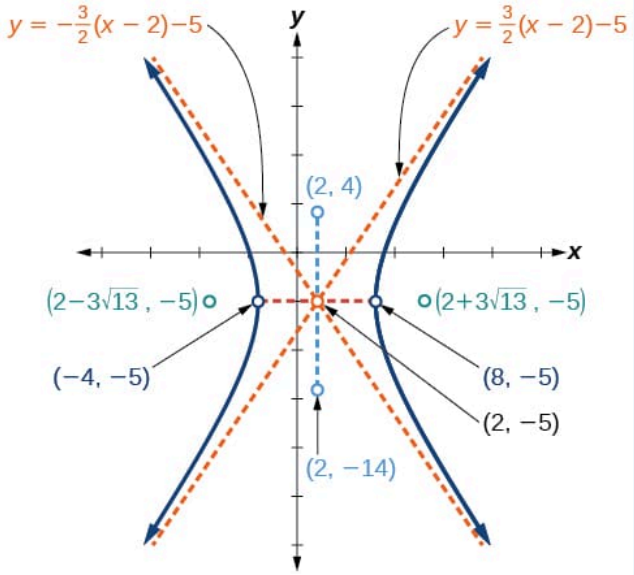


Figure 14.4.3.10: A horizontal hyperbola centered at (2, -5) with Vertices at (-4, -5) and (8, 5) and Foci at $(2-3\sqrt{13}, -5)$ and $(2+3\sqrt{13}, -5)$. Also shown are the slant asymptotes, $y = \frac{3}{2}(x-2)-5$ and $y = -\frac{3}{2}(x-2)-5$. The points (2, -14), (2, 4) and (0, 0) are labeled.

Exercise 14.4.3.5

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^2}{100} - \frac{(x-3)^2}{64} = 1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.

Answer available at the end of this section.

Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength (Figure 14.4.3.12). For example, a 500-foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!



Figure 14.4.3.12: Cooling towers at the Drax[1] power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)

The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 14.4.3.6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

Example 14.4.3.6: Solving Applied Problems Involving Hyperbolas

The design layout of a cooling tower is shown in Figure 14.4.3.13. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.

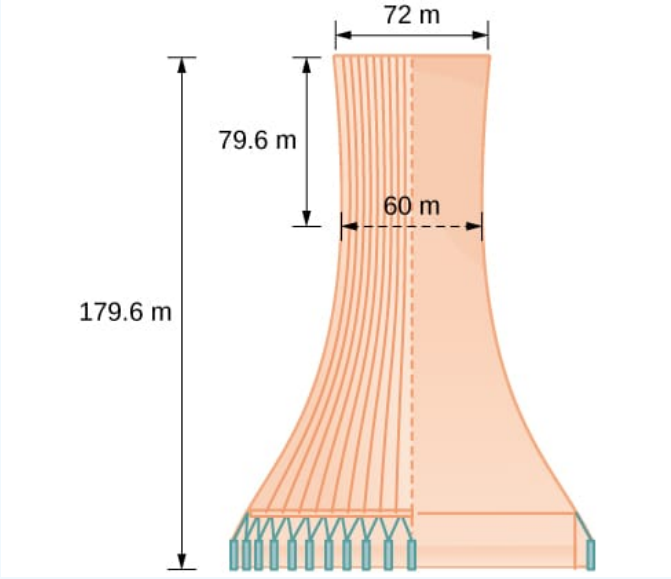


Figure 14.4.3.13: Project design for a natural draft cooling tower

Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

Solution

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of a^2 and b^2 to complete the model.

First, we find a^2. Recall that the length of the transverse axis of a hyperbola is $2a$. This length is represented by the distance where the sides are closest, which is given as 65.3 meters. So, $2a = 60$. Therefore, $a = 30$ and $a^2 = 900$.

To solve for b^2,we need to substitute for x and y in our equation using a known point. To do this, we can use the dimensions of the tower to find some point (x, y) that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the y-axis bisects the tower, our x-value can be represented by the radius of the top, or 36 meters. The *y*-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 && \text{Standard form of horizontal hyperbola.} \\ b^2 &= \frac{y^2}{\frac{x^2}{a^2} - 1} && \text{Isolate } b^2 \\ &= \frac{(79.6)^2}{\frac{(36)^2}{900} - 1} && \text{Substitute for } a^2,\ x,\ \text{and } y \\ &\approx 14400.3636 && \text{Round to four decimal places} \end{aligned}$$

The sides of the tower can be modeled by the hyperbolic equation

$$\frac{x^2}{900} - \frac{y^2}{14400.3636} = 1, \text{or } \frac{x^2}{30^2} - \frac{y^2}{120.0015^2} = 1$$

Exercise 14.4.3.6

A design for a cooling tower project is shown in Figure 14.4.3.14. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola—indicated by the intersection of dashed perpendicular lines in the figure—is the origin of the coordinate plane. Round final values to four decimal places.

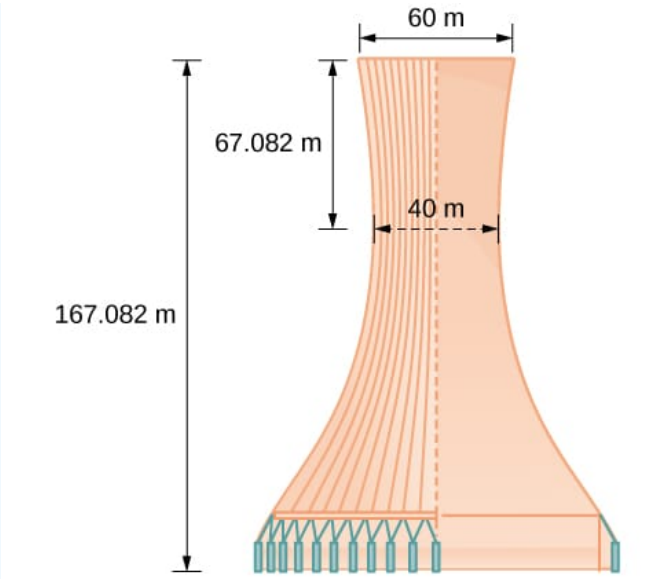


Figure 14.4.3.14: Project design for a natural draft cooling tower. The overall height is 167.082 meters. The diameter at the top is 60 meters, and at their closest, 79.6 meters from the top, the sides are 60 meters apart.

Answer available at the end of this section.

Media

Access these online resources for additional instruction and practice with hyperbolas.

- Conic Sections: The Hyperbola Part 1 of 2
- Conic Sections: The Hyperbola Part 2 of 2
- Graph a Hyperbola with Center at Origin
- Graph a Hyperbola with Center not at Origin

Key Equations

Hyperbola, center at origin, transverse axis on *x*-axis	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Hyperbola, center at origin, transverse axis on *y*-axis	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Hyperbola, center at (h,k),transverse axis parallel to *x*-axis	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$
Hyperbola, center at (h,k),transverse axis parallel to *y*-axis	$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$

Key Concepts

- A hyperbola is the set of all points (x,y) in a plane such that the difference of the distances between (x,y) and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci. See Example 14.4.3.1.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form. See Example 14.4.3.2 and Example 14.4.3.3.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola. See Example 14.4.3.4 and Example 14.4.3.5.

- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides. See Example $14.4.3.6$.

Contributors and Attributions

- Template:ContribOpenStaxPreCalc

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[1]Drax power station is a UK renewable power station focusing on biomass power.

Answers

Exercise $14.4.3.1$

Vertices: $(\pm 3, 0)$; Foci: $(\pm\sqrt{34}, 0)$

Exercise $14.4.3.2$

$\frac{y^2}{4} - \frac{x^2}{16} = 1$

Exercise $14.4.3.3$

$\frac{(y-3)^2}{25} + \frac{(x-1)^2}{144} = 1$

Exercise $14.4.3.4$

vertices: $(\pm 12, 0)$; co-vertices: $(0, \pm 9)$; foci: $(\pm 15, 0)$; asymptotes: $y = \pm\frac{3}{4}x$;

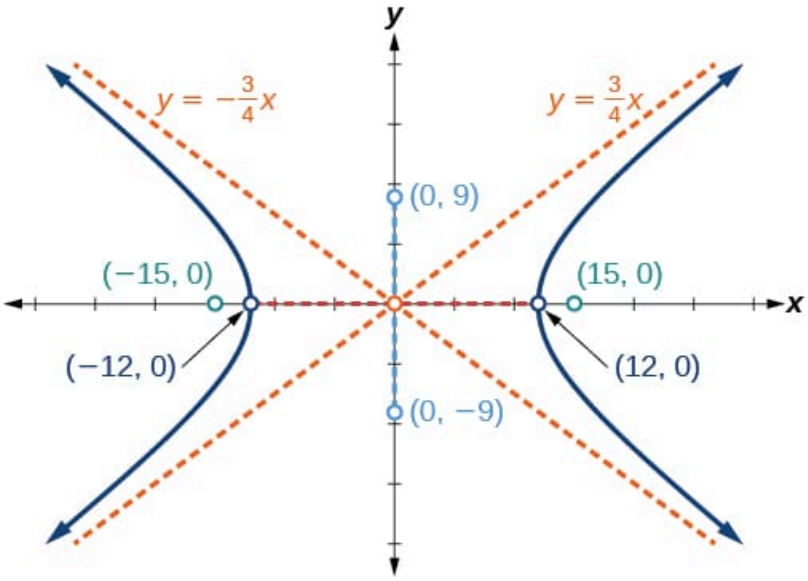


Figure $14.4.3.9$: A horizontal hyperbola centered at (0, 0) in the x-y coordinate system with Vertices at (-12, 0) and (12, 0) and Foci at (-15, 0) and (15, 0). Also shown are the slant asymptotes, $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The points (0, 9) (0, -9) and (0, 0) are labeled.

Exercise $14.4.3.5$

center: $(3, -4)$; vertices: $(3, -14)$ and $(3, 6)$; co-vertices: $(-5, -4)$; and $(11, -4)$; foci: $(3, -4 - 2\sqrt{41})$ and $(3, -4 + 2\sqrt{41})$; asymptotes: $y = \pm\frac{5}{4}(x-3) - 4$

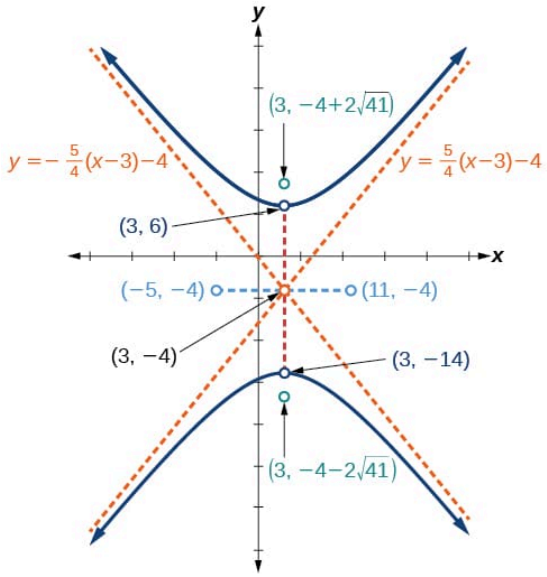


Figure 14.4.3.11

Exercise 14.4.3.6

The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^2}{400} - \frac{y^2}{3600} = 1$ or $\frac{x^2}{20^2} - \frac{y^2}{60^2} = 1$.

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14.4.4: The Ellipse

Learning Objectives

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C., shown in Figure 14.4.4.1, is such a room. It is an oval-shaped room called a *whispering chamber*[1] because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.



Figure 14.4.4.1: The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)

Writing Equations of Ellipses in Standard Form

A conic section, or conic, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape, as shown in Figure 14.4.4.2.

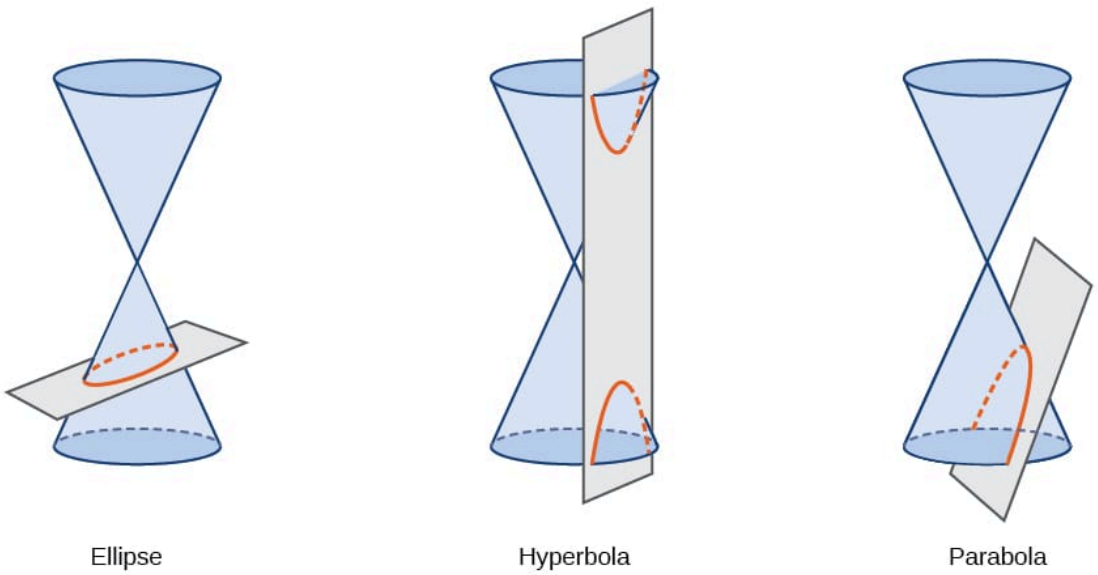


Figure 14.4.4.2: Generating conic sections by intersecting a plane through a circular cone. Missing here is the circle. *How would you do that one?*

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter, we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An ellipse is the set of all points (x, y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus(plural: foci).

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse. See Figure 14.4.4.3.

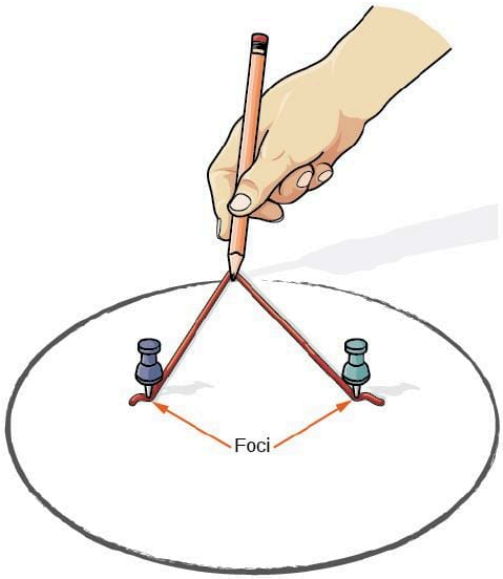


Figure 14.4.4.3: This figure shows two thumbtacks stuck in a piece of paper with a slack piece of string between them. A pencil pulls the string taught and by moving around, draws an ellipse.

Every ellipse has two axes of symmetry. The longer axis is called the major axis, and the shorter axis is called the minor axis. Each endpoint of the major axis is the vertex of the ellipse (plural: vertices), and each endpoint of the minor axis is a co-vertex of

the ellipse. The center of an ellipse is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci (Figure 14.4.4.4).

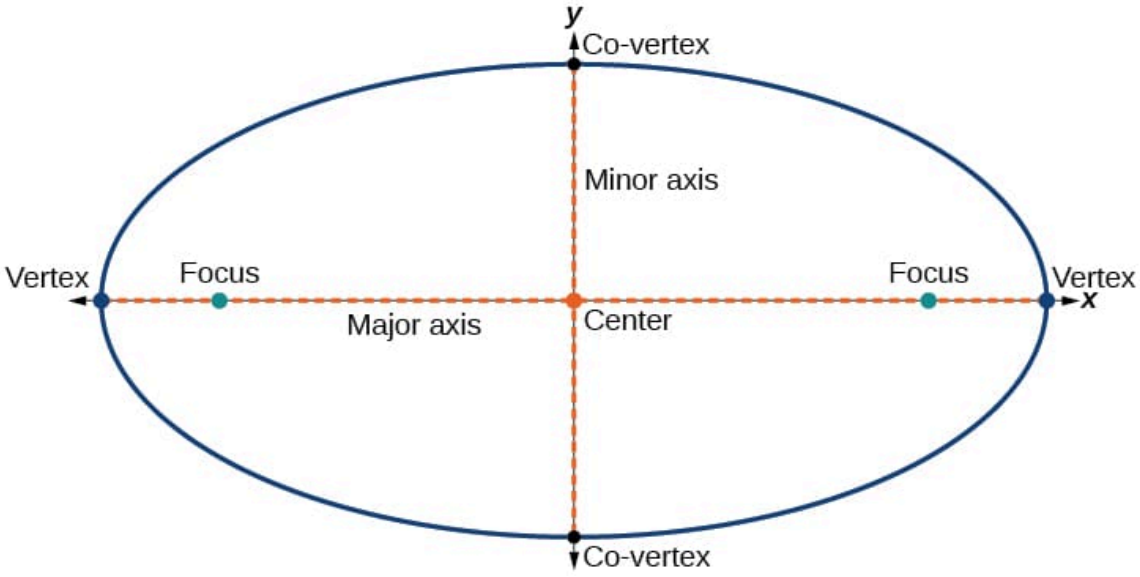


Figure 14.4.4.4: A horizontal ellipse centered at (0, 0) in the x-y coordinate system with the ellipse properties labeled herein.

In this section, we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the x- and y-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

Deriving the Equation of an Ellipse Centered at the Origin

To derive the equation of an **ellipse** centered at the origin, we begin with the foci $(-c, 0)$ and $(c, 0)$. The ellipse is the set of all points (x, y) such that the sum of the distances from (x, y) to the foci is constant, as shown in Figure 14.4.4.5.

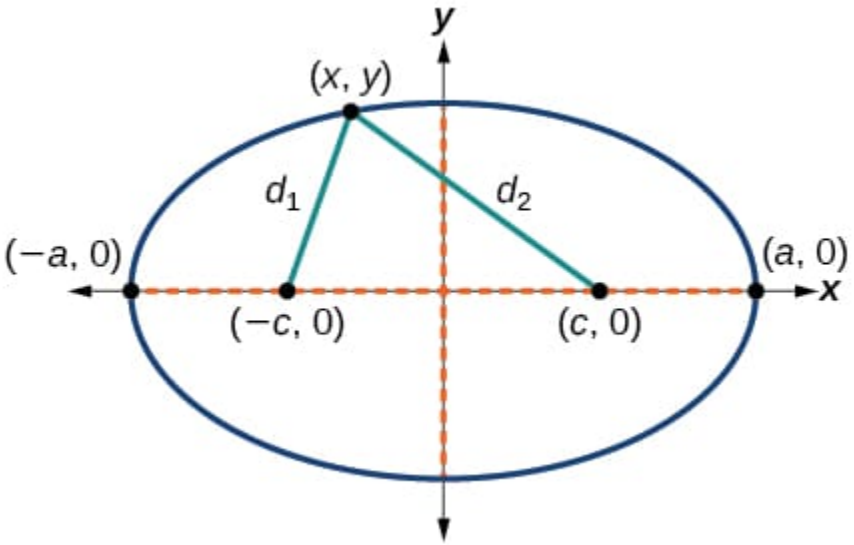


Figure 14.4.4.5: A horizontal ellipse centered at (0, 0) in the x-y coordinate system, with Vertices at (-a, 0) and (a, 0) and Foci at (-c, 0) and (c, 0). Lines of length d1 and d2 connect a point (x, y) on the ellipse to the two Foci.

If $(a,0)$ is a vertex of the ellipse, the distance from $(-c,0)$ to $(a,0)$ is $a-(-c)=a+c$. The distance from $(c,0)$ to $(a,0)$ is $a-c$. The sum of the distances from the foci to the vertex is

$$(a+c)+(a-c)=2a$$

If (x,y) is a point on the ellipse, then we can define the following variables:

- $d_1=$ the distance from $(-c,0)$ to (x,y)
- $d_2=$ the distance from $(c,0)$ to (x,y)

By the definition of an ellipse, d_1+d_2 is constant for any point (x,y) on the ellipse. We know that the sum of these distances is $2a$ for the vertex $(a,0)$. It follows that $d_1+d_2=2a$ for any point on the ellipse. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$d_1+d_2=2a \tag{14.4.4.1}$$

Now using the distance formula

$$\sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a \tag{14.4.4.2}$$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a \tag{14.4.4.3}$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2} \tag{14.4.4.4}$$

$$(x+c)^2+y^2=\left[2a-\sqrt{(x-c)^2+y^2}\right]^2 \tag{14.4.4.5}$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \tag{14.4.4.6}$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2 \tag{14.4.4.7}$$

$$2cx=4a^2-4a\sqrt{(x-c)^2+y^2}-2cx \tag{14.4.4.8}$$

Isolate the radical as you would normally do in these type of problems

$$4cx-4a^2=-4a\sqrt{(x-c)^2+y^2} \tag{14.4.4.9}$$

$$cx-a^2=-a\sqrt{(x-c)^2+y^2} \tag{14.4.4.10}$$

Square both sides

$$\left[cx-a^2\right]^2=a^2\left[\sqrt{(x-c)^2+y^2}\right]^2 \tag{14.4.4.11}$$

$$c^2x^2-2a^2cx+a^4=a^2(x^2-2cx+c^2+y^2) \tag{14.4.4.12}$$

$$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2 \tag{14.4.4.13}$$

$$a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2 \tag{14.4.4.14}$$

$$x^2(a^2-c^2)+a^2y^2=a^2(a^2-c^2) \tag{14.4.4.15}$$

Set $b^2=a^2-c^2$ to simplify

$$x^2b^2+a^2y^2=a^2b^2 \tag{14.4.4.16}$$

$$\frac{x^2b^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2} \tag{14.4.4.17}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{14.4.4.18}$$

Thus, the standard equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.This equation defines an ellipse centered at the origin. If $a>b$,the ellipse is stretched further in the horizontal direction, and if $b>a$, the ellipse is stretched further in the vertical direction.

Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the **ellipse** are its center, **vertices**, **co-vertices**, **foci**, and lengths and positions of the **major and minor axes**. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(0,0)$

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the x-*axis* is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{14.4.4.19}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a,0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0,\pm b)$
- the coordinates of the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. See Figure $14.4.4.6a$.

The standard form of the equation of an ellipse with center $(0,0)$ and major axis on the y-*axis* is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \tag{14.4.4.20}$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(0,\pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(\pm b,0)$
- the coordinates of the foci are $(0,\pm c)$, where $c^2=a^2-b^2$. See Figure $14.4.4.6b$.

Note that the vertices, co-vertices, and foci are related by the equation $c^2=a^2-b^2$. When we are given the coordinates of the foci and vertices of an ellipse, we can use this relationship to find the equation of the ellipse in standard form.

How to: Given the vertices and foci of an ellipse centered at the origin, write its equation in standard form

1. Determine whether the major axis lies on the *x*- or *y*-axis.

- If the given coordinates of the vertices and foci have the form $(\pm a, 0)$ and $(\pm c, 0)$ respectively, then the major axis is the *x*-axis. Use the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(\pm c, 0)$,respectively, then the major axis is the *y*-axis. Use the standard form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

2. Use the equation $c^2 = a^2 - b^2$, along with the given coordinates of the vertices and foci, to solve for b^2.
3. Substitute the values for a^2 and b^2 into the standard form of the equation determined in Step 1.

Example 14.4.4.1: Writing the Equation of an Ellipse Centered at the Origin in Standard Form

What is the standard form equation of the ellipse that has vertices $(\pm 8, 0)$ and foci $(\pm 5, 0)$?

Solution

The foci are on the x-axis, so the major axis is the x-axis. Thus, the equation will have the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The vertices are $(\pm 8, 0)$,so $a = 8$ and $a^2 = 64$.

The foci are $(\pm 5, 0)$,so $c = 5$ and $c^2 = 25$.

We know that the vertices and foci are related by the equation $c^2 = a^2 - b^2$. Solving for b^2, we have:

$$\begin{aligned} c^2 &= a^2 - b^2 \\ 25 &= 64 - b^2 \qquad \text{Substitute for } c^2 \text{ and } a^2 \\ b^2 &= 39 \qquad \text{Solve for } b^2 \end{aligned}$$

Now we need only substitute $a^2 = 64$ and $b^2 = 39$ into the standard form of the equation. The equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{39} = 1$.

Exercise 14.4.4.1

What is the standard form equation of the ellipse that has vertices $(0, \pm 4)$ and foci $(0, \pm\sqrt{15})$?

Answer available at the end of this section.

Q&A

Can we write the equation of an ellipse centered at the origin given coordinates of just one focus and vertex?

Yes. Ellipses are symmetrical, so the coordinates of the vertices of an ellipse centered around the origin will always have the form $(\pm a, 0)$ or $(0, \pm a)$. Similarly, the coordinates of the foci will always have the form $(\pm c, 0)$ or $(0, \pm c)$. Knowing this, we can use a and c from the given points, along with the equation $c^2 = a^2 - b^2$, to find b^2.

Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an **ellipse** can be translated. If an ellipse is translated h units horizontally and k units vertically, the center of the ellipse will be (h, k). This **translation** results in the standard form of the equation we saw previously, with x replaced by $(x - h)$ and *y* replaced by $(y - k)$.

STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER (H, K)

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the x-axis is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \tag{14.4.4.21}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$,where $c^2 = a^2 - b^2$. See Figure $14.4.4.7a$.

The standard form of the equation of an ellipse with center (h, k) and major axis parallel to the y-axis is

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \tag{14.4.4.22}$$

where

- $a > b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^2 = a^2 - b^2$. See Figure $14.4.4.7b$.

Just as with ellipses centered at the origin, ellipses that are centered at a point (h, k) have vertices, co-vertices, and foci that are related by the equation $c^2 = a^2 - b^2$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

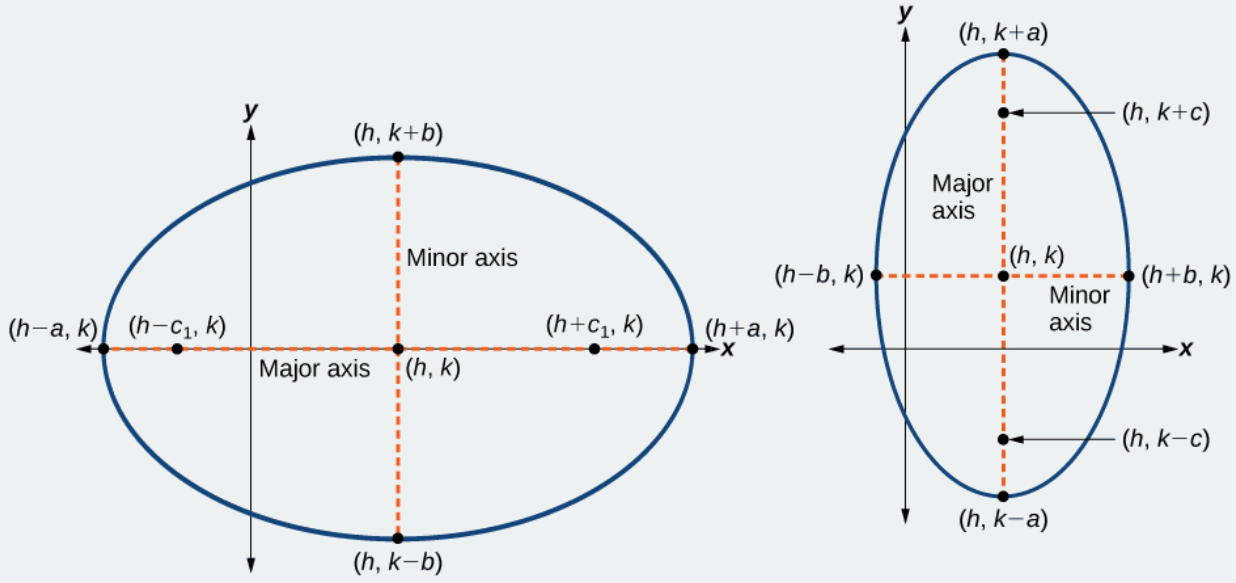


Figure $14.4.4.7$: (a) Horizontal ellipse with center (h, k) (b) Vertical ellipse with center (h, k)

How to: Given the vertices and foci of an ellipse not centered at the origin, write its equation in standard form

1. Determine whether the major axis is parallel to the x- or y-axis.
   - If the *y*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the x-axis. Use the standard form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
   - If the *x*-coordinates of the given vertices and foci are the same, then the major axis is parallel to the *y*-axis. Use the standard form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
2. Identify the center of the ellipse (h,k) using the midpoint formula and the given coordinates for the vertices.
3. Find a^2 by solving for the length of the major axis, $2a$, which is the distance between the given vertices.
4. Find c^2 using h and k, found in Step 2, along with the given coordinates for the foci.
5. Solve for b^2 using the equation $c^2=a^2-b^2$.
6. Substitute the values for h, k, a^2, and b^2 into the standard form of the equation determined in Step 1.

Example 14.4.4.2: Writing the Equation of an Ellipse Centered at a Point Other Than the Origin

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$?

Solution

The x-coordinates of the vertices and foci are the same, so the major axis is parallel to the y-axis. Thus, the equation of the ellipse will have the form

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

First, we identify the center, (h,k). The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$. Applying the midpoint formula, we have:

$$\begin{aligned}(h,k)&=\left(\frac{-2+(-2)}{2},\frac{-8+2}{2}\right)\\&=(-2,-3)\end{aligned}$$

Next, we find a^2. The length of the major axis, $2a$, is bounded by the vertices. We solve for a by finding the distance between the *y*-coordinates of the vertices.

$$\begin{aligned}2a&=2-(-8)\\2a&=10\\a&=5\end{aligned}$$

So $a^2=25$.

Now we find c^2. The foci are given by $(h,k\pm c)$. So, $(h,k-c)=(-2,-7)$ and $(h,k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for c.

$$\begin{aligned}k+c&=1\\-3+c&=1\\c&=4\end{aligned}$$

So $c^2=16$.

Next, we solve for b^2 using the equation $c^2=a^2-b^2$.

$$c^2 = a^2 - b^2$$
$$16 = 25 - b^2$$
$$b^2 = 9$$

Finally, we substitute the values found for h, k, a^2, and b^2 into the standard form equation for an ellipse:

$$\frac{(x+2)^2}{9} + \frac{(y+3)^2}{25} = 1$$

Exercise 14.4.4.2

What is the standard form equation of the ellipse that has vertices $(-3, 3)$ and $(5, 3)$ and foci $(1 - 2\sqrt{3}, 3)$ and $(1 + 2\sqrt{3}, 3)$?

Answer available at the end of this section.

Graphing Ellipses Centered at the Origin

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b \text{ for horizontal ellipses}$$

and

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b \text{ for vertical ellipses}$$

How to: Given the standard form of an equation for an ellipse centered at $(0, 0)$, sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, then
     - the major axis is the x-axis
     - the coordinates of the vertices are $(\pm a, 0)$
     - the coordinates of the co-vertices are $(0, \pm b)$
     - the coordinates of the foci are $(\pm c, 0)$
   - If the equation is in the form $x^2b^2 + y^2a^2 = 1$,where $a > b$, then
     - the major axis is the y-axis
     - the coordinates of the vertices are $(0, \pm a)$
     - the coordinates of the co-vertices are $(\pm b, 0)$
     - the coordinates of the foci are $(0, \pm c)$
2. Solve for c using the equation $c^2 = a^2 - b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Example 14.4.4.3: Graphing an Ellipse Centered at the Origin

Graph the ellipse given by the equation, $\frac{x^2}{9} + \frac{y^2}{25} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $25 > 9$,the major axis is on the y-axis. Therefore, the equation is in the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$,where $b^2 = 9$ and $a^2 = 25$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(0, \pm a) = (0, \pm\sqrt{25}) = (0, \pm 5)$
- the coordinates of the co-vertices are $(\pm b, 0) = (\pm 9, 0) = (\pm 3, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^2 = a^2 - b^2$ Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{25 - 9} \\ &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse. See Figure 14.4.4.8.

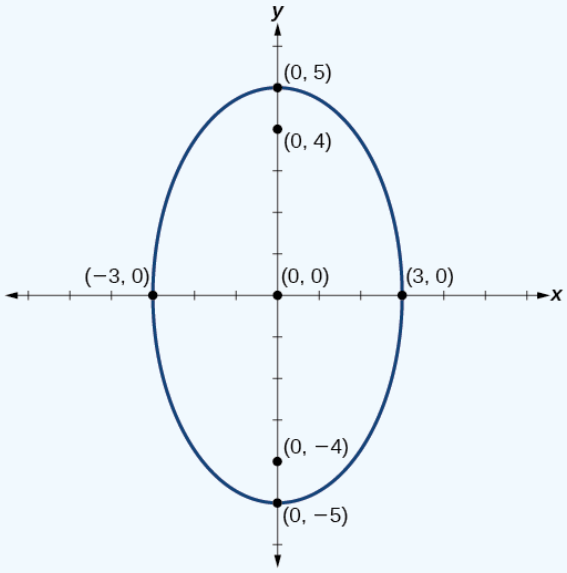


Figure 14.4.4.8: A vertical ellipse centered at (0, 0) in the x-y coordinate system, with vertices at (0,5) and (0,-5), co-vertices at (3, 0) and (negative 3, 0), and foci at (0, 4) and (0,-4).

Exercise 14.4.4.3

Graph the ellipse given by the equation $\frac{x^2}{36} + \frac{y^2}{4} = 1$. Identify and label the center, vertices, co-vertices, and foci.

Answer available at the end of this section.

Example 14.4.4.4: Graphing an Ellipse Centered at the Origin from an Equation Not in Standard Form

Graph the ellipse given by the equation $4x^2 + 25y^2 = 100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Solution

First, use algebra to rewrite the equation in standard form.

$$4x^2 + 25y^2 = 100$$
$$\frac{4x^2}{100} + \frac{25y^2}{100} = \frac{100}{100}$$
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Next, we determine the position of the major axis. Because $25 > 4$, the major axis is on the x-axis. Therefore, the equation is in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(0, 0)$
- the coordinates of the vertices are $(\pm a, 0) = (\pm\sqrt{25}, 0) = (\pm 5, 0)$
- the coordinates of the co-vertices are $(0, \pm b) = (0, \pm\sqrt{4}) = (0, \pm 2)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$c = \pm\sqrt{a^2 - b^2}$$
$$= \pm\sqrt{25 - 4}$$
$$= \pm\sqrt{21}$$

Therefore the coordinates of the foci are $(\pm\sqrt{21}, 0)$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

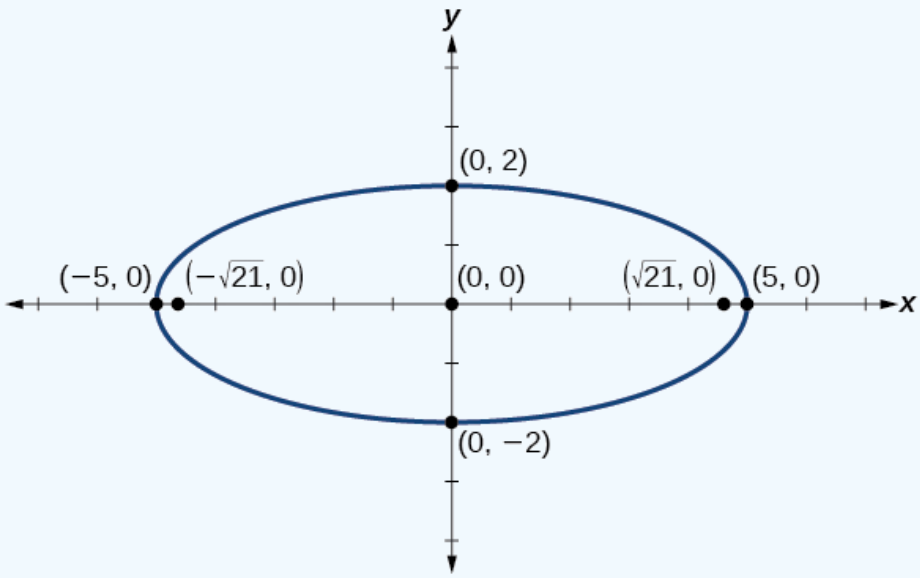


Figure 14.4.4.10: A horizontal ellipse centered at (0, 0) with vertices at (5, 0) and (-5, 0), co-vertices at (0, 2) and (0, -2), and foci at $(\sqrt{21}, 0)$ and $(-\sqrt{21}, 0)$.

Exercise 14.4.4.4

Graph the ellipse given by the equation $49x^2 + 16y^2 = 784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.

Answer available at the end of this section.

Graphing Ellipses Not Centered at the Origin

When an **ellipse** is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, (h, k),we use the standard forms $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b$ for horizontal ellipses and

$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, $a>b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

How to: Given the standard form of an equation for an ellipse centered at (h,k), sketch the graph.

1. Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.
   - If the equation is in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a>b$, then
     - the center is (h,k)
     - the major axis is parallel to the x-axis
     - the coordinates of the vertices are $(h\pm a,k)$
     - the coordinates of the co-vertices are $(h,k\pm b)$
     - the coordinates of the foci are $(h\pm c,k)$
   - If the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $a>b$, then
     - the center is (h,k)
     - the major axis is parallel to the y-axis
     - the coordinates of the vertices are $(h,k\pm a)$
     - the coordinates of the co-vertices are $(h\pm b,k)$
     - the coordinates of the foci are $(h,k\pm c)$
2. Solve for c using the equation $c^2=a^2-b^2$.
3. Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.

Example 14.4.4.5: Graphing an Ellipse Centered at (h,k)

Graph the ellipse given by the equation, $\frac{(x+2)^2}{4}+\frac{(y-5)^2}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

Solution

First, we determine the position of the major axis. Because $9>4$, the major axis is parallel to the y-axis. Therefore, the equation is in the form $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$, where $b^2=4$ and $a^2=9$. It follows that:

- the center of the ellipse is $(h,k)=(-2,5)$
- the coordinates of the vertices are $(h,k\pm a)=(-2,5\pm\sqrt{9})=(-2,5\pm3)$, or $(-2,2)$ and $(-2,8)$
- the coordinates of the co-vertices are $(h\pm b,k)=(-2\pm\sqrt{4},5)=(-2\pm2,5)$, or $(-4,5)$ and $(0,5)$
- the coordinates of the foci are $(h,k\pm c)$, where $c^2=a^2-b^2$. Solving for c,we have:

$$\begin{aligned}c&=\pm\sqrt{a^2-b^2}\\&=\pm\sqrt{9-4}\\&=\pm\sqrt{5}\end{aligned}$$

Therefore, the coordinates of the foci are $(-2,5-\sqrt{5})$ and $(-2,5+\sqrt{5})$.

Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.

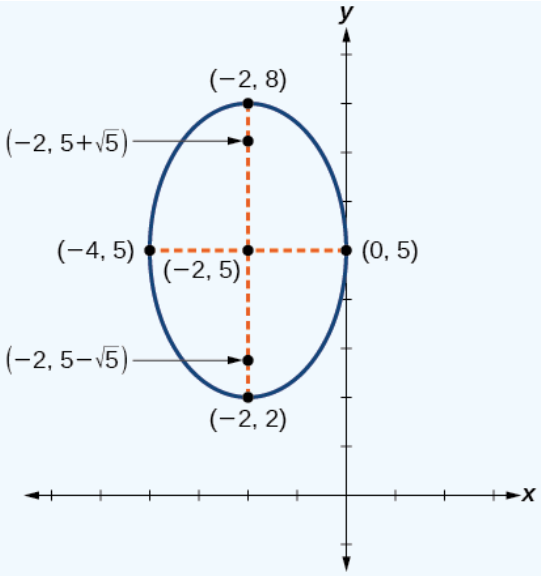


Figure 14.4.4.12: A vertical ellipse centered at (negative 2, 5) with vertices at (-2, 2) and (-2, 8), co-vertices at (0, 5) and (-4, 5), and foci at $(-2, 5+\sqrt{5})$ and $(2, 5-\sqrt{5})$. The Major and Minor Axes, connecting the Vertices and Co-Vertices respectively, are shown.

Exercise 14.4.4.5

Graph the ellipse given by the equation $\frac{(x-4)^2}{36}+\frac{(y-2)^2}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Answer available at the end of this section.

How to: Given the general form of an equation for an ellipse centered at (h, k), express the equation in standard form.

1. Recognize that an ellipse described by an equation in the form $ax^2+by^2+cx+dy+e=0$ is in general form.
2. Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
3. Factor out the coefficients of the x^2 and y^2 terms in preparation for completing the square.
4. Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_1(x-h)^2+m_2(y-k)^2=m_3$, where m_1, m_2,and m_3 are constants.
5. Divide both sides of the equation by the constant term to express the equation in standard form.

Example 14.4.4.6: Graphing an Ellipse Centered at (h, k) by First Writing It in Standard Form

Graph the ellipse given by the equation $4x^2+9y^2-40x+36y+100=0$. Identify and label the center, vertices, co-vertices, and foci.

Solution

We must begin by rewriting the equation in standard form.

$$4x^2+9y^2-40x+36y+100=0$$

Group terms that contain the same variable, and move the constant to the opposite side of the equation.

$$(4x^2-40x)+(9y^2+36y)=-100$$

Factor out the coefficients of the squared terms.

$$4(x^2 - 10x) + 9(y^2 + 4y) = -100$$

Complete the square twice. Remember to balance the equation by adding the same constants to each side.

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 100 + 36$$

Rewrite as perfect squares.

$$4(x - 5)^2 + 9(y + 2)^2 = 36$$

Divide both sides by the constant term to place the equation in standard form.

$$\frac{(x - 5)^2}{9} + \frac{(y + 2)^2}{4} = 1$$

Now that the equation is in standard form, we can determine the position of the major axis. Because $9 > 4$, the major axis is parallel to the x-axis. Therefore, the equation is in the form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 4$. It follows that:

- the center of the ellipse is $(h, k) = (5, -2)$
- the coordinates of the vertices are $(h \pm a, k) = (5 \pm \sqrt{9}, -2) = (5 \pm 3, -2)$, or $(2, -2)$ and $(8, -2)$
- the coordinates of the co-vertices are $(h, k \pm b) = (5, -2 \pm \sqrt{4}) = (5, -2 \pm 2)$, or $(5, -4)$ and $(5, 0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$. Solving for c, we have:

$$\begin{aligned} c &= \pm\sqrt{a^2 - b^2} \\ &= \pm\sqrt{9 - 4} \\ &= \pm\sqrt{5} \end{aligned}$$

Therefore, the coordinates of the foci are $(5 - \sqrt{5}, -2)$ and $(5 + \sqrt{5}, -2)$.

Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse as shown in Figure 14.4.4.14.

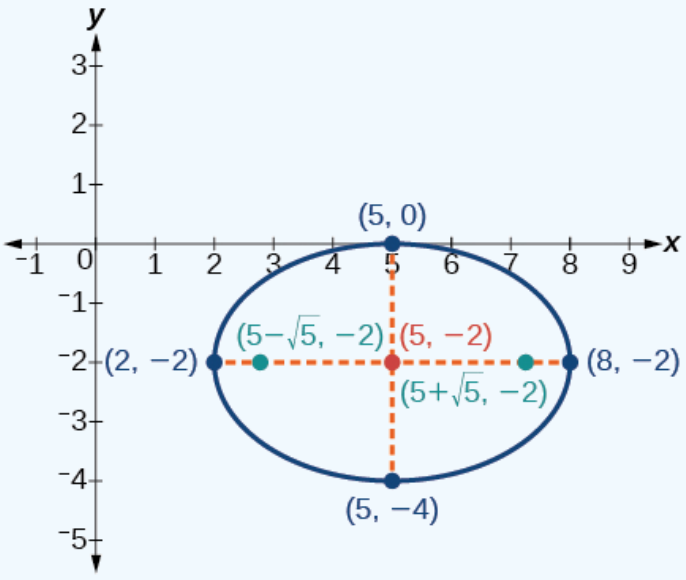


Figure 14.4.4.14: A horizontal ellipse centered at (5, -2) with vertices at (2, -2) and (8, -2), co-vertices at (5, 0) and (5, -4), and foci at $5 + \sqrt{5}, -2)$ and $(5 - \sqrt{5}, -2)$. The Major and Minor Axes, connecting the Vertices and Co-Vertices respectively, are shown.

Exercise 14.4.4.6

Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.

$$4x^2 + y^2 - 24x + 2y + 21 = 0$$

Answer available at the end of this section.

Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus (Figure 14.4.4.15). In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci—about 43 feet apart—can hear each other whisper.

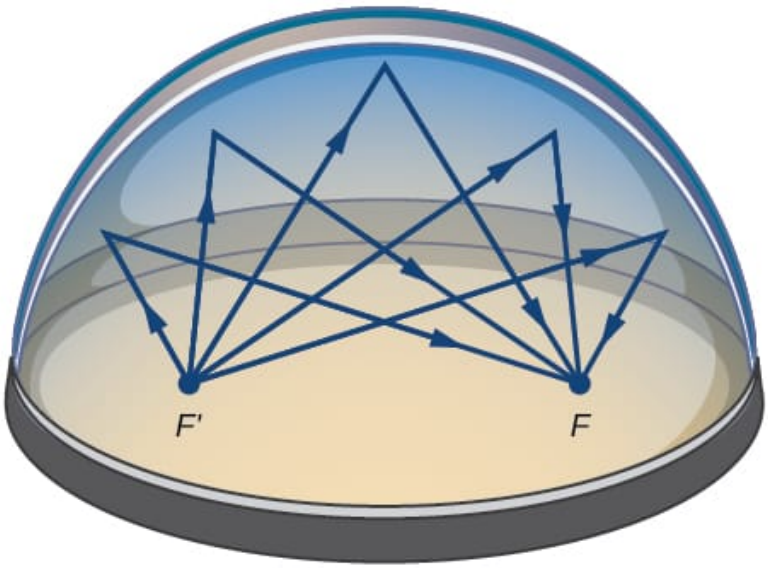


Figure 14.4.4.15: Sound waves are reflected between foci in an elliptical room, called a whispering chamber.

Example 14.4.4.7: Locating the Foci of a Whispering Chamber

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long as shown in Figure 14.4.4.16.

a. What is the standard form of the equation of the ellipse representing the outline of the room? ***Hint:*** assume a horizontal ellipse, and let the center of the room be the point $(0, 0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.

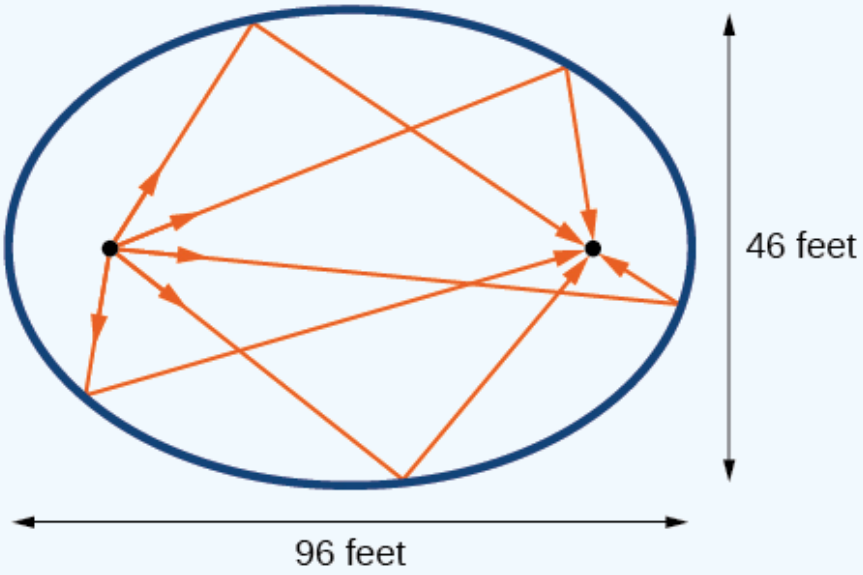


Figure 14.4.4.16: A horizontal ellipse with rays originating at each focus and going through the other focus.

Solution

a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b$. We know that the length of the major axis, $2a$, is longer than the length of the minor axis, $2b$. So the length of the room, 96, is represented by the major axis, and the width of the room, 46, is represented by the minor axis.

   Therefore, the equation of the ellipse is $\frac{x^2}{2304}+\frac{y^2}{529}=1$

   - Solving for a, we have $2a=96$, so $a=48$, and $a^2=2304$.
   - Solving for b, we have $2b=46$, so $b=23$, and $b^2=529$.

b. To find the distance between the senators, we must find the distance between the foci, $(\pm c,0)$, where $c^2=a^2-b^2$. Solving for c,we have:

$$\begin{aligned} c^2 &= a^2-b^2 \\ c^2 &= 2304-529 \quad \text{Substitute using the values found in part } (a) \\ c &= \pm\sqrt{2304-529} \quad \text{Take the square root of both sides.} \\ c &= \pm\sqrt{1775} \quad \text{Subtract.} \\ c &\approx \pm 42 \quad \text{Round to the nearest foot.} \end{aligned}$$

The points $(\pm 42,0)$ represent the foci. Thus, the distance between the senators is $2(42)=84$ feet.

Exercise 14.4.4.7

Suppose a whispering chamber is 480 feet long and 320 feet wide.

a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

Answer a available at the end of this section.

Answer b available at the end of this section.

Media

Access these online resources for additional instruction and practice with ellipses.

- Conic Sections: The Ellipse
- Graph an Ellipse with Center at the Origin
- Graph an Ellipse with Center Not at the Origin

Key Equations

Horizontal ellipse, center at origin	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$
Vertical ellipse, center at origin	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1,\ a>b$
Horizontal ellipse, center (h,k)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b$
Vertical ellipse, center (h,k)	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b$

Key Concepts

- An ellipse is the set of all points (x,y) in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form. See Example $14.4.4.1$ and Example $14.4.4.2$.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse. See Example $14.4.4.3$ and Example $14.4.4.4$.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse. See Example $14.4.4.5$ and Example $14.4.4.6$.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci. See Example $14.4.4.7$.

Contributors and Attributions

- Template:ContribOpenStaxPreCalc

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[1]Other whispering chambers, also known as whispering galleries, include the Mapparium in Boston (which is also a interesting map of the world circa 1934) and St. Louis Union Station Hotel (where it is known as a whispering arch).

Answers

Exercise $14.4.4.1$

$x^2+\frac{y^2}{16}=1$

Exercise $14.4.4.2$

$\frac{(x-1)^2}{16}+\frac{(y-3)^2}{4}=1$

Exercise $14.4.4.3$

center: $(0,0)$; vertices: $(\pm 6,0)$; co-vertices: $(0,\pm 2)$; foci: $(\pm 4\sqrt{2},0)$

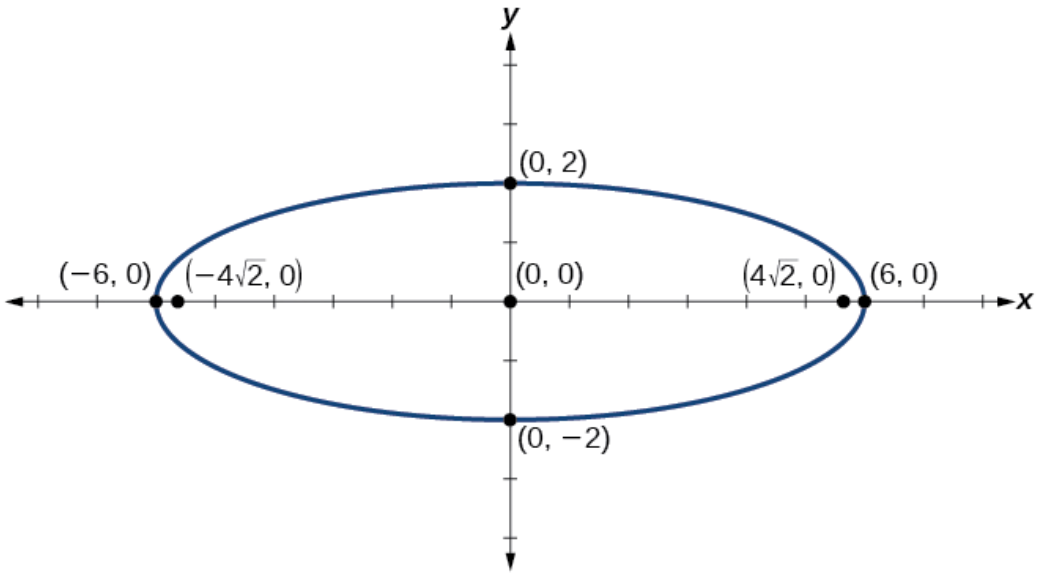


Figure 14.4.4.9: A horizontal ellipse centered at (0, 0) in the x y coordinate system, with vertices at (-6,0) and (6,0), co-vertices at (0, 2) and (0, -2), and foci at $(-4\sqrt{2}, 0)$ and $(4\sqrt{2}, 0)$.

Exercise 14.4.4.4

Standard form: $\frac{x^2}{16} + \frac{y^2}{49} = 1$; center: $(0, 0)$; vertices: $(0, \pm 7)$; co-vertices: $(\pm 4, 0)$; foci: $(0, \pm\sqrt{33})$

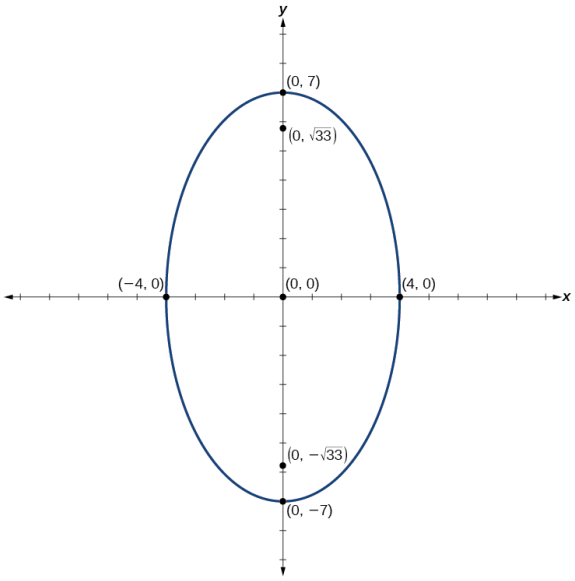


Figure 14.4.4.11: A vertical ellipse centered at (0, 0) in the x-y coordinate system, with vertices at (0, 7) and (0, -7), co-vertices at (4, 0) and (-4, 0), and foci at $(0, \sqrt{33})$ and $(0, -\sqrt{33})$.

Exercise 14.4.4.5

Center: $(4, 2)$; vertices: $(-2, 2)$ and $(10, 2)$; co-vertices: $(4, 2 - 2\sqrt{5})$ and $(4, 2 + 2\sqrt{5})$; foci: $(0, 2)$ and $(8, 2)$

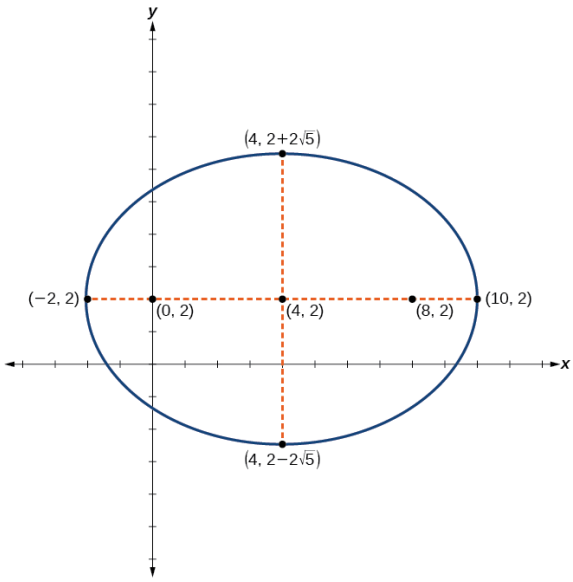


Figure 14.4.4.13: A horizontal ellipse centered at (4, 2) with vertices at (-2, 2) and (10, 2), co-vertices at $(4, 2+2\sqrt{5})$ and $(4, 2-2\sqrt{5})$, and foci at (0, 2) and (8, 2).

Exercise 14.4.4.6

$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{16}=1$; center: $(3,-1)$; vertices: $(3,-5)$ and $(3,3)$; co-vertices: $(1,-1)$ and $(5,-1)$; foci: $(3,-1-2\sqrt{3})$ and $(3,-1+2\sqrt{3})$

Exercise 14.4.4.7

$\frac{x^2}{57,600}+\frac{y^2}{25,600}=1$

Exercise 14.4.4.7

The people are standing 358 feet apart.

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- 15.7: Basic data modeling by Chuck Severance is licensed CC BY-NC-SA 4.0. Original source: https://www.py4e.com/html3.

14.4.5: Hyperbolic Functions

Learning Objectives

- Apply the formulas for derivatives and integrals of the hyperbolic functions.
- Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
- Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions previously, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine[1] and hyperbolic cosine[1] are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{14.4.5.1}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{14.4.5.2}$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in Figure $14.4.5.1$.

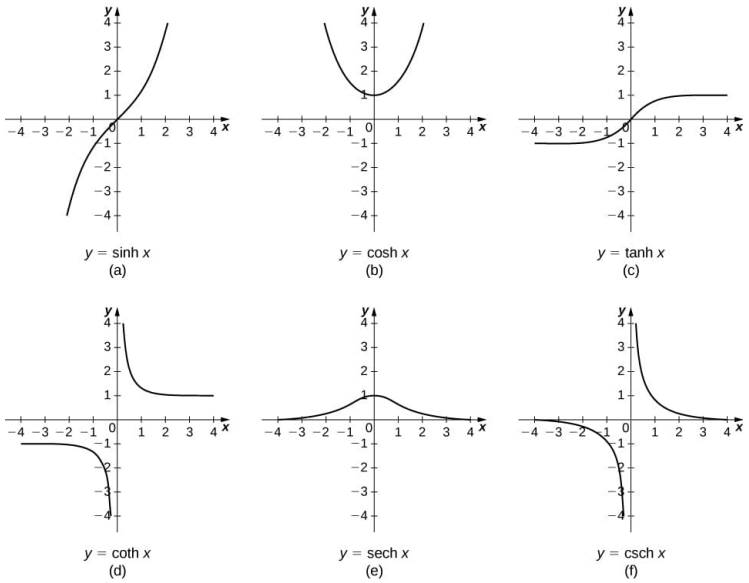


Figure $14.4.5.1$: Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] \\ &= \cosh x.\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x. \tag{14.4.5.3}$$

We summarize the differentiation formulas for the hyperbolic functions in Table 14.4.5.1.

Table 14.4.5.1*: Derivatives of the Hyperbolic Functions*

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match:

$$\frac{d}{dx}\sin x = \cos x \tag{14.4.5.4}$$

and

$$\frac{d}{dx}\sinh x = \cosh x.$$

The derivatives of the cosine functions, however, differ in sign:

$$\frac{d}{dx}\cos x = -\sin x,$$

but

$$\frac{d}{dx}\cosh x = \sinh x.$$

As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions. These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\int \sinh u\, du = \cosh u + C \tag{14.4.5.5}$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \tag{14.4.5.6}$$

$$\int \cosh u\, du = \sinh u + C \tag{14.4.5.7}$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech}\ u + C - \operatorname{csch} u + C \tag{14.4.5.8}$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C \tag{14.4.5.9}$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C \tag{14.4.5.10}$$

Example 14.4.5.1: Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}(\sinh(x^2))$
b. $\frac{d}{dx}(\cosh x)^2$

Solution:

Using the formulas in Table 14.4.5.1 and the chain rule, we get

a. $\frac{d}{dx}(\sinh(x^2)) = \cosh(x^2) \cdot 2x$
b. $\frac{d}{dx}(\cosh x)^2 = 2\cosh x \sinh x$

Exercise 14.4.5.1

Evaluate the following derivatives:

a. $\frac{d}{dx}(\tanh(x^2 + 3x))$
b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Hint available at the end of this section.

Example 14.4.5.2: Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x\cosh(x^2)dx$
b. $\int \tanh x\ dx$

Solution

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\ dx$ and

$$\int x\cosh(x^2)dx = \int \frac{1}{2}\cosh u\, du$$
$$= \frac{1}{2}\sinh u + C$$
$$= \frac{1}{2}\sinh(x^2) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x\ dx$ and

$$\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x}\, dx$$
$$= \int \frac{1}{u} du$$
$$= \ln|u| + C$$
$$= \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain

$$\int \tanh x\, dx = \ln(\cosh x) + C.$$

Exercise 14.4.5.2

Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x\ dx$

b. $\int \text{sech}^{\,2}(3x)\ dx$

Hint available at the end of this section.

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in Table 14.4.5.2.

Table 14.4.5.2: Domains and Ranges of the Inverse Hyperbolic Functions

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\cosh^{-1} x$	$(1,\infty)$	$[0,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\coth^{-1} x$	$(-\infty,1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\text{sech}^{-1} x$	$(0,1)$	$[0,\infty)$
$\text{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

The graphs of the inverse hyperbolic functions are shown in the following figure.

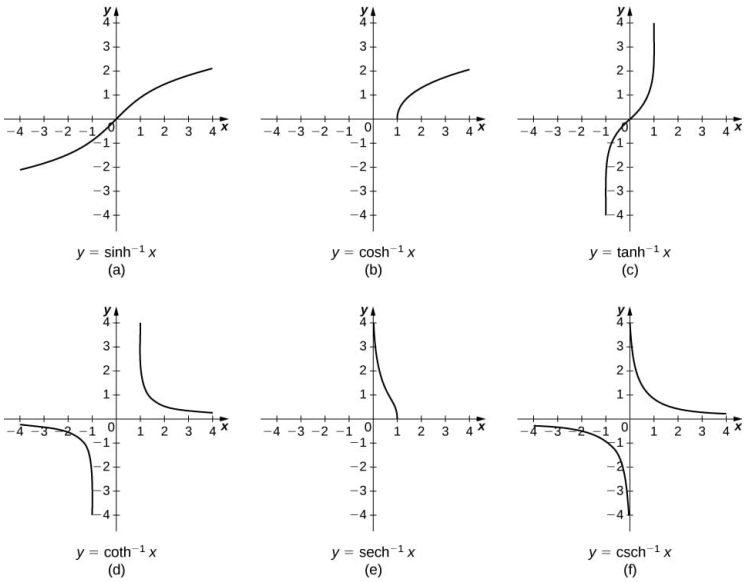


Figure 14.4.5.3: Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$y = \sinh^{-1} x \tag{14.4.5.11}$$

$$\sinh y = x \tag{14.4.5.12}$$

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \tag{14.4.5.13}$$

$$\cosh y\frac{dy}{dx} = 1. \tag{14.4.5.14}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$.Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}. \tag{14.4.5.15}$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in Table 14.4.5.3.

Table 14.4.5.3: Derivatives of the Inverse Hyperbolic Functions

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$
$\text{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\text{csch}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/(1 - x^2)$, we need to select the proper

antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1} u + C \tag{14.4.5.16}$$

$$\int \frac{1}{u\sqrt{1-u^2}}du = -\text{sech}^{-1}|u| + C \tag{14.4.5.17}$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1} u + C \tag{14.4.5.18}$$

$$\int \frac{1}{u\sqrt{1+u^2}}du = -\text{csch}^{-1}|u| + C \tag{14.4.5.19}$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1} u + C & \text{if } |u| < 1 \\ \coth^{-1} u + C & \text{if } |u| > 1 \end{cases} \tag{14.4.5.20}$$

Example 14.4.5.3: Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$
b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

Solution

Using the formulas in Table 14.4.5.3 and the chain rule, we obtain the following results:

a. $\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$
b. $\frac{d}{dx}(\tanh^{-1} x)^2 = \frac{2(\tanh^{-1} x)}{1-x^2}$

Exercise 14.4.5.3

Evaluate the following derivatives:

a. $\frac{d}{dx}(\cosh^{-1}(3x))$
b. $\frac{d}{dx}(\coth^{-1} x)^3$

Hint available at the end of this section.

Example 14.4.5.4: Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$
b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution

We can use u**-substitution** in both cases.

Let $u = 2x$. Then, $du = 2\ dx$ and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,dx &= \int \frac{1}{2\sqrt{u^2-1}}\,du \\ &= \frac{1}{2}\cosh^{-1} u + C \\ &= \frac{1}{2}\cosh^{-1}(2x) + C.\end{aligned}$$

Let $u = 3x$. Then, $du = 3\ dx$ and we obtain

$$\begin{aligned}\int \frac{1}{2x\sqrt{1-9x^2}}dx &= \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|u| + C \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C\end{aligned}$$

Exercise 14.4.5.4

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Hint available at the end of this section.

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.



Figure 14.4.5.3: Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a \cdot \cosh(x/a)$ are catenaries. Figure 14.4.5.4 shows the graph of $y = 2\cosh(x/2)$.

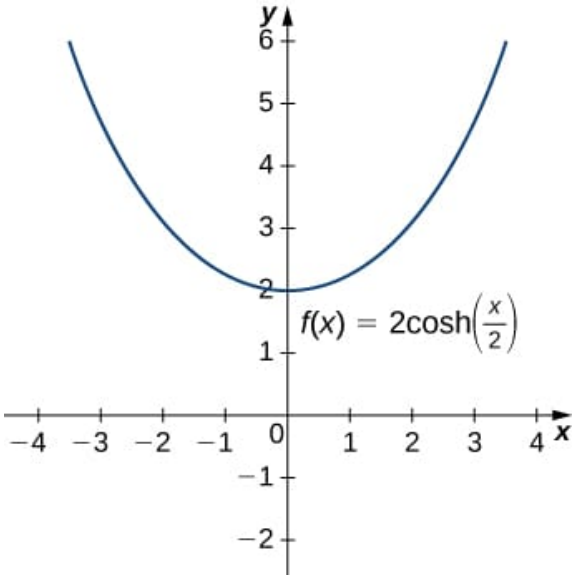


Figure 14.4.5.4: A hyperbolic cosine function forms the shape of a catenary.

Example 14.4.5.5: Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 6.4 that the formula for arc length is

$$\underbrace{\int_a^b \sqrt{1 + [f'(x)]^2}\,dx}_{\text{Arc Length}}.$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then the arc length is

$$\int_a^b \sqrt{1+[f'(x)]^2}dx = \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}dx.$$

Now recall that

$$1+\sinh^2 x = \cosh^2 x,$$

so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right)dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} \\ &= 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] \\ &= 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586 \text{ ft}.\end{aligned}$$

Exercise 14.4.5.5:

Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

Answer available at the end of this section.

Key Concepts

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

Glossary

catenary
A curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary.

Contributors and Attributions

- Template:ContribOpenStaxCalc

---

[1]Note how this differs from our definition sine and cosine where here there is a reference to complex numbers. Recall $sin(\theta)$

$= \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

Answers

Exercise 14.4.5.1

Use the formulas in Table 14.4.5.1 and apply the chain rule as necessary.

$\frac{d}{dx}(\tanh(x^2+3x)) = (\operatorname{sech}^2(x^2+3x))(2x+3)$

$\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3}\cosh x$

Exercise 14.4.5.2

Use the formulas above and apply u-substitution as necessary.

$\int \sinh^3 x \cosh x \; dx = \frac{\sinh^4 x}{4} + C$

$\int \operatorname{sech}^2(3x) \; dx = \frac{\tanh(3x)}{3} + C$

Exercise 14.4.5.3

Use the formulas in Table 14.4.5.3 and apply the chain rule as necessary.

$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$

$\frac{d}{dx}(\coth^{-1} x)^3 = \frac{3(\coth^{-1} x)^2}{1-x^2}$

Exercise 14.4.5.4

Use the formulas above and apply u-substitution as necessary.

$\int \frac{1}{\sqrt{x^2-4}} dx = \cosh^{-1}(\frac{x}{2}) + C$

$\int \frac{1}{\sqrt{1-e^{2x}}} dx = -\operatorname{sech}^{-1}(e^x) + C$

Exercise 14.4.5.5**:**

52.95 ft

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14.4.6: Coordinate Systems

Cartesian Coordinate System

While the **rectangular** (also called **Cartesian**) coordinate system is the most common, certain problems are easier to analyze in alternate coordinate systems. A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangular "box."

List of some other Coordinate Systems

The choice of a coordinate system depends on the shapes that are involved in the problem at hand. There are many coordinate systems[1] including the Cartesian coordinate system, the cylindrical coordinate system, the spherical coordinate system, the ellipsoidal coordinate system, the prolate spheroidal coordinate system, and the oblate spheroidal coordinate system. In addition thee are coordinate systems that rather than focusing on shape are adapted to the subject. Examples of coordinate systems of this type are astronomical coordinate systems (based off spherical coordinate system), canonical coordinate system (mechanics: phase space), and Schwarzschild coordinate systems (relativistic: akin to a modified spherical coordinate system).

Cylindrical Coordinate System

Here we will only discuss the main coordinate system the Cartesian coordinate system (which you already know), the cylindrical coordinate system, and the spherical coordinate system. First we start with the "2D cylindrical coordinate system" we call **polar coordinates.**

In two dimensions you may already be familiar with polar coordinates. In this system, each point in the plane is identified by a pair of numbers (r, θ). The number θ measures the angle between the positive x-axis and a vector with tail at the origin and head at the point, as shown in Figure $14.4.6.1$; the number r measures the distance from the origin to the point. Either of these may be negative; a negative θ indicates the angle is measured clockwise from the positive x-axis instead of counter-clockwise, and a negative r indicates the point at distance $|r|$ in the opposite of the direction given by θ. Figure $14.4.6.1$ also shows the point with rectangular coordinates $(1, \sqrt{3})$ and polar coordinates $(2, \pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

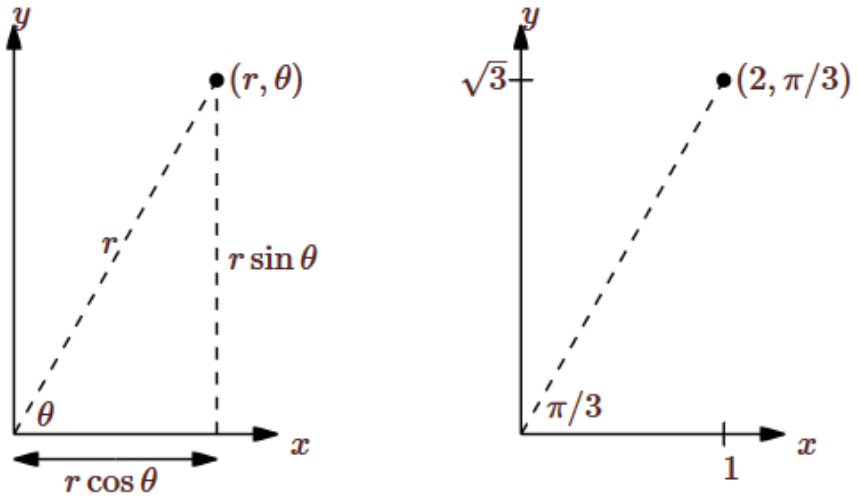


Figure $14.4.6.1$***:*** *Polar coordinates: the general case and the point with rectangular coordinates* $(1, \sqrt{3})$.

We can extend polar coordinates to three dimensions simply by adding a z coordinate; this is called **cylindrical coordinates**. Each point in three-dimensional space is represented by three coordinates (r, θ, z) in the obvious way: this point is z units above or below the point (r, θ) in the x-y plane, as shown in Figure 14.4.6.5. The point with rectangular coordinates $(1, \sqrt{3}, 3)$ and cylindrical coordinates $(2, \pi/3, 3)$ is also indicated in Figure 14.4.6.2.

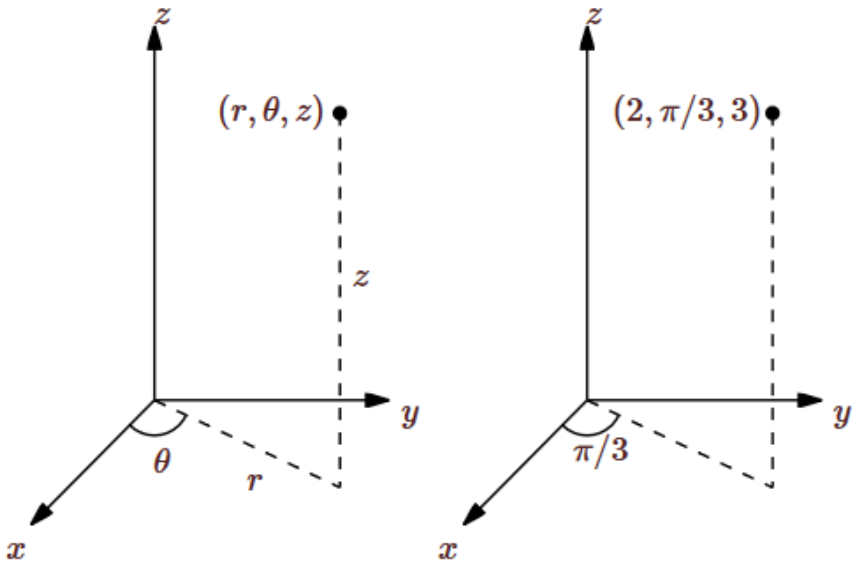


Figure 14.4.6.2***:*** *Cylindrical coordinates: the general case and the point with rectangular coordinates* $(1, \sqrt{3}, 3)$.

Some figures with relatively complicated equations in rectangular coordinates will be represented by simpler equations in cylindrical coordinates. For example, the cylinder in Figure 14.4.6.3 has equation $x^2 + y^2 = 4$ in rectangular coordinates, but equation $r = 2$ in cylindrical coordinates.

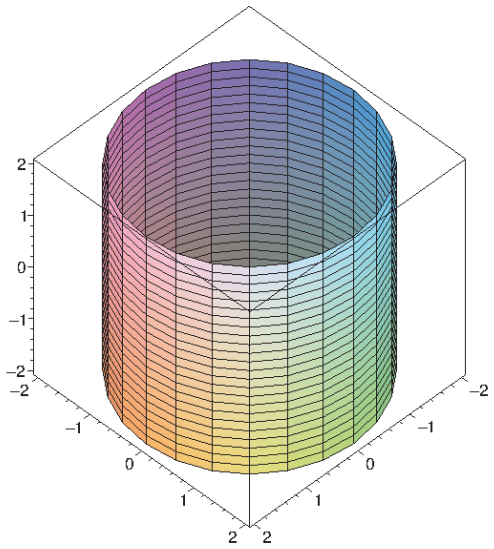


Figure 14.4.6.3***:*** *The cylinder* $r = 2$.

Given a point (r, θ) in polar coordinates, it is easy to see (Figure 14.4.6.1) that the rectangular coordinates of the same point are $(r\cos\theta, r\sin\theta)$, and so the point (r, θ, z) in cylindrical coordinates is $(r\cos\theta, r\sin\theta, z)$ in rectangular coordinates. This means it is usually easy to convert any equation from rectangular to cylindrical coordinates: simply substitute

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \qquad (14.4.6.1)$$

and leave z alone. For example, starting with $x^2 + y^2 = 4$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ gives

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 4 \\ r^2(\cos^2\theta + \sin^2\theta) &= 4 \\ r^2 &= 4 \\ r &= 2. \end{aligned} \tag{14.4.6.2}$$

Of course, it's easy to see directly that this defines a cylinder as mentioned above.

Let's try this with octave as a quick script example of generating what is above.

```
[rflux,thetaflux] = meshgrid(linspace(0,0.7,30),linspace(0,2*pi,30));
flux = abs(2-sqrt(2)*rflux);
[r,theta] = meshgrid(2.*ones(1,30),linspace(0,2*pi,30));
surf(r.*cos(theta),r.*sin(theta),flux);
```

Cylindrical coordinates are an obvious extension of polar coordinates to three dimensions, but the use of the z coordinate means they are not as closely analogous to polar coordinates as another standard coordinate system. In polar coordinates, we identify a point by a direction and distance from the origin; in three dimensions we can do the same thing, in a variety of ways. The question is: how do we represent a direction? One way is to give the angle of rotation, θ, from the positive x axis, just as in cylindrical coordinates, and also an angle of rotation, ϕ, from the positive z axis.

Spherical Coordinate System

Roughly speaking, θ is like longitude and ϕ is like latitude. (Earth longitude is measured as a positive or negative angle from the prime meridian, and is always between 0 and 180 degrees, east or west; θ can be any positive or negative angle, and we use radians except in informal circumstances. Earth latitude is measured north or south from the equator; ϕ is measured from the north pole down.) This system is called **spherical coordinates**; the coordinates are listed in the order (ρ, θ, ϕ), where ρ is the distance from the origin, and like r in cylindrical coordinates it may be negative. The general case and an example are pictured in Figure 14.4.6.4; the length marked r is the r of cylindrical coordinates.

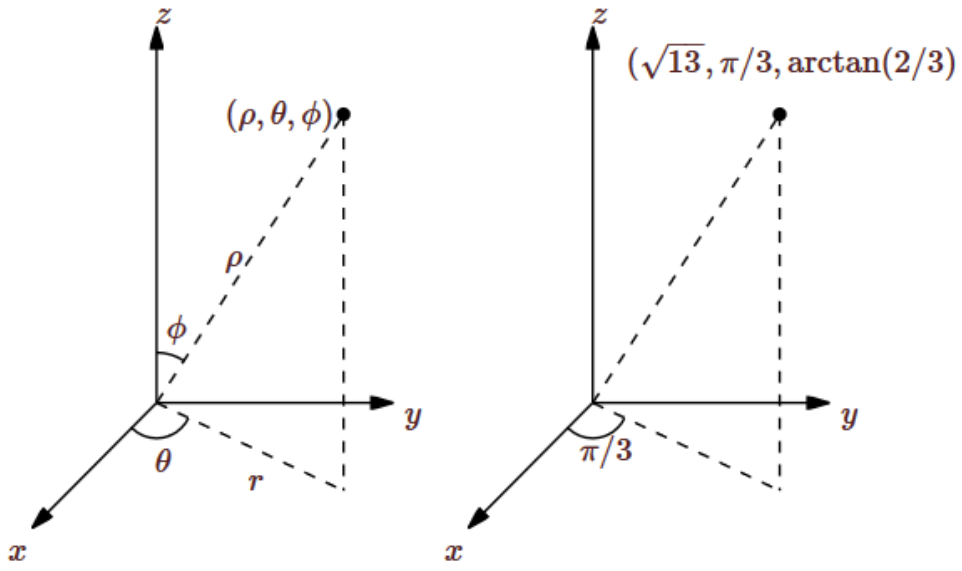


Figure 14.4.6.4***:*** *Spherical coordinates: the general case and the point with rectangular coordinates* $(1, \sqrt{3}, 3)$.Example 14.4.6.1

As the cylinder had a simple equation in cylindrical coordinates, so does the sphere in spherical coordinates: $\rho = 2$ is the sphere of radius 2.

Solution

If we start with the Cartesian equation of the sphere and substitute, we get the spherical equation:

$$\begin{aligned} x^2+y^2+z^2 &= 2^2 \\ \rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi &= 2^2 \\ \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)+\rho^2\cos^2\phi &= 2^2 \\ \rho^2\sin^2\phi+\rho^2\cos^2\phi &= 2^2 \\ \rho^2(\sin^2\phi+\cos^2\phi) &= 2^2 \\ \rho^2 &= 2^2 \\ \rho &= 2 \end{aligned}$$

Example 14.4.6.2

Find an equation for the cylinder $x^2+y^2=4$ in spherical coordinates.

Solution

Proceeding as in the previous example:

$$\begin{aligned} x^2+y^2 &= 4 \\ \rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta &= 4 \\ \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) &= 4 \\ \rho^2\sin^2\phi &= 4 \\ \rho\sin\phi &= 2 \\ \rho &= \frac{2}{\sin\phi} \end{aligned}$$

¬ So as an engineer when would you likely use the cylindrical coordinate system?

Since the cylindrical coordinate system would be best for cylindrical type objects, you would use this coordinate system for pipes (since in general pipes are cylindrical) and wires (electrical engineering). These two types of objects are ubiquitous in our society. Note you do not have to use a cylindrical coordinate system, it is just a matter of convenience. You can use any coordinate system you want. Of course it might make your problem unnecessarily difficult.

¬ So as a scientist when would you likely use a spherical coordinate system?

Since a spherical coordinate system would be best for spherical type objects it would be best to use this system for astronomy as the sky is replete with spherical objects (or close to spherical objects - you might want to use one of the slightly modified spherical coordinate systems for really detailed studies). Note engineers might use a spherical coordinate system for any ball like object, say ball bearings. Of course the use of said system is only to make problems easier. You can always use whatever coordinate system you want. Just be sure to be consistent.

How about a program to make a sphere using spherical coordinate ideas? Let us try it, eh?

```
[theta,phi] = meshgrid(linspace(0,pi,30),linspace(0,2*pi,30));
r = 2.*ones(1,30);
surf(r.*cos(phi).*sin(theta),r.*sin(phi).*sin(theta),r.*cos(theta));
```

Interestingly this looks more prolate then spherical but keep on eye on the actual scales which is 2 to -2. If you run this in your own copy of octave there will be controls for you to rotate the image.

Let us really go all out and try a prolate sphere. This is beyond the discussion but it is interesting as a program example.

```
[nu,phi] = meshgrid(linspace(0,pi,30),linspace(0,2*pi,30));
a = 1.*ones(1,30);
mu = 0.4*ones(1,30);
surf(a.*sinh(mu).*cos(phi).*sin(nu),a.*sinh(mu).*sin(phi).*sin(nu),a.*cosh(mu).*cos(nu));
```

```
    axis([-1.2 1.2 -1.2 1.2 -1.2 1.2]);
```

This is probably over exaggerated but it gives you a sense of a prolate sphere. The Earth is a oblate sphere, why don't you try using the code here to help you. Use actual data to make your image realistic.

That is enough for this subject in this book/course, let us move on to the next section.

Contributors

- Template:Guichard
- Template:ContribMarshall
- Additional material and changes with a minor amount of deletions to fit with the changes. Scott Johnson, Scott Sinex, and Joshua Halpern

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[1]Here we focus on the 3D coordinate systems with some mention of 2D coordinate systems. A full explanation will also include 1D coordinate systems, etc.

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14.5: Scalars, vectors, and tensors

Scalars, Vectors, and Tensors

When dealing with physical quantities there are expressions that imbue on the quantities more information, these are called tensors (which includes vectors). The concept of vectors were developed in the 1800s and then tensors were developed in the 1900s, so yes they are a new concept if you consider new more then a century ago. Tensors were very important in the development of engineering and physics. *So what is a scalar, vector, and tensor in simple terms?*

- Scalar - a single quantity with no direction
  - Example: My house is 50 km from here
  - Physics: 50 km
  - Tensor of rank 0
- Vector - a magnitude and direction
  - Example: My house is 50 km NNE from here
  - Physics' symbol example: $\vec{x}$
  - Tensor of rank 1
- Tensor - a magnitude with a "stress" associated with it
  - Example: My house has a moment of inertia of I_{ij} for i = x,y,z and j=x,y,z (see matrix below)
  - Physics' symbol example: $\overleftrightarrow{x}$ (note that there are many different expressions for tensor, the authors prefer the double arrow)
  - Tensor of rank 2 or above (yes, engineers do use tensors but in general not until junior or senior year; engineers generally only go to rank 2 but scientists go higher)

For tensors of rank 1 and higher a matrix (or vector if you wish) can be used to express them. The matrix expression gets more complicated for rank 3 and 4 and will not be discussed herein (but you can research on your own if you like).

For a vector we can express it as

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} i \\ j \\ k \end{pmatrix} \tag{14.5.1}$$

For a tensor of rank 2 (moment of inertia) we can express it as

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{14.5.2}$$

The math of tensors (scalars,vectors, and tensors)

The mathematics of scalars is obvious and we will not discuss that here, but the math of vectors is less obvious. There are graphical ways to do the addition of vectors, but here we will only discuss the arithmetic way (we will do some graphical methods in the coming sections).

Let us define the a vector A that will consist of three components in Cartesian coordinate system (x,y,z). When defining vectors we define unit vectors as one unit in magnitude of that particular vector (so the equivalent of 1 in scalar form). Here we will define the unit vectors in the x,y,z respectively as $\hat{i}$, $\hat{j}$, and $\hat{k}$. As an example, a vector going in the x-direction of magnitude 5 km would be represented as $\vec{A} = 5\hat{i}$. To add to the vector A we will define the vector B as well.

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{14.5.3}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \tag{14.5.4}$$

unit vectors are added and associated with the unit vector:

$$\vec{C} = (A_x + B_x)\,\hat{i} + (A_y + B_y)\,\hat{j} + (A_z + B_z)\,\hat{k} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k} \tag{14.5.5}$$

The more interesting operations come with what could be viewed as multiplication of vectors. This is because there are at least three ways to "multiply" the vectors: the dot product, the cross product, and the dyadic vector product.

The **dot product** is the product of two vectors and produces a scalar. From a component view the main rules are that the dot product of same unit vectors are equal to one and different unit vectors are zero. So $\hat{i} \cdot \hat{i} = 1$ and the like and $\hat{i} \cdot \hat{j} = 0$. The dot product the vector A and vector B:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{14.5.6}$$

The dot product is known as a scalar product and is invariant (independent of coordinate system).

An example of a dot product in physics is mechanical work which is the dot product of force and distance:

$$W = \vec{F} \cdot \vec{d} \tag{14.5.7}$$

The **cross product** is the product of two vectors and produce a vector. From a component view the main rules are that the cross product of the same unit vectors are equal to zero and different unit vectors are the cyclic. So $\hat{i} \times \hat{i} = 0$ and the like and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{i} = -\hat{k}$ and the like.

$$\vec{A} \times \vec{B} = (A_y B_z - A_y B_z)\hat{i} + (A_x B_z - A_z B_x)\hat{j} + (A_x B_y - A_y B_x)\hat{k} \tag{14.5.8}$$

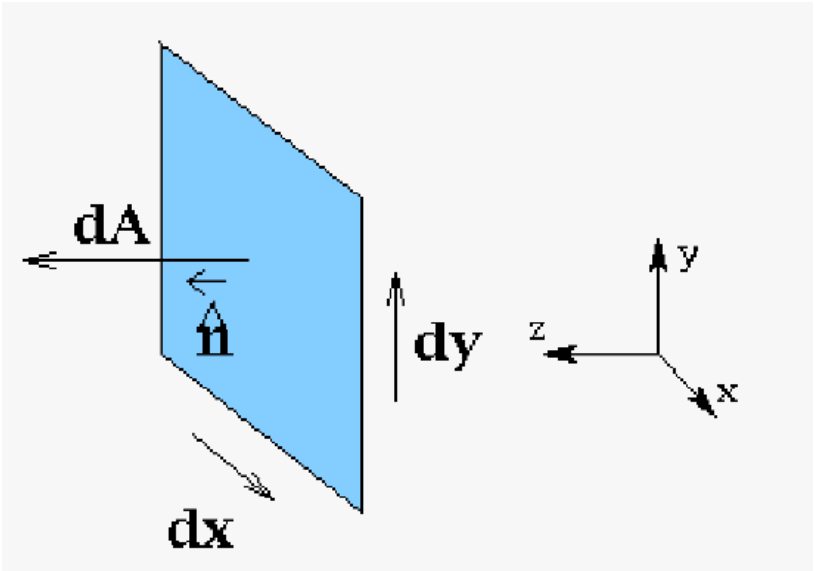
or using Levi-Civita symbols and using $\vec{x}_i$, A_i, B_i formalism instead of $\vec{i}$, $\vec{j}$, $\vec{k}$ formalism.

$$\vec{A} \times \vec{B} = \epsilon_{ijk}\vec{x}_i A_j B_k \tag{14.5.9}$$

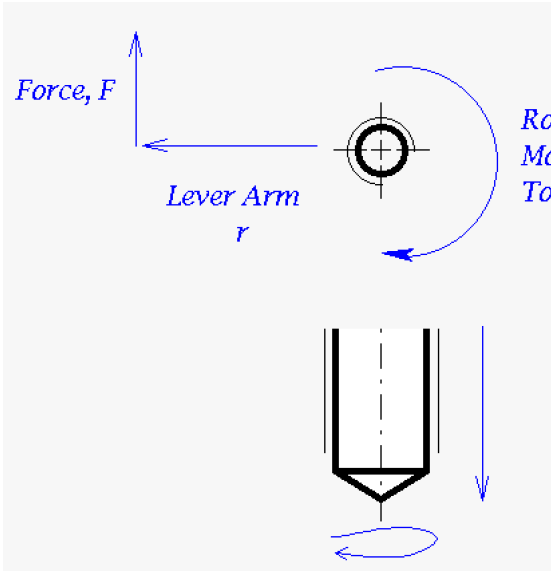
where ϵ_{ijk} is **zero** if 2 or more indices are the same and **one** if the indices are a cyclic (clockwise depending on reference) permutation and **negative one** if they are an anticyclic (counter-clockwise) permutation.

The cross product can be though of as a measurement of "co-alignment" of vectors. Some examples:

- Cross product is a directional area (this is very useful): $d\vec{A} = \hat{n}dA = d\vec{x} \times d\vec{y}$ where $\hat{n}$ is the unit normal vector (to the Area)
- Cross product is moment of force (torque): $\vec{\tau} = \vec{r} \times \vec{F}$



Directional vector x and y can when taken as a cross product results in a directional area which is normal to the directional vectors. The normal vector is shown for reference.



A lever arm (wrench) has a force on it that rotates th piece of wood).

The **dyadic cross product** is the product of two vectors and produce a tensor (rank 2). The best way to look at this is through matrices.

$$\overleftrightarrow{C} = \vec{A} \otimes \vec{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \begin{pmatrix} B_x & B_y & B_z \end{pmatrix} \hat{c} = \begin{pmatrix} A_x Bx & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix} \hat{c} \tag{14.5.10}$$

An example of a tensor product would be found in the Navier-Stokes equations which is used in fluid motion (air flow around wings, etc. would be an engineering application). This concept is beyond the scope of these work, so we will stop here on the idea of multiplication of vectors.

If you can multiple can you divide, right? Well technically yes but in this case the definable quantities are not unique. If there is some physics that involves division in this realm it probably is better going to the slightly older quaternion system (the vector system offered some improvements to quaternions, but quaternions are still used today especially with regards to rotation).

Fields

The idea of fields can lead to confusion when first learning the idea of vectors. Fields are not vectors or tensors but may contain them or be derivable from them.

- A scalar field is a field of scalar points
  - Measurements of the amount of rain in a field at certain points would be an example of a field of scalar points
  - This is NOT a vector
- A vector field is a field of vectors
  - This is not a tensor
- A tensor field is a field of tensors

Important differentials with vectors

We add this for completeness without going into detail so as to introduce a concept for latter use.

There are differential transforms that involve vectors as well. This operations (or transforms) are very important in describing physical process that are best described with vectors. They are applicable to engineers, scientists, economists, etc.

We will start with the gradient of a "scalar" function which produces a "vector" function. The gradient is exactly like it is in regular English (going up a steep hill has a large gradient and going up a slow rising hill has a small gradient). In this context it is a vector measurement of the change of a "scalar" function. Given a function f(x,y,z) the gradient is $\vec{\nabla} f$. The ∇ represents the component partial differential of the function which you will learn later in your academic career. The vector symbol is used to indicate that each component will be associate with a unit vector. Examples: **force** is the gradient of potential energy and the **electric field** is a gradient of the electric potential field.

The divergence of a vector function produces a scalar function. Here again regular English applies as this operation (transform) gives a result that describes divergence of a vector function. This is very useful in measuring the characteristics of a sink or source (or if you have a sink or source at all). Given a vector function the divergence is $\vec{\nabla} \cdot \vec{F}$. The most famous example of this are the divergence of the electric field being related to the charge density and the divergence of the magnetic field being equal to zero. *How are these famous?* They are two of Maxwell's equations (in differential form). Which simplistically says there are charges that produce an electric field, but no magnetic "charges."

The curl of a vector function produces a vector function. Here again regular English applies as this operation (transform) gives a result that describes the curl (or circular density) of a vector function. This gives an idea of rotational nature of different fields. Given a vector function the curl is $\vec{\nabla} \times \vec{F}$. The most famous example of this is the curl of the electric field being related to the magnetic field and visa versa. *How are these famous?* They are the other two of Maxwell's four equations.

Note in the divergence the dot product is used and in the curl the cross product is used as defined previously. The differential extensions are clearly very important and will be fully introduced in later courses.

Lastly there is a more complicated operation (transform) called the Laplacian that has various forms: one for scalars, one for

vectors, and one for tensors of higher rank. This uses the nabla[1] ∇ and both the dot product and cross product to produce a measure of distribution or diffusion. Since not all engineers will need to be familiar with this so we will stop here.

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[1]Nabla is Ancient Greek for Phoenician harp as it somewhat represents a harp. This operator is called the Del operator.

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14.6: Calculus

Small Pebble

Calculus (derived from Latin meaning "small pebble" {calx - limestone, chalk, small stone}) is a method of calculating with small stones which in modern times translates into a method of computation with specific rules. An interesting note is that calculus as a small stone (like kidney stones) is still used in medicine today. So why is a pebble the origin of the basis of our modern mathematical technique? Because small pebbles were used for accounting on a counting board (abacus - Latin) in ancient Rome. The origin of these counting boards are not fully known but the language is from ancient Rome/Greek/Phoenician. It is known that ancient Egyptians had counting boards but some believe it is older still.

Historically calculus is the wrong word as the calculus (infinitesimal to be precise) of today derives from geometry not calculus (simple math), but it was the formal name given by Leibniz and it stuck. For completeness sake there are many types of calculus in mathematics including infinitesimal calculus, predictive calculus, calculus of finite differences, event calculus (computer science), π-calculus (computer science), μ-calculus (computer science), λ-calculus(computer science), etc. Here most of the types of calculus are more reflective of the method of calculating definition.

In engineering we normally view calculus as infinitesimal calculus which includes differential calculus, integral calculus, fractional calculus, calculus of variations, and tensor calculus (see vector discussion). In the sections following we will only concern ourselves with integral and differential calculus as they are appropriate for undergraduate engineering. When we refer to calculus from here on, we will be referring to the aforementioned calculus.

Calculus was independently developed by Isaac Newton and Gottfried Leibniz (they used different methods that assure us that their work was independent). It should be noted that early cultures (ancient Egypt, ancient Greece, and ancient Chinese) used versions of something close to calculus in arcs, circles, and various other attempts to do a form of integral calculus. All of these attempts however were more akin to numerical methods[1] rather than the formal differential calculus of Newton and Leibniz. The method of exhaustion (normally credit to ancient Greece, however started in a rudimentary fashion ancient Egypt) is a famous example of "pre-calculus" that leads to limits, differentials, and area calculations (integral-like) however this method is closer to numerical methods then calculus and does not connect the differential to integral calculus. Still it was a step towards calculus.

Calculus

So to continue the theme of step-by-step development (in previous chapters), Newton's and Leibniz's calculus depended on the ancient Egyptians, the ancient Greeks, and a number of other people including Isaac Barrow (Newton's mentor), John Wallis, and Pierre de Fermat. Like other great achievements described in this course, Newton's and Leibniz's amazing achievements were build on the foundations of many mathematicians, scientists, and engineers. To be sure, Newton and Leibniz did reach the final step to have ***true*** calculus, but it was not in complete isolation[2].

Leibniz's work was more mathematical and theoretical (which lead to some interesting ideas that were akin to Einstein's viewpoint on space and time) where as Newton's work was more for practical purposes. For that reason we will focus on Newton's calculus (except we will still stick to Leibniz's symbols which are the symbols of our present day calculus) because it fits more with engineering philosophy.

Newton was concerned with practical issues with motion, in particular planetary motion, as non-calculus methods did not work well. Small little errors are introduced when using a numerical methods type of calculation that build-up with each successive iteration which accumulate because of propagation error. When dealing with acceleration and velocity in planetary motion this error will lead to problematic results, hence the need for a ***true*** calculus.

Infinitesimal

Calculus depends on the idea of an "infinitesimal." There is no precise definition of an infinitesimal but symbols that evoke the idea of an infinitesimal are ϵ,δ, and dt. The first two symbols are usually used in mathematical analysis (or sometimes engineering numerical methods class) and the last symbol is the one used in your engineering, science, calculus classes and herein.

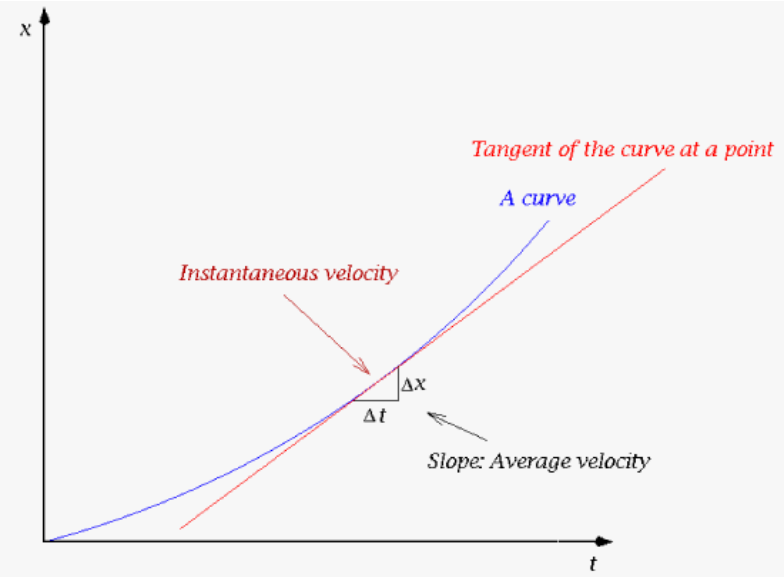
The idea of the infinitesimal is really just the idea of incredibly small which we express either as $dt = \lim_{\Delta t \to 0} \Delta t$ or $dt = \lim_{\Delta t \to \infty} \frac{1}{\Delta t}$. Again this is more an idea not a rigorous precise definition because there is no way to precisely say what is "small." Here we see an aspect of art creeping into mathematics where we have a representation of something we can't precisely express.

Derivative

Total derivative

So let's go to the "definition" of the derivative (total derivative in this case). Basically it is just the tangent of a point as defined by successively reducing the rise over run (slope). The slope in the example presented here would be the average velocity but if we reduce it to a derivative it would be the instantaneous velocity. So the limit of differences on a curve goes to a differential[3] form that define our instantaneous velocity.

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{14.6.1}$$



A depiction of the idea of a derivative with the tangent of a curve at a point being the instantaneous velocity or the derivative of x over time.

Partial Derivative

What about a function with two variables or more variables? In this case we need to move to partial derivatives. So if we have a function, f(x,y) then the differential of the function is a combination of partial derivatives.

$$df(x,y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{14.6.2}$$

It is easy to see that if we suddenly remove the dependency on y and just have f(x) then the above equation reduces to $\frac{df}{dx} = \frac{\partial f}{\partial x}$. A partial derivative acts differently then a total derivative only if our function is dependent on two or more variables. Also, if our function is dependent on two or more variable then you **must** use partial derivatives. In the above example the x position only depends on time so $v = \frac{dx}{dt}$ is correct. But if we had a function that depends on time and position like velocity is in some formalisms in fluid dynamics then we would need a partial derivative to define acceleration $a = \frac{\partial v}{\partial t}$ with some other components that we will leave for later classes in your future.

Integral

Riemann sum

Integrals[4] can be "defined" in a similar way to derivatives at least if we are finding an area or something similar.

$$F(x) = \int_a^b f(x)dx = \lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x \tag{14.6.3}$$

or in a manner more fitting for numerical methods.

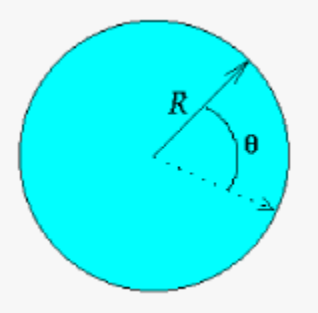
$$F = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \tag{14.6.4}$$

There are many versions of this depending how you wish to define the Δx which could be $x_{i+1} - x_i$, $\frac{x_i - x_{i-1}}{2}$, etc. In numerical methods to improve accuracy this usually gets even more complex. This is one of the reasons a numerical methods class or section of a class (say in Signals and Systems) is needed for a full understanding of engineering and science.

Quadrature

The other idea is integration or quadrature which is the idea of finding the exact area of objects. As an example the area of a circle depends on its radius, R and is best done in polar coordinates, r and θ.

$$A = \int_0^{2\pi} \int_0^R r dr d\theta \tag{14.6.5}$$



With integration we can get the exact area of a circle by integrating from through each "infinitesimal" section of θ over the 2π circumference while integrating from the center of the circle to the outer edge.

$$A = \theta|_0^{2\pi} \int_0^R r dr \tag{14.6.6}$$

$$A = 2\pi \int_0^R r dr \tag{14.6.7}$$

$$A = 2\pi \frac{1}{2} r^2|_0^R \tag{14.6.8}$$

$$A = \pi R^2 \tag{14.6.9}$$

The following theorems define calculus in a formal sense. While you might see these as not useful in your engineering applications, they are important in building essential concepts in numerical methods.

Fundamental Theorems of Calculus

- If f(x) is continuous on a closed interval [a,b] and if F is the anti-derivative we have the an integral.

$$\int_a^b f(t)dt = F(b) - F(a) \tag{14.6.10}$$

- If f(x) is continuous on the open interval (a,b) then at any point θ, F can be defined as an integral of the function f.

$$F(x) = \int_\theta^x f(\tau)d\tau \tag{14.6.11}$$

$$F'(x) = f(x) \tag{14.6.12}$$

Mean-Value Theorem

- If f(x) id differentiable on the open interval (a,b) and continuous on the closed interval [a,b], then there must exist at least one point, say θ in (a,b) such that

$$f'(\theta) = \frac{f(b) - f(a)}{b - a} \tag{14.6.13}$$

From these theorems we can make some other statements (equations) that are useful. For the anti-derivative we have an equation that represents a mean value as $F(b) - F(a) = F'(\theta)(b - a)$. For an estimate of the integral of the function f(x) we have $\int_a^b f(\tau)d\tau = f(\theta)(b - a)$ where θ is an average value of f(t) on [a,b]. And for the integral of a function that can be factored into the function f(x) and another function, say g(x), we can say $\int_a^b f(\tau)g(\tau)d\tau = f(\theta)\int_a^b g(\tau)d\tau$ where $f(\theta)$ is a g(t)-weighted average of f(t) on [a,b].

The theorems help to establish the theory of calculus which you will learn in your calculus classes. From here we will go on to the sections that will detail some practical applications of this (as is suitable for a introduction class).

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[1]Numerical methods (see parachute person) are a computational method that approximates more rigorous but intractable mathematical equations. This leads to a fun fact relating to ancient Egypt: Ancient Egyptian mathematics were a form of *binary* mathematics (just like how electronic computers work) and practical calculus on computers in the present are accomplished by numerical methods. Note the numerical methods in the present are somewhat different from numerical methods of the past in that the theories of calculus and the electronic computers have modified the methods (to improve them).

[2]While the beginning of Newton's fluxions (his term for calculus) was build on others work he did actually finished up his theories in isolation from the plague (just like people isolated themselves from the COVID pandemic). The fluxion is what we now call a time derivative. The time derivative symbol is the only symbol of Newton's method that is still in use.

[3]A quick note on notation. Derivatives can be expressed with certain "short-cut" symbols (the apostrophe and the dot):

$$f'(x) = \frac{df(x)}{dx} \tag{14.6.14}$$

$$\dot{f}(t) = \frac{df(t)}{dt} \tag{14.6.15}$$

Here note that the apostrophe is for derivatives not time-based, where as the dot is used for time derivatives only. The dot derivative is the only symbol of Newton's that is still used today.

[4]It should be noted that there are many types of integrals. Here we are just discussing the basic integral you learn when you first start calculus. Other types of integrals are line integral (scalar fields and vector fields), contour integral (complex plane), surface integral (scalar fields and vector fields), volume integral, path integral (arcs), etc. The path integral has multiple definitions of which one is the line integral. The definitions the at least one of the authors use distinguishes line integral for path integral. See Wolfram's Mathworld for details. These integrals will be in later classes after you finish your freshman calculus and sophomore differential equations.

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14.7: INFINITESIMAL CALCULUS FOR DERIVATIVES

This chapter discussing the calculus of differentiation in a very brief manner. Proofs are limited and applications are more emphasized here. Most of the sections here are from other LibreTexts courses/books rather than the main authors of this course. Tables, applications, and tidbits are herein and as such this does not replace a calculus course where you will learn most of the theory properly. Here we are really only concerned with how derivatives are used in engineering and science. While the basics should be understood in this chapter, it is not necessary to understand the full scope presented here as you will have that opportunity in calculus class proper.

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

We previously defined a derivative in the sense of an application derivative. Not a complete true definition but a practical one.

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{14.7.1}$$

For a true definition of derivative we need to look to the fundamental theorems of calculus, but we can still improve on the practical derivative by making it general.

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{14.7.2}$$

This is akin to the definition you would see in numerical methods, though various "improvements" are done to ensure a better answer since we can't actually have the "step size" go to zero.

For an example we will use the previous derivative definition to solve for the $\sqrt{x}$.

$$f(x) = \sqrt{x} \tag{14.7.3}$$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{14.7.4}$$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \tag{14.7.5}$$

Now let us do a standard trick of math: $(a+b)(a-b) = a^2 - b^2$. Where we square the square roots and "remove" the middle term which would still have square roots if we did not use this trick.

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \tag{14.7.6}$$

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{(x+h) - (x)}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \tag{14.7.7}$$

$$\frac{f(x)}{dx} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \tag{14.7.8}$$

$$\frac{f(x)}{dx} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \tag{14.7.9}$$

Now we can take the limit of h going to zero easily but just setting it to zero. The final answer is

$$\frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}} \tag{14.7.10}$$

Is this right? Let's try this again with the standard table definition of how to do this: that is multiply by the exponent and then subtract one from the exponent.

$$\frac{d(x^{\frac{1}{2}})}{dx} \tag{14.7.11}$$

$$\frac{1}{2}x^{-\frac{1}{2}} \tag{14.7.12}$$

Which is the same as the solution using the limit...ok looks good. Let's start looking at some applications, but first let us look at a standard table of derivatives.

14.7.1: Table of Derivatives

14.7.2: Derivatives of Exponential and Logarithmic Functions

14.7.3: Maxima and Minima

14.7.4: Derivatives and the Shape of a Graph

14.7.5: L'Hôpital's Rule

14.7.6: Newton's Method

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14.7.1: Table of Derivatives

This table of derivatives is from another open textbook in Libretexts and OpenStax.

Common derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f\,'(x) + g\,'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f\,'(x)g(x) + f(x)g\,'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf\,'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f\,'(x) - g\,'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

Chain rule

8. $\frac{d}{dx}[f(g(x))] = f\,'(g(x)) \cdot g\,'(x)$

or

8. $\frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds}$

we will see this later when we note that acceleration is $a = \frac{d^2s}{dt^2}$

$$a = \frac{d^2s}{dt^2} = \frac{ds}{dt}\frac{d(\frac{ds}{dt})}{ds} = v\frac{dv}{ds} \tag{14.7.1.1}$$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$

13. $\frac{d}{dx}(\cot x) = -\csc^2 x$

14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

17. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$

18. $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$

19. $\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$

20. $\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$

22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

23. $\frac{d}{dx}(b^x) = b^x \ln b$

24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Less Common Derivatives

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$

26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

28. $\frac{d}{dx}(\cosh x) = \sinh x$

29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\dfrac{d}{dx}\left(\operatorname{arcsinh} x\right)=\dfrac{1}{\sqrt{x^2+1}}$

32. $\dfrac{d}{dx}\left(\operatorname{arctanh} x\right)=\dfrac{1}{1-x^2}\quad (|x|<1)$

33. \(\quad \dfrac{d}{dx}\left(\text{arcsech} \,x\right)=\dfrac{-1}{x\sqrt{1-x^2}}\quad (0

34. $\dfrac{d}{dx}\left(\operatorname{arccosh} x\right)=\dfrac{1}{\sqrt{x^2-1}}\quad (x>1)$

35. $\dfrac{d}{dx}\left(\operatorname{arccoth} x\right)=\dfrac{1}{1-x^2}\quad (|x|>1)$

36. $\dfrac{d}{dx}\left(\operatorname{arccsch} x\right)=\dfrac{-1}{|x|\sqrt{1+x^2}}\quad (x\neq 0)$

For other derivatives you could use the computational intelligence website Wolframalpha.

Contributors and Attributions

- Template:ContribOpenStaxCalc
- Modified to change inverse trig notation by Paul Seeburger (Monroe Community College)

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14.7.2: Derivatives of Exponential and Logarithmic Functions

Learning Objectives

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x, b > 0,$ is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \frac{1}{b^n}$, for a positive integer n, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of br where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x, b > 0$, we may interpret b^r as $\lim_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, 4^{3.1} < 4^\pi < 4^{3.2}, 4^{3.14} < 4^\pi < 4^{3.15},$$

$$4^{3.141} < 4^\pi < 4^{3.142}, 4^{3.1415} < 4^\pi < 4^{3.1416}, \ldots .$$

As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x,\ b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure provides graphs of the functions $y = 2^x,\ y = 3^x,\ y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function.**

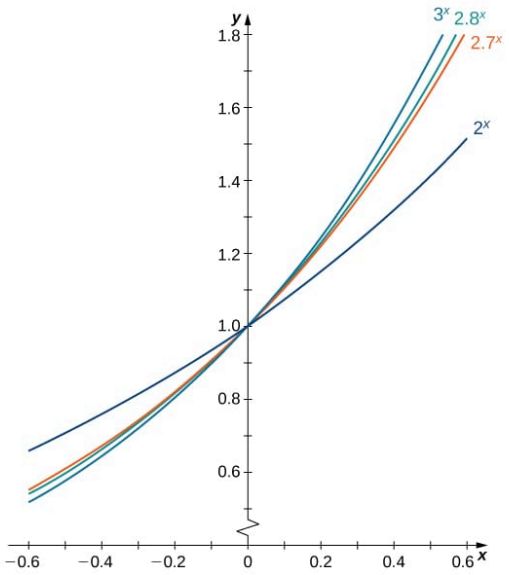


Figure $14.7.2.1$: The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$

to obtain the estimate

\[\frac{b^{−0.00001}−1}{−0.00001}

See the following table.

Table : Estimating the value of e

b	\(\frac{b^{−0.00001}−1}{−0.00001}	b	\(\frac{b^{−0.00001}−1}{−0.00001}
2	\(0.693145	2.718	\(1.000002
2.7	\(0.993247	2.719	\(1.000259
2.71	\(0.996944	2.72	\(1.000627
2.718	\(0.999891	2.8	\(1.029614
2.7182	\(0.999965	3	\(1.098606

The evidence from the table suggests that \(2.7182

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

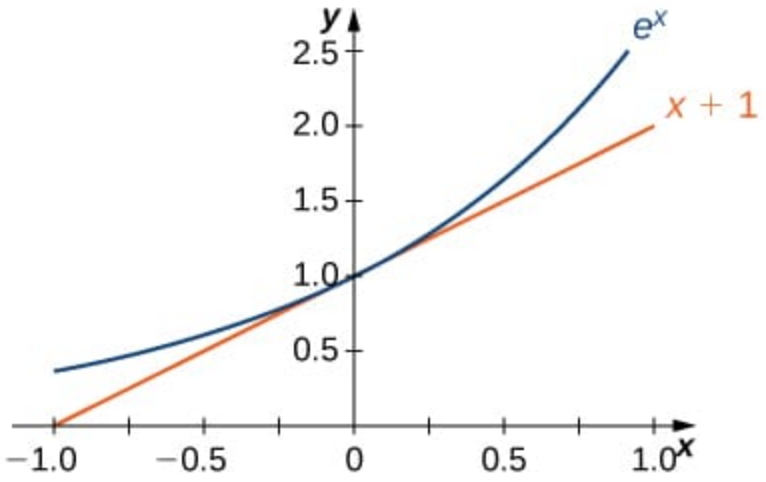


Figure 14.7.2.2: The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x,\ b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

Turning to $B'(x)$, we obtain the following.

Erroneous nesting of equation structures

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at $0, B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x,\ E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x,\ b > 0$, will be derived later.)

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x. \tag{14.7.2.1}$$

In general,

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} g'(x) \tag{14.7.2.2}$$

Example 14.7.2.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

$$f'(x) = e^{\tan(2x)} \frac{d}{dx}(\tan(2x)) = e^{\tan(2x)} \sec^2(2x) \cdot 2$$

Example 14.7.2.2: Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$\begin{aligned} y' &= \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}(2x^2 - 1)}{x^2} && \text{Simplify.} \end{aligned}$$

Exercise 14.7.2.1

Find the derivative of $h(x) = xe^{2x}$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.7.2.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$A'(t) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Exercise 14.7.2.2

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Hint available at the end of this section.

Answer available at the end of this section.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Definition: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}. \tag{14.7.2.3}$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g\,'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)} g'(x). \tag{14.7.2.4}$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1. \tag{14.7.2.5}$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}. \tag{14.7.2.6}$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}. \tag{14.7.2.7}$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$

is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}. \tag{14.7.2.8}$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)} g'(x). \tag{14.7.2.9}$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure.

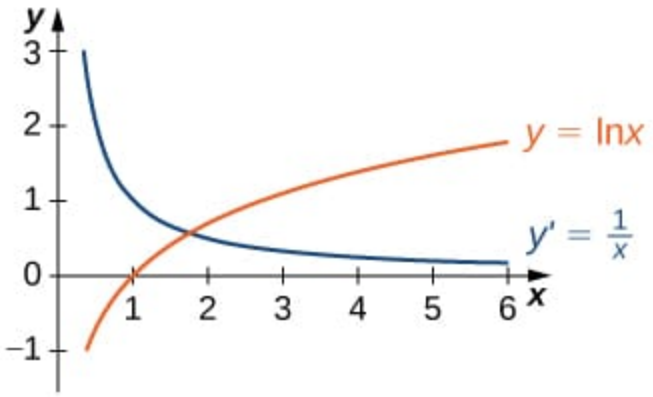


Figure 14.7.2.3: The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$

Example 14.7.2.4: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution

Use Equation 14.7.2.9 directly.

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot (3x^2 + 3) \qquad \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x).$$

$$= \frac{3x^2 + 3}{x^3 + 3x - 4} \qquad \text{Rewrite.}$$

Example 14.7.2.5: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = 2 \ln x + \ln(\sin x) - \ln(2x + 1) \qquad \text{Apply properties of logarithms.}$$

$$f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x + 1} \qquad \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x).$$

Exercise 14.7.2.3

Differentiate: $f(x) = \ln(3x + 2)^5$.

Hint available at the end of this section.

Answer available at the end of this section.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0,\ b \neq 1$.

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{14.7.2.10}$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{14.7.2.11}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b. \tag{14.7.2.12}$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b \tag{14.7.2.13}$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{14.7.2.14}$$

The derivative in Equation 14.7.2.11 now follows from the chain rule.

If $y = b^x$. then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b. \tag{14.7.2.15}$$

The more general derivative (Equation 14.7.2.13) follows from the chain rule.

□

Example 14.7.2.6: Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and Note.

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2} \qquad \text{Simplify.}$$

Example 14.7.2.7: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using Equation, we see that

$$\frac{dy}{dx} = \frac{3}{(3x + 1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\frac{dy}{dx}\Big|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Exercise 14.7.2.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Hint available at the end of this section.

Answer available at the end of this section.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 14.7.2.8: Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$\ln y = \ln (2x^4+1)^{\tan x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \tan x \ln(2x^4+1)$	Step 2. Expand using properties of logarithms.
$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x$	Step 3. Differentiate both sides. Use the product rule on the right.
$\frac{dy}{dx} = y \cdot (\sec^2 x \ln(2x4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = (2x^4+1)^{\tan x}(\sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$	Step 5. Substitute $y = (2x^4+1)^{\tan x}$.

Example 14.7.2.9: Extending the Power Rule

Find the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$\ln y = \ln \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$	Step 5. Substitute $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Exercise 14.7.2.5

Use logarithmic differentiation to find the derivative of $y = x^x$.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.7.2.6

Find the derivative of $y = (\tan x)^\pi$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x,\ b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \dfrac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.

- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g\,'(x)$$

- **Derivative of the natural logarithmic function**

$$\frac{d}{dx}\left(\ln g(x)\right) = \frac{1}{g(x)}g\,'(x)$$

- **Derivative of the general exponential function**

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g\,'(x)\ln b$$

- **Derivative of the general logarithmic function**

$$\frac{d}{dx}\left(\log_b g(x)\right) = \frac{g'(x)}{g(x)\ln b}$$

Glossary

logarithmic differentiation
: is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Contributors and Attributions

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Answers

Exercise 14.7.2.1
Don't forget to use the product rule.

Exercise 14.7.2.1
$h\,'(x) = e^{2x} + 2xe^{2x}$

Exercise 14.7.2.2
Find $A\,'(4)$.

Exercise 14.7.2.2
996

Exercise 14.7.2.3
Use a property of logarithms to simplify before taking the derivative.

Exercise 14.7.2.3
$f\,'(x) = \frac{15}{3x+2}$

Exercise 14.7.2.4
Evaluate the derivative at $x = 2$.

Exercise 14.7.2.4
$9\ln(3)$

Exercise 14.7.2.5
Follow the problem solving strategy.

Exercise 14.7.2.5
Solution: $\frac{dy}{dx} = x^x(1+\ln x)$

Exercise 14.7.2.6

Use the power rule (since the exponent π is a constant) and the chain rule.

Exercise 14.7.2.6

$y' = \pi(\tan x)^{\pi-1} \sec^2 x$

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14.7.3: Maxima and Minima

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty, f(x) \to \infty$.FN Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure $14.7.3.1$).

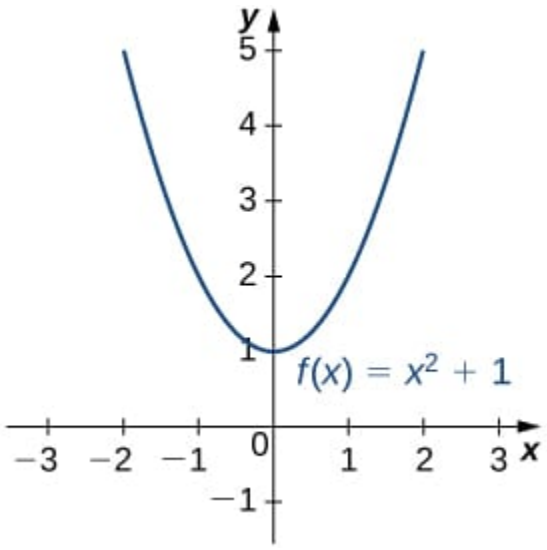


Figure $14.7.3.1$: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2+1} = f(x) \tag{14.7.3.1}$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 14.7.3.2(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 14.7.3.2 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

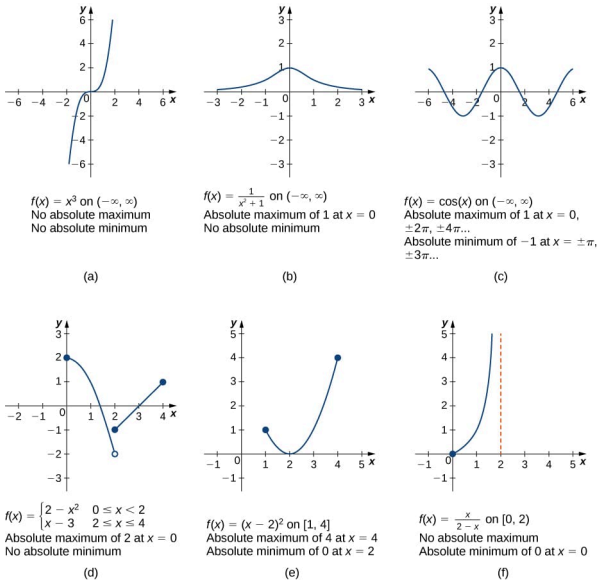


Figure 14.7.3.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 14.7.3.1: Extreme Value Theorem

If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

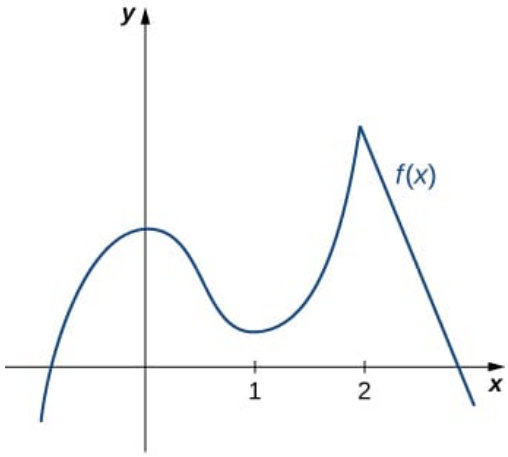
The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure 14.7.3.2 (d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure 14.7.3.3. The graph can be described as two mountains with a valley in the middle. The

absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 14.7.3.3: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 14.7.3.3 again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 14.7.3.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}. \tag{14.7.3.2}$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{14.7.3.3}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}. \tag{14.7.3.4}$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 14.7.3.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

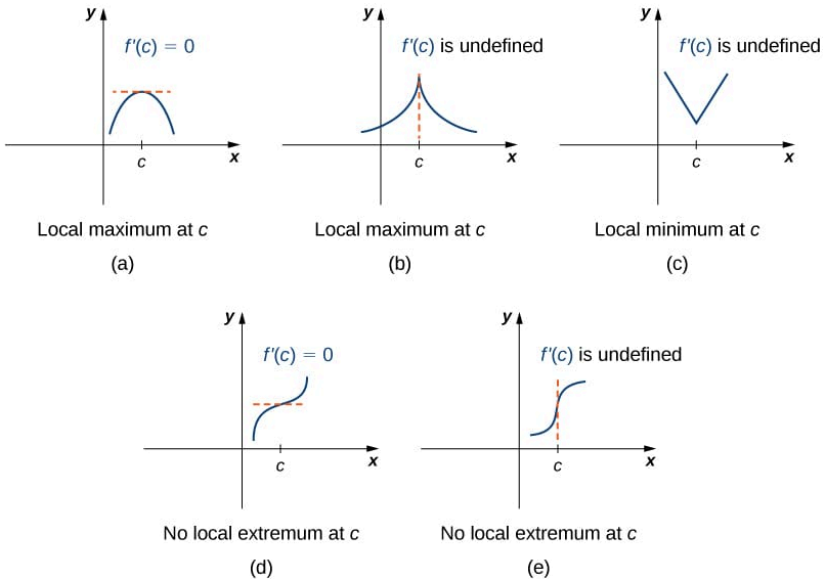


Figure 14.7.3.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 14.7.3.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 14.7.3.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

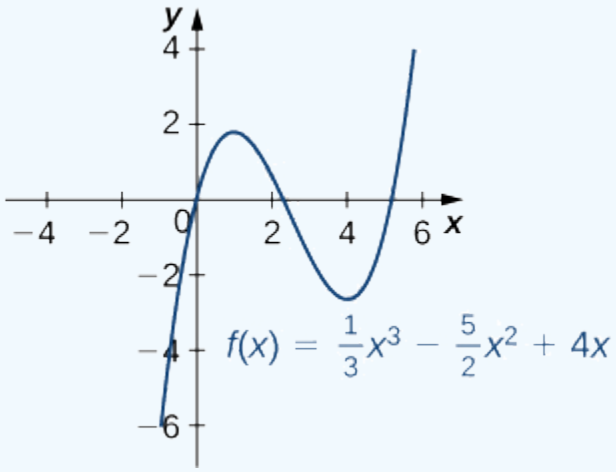


Figure 14.7.3.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure $14.7.3.6$, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.

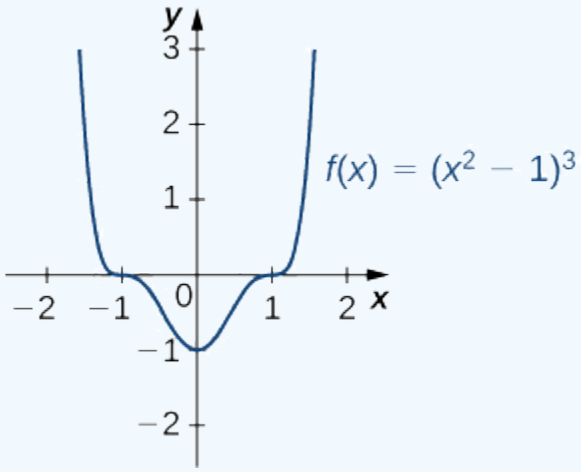


Figure $14.7.3.6$: This function has three critical points: x=0, x=1, and x=−1. The function has a local (and absolute) minimum at x=0, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is

$$f'(x) = \frac{(1+x^24)-4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure $14.7.3.7$, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I , then f has a local extremum at c.)

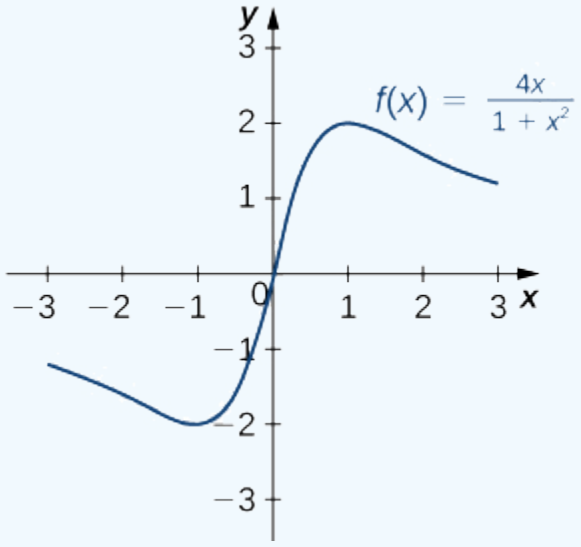


Figure $14.7.3.7$: This function has an absolute maximum and an absolute minimum.

Exercise 14.7.3.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint available at the end of this section.

Answer available at the end of this section.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 14.7.3.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 14.7.3.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From Note, the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 14.7.3.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure $14.7.3.8$.

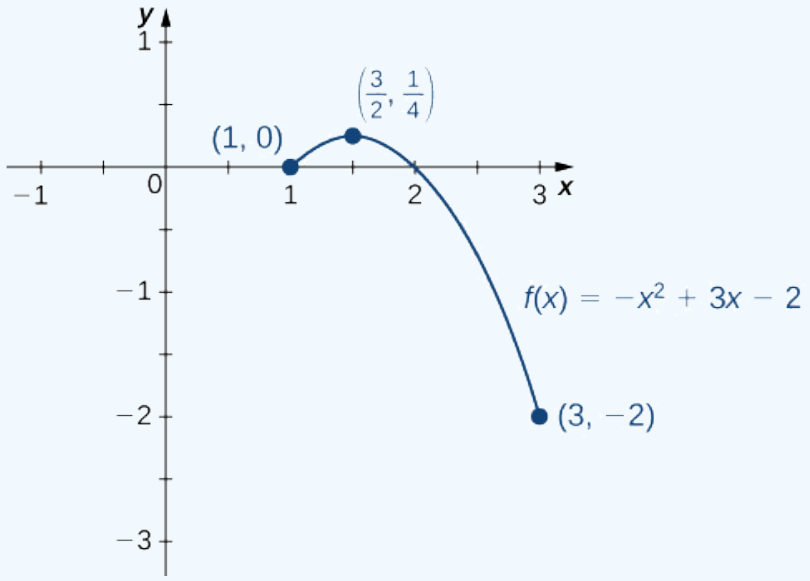


Figure $14.7.3.8$: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3\,(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \tfrac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval [0, 2] is zero, and it occurs at $x = 0$. The absolute minimum is –2, and it occurs at $x = 1$ as shown in Figure 14.7.3.9.

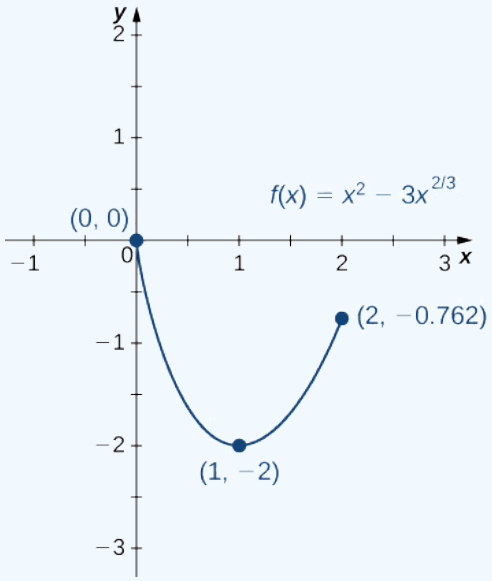


Figure 14.7.3.9: This function has an absolute maximum at an endpoint of the interval.

Exercise 14.7.3.2

Find the absolute maximum and absolute minimum of $(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint available at the end of this section.

Answer available at the end of this section.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Glossary

absolute extremum
: if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum
: if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum
: if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point
: if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem
: if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem
: if f has a local extremum at c, then c is a critical point of f

local extremum
: if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum
: if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum
: if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.7.3.1

Calculate $f'(x)$.

Exercise 14.7.3.1

$x = -23, x = 1$

Exercise 14.7.3.2

Look for critical points. Evaluate f at all critical points and at the endpoints.

Exercise 14.7.3.2

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

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14.7.4: Derivatives and the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function \(f\) has a local extremum at a point \(c\), then \(c\) must be a critical point of \(f\). However, a function is not guaranteed to have a local extremum at a critical point. For example, \(f(x)=x^3\) has a critical point at \(x=0\) since \(f'(x)=3x^2\) is zero at \(x=0\), but \(f\) does not have a local extremum at \(x=0\). Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary \(3\) of the Mean Value Theorem showed that if the derivative of a function is positive over an interval \(I\) then the function is increasing over \(I\). On the other hand, if the derivative of the function is negative over an interval \(I\), then the function is decreasing over \(I\) as shown in the following figure.

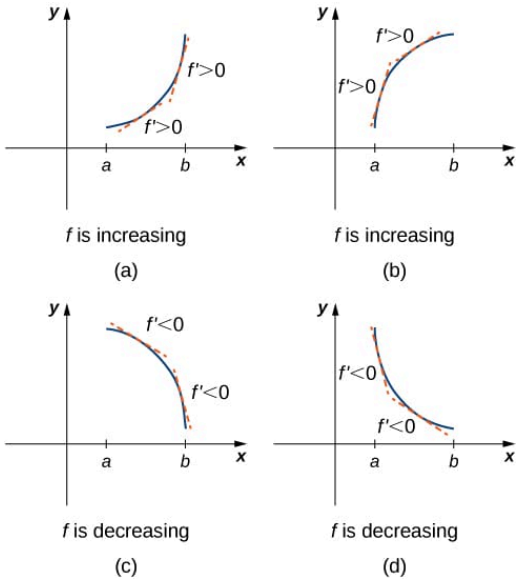


Figure \(\PageIndex{1}\): In four separate graphs a function f(x) is graphed. a) In the first graph, the function is increasing from x = a to x = b increasing faster as we approach b. In this graph f' > 0. b) In the second graph, the function is, also, increasing from x = a to x = b but is rising slower as it reaches b. In this graph f' > 0 even though the slope is decreasing. c) In this third graph, the function is decreasing from x = a to x = b decreasing faster as we approach b. In this graph f' < 0. d) In this forth graph, the function is decreasing from x = a to x = b but is decreasing slower as it reaches b. In this graph f' < 0 even though the slope is increasing.

A continuous function \(f\) has a local maximum at point \(c\) if and only if \(f\) switches from increasing to decreasing at point \(c\). Similarly, \(f\) has a local minimum at \(c\) if and only if \(f\) switches from decreasing to increasing at \(c\). If \(f\) is a continuous function over an interval \(I\) containing \(c\) and differentiable over \(I\), except possibly at \(c\), the only way \(f\) can switch from increasing to decreasing (or vice versa) at point \(c\) is if \(f'\) changes sign as \(x\) increases through \(c\). If

\(f\) is differentiable at \(c\), the only way that \(f'\). can change sign as \(x\) increases through \(c\) is if \(f'(c)=0\). Therefore, for a function \(f\) that is continuous over an interval \(I\) containing \(c\) and differentiable over \(I\), except possibly at \(c\), the only way \(f\) can switch from increasing to decreasing (or vice versa) is if \(f'(c)=0\) or \(f'(c)\) is undefined. Consequently, to locate local extrema for a function \(f\), we look for points \(c\) in the domain of \(f\) such that \(f'(c)=0\) or \(f'(c)\) is undefined. Recall that such points are called critical points of \(f\).

Note that \(f\) need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure \(\PageIndex{2}\), we show that if a continuous function \(f\) has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if \(f\) has a local extremum at a critical point, then the sign of \(f'\) switches as \(x\) increases through that point.

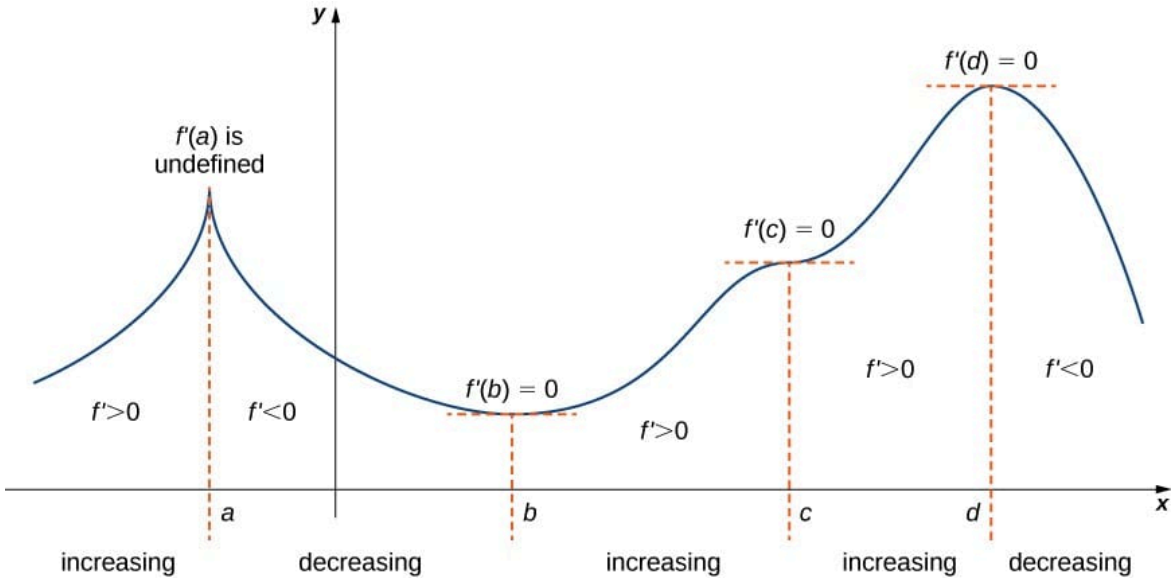


Figure \(\PageIndex{2}\): The function \(f\) has four critical points: \(a,b,c\),and \(d\). The function \(f\) has local maxima at \(a\) and \(d\), and a local minimum at \(b\). The function \(f\) does not have a local extremum at \(c\). The sign of \(f'\) changes at all local extrema.

Using Figure \(\PageIndex{2}\), we summarize the main results regarding local extrema.

- If a continuous function \(f\) has a local extremum, it must occur at a critical point \(c\).
- The function has a local extremum at the critical point \(c\) if and only if the derivative \(f'\) switches sign as \(x\) increases through \(c\).
- Therefore, to test whether a function has a local extremum at a critical point \(c\), we must determine the sign of \(f'(x)\) to the left and right of \(c\).

This result is known as the **first derivative test.**

First Derivative Test

Suppose that \(f\) is a continuous function over an interval \(I\) containing a critical point \(c\). If \(f\) is differentiable over \(I\), except possibly at point \(c\), then \(f(c)\) satisfies one of the following descriptions:

i. If \(f'\) changes sign from positive when \(xc\), then \(f(c)\) is a local maximum of \(f\).
ii. If \(f'\) changes sign from negative when \(xc\), then \(f(c)\) is a local minimum of \(f\).
iii. If \(f'\) has the same sign for \(xc\), then \(f(c)\) is neither a local maximum nor a local minimum of \(f\)

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example \(\PageIndex{1}\): Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for \(f(x)=x^3−3x^2−9x−1.\) Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is \(f'(x)=3x^2−6x−9.\) To find the critical points, we need to find where \(f'(x)=0.\) Factoring the polynomial, we conclude that the critical points must satisfy

\[3(x^2−2x−3)=3(x−3)(x+1)=0. \nonumber\]

Therefore, the critical points are \(x=3,−1.\) Now divide the interval \((−∞,∞)\) into the smaller intervals \((−∞,−1),(−1,3)\) and \((3,∞).\)

Step 2. Since \(f'\) is a continuous function, to determine the sign of \(f'(x)\) over each subinterval, it suffices to choose a point over each of the intervals \((−∞,−1),(−1,3)\) and \((3,∞)\) and determine the sign of \(f'\) at each of these points. For example, let's choose \(x=−2\), \(x=0\), and \(x=4\) as test points.

Table: \(\PageIndex{1}\): First Derivative Test for \(f(x)=x^3−3x^2−9x−1.\)

Interval	Test Point	Sign of \(f'(x)=3(x−3)(x+1)\) at Test Point	Conclusion
\((−∞,−1)\)	\(x=−2\)	(+)(−)(−)=+	\(f\) is increasing.
\((−1,3)\)	\(x=0\)	(+)(−)(+)=+	\(f\) is increasing.
\((3,∞)\)	\(x=4\)	(+)(+)(+)=+	\(f\) is increasing.

Step 3. Since \(f'\) switches sign from positive to negative as \(x\) increases through \(1\), \(f\) has a local maximum at \(x=−1\). Since \(f'\) switches sign from negative to positive as \(x\) increases through \(3\), \(f\) has a local minimum at \(x=3\). These analytical results agree with the following graph.

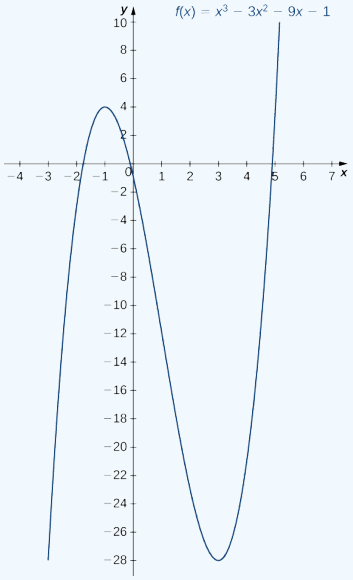


Figure \(\PageIndex{3}\): The function \(f\) has a maximum at \(x=−1\) and a minimum at \(x=3\)

Exercise \(\PageIndex{1}\)

Use the first derivative test to locate all local extrema for \(f(x)=−x^3+\frac{3}{2}x^2+18x.\)

Hint available at the end of this section.

Answer available at the end of this section.

Example \(\PageIndex{2}\): Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for \(f(x)=5x^{1/3}–x^{5/3}.\) Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

\[f'(x)=\frac{5}{3}x^{–2/3}–\frac{5}{3}x^{2/3}=\frac{5}{3x^{2/3}}–\frac{5x^{2/3}}{3}=\frac{5–5x^{4/3}}{3x^{2/3}}=\frac{5(1–x^{4/3})}{3x^{2/3}}.\nonumber\]

The derivative \(f'(x)=0\) when \(1–x^{4/3}=0.\) Therefore, \(f'(x)=0\) at \(x=±1\). The derivative \(f'(x)\) is undefined at \(x=0.\) Therefore, we have three critical points: \(x=0\), \(x=1\), and \(x=–1\). Consequently, divide the interval \((–∞,∞)\) into the smaller intervals \((–∞,–1),\,(–1,0),\,(0,1)\), and \((1,∞)\).

Step 2: Since \(f'\) is continuous over each subinterval, it suffices to choose a test point \(x\) in each of the intervals from step 1 and determine the sign of \(f'\) at each of these points. The points \(x=–2,\,x=–\frac{1}{2},\,x=\frac{1}{2}\), and \(x=2\) are test points for these intervals.

Table: \(\PageIndex{2}\): First Derivative Test for \(f(x)=5x^{1/3}–x^{5/3}.\)

Interval	Test Point	Sign of \(f'(x)=\frac{5(1–x^{4/3})}{3x^{2/3}}\) at Test Point	Conclusion
\((–∞,–1)\)	\(x=–2\)	\(\frac{(+)(–)}{+}=–\)	\(f\) is decreasing.
\((–1,0)\)	\(x=–\frac{1}{2}\)	\(\frac{(+)(+)}{+}=+\)	\(f\) is increasing.
\((0,1)\)	\(x=\frac{1}{2}\)	\(\frac{(+)(+)}{+}=+\)	\(f\) is increasing.
\((1,∞)\)	\(x=2\)	\(\frac{(+)(–)}{+}=–\)	\(f\) is decreasing.

Step 3: Since \(f\) is decreasing over the interval \((–∞,–1)\) and increasing over the interval \((–1,0)\), \(f\) has a local minimum at \(x=–1\). Since \(f\) is increasing over the interval \((–1,0)\) and the interval \((0,1)\), \(f\) does not have a local extremum at \(x=0\). Since \(f\) is increasing over the interval \((0,1)\) and decreasing over the interval \((1,∞)\), \(f\) has a local maximum at \(x=1\). The analytical results agree with the following graph.

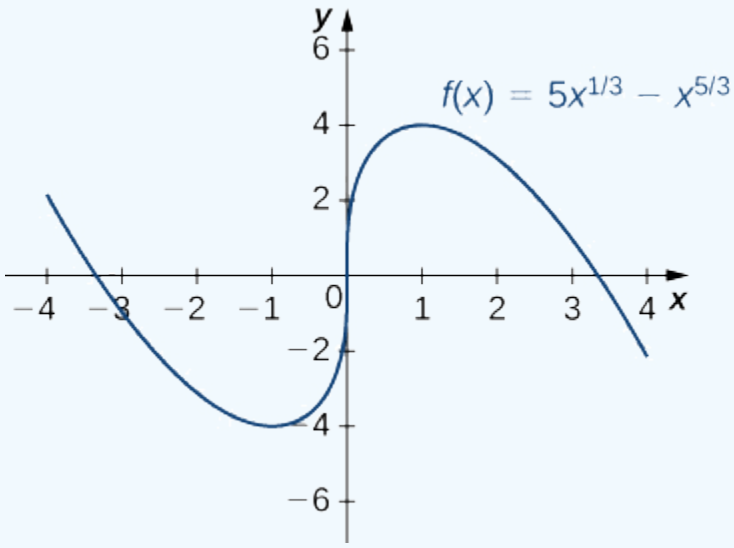


Figure \(\PageIndex{4}\): The function \(f\) has a local minimum at \(x=–1\) and a local maximum at \(x=1\)

Exercise \(\PageIndex{2}\)

Use the first derivative test to find all local extrema for \((x)=\dfrac{3}{x−1}\).

Hint available at the end of this section.

Answer available at the end of this section.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure \(\PageIndex{5}\) shows a function \(f\) with a graph that curves upward. As \(x\) increases, the slope of the tangent line increases. Thus, since the derivative increases as \(x\) increases, \(f'\) is an increasing function. We say this function \(f\) is concave up. Figure \(\PageIndex{4b}\) shows a function \(f\) that curves downward. As \(x\) increases, the slope of the tangent line decreases. Since the derivative decreases as \(x\) increases, \(f'\) is a decreasing function. We say this function \(f\) is concave down.

Definition: concavity test

Let \(f\) be a function that is differentiable over an open interval \(I\). If \(f'\) is increasing over \(I\), we say \(f\) is **concave up** over \(I\). If \(f'\) is decreasing over \(I\), we say \(f\) is **concave down** over \(I\).

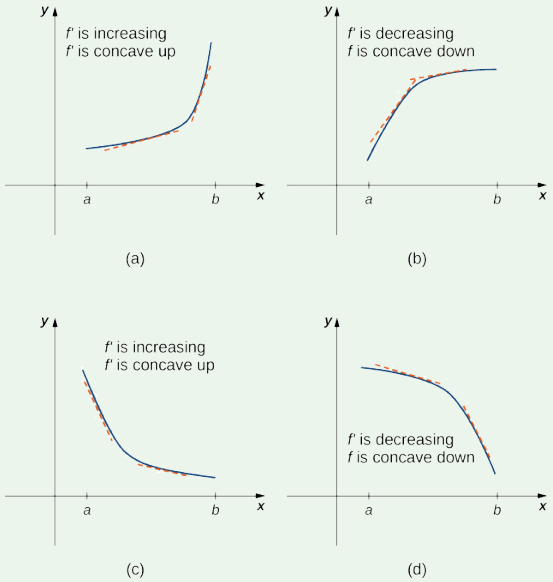


Figure \(\PageIndex{5}\): (a), (c) Since \(f'\) is increasing over the interval \((a,b)\), we say \(f\) is concave up over \((a,b). (b), (d)\) Since \(f'\) is decreasing over the interval \((a,b)\), we say \(f\) is concave down over \((a,b).\)

In general, without having the graph of a function \(f\), how can we determine its concavity? By definition, a function \(f\) is concave up if \(f'\) is increasing. From Corollary \(3\), we know that if \(f'\) is a differentiable function, then \(f'\) is increasing if its derivative \(f''(x)>0\). Therefore, a function \(f\) that is twice differentiable is concave up when \(f''(x)>0\). Similarly, a function \(f\) is concave down if \(f'\) is decreasing. We know that a differentiable function \(f'\) is decreasing if its derivative \(f''(x)<0\). Therefore, a twice-differentiable function \(f\) is concave down when \(f''(x)<0\). Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x)>0$ for all $x\in I$, then f is concave up over I
ii. If $f''(x)<0$ for all $x\in I,$ then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure $\PageIndex{6}$). However, a continuous function can switch concavity only at a point x if $f''(x)=0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x)=0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x)=0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a,f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a, \,f(a))$ is an *inflection point* of f.

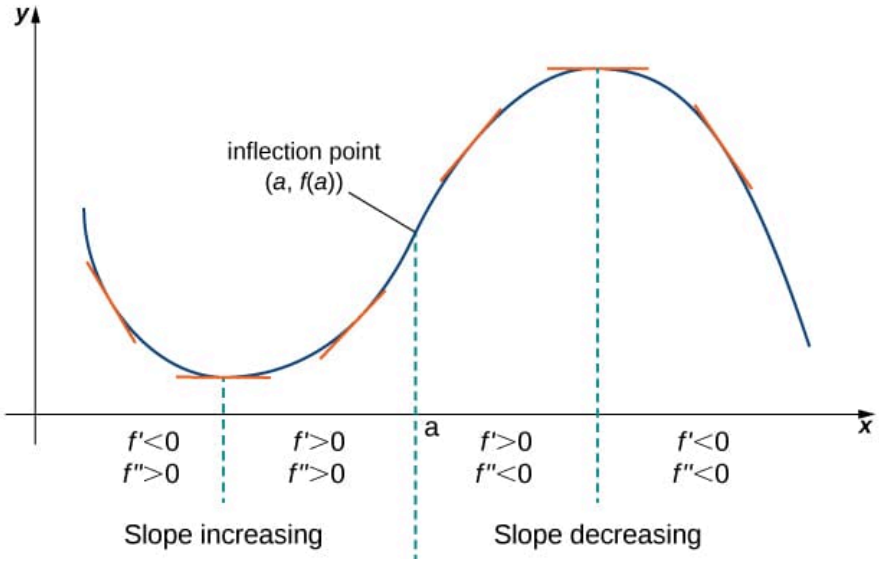


Figure $\PageIndex{6}$: Since $f''(x)>0$ for xa, the function f is concave down over the interval $(a,∞)$. The point $(a,f(a))$ is an inflection point of f.

Example $\PageIndex{3}$: Testing for Concavity

For the function $f(x)=x^3−6x^2+9x+30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x).$ The first derivative is $f'(x)=3x^2−12x+9,$ so the second derivative is $f''(x)=6x−12.$ If the function changes concavity, it occurs either when $f''(x)=0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x)=0$. Solving the equation $6x−12=0$, we see that $x=2$ is the only place where f could change concavity. We now test points over the intervals $(−∞,2)$ and $(2,∞)$ to determine the concavity of f. The points $x=0$ and $x=3$ are test points for these intervals.

Table: \(\PageIndex{3}\): Test for Concavity for \(f(x)=x^3−6x^2+9x+30.\)

Interval	Test Point	Sign of \(f''(x)=6x−12\) at Test Point	Conclusion
\((−∞,2)\)	\(x=0\)	−	\(f\) is concave down
\((2,∞)\)	\(x=3\)	+	\(f\) is concave up

We conclude that \(f\) is concave down over the interval \((−∞,2)\) and concave up over the interval \((2,∞)\). Since \(f\) changes concavity at \(x=2\), the point \((2,f(2))=(2,32)\) is an inflection point. Figure \(\PageIndex{7}\) confirms the analytical results.

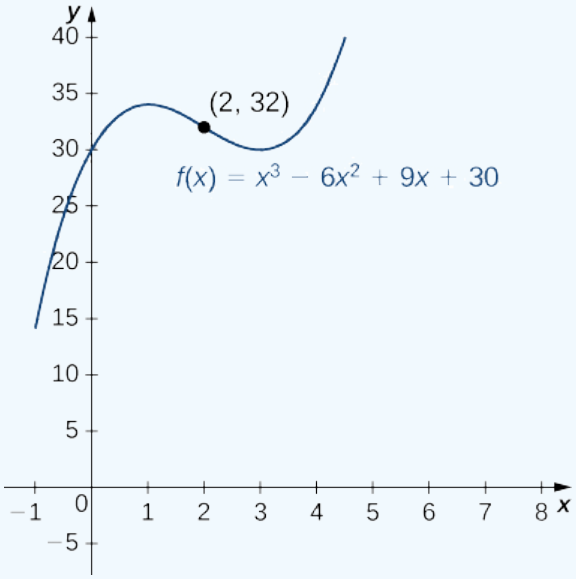


Figure \(\PageIndex{7}\): The given function has a point of inflection at \((2,32)\) where the graph changes concavity.

Exercise \(\PageIndex{3}\)

For \(f(x)=−x^3+\frac{3}{2}x^2+18x\), find all intervals where \(f\) is concave up and all intervals where \(f\) is concave down.

Hint available at the end of this section.

Answer available at the end of this section.

We now summarize, in Table \(\PageIndex{4}\), the information that the first and second derivatives of a function \(f\) provide about the graph of \(f\), and illustrate this information in Figure \(\PageIndex{8}\).

Table: \(\PageIndex{4}\): What Derivatives Tell Us about Graphs

Sign of \(f'\)	Sign of \(f''\)	Is \(f\) increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

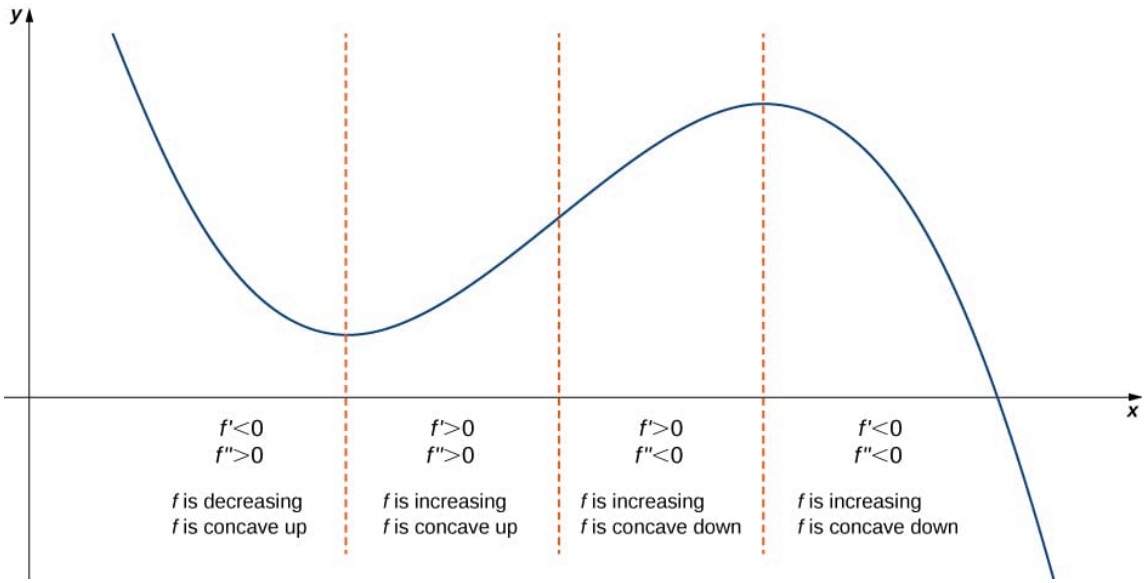


Figure \(\PageIndex{8}\):Consider a twice-differentiable function over an open interval \(I\). If \(f'(x)>0\) for all \(x∈I\), the function is increasing over \(I\). If \(f'(x)<0\) for all \(x∈I\), the function is decreasing over \(I\). If \(f''(x)>0\) for all \(x∈I\), the function is concave up. If \(f''(x)<0\) for all \(x∈I\), the function is concave down on \(I\).

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let \(f\) be a twice-differentiable function such that \(f'(a)=0\) and \(f''\) is continuous over an open interval \(I\) containing \(a\). Suppose \(f''(a)<0\). Since \(f''\) is continuous over \(I, f''(x)<0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is a decreasing function over \(I\). Since \(f'(a)=0\), we conclude that for all \(x∈I, \,f'(x)>0\) if \(xa\). Therefore, by the first derivative test, \(f\) has a local maximum at \(x=a\).

On the other hand, suppose there exists a point \(b\) such that \(f'(b)=0\) but \(f''(b)>0\). Since \(f''\) is continuous over an open interval \(I\) containing \(b\), then \(f''(x)>0\) for all \(x∈I\) (Figure \(\PageIndex{9}\)). Then, by Corollary \(3\), \(f'\) is an increasing function over \(I\). Since \(f'(b)=0\), we conclude that for all \(x∈I\), \(f'(x)<0\) if \(x0\) if \(x>b\). Therefore, by the first derivative test, \(f\) has a local minimum at \(x=b.\)

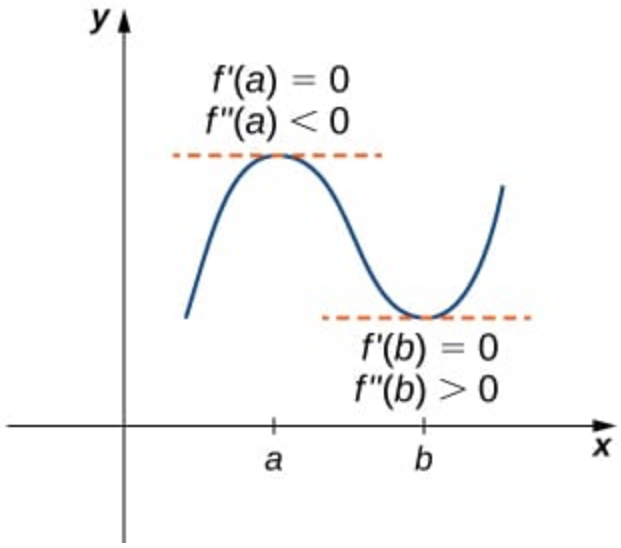


Figure \(\PageIndex{9}\): Consider a twice-differentiable function \(f\) such that \(f''\) is continuous. Since \(f'(a)=0\) and \(f''(a)<0\), there is an interval \(I\) containing \(a\) such that for all \(x\) in \(I\), \(f\) is increasing if \(xa\). As a result, \(f\) has a local maximum at \(x=a\). Since \(f'(b)=0\) and \(f''(b)>0\), there is an interval \(I\) containing \(b\) such that for all \(x\) in \(I\), \(f\) is decreasing if \(xb\). As a result, \(f\) has a local minimum at \(x=b\).

Second Derivative Test

Suppose \(f'(c)=0\) and \(f''\) is continuous over an interval containing \(c\).

i. If \(f''(c)>0\), then \(f\) has a local minimum at \(c\).
ii. If \(f''(c)<0\), then \(f\) has a local maximum at \(c\).
iii. If \(f''(c)=0,\) then the test is inconclusive.

Note that for case iii. when \(f''(c)=0\), then \(f\) may have a local maximum, local minimum, or neither at \(c\). For example, the functions \(f(x)=x^3, \; f(x)=x^4,\) and \(f(x)=−x^4\) all have critical points at \(x=0\). In each case, the second derivative is zero at \(x=0\). However, the function \(f(x)=x^4\) has a local minimum at \(x=0\) whereas the function \(f(x)=−x^4\) has a local maximum at \(x\), and the function \(f(x)=x^3\) does not have a local extremum at \(x=0\).

Let's now look at how to use the second derivative test to determine whether \(f\) has a local maximum or local minimum at a critical point \(c\) where \(f'(c)=0.\)

Example \(\PageIndex{4}\): Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for \(f(x)=x^5−5x^3.\)

Solution

To apply the second derivative test, we first need to find critical points \(c\) where \(f'(c)=0\). The derivative is \(f'(x)=5x^4−15x^2\). Therefore, \(f'(x)=5x^4−15x^2=5x^2(x^2−3)=0\) when \(x=0,\,±\sqrt{3}\).

To determine whether \(f\) has a local extremum at any of these points, we need to evaluate the sign of \(f''\) at these points. The second derivative is

\(f''(x)=20x^3−30x=10x(2x^2−3).\)

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether \(f\) has a local maximum or local minimum at any of these points.

Table: \(\PageIndex{5}\): Second Derivative Test for \(f(x)=x^5−5x^3.\)

\(x\)	\(f''(x)\)	Conclusion
\(−\sqrt{3}\)	\(−30\sqrt{3}\)	Local maximum
\(0\)	\(0\)	Second derivative test is inconclusive
\(\sqrt{3}\)	\(30\sqrt{3}\)	Local minimum

By the second derivative test, we conclude that \(f\) has a local maximum at \(x=−\sqrt{3}\) and \(f\) has a local minimum at \(x=\sqrt{3}\). The second derivative test is inconclusive at \(x=0\). To determine whether \(f\) has a local extrema at \(x=0,\) we apply the first derivative test. To evaluate the sign of \(f'(x)=5x^2(x^2−3)\) for \(x∈(−\sqrt{3},0)\) and \(x∈(0,\sqrt{3})\), let \(x=−1\) and \(x=1\) be the two test points. Since \(f'(−1)<0\) and \(f'(1)<0\), we conclude that \(f\) is decreasing on both intervals and, therefore, \(f\) does not have a local extrema at \(x=0\) as shown in the following graph.

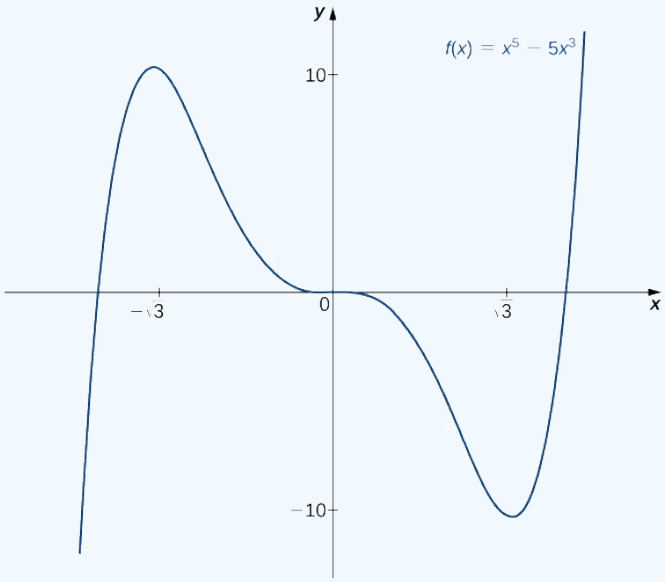


Figure \(\PageIndex{10}\):The function \(f\) has a local maximum at \(x=−\sqrt{3}\) and a local minimum at \(x=\sqrt{3}\)

Exercise \(\PageIndex{4}\)

Consider the function \(f(x)=x^3−(\frac{3}{2})x^2−18x\). The points \(c=3,\,−2\) satisfy \(f'(c)=0\). Use the second derivative test to determine whether \(f\) has a local maximum or local minimum at those points.

Hint available at the end of this section.

Answer available at the end of this section.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as \(x→±∞.\) At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If \(c\) is a critical point of \(f\) and \(f'(x)>0\) for \(xc\), then \(f\) has a local maximum at \(c\).
- If \(c\) is a critical point of \(f\) and \(f'(x)<0\) for \(x0\) for \(x>c,\) then \(f\) has a local minimum at \(c\).
- If \(f''(x)>0\) over an interval \(I\), then \(f\) is concave up over \(I\).
- If \(f''(x)<0\) over an interval \(I\), then \(f\) is concave down over \(I\).
- If \(f'(c)=0\) and \(f''(c)>0\), then \(f\) has a local minimum at \(c\).
- If \(f'(c)=0\) and \(f''(c)<0\), then \(f\) has a local maximum at \(c\).

- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down
: if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up
: if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity
: the upward or downward curve of the graph of a function

concavity test
: suppose f is twice differentiable over an interval I; if $f''>0$ over I, then f is concave up over I; if $f''<$ over I, then f is concave down over I

first derivative test
: let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point
: if f is continuous at c and f changes concavity at c, the point $(c,f(c))$ is an inflection point of f

second derivative test
: suppose $f'(c)=0$ and f'' is continuous over an interval containing c; if $f''(c)>0$, then f has a local minimum at c; if $f''(c)<0$, then f has a local maximum at c; if $f''(c)=0$, then the test is inconclusive

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise $\PageIndex{1}$
Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Exercise $\PageIndex{1}$
f has a local minimum at -2 and a local maximum at 3.

Exercise $\PageIndex{2}$
The only critical point of f is $x=1.$

Exercise $\PageIndex{2}$
f has no local extrema because f' does not change sign at $x=1$.

Exercise $\PageIndex{3}$
Find where $f''(x)=0$

Exercise $\PageIndex{3}$
f is concave up over the interval $(-\infty,\frac{1}{2})$ and concave down over the interval $(\frac{1}{2},\infty)$

Exercise $\PageIndex{4}$
$f''(x)=6x-3$

Exercise $\PageIndex{4}$
f has a local maximum at -2 and a local minimum at 3.

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14.7.5: L'Hôpital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}. \tag{14.7.5.1}$$

If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2 \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}. \tag{14.7.5.2}$$

However, what happens if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and

g such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and such that $g\,'(a) \neq 0$ For x near a,we can write

$$f(x) \approx f(a) + f'(a)(x-a) \tag{14.7.5.3}$$

and

$$g(x) \approx g(a) + g'(a)(x-a). \tag{14.7.5.4}$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}. \tag{14.7.5.5}$$

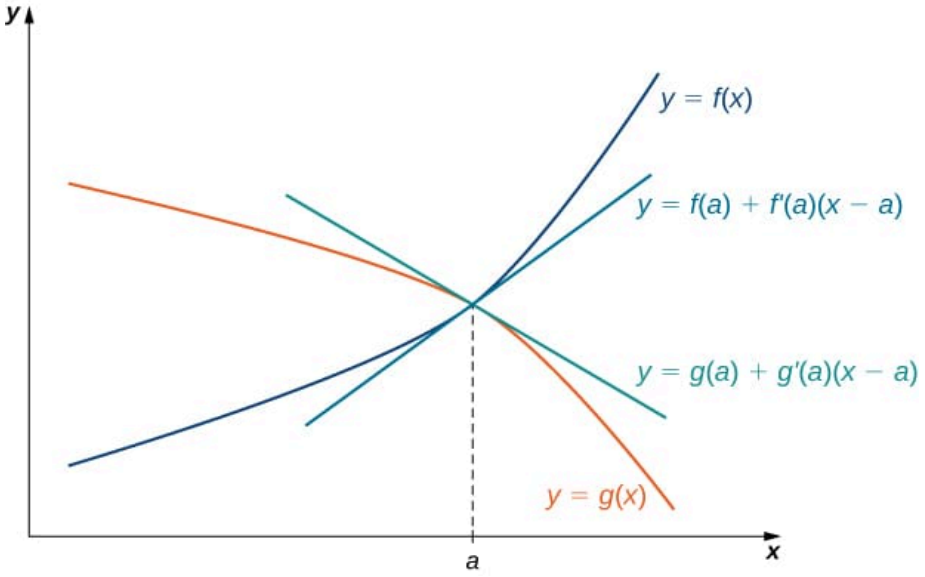


Figure 14.7.5.1: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x\to a} f(x) = 0$. Similarly, $g(a) = \lim_{x\to a} g(x) = 0$. If we also assume that $f\,'$ and $g\,'$ are continuous at $x = a$, then $f\,'(a) = \lim_{x\to a} f\,'(x)$ and $g\,'(a) = \lim_{x\to a} g\,'(x)$. Using these ideas, we conclude that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{14.7.5.6}$$

Note that the assumption that $f\,'$ and $g\,'$ are continuous at a and $g\,'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{14.7.5.7}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a} f'(x)}{\lim\limits_{x\to a} g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 14.7.5.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$
b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$
c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$
d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a.. Since the numerator $1-\cos x \to 0$ and the denominator $x \to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\left(1-\cos x\right)}{\frac{d}{dx}\left(x\right)} = \lim_{x\to 0}\frac{\sin x}{1} = \frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0} 1} = \frac{0}{1} = 0.$$

b. As $x \to 1$, the numerator $\sin(\pi x) \to 0$ and the denominator $\ln(x) \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x}=\lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}$$
$$=\lim_{x\to 1}(\pi x)\cos(\pi x)$$
$$=(\pi\cdot 1)(-1)=-\pi.$$

c. As $x\to\infty$, the numerator $e^{1/x}-1\to 0$ and the denominator $\frac{1}{x}\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{e^{1/x}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}e^{1/x}=e^0=1.$$

d. As $x\to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}=\lim_{x\to 0}\frac{\cos x-1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x\to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0}\frac{\cos x-1}{2x}=\lim_{x\to 0}\frac{-\sin x}{2}=0.$$

Therefore, we conclude that

$$\lim_{x\to 0}\frac{\sin x-x}{x^2}=0.$$

Exercise 14.7.5.1

Evaluate

$$\lim_{x\to 0}\frac{x}{\tan x}.$$

Hint available at the end of this section.

Answer available at the end of this section.

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞. is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a}f(x)=\infty$ (or $-\infty$) and $\lim_{x\to a}g(x)=\infty$ (or $-\infty$). Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{14.7.5.8}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a=\infty$ or $-\infty$, or the limit is one-sided.

Example 14.7.5.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty} \dfrac{3x+5}{2x+1}$

b. $\lim_{x\to 0^+} \dfrac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2x+1}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2x+1}=\frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to 0^+}\ln x=-\infty$ and $\lim_{x\to 0^+}\cot x=\infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=\lim_{x\to 0^+}\frac{1/x}{-\csc^2 x}=\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}.$$

Now as $x\to 0^+, \csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $cscx$ to write

$$\lim_{x\to 0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to 0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to 0^+}\sin^2 x=0$ and $\lim_{x\to 0^+}x=0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to 0^+}\frac{\sin^2 x}{-x}=\lim_{x\to 0^+}\frac{2\sin x\cos x}{-1}=\frac{0}{-1}=0.$$

We conclude that

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}=0.$$

Exercise 14.7.5.2

Evaluate

$$\lim_{x\to\infty}\frac{\ln x}{5x}.$$

Hint available at the end of this section.

Answer available at the end of this section.

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient f(x)g(x), it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 14.7.5.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x\to 1}\frac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5)=6$ and $\lim_{x\to 1}(3x+4)=7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\frac{d}{dx}(x^2+5)}{\frac{d}{dx}(3x+4)}.$$

Exercise 14.7.5.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\frac{\cos x}{x}$ by other means.

Hint available at the end of this section.

Answer available at the end of this section.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way

that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0\cdot\infty$ to denote the form that arises in this situation. The expression $0\cdot\infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x\to\infty$. For example, let n be a positive integer and consider

$$f(x)=\frac{1}{(x^n+1)} \text{ and } g(x)=3x^2.$$

As $x\to\infty, f(x)\to 0$ and $g(x)\to\infty$. However, the limit as $x\to\infty$ of $f(x)g(x)=\frac{3x^2}{(x^n+1)}$ varies, depending on n. If $n=2$, then $\lim_{x\to\infty} f(x)g(x)=3$. If $n=1$, then $\lim_{x\to\infty} f(x)g(x)=\infty$. If $n=3$, then $\lim_{x\to\infty} f(x)g(x)=0$. Here we consider another limit involving the indeterminate form $0\cdot\infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 14.7.5.4: Indeterminate Form of Type $0\cdot\infty$

Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution

First, rewrite the function $x\ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x\ln x=\frac{\ln x}{1/x}$$

we see that $\ln x\to-\infty$ as $x\to 0^+$ and $\frac{1}{x}\to\infty$ as $x\to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{\frac{d}{dx}\left(\ln x\right)}{\frac{d}{dx}\left(1/x\right)}=\lim_{x\to 0^+}\frac{1/x^2}{-1/x}=\lim_{x\to 0^+}(-x)=0.$$

We conclude that

$$\lim_{x\to 0^+} x\ln x=0.$$

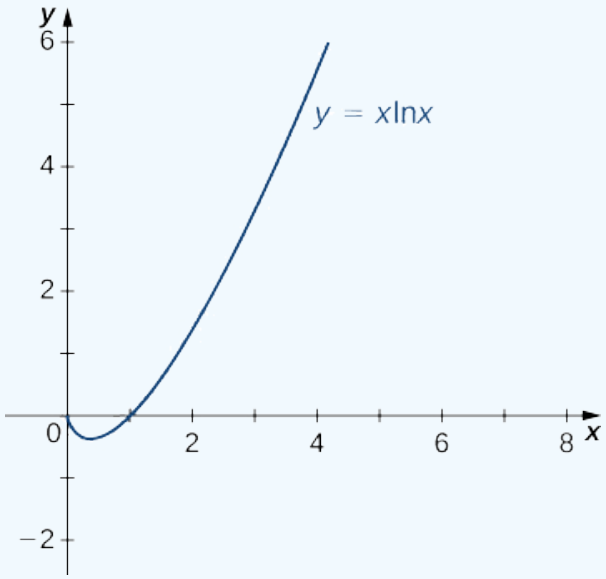


Figure 14.7.5.2: Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

Exercise 14.7.5.4

Evaluate

$$\lim_{x \to 0} x \cot x.$$

Hint available at the end of this section.

Answer available at the end of this section.

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty, f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 14.7.5.5: Indeterminate Form of Type $\infty - \infty$

Evaluate

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right).$$

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(sec^2 x) - 2x \to 1$ and $x^2 sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Exercise 14.7.5.5

Evaluate $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Hint available at the end of this section.

Answer available at the end of this section.

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions $0^0, \infty^0$, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}. \tag{14.7.5.9}$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x)). \tag{14.7.5.10}$$

Therefore,

$$\lim_{x \to a} [\ln(y)] = \lim_{x \to a} [g(x) \ln(f(x))]. \tag{14.7.5.11}$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a} \ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L. \tag{14.7.5.12}$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L, \tag{14.7.5.13}$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L. \tag{14.7.5.14}$$

Example 14.7.5.6: Indeterminate Form of Type ∞^0

Evaluate

$$\lim_{x\to\infty} x^{1/x}.$$

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x\to\infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty} y\right) = 0,$$

which leads to

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,

$$\lim_{x\to\infty} x^{1/x} = 1.$$

Exercise 14.7.5.6

Evaluate

$$\lim_{x\to\infty} x^{1/\ln(x)}.$$

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.7.5.7: Indeterminate Form of Type 0^0

Evaluate

$$\lim_{x\to 0^+} x^{\sin x}.$$

Solution

Let

$$y = x^{\sin x}.$$

Therefore,

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We now evaluate $\lim_{x\to 0^+} \sin x \ln x$. Since $\lim_{x\to 0^+} \sin x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{1/\ln x} = \lim_{x\to 0^+} \frac{\cos x}{-1/(x(\ln x)^2)} = \lim_{x\to 0^+} (-x(\ln x)^2 \cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x,}$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $cscx = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x\to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x)\right] = \left(\lim_{x\to 0^+} \frac{\sin x}{x}\right) \cdot \left(\lim_{x\to 0^+} (-\tan x)\right) = 1 \cdot 0 = 0.$$

We conclude that $\lim_{x\to 0^+} \ln y = 0$. Therefore, $\ln\left(\lim_{x\to 0^+} y\right) = 0$ and we have

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x\to 0^+} x^{\sin x} = 1.$$

Exercise 14.7.5.7

Evaluate $\lim_{x\to 0^+} x^x$.

Hint available at the end of this section.

Answer available at the end of this section.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 14.7.5.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 14.7.5.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty.$$

or, equivalently

$$\lim_{x\to\infty x} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 14.7.5.2. In fact,

$$\lim_{x\to\infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 14.7.5.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \tag{14.7.5.15}$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = M, \tag{14.7.5.16}$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 14.7.5.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}.$$

Since $\lim_{x\to\infty} 2x = \infty$ and $\lim_{x\to\infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 14.7.5.3 and Table 14.7.5.3)

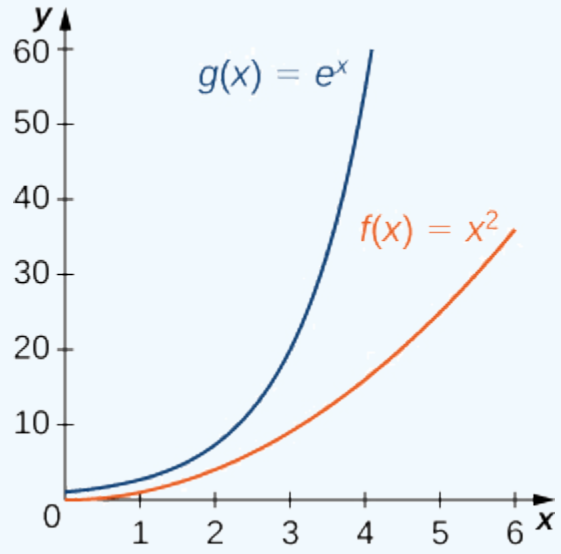


Figure 14.7.5.3: An exponential function grows at a faster rate than a power function.

Table 14.7.5.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400

e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \frac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure $14.7.5.4$ and Table $14.7.5.4$).

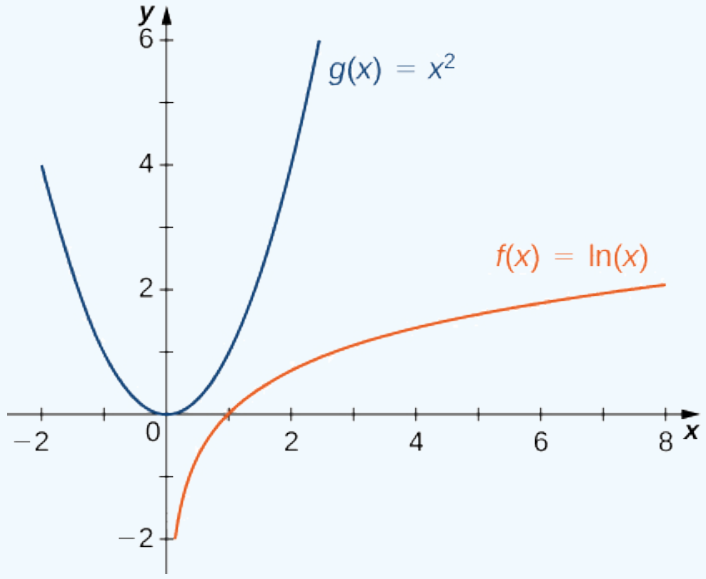


Figure $14.7.5.4$: A power function grows at a faster rate than a logarithmic function.

Table $14.7.5.4$: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.10
x^2	100	10,000	1,000,000	100,000,000

Example $14.7.5.9$: Comparing the Growth Rates of x^{100} and 2^x

Compare the growth rates of x^{100} and 2^x.

Hint: Apply L'Hôpital's rule to $x^{100}/2^x$

Solution

The function 2^x grows faster than x^{100}.

Using the same ideas as in Example a. it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure $14.7.5.5$ and Table $14.7.5.5$, we compare e^x with x^3 and x^4 as $x \to \infty$.

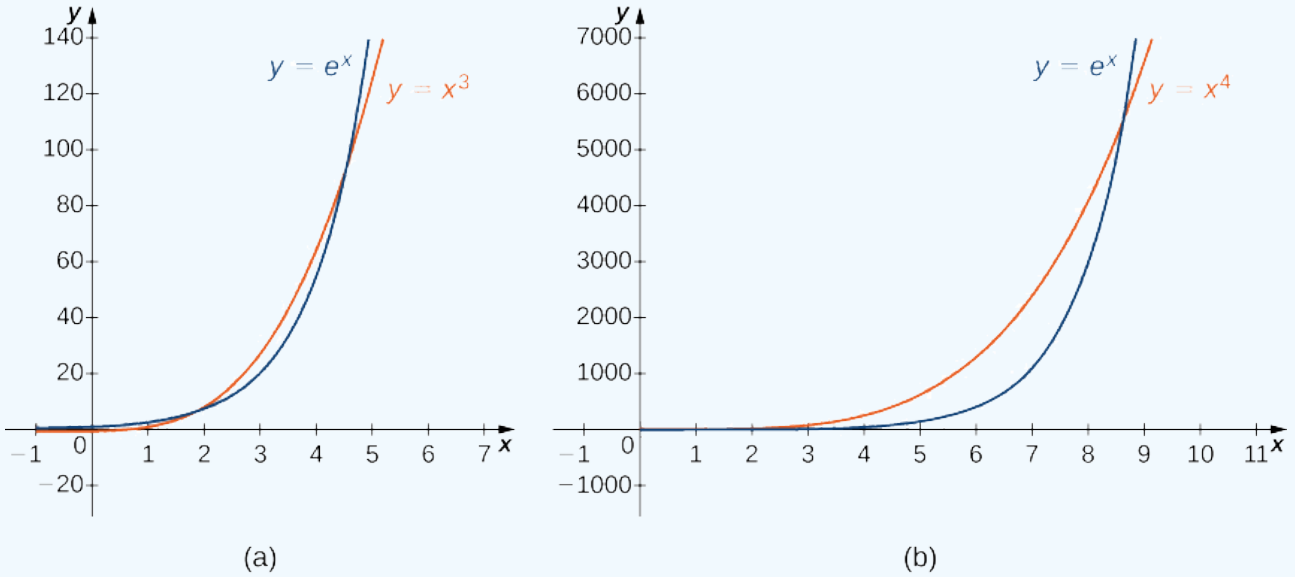


Figure 14.7.5.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 14.7.5.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,326,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 14.7.5.6 and Table, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

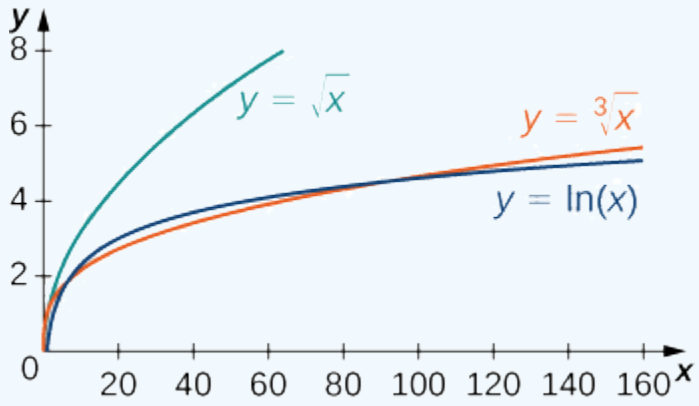


Figure 14.7.5.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 14.7.5.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.
- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}$,$\infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L'Hôpital's rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.7.5.1

$\frac{d}{dx}\big(\tan x\big) = \sec^2 x$

Exercise 14.7.5.1

1

Exercise 14.7.5.2

$\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$

Exercise 14.7.5.2

0

Exercise 14.7.5.3

Determine the limits of the numerator and denominator separately.

Exercise 14.7.5.3

$\lim_{x \to 0^+} \cos x = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Exercise 14.7.5.4

Write $x \cot x = \frac{x \cos x}{\sin x}$

Exercise 14.7.5.4

1

Exercise 14.7.5.5

Rewrite the difference of fractions as a single fraction.

Exercise 14.7.5.5

0

Exercise 14.7.5.6

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Exercise 14.7.5.6

e

Exercise 14.7.5.7

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Exercise 14.7.5.7

1

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14.7.6: Newton's Method

Learning Objectives

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0$. For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. There are many numerical method approximation methods to solve these problems divided into two types bounded methods and open methods. Bounded methods are known for always converging but slow. Open methods are fast but sometimes don't converge. In practice both methods are used in one routine. Full explanation of these methods are done in numerical methods (or signals and systems).

In this section, we take a look at a technique that provides a very efficient way of approximating the **zeroes of functions** called the Newton's method which is an open method. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x) = 0$. If f is the first-degree polynomial $f(x) = ax + b$, then the solution of $f(x) = 0$ is given by the formula $x = -\dfrac{b}{a}$. If f is the second-degree polynomial $f(x) = ax^2 + bx + c$, the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree **3** or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree **5** or greater, it is known that no such formulas exist. For example, consider the function

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0$. Similar difficulties exist for non-polynomial functions. For example, consider the task of finding solutions of $tan(x) - x = 0$. No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0$. By sketching a graph of f, we can estimate a root of $f(x) = 0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f\,'(x_0) \neq 0$, this tangent line intersects the x-axis at some point $(x_1, 0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1. If $f\,'(x_1) \neq 0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: $x_0,\ x_1,\ x_2,\ \ldots$. Typically, the numbers $x_0,\ x_1,\ x_2,\ \ldots$ quickly approach an actual root $x*$, as shown in the following figure.

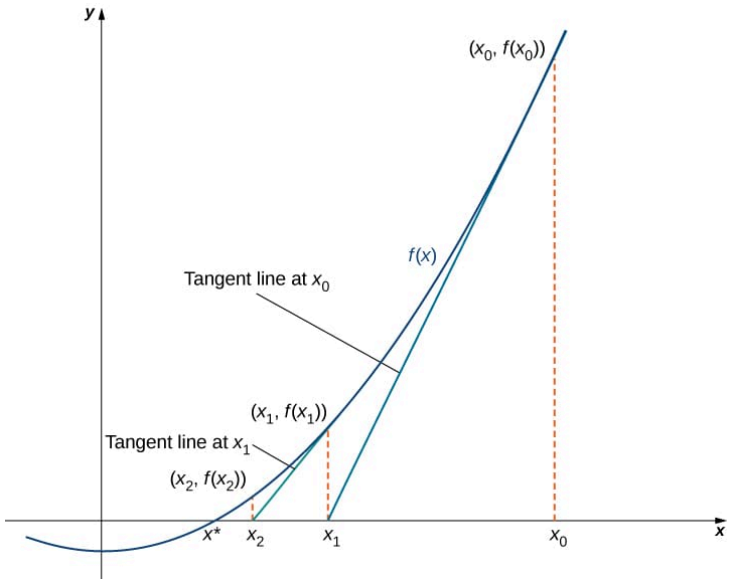


Figure 14.7.6.1:The approximations $x_0,\ x_1,\ x_2,\ \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0,\ x_1,\ x_2,\ \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0, x_n$ satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{14.7.6.1}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 14.7.6.1: Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1,\ x_2,\ x_3,\ x_4$, and x_5.

Solution

From Figure 14.7.6.2, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use Equation 14.7.6.1. Since $f(x) = x^3 - 3x + 1$, the derivative is $f\,'(x) = 3x^2 - 3$. Using Equation 14.7.6.1 with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2, we use Equation with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

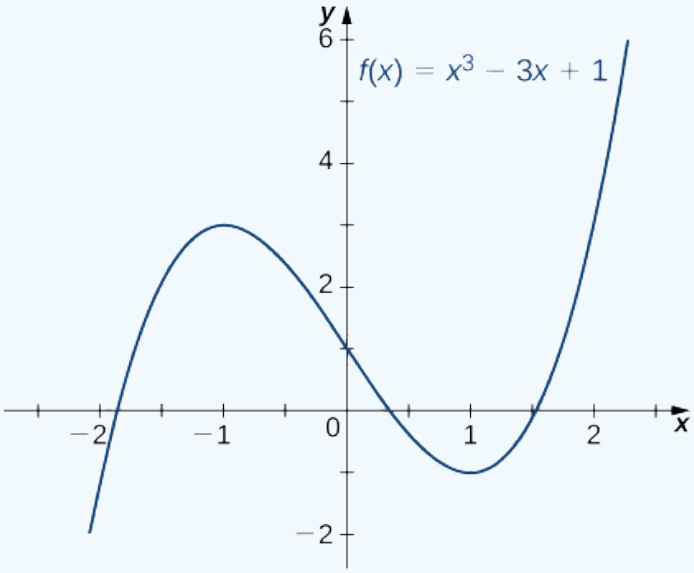


Figure 14.7.6.2: The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

Exercise 14.7.6.1

Letting $x_0 = 0$, let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example 14.7.6.2: Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (Figure 14.7.6.3). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$).

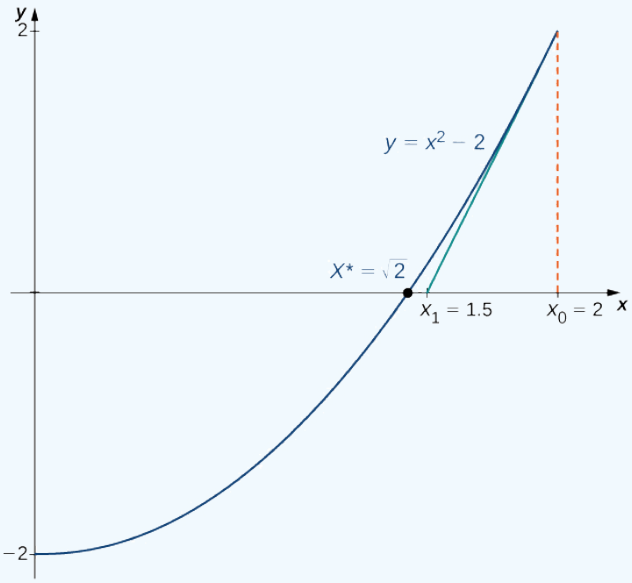


Figure 14.7.6.3: We can use Newton's method to find $\sqrt{2}$.

Solution

For $f(x) = x^2 - 2,\ \ f\,'(x) = 2x$. From 14.7.6.1, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \tfrac{1}{2}\left(x_0 + \tfrac{2}{x_0}\right) = \tfrac{1}{2}\left(2 + \tfrac{2}{2}\right) = 1.5$$

$$x_2 = \tfrac{1}{2}\left(x_1 + \tfrac{2}{x_1}\right) = \tfrac{1}{2}\left(1.5 + \tfrac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$x_1 = 1.5$$

$$x_2 \approx 1.416666667$$

$$x_3 \approx 1.414215686$$

$$x_4 \approx 1.414213562$$

$$x_5 \approx 1.414213562.$$

Since we obtained the same value for x_4 and x_5, it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.

Exercise 14.7.6.2

Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation 14.7.6.1 as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an iterative process. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure 14.7.6.4). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example 14.7.6.3, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

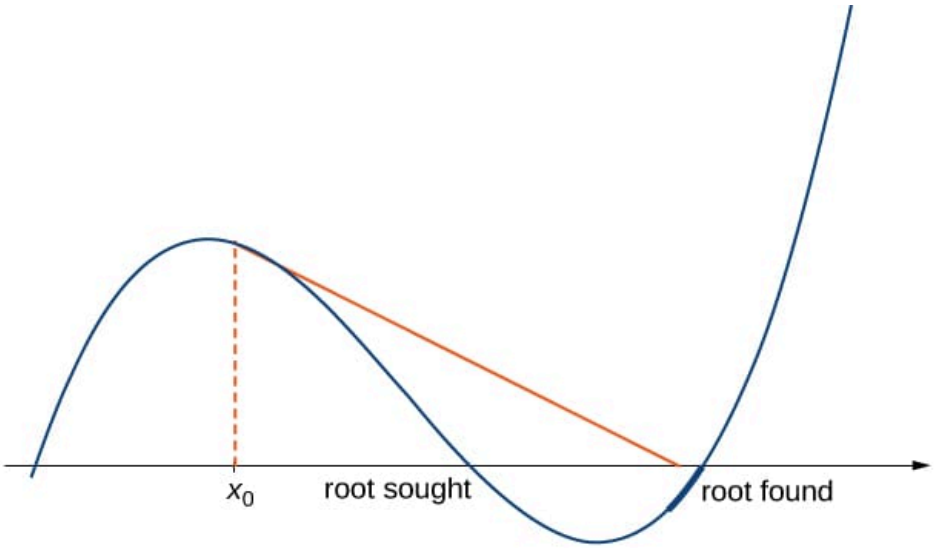


Figure 14.7.6.4: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 14.7.6.3: When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1,\ x_2,\ \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f\,'(x) = 3x^2 - 2$.Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0,\ x_1,\ x_2,\ \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (Figure 14.7.6.5). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

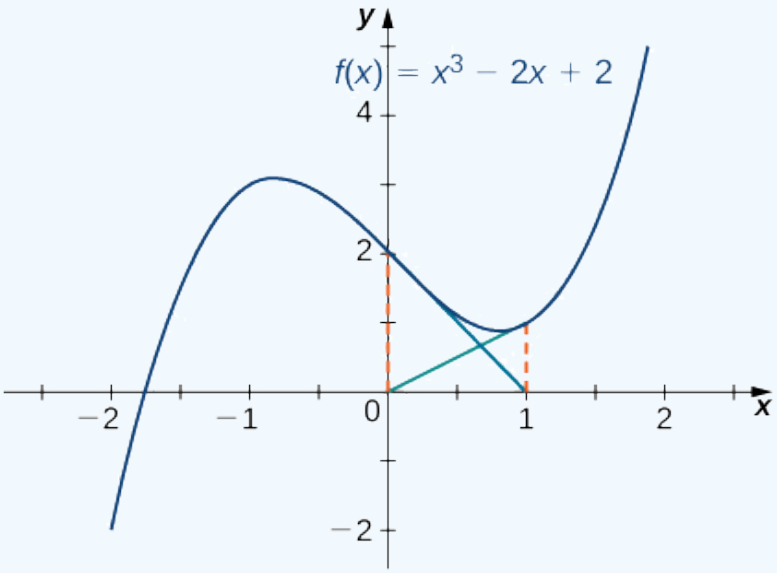


Figure 14.7.6.5: The approximations continue to alternate between 0 and 1 and never approach the root of f.

Exercise 14.7.6.3

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Hint available at the end of this section.

Answer available at the end of this section.

From Example 14.7.6.3, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative

process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0,\ x_1,\ x_2,\ \ldots,\ x_n,\ \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In Example 14.7.6.4, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example 14.7.6.4: Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. Make a conjecture about what happens to this list of numbers $x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots$ as $n \to \infty$. If the list of numbers $x_1,\ x_2,\ x_3,\ \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

- $x_1 = \frac{1}{2}(0) + 4 = 4$
- $x_2 = \frac{1}{2}(4) + 4 = 6$
- $x_3 = \frac{1}{2}(6) + 4 = 7$
- $x_4 = \frac{1}{2}(7) + 4 = 7.5$
- $x_5 = \frac{1}{2}(7.5) + 4 = 7.75$
- $x_6 = \frac{1}{2}(7.75) + 4 = 7.875$
- $x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$
- $x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$
- $x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$

From this list, we conjecture that the values x_n approach 8.

Figure 14.7.6.6 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ approaches $x * = 8$ as $n \to \infty$.

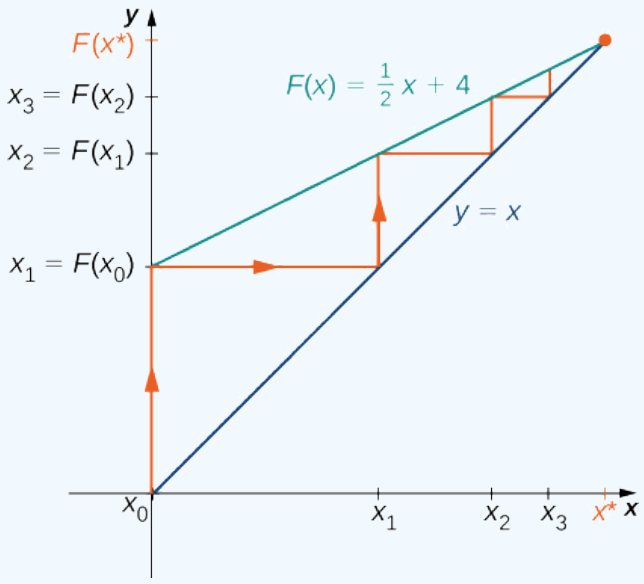


Figure 14.7.6.6: This iterative process approaches the value $x * = 8$.

Exercise 14.7.6.4

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find $x_1,\ x_2,\ x_3,\ x_4,\ x_5$. Make a conjecture about what happens to the list of numbers $x_1,\ x_2,\ x_3,\ \ldots\ x_n,\ \ldots$ as $n \to \infty$.

Hint available at the end of this section.

Answer available at the end of this section.

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example 14.7.6.4 that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2–cycles, 4–cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of **chaos** is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure 14.7.6.7), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

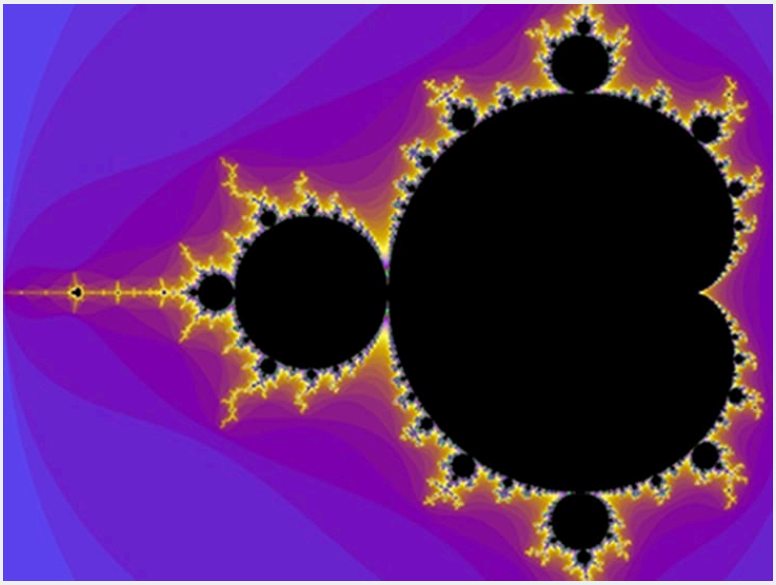


Figure 14.7.6.7: The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1 - x) \tag{14.7.6.2}$$

where $x \in [0, 1]$ and $r > 0$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a cobweb diagram. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

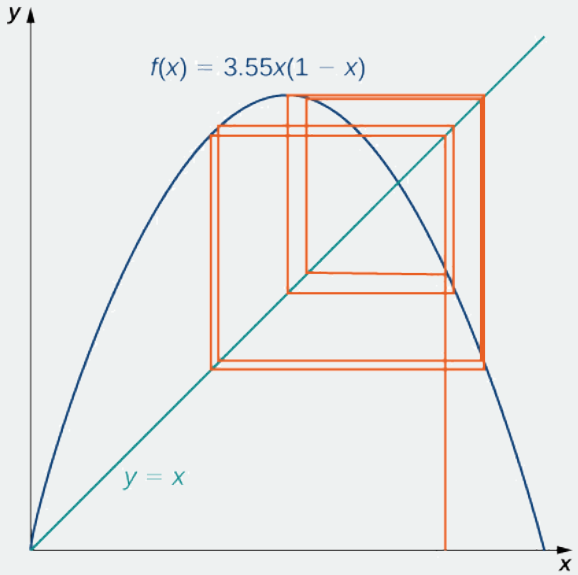


Figure $14.7.6.8$: A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, 2–cycle, 4–cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Key Concepts

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1,\ x_2,\ x_3,\ \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0,\ x_1,\ x_2,\ \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

Glossary

iterative process
: process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Newton's method
: method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.7.6.1

Use Equation 14.7.6.1.

Exercise 14.7.6.1

$x_1 \approx 0.33333333$
$x_2 \approx 0.347222222$

Exercise 14.7.6.2

For $f(x) = x^2 - 3$, Equation 14.7.6.1 reduces to $x_n = \frac{x_{n-1}}{2} + \frac{3}{2x_{n-1}}$.

Exercise 14.7.6.2

$x_1 = 2$
$x_2 = 1.75$

Exercise 14.7.6.3

Use Equation 14.7.6.1.

Exercise 14.7.6.3

$x_1 \approx -1.842105263$
$x_2 \approx -1.772826920$

Exercise 14.7.6.4

Consider the point where the lines $y = x$ and $y = F(x)$ intersect.

Exercise 14.7.6.4

$x_1 = 6,\ \ x_2 = 8,\ \ x_3 = \frac{26}{3},\ \ x_4 = \frac{80}{9},\ \ x_5 = \frac{242}{27};\ \ x^* = 9$

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14.8: INFINITESIMAL CALCULUS FOR INTEGRATION

This chapter discussing the calculus of integration in a very brief manner. Proofs are limited and applications are more emphasized here. Most of the sections here are from other Libretexts courses/books rather than the main authors of this course. Tables, applications, and tidbits are herein but nothing more and as such this does not replace a calculus course where you will learn most of the theory properly. Here we are really only concerned with how integrals are used in engineering and science. While you should try and understand the basics in this chapter, it is not necessary to understand the full scope presented here as you will have that opportunity in your calculus class proper.

As we stated previously, integrals can be defined by a Riemann sum.

$$F(x) = \int_a^b f(x)dx = \lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(x_i)\Delta x \tag{14.8.1}$$

This is not a perfect definition, but is sufficient for our proposes here. A true integral is actually an antiderivative, so the derivative is the place to look for the idea on an integral. We will leave the complexities of integration for your calculus class and in the preceding sections give a table of integrals and some application.

Finally, the following Wolfram Demonstration exemplifies the Riemann sum. You can change the number of boxes (n) to reduce your Δx but cannot, obviously, have $||\Delta x||$ approach zero. With this you can imagine it though.

Interactive Content

This interactive element is not available in this format. Visit the link or scan the QR code to access it online.

View this Interactive Content online

Example of Riemann Sums using a Wolfram Demonstration Project. The controls are useable for you to try to reduces the Δx. Phil Ramsden, "Riemann Sums: A Simple Illustration", http://demonstrations.wolfram.com/RiemannSumsASimpleIllustration/,Wolfram Demonstrations Project, Published: March 7 2011

14.8.1: Table of Integrals

14.8.2: Substitution

14.8.3: Applications of Integration

14.8.3.1: Prelude to Applications of Integration

14.8.3.2: Volumes of Revolution - Cylindrical Shells

14.8.3.3: Arc Length of a Curve and Surface Area

14.8.3.4: Physical Applications of Integration

14.8.3.5: Moments and Centers of Mass

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14.8.1: Table of Integrals

This table of integrals is from another open textbook in Libretexts and OpenStax.

Most Common Integrals

Basic Integrals

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{dx}{x} = \ln |x| + C$
3. $\int e^x \, dx = e^x + C$

or more generally

3. $\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$
4. $\int a^x \, dx = \frac{a^x}{\ln a} + C$
5. $\int \sin x \, dx = -\cos x + C$
6. $\int \cos x \, dx = \sin x + C$
7. $\int \sec^2 x \, dx = \tan x + C$
8. $\int \csc^2 x \, dx = -\cot x + C$
9. $\int \sec x \tan x \, dx = \sec x + C$
10. $\int \csc x \cot x \, dx = -\csc x + C$
11. $\int \tan x \, dx = \ln |\sec x| + C$
12. $\int \cot x \, dx = \ln |\sin x| + C$
13. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$
14. $\int \csc x \, dx = \ln |\csc x - \cot x| + C$
15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \left(\frac{x}{a} \right) + C$
16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \left(\frac{x}{a} \right) + C$

17. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{arcsec}\left(\frac{|x|}{a}\right) + C$

Trigonometric Integrals

18. $\int \sin^2 x \ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

19. $\int \cos^2 x \ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

20. $\int \tan^2 x \ dx = \tan x - x + C$

21. $\int \cot^2 x \ dx = -\cot x - x + C$

22. $\int \sin^3 x \ dx = -\frac{1}{3}(2 + \sin^2 x)\cos x + C$

23. $\int \cos^3 x \ dx = \frac{1}{3}(2 + \cos^2 x)\sin x + C$

24. $\int \tan^3 x \ dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

25. $\int \cot^3 x \ dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

26. $\int \sec^3 x \ dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

27. $\int \csc^3 x \ dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$

28. $\int \sin^n x \ dx = \frac{-1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \ dx$

29. $\int \cos^n x \ dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \ dx$

30. $\int \tan^n x \ dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x \ dx$

31. $\int \cot^n x \ dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x \ dx$

32. $\int \sec^n x \ dx = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x \ dx$

33. $\int \csc^n x \ dx = \frac{-1}{n-1}\cot x \csc^{n-2} x + \frac{n-2}{n-1}\int \csc^{n-2} x \ dx$

34. $\int \sin ax \sin bx \ dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} + C$

35. $\int \cos ax \cos bx \ dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} + C$

36. $\int \sin ax \cos bx \ dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)} + C$

37. $\int x \sin x \ dx = \sin x - x\cos x + C$

38. $\int x\cos x\ dx = \cos x + x\sin x + C$

39. $\int x^n \sin x\ dx = -x^n \cos x + n\int x^{n-1}\cos x\ dx$

40. $\int x^n \cos x\ dx = x^n \sin x - n\int x^{n-1}\sin x\ dx$

41. $\int \sin^n x\cos^m x\ dx =$ use the methods for powers of sine and cosine

Exponential and Logarithmic Integrals

42. $\int xe^{ax}\ dx = \frac{1}{a^2}(ax-1)e^{ax} + C$

Example of integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

43. $\int x^n e^{ax}\ dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\ dx$

44. $\int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

45. $\int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

46. $\int \ln x\ dx = x\ln x - x + C$

47. $\int x^n \ln x\ dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1] + C$

48. $\int \frac{1}{x\ln x}\ dx = \ln|\ln x| + C$

Less Common Integrals

Hyperbolic Integrals

49. $\int \sinh x\ dx = \cosh x + C$

50. $\int \cosh x\ dx = \sinh x + C$

51. $\int \tanh x\ dx = \ln\cosh x + C$

52. $\int \coth x\ dx = \ln|\sinh x| + C$

53. $\int \operatorname{sech}\ x\ dx = \arctan|\sinh x| + C$

54. $\int \operatorname{csch}\ x\ dx = \ln|\tanh\frac{1}{2}x| + C$

55. $\int \operatorname{sech}^2 x \; dx = \tanh \; x + C$

56. $\int \operatorname{csch}^2 x \; dx = -\coth \; x + C$

57. $\int \operatorname{sech} \; x \tanh x \; dx = -\operatorname{sech} \; x + C$

58. $\int \operatorname{csch} \; x \coth x \; dx = -\operatorname{csch} \; x + C$

Inverse Trigonometric Integrals

59. $\int \arcsin x \; dx = x \arcsin x + \sqrt{1 - x^2} + C$

60. $\int \arccos x \; dx = x \arccos x - \sqrt{1 - x^2} + C$

61. $\int \arctan x \; dx = x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$

62. $\int x \arcsin x \; dx = \frac{2x^2 - 1}{4} \arcsin x + \frac{x\sqrt{1 - x^2}}{4} + C$

63. $\int x \arccos x \; dx = \frac{2x^2 - 1}{4} \arccos x - \frac{x\sqrt{1 - x^2}}{4} + C$

64. $\int x \arctan x \; dx = \frac{x^2 + 1}{2} \arctan x - \frac{x}{2} + C$

65. $\int x^n \arcsin x \; dx = \frac{1}{n+1} \left[x^{n+1} \arcsin x - \int \frac{x^{n+1} \, dx}{\sqrt{1 - x^2}} \right], \quad n \neq -1$

66. $\int x^n \arccos x \; dx = \frac{1}{n+1} \left[x^{n+1} \arccos x + \int \frac{x^{n+1} \, dx}{\sqrt{1 - x^2}} \right], \quad n \neq -1$

67. $\int x^n \arctan x \; dx = \frac{1}{n+1} \left[x^{n+1} \arctan x - \int \frac{x^{n+1} \, dx}{1 + x^2} \right], \quad n \neq -1$

Integrals with Fractions and Square roots

Integrals Involving $a^2 + x^2$, $a > 0$

68. $\int \sqrt{a^2 + x^2} \; dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$

69. $\int x^2 \sqrt{a^2 + x^2} \; dx = \frac{x}{8} (a^2 + 2x^2) \sqrt{a^2 + x^2} - \frac{a^4}{8} \ln \left(x + \sqrt{a^2 + x^2} \right) + C$

70. $\int \frac{\sqrt{a^2 + x^2}}{x} \; dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right| + C$

71. $\int \frac{\sqrt{a^2 + x^2}}{x^2} \; dx = -\frac{\sqrt{a^2 + x^2}}{x} + \ln \left(x + \sqrt{a^2 + x^2} \right) + C$

72. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right) + C$

73. $$\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = \frac{x}{2}\left(\sqrt{a^2+x^2}\right) - \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + C$$

74. $$\int \frac{dx}{x\sqrt{a^2+x^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+x^2}+a}{x}\right| + C$$

75. $$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$$

76. $$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

Integrals Involving $x^2 - a^2$, $a > 0$

77. $$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

78. $$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

79. $$\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|} + C$$

80. $$\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = -\frac{\sqrt{x^2-a^2}}{x} + \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

81. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

82. $$\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

83. $$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

84. $$\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2\sqrt{x^2-a^2}} + C$$

Integrals Involving $a^2 - x^2$, $a > 0$

85. $$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

86. $$\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a} + C$$

87. $$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$$

88. $$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = \frac{-1}{x}\sqrt{a^2-x^2} - \arcsin\frac{x}{a} + C$$

89. $$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{1}{2}\left(-x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + C$$

90. $$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$$

91. $$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{1}{a^2 x}\sqrt{a^2-x^2} + C$$

92. $$\int \left(a^2-x^2\right)^{3/2}\, dx = -\frac{x}{8}\left(2x^2-5a^2\right)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a} + C$$

93. $$\int \frac{dx}{(a^2-x^2)^{3/2}} = -\frac{x}{a^2\sqrt{a^2-x^2}} + C$$

Integrals Involving $a^2 - x^2$, $x^2 < a^2$

94. $$\int \frac{dx}{(a^2-x^2)} = \frac{1}{a}\tanh^{-1}\frac{x}{a}$$

95. $$\int \frac{x^3 dx}{(a^2-x^2)^2} = \frac{x}{2(a^2-x^2)} - \frac{1}{2}\ln(a^2-x^2)$$

96. $$\int \frac{x dx}{(a^2-x^2)^n} = \frac{1}{2(n-1)(a^2-x^2)^{n-1}}$$

Integrals Involving $2ax - x^2$, $a > 0$

97. $$\int \sqrt{2ax-x^2}\, dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\arccos\left(\frac{a-x}{a}\right) + C$$

98. $$\int \frac{dx}{\sqrt{2ax-x^2}} = \arccos\left(\frac{a-x}{a}\right) + C$$

99. $$\int x\sqrt{2ax-x^2}\, dx = \frac{2x^2-ax-3a^2}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\arccos\left(\frac{a-x}{a}\right) + C$$

100. $$\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{ax} + C$$

Integrals Involving $a + bx$, $a \neq 0$

101. $$\int \frac{x}{a+bx}\, dx = \frac{1}{b^2}(a+bx-a\ln|a+bx|) + C$$

102. $$\int \frac{x^2}{a+bx}\, dx = \frac{1}{2b^3}\left[(a+bx)^2 - 4a(a+bx) + 2a^2\ln|a+bx|\right] + C$$

103. $$\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$$

104. $$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

105. $$\int \frac{x}{(a+bx)^2}\, dx = \frac{a}{b^2(a+bx)} + \frac{1}{b^2}\ln|a+bx| + C$$

106. $$\int \frac{x}{x(a+bx)^2}\, dx = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$$

107. $$\int \frac{x^2}{(a+bx)^2}\, dx = \frac{1}{b^3}\left(a+bx - \frac{a^2}{a+bx} - 2a\ln|a+bx|\right) + C$$

108. $$\int x\sqrt{a+bx}\, dx = \frac{2}{15b^2}(3bx-2a)(a+bx)^{3/2} + C$$

109. $$\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx}+C$$

110. $$\int \frac{x^2}{\sqrt{a+bx}}\,dx = \frac{2}{15b^3}(8a^2+3b^2x^2-4abx)\sqrt{a+bx}+C$$

111. $$\int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|+C, & \text{if } a>0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}}+C, & \text{if } a<0 \end{cases}$$

112. $$\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx}+a\int \frac{dx}{x\sqrt{a+bx}}$$

113. $$\int \frac{\sqrt{a+bx}}{x^2}\,dx = -\frac{\sqrt{a+bx}}{x}+\frac{b}{2}\int \frac{dx}{x\sqrt{a+bx}}$$

114. $$\int x^n\sqrt{a+bx}\,dx = \frac{2}{b(2n+3)}\left[x^n(a+bx)^{3/2}-na\int x^{n-1}\sqrt{a+bx}\,dx\right]$$

115. $$\int \frac{x^n}{\sqrt{a+bx}}\,dx = \frac{2x^n\sqrt{a+bx}}{b(2n+1)}-\frac{2na}{b(2n+1)}\int \frac{x^{n-1}}{\sqrt{a+bx}}\,dx$$

116. $$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{a(n-1)x^{n-1}}-\frac{b(2n-3)}{2a(n-1)}\int \frac{dx}{x^{n-1}\sqrt{a+bx}}$$

For other integrals you could use the computational intelligence website Wolframalpha.

Contributors and Attributions

- Template:ContribOpenStaxCalc
- Inverse trig notation changes by Paxl Seebxrger (Monroe Community College)
- Change u to x as the u for students was confusing by Scott Johnson (PGCC)

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14.8.2: Substitution

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

In this section we examine *integration by substitution*, which while we could have put it in the table of integrals is best served by its own section. This method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\big[g(x)\big]g'(x)\ dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{14.8.2.1}$$

we assign the following to the functions to produce the integrand form:

$$f(x)=x^3$$

$$g(x)=x^2-3.$$

$$g'(x)=2x.$$

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du\\ &= F(u)+C\\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big]=F'(g(x))g'(x)=f[g(x)]g'(x). \tag{14.8.2.2}$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C. \tag{14.8.2.3}$$

If we now substitute $u=g(x)$, and $du=g'(x)\ dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C. \tag{14.8.2.4}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\ dx$.

Rewrite the integral (Equation 14.8.2.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g\,'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g\,'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 14.8.2.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\ dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\ dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right) 5(3x^2 + 4)^4 6x$$
$$= 6x(3x^2 + 4)^4.$$

This is exactly the expression we started with inside the integrand.

Exercise 14.8.2.1

Use substitution to find the antiderivative of $\int 3x^2(x^3 - 3)^2 \, dx$.

Hint available at the end of this section.

Answer available at the end of this section.

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example 14.8.2.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2 - 5}\, dz.$$

Solution

Rewrite the integral as $\int z(z^2 - 5)^{1/2} \, dz$. Let $u = z^2 - 5$ and $du = 2z \, dz$. Now we have a problem because $du = 2z \, dz$ and the original expression has only $z \, dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$

$$du = 2z\, dz$$

$$\frac{1}{2}du = \frac{1}{2}(2z)\, dz = z\, dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2 - 5)^{1/2}\, dz = \frac{1}{2}\int u^{1/2}\, du.$$

Integrate the expression in u:

$$\frac{1}{2}\int u^{1/2}\,du = \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(z^2-5)^{3/2} + C$$

Exercise 14.8.2.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\ dx$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.2.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\ dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\ dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

Exercise 14.8.2.3

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}\ dt$.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.8.2.4

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t \; dt$.

Hint available at the end of this section.

Answer available at the end of this section.

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 14.8.2.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\, dx.$$

Solution

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u :

$$\int \frac{x}{\sqrt{x-1}}\, dx = \int \frac{u+1}{\sqrt{u}}\, du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) du = \int \left(u^{1/2} + u^{-1/2}\right) du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}
\int (u^{1/2} + u^{-1/2})\, du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\
&= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\
&= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \tag{14.8.2.5}$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{14.8.2.6}$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 14.8.2.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2\ dx$. Since the original function includes one factor of x^2 and $du = 6x^2\ dx$, multiply both sides of the du equation by $1/6$. Then,

$$\begin{aligned} du &= 6x^2\,dx \\ \text{becomes}\quad \frac{1}{6}du &= x^2\,dx.\end{aligned}$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1$, and when $x = 1,\ \ u = 1 + 2(1) = 3$.

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3 \\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}.\end{aligned}$$

Exercise 14.8.2.5

Use substitution to evaluate the definite integral $\int_{-1}^{0} y(2y^2-3)^5 \, dy$.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.8.2.6

Use substitution to evaluate $\int_{0}^{1} x^2 \cos\left(\frac{\pi}{2}x^3\right) \, dx$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.2.6: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_{0}^{1} xe^{4x^2+3} \, dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x \; dx$. To adjust the limits of integration, we note that when $x = 0, \; u = 3$, and when $x = 1, \; u = 7$. So our substitution gives

$$\begin{aligned} \int_{0}^{1} xe^{4x^2+3} \, dx &= \frac{1}{8}\int_{3}^{7} e^u \, du \\ &= \frac{1}{8}e^u\Big|_{3}^{7} \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568 \end{aligned}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 14.8.2.7.

Example 14.8.2.7: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_{0}^{\pi/2} \cos^2\theta \, d\theta.$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\ d\theta$, or $\dfrac{1}{2}\ du = d\theta$. Also, when $\theta = 0,\ \ u = 0$, and when $\theta = \pi/2,\ \ u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\theta} \\ &= \left(\frac{\pi}{4}-0\right) + (0-0) = \frac{\pi}{4}\end{aligned}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Glossary

change of variables
: the substitution of a variable, such as u, for an expression in the integrand

integration by substitution
: a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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Answers

Exercise 14.8.2.1

Let $u = x^3 - 3$.

Exercise 14.8.2.1

$$\int 3x^2(x^3-3)^2 \, dx = \frac{1}{3}(x^3-3)^3 + C$$

Exercise 14.8.2.2

Multiply the du equation by $\frac{1}{3}$.

Exercise 14.8.2.2

$$\int x^2(x^3+5)^9 \, dx = \frac{(x^3+5)^{10}}{30} + C$$

Exercise 14.8.2.3

Use the process from Example 14.8.2.3 to solve the problem.

Exercise 14.8.2.3

$$\int \frac{\cos t}{\sin^2 t} \, dt = -\frac{1}{\sin t} + C$$

Exercise 14.8.2.4

Use the process from Example 14.8.2.3 to solve the problem.

Exercise 14.8.2.4

$$\int \cos^3 t \sin t \, dt = -\frac{\cos^4 t}{4} + C$$

Exercise 14.8.2.5

Use the steps from Example 14.8.2.5 to solve the problem.

Exercise 14.8.2.5

$$\int_{-1}^{0} y(2y^2-3)^5 \, dy = \frac{91}{3}$$

Exercise 14.8.2.6

Use the process from Example 14.8.2.5 to solve the problem.

Exercise 14.8.2.6

$$\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right) \, dx = \frac{2}{3\pi} \approx 0.2122$$

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14.8.3: APPLICATIONS OF INTEGRATION

In this section, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

14.8.3.1: Prelude to Applications of Integration

14.8.3.2: Volumes of Revolution - Cylindrical Shells

14.8.3.3: Arc Length of a Curve and Surface Area

14.8.3.4: Physical Applications of Integration

14.8.3.5: Moments and Centers of Mass

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14.8.3.1: Prelude to Applications of Integration

The Hoover Dam, a concrete arch-gravity dam finished in 1936, is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.



Figure 14.8.3.1.1: Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia).

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14.8.3.2: Volumes of Revolution - Cylindrical Shells

Learning Objectives

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, an example method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method which is not discussed in this course/book but should be discussed in you proper calculus course. Note that with the disk and washer methods, we would integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the **x-axis**, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure $14.8.3.2.1a$. We then revolve this region around the y-axis, as shown in Figure $14.8.3.2.1b$. Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the **x-axis** or a line parallel to it.

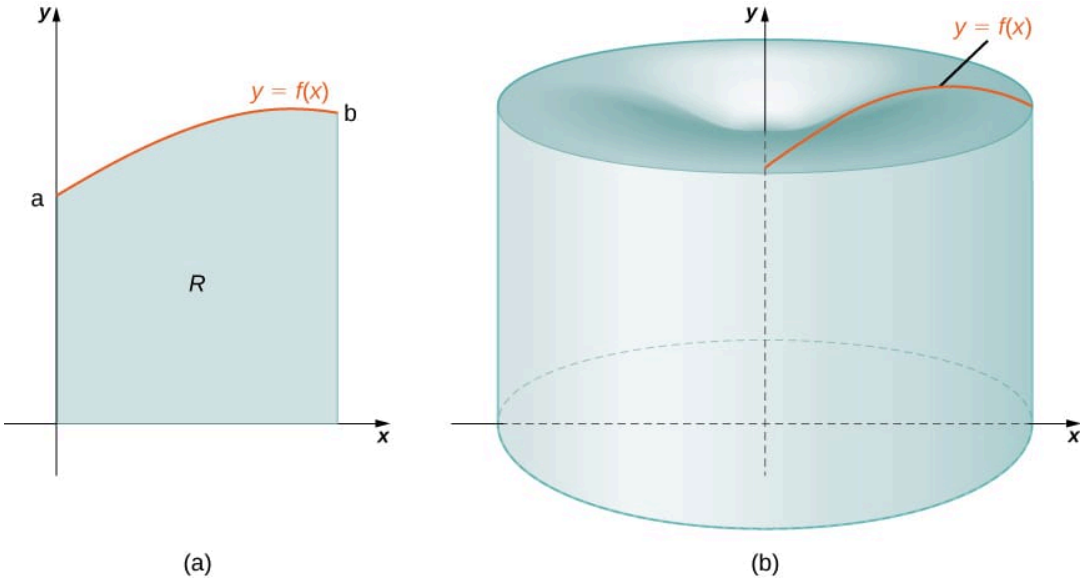


Figure $14.8.3.2.1$: (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the **y-axis.**

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \ldots, x_n$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure $14.8.3.2.2a$. When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Figure $14.8.3.2.2$.

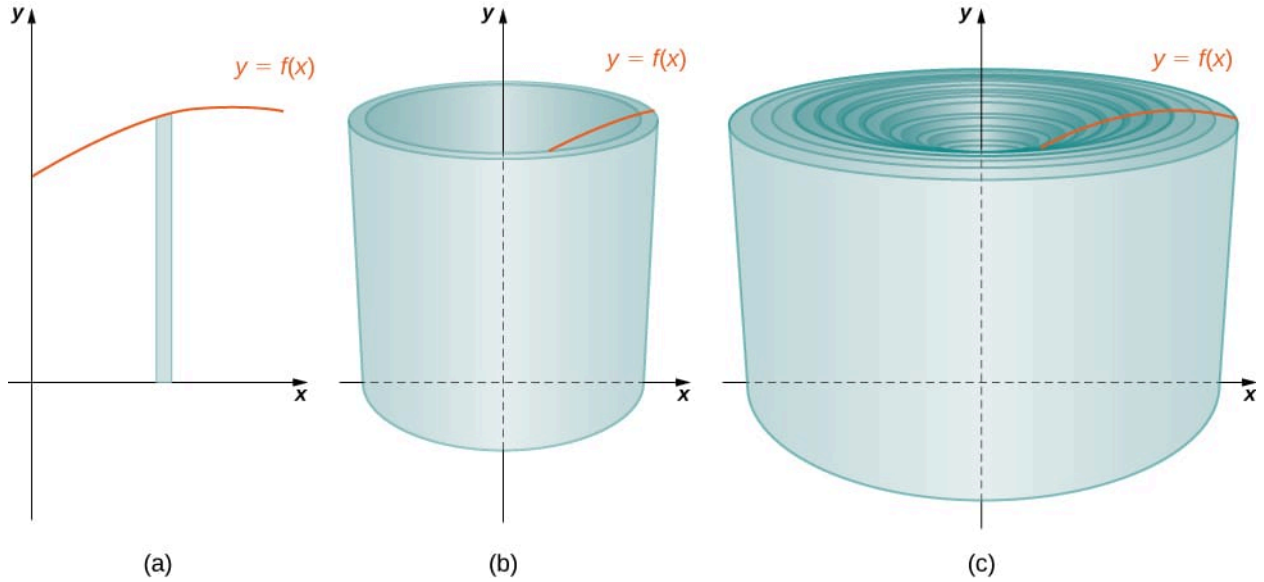


Figure 14.8.3.2.2: (a) A representative rectangle. (b) When this rectangle is revolved around the **y-axis**, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 14.8.3.2.3.

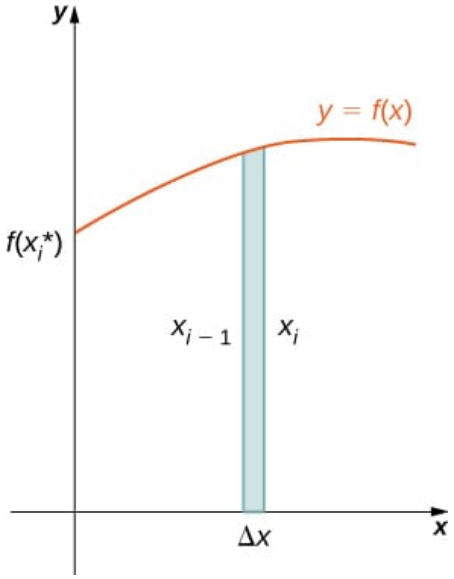


Figure 14.8.3.2.3: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{shell} = 2\pi\, f(x_i^*) \left(\frac{x_i + x_{i-1}}{2} \right) \Delta x. \tag{14.8.3.2.1}$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{shell} \approx 2\pi\, f(x_i^*) x_i^*\, \Delta x. \tag{14.8.3.2.2}$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure 14.8.3.2.4).

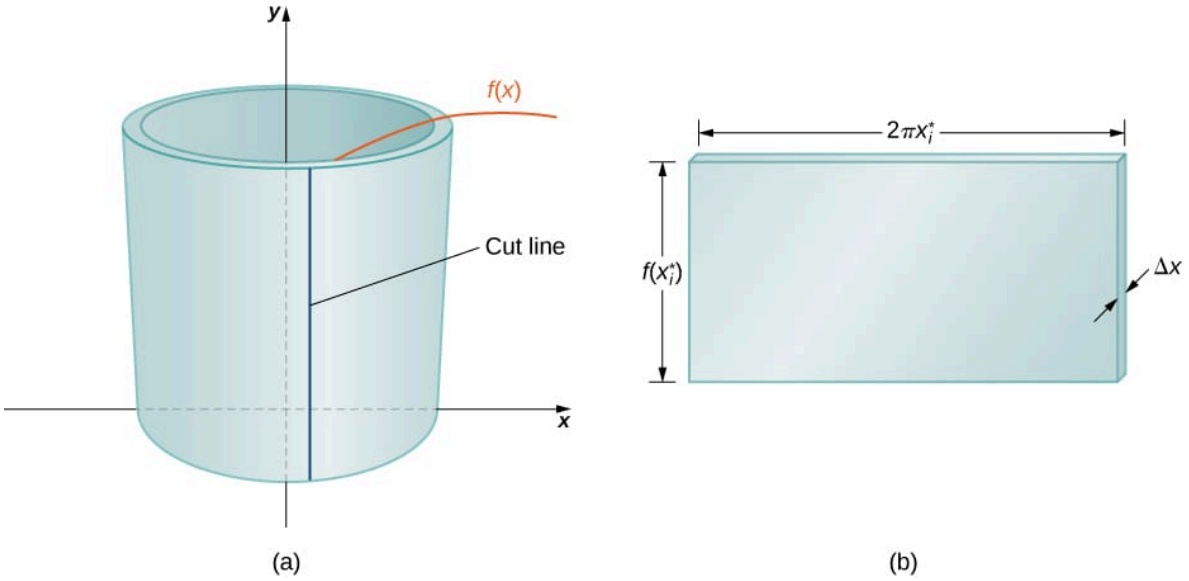


Figure 14.8.3.2.4: (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (Figure). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{shell} \approx f(x_i^*)(2\pi\, x_i^*)\, \Delta x, \tag{14.8.3.2.3}$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\, \Delta x). \tag{14.8.3.2.4}$$

Here we have another Riemann sum, this time for the function $2\pi\ x\ f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\, \Delta x) = \int_a^b (2\pi\, x\, f(x))\, dx. \tag{14.8.3.2.5}$$

This leads to the following rule for the *method of cylindrical shells*.

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the **x-axis,** on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx. \tag{14.8.3.2.6}$$

Now let's consider an example.

Example 14.8.3.2.1: The Method of Cylindrical Shells I

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the **x-axis** over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in Figure 14.8.3.2.5.

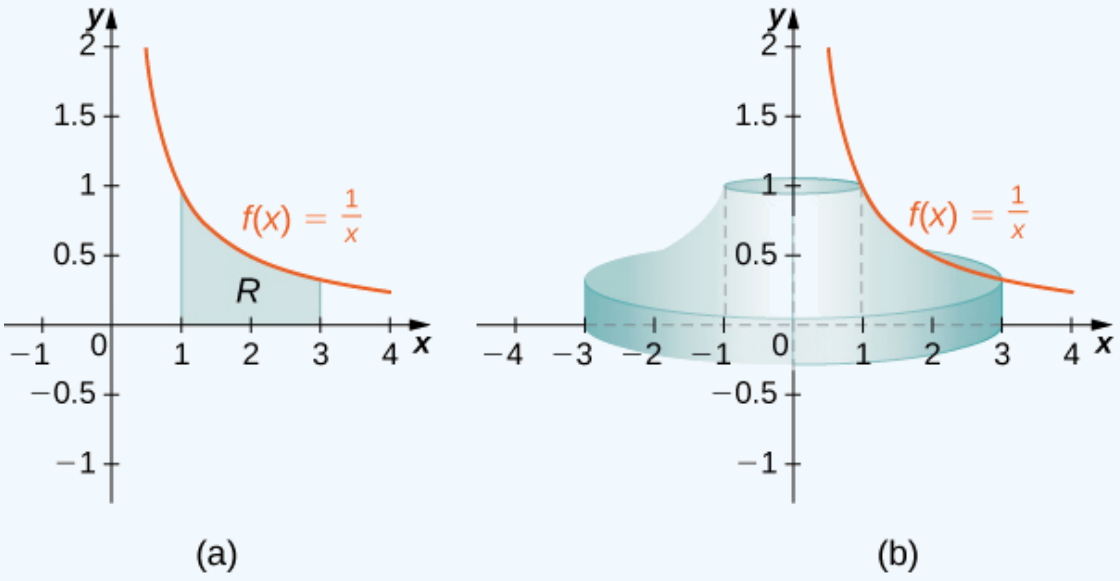


Figure 14.8.3.2.5: (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

Figure 14.8.3.2.5 (c) Visualizing the solid of revolution with CalcPlot3D.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi\, x\, f(x))\, dx \\ &= \int_1^3 \left(2\pi\, x \left(\frac{1}{x}\right)\right) dx \\ &= \int_1^3 2\pi\, dx \\ &= 2\pi\, x\Big|_1^3 = 4\pi \text{ units}^3. \end{aligned}$$

Exercise 14.8.3.2.1

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.3.2.2: The Method of Cylindrical Shells II

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y**-axis.**

Solution

First graph the region R and the associated solid of revolution, as shown in Figure 14.8.3.2.6.

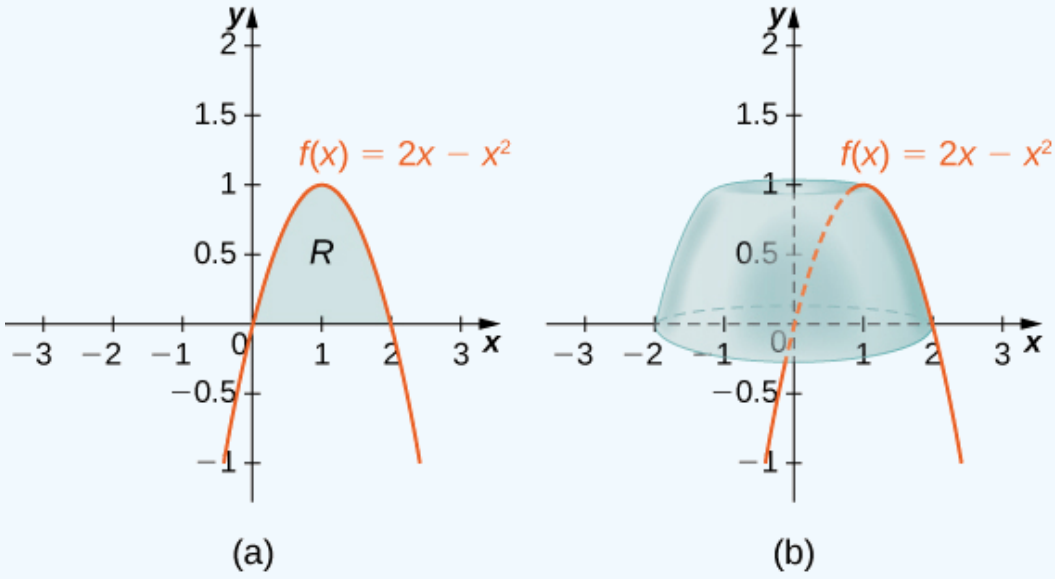


Figure 14.8.3.2.6: (a) The region R *under the graph of* $f(x) = 2x - x^2$ *over the interval* $[0, 2]$. *(b) The volume of revolution obtained by revolving* R *about the* y*-axis.*

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi\, x\, f(x))\, dx \\ &= \int_0^2 (2\pi\, x(2x - x^2))\, dx \\ &= 2\pi \int_0^2 (2x^2 - x^3)\, dx \\ &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right] \Bigg|_0^2 \\ &= \frac{8\pi}{3} \text{ units}^3 \end{aligned}$$

Exercise 14.8.3.2.2

Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint available at the end of this section.

Answer available at the end of this section.

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the x-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi\, y\, g(y))\, dy. \tag{14.8.3.2.7}$$

Example 14.8.3.2.3: The Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in Figure 14.8.3.2.7.

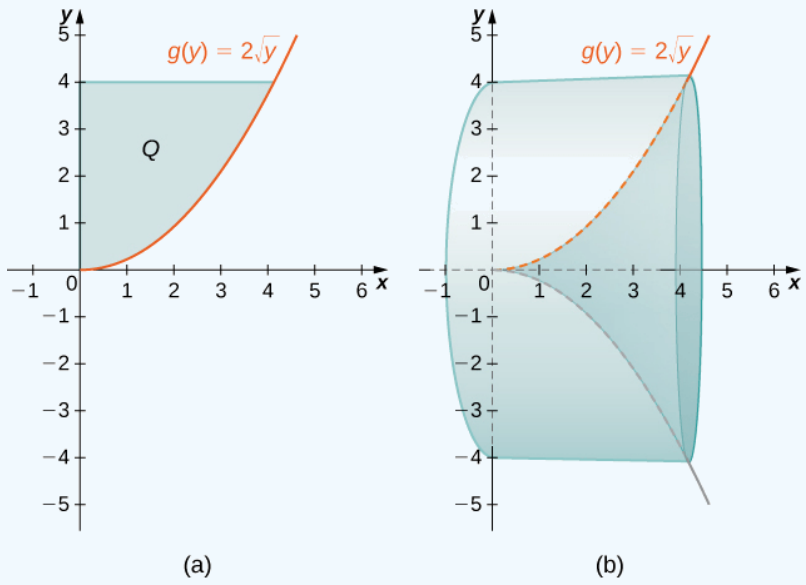


Figure 14.8.3.2.7: (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi\, y\, g(y))\, dy \\ &= \int_0^4 (2\pi\, y(2\sqrt{y}))\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5}\right] |_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

Exercise 14.8.3.2.3

Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Hint available at the end of this section.

Answer available at the end of this section.

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*) \left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{shell} &= 2\pi\, f(x_i^*)(\frac{(x_i + k) + (x_{i-1} + k)}{2})((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi\, f(x_i^*) \left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right) \Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{shell} \approx 2\pi(x_i^* + k) f(x_i^*) \Delta x. \tag{14.8.3.2.8}$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k) f(x)) dx. \tag{14.8.3.2.9}$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 14.8.3.2.4: A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 14.8.3.2.8.

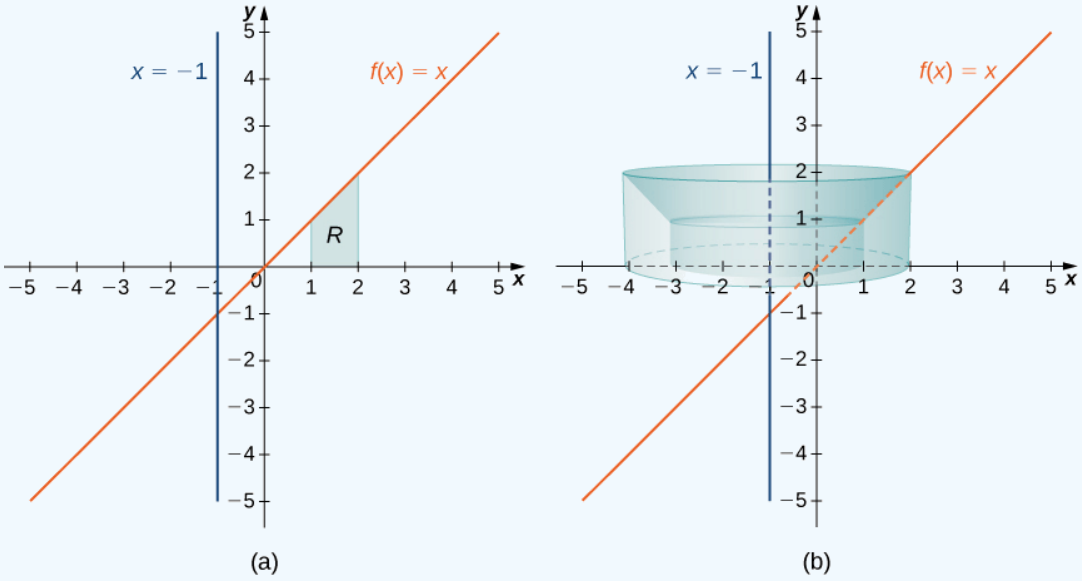


Figure 14.8.3.2.8: (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)f(x)\,dx \\ &= \int_1^2 2\pi(x+1)x\,dx = 2\pi\int_1^2 x^2 + x\,dx \\ &= 2\pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{23\pi}{3}\ \text{units}^3 \end{aligned}$$

Exercise 14.8.3.2.4

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

Hint available at the end of this section.

Answer available at the end of this section.

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 14.8.3.2.5: A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 14.8.3.2.9.

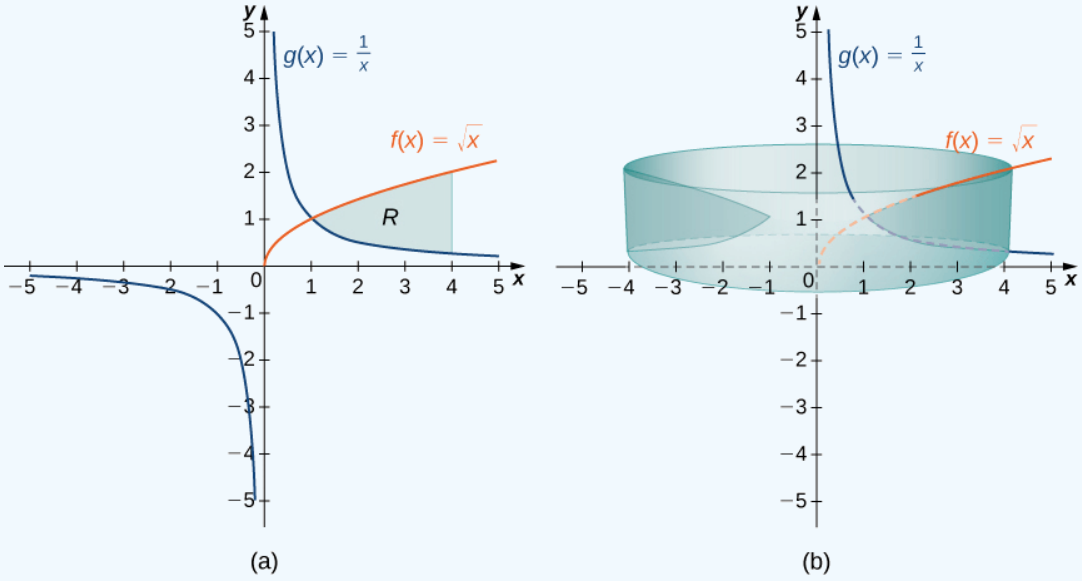


Figure 14.8.3.2.9: (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi\, x(f(x) - g(x)))\, dx$$

$$= \int_1^4 (2\pi\, x(\sqrt{x} - \frac{1}{x}))\, dx = 2\pi \int_1^4 (x^{3/2} - 1) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - x \right] \Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.$$

Exercise 14.8.3.2.5

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Which Method Should We Use?

There are several methods for finding the volume of a solid of revolution, but *how do we know which method to use?* It often comes down to a choice of which integral is easiest to evaluate. Figure 14.8.3.2.10 describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the *x*-axis

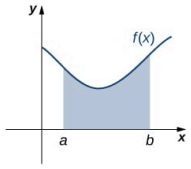
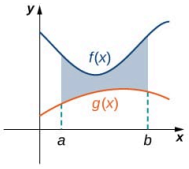
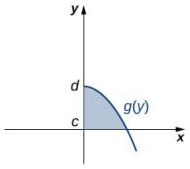
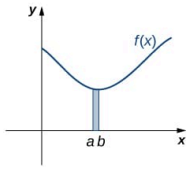
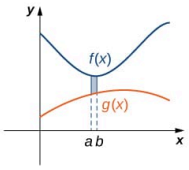
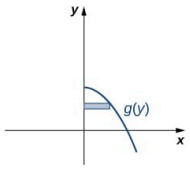
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 14.8.3.2.10

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 14.8.3.2.6: Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x, y = 2 - x$, and the x-axis.
b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a.

First, sketch the region and the solid of revolution as shown.

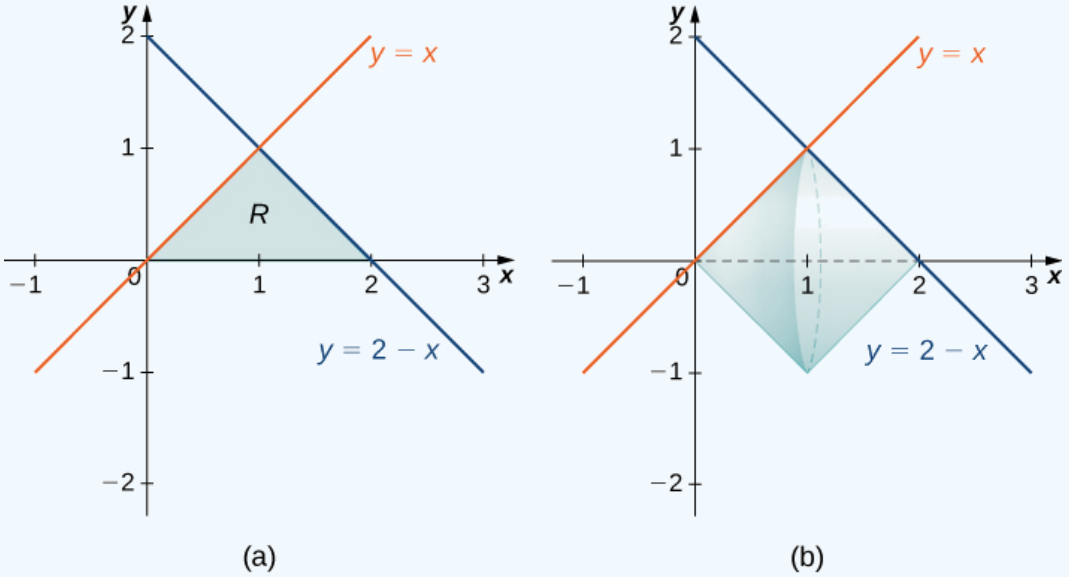


Figure 14.8.3.2.11: (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \pi x^2 \, dx + \int_1^2 \pi (2 - x)^2 \, dx.$$

If we used the shell method instead, we would use functions of y to represent the curves, producing

$$V = \int_0^1 2\pi y[(2 - y) - y] \, dy = \int_0^1 2\pi y[2 - 2y] \, dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b.

First, sketch the region and the solid of revolution as shown.

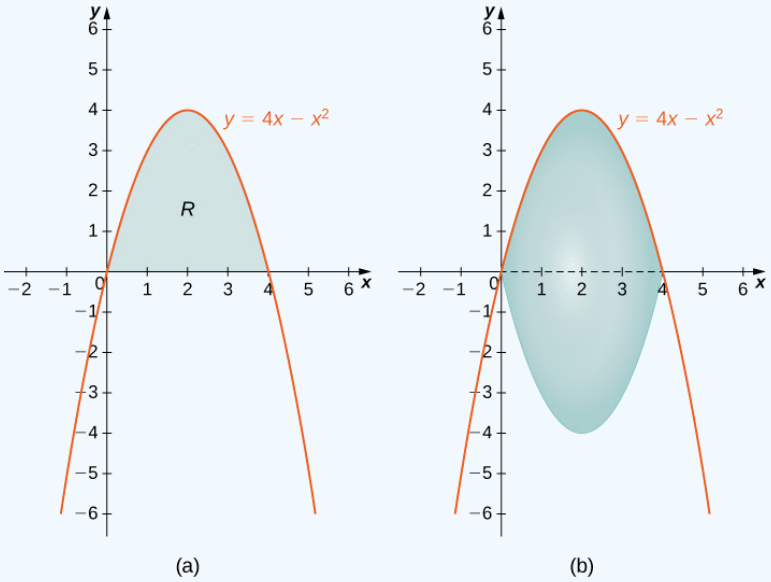


Figure 14.8.3.2.12: (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 \, dx$$

Exercise 14.8.3.2.6

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b (2\pi \, x \, f(x)) \, dx$$

Glossary

method of cylindrical shells
: a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.8.3.2.1

Use the procedure from Example 14.8.3.2.1.

Exercise 14.8.3.2.1

$\frac{15\pi}{2}$ units^3

Exercise 14.8.3.2.2

Use the process from Example 14.8.3.2.2.

Exercise 14.8.3.2.2

8π units^3

Exercise 14.8.3.2.3

Use the process from Example 14.8.3.2.3.

Exercise 14.8.3.2.3

12π units^3

Exercise 14.8.3.2.4

Use the process from Example 14.8.3.2.4.

Exercise 14.8.3.2.4

$\frac{11\pi}{6}$ units^3

Exercise 14.8.3.2.5

Hint: Use the process from Example 14.8.3.2.5.

Exercise 14.8.3.2.5

$\frac{\pi}{6}$ units^3

Exercise 14.8.3.2.6

Sketch the region and use Figure 14.8.3.2.12 to decide which integral is easiest to evaluate.

Exercise 14.8.3.2.6

Use the method of washers;

$$V = \int_{-1}^{1} \pi \left[\left(2 - x^2\right)^2 - \left(x^2\right)^2 \right] dx$$

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14.8.3.3: Arc Length of a Curve and Surface Area

Learning Objectives

- Determine the length of a curve, $y = f(x)$, between two points.
- Determine the length of a curve, $x = g(y)$, between two points.
- Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f\,'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **smooth.** (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure $14.8.3.3.1$ depicts this construct for $n = 5$.

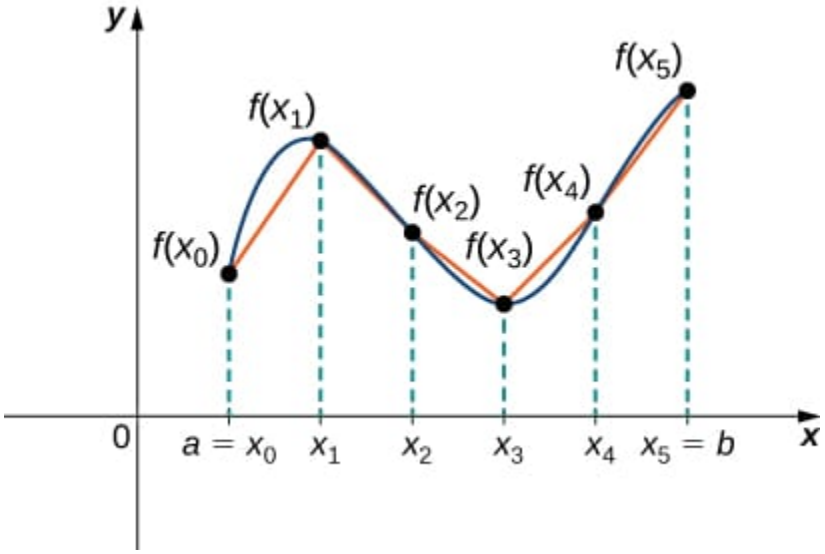


Figure $14.8.3.3.1$: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure $14.8.3.3.2$. Note that some (or all) Δy_i may be negative.

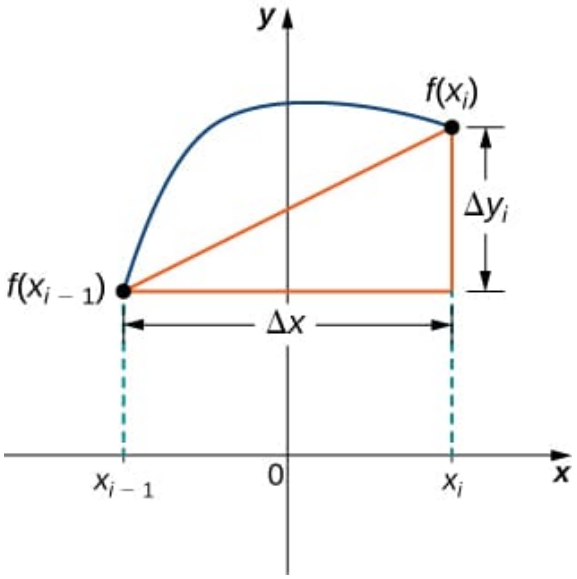


Figure 14.8.3.3.2: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We can also write this as

$$\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f\,'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x\sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\text{Arc Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}dx.\end{aligned}$$

We summarize these findings in the following theorem.

Arc Length for $y = f(x)$

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{14.8.3.3.1}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 14.8.3.3.1: Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}dx \\ &= \int_0^1 \sqrt{1 + 9x}dx.\end{aligned}$$

Substitute $u = 1 + 9x$. Then, $du = 9dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1 + 9x}dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1 + 9x}9dx \\ &= \frac{1}{9}\int_1^{10} \sqrt{u}du \\ &= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\,|_1^{10} = \frac{2}{27}[10\sqrt{10} - 1] \\ &\approx 2.268 units.\end{aligned}$$

Exercise 14.8.3.3.1

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Hint available at the end of this section.

Answer available at the end of this section.

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 14.8.3.3.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$
$$= \int_1^3 \sqrt{1+4x^2}\,dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1+4x^2}\,dx \approx 8.26815.$$

Exercise 14.8.3.3.2

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Hint available at the end of this section.

Answer available at the end of this section.

Arc Length of the Curve $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 14.8.3.3.3 shows a representative line segment.

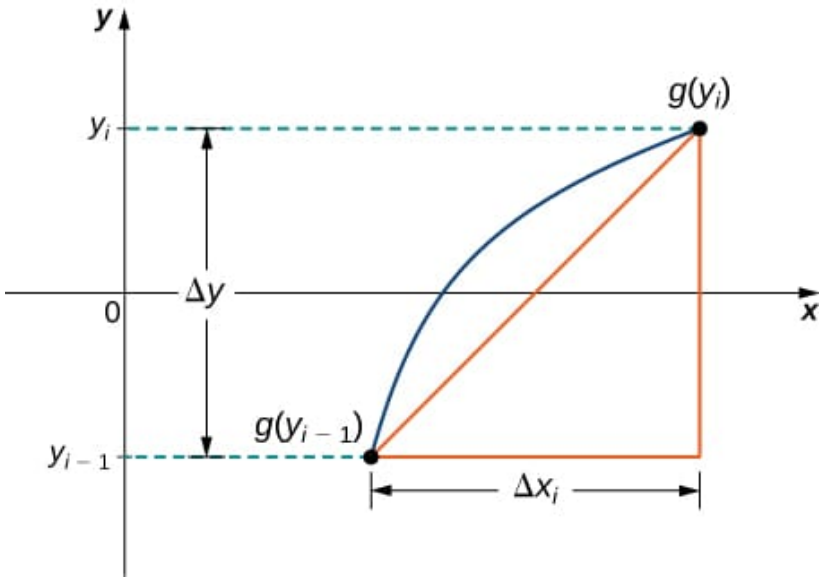


Figure 14.8.3.3.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$\sqrt{(\Delta y)^2 + (\Delta x_i)^2}, \tag{14.8.3.3.2}$$

which can also be written as

$$\Delta y \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}. \tag{14.8.3.3.3}$$

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

Arc Length for x = g(y)

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{14.8.3.3.4}$$

Example 14.8.3.3.3: Calculating the Arc Length of a Function of y

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution

We have $g\,'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\begin{aligned} \text{Arc Length} &= \int_c^d \sqrt{1 + [g'(y)]^2}\, dy \\ &= \int_1^2 \sqrt{1 + 81y^4}\, dy. \end{aligned}$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1 + 81y^4}\, dy \approx 21.0277.$$

Exercise 14.8.3.3.3

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Hint available at the end of this section.

Answer available at the end of this section.

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

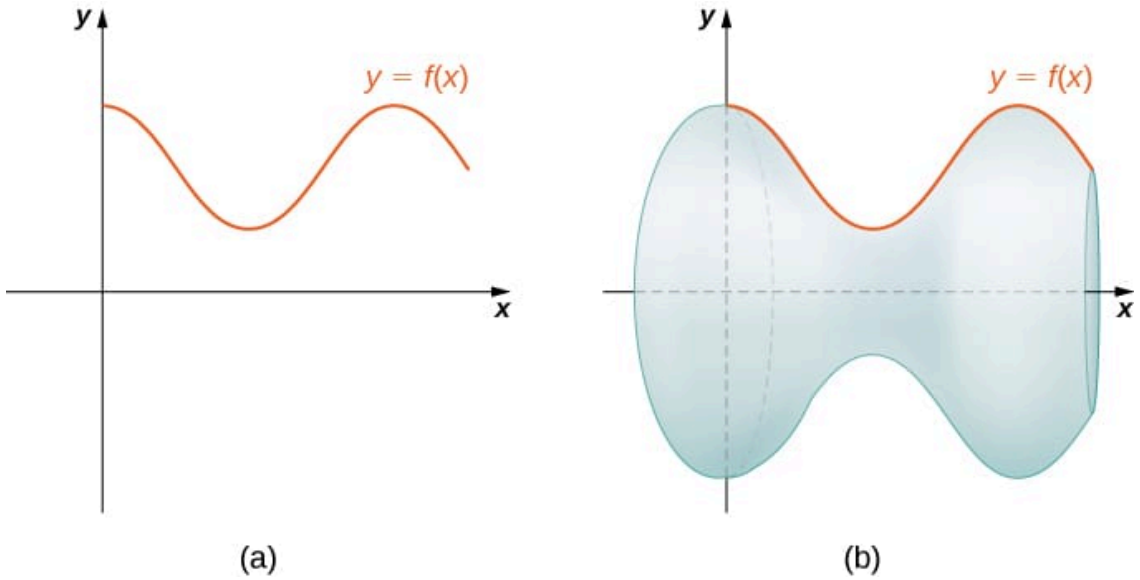


Figure 14.8.3.3.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x-axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

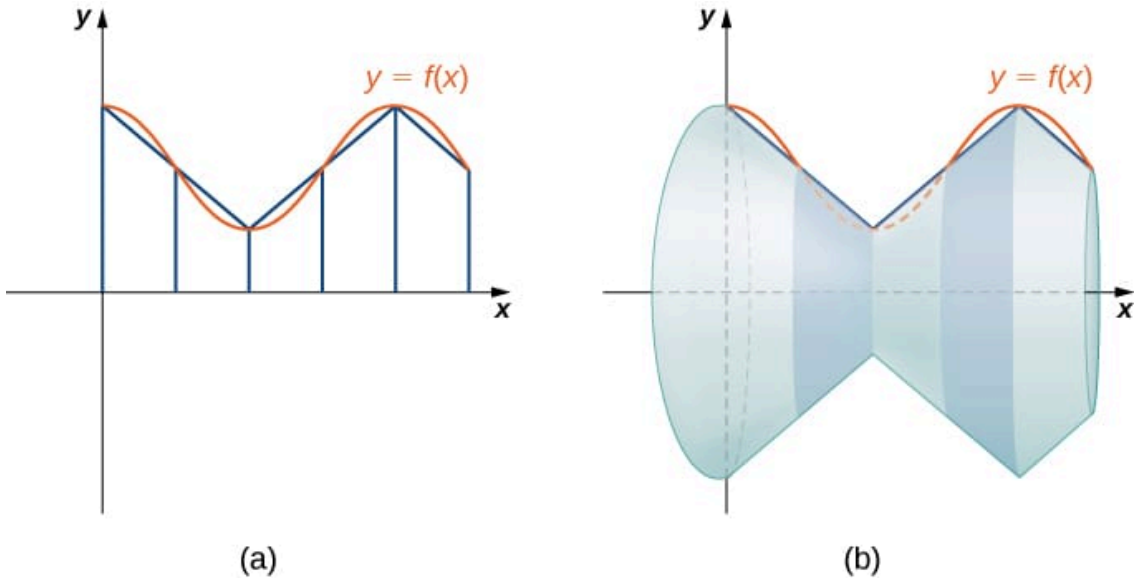


Figure 14.8.3.3.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the $x-axis$.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

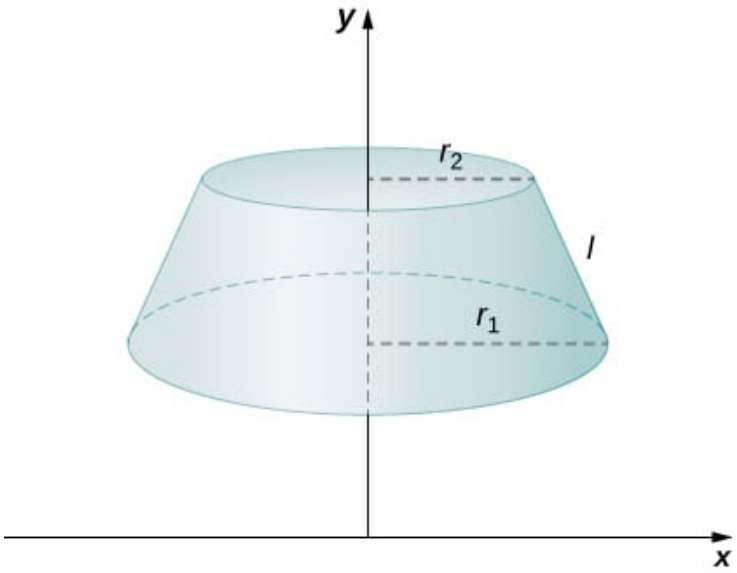


Figure 14.8.3.3.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi rs, \tag{14.8.3.3.5}$$

where r is the radius of the base of the cone and s is the **slant height** (Figure 14.8.3.3.7).

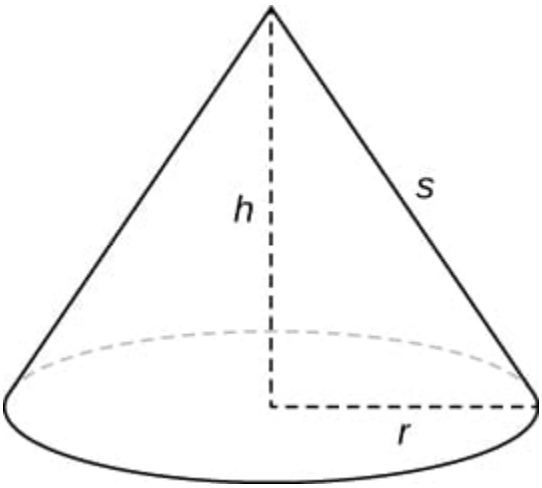


Figure 14.8.3.3.7: The lateral surface area of the cone is given by πrs.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 14.8.3.3.8).

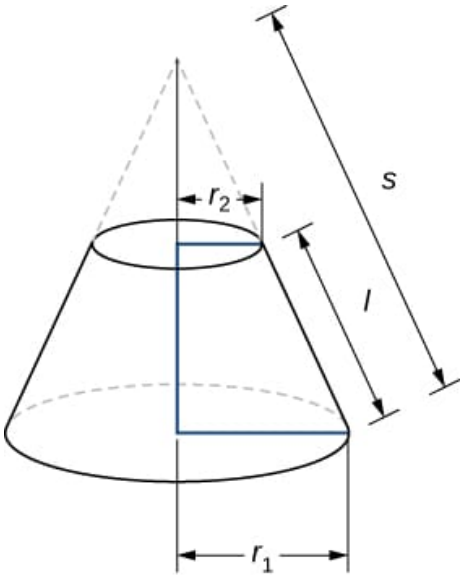


Figure 14.8.3.3.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s} \tag{14.8.3.3.6}$$

Solving for s, we get =s–ls

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1(s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2)s \\ \frac{r_1 l}{r_1 - r_2} &= s \end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\ &= \pi r_1 s - \pi r_2(s-l) \\ &= \pi r_1\left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2\left(\frac{r_1 l}{r_1 - r_2 - l}\right) \\ &= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r2_l}{r_1 - r_2} + \frac{\pi r_2 l(r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\ &= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\ &= \pi(r_1 + r_2)l. \end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the $x - axis$. A representative band is shown in the following figure.

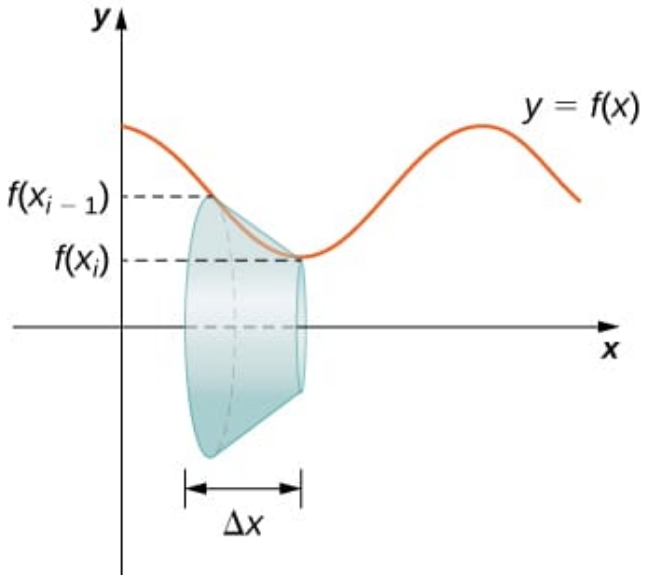


Figure 14.8.3.3.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi(f(x_{i-1}) + f(x_i))\sqrt{\Delta x^2 + (\Delta yi)^2} \\ &= \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + (\frac{\Delta y_i}{\Delta x})^2} \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f\,'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi(f(x_{i-1}) + f(x_i))\Delta x\sqrt{1 + (f'(x_i^*))^2}$$

Furthermore, since$f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x[i]$ such that \(f(x^{**}_i)=(1/2)[f(xi–1)+f(xi)],

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + (f'(x_i^*))^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + (f'(x_i^*))^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let n $\to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x^{**}_i\) are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\text{Surface Area} = \lim_{n\to\infty}\sum_{i=1} n^2\pi f(x_i^{**})\Delta x\sqrt{1+(f'(x_i^*))^2}$$

$$= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the $y-axis$. These findings are summarized in the following theorem.

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a,b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})dx \tag{14.8.3.3.7}$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c,d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the $y-axis$ is given by

$$\text{Surface Area} = \int_c^d (2\pi g(y)\sqrt{1+(g'(y))^2}dy \tag{14.8.3.3.8}$$

Example 14.8.3.3.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x)=\sqrt{x}$ over the interval $[1,4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure 14.8.3.3.10.

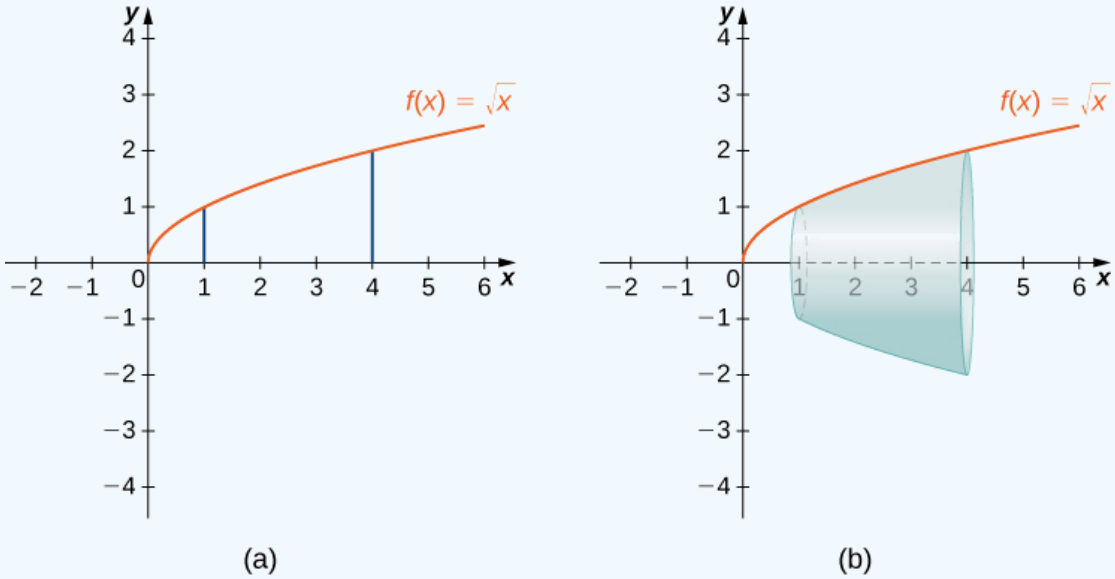


Figure 14.8.3.3.10: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x)=\sqrt{x}$. Then, $f\,'(x)=1/(2\sqrt{x})$ and $(f'(x))^2=1/(4x)$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2}dx \\ &= \int_1^4 (\sqrt{2\pi\sqrt{x}1+\frac{1}{4x}})dx \\ &= \int_1^4 (2\pi\sqrt{x+14}dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1, u = 5/4$, and when $x = 4, u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 (2\pi\sqrt{x+\frac{1}{4}})dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17}-5\sqrt{5}] \approx 30.846\end{aligned}$$

Exercise 14.8.3.3.4

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.8.3.3.5: Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \frac{[}{3}]3x$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

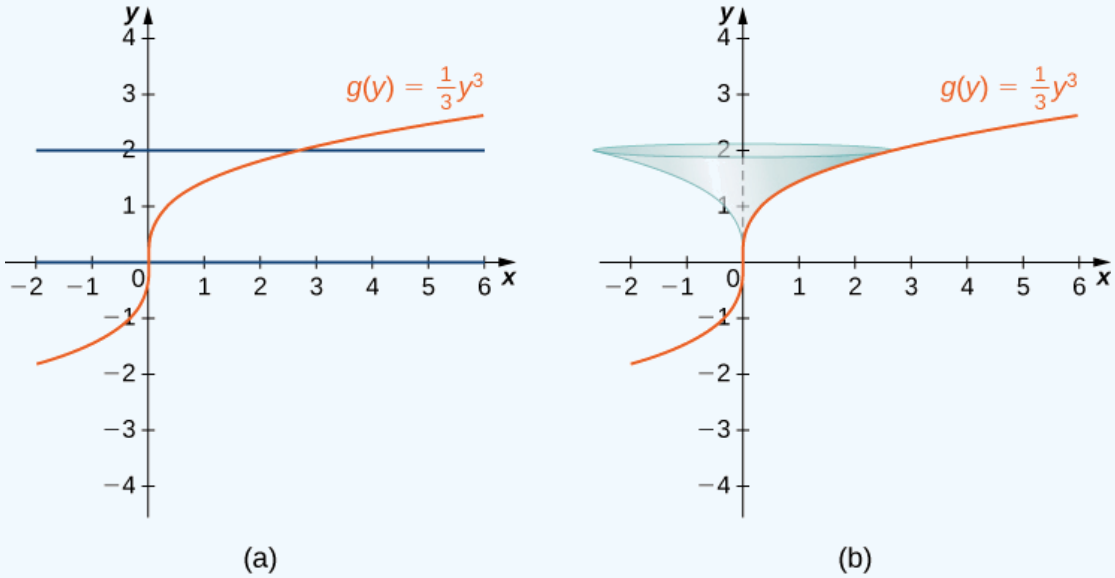


Figure 14.8.3.3.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d (2\pi g(y)\sqrt{1+(g'(y))^2})dy \\ &= \int_0^2 (2\pi(\frac{1}{3}y^3)\sqrt{1+y^4})dy \\ &= \frac{2\pi}{3}\int_0^2 (y^3\sqrt{1+y^4})dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3dy$. When $y = 0, u = 1$, and when $y = 2, u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 (y^3\sqrt{1+y^4})dy &= \frac{2\pi}{3}\int_1^{17}\frac{1}{4}\sqrt{u}du \\ &= \frac{\pi}{6}[\frac{2}{3}u^{3/2}]\,|_1^{17} = \frac{\pi}{9}[(17)^{3/2}-1] \approx 24.118.\end{aligned}$$

Exercise 14.8.3.3.5

Let $g(y) = \sqrt{9-y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

Key Equations

- **Arc Length of a Function of x**

  Arc Length $= \int_a^b \sqrt{1+[f'(x)]^2}dx$

- **Arc Length of a Function of y**

  Arc Length $= \int_c^d \sqrt{1+[g'(y)]^2}dy$

- **Surface Area of a Function of x**

  Surface Area $= \int_a^b (2\pi f(x)\sqrt{1+(f'(x))^2})dx$

Glossary

arc length
: the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

frustum
: a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

surface area
: the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise 14.8.3.3.1

Use the process from the previous example. Don't forget to change the limits of integration.

Exercise 14.8.3.3.1

$$\frac{1}{6}(5\sqrt{5}-1) \approx 1.697$$

Exercise 14.8.3.3.2

Use the process from the previous example.

Exercise 14.8.3.3.2

$$\text{Arc Length} \approx 3.8202 \tag{14.8.3.3.9}$$

Exercise 14.8.3.3.3

Use the process from the previous example.

Exercise 14.8.3.3.3

$$\text{Arc Length} = 3.15018$$

Exercise 14.8.3.3.4

Use the process from the previous example.

Exercise 14.8.3.3.4

$$\frac{\pi}{6}(5\sqrt{5}-3\sqrt{3}) \approx 3.133 \tag{14.8.3.3.10}$$

Exercise 14.8.3.3.5

Use the process from the previous example.

Exercise 14.8.3.3.5

12π

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14.8.3.4: Physical Applications of Integration

Learning Objectives

- Determine the mass of a one-dimensional object from its linear density function.
- Determine the mass of a two-dimensional circular object from its radial density function.
- Calculate the work done by a variable force acting along a line.
- Calculate the work done in pumping a liquid from one height to another.
- Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure $14.8.3.4.1$). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

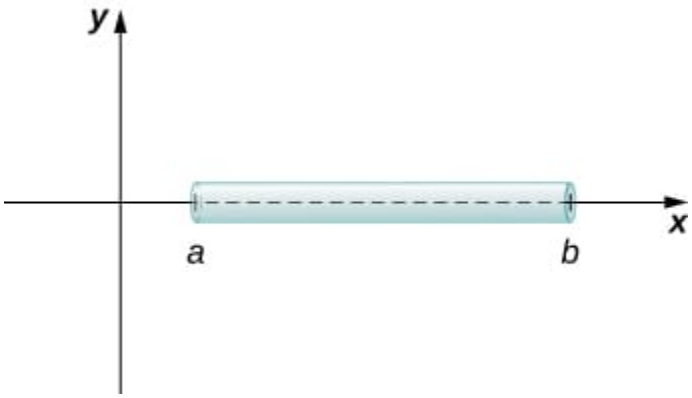


Figure $14.8.3.4.1$: We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear density function, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure $14.8.3.4.2$ shows a representative segment of the rod.

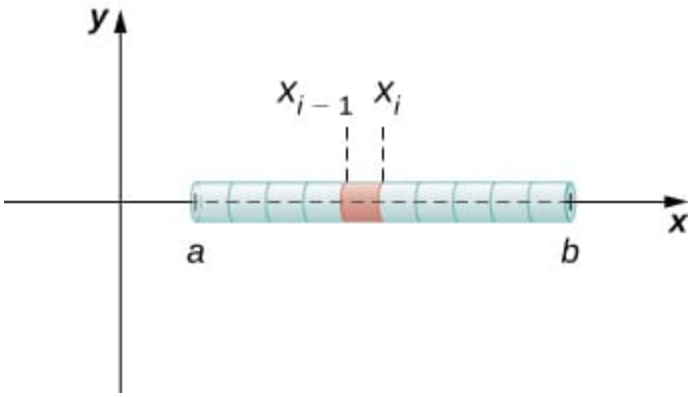


Figure 14.8.3.4.2: A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$\begin{aligned} m_i &\approx \rho(x_i^*)(x_i - x_{i-1}) \\ &= \rho(x_i^*)\Delta x. \end{aligned}$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$\begin{aligned} m &= \sum_{i=1}^{n} m_i \\ &\approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x. \end{aligned}$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x \\ &= \int_a^b \rho(x)dx. \end{aligned}$$

We state this result in the following theorem.

Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{14.8.3.4.1}$$

We apply this theorem in the next example.

Example 14.8.3.4.1: Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying Equation 14.8.3.4.1 directly, we have

$$\begin{aligned} m &= \int_a^b \rho(x)dx \\ &= \int_{\pi/2}^{\pi} \sin x\, dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} \\ &= 1. \end{aligned}$$

Exercise 14.8.3.4.1

Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

Hint available at the end of this section.

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called **area density**), and further assume the density varies only along the disk's radius (called **radial density**). We orient the disk in the $xy - plane$, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

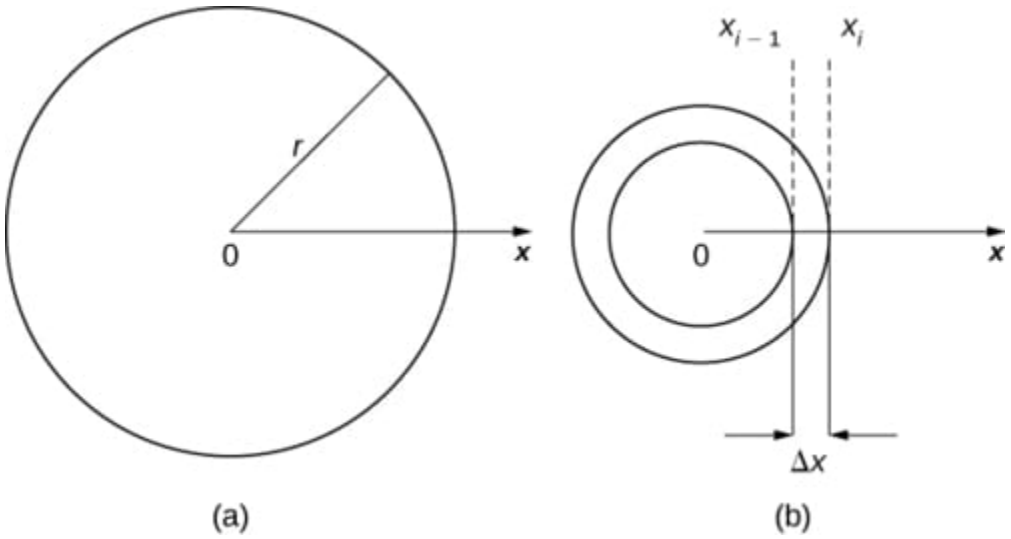


Figure 14.8.3.4.3: (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi[x_i^2 - x_{i-1}^2] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x \\ &= \int_0^r 2\pi x \rho(x) dx. \end{aligned}$$

We summarize these findings in the following theorem.

Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \tag{14.8.3.4.2}$$

Example 14.8.3.4.2: Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying Equation 14.8.3.4.2, we find

$$\begin{aligned} m &= \int_0^r 2\pi x \rho(x) dx \\ &= \int_0^4 2\pi x\sqrt{x} dx = 2\pi \int_0^4 x^{3/2} dx \\ &= 2\pi \frac{2}{5} x^{5/2} \,|_0^4 = \frac{4\pi}{5}[32] \\ &= \frac{128\pi}{5}. \end{aligned}$$

Exercise 14.8.3.4.2

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Hint available at the end of this section.

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force

and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the joule. Both are defined as kilograms times meters squared over seconds squared $(kg \cdot m^2/s^2)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point x_i, we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x. \tag{14.8.3.4.3}$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x. \tag{14.8.3.4.4}$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x \\ &= \int_a^b F(x)dx. \end{aligned}$$

Thus, we can define work as follows.

Definition: Work

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the work done on the object is

$$W = \int_a^b F(x)dx. \tag{14.8.3.4.5}$$

Note that if F is constant, the integral evaluates to $F \cdot (b - a) = F \cdot d$, which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (Figure $14.8.3.4.4$).

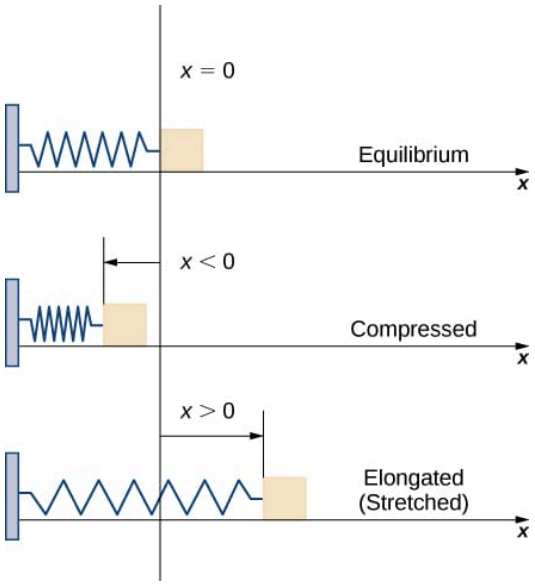


Figure 14.8.3.4.4: A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the spring constant and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 14.8.3.4.3: The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$\begin{aligned} W &= \int_a^b F(x)dx \\ &= \int_0^{0.5} 50x\,dx \\ &= 25x^2\Big|_0^{0.5} \\ &= 6.25. \end{aligned}$$

The work done to stretch the spring is 6.25 J.

Exercise 14.8.3.4.3

Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Hint available at the end of this section.

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure $14.8.3.4.5$).

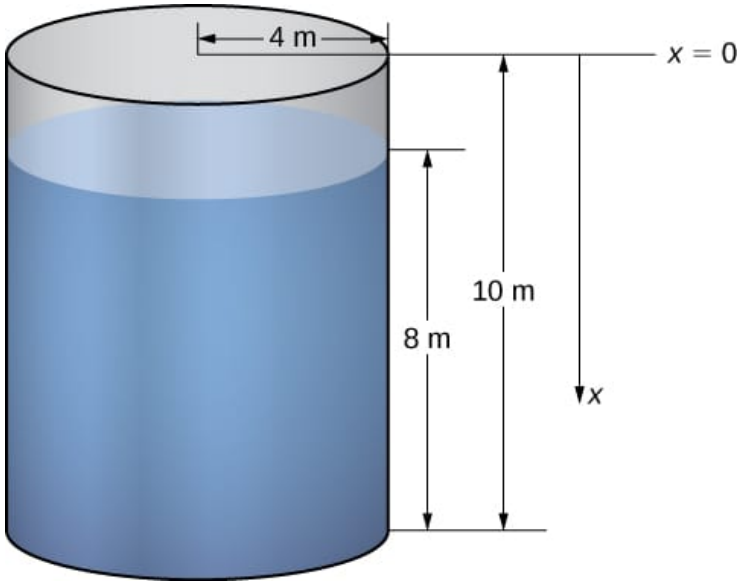


Figure $14.8.3.4.5$: How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure $14.8.3.4.6$ shows a representative layer.

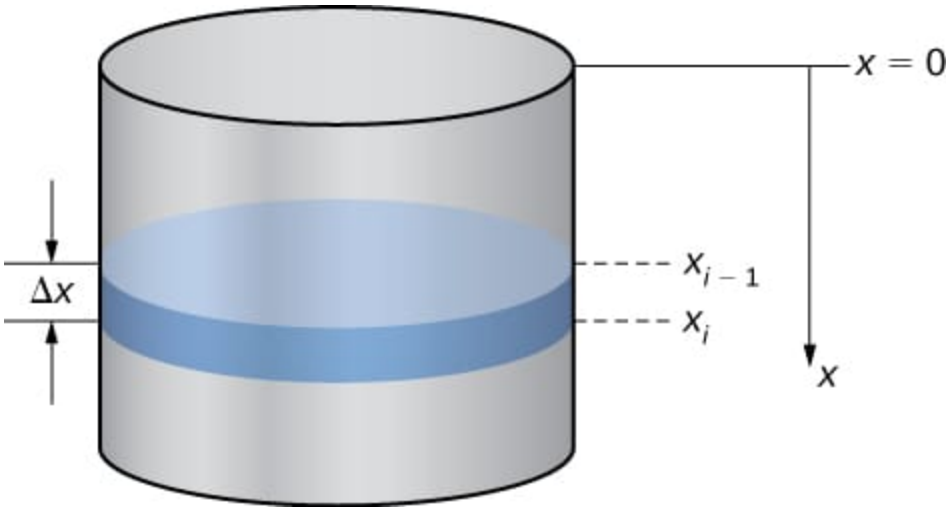


Figure 14.8.3.4.6: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\ \mathrm{N/m^3}$, or $62.4\ \mathrm{lb/ft^3}$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the i^{th} layer of water W_i is approximately

$$W_i \approx 156,800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$\begin{aligned} W &= \sum_{i=1}^{n} W_i \\ &\approx \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x. \end{aligned}$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x \\ &= 156,800\pi \int_2^{10} x dx \\ &= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} = 7,526,400\pi \approx 23,644,883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 14.8.3.4.4: A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in Figure 14.8.3.4.7. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

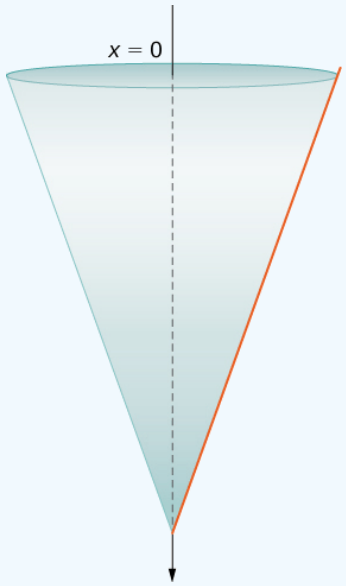


Figure 14.8.3.4.7: A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, 8]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (Figure 14.8.3.4.8).

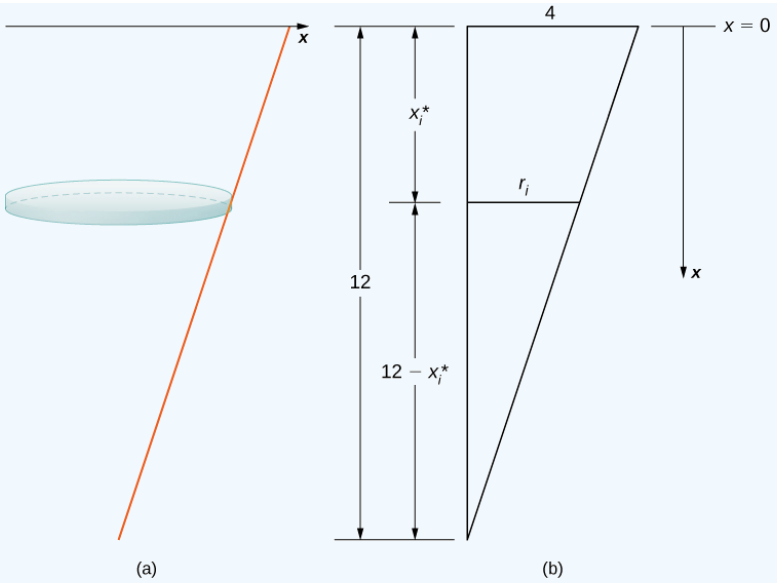


Figure 14.8.3.4.8: Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \qquad \text{(step 1)} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 2)}$$

The weight-density of water is 62.4lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{(step 3)}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 5)}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 6)}$$

Taking the limit as $n \to \infty$, we obtain

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4\pi x_i^*(4 - \frac{x_i^*}{3})^2 \Delta x$$

$$= \int_0^8 62.4\pi x \left(4 - \frac{x}{3}\right)^2 dx$$

$$= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx$$

$$= 62.4\pi \left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Big|_0^8 = 10,649.6\pi \approx 33,456.7.$$

It takes approximately $33,450$ ft-lb of work to empty the tank to the desired level.

Exercise 14.8.3.4.4

A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hint available at the end of this section.

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called **pascals.**

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth s (Figure $14.8.3.4.9$). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As$, where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s$.

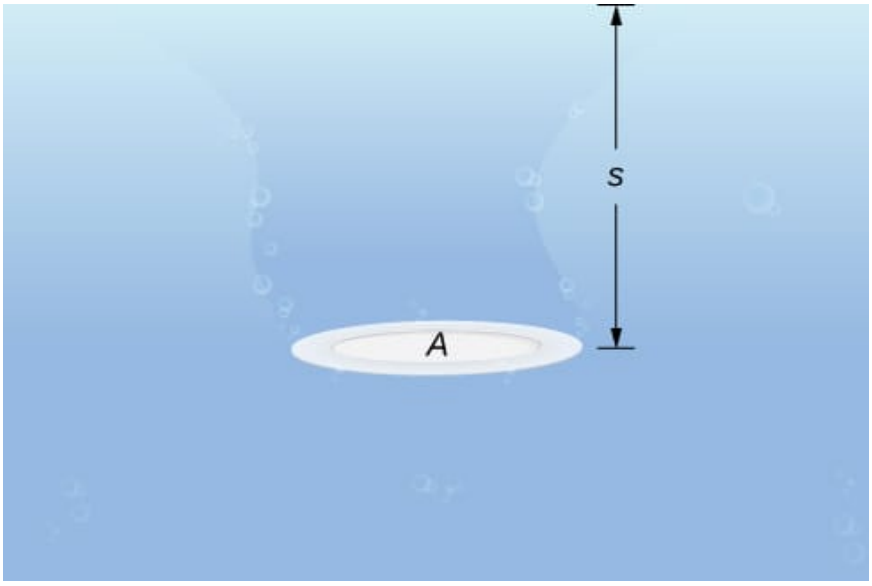


Figure $14.8.3.4.9$: A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force

exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the x-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point x.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (Figure 14.8.3.4.10).

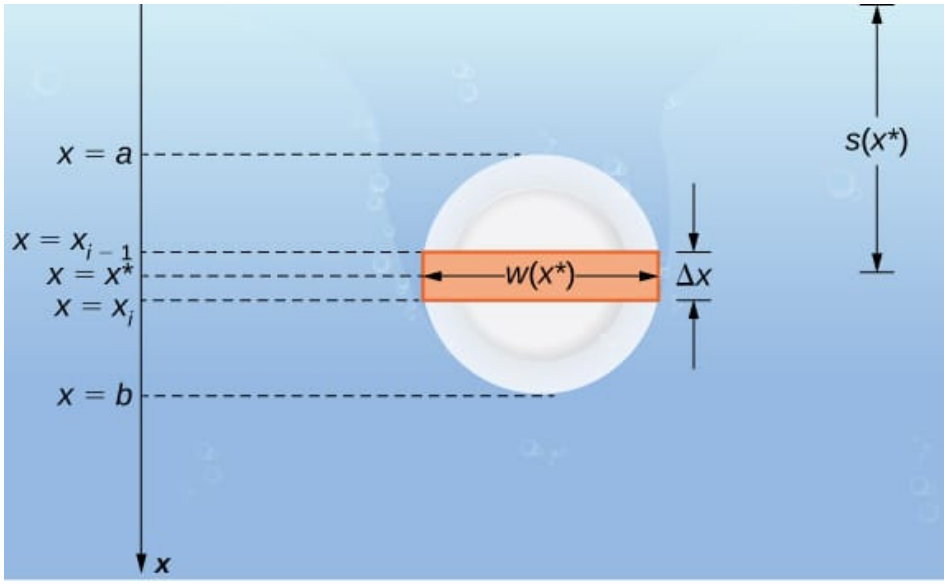


Figure 14.8.3.4.10: A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho As = \rho[w(x_i^*)\Delta x]s(x_i^*). \tag{14.8.3.4.6}$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*). \tag{14.8.3.4.7}$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{14.8.3.4.8}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 14.8.3.4.8 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.

3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\ \text{lb/ft}^3$, or $9800\ \text{N/m}^3$.
4. Use the equation to calculate the total force.

Example 14.8.3.4.5: Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 14.8.3.4.11 shows the trough and a more detailed view of one end.

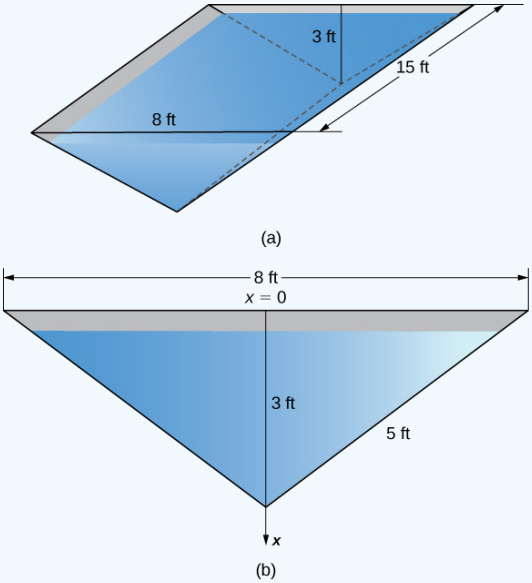


Figure 14.8.3.4.11: (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is $62.4\ \text{lb/ft}^3$ (step 3), so applying Equation 14.8.3.4.8, we obtain

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\,dx \\ &= \int_0^3 62.4\left(8 - \frac{8}{3}x\right) x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right) dx \\ &= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8. \end{aligned}$$

The water exerts a force of 748.8 lb on the end of the trough (step 4).

Exercise 14.8.3.4.5

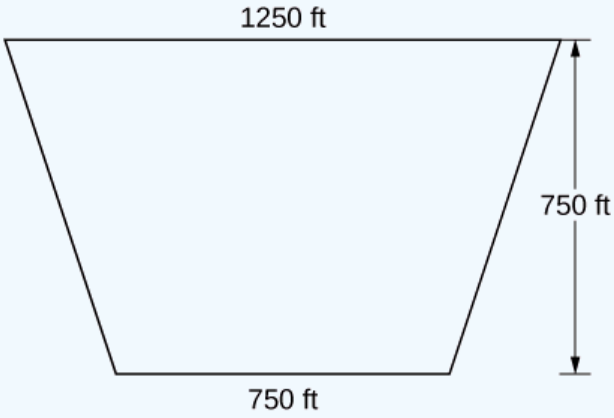
A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Hint available at the end of this section.

Example 14.8.3.4.6: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).





When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

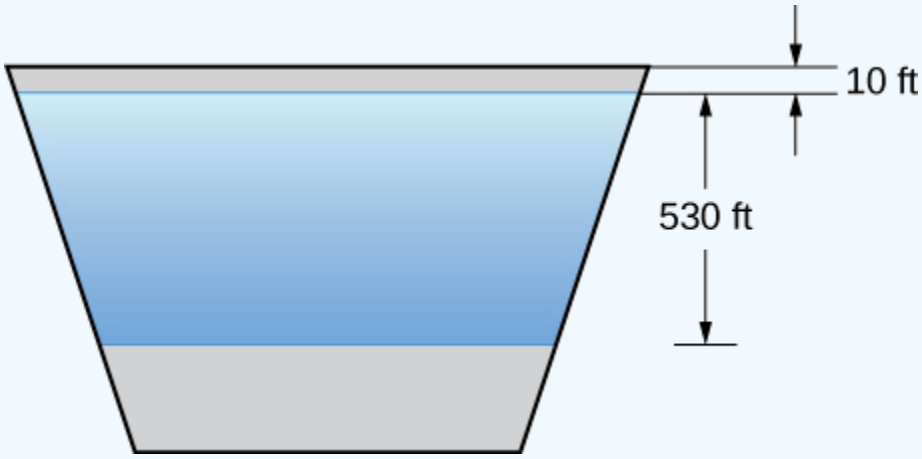


Figure 14.8.3.4.12: A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution:

a.

We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

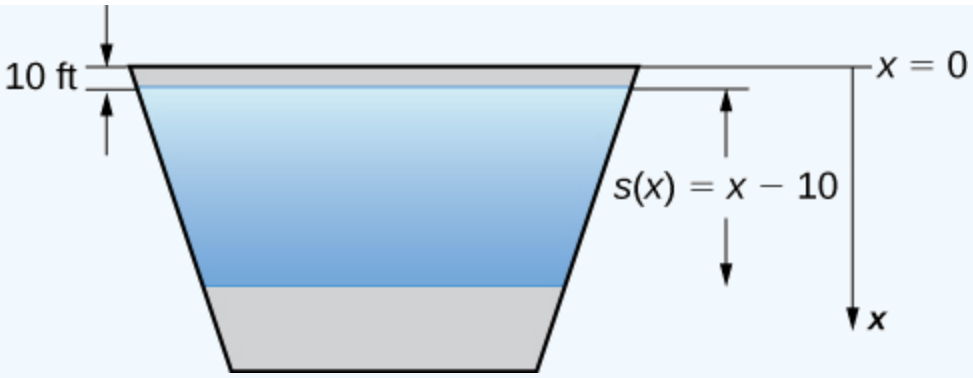


Figure 14.8.3.4.13: We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

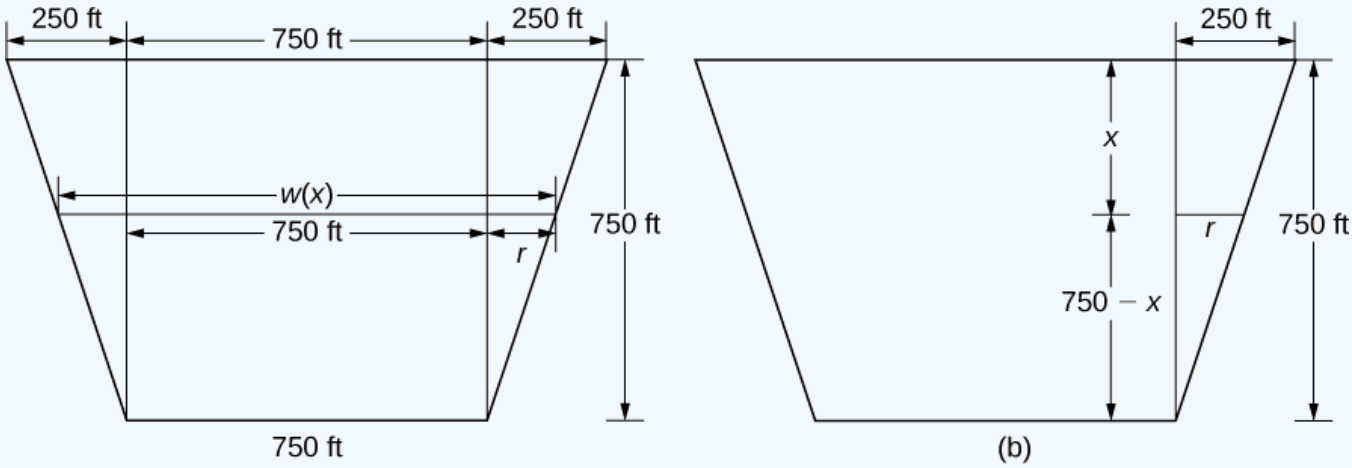


Figure 14.8.3.4.14: We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \qquad \text{(step 2)}$$

Using a weight-density of 62.4lb/ft^3 (step 3) and applying Equation 14.8.3.4.8, we get

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\, dx \\ &= \int_{10}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 10)\, dx \\ &= 62.4 \int_{10}^{540} -\frac{2}{3}[x^2 - 1885x + 18750]\, dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Big|_{10}^{540} \approx 8,832,245,000\,\text{lb} = 4,416,122.5\,\text{t}. \end{aligned}$$

Note the change from pounds to tons ($2000\text{lb} = 1$ ton) (step 4). This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$F = \int_a^b \rho w(x) s(x)\, dx$$
$$= \int_{135}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 135)\, dx$$
$$= -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x - 1875)(x - 135)\, dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x^2 - 2010x + 253125)\, dx$$
$$= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Big|_{135}^{540} \approx 5,015,230,000\,\text{lb} = 2,507,615\,\text{t}.$$

Exercise 14.8.3.4.6

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Hint available at the end of this section.

Key Concepts

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

Key Equations

- **Mass of a one-dimensional object**

  $$m = \int_a^b \rho(x) dx$$

- **Mass of a circular object**

  $$m = \int_0^r 2\pi x \rho(x) dx$$

- **Work done on an object**

  $$W = \int_a^b F(x) dx$$

- **Hydrostatic force on a plate**

  $$F = \int_a^b \rho w(x) s(x) dx$$

Glossary

density function
: a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

Hooke's law
: this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where k is a constant

hydrostatic pressure
: the pressure exerted by water on a submerged object

work
: the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

Contributors and Attributions

- Template:ContribOpenStaxCalc

Answers

Exercise $14.8.3.4.1$
Use the process from the previous example.
$70/3$

Exercise $14.8.3.4.2$
Use the process from the previous example.
24π

Exercise $14.8.3.4.3$
Use the process from the previous example. Be careful with units.
8 ft-lb

Exercise $14.8.3.4.4$
Use the process from the previous example.
Approximately $43,255.2$ ft-lb

Exercise $14.8.3.4.5$
Follow the problem-solving strategy and the process from the previous example.
$156,800$ N

Exercise $14.8.3.4.6$
Change the depth function, $s(x)$, and the limits of integration.
Approximately 7,164,520,000 lb or 3,582,260 t

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14.8.3.5: Moments and Centers of Mass

Learning Objectives

- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids,* under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure $14.8.3.5.1a$. Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure $14.8.3.5.1b$.

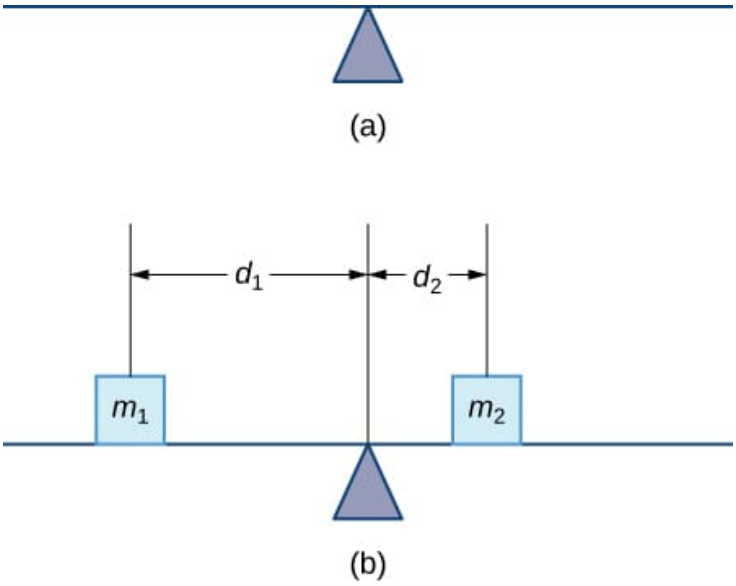


Figure $14.8.3.5.1$: (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$

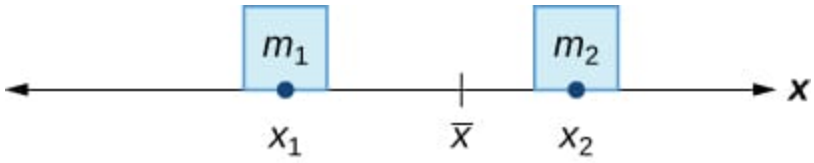


Figure 14.8.3.5.2: The center of mass $\overline{x}$ is the balance point of the system.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 14.8.3.5.2). The center of mass, $\overline{x}$, is the point where the fulcrum should be placed to make the system balance.

Thus, we have

$$\begin{aligned} m_1|x_1 - \overline{x}| &= m_2|x_2 - \overline{x}| \\ m_1(\overline{x} - x_1) &= m_2(x_2 - \overline{x}) \\ m_1\overline{x} - m_1x_1 &= m_2x_2 - m_2\overline{x} \\ \overline{x}(m_1 + m_2) &= m_1x_1 + m_2x_2 \end{aligned}$$

or

$$\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{14.8.3.5.1}$$

The expression in the numerator of Equation 14.8.3.5.1, $m_1x_1 + m_2x_2$, is called the *first moment of the system* with respect to the origin. If the context is clear, we often drop the word first and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if n masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\overline{x} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} \tag{14.8.3.5.2}$$

Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the *moment of the system* with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_ix_i \tag{14.8.3.5.3}$$

and the *center of mass of the system* is given by

$$\overline{x} = \frac{M}{m}. \tag{14.8.3.5.4}$$

We apply this theorem in the following example.

Example 14.8.3.5.1: Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

- $m_1 = 30\ kg$, placed at $x_1 = -2m$
- $m_2 = 5\ kg$, placed at $x_2 = 3m$
- $m_3 = 10\ kg$, placed at $x_3 = 6m$
- $m_4 = 15\ kg$, placed at $x_4 = -3m$.

Solution

Find the moment of the system with respect to the origin and find the center of mass of the system.

First, we need to calculate the moment of the system (Equation 14.8.3.5.3):

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 \\ &= -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 \\ &= 60\, kg \end{aligned}$$

Then we have (from Equation 14.8.3.5.4)

$$\overline{x} - = \frac{M}{m} = -\frac{30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

Exercise 14.8.3.5.1

Suppose four point masses are placed on a number line as follows:

- $m_1 = 12\ kg$ placed at $x_1 = -4m$
- $m_2 = 12\ kg$ placed at $x_2 = 4m$
- $m_3 = 30\ kg$ placed at $x_3 = 2m$
- $m_4 = 6\ kg$, placed at $x_4 = -6m$.

Find the moment of the system with respect to the origin and find the center of mass of the system.

Hint available at the end of this section.

Answer available at the end of this section.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the x-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the y-axis is given by

$$M_y = m_1 x_1. \tag{14.8.3.5.5}$$

Notice that the x-coordinate of the point is used to calculate the moment with respect to the y-axis, and vice versa. The reason is

that the x-coordinate gives the distance from the point mass to the y-axis, and the y-coordinate gives the distance to the x-axis (see the following figure).

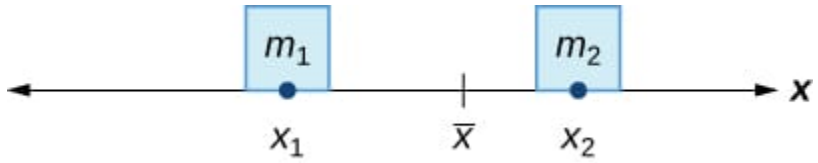


Figure 14.8.3.5.3: Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the xy-plane, we can use the moments with respect to the x- and y-axes to calculate the x- and y-coordinates of the center of mass of the system.

Center of Mass of Objects in a Plane

Let m_1, m_2, ..., m_n be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x- and y-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{14.8.3.5.6}$$

and

$$M_y = \sum_{i=1}^{n} m_i x_i. \tag{14.8.3.5.7}$$

Also, the coordinates of the center of mass $(\overline{x}, \overline{y})$ of the system are

$$\overline{x} = \frac{M_y}{m} \tag{14.8.3.5.8}$$

and

$$\overline{y} = \frac{M_x}{m}. \tag{14.8.3.5.9}$$

The next example demonstrates how to the center of mass formulas (Equations 14.8.3.5.6 - 14.8.3.5.9) may be applied.

Example 14.8.3.5.2: Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy-plane as follows (assume coordinates are given in meters):

- $m_1 = 2\ kg$ placed at $(-1, 3)$,
- $m_2 = 6\ kg$ placed at $(1, 1)$,
- $m_3 = 4\ kg$ placed at $(2, -2)$.

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,kg.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\overline{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

and

$$\overline{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

Exercise 14.8.3.5.2

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

- $m_1 = 5\ kg$, placed at $(-2, -3)$,
- $m_2 = 3\ kg$, placed at $(2, 3)$,
- $m_3 = 2\ kg$, placed at $(-3, -2)$.

Find the center of mass of the system.

Hint available at the end of this section.

Answer available at the end of this section.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

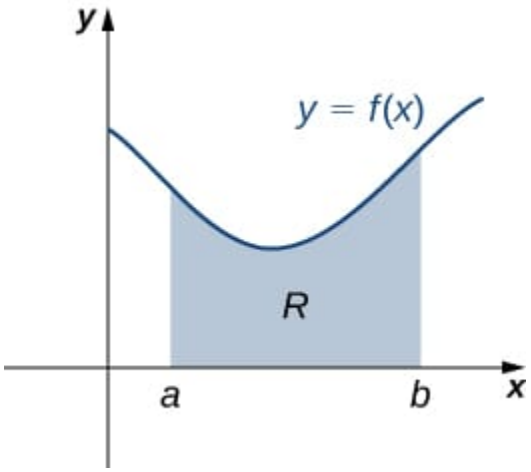


Figure 14.8.3.5.4: A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the *x*-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \ldots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2, \ldots, n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.

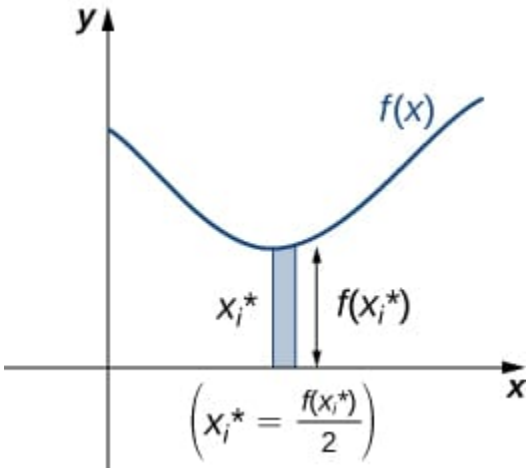


Figure 14.8.3.5.5: A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x. \tag{14.8.3.5.10}$$

Equation 14.8.3.5.10 is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x \\ &= \rho \int_a^b f(x)dx. \end{aligned}$$

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $(x_i^*, (f(x_i^*))/2)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $(f(x_i^*))/2$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho([f(x_i^*)]^2/2)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$\begin{aligned} M_x &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{[f(x_i^*)]^2}{2}\Delta x \\ &= \rho \int_a^b \frac{[f(x)]^2}{2}dx. \end{aligned}$$

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is x_i^*. Then the moment of the lamina with respect to the *y*-axis is given by

$$\begin{aligned} M_y &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i^* f(x^*)i)\Delta x \\ &= \rho \int_a^b x f(x)dx. \end{aligned}$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\overline{x} = M_y/m$ and $\overline{y} = M_x/m$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\overline{x}$ and $\overline{y}$ are calculated.

We summarize these findings in the following theorem.

Center of Mass of a Thin Plate in the xy-Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{14.8.3.5.11}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2}dx \tag{14.8.3.5.12}$$

and

$$M_y = \rho \int_a^b x f(x)dx. \tag{14.8.3.5.13}$$

iii. The coordinates of the center of mass $(\overline{x}, \overline{y})$ are

$$\overline{x} = \frac{M_y}{m} \tag{14.8.3.5.14}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{14.8.3.5.15}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 14.8.3.5.3: Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

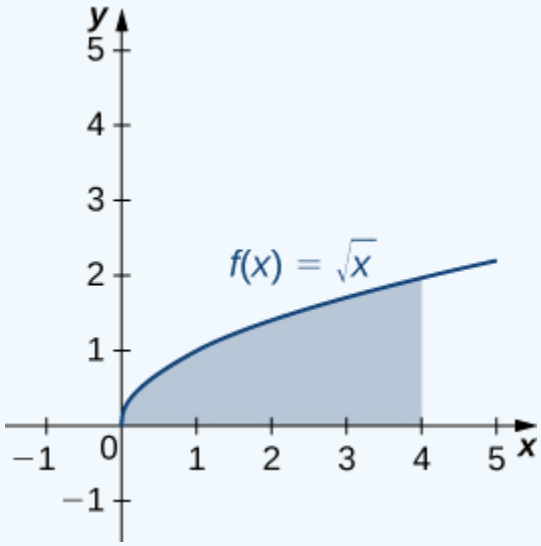


Figure 14.8.3.5.6: Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass (Equation 14.8.3.5.11):

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_0^4 \sqrt{x}dx \\ &= \frac{2}{3}x^{3/2}\,|_0^4 \\ &= \frac{2}{3}[8-0] \\ &= \frac{16}{3}. \end{aligned}$$

Next, we compute the moments (Equation 14.8.3.5.14):

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \int_0^4 \frac{x}{2} dx \\ &= \frac{1}{4} x^2 \,|_0^4 \\ &= 4 \end{aligned}$$

and (Equation 14.8.3.5.13):

$$\begin{aligned} M_y &= \rho \int_a^b x f(x) dx \\ &= \int_0^4 x\sqrt{x} dx \\ &= \int_0^4 x^{3/2} dx \\ &= \frac{2}{5} x^{5/2} \,|_0^4 \\ &= \frac{2}{5}[32 - 0] \\ &= \frac{64}{5}. \end{aligned}$$

Thus, we have (Equation 14.8.3.5.14):

$$\begin{aligned} \overline{x} &= \frac{M_y}{m} \\ &= \frac{64/5}{16/3} \\ &= \frac{64}{5} \cdot \frac{3}{16} \\ &= \frac{12}{5} \end{aligned}$$

and (Equation 14.8.3.5.15):

$$\begin{aligned} \overline{y} &= \frac{M_x}{y} \\ &= \frac{4}{16/3} \\ &= 4 \cdot \frac{3}{16} \\ &= \frac{3}{4}. \end{aligned}$$

The centroid of the region is $(12/5, 3/4)$.

Exercise 14.8.3.5.3

Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the *x*-axis over the interval $[0, 2]$. Find the centroid of the region.

Hint available at the end of this section.

Answer available at the end of this section.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the *x*-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in Figure $14.8.3.5.7$.

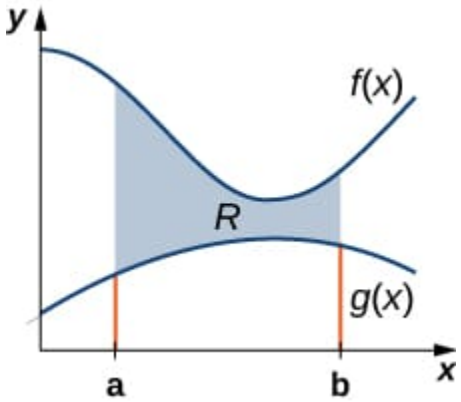


Figure $14.8.3.5.7$: A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in Figure $14.8.3.5.8$.

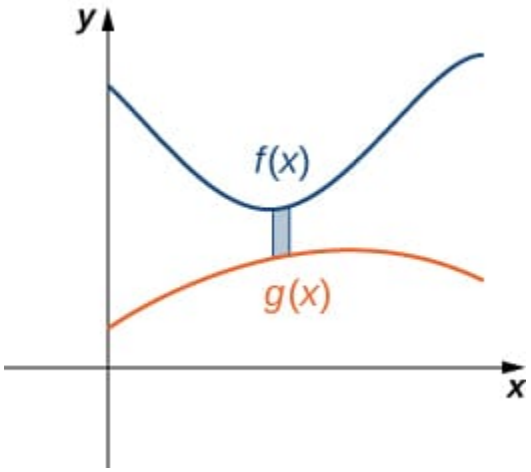


Figure $14.8.3.5.8$: A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $(x_i^*, (f(x_i^*) + g(x_i^*))/2)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the *y*-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho[f(x_i^*) - g(x_i^*)]\Delta x$, by the distance of the centroid from the x-axis, $(f(x_i^*) + g(x_i^*))/2$, which gives $\rho(1/2)[f(x_i^*)]^2 - [g(x_i^*)]^2\Delta x$. Summarizing these findings, we arrive at the following theorem.

Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]dx. \tag{14.8.3.5.16}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b 12([f(x)]^2 - [g(x)]^2)dx \tag{14.8.3.5.17}$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx. \tag{14.8.3.5.18}$$

iii. The coordinates of the center of mass $\overline{x}, \overline{y})$ are

$$\overline{x} = \frac{M_y}{m} \tag{14.8.3.5.19}$$

and

$$\overline{y} = \frac{M_x}{m} \tag{14.8.3.5.20}$$

We illustrate this theorem in the following example.

Example 14.8.3.5.4: Finding the Centroid of a Region Bounded by Two Functions

Let *R* be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

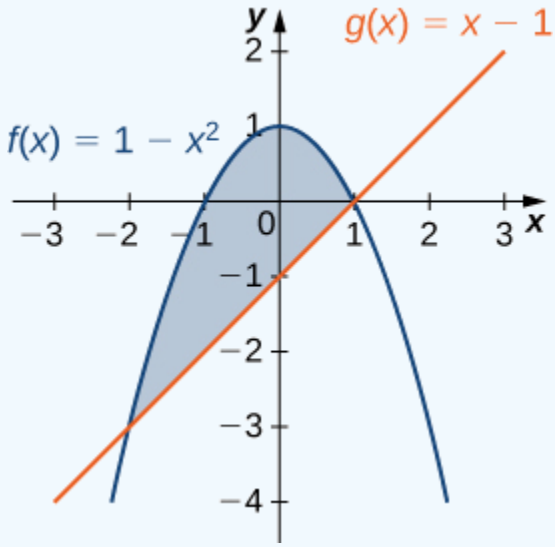


Figure 14.8.3.5.9: Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from –2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^1 [1 - x^2 - (x-1)]dx \\ &= \int_{-2}^1 (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right] |_{-2}^1 \\ &= \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] \\ &= \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx \\ &= \frac{1}{2}\int_{-2}^1 ((1-x^2)^2 - (x-1)^2)dx \\ &= \frac{1}{2}\int_{-2}^1 (x^4 - 3x^2 + 2x)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right] |_{-2}^1 \\ &= -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^1 x[(1-x^2) - (x-1)]dx \\ &= \int_{-2}^1 x[2 - x^2 - x]dx \\ &= \int_{-2}^1 (2x - x^4 - x^2)dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right] |_{-2}^1 \\ &= -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \overline{x} &= \frac{M_y}{m} \\ &= -\frac{9}{4} \cdot \frac{2}{9} \\ &= -\frac{1}{2} \end{aligned}$$

and

$$\begin{aligned}\overline{y} &= \frac{M_x}{y} \\ &= -\frac{27}{10} \cdot \frac{2}{9} \\ &= -\frac{3}{5}.\end{aligned}$$

The centroid of the region is $(-(1/2), -(3/5))$.

Exercise 14.8.3.5.4

Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

Hint available at the end of this section.

Answer available at the end of this section.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 14.8.3.5.5: Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure

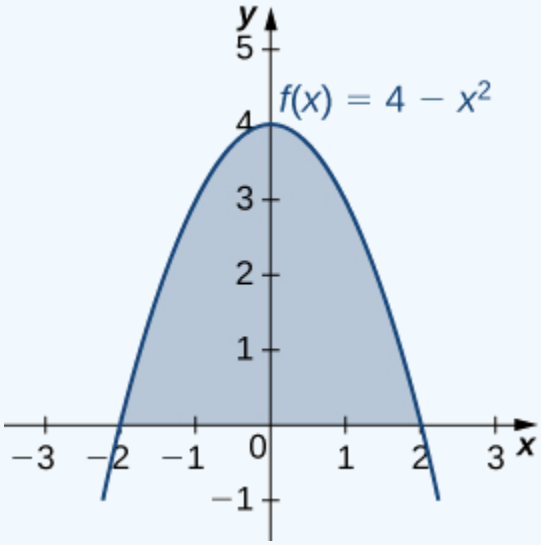


Figure 14.8.3.5.10: We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the *y*-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\overline{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} (4-x^2)dx \\ &= \left[4x - \frac{x^3}{3}\right] |_{-2}^{2} \\ &= \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^{2} \left[4-x^2\right]^2 dx = \frac{1}{2} \int_{-2}^{2} (16 - 8x^2 + x^4)dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right] |_{-2}^{2} = \frac{256}{15} \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

Exercise 14.8.3.5.5

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Hint available at the end of this section.

Answer available at the end of this section.

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).



Figure 14.8.3.5.11: The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (Figure 14.8.3.5.12).

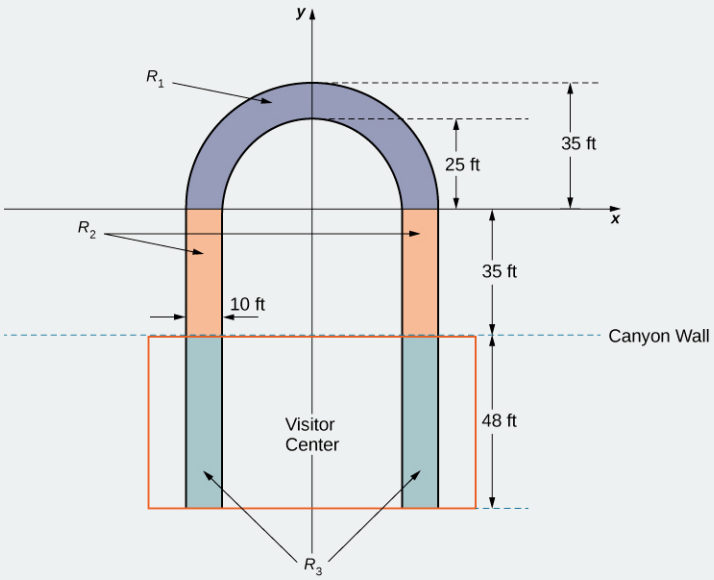


Figure 14.8.3.5.12: We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32 \ \ ft/sec^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3.Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the *theorem of Pappus for volume,* which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance *d* traveled by the centroid of R.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)] \ dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x -coordinate of the centroid, $\overline{x}$, which is

$$x = \frac{M_y}{m}, \tag{14.8.3.5.21}$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \tag{14.8.3.5.22}$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx. \tag{14.8.3.5.23}$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx} \tag{14.8.3.5.24}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx. \tag{14.8.3.5.25}$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx. \tag{14.8.3.5.26}$$

So,

$$V = d \cdot A \tag{14.8.3.5.27}$$

and the proof is complete.

□

Example 14.8.3.5.6: Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Solution

The region and torus are depicted in the following figure.

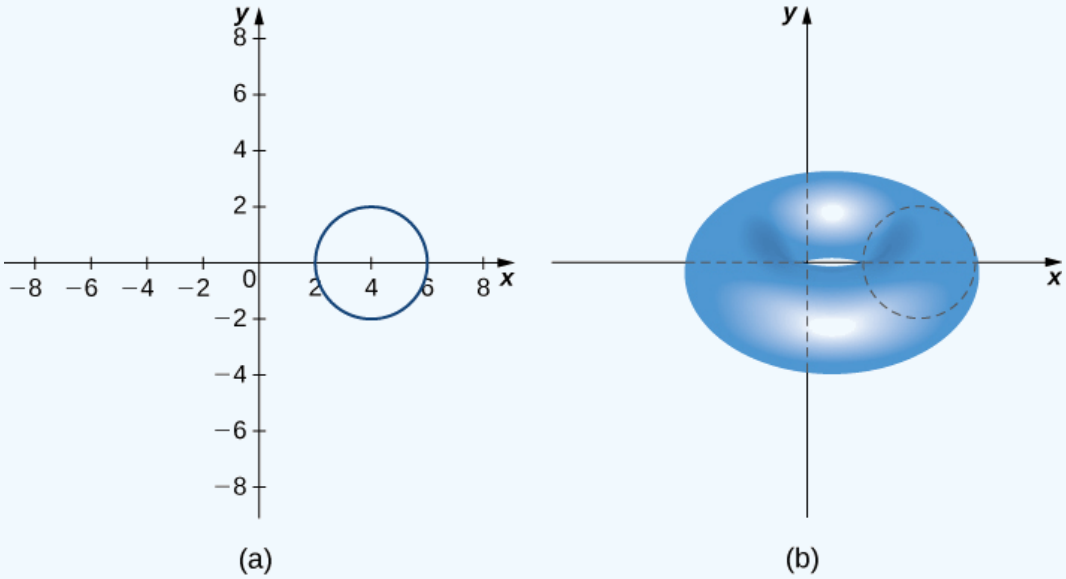


Figure 14.8.3.5.13: Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi \text{ units}^2$. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

Exercise 14.8.3.5.6

Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\ dx$ and $M_y = \rho \int_a^b x f(x)\ dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x -axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

Key Equations

- **Mass of a lamina**

  $$m = \rho \int_a^b f(x) dx$$

- **Moments of a lamina**

  $$M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\ dx \text{ and } M_y = \rho \int_a^b x f(x)\ dx$$

- **Center of mass of a lamina**

  $$\overline{x} = \frac{M_y}{m} \text{ and } \overline{y} = \frac{M_x}{m}$$

Glossary

center of mass
: the point at which the total mass of the system could be concentrated without changing the moment

centroid
: the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

lamina
: a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

moment
: if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\ dx$ and $M_y = \rho \int_a^b x f(x)\ dx$, respectively

symmetry principle
: the symmetry principle states that if a region R is symmetric about a line I, then the centroid of R lies on I

theorem of Pappus for volume
: this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

Contributors and Attributions

- Template:ContribOpenStaxCalc

NB: The hollow square □ means Q.E.D. or "quad erat demonstrandum." Sometimes a filled in square is used ■. These squares are sometimes referred to as a tombstone.

Answers

Exercise 14.8.3.5.1
Use the process from the previous example.

Exercise 14.8.3.5.1
$M = 24, \overline{x} = \frac{2}{5}m$

Exercise 14.8.3.5.2
Use the process from the previous example.

Exercise 14.8.3.5.2
$(-1, -1)$ m

Exercise 14.8.3.5.3
Use the process from the previous example.

Exercise 14.8.3.5.3
The centroid of the region is $(3/2, 6/5)$.

Exercise 14.8.3.5.4
Use the process from the previous example.

Exercise 14.8.3.5.4
The centroid of the region is $(1, 13/5)$.

Exercise 14.8.3.5.5
Use the process from the previous example.

Exercise 14.8.3.5.5
The centroid of the region is $(0, 2/5)$.

Exercise 14.8.3.5.6
Use the process from the previous example.

Exercise 14.8.3.5.6
$6\pi^2$ units^3

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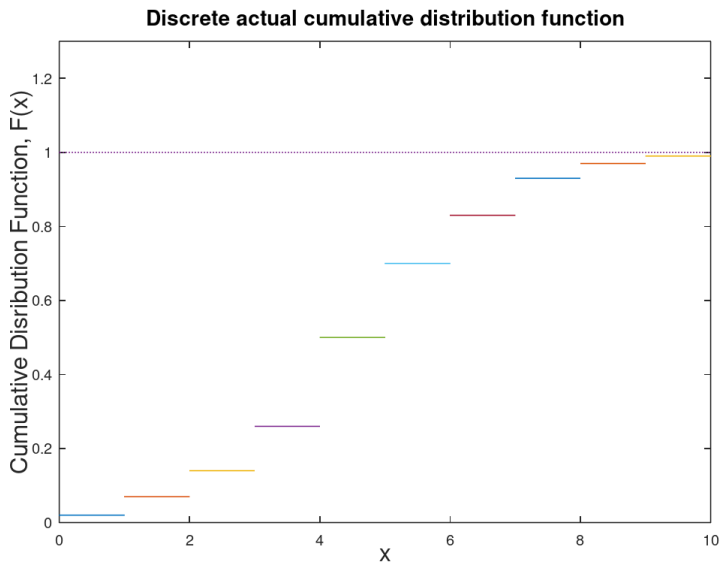
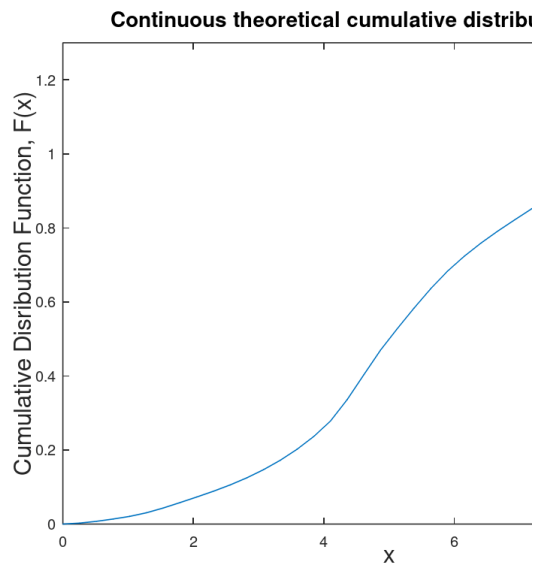
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14.9: Statistics and Probability

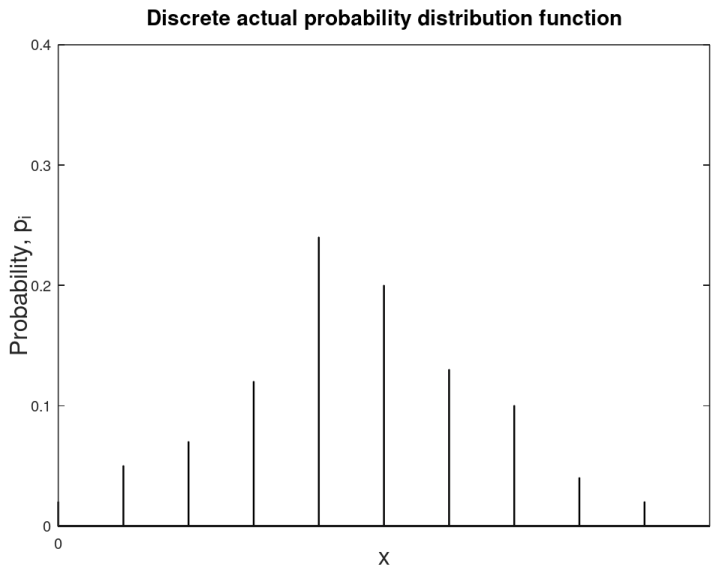
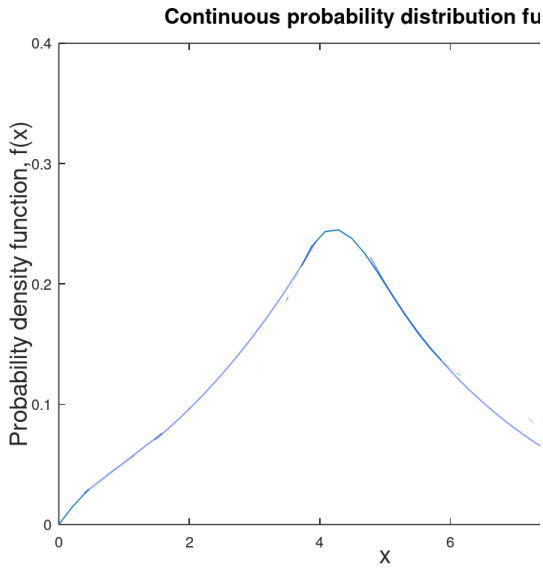
Statistics is the collecting and analyzing of numerical data for the purpose of inferring results from representative samples sometimes referred to as statistical analysis. Probability quantifies how likely an event is to occur given certain conditions. Given a random variable R we can define some basic principals of probability. P(R) will represent the probability of a random event R will occur.

- $P(R) \geq 0$
- $\sum_i P(R_i) = 1.0$
- For random variables Q and R: $P(Q \cup R) = P(Q) + P(R) - P(Q \cap R)$
- For random variables Q and R: If the events do not take place in at once then $P(Q \cap R) = 0$
- For random variables Q and R: Bayes Theorem: $P(Q|R) = \frac{P(R|Q)P(Q)}{P(R)}$ where $(Q|R)$ means Q given R

Here let us assume there is a random variable R which is associated with a number or a range of numbers within a given probability distribution. Lets assume we have we have x_i numbers (let say the score for a quiz that is worth 10 points). Let us define a function F(x) which will be our cumulative distribution function (CDF). The CDF will be the probability that a random variable is less than a specific value of x (in this case 10). The range of F(x) is from 0 to 1 (which is a bit obvious, eh?). Theoretically $F(-\infty)$ = 0 and increases with each larger value of x (because it is cumulative so for instance in this example CDF of $x \leq 10 = 1$ and CDF of $x \leq 9$ is close to 1 but not quite).

Discrete	Continuous
	
Discrete cumulative distribution function (CDF) which is determined through data either collected or experimental. View this graph as a step graph so the end of the graph the final value is one.	Continuous CDF which is theoretical and derived from determine the probability that R lies between x_1 and x formula: $P(x_1 \leq R < x_2) = F(x_2) - F(x_1)$

Of more use is the probability distribution function (PDF) which is the particular probability of the random variable R at x_i.

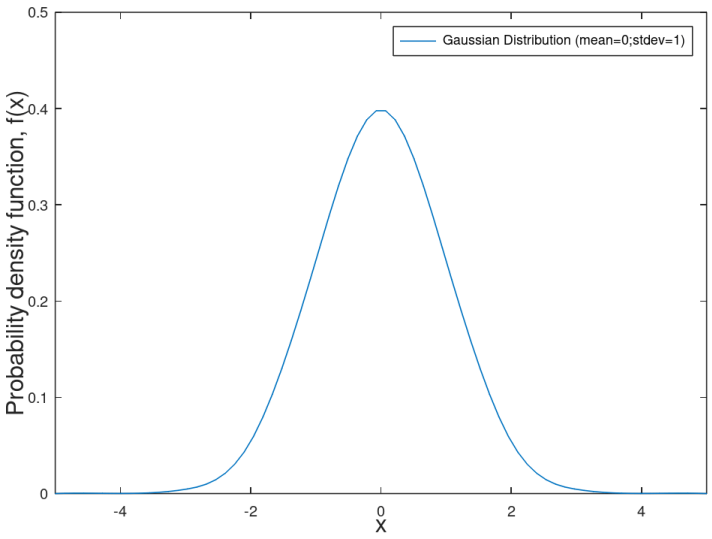
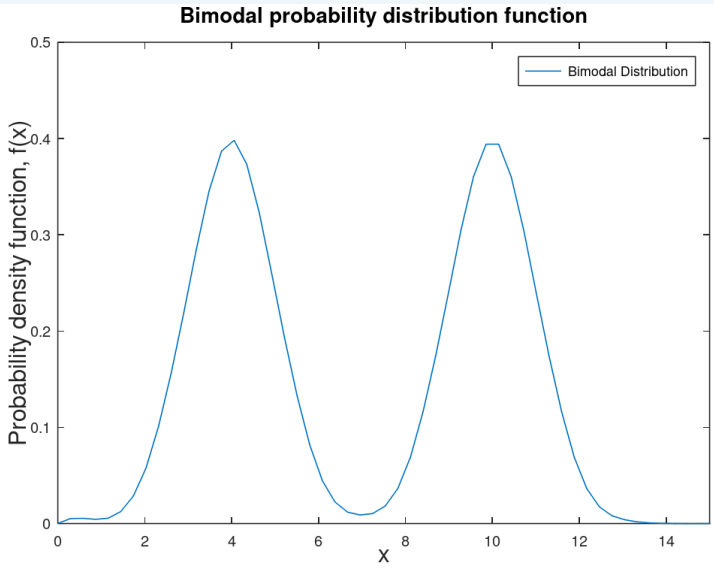
Discrete	Continuous
	
Discrete probability distribution function (PDF) which is based of collected or experimental data as is the discrete CDF (which in this example is a difference between each level of x in the discrete CDF above). This is properly referred to as the probability mass function (PMF), though discrete PDF is more descriptive.	Continuous PDF which is theoretical and derived from determine the probability that R lies between x_1 and the curve between those values or $P(x_1 \leq R < x_2) =$ technically referred to as the probability *density* funct also referred to as the probability distribution functio descriptive and can include the CDF and PMF).

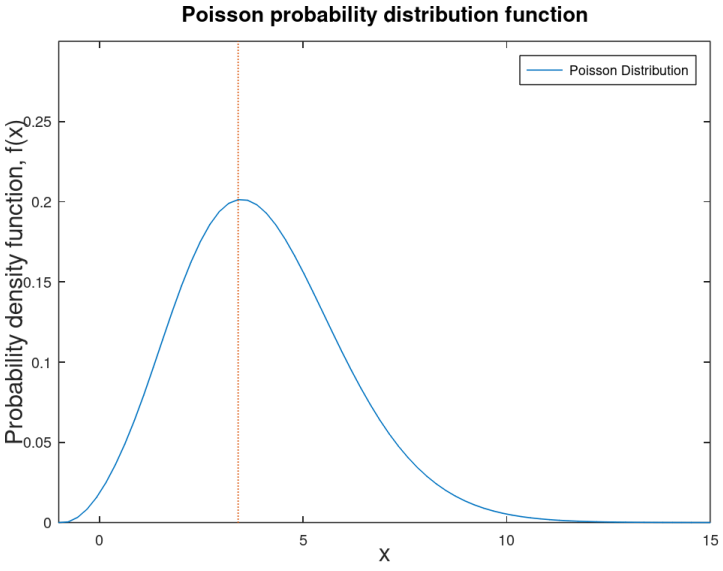
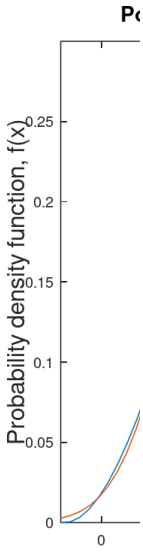
With the PDF we can define quantities like mean and variance.

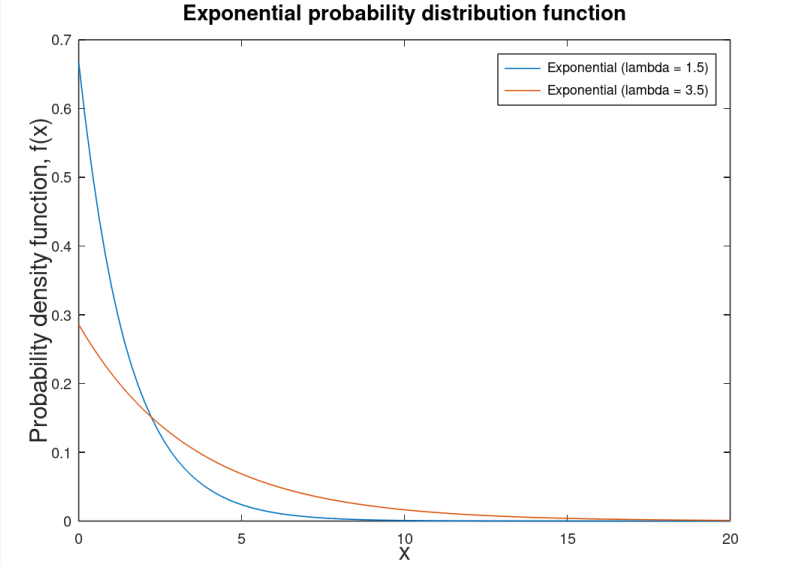
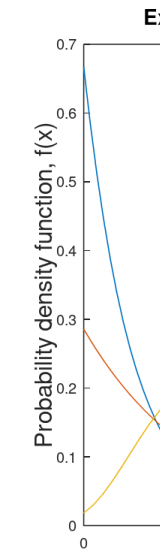
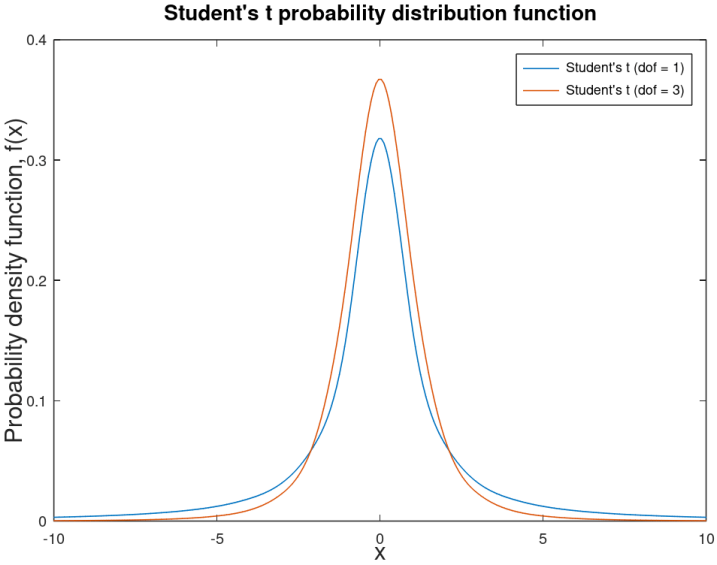
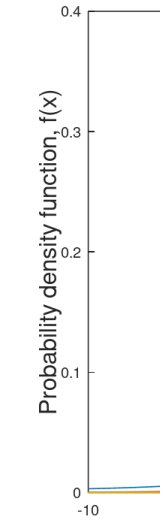
Discrete (mean and variance)	Continuous (expected mean and variance)
$\overline{x} = \sum_{i=1}^{n} x_i p_i$	$\mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$
$\sigma^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2 \cdot p_i$	$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx$

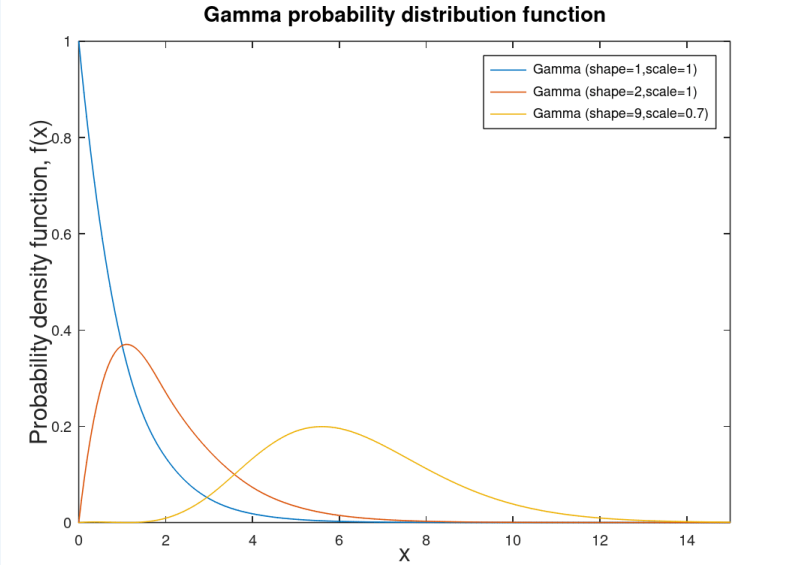
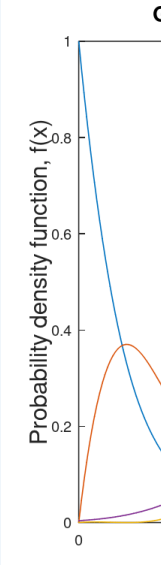
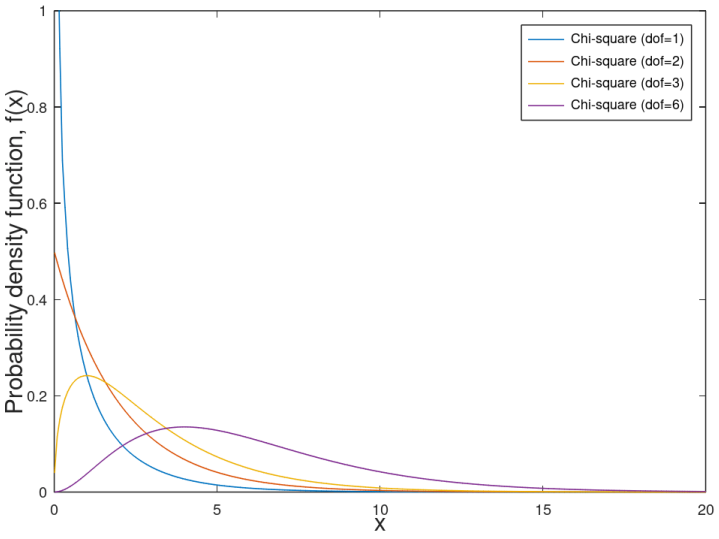
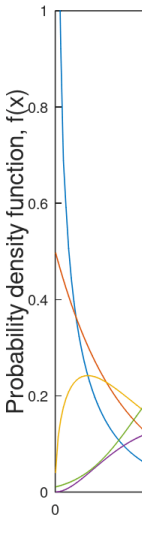
Here you can view p_i as a kind of weight. When you take an average you are taking a mean with a weight of $\frac{1}{n}$ where n is the number of data points. f(x) can be viewed as kind of a continuous weight. *But what weight to use?* That is the "science" to this as there are many statistical distributions that could be used which depends on the system (or experiment).

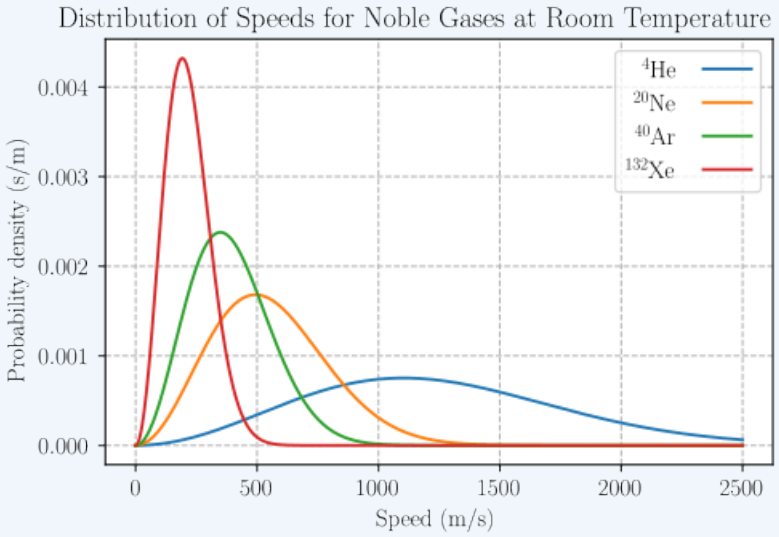
Here we present a table of statistical distributions, statistical coefficients, and statistical models that are used in science and engineering.

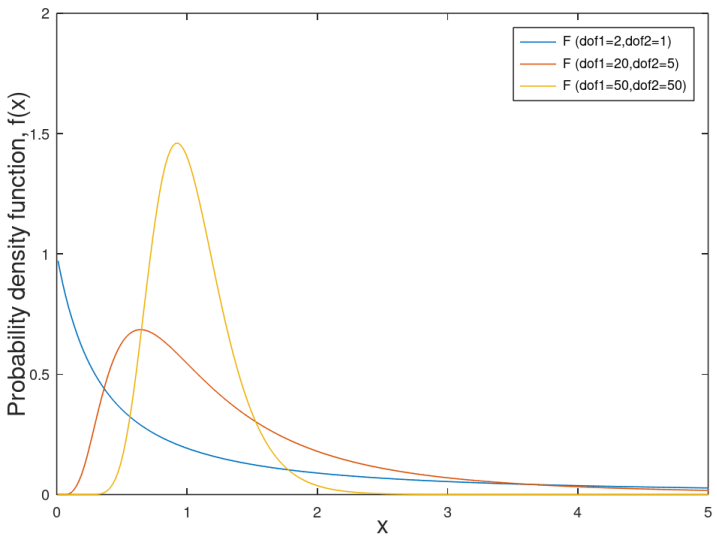
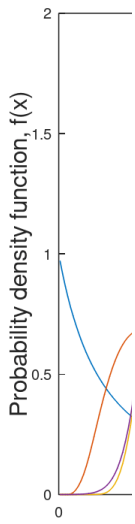
Distribution	Uses	Graph of PDF or a comment	Comparis
Gaussian distribution (or error distribution or normal distribution)	Random error. All distributions as n goes to infinity are Gaussian		This is th
Bimodal distribution	Usually two normal distribution side-by-side, typical of student test grades and heights of humans. (could have more then two distribution then that would be multimodal distribution).		Gaussian dis (male huma

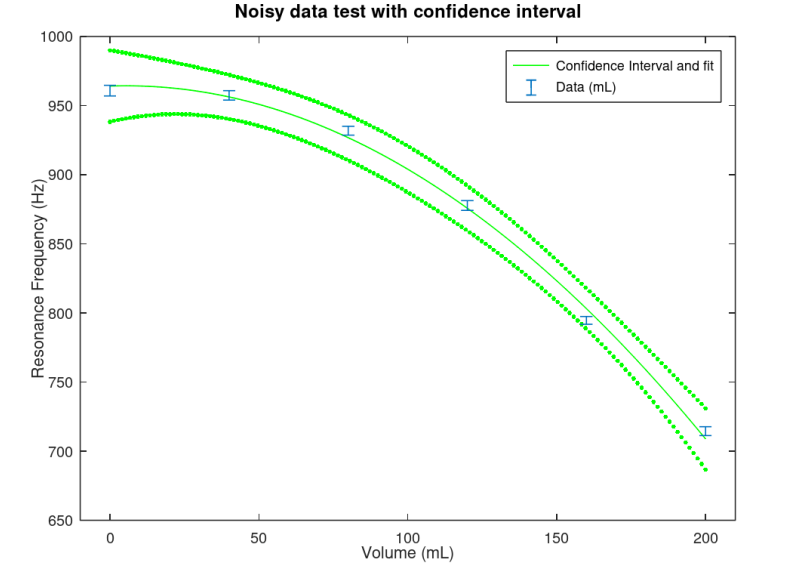
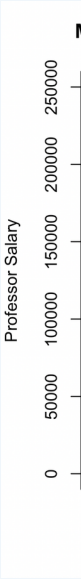
Distribution	Uses	Graph of PDF or a comment	Comparis
Poisson distribution	Counting distribution. Photons on a CCD are described by a Poisson distribution. The graphs at left are the Poisson PDF and a comparison of the Poisson PDF to the Gaussian PDF. Note: counting distributions are integer-based (discrete) so the graph on the right is smoothed for comparison purposes and integration purposes. Discrete distributions are technically PMF (probability mass functions) which are the discrete versions of PDFs.		

Distribution	Uses	Graph of PDF or a comment	Comparis
Exponential distribution	Related to the Poisson distribution[1] and special case of the Gamma distribution. Used in reliability calculations in engineering. Helps determine the probability of an event's occurrence. Note: Lambda is the rate of events in a particular interval.		
Student's t-distribution	A distribution for small statistics[2] and unknown standard deviation. Degrees of freedom (dof) are the number of observations minus one. As the degrees of freedom approaches infinity this distribution becomes the Gaussian distribution (as do all distributions). This distribution is associated with the T-Test.		

Distribution	Uses	Graph of PDF or a comment	Comparis
Gamma distribution	Distribution for modeling errors in different distributions. This distribution is a two parameter distribution that is more a mathematical construct of many other distributions, like chi-square distribution.		
Chi-square distribution	Special case of gamma distribution. Used for confidence intervals and hypothesis testing (inferential statistics). One of the most used distributions in science, economics, and engineering.		

Distribution	Uses	Graph of PDF or a comment	Comparis
Maxwellian distribution	Distribution used for velocity distribution in a kinetic gas in physics.	Maxwellian-Boltzmann Distribution[3] 	
Kappa distribution	A replacement for Maxwellian distribution in certain conditions (non-equilibrium; nonextensive). Useful in statistical mechanics. Maxwellian distribution is special case of the Kappa distribution.	Commonly used for space plasma because space plasma is not in thermal equilibrium. Also used in economics and personal finance.	View this as a d deformation is
Cohen's Kappa coefficient	Used to assess the agreement of two raters or to assess classification models. Useful in citizen science like eBird or Zooniverse.	Not a distribution.	Not a distributi

Distribution	Uses	Graph of PDF or a comment	Comparis
ANOVA	Analysis of variance	ANOVA[4] is a set of models that model the random aspects of a system from its systematic aspects. (One-way ANOVA) Similar tests are the T-Test or Kruskal–Wallis test (which is a modified ANOVA test). Other tests, tests variances instead of means like the F-Test.	Not a distributi two systems (e squares (varian between two sy a difference be
Factorial ANOVA	N-factor or N-way ANOVA (one of the most common examples is a two-way ANOVA).	More then one independent variable ANOVA. Two-way would be two independent variables, three-way would be three independent variable, etc.	ANOVA is used N or more inde
F distribution	Distribution associated with analysis of variance (ANOVA) and F-Test. This distribution can be described as skewed[5].		

Distribution	Uses	Graph of PDF or a comment	Comparis
Confidence interval or confidence contour	Given that a data's probability distribution is known or can be approximated, the confidence interval or contour line (2D) represents the probability of the data being inside the interval or contour line. The confidence level is specified by the analyst (95% is a typical value). This is not a distribution.		Confid 

This concludes the brief introduction to statistics and probability. It is highly encouraged for you to take a calculus-based probability course in you junior or senior year of university. Many details left out of this brief introduction will be filled in. The next discussion will be about the all important subject of differential equations in brief.

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[1]Technically the Poisson point process which is a mathematical structure of random variables. It looks remarkable like queueing theory for arrival times, etc. From an engineering point of view it is mostly important for reliability.

[2]It is important to understand that small statistics can be biased, in particular standard deviations (and hence our "normal" error) can be smaller with a small sample compared to a larger sample. Student t would work to resolve that bias by weighing the points differently.

[3]Public domain image of speeds of noble gases at room temperature from Wikipedia.

[4]Using ANOVA with the R programming language (S programming language made new "again." This language was influenced by the one of the most unusual languages - APL) is described in the Libretexts' course: Learning Statistics with R - A tutorial for Psychology Students and other Beginners by Navarro.

[5]Skewness is a statistical measurement like mean that describes how your data or PDF is skewed in relationship to a normal (Gaussian) curve. This is different from Kurtosis which describes the strength of your tail in your data or PDF compared to a normal (Gaussian) curve.

[6]This is a confidence contour using the R programming language (S programming language made new again) and the car

package with the carData package that contains data (which we used here) that are licenses under GNU General Public License v2.0. R is a language that centers around statistical analysis, so it is worth taken a look at...it's free so go for it.

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14.10: DIFFERENTIAL EQUATIONS

Differential Equations

When an equation is produced with differentials in it it is called a differential equation. The final force equation produced for parachute person based of physics is a differential equation.

Integral and Integro-differential equation

If an equation instead has integrals then it is an integral equation and if an equation has both derivatives and integrals it is known as an integro-differential equation. Integral equations and integro-differential equations can be converted into differential equations to be solved or alternatively you can use Laplace equations to solve the equations.

Difference equation

Many differential equations are solvable analytically however when the complexity of a system increases it is usually an intractable problem to solve differential equations and this leads us to using numerical methods. In this case the differential equations reduce down to a difference equation. We used numerical methods for parachute person but we did not need to in that particular case as it is easily solvable analytically, it was more of an academic exercise.

From parachute person let us review the differential equation and the difference equation that was generated from basic physics.

$m\frac{dv}{dt} = mg - cv^2$	$v_{i+1} = v_i + (g - \frac{c}{m}(v_i)^2)(t_{i+1} - t_i)$
Differential equation from parachute person. This is a first order differential equation.	*Difference* equation from parachute person.

Integro-differential equation and RLC circuit

To complete this initial discussion we look at electrical engineering and the ubiquitous RLC circuit is defined by an integro-differential equation if we use Kirchhoff's voltage law. This can be converted to a differential equation as show in the table below.

$-Ri(t) - L\frac{di(t)}{dt} - \frac{1}{C}\int_{-\infty}^{t} i(t')dt' + V(t) = 0$	$RC\frac{dv_c(t)}{dt} + LC\frac{d^2v_c(t)}{dt} + v_c(t) = V(t)$
Assume a current of i(t) produced with a voltage V(t) we get this integro-differential equation for a serial RLC circuit.	The equation to the left is converted into a differential equation by specifying the current in the capacitor as $C\frac{dv_c(t)}{dt}$ where $v_c(t)$ is the voltage across the capacitor. The current in the capacitor would be dthe current for the whole circuit.

First-order differential equation

First order systems are divided into natural response and forced response parts.

- Natural response is called a homogeneous solution or sometimes a complementary solution, however we believe the natural response name gives a more physical connection to the idea.
- Force response is called a particular solution in mathematics. Again force response as more of a physical connection.

Let us take an simple first-order differential equation as an example.

$$\tau\frac{dx(t)}{dt} + x(t) = K_s f(t) \tag{14.10.1}$$

We solve this problem in two parts, the natural response part and then the force response part.

- Natural response part

$$\tau\frac{dx(t)}{dt}+x(t)=0 \tag{14.10.2}$$

$$\frac{dx_n(t)}{dt}=-\frac{x_n(t)}{\tau} \tag{14.10.3}$$

Let's rewrite this in order to integrate.

$$\frac{dx_n(t)}{x_n(t)}=-\frac{dt}{\tau} \tag{14.10.4}$$

$$\int\frac{dx_n(t)}{x_n(t)}=-\int\frac{dt}{\tau} \tag{14.10.5}$$

$$x_n(t)=e^{-\frac{t}{\tau}}e^c \tag{14.10.6}$$

$$x_n(t)=\alpha e^{-\frac{t}{\tau}} \tag{14.10.7}$$

- Force response part
  - For a general function f(t) there would be many solutions so...
  - We will assume a constant force

$$\tau\frac{dx_f(t)}{dt}+x_f(t)=K_sF \tag{14.10.8}$$

The solution to this is obvious as the derivative of a constant is zero so we just set $x_f(t)$ to K_sF. Therefore $x_f(t)=K_sF$ for $t\geq 0$.

- Full solution

$$x(t)=x_n(t)+x_f(t)=\alpha e^{-\frac{t}{\tau}}+K_sF \tag{14.10.9}$$

Force equation idea versus mathematical idea

This is a defense of the idea of using natural and force response as opposed to the more mathematical definitions (which is appropriate in a pure math course, but this is engineering/science class).

A homogeneous differential equation of order n is

$$f_n(x)y^{(n)}+f_{n-1}(x)y^{n-1}\ldots f_1(x)y'+f_0(x)y=0 \tag{14.10.10}$$

where y^n is the n_{th} derivative of the function y.

A non-homogeneous differential equation of order n is

$$f_n(x)y^{(n)}+f_{n-1}(x)y^{n-1}\ldots f_1(x)y'+f_0(x)y=g(x) \tag{14.10.11}$$

The solution to the non-homogeneous equation is

$$y(x)=y_c(x)+y_p(x) \tag{14.10.12}$$

where $y_c(x)$ is the complementary solution of the homogenous differential equation and where $y_p(x)$ is the particular solutions based off g(x).

The term complementary is for the solution and clearly means that it complements the full solution.

We, however, like to take a physical interpretation and call the complementary solution a **natural solution** and the particular solution a **forced solution**. However it should be noted that this is contrary to mathematical definitions (natural means something else in mathematics). Under this terminology the solution to the non-homogeneous equation is

$$y(x) = y_n(x) + y_f(x) \tag{14.10.13}$$

where $y_n(x)$ is the natural (or unforced) solution of the homogenous differential equation and where $y_f(x)$ is the forced solutions based off g(x).

The idea for these terms comes from the idea of a force equation for a spring-mass-damper system.

$$m\ddot{x} + B\ddot{x} + kx = K_s F(x) \tag{14.10.14}$$

where m is mass, B is the damping coefficient, and k is the spring constant **and** $m\ddot{x}$ is the mass force, $B\ddot{x}$ is the damper force, and kx is the spring force (Hooke's law).

Natural solution, complementary solution, and homogeneous solution to a homogeneous differential equation are all equally valid. Forced solution and particular solution are as well equally valid. There is no need for a debate, just some understanding that there are different definitions.

14.10.1: First-order Differential Equations

14.10.2: Second-order Differential Equations

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14.10.1: First-order Differential Equations

General first-order differential equation

We consider the general first-order differential equation:

$$\tau\frac{dy(t)}{dt}+y(t)=x(t) \tag{14.10.1.1}$$

The general solution is given by:

$$y(t)=y_0e^{-(t-t_0)/\tau}+\frac{e^{-(t-t_0)/\tau}}{\tau}\int_{t_0}^{t}x\left(t'\right)e^{(t'-t_0)/\tau}dt' \tag{14.10.1.2}$$

where $y_0=y\left(t-t_0\right)$. Note that t' is used to be distinguished from the upper limit t of the integral.

To obtain the general solution, begin with the first order differential equation:

$$\tau\frac{dy(t)}{dt}+y(t)=x(t) \tag{14.10.1.3}$$

Divide both sides by τ:

$$\frac{dy(t)}{dt}+\frac{1}{\tau}y(t)=\frac{1}{\tau}x(t) \tag{14.10.1.4}$$

Rewriting in condensed form using the integrating factor $e^{-\frac{t}{\tau}}$:

$$e^{-\frac{t}{\tau}}\frac{d}{dt}\left[e^{\frac{t}{\tau}}y(t)\right]=\frac{1}{\tau}x(t) \tag{14.10.1.5}$$

Notice how a chain differentiation will returns to previous form

Simplify:

$$\frac{d}{dt}\left[e^{t/\tau}y(t)\right]=\frac{1}{\tau}x(t)e^{t/\tau} \tag{14.10.1.6}$$

Integrate both sides:

$$e^{t/\tau}y(t)-e^{t_0/\tau}y\left(t_0\right)=\frac{1}{\tau}\int_{t_0}^{t}x\left(t'\right)e^{t'/\tau}dt' \tag{14.10.1.7}$$

Solve for *y*(*t*):

$$y(t)=y_0e^{-(t-t_0)/\tau}+\frac{e^{-t/\tau}}{\tau}\int_{t_0}^{t}x\left(t'\right)e^{t'/\tau}dt' \tag{14.10.1.8}$$

Example 14.10.1.1: Solutions of First Order Differential Equations

Consider:

$$\frac{dy(t)}{dt}=x(t) \tag{14.10.1.9}$$

Multiplying both sides by *dt* gives:

$$\int_{t_0}^{t} \frac{dydt}{dt} = \int_{t_0}^{t} x\left(t'\right) dt' \tag{14.10.1.10}$$

The general solution is given as:

$$y(t) = y_0 + \int_{t_0}^{t} x\left(t'\right) dt' \tag{14.10.1.11}$$

Now Consider:

$$\frac{dy(t)}{dt} = -ay(t) \tag{14.10.1.12}$$

Dividing both sides by y(t) gives:

$$\frac{1}{y(t)} \frac{dy(t)}{dt} = -a \tag{14.10.1.13}$$

which can be rewritten as:

$$\frac{d}{dt}[\ln(y(t))] = -a \tag{14.10.1.14}$$

Multiplying both sides by dt, integrating, and setting both sides of the equation as exponents to the exponential function gives the general solution:

$$y(t) = y_0 e^{-a(t-t_0)} \tag{14.10.1.15}$$

Now Consider:

$$\frac{dy(t)}{dt} = -ay(t) + x(t) \tag{14.10.1.16}$$

The detailed solution steps are as follows:

1. Separate y(t) and x(t) terms

$$\frac{dy(t)}{dt} + ay(t) = x(t) \tag{14.10.1.17}$$

2. Rewrite the left hand side in condensed form using the "integrating factor" e^{-at}

$$e^{-at} \frac{d\left(e^{at} y(t)\right)}{dt} = x(t) \tag{14.10.1.18}$$

Notice how a chain differentiation will return the LHS to the form written in step 1

3. Divide both sides by e^{-at}

$$\frac{d\left(e^{at} y(t)\right)}{dt} = e^{at} x(t) \tag{14.10.1.19}$$

4. Multiply both sides by *dt* and integrate

$$e^{at} y(t) - e^{at_0} y_0 = \int_{t_0}^{t} x\left(t'\right) e^{at'} dt' \tag{14.10.1.20}$$

The general solution is given as:

$$y(t) = y_0 e^{-a(t-t_0)} + e^{-at} \int_{t_0}^{t} x\left(t'\right) e^{a(t'-t_0)} dt' \tag{14.10.1.21}$$

Step Function Simplification

$$\tau\frac{dy(t)}{dt}+y(t)=x(t) \tag{14.10.1.22}$$

Using the step function we set $x(t)=\Theta(t-t_0)$ and also set $y_0=0$

The step function is $x(t)=1$ for $t\geq t_0$ and $x(t)=0$ otherwise:

The previously derived general solution:

$$y(t)=y_0e^{-(t-t_0)/\tau}+\frac{e^{-(t-t_0)/\tau}}{\tau}\int_{t_0}^{t}x\left(t'\right)e^{(t'-t_0)/\tau}dt' \tag{14.10.1.23}$$

$$y(t)=\frac{e^{-(t-t_0)/\tau}}{\tau}\int_{t_0}^{t}\Theta(t-t_0)e^{(t'-t_0)/\tau}dt' \tag{14.10.1.24}$$

$$y(t)=\frac{e^{-(t-t_0)/\tau}}{\tau}\int_{t_0}^{t}e^{(t'-t_0)/\tau}dt' \tag{14.10.1.25}$$

$$y(t)=\frac{e^{-(t-t_0)/\tau}}{\tau}\left(\tau\left[e^{-(t'-t_0)/\tau}\right]\right)\Big|_{t_0}^{t} \tag{14.10.1.26}$$

$$y(t)=\frac{e^{-(t-t_0)/\tau}}{\tau}\left(\tau\left[e^{-(t-t_0)/\tau}-1\right]\right) \tag{14.10.1.27}$$

$$y(t)=\left(\left[e^{-2(t-t_0)/\tau}-e^{-(t-t_0)/\tau}\right]\right) \tag{14.10.1.28}$$

$$y(t)=e^{-\frac{(t-t_0)}{\tau}} \tag{14.10.1.29}$$

The quantity τ is a time constant that reflects when $y(t)$ drops to $\frac{1}{e}$ of its original value.

Example 14.10.1.2

Problem Statement:

Consider the differential equation

$$\frac{dy(t)}{dt}=-0.5y(t)+x(t) \tag{14.10.1.30}$$

where $x(t)=2+0.01t$

Assuming $y_0=0$, what is the behavior of $y(t)$?

Solution

General solution was derived previously as:

$$y(t)=y_0e^{-a(t-t_0)}+e^{-at}\int_{t_0}^{t}x\left(t'\right)e^{a(t'-t_0)}dt' \tag{14.10.1.31}$$

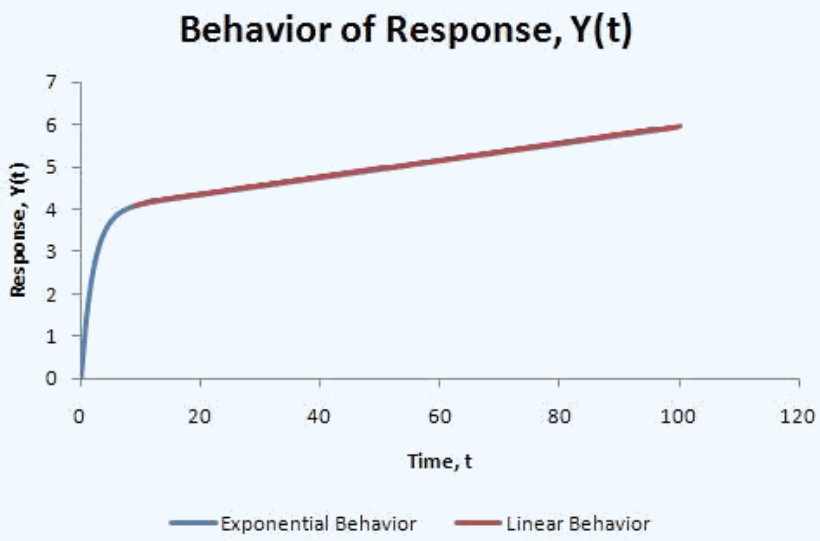
For $a=0.05$ and $y_0=0$, while setting $t_0=0$, the solution reduces to:

$$y(t)=e^{-0.5t}\int_{0}^{t}\left(2+0.01t'\right)e^{0.5(t')}dt' \tag{14.10.1.32}$$

Using the integral table in this chapter and simplifying gives the following:

$$y(t)=\frac{\left(e^{-0.5t}\right)\left(\left(x+198\right)\left(e^{0.5t}\right)-198\right)\right)}{50} \tag{14.10.1.33}$$

Plotting this function displays the following behavior:



As can be seen clearly from the graph, initially the systemic response shows more exponential characteristics. However, as time progresses, linear behavior dominates.

Parachute Person

If you remember in the section on parachute person we developed a first order differential equation.

$$\frac{dv}{dt} + \frac{c}{m}v^2 = g \tag{14.10.1.34}$$

This is a form of a Riccate[1] differential equation, albeit a simple one. It is solvable[2] but requires a known particular solution (which we could assume easily given that g is a constant), however it is easier to solve this just with an Euler approximation as was done in the parachute person section. Furthermore, this problem becomes more complex as you add more real world components to it, up to and including the addition of Navier-Stokes equation[3]. So a numerical approximation is the better method to follow for parachute person and is worth learning.

---

[1]Which is known for leading to a differential equation solution to the time independent Schrödinger equation, not parachute person.

[2]If we write the equation in the form $\frac{dv}{dt} = \left(\sqrt{\frac{m}{c}g}\right)^2 - v^2$ and then rewrite so we can integrate as $\frac{m}{c}\frac{dv}{\left(\sqrt{\frac{m}{c}g}\right)^2 - v^2} = dt$. This will eventually lead to a solution (with some assumptions and an integral table): $v = \sqrt{\left(\frac{m}{c}g\right)}\tanh\left(g\sqrt{\frac{c}{m}}t\right)$.

[3]The Navier-Stokes equation is for fluid motion. Air is considered a fluid. Navier-Stokes will be discussed later in this course/book, but only briefly as this is a higher level subject.

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14.10.2: Second-order Differential Equations

Second-order differential equations

We consider the general Second-order differential equation:

$$\tau^2 \frac{d^2Y(t)}{dt^2} + 2\zeta\tau\frac{dY(t)}{dt} + Y(t) = X(t) \tag{14.10.2.1}$$

If you expand the previous Second-order differential equation:

$$\tau_1\tau_2\frac{d^2Y(t)}{dt^2} + (\tau_1 + \tau_2)\frac{dY(t)}{dt} + Y(t) = X(t) \tag{14.10.2.2}$$

$$\left(\tau_1\frac{d}{dt} + 1\right)\left(\tau_2\frac{d}{dt} + 1\right)(Y(t) = X(t) \tag{14.10.2.3}$$

where:

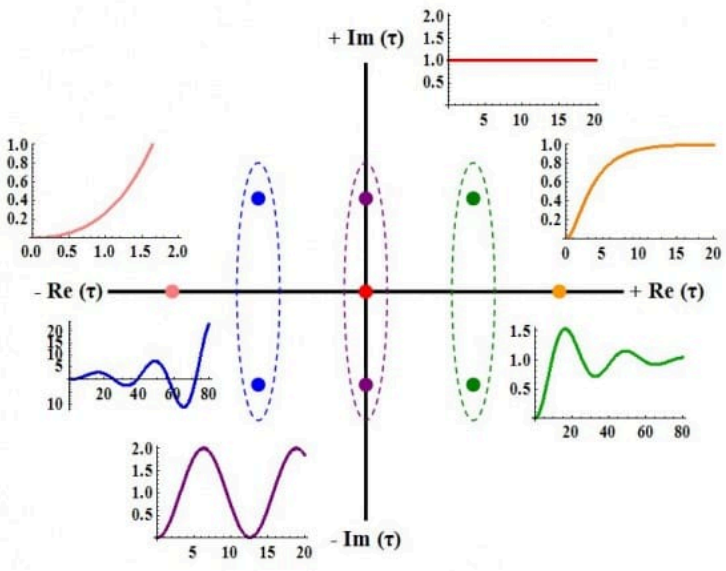
$$\tau = \sqrt{\tau_1\tau_2} \tag{14.10.2.4}$$

$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1\tau_2}} \tag{14.10.2.5}$$

Expansion of the differential equation allows you to guess what the shape of the solution (Y(t)) will look like when X(t)=1.

The following rules apply when τ_1 = Re(τ_1)+ i*Im(τ_1) and τ_2 = Re(τ_2)+ i*Im(τ_2):

- If Re(τ_1) and Re(τ_2) are both positive, and there are NO imaginary parts, $Y(t)$ will exponentially decay (overdamped). Yellow in graph.
- If Re(τ_1) and Re(τ_2) are both positive, and there ARE imaginary parts, $Y(t)$ will oscillate until it reaches steady state (underdamped). Green in graph.
- If Re(τ_1) and Re(τ_2) are both negative, and there are NO imaginary parts, $Y(t)$ will exponentially grow (unstable). Pink in graph.
- If Re(τ_1) and Re(τ_2) are both negative, and there ARE imaginary parts, $Y(t)$ will oscillate and grow exponentially (unstable). Blue in graph.
- If Re(τ_1) and Re(τ_2) are both zero, and there ARE imaginary parts, $Y(t)$ will oscillate and neither grow nor decay. Purple in graph.
- If τ_1 and τ_2 are both zero, $Y(t)$ is equal to X(t). Red in graph.



Solution of the General Second-Order System (When X(t)= θ(t))

The solution for the output of the system, $Y(t)$, can be found in the following section, if we assume that the input, $X(t)$, is a step function $\theta(t)$. The solution will depend on the value of ζ. If ζ is less than one, $Y(t)$ will be underdamped. This means that the output will overshoot and oscillate. If ζ is equal to one, $Y(t)$ will be critically damped. This means that the output will reach the steady state value quickly, without overshoot or oscillation. If ζ is greater than one, $Y(t)$ will be overdamped. This means that the output will not reach the steady state value as quickly as a critically damped system, but there will be no overshoot or oscillation[1].

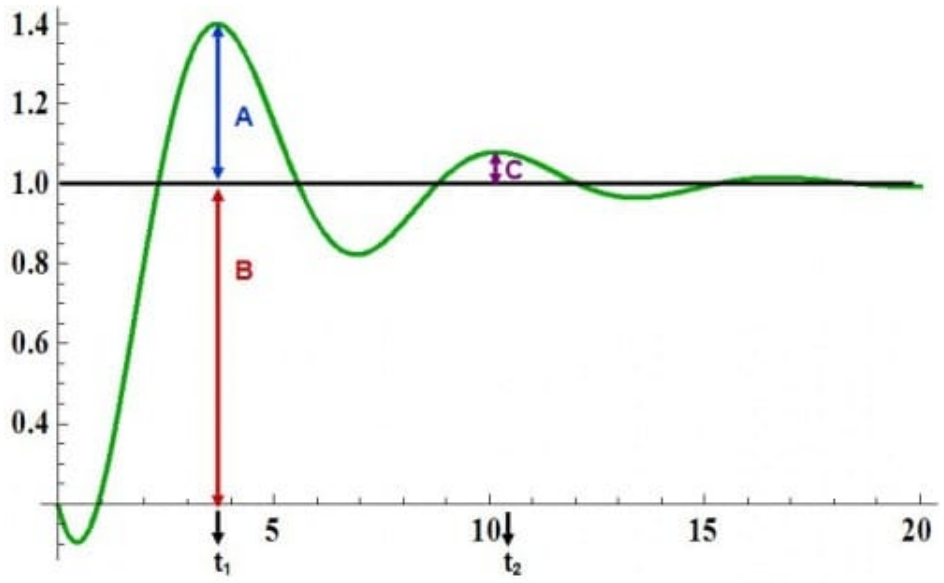
Underdamped (ζ<1)

If ζ < 1, the solution is:

$$Y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin\left(\sqrt{(1-\zeta^2)}\frac{t}{\tau} + \phi\right) \tag{14.10.2.6}$$

where:

$$\phi = -\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{14.10.2.7}$$



The decay ratio (C/A) can be calculated using the following equation:

$$\frac{C}{A} = e^{-2\pi\zeta/\sqrt{1-\zeta^2}} \tag{14.10.2.8}$$

The overshoot (A/B) can be calculated using the following equation:

$$\frac{A}{B} = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \tag{14.10.2.9}$$

The period (T) and the frequency (ω) are the following:

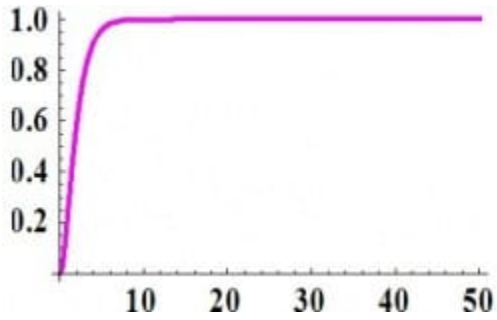
$$T = t_2 - t_1 = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \tag{14.10.2.10}$$

$$\omega = \frac{2\pi}{T} = \frac{\sqrt{1-\zeta^2}}{\tau} \tag{14.10.2.11}$$

Critically Damped (ζ=1)

If ζ = 1, the solution is:

$$Y(t) = 1 - \left(1 + \frac{t}{\tau}\right) e^{-t/\tau} \tag{14.10.2.12}$$



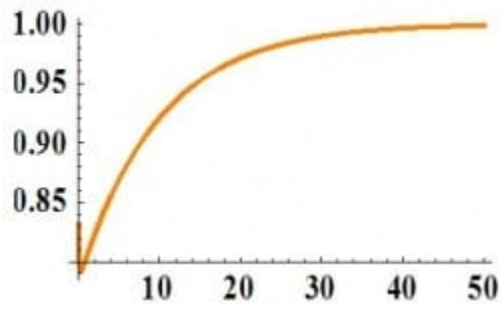
Overdamped (ζ>1)

If $\zeta > 1$, the solution is:

$$Y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sinh\left(\sqrt{(\zeta^2-1)}\frac{t}{\tau} + \phi\right) \tag{14.10.2.13}$$

where:

$$\phi = -\tanh^{-1}\left(\frac{\sqrt{\zeta^2-1}}{\zeta}\right) \tag{14.10.2.14}$$



Example 14.10.2.1

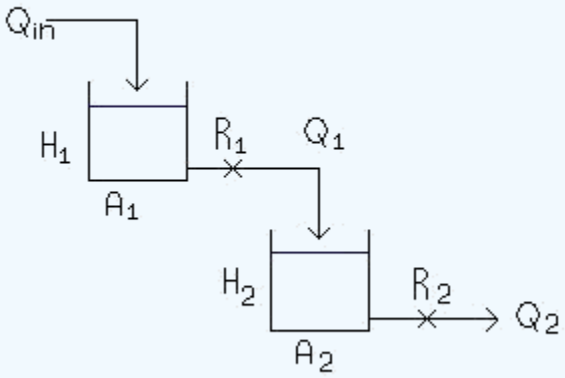
Given:

- $A_1 = 1\ m^2$
- $A_2 = 1.5\ m^2$
- $R_1 = 0.25\ s/m^2$
- $R_2 = 0.75\ s/m^2$

where:

- A is the area of the tank
- Q is the volumetric flow rate
- R is the resistance to the flow of the stream exiting the tank
- H is the height of liquid in the tank

develop an expression describing the response of H_2 to Q_{in}. Determine if the system is over, under or critically damped and determine what the graph of the expression would look like using the complex τ plane above. A diagram of the system is shown below:



Solution

Performing a mass balance on each tank:

$$A_1 \frac{dH_1}{dt} = Q_{in} - \frac{H_1}{R_1} \tag{14.10.2.15}$$

$$A_2 \frac{dH_2}{dt} = \frac{H_1}{R_1} - \frac{H_2}{R_2} \tag{14.10.2.16}$$

where the left hand terms account for the accumulation in the tank and the right hand terms account for the flow in the entering and exiting streams

Let $\tau_1 = R_1A_1$ and $\tau_2 = R_2A_2$

Equations 14.10.2.15 and 14.10.2.16 now become

$$\tau_1 \frac{dH_1}{dt} = R_1 Q_{in} - H_1 \tag{14.10.2.17}$$

$$\tau_2 \frac{dH_2}{dt} = \frac{R_2}{R_1} H_1 - H_2 \tag{14.10.2.18}$$

Put like terms on the same side and factor

$$\left(\tau_1 \frac{d}{dt} + 1\right) H_1 = R_1 Q_{in} \tag{14.10.2.19}$$

$$\left(\tau_2 \frac{d}{dt} + 1\right) H_2 = \frac{R_2}{R_1} H_1 \tag{14.10.2.20}$$

Apply $\left(\tau_1 \frac{d}{dt} + 1\right)$ operator from Equation 14.10.2.19 to Equation 14.10.2.20

$$\left(\tau_1 \frac{d}{dt} + 1\right)\left(\tau_2 \frac{d}{dt} + 1\right) H_2 = \left(\tau_1 \frac{d}{dt} + 1\right) \frac{R_2}{R_1} H_1 \tag{14.10.2.21}$$

The $\left(\tau_1 \frac{d}{dt} + 1\right) H_1$ term from the left hand portion of Equation 14.10.2.19 can be substituted into the right hand side of Equation 14.10.2.21

$$\left(\tau_1 \frac{d}{dt} + 1\right)\left(\tau_2 \frac{d}{dt} + 1\right) H_2 = R_1 Q_{in} \frac{R_2}{R_1} \tag{14.10.2.22}$$

$$\left(\tau_1 \frac{d}{dt} + 1\right)\left(\tau_2 \frac{d}{dt} + 1\right) H_2 = Q_{in} R_2 \tag{14.10.2.23}$$

This expression shows the response of H_2 to Q_{in} as a second order solution like those pictured above. Here Y(t)=H_2 and X(t)=R_2 Q_{in}

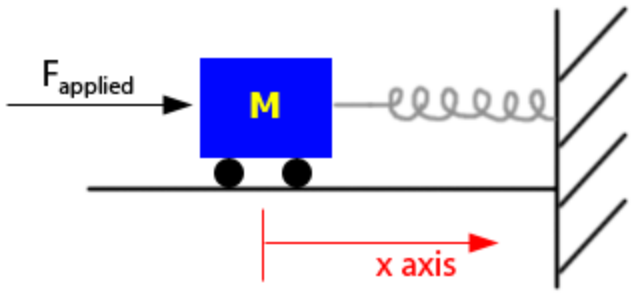
$$\zeta = \frac{\tau_1 + \tau_2}{2\sqrt{\tau_1 \tau_2}} = \frac{(0.25 * 1) + (0.75 * 1.5)}{2\sqrt{(0.25 * 1)(0.75 * 1.5)}} = 1.296 \tag{14.10.2.24}$$

Overdamped.

Both values of τ are positive real numbers, and the behavior of the graph of the equation can be found on the complex τ plane above.

Example 14.10.2.2: Analogy to Physics - Spring System

It might be helpful to use a spring system as an analogy for our second order systems.



From Newton's second law of motion,

$$F = ma \tag{14.10.2.25}$$

where:

- *F* is Force
- *m* is mass
- *a* is acceleration

For the spring system, this equation can be written as:

$$F_{\text{applied}} - F_{\text{friction}} - F_{\text{restoring}} = mx'' \tag{14.10.2.26}$$

where *x*" is the acceleration of the car in the x-direction

$$F_{\text{applied}} - fx - kx = mx'' \tag{14.10.2.27}$$

where:

- *k* is the spring constant, which relates displacement of the object to the force applied
- *f* is the frequency of oscillation

$$\frac{m}{k}x'' + \frac{f}{k}x' + x = F_{applied} \tag{14.10.2.28}$$

As you can see, this equation resembles the form of a second order equation[2]. The equation can be then thought of as:

$$\mathrm{T}^2X'' + 2\zeta \mathrm{T}X' + X = F_{\text{applied}} \tag{14.10.2.29}$$

$$\tau = \sqrt{\frac{m}{k}} \tag{14.10.2.30}$$

$$\zeta = \frac{f}{2\sqrt{mk}} \tag{14.10.2.31}$$

Because of this, the spring exhibits behavior like a second order differential equations:

- If $\zeta > 1$ or $f > 2\sqrt{mk}$ it is overdamped
- If $\zeta = 1$ or $f = 2\sqrt{mk}$ it is critically damped
- If $\zeta < 1$ or $f < 2\sqrt{mk}$ it is underdamped

[1]Note that there is no "good" or "bad" associated with overdamped, critically damped, and underdamped. Any of these conditions could be desired (or not desired) depending on the applications.

[2]While this equation can be thought of as a mechanical or civil engineering equation, a similar equation exists in electrical engineering with capacitors, inductors, and resistors. See the signals and systems (control systems) section herein.

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14.11: MECHANICS

Mechanical deals with forces, motion, and tendency for motion. Mechanics is founded on the conservation laws of physics. Mechanics describes motion with direction ($\vec{x}$), velocity ($\vec{v}$), acceleration ($\vec{a}$), momentum ($\vec{p}$), and force ($\vec{F}$) and the analogous in angular motion (to be discussed in kinematics section). Deformation of bodies (such as stress and strain) is included in mechanics as well. Statics, dynamics, and solids make up the major sections of mechanics.

The concept of motion is fraught with the idea of whether an idea is real or not because most of it is abstract ideas. If we feel something do we really feel it type argument. Here if an effect can be felt in some way then it is real and the discussions herein are about real effects.

Velocity is defined by directional speed which is the distance traveled over the time it took, acceleration is defined by the change in velocity over the change in time, momentum is a product of the velocity and a quantity of matter we call mass (an intrinsic part of a body that is **not** weight), force is an interaction that cause an object to either actually move or change direction or try to move or change direction (in the case of statics where we define force but it isn't actually moving).

For mechanical and civil (and related disciplines) engineers a set of courses are taken to fully understand this for the propose of making functional machines and structures. While this topic is the same topic that is found in physics there are techniques not generally introduced in physics that help engineers in the design process which will will briefly introduce here.

14.11.1: Statics

14.11.2: Dynamics

14.11.2.1: Kinematics

14.11.2.2: Kinetics

14.11.3: Hydraulics

14.11.3.1: Hydrostatics

14.11.3.2: Hydrodynamics

14.11.4: Solid Mechanics

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14.11.1: Statics



Original columns that held up the capitol dome now located in the National Arboretum, Washington DC. Holding up the capitol dome would be a statics problem in mechanics.

Statics

Statics is the study of objects under different forces when the body is at rest in equilibrium. From an equation point of view this means that the sum of forces are equal to zero ***and the sum of moments*** (negative torque[1], see description in vectors) are equal to zero. Assuming that we are in a rectangular coordinate system, the vectors have been decomposed to summable parts, and we have an object that has two pivot points (A and B) we can write the equations that define a static situation as the following:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \tag{14.11.1.1}$$

and

$$\sum M_A = 0 \qquad \sum M_B = 0 \tag{14.11.1.2}$$

When moment is involved one of the biggest hurdle for students is what defines a pivot point. Below, we present three examples in the hopes of helping the student with this hurdle.

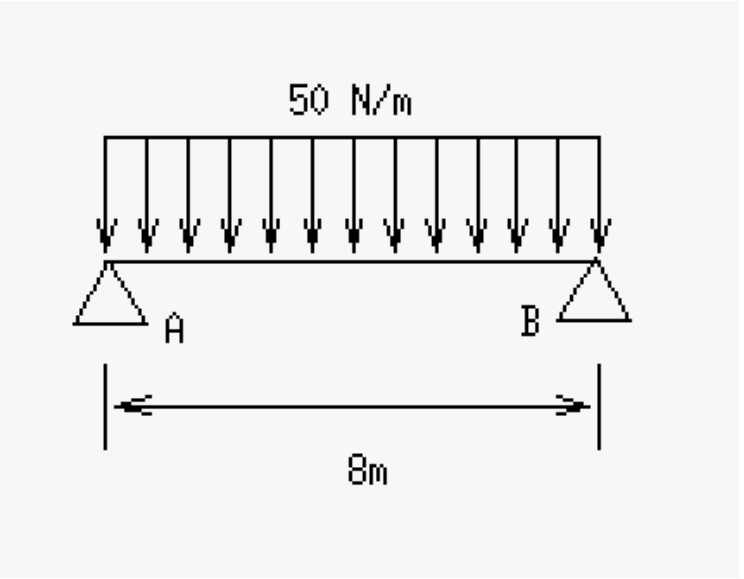
Procedure

First let us establish rules for solving problems in mechanics when dealing with forces (these are the same rules in physics class).

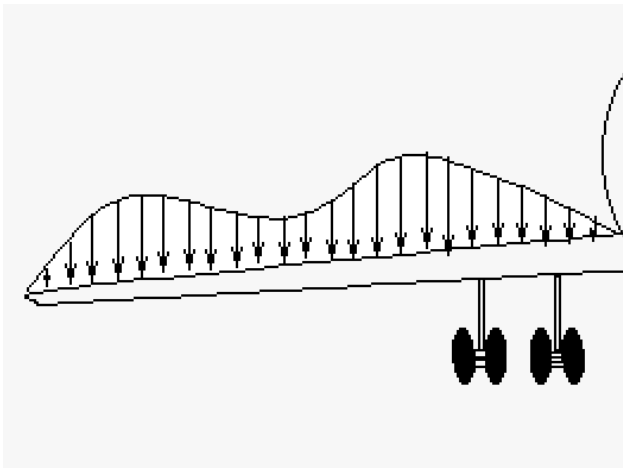
- **First step** is to draw your coordinate directions (which way does "x" go...which way does "y" go...which way does "z" go). Note if there is just one dimension then just use one coordinate, etc.
- **Second step** is to set up your **force and moment diagram.**
- **Third step** is depending on the situation get the component parts of the various forces.
- **Forth step** is to establish the force and moment equations.
- **Fifth step** is to use your statics equations (or dynamics equations).

An important idea that is always used in engineering class but is only sometimes used in physics class, is the idea of a distributed load. A distributed load (units N/m) is like a one dimensional pressure[2]. The figures below represent a simple expression of

distributed load and a more complex expression of distributed load. The idea is that the forces are not pin-point but instead distributed throughout a system. A good way to think about this is ice on an airplane wing. Ice on a wing is likely to have different forces at different points which we can represent using a distributed load. We won't go into that much detail on this (as this would eventually lead to some form of finite element analysis which is beyond this course). We will just introduce the concept and do simple examples (which are not as good as doing this correctly, but are good for conceptual understanding).



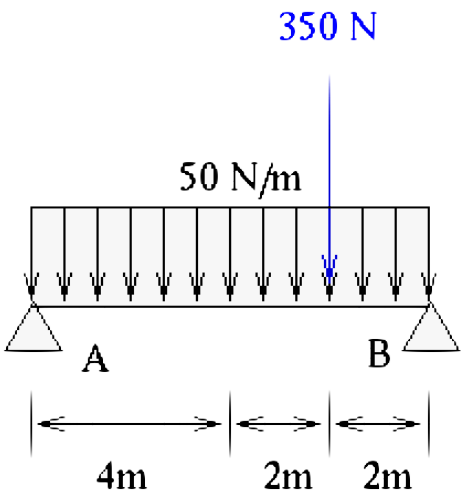
This is the simplest example of a distributed load. As a rectangle with uniform load this can be converted to a "single-point" force by using the center of mass[3] (which we will call the **distributed load force** herein). In this example we could replace the load with a force of 400 N centered between points A and B (4 meters in from either point).



This is a more complex example of a distributed load. This snow and ice. In a real world situation loads will not accom case we could approximate this shape with two semi-circle accuracy we could use a system similar to the trapezoidal r

Example 1

Calculate the reactive force at support B pictured in the figure below.

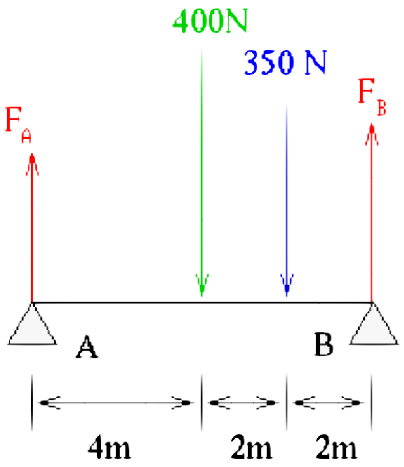


Solving a distributed load problem

We are given a problem above that has a distributed load along with a single force applied at one point, so *how do we solve this?*

The distributed load is 50 N/m which you could envision as a one dimensional pressure along the beam. The single force point is 350 N.

Let us figure out the reaction force at B. First let us convert the distributed load into a single force F_D at its center of mass (we can do this for this problem rather easily, but it may not be so easy for other problems). $F_D = (50\frac{N}{m})(8m) = 400N$ which will be placed (centered) at 4 m. The forces are shown below for this particular problem (force diagram - always do that first). Since reaction force is up in this figure we will assume the coordinate system (say y) will be up as well so the reaction forces are positive.



Since the system is static we consider the following four equations for this problem:

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_A = 0 \qquad \sum M_B = 0 \tag{14.11.1.3}$$

The sum of the forces must be equal to zero (or the problem is not be static). Since we said we will use the y-direction for up then the second force equation is all we need to consider in this problem. This gives us

$$F_A + F_B - 400 - 350 = 0 \tag{14.11.1.4}$$

This gives us the reaction force of B in terms of A. However we do not have another force equation to be able to solve this (one equation, two unknowns is not useful, of course). But we can look at the moment about A. *Why A and not B?* Because B has the reaction force itself which will not be involved in the moment about B. For moment we have $M = F \times \text{distance}$.

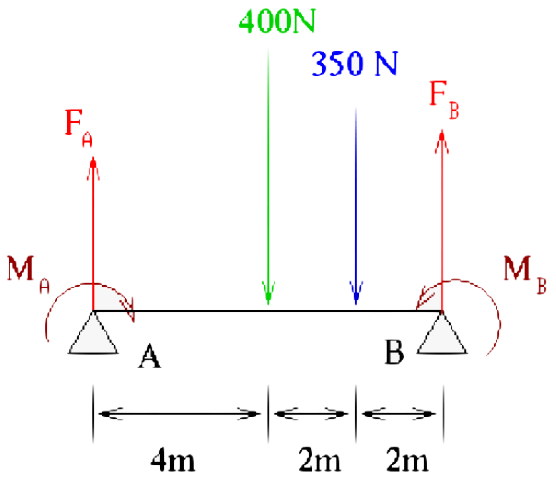
The moment about A is then

$$F_B \times 8 - 350 \times 6 - 400 \times 4 = 0 \tag{14.11.1.5}$$

$$8F_B = 3700 \tag{14.11.1.6}$$

$$F_B = 462.5\ N \tag{14.11.1.7}$$

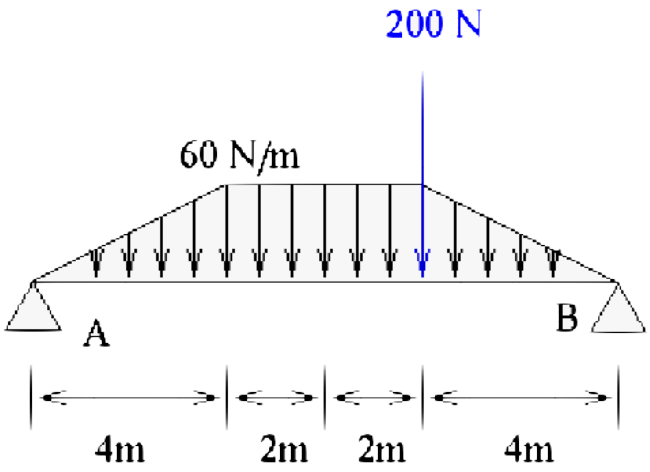
You might note that this makes sense from just a thought experiment point of view where all the forces are divided up with appropriate "weights" to "equalize" the reaction forces. The forces with the moments are shown below.



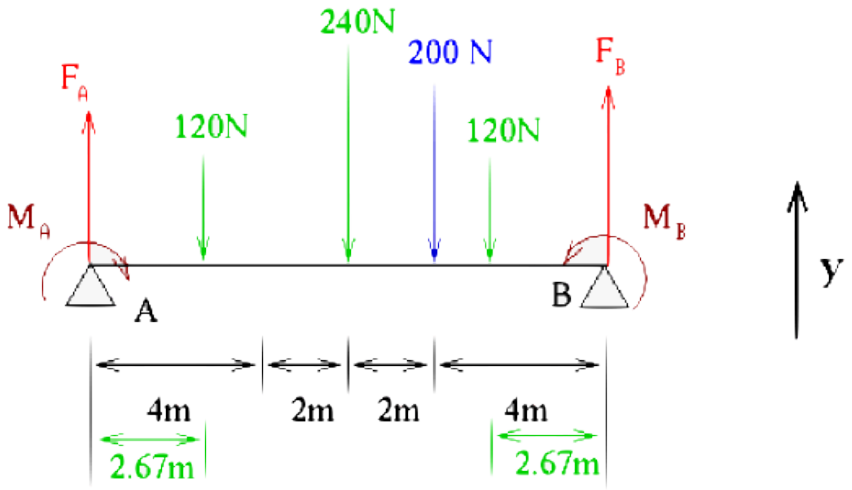
Example 2

Another distributed load problem

Calculate the reactive force at support B pictured in the figure below. Try to solve this on your own first. You will have to think about it.



¬ Force and moment diagram (with coordinate system)



Here we have the center of mass forces based off the distributed load in green, plus were they are placed in the diagram. Also the moments are shown. The forces and positions are calculated based off the shape of the portion of the distributed load.

¬ Forces and moments defined

The key here is to take the distributed load and break it up into areas that we can get the center of mass force and placement. Note that if we were figuring out the area of the distributed load above we would split it up to a two triangles and a rectangle. Well that simple arithmetic way of getting area is exactly what we do for the center of mass force. So there are three center of mass forces, one a rectangle which is $F_{rect} = 60\frac{N}{m} \times 4\ m = 240\ N$, and two triangles that have the same center of mass force which is $F_{\Delta} = \frac{1}{2} \times 60\frac{N}{m} \times 4\ m = 120\ N$. Now that we have the values we need to know where to place them. The rectangle is the easiest as that should be right at the center of the rectangle or 6 m from support A (or B). The triangles should be at a distance that would represent the center of gravity (where to place the triangle on your figure so it will balance) which is $\frac{1}{3}$ the distance from the right angle vertex (in the x-direction). Therefore one of the forces should be place 2.67 m from support A and one 2.67 m from support B. This is reflected in the previous figure.

¬ Solution

Since this is a statics problem we establish the following equations.

$$\sum F_y = 0 \qquad \sum M_A = 0 \qquad \sum M_B = 0 \tag{14.11.1.8}$$

For force equation it is clear we will have $F_A + F_B - 120 - 240 - 200 - 120 = 0$ which has two unknowns with only one equation, so we should look to the moment equations to solve this. Since we want the reaction force on support B we would use the moment equation for support A.

$$F_B \times 12 - 120 \times 9\frac{1}{3} - 200 \times 8 - 240 \times 6 - 120 \times 2\frac{2}{3} = 0 \tag{14.11.1.9}$$

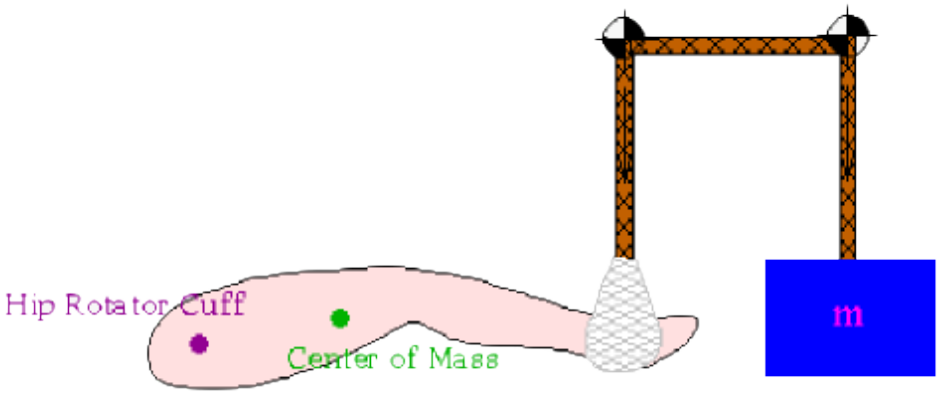
$$12F_B = 4480 \tag{14.11.1.10}$$

$$F_B = 373\frac{1}{3}\ N \qquad \checkmark \tag{14.11.1.11}$$

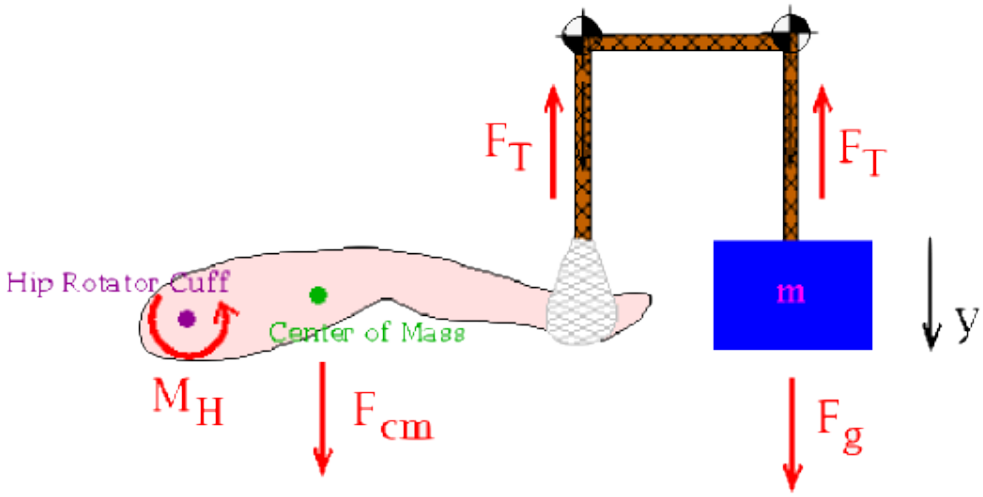
Example 3

A different type of problem

An ice skater had a bit of a spill and sprained her ankle. The doctor decided that the leg needed to be raised a bit to enhance healing. *What mass would be needed to suspend her leg in a simple pulley configuration described in the figure?* The skater weighs 60 kg and the average percentage weight of a leg is about 18% of the total weight of a person. The center of mass (CM) is about 26 cm from the hip rotator cuff and the ankle support is about 73 cm from the hip rotator cuff.



¬ Force and moment diagram (with coordinate system)



Here we have gravitational forces and tension forces. Also there is a moment about the hip rotator cuff.

¬ Forces and moments defined

In order to solve this problem we need to note that this is a statics problem as there is no motion (the leg is suspended not being picked up). Only one force needs to be defined and that is $F = mg$.

¬ Solution

Since this is a statics problem we use the following equations.

$$\sum F_m = 0 \qquad \sum M_H = 0 \tag{14.11.1.12}$$

Which leads to

$$F_g - F_T = 0 \tag{14.11.1.13}$$

$$F_g = F_T \tag{14.11.1.14}$$

Noting that the tension is the same on both sides of the pulley system we can utilize the moment equation.

$$0.26F_{cm} - 0.73F_T = 0 \tag{14.11.1.15}$$

$$0.26F_{cm} - 0.73F_g = 0 \tag{14.11.1.16}$$

$$0.26m_{cm}g = 0.73mg \tag{14.11.1.17}$$

$$m = \frac{0.26}{0.73}m_{cm} \tag{14.11.1.18}$$

The center of mass needs to be calculated as it is not given as a single number. It is a simple problem in itself.

$$m_{cm} = 0.18 \times 60 = 10.8\ kg \tag{14.11.1.19}$$

Plugging the center of mass in gives us the solution to the problem.

$$m = 3.8\ kg \qquad \checkmark \tag{14.11.1.20}$$

Next we consider dynamics.

---

[1]Torque and moment in physics are considered to be the same thing, but in mechanical engineering they are the opposite of each other. That is $\vec{M} = \vec{F} \times \vec{r} \to Fr$ and $\vec{\tau} = \vec{r} \times \vec{F} \to rF$. The reason behind this is because of how each discipline uses the moment (it is more particular to mechanical engineering). There are differences between physics, mathematics, and engineering sometimes that amount to just a small thing (like one being the negative of another) so you need to be careful in what discipline you are talking in.

[2]Pressure has units of $\frac{N}{m^2}$ which is force per area whereas a distributed load would be force per length. Stress, shear stress, and strain have the same units of pressure. While the units are the same the ideas are different with pressure being external force and stress and strain being internal. For completeness a pressure gradient $-\vec{\nabla}P$ would have units of $\frac{N}{m^3}$. Pressure gradients are normally used in fluid mechanics (Cauchy momentum equation or Navier-Stokes momentum equation which are used in advanced classes though might be used in a simpler form in computer class).

[3]Center-of-mass and center-of-gravity are two different concepts that are equal in most situations.

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14.11.2: DYNAMICS

Dynamics is the study of objects when the body is in motion.

Dynamics is divided into kinematics and kinetics. Kinematics refers to motion without a reference to force. Kinetics refers to motion with a reference to force.

14.11.2.1: Kinematics

14.11.2.2: Kinetics

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14.11.2.1: Kinematics

Kinematics describes the motion of objects without reference to force. Specifically, kinematics means the science of motion and is derived from the same Greek word that gives us cinematic (cinema - which is pictures in motion, so makes sense, eh?). Here we discuss the very basics of kinematics to give a starting idea of what it is about.

We start with the idea that we have a length (in vector form that represents the three elements to define a point in 3D space).

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{14.11.2.1.1}$$

where here we can take x_1, x_2, and x_3 to be x, y, and z if we wish. From here on we will use the traditional $\vec{s}$ rather than $\vec{x}$ as it is more general.

Linear World

To get velocity we take the derivative.

$$\vec{v} = \frac{d\vec{s}}{dt} \tag{14.11.2.1.2}$$

To get acceleration we take the derivative again.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{s}^2}{dt^2} = \vec{v}\frac{d\vec{v}}{ds} \tag{14.11.2.1.3}$$

To continue this though process, for the derivative of acceleration we get jerk, the derivative of jerk gets snap, the derivative of snap gets crackle, the derivative of crackle gets pop, etc. While jerk, snap, crackle, and pop are important in real world engineering and physics we will not discuss them here except for brief mentions; feel free to learn about them yourself.

From these initial equations and the assumption that **acceleration is constant** we can produce the important kinematic equations by integrating the various initial equations above

$$v = v_0 + at \tag{14.11.2.1.4}$$

$$s = s_0 + v_o t + \frac{at^2}{2} \tag{14.11.2.1.5}$$

$$v^2 = v_0^2 + 2a(s - s_0) \tag{14.11.2.1.6}$$

It is possible to produce kinematic equations with **non-constant acceleration**, but those would be in terms of constant jerk (or snap or crackle or pop). Just follow the same principle of integration, but instead use jerk (or snap or crackle or pop). *Are these useful?* Yes, in real life we do have jerk, etc. so we might want different kinematic equations.

Angular World

To get angular velocity we take the derivative of the angular "length".

$$\vec{\omega} = \frac{d\vec{\theta}}{dt} \tag{14.11.2.1.7}$$

To get angular acceleration we take the derivative again.

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d\vec{\theta}^2}{dt^2} = \vec{\omega}\frac{d\vec{\omega}}{d\theta} \tag{14.11.2.1.8}$$

To continue this though process, for the derivative of angular acceleration we get angular jerk, the derivative of angular jerk gets angular snap, the derivative of angular snap gets angular crackle, the derivative of angular crackle gets angular pop, etc. Again these are important in the "real world," but they are fairly advanced and are best studied much later in your academic career.

From these initial equations and the assumption that angular acceleration is constant we can produce the important angular kinematic equations by integrating the various initial equations above

$$\omega = \omega_0 + \alpha t \tag{14.11.2.1.9}$$

$$\theta = \theta_0 + \omega_o t + \frac{\alpha t^2}{2} \tag{14.11.2.1.10}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{14.11.2.1.11}$$

It is possible to produce kinematic equations with non-constant angular acceleration by following the same idea as in the linear world mini-section.

Linear and angular world

In general the linear and angular world are tethered together. We saw this in statics in considering both the sum of forces and sum of moments in the examples. In kinematics there are equations the incorporate both concepts. We find there use especially relevant in orbits of planetary object. Here we will point out the basic "first principle" equations. But note that since we are talking circular motion we will need to employ the cross product (see vector section).

$$\vec{v_n} = 0 \tag{14.11.2.1.12}$$

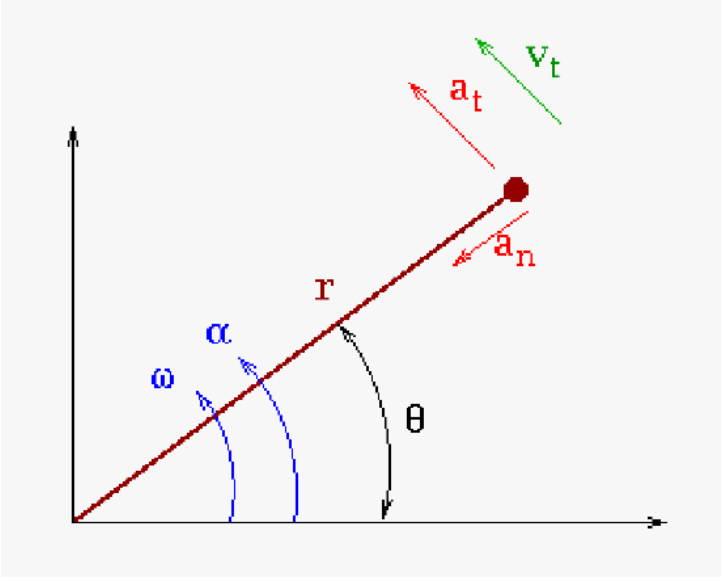
$$\vec{v_t} = \vec{\omega} \times \vec{r} = \vec{v} \tag{14.11.2.1.13}$$

$$\vec{a} = \vec{a_t} + \vec{a_n} \tag{14.11.2.1.14}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(\vec{\omega} \times \vec{r})}{dt} = \vec{r} \times \frac{d\vec{\omega}}{dt} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \vec{r} \times \vec{\alpha} + \vec{\omega} \times \vec{v} \tag{14.11.2.1.15}$$

$$\vec{a_t} = \vec{r} \times \vec{\alpha} \tag{14.11.2.1.16}$$

$$\vec{a_n} = \vec{\omega} \times \vec{v} \tag{14.11.2.1.17}$$



Cartoon description of the relationship between the linear and angular world. The subscript t indicates tangential motion and the subscript n indicates normal motion. Note the normal acceleration is sometimes referred to as centripetal acceleration. In physics class the centrifugal force[1] and the gravitational force (which some refer to as centripetal force though this is more Newtonian rather than Einsteinian) are used to explain the stability of orbits, though this is not exactly true with real orbits for a number of reasons it is a very good approximation. Sometimes the tangential velocity and centripetal acceleration is used to explain the stability of the orbits: that description is more visually satisfying and allows for a better understanding of the stability of orbits.

Giving a right angle between $\vec{a_t}$ and $\vec{a_n}$ in the circular orbit normally introduced in physics class we can just express the equations more simply as

$$v_t = r\omega = v \tag{14.11.2.1.18}$$

$$a_t = r\alpha = \frac{dv}{dt} \tag{14.11.2.1.19}$$

$$a_n = r\omega^2 = \frac{v^2}{r} \tag{14.11.2.1.20}$$

Simple harmonic motion can also have an expression combining the linear world with the angular world.

$$a = \frac{dv}{dt} = -\omega^2 r \tag{14.11.2.1.21}$$

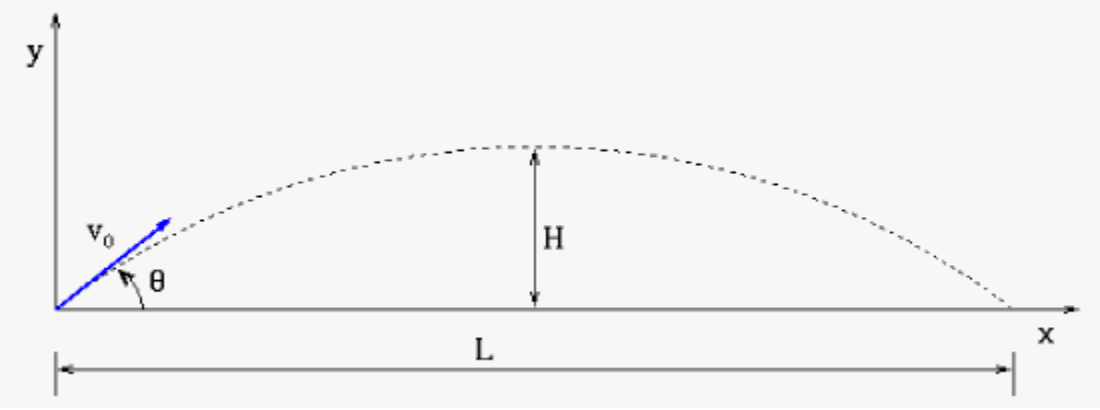
This is useful in simple simulations were simple harmonic to represent the motion of molecules (or atoms).

This is all that is going to be said for this brief introduction, a simple example (good for a test question) will be given then we will go onto kinetics.

Example

The example for this section will not involve the angular portion (we will save that for your physics class).

Consider an idealized projectile motion on Mars. Find expressions (equations) for the range L and the maximum height H in terms of the initial velocity v_0 and angle θ.



Some visual aid for the problem stated herein. The blue area represents the projectile that has originally launched with a velocity of v_0. Of course real world effects (wind and the acceleration to get to the initial velocity could be added to make this problem even more interesting, however the result won't be that different from this example unless we are talking about very strong winds).

Answer

We start with the two kinematic equations

$$s_x = s_{0x} + v_{0x}t + \frac{1}{2}a_x t^2 \qquad s_y = s_{0y} + v_{0y}t + \frac{1}{2}a_y t^2 \tag{14.11.2.1.22}$$

and we note that $a_x = 0$ and $a_y = -0.377g$. We also need to express the velocity in terms of its components in x and y, therefore $v_{0x} = v_0 cos(\theta)$ and $v_{0y} = v_0 sin(\theta)$ which leads to the equations

$$s_x = v_0 cos(\theta)t \qquad s_y = v_0 sin(\theta)t - \frac{0.377gt^2}{2} \tag{14.11.2.1.23}$$

Notice that s_y equals 0 at one end and L at the other end. Therefore we can use this to determine the time it takes to get to L by using the s_y equation which leads to the equation

$$0 = v_0 sin(\theta)t - \frac{0.377gt^2}{2} \tag{14.11.2.1.24}$$

which has two solution, $t = 0$ which is *trivial* and $t = \frac{2v_0 sin(\theta)}{0.377g}$. Therefore plugging in our non-trivial t into s_x when it equal L we get

$$L = v_0 cos(\theta)\frac{2v_0 sin(\theta)}{0.377g} \tag{14.11.2.1.25}$$

$$L = \frac{v_0^2 sin(2\theta)}{0.377g} \qquad \checkmark \tag{14.11.2.1.26}$$

And plugging in one-half of the time, $t = \frac{v_0 sin(\theta)}{0.377g}$, to get to L into s_y and setting it equal to the height, H, we get

$$H = v_0\left(\frac{v_0 sin(\theta)}{0.377g}\right) - \frac{0.377g}{2}\left(\frac{v_0 sin(\theta)}{0.377g}\right)^2 \tag{14.11.2.1.27}$$

$$H = \frac{1}{2}\left(\frac{v_0^2 sin^2(\theta)}{0.377g}\right) \qquad \checkmark \tag{14.11.2.1.28}$$

---

[1]Centrifugal force is an apparent force like the Coriolis force and are **real** forces. There is an argument that centrifugal force is not real and fictitious, but this depends on what you call a force (or anything for that matter). Almost all the items we define in physics are a concept and could be said to be not real or real depending on a particular view point (think on this - *is mass real? Mass is NOT weight so be careful how you approach this.).* There is an argument if you feel it then it is real, therefore reactive centrifugal force (like in a car when you turn) would be real. Coriolis force under this argument is also real. But each of these forces can be reformulated with other "real" forces however this is a circular argument (like saying A+B = C therefore C is not real, but we could say A-C=B therefore B is not real). And if this argument is used then *centripetal* force is not real as well as there always is another "real" force (like gravity for instance). Actually there is a stronger argument for *centripetal* force not being real then for centrifugal force and Coriolis force, however this goes far beyond the scope of this class. An attempted fix of this controversy is to just say centrifugal and Coriolis are apparent forces. However if we take a deeper look using frames of reference (graduate level physics) then indeed these forces are **real**.

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14.11.2.2: Kinetics

Kinetics describes the motion of objects with reference to force (and/or moment). Specifically, kinetics means forcing into motion. Here we discuss there very basics of kinetics to give a starting idea of what it is about.

Before moving to force we define linear momentum as $\vec{p} = m\vec{v}$ and angular momentum as $\vec{L} = I\vec{\omega}$. Both linear momentum and angular momentum are conserved in an isolated system (conservation laws will be discussed later in this course). Linear momentum ("impetus" which should not be confused with impulse) is a developed concept that is believed to have started with Aristotle, drastically improved by Philoponus (6th century) and by Ibn Sina (9th century, also known as Avicenna; not to be confused with Ibn Sahl), then refined by Buridan (12th century), and reaching its understood form of today by Newton (15th century).[1]

Kinetics is the study of objects under different forces when the body is not at rest (i.e. dynamic instead of static). From an equation point of view this means that the sum of forces are equal to the time derivative of momentum ***and the sum of moments*** are equal to the time derivative of angular momentum.

$$\sum \vec{F} = \frac{d\vec{p}}{dt} \tag{14.11.2.2.1}$$

and

$$\sum \vec{M} = \frac{d\vec{L}}{dt} \tag{14.11.2.2.2}$$

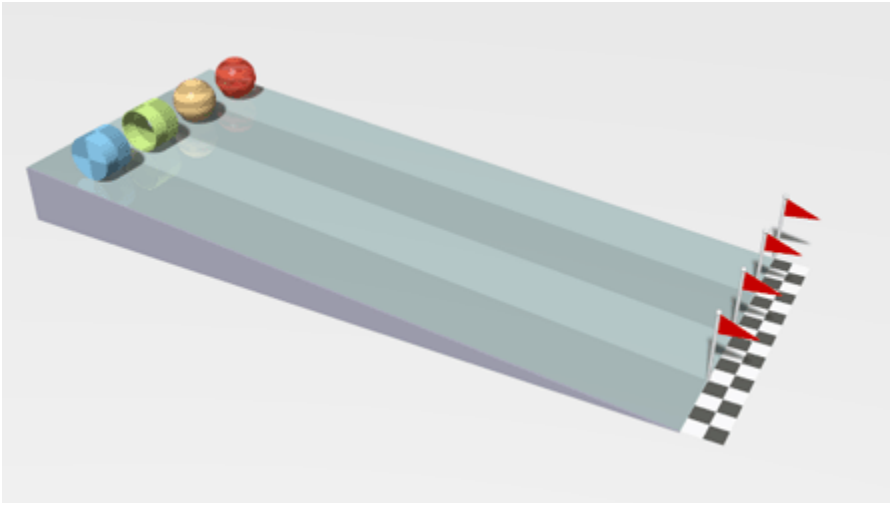
If we assume mass is constant and shape is constant then we can rewrite these equations as

$$\sum \vec{F} = m\vec{a} \tag{14.11.2.2.3}$$

and

$$\sum \vec{M} = I\vec{\alpha} \tag{14.11.2.2.4}$$

Where I is the moment of inertia (SI units of kg-m^2) and is the angular equivalent of mass (m with SI units of kg). Note mass and moment of inertia are not the same.



This is a simulation of four objects with the same mass but different moment of inertia. If these balls were sliding down the ramp ("linear world") they would all get to the bottom at the same time, but because they are rolling down the ramp ("angular world") they get to the bottom at different times due to their different moment of inertia. The brownish-yellow ball is a solid sphere which would have the greatest moment of inertia, the blue object is a solid cylinder, the red ball is a spherical shell, and the greenish object is a ring. Note that gravitational force is still dependent on the mass of the objects and not the moment of inertia. The solution to this problem mathematically is easier to do using energy in combination with kinematics rather then the methods presented in this section. From Wikipedia: public domain creation of Lucas Vieira: Rolling racers.

This section will not discuss the equation that deals with angular momentum as that is more suitable for an advanced course, but some brief remarks will be made for completeness.

Newton's Law of Universal Gravitation

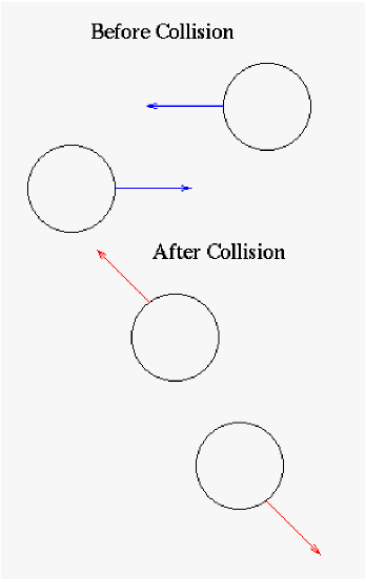
The law of gravitational force is an very accurate approximation of the forces between two bodies. Before Eddington's verification of Einstein's gravitational theories it was believed to be exact and is for all intensive purposes exact in most situations. The law in its modern form between two objects of m_1 and m_2 is

$$\vec{F_{12}} = \frac{Gm_1m_2}{r_{12}^2}\hat{r_{12}} \tag{14.11.2.2.5}$$

where G is the universal gravitational constant (SI units of $6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$ and the direction of the force is defined by $\hat{r_{12}}$.

Newton's laws of motion

- 1st law: An object at rest stays at rest; an object i motion stays in motion assuming no external forces (unbalanced forces)
- 2nd law: Force equals the time derivative of momentum: $\vec{F} = \frac{d\vec{p}}{dt}$ or where mass is constant $\vec{F} = m\vec{a}$
- 3rd law: When two objects interact (say by collision) they have equal, collinear, and opposite reactions



Cartoon of two object colliding to show the collinear reaction portion of Newton's 3rd law. Here two balls glance each other which might happen in planetary interactions.

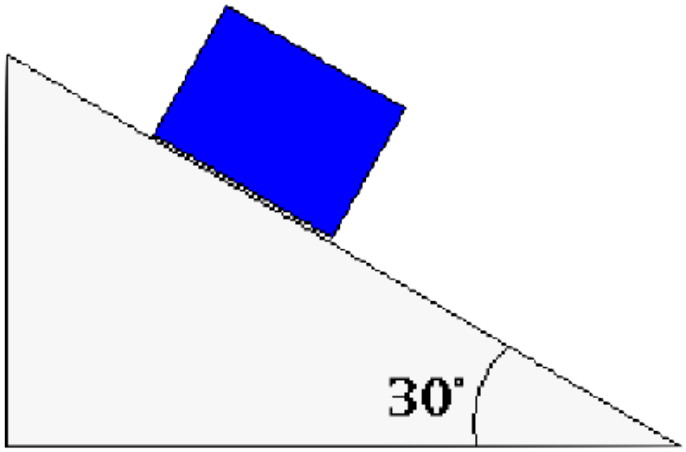
So far we have been solving mechanics problems with kinematics and kinetics, but some of these problems can be solved using energy.

Examples

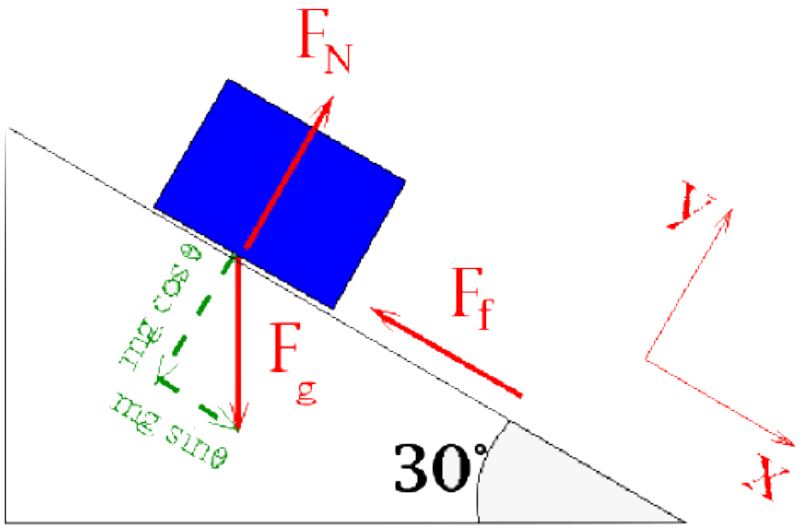
How far does a block slide on a 30 degree inclined plane in t = 10 seconds if the coefficient of friction is $\mu_k = 0.4$?

Since the block is moving we recognize this as a dynamic problem, but there maybe more to this problem then what we first observe. At any rate we should always methodically follow the following steps.

- **First step** is to draw your coordinate directions (which way does "x" go...which way does "y" go).
- **Second step** in a dynamic physics problem is to set up your **force diagram**.
- **Third step** is depending on the situation get the component parts of the various forces.
- **Forth step** is to establish the force equations.
- **Fifth step** is to use your dynamics equations (or statics equations).



¬ Sliding block force diagram (with coordinate system)



Here the coordinate system is at an angle so that only one force has to be converted into its component parts, x is also going down the ramp as that is the direction the block will go. The forces are F_N which is the normal force, F_f which is the sliding friction force (we assume the block is in motion as soon as the problem starts; this is not

quite real world, but good enough), and F_g which is gravitational force which you probably surmised already. The gravitational force is broken up into it's component parts (in green) to better solve the problem. Note that frictional force is proportional to normal force.

¬ Forces defined

Since we are going to combine forces only in their respective directions we can dispense with the vector notation. We first establish the equation between normal force and frictional force using the coefficient of friction as the proportionality constant.

$$F_f = \mu_k F_N \tag{14.11.2.2.6}$$

¬ Solution

Now we have an interesting system here as one part of the problem is static (the y-direction) and one part is dynamic (the x-direction), so following the rules we have established previously we have

$$\sum F_y = 0 \qquad \sum F_x = ma \tag{14.11.2.2.7}$$

were we have assumed the mass is constant. This leads to the following two equations

$$F_N - mgcos(\theta) = 0 \qquad mgsin(\theta) - F_f = ma \tag{14.11.2.2.8}$$

where the gravitational force components have been included in the equations. Solving for the static equation we get

$$F_N = mgcos(\theta) \tag{14.11.2.2.9}$$

which can be substituted into the second equation with the addition of the frictional force equation

$$mgsin(\theta) - \mu_k mgcos(\theta) = ma \tag{14.11.2.2.10}$$

$$a = gsin(\theta) - \mu_k gcos(\theta) \tag{14.11.2.2.11}$$

where it might be noted that the mass was not needed to solve this type of problem (note that the mass is irrelevant in the case of rolling objects, as well, as mass is part of the moment of inertia, however, the shape of the object will matter).

Plugging in the values presented in the initial problem we have $a = 1.5052 \ \frac{m}{sec^2}$ but this does not solve the problem as we want the distance traveled. Since a is constant we can use one of the kinematic equations.

$$s_x = s_{0x} + v_{0x} + \frac{1}{2} a_x t^2 \tag{14.11.2.2.12}$$

where s_{0x} and v_{0x} will be reasonably assumed to equal zero. Plugging in the acceleration and the time we get $s_x = 75.3m$ ✓.

Momentum and Impulse Force (or just Impulse)

Momentum is a property of a body which can be interpreted to mean what force is needed to change a body's motion. An impulse is the change of momentum[2] ($mv_2 - mv_1$ or $F_{\Delta t} \times \Delta t$) of a body usually used to describe collision. J is the common symbol for impulse force (because it is not really a force per say as it has units of force-time). The impulse force[3] is very useful in problems of collision and quick pushes and pulls of objects (like closing a car door). One problem in engineering is to minimize the impact of impulse force. In order to do this you could reduce the force, but that usually isn't possible in most cases, so increasing the duration of the time is usually where the engineer looks to. Car impacts during a crash are a good example where lengthening the duration of the crash (through crumble zones) actually reduces the impact of the impulse force.

How is change of momentum and impulse force equal? Let us use calculus[4] to prove the point, $J = \int F dt = \int \frac{dp}{dt} dt = \int dp \ dt$ which clearly shows the $J = \Delta p$. Of course your physics class will go into much more detail on this.

Next we consider hydraulics and its similarity to the latest topics herein.

---

[1]There is an argument for other scientists to be the originator of the final form but Newton's idea of a "quantity of matter" (i.e. mass) it is believed by the authors justifies Newton as the originator of the final form. Note the term momentum was not used until the 16th century. The student can research this on their own and decide if their opinion differs. Clearly our final ideas of everything evolve and always are due to many people, never just one.

[2]Here we are describing it in terms of force and momentum but you could also look at this concept (as you can most mechanically situations) in terms of energy using work ($F_{\Delta t}d$) and kinetic energy ($\frac{1}{2}mv^2$).

[3]There is the concept of impact force with units of force which is another way of looking at the impulse force (with units of momentum). The physics of karate usually refers impact force rather than impulse.

[4]In general it is always easier to use calculus. You will see this as you move further into your courses.

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14.11.3: HYDRAULICS

Hydraulics is mechanics with liquids[1] instead of solids (solid mechanics). Just like mechanics, hydraulics separates into hydrostatics and hydrodynamics[2]. The concepts are identical to solid mechanics. Statics when the system is not in motion, dynamics when it is in motion. Both work-energy concepts (Bernoulli) and force-momentum concepts (Archimedes) to describe motion are used in the hydraulics world. This is a matter of convenience, as each method can mostly produce the same results. For hydrostatics we still use the force-momentum concept, but in order to truly describe hydrostatics accurately we should do this in the context of the stress tensor[3]. Herein we will stick to Archimedes' principle named after Archimedes[4] of Syracuse and Bernoulli's principle.

To describe motion using the work-energy concepts we will need to use pressure (P), volume (V), velocity $\vec{v}$, density (ρ), mass (m), drag coefficient (c_d), and height (z or h).

---

[1]Fluids come in two types: incompressible and compressible. Incompressible fluids are considered liquids and compressible fluids are gases. Hydraulics usually deals with the incompressible fluids, though that depends on the specific definition of hydraulics.

[2]Hydraulics can be defined as mechanics with fluids. In that case, hydraulics has an additional subject of aerodynamics (for compressible fluids).

[3]Not herein though as it is a little advanced for an introductory course. Though we will point you into that direction if you wish to do this on your own.

[4]There were many Archimedes in Ancient Greece with a number of them being noteworthy. This Archimedes is the polymath you know as the "Leonardo da Vinci of Ancient Greece."

14.11.3.1: Hydrostatics

14.11.3.2: Hydrodynamics

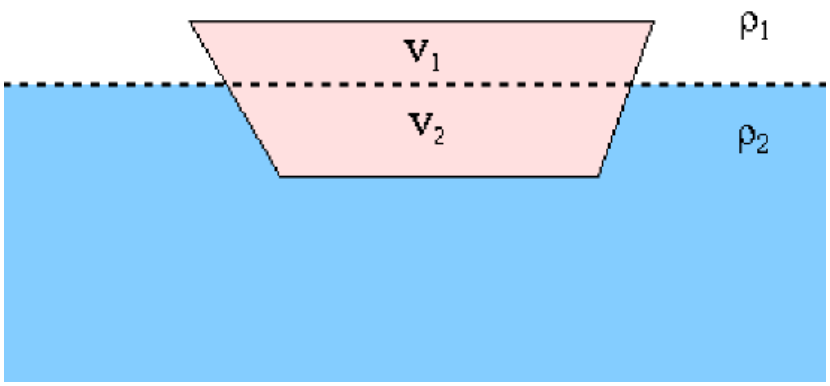
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14.11.3.1: Hydrostatics

Hydrostatics is applicable to fluid systems that are not in motion. One of the most common ideas in hydrostatics is buoyancy. This is where an object either floats, or is suspended, or sinks in a fluid. Buoyancy, generally, depends on the fluid being incompressible (liquid).

The force of buoyancy is the main idea for this subject which is dependent on the densities of the fluid the object is partially submerged in and the fluid the object is protruding into. The force equation we would be interested in is[1] $F = mg = \rho V g = g\rho V$ in the context of displacement. Let us say we have an object of volume, V, that is "floating" in a fluid with density ρ_1 in an "atmosphere" of ρ_2. We would first split the object into two parts with volume V_2 representing the volume of the submerged portion of the object and V_1 being the volume of the object in the "atmosphere." Then from there we can assert the force of buoyancy as $F_b = g\rho_1 V_1 + g\rho_2 V_2$, however we will assume that the "atmosphere" is a gas giving us the final form for the force of buoyancy as

$$F_b = g\rho_2 V_2 \tag{14.11.3.1.1}$$



This is a cartoon of an object (a boat, say) floating in a fluid of density ρ_2 and in a fluid of gas of density ρ_1. The force of buoyancy would be based off the displaced weights of each fluid. The displaced weight idea forms the basis of Archimedes' principle.

If we assume the density of our object is ρ_o we can determine from this idea if our object will float ($\rho_2 > \rho_o$), sink ($\rho_o > \rho_2$), or be suspending in the liquid ($\rho_2 = \rho_o$). You can also use these ideas to measure the density of objects.

There are many other concepts in hydrostatics but for this introduction we will leave it here and move onto hydrodynamics.

---

[1]Technically we should start with a stress tensor known as a Cauchy stress tensor for the force expressed as $\vec{F} = -\vec{\nabla} \cdot \overleftrightarrow{\sigma}$ (this form of force is typically used in simulations were we use $\vec{F} = -\vec{\nabla} V$ where V is the potential that is defined by the application). Note that the divergence of a tensor is a vector (like the gradient of a vector), unlike the divergence of a vector which is a scalar.

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14.11.3.2: Hydrodynamics

Hydrodynamics is applicable to systems that are in motion due to or part of a liquid medium. The more important concepts are conservation of mass law and the conservation of energy law which gives you the continuity equation and Bernoulli's equation, respectively.

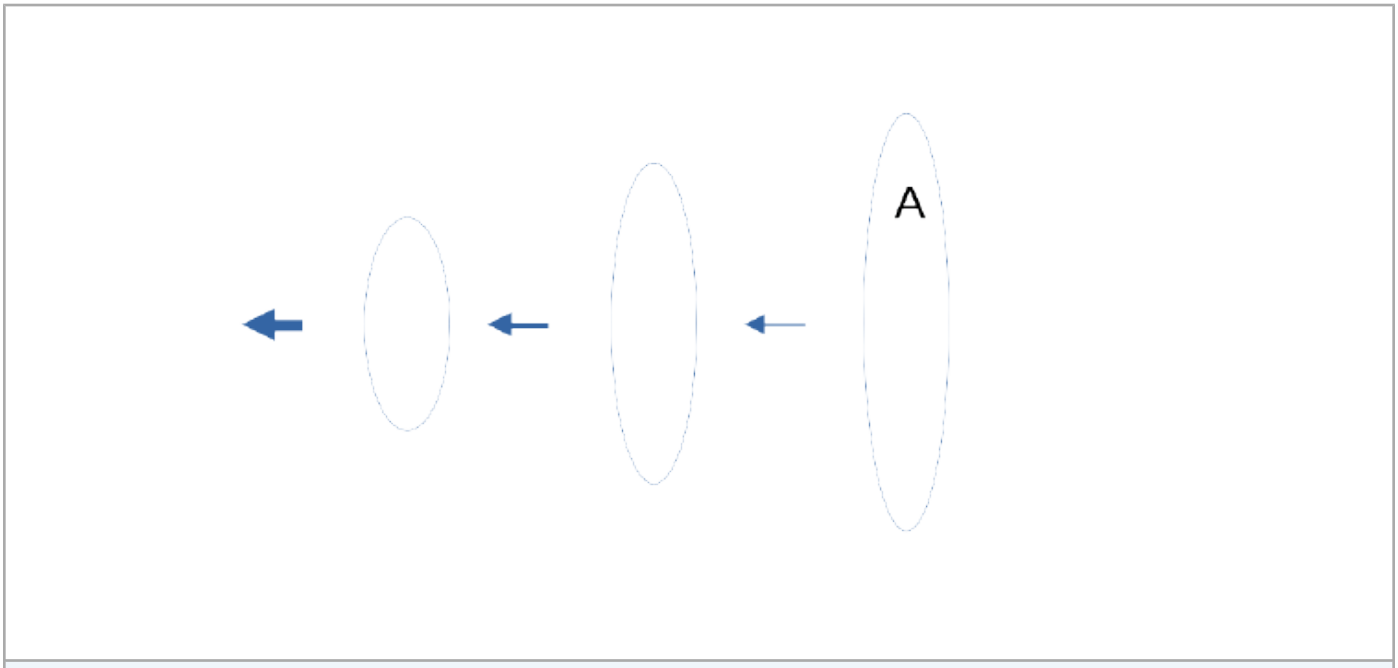
Continuity Equation

Conservation of mass leads to the continuity equation which we express[1] as

$$A_1 v_1 \rho_1 = A_2 v_2 \rho_2 \tag{14.11.3.2.1}$$

where A is the cross sectional area (with a direction normal to the A). In situations with where the density is constant we can see this equation in action with say a hose (where we restrict the area to increase the velocity).

$$A_1 v_1 = A_2 v_2 \tag{14.11.3.2.2}$$



Cartoon to visually show the flow of liquid as the Area is restricted the fluid's velocity increases.

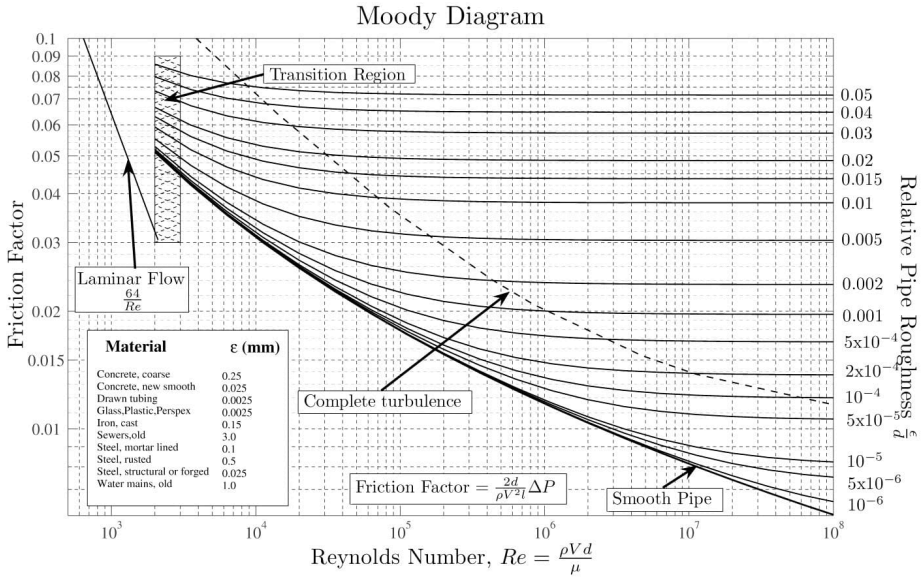
The equation that is most associated with hydrodynamics is Bernoulli's equation. This principal is really an application of the conservation of energy[2].

$$P_1 V_1 + m_1 g z_1 + \frac{m_1 v_1^2}{2} = P_2 V_2 + m_2 g z_2 + \frac{m_2 v_2^2}{2} \tag{14.11.3.2.3}$$

Where PV is a work energy, $\frac{mv^2}{2}$ is kinetic energy, and mgz is potential energy.

If you assume density is equal in a steady flow you can rewrite the above equation into its more traditional form. This equation assumes that the fluid is non-viscous, irrotation, and completely incompressible. In addition it assumes no friction loss. Bernoulli's equation is used to design flow meters and is used to partially explain airplane flight though other factors including turbulent flow is an important aspect that it does not handle. Bernoulli's equation is a good approximation for most hydrodynamic flow, but other equations and concepts must for more accurate calculations. In engineering the Bernoulli's equation is usually supplemented (or replaced) with a Moody Chart/Diagram to handle concepts the Bernoulli's equation is not equipped to handle. Another method

would be to use the Navier-Stokes equation (which can be intractable in certain instances and requires approximation methods - Numerical Methods!).



A Moody diagram is empirical and takes into account a number of concepts that the Bernoulli's equation does not consider. This image is from Wikipedia from Donebythesecondlaw (CC BY-SA 3.0)

Momentum and fluids

In general it is important to understand momentum and force for fluid motion. Following the dynamics principles described previously we have $\sum \vec{F} = \frac{d\vec{p}}{dt}$, but unlike previously we can not assume the mass is constant. So following some basics rules of calculus we can rewrite this as $\sum \vec{F} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$ or using Newton's notation as $\sum \vec{F} = \dot{m}v + m\dot{v}$. The force can be from pressure-volume forces, gravity forces, convection forces, various frictional forces, etc. However if we include these forces found in fluids we need to modify our original equation to include other courses of momentum. A general momentum equation can be formed for a pipe that has the same volume of fluid (so $\dot{v} = 0$) flowing through it:

$$\sum \vec{F} = \dot{m}\left(v_{out} - v_{in}\right) \tag{14.11.3.2.4}$$

A similar equation can be written for a general angular momentum equation.

The continuity equation is derived from the idea of the time varying mass (and moment of inertia) and basically states that the mass (and moment of inertia) must be conserved over time and space.

The idea presented here can be taken into energy-work space with regards to changing energy. You will note that if energy[3] has a dynamics component that changes over time you could image your kinetic energy varying as $\frac{d(\frac{1}{2}mv^2)}{dt}$ which would would expand out to $\frac{1}{2}\left(\dot{m}v^2 + 2v\dot{v}\right)$ and your potential energy varying as $\frac{d(mgh)}{dt}$ which would become (assuming gravity and height are not varying with time) $gh\dot{m}$. Again we would have a time varying mass (and moment of inertia) which will lead us to a energy-work type equation similar to the general momentum equation. These concepts will be completed in your junior or senior year when you have had your full complement of calculus and differential equations.

Let us look at the Navier-Stokes equation which is based off a general momentum equation.

Navier-Stokes Equations

Navier-Stokes equations[4] consists of a continuity, momentum, and energy equation for viscous fluids that takes into account. By the very nature of the equation frictional forces are included along with other aspects missed by the Bernoulli's equation. The actual equations are dynamic differential equations. The main equation is a force equals mass times acceleration expressed as

$-V\vec{\nabla}P + mg + \mu V\nabla^2\vec{v} = m(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v})$ where the left side of the equation includes pressure-volume force, gravity force, and a Stoke's flow like force (where μ is the dynamic viscosity) and the right side of the equation is mass times acceleration[5] plus a term that represents internal stress forces which takes into account friction/viscosity[6]. Generally this equation is written in terms of density:

$$-\vec{\nabla}P + \rho g + \mu\nabla^2\vec{v} = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) \qquad (14.11.3.2.5)$$

The continuity and energy equation are discussed in the above section enough to give you a general idea and while we have not done a full derivation we hope you can at least get the idea of how these are derived from what you know from general mechanics. A full understanding of calculus and differential equations would be needed to complete this discussion, so for now we will leave it at this.

Darcy's law, vorticity transport, etc. are derivations of and/or extensions of Navier-Stokes equations. There are many "versions" of Navier-Stokes equations usually adding a term from Cauchy's momentum equation. For the most complex situations the Navier-Stokes equations is intractable and numerical methods will need to be used to solve the equations (which is the case for most very complex situations).

Reynolds Number

An important concept in hydrodynamics is the Reynolds number[7] is defined as the inertial forces to viscous forces. This number defines a clear demarcation between laminar flow and turbulent flow (which is important in aerodynamics). Reynolds number is dimensionless and can be expressed as $Re = \frac{\rho v l}{\mu}$ where l is the length of the characterization range. Reynolds number below 2000 is laminar flow and greater than 4000[8] is turbulent flow. Note that between 2000 and 4000 it gets a bit murky between laminar and turbulent flow with a mix of the two to one degree or another. Because of media reports on turbulent flow and the downing of aircrafts there is a general impression that turbulent flow is bad, but this is not the case. Laminar and turbulent flow are neither good or bad but descriptions that maybe bad in certain circumstances and good in others. What is good and bad is more related to the application, not the flow characteristic.

Turbulent flow is a mixed blessing for flight as it is needed for flight, but also dangerous for flight. In general you do not want turbulent flow on the top of a wing which you can get if you take too steep an ascent however other turbulent conditions will help flight. When a nurse or doctor is measuring your blood pressure (auscultation method) what they listen for is the sound of turbulent flow changing to laminar flow (which is silent).

Bernoulli's equation can only be used in a laminar flow region which would mean a Reynolds number less than 2000. For Reynolds number greater than 2000 either a Moody chart or Navier-Stokes equation is more appropriate.

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[1]Technically the principals here should be described by differential equations with the divergence idea we discussed previously, but for this introduction we will stick to the simple expressions. This is true for the continuity equation as well as Bernoulli's equation.

[2]Conservation of energy is important in electrical circuits as well and leads to the most important equations in that discipline. You will find that all the important equations of engineering (and physics) come from the conservation laws (especially energy, mass, and momentum in an engineering context).

[3]Note that energy is related to momentum by the equation $E = \frac{p^2}{2m}$.

[4]The Navier-Stokes equation is a special case of the Cauchy's momentum equation which is a tensor-based differential equation derived from the standard equation of motion $\vec{F} = \frac{d\vec{p}}{dt}$. As such this includes pressure, stress (the Cauchy stress tensor), and mass flow which is more general then just fluid motion.

[5]Here we use a partial derivative of the velocity with respect to time (because of the energy formalism used here) but if it helps you see this better just use a total derivative of the velocity with respect to time. In the energy formalism, technically Lagrangian formalism, velocity depends on time and position, v(t,x,y,z).

[6]Which is mass times what is called convective acceleration. Convective acceleration is acceleration in a fluid that is convective (see thermodynamics section).

[7]This number would be more appropriately called Stokes number but that name is designated for something else Stokes did so Reynolds number it is...

[8]Others put this number at 3000 or 3500, but since there still maybe a small bit of laminar flow in those regions we will stick with 4000 here, but please do your own research.

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14.11.4: Solid Mechanics

Solids

This section is on solid mechanics which is just an extension of mechanics to actual materials.[1]

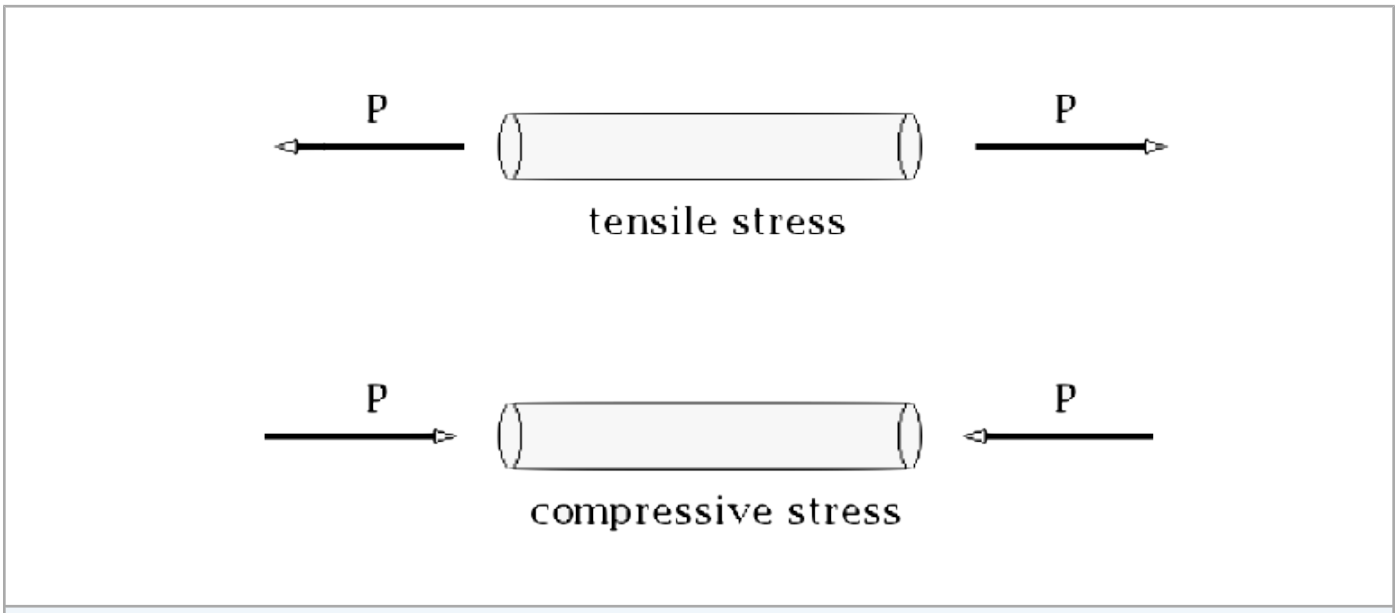
Continuum mechanics

Solid mechanics is continuum mechanics where objects are not viewed as individual particles but as a whole system from which mechanical properties can be described to. Technically we do that for statics and dynamics as well but in this subject it is viewed more in a "real world" sense, like a solid, not an ideal beam for instance, etc.

Here we will cover only a couple of aspects to give the student an idea of what is taught herein. Many aspects of this subject will not be broached here, just some essential ideas.

Stress and Strain

Stress is an intensity of pulling or pressing defined by the following equation: $\sigma = \frac{\vec{P}}{A}$ where $\vec{P}$ is force[2] and A is the cross sectional area. There are two types of linear[3] stress tensile stress (pulling) and compressive stress (pressing).

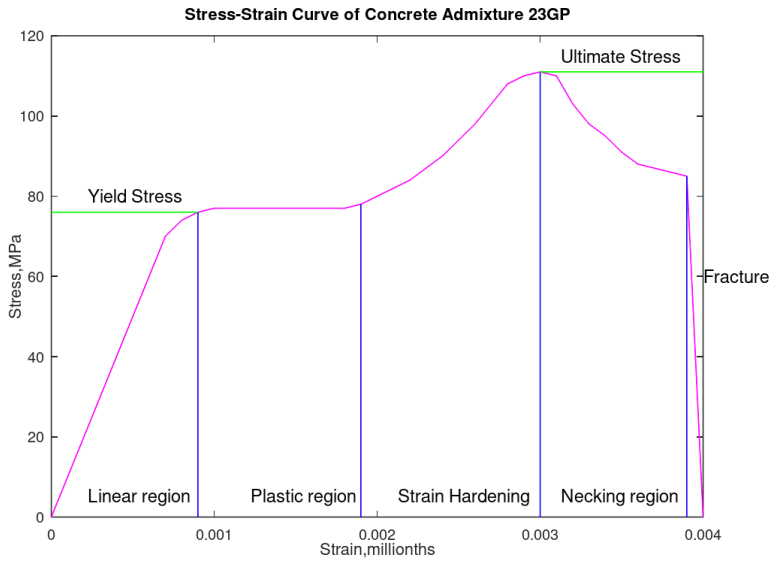


Cartoon of the two types of linear stress where $\vec{P}$ is a force.

Strain defines the change in length compared to the original length when stress is applied to a material. The simplest equation for strain is defined as $\epsilon = \frac{\delta}{L}$ where δ is elongation and L is length.

Stress-Strain Curve

After defining stress and strain we can define one of the most important curves for materials: the stress-strain curve. This curve points out how the material behaves under stress and is useful when considering what materials you would use for a particular application.



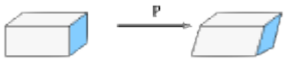
This is a fictitious stress-strain diagram to show the various regions of a stress-strain diagram might have for different materials. For academic purposes this curve includes all of them which probably would not happen for most materials. Note this curve does not represent ringing in the plastic region which might be seen in some materials. Each region describes a particular way the material behaves. In general you would want the material to perform in the linear region.

In order: The linear region is the region where as stress is applied strain rises linearly, this region is the region that is generally useable for structures. Yield stress is the minimum stress where the material will deform without a sizeable load (stress). The plastic region is the region where the material will deform with very little load (stress). That is with a little load the material will change substantially (imagine being on a bridge where that would occur...scary). Strain hardening is a region where there is resistance to strain and is accompanied with a crystalline change (which could be useful in making a new material or not). The ultimate stress is the largest stress the material will tolerate before starting to degrade. The necking region is when the material starts to break in the form of a neck (think of a rubber band that you are pulling hard and it starts to break). Finally the fracture region is when the system will fracture (break).

Other topics

An outline of some other topics in solid mechanics will help to show what this subject is about. Because a stress really is a tensor this topic cannot be fully fleshed out until junior or senior year of college.

- Tension and compression
  - This is an extension of the stress-strain idea
  - Hooke's law and Young's modulus would be introduced here
- Thermal stress
  - Temperature change can cause stress as well
- Bending stress
  - This leads to bending of beams which depend on second moments of area, cross sectional area (your beam might be a hollow cylinder or a solid cylinder for example), and centriod (eg. center of mass or gravity)
- Shear stress and shear strain
  - Stress and strain for when one part of the object is stressed but not the other part of the object



- Shear stress and strain due to torsion
  - Rotational
- Buckling

    - Upright beams, struts, etc.
    - Euler's formula which describes critical buckling
- Euler-Bernoulli beam theory (more then just Euler's formula)
    - This theory is for beam deflection and load
        - Static Euler-Bernoulli equation
        - Dynamic Euler-Lagrange equation
            - Basically a minimization of energy function
            - Solving this is basically an optimization problem (which for intractable problems is a numerical methods problem)
- Stress tensor

Nuts and Bolts

Sometimes in solids some useful parts that could be described as technician type parts are looked at as well. Again we will outline this information.

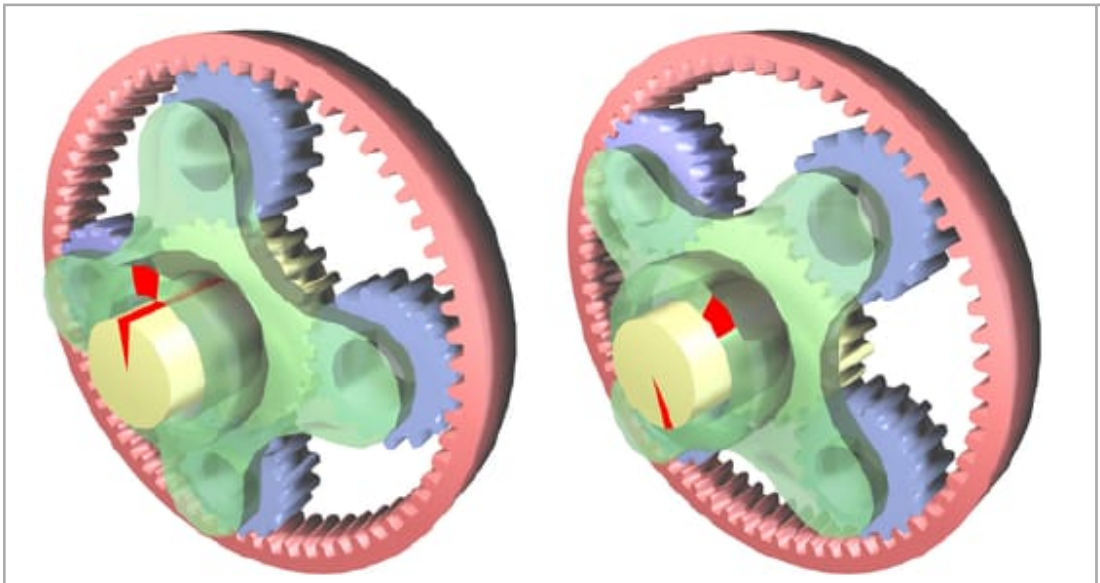
- Screws and bolts
- Axles and shafts considerations
    - Bearing stress
    - Deflection
    - Vibration
- Friction-locked joints types
    - Clamped joint
        - Where the shaft is clamped shut with a bolt on the end
    - Taper joint
        - For clamping tapered shafts
    - Clamping sleeve
        - Circular sleeve commonly used in cryostats, vacuum pumps, etc.
        - Friction locked and tightened with hex bolts very similar to a tire on a car

Springs

- Springs
    - Spring rate is a quantity that describes the amount of deflection of a spring given a certain amount of weight
        - Low spring rates are "soft" springs
        - High spring rates are "hard" springs
        - For springs in parallel between two surfaces the total spring rate is the sum of all the spring rates (like capacitors)
        - For springs in series between many surfaces the reciprocal of the total spring rate is the sum of the reciprocal of all the spring rates (like capacitors)
    - Types of spring
        - Springs that are like diving boards
            - Rectangular, trapezoidal, and triangular springs
        - Leaf springs
            - A number of the springs like diving boards stacked together to form a spring
            - Used in wheeled vehicles
        - Ring spring
            - Combination of many springs
            - Springs called either inner and outer springs
            - Inner spring compresses while outer spring expands during axial compression
            - Kind of like a telescoping spring
        - Torsion springs
            - Spring that works rotationally rather than linearly
            - Types
                - Coaxial
                - Bar
            - Torque rather than force would be a consideration
            - Also shear stress is an issue

Gears

- Gears
  - Involute gear
    - Gears with teeth (what you normally think of when you think of gears)
    - Gear ratio is the amount of turns on smaller gear (pinion) with one turn of the larger gear (gear)
      - This assumes the gears are messed together
      - Gear ratio is the number of teeth on the larger gear divided by the number of teeth on the smaller gear $G = \frac{n_g}{n_p}$
  - Bevel gear
    - Cone shape gears that are normally messed at right angles to each other
    - Teeth can be straight or spiral
  - Epicyclic gearing (planetary gear set)
    - A system of involute gears that have an outer ring gear (teeth on the inside of a circular gear) with one or more gears ("planent gears") inside the ring gear meshed with a center gear ("sun gear")
    - Large gear ratio in compact space
    - Pencil sharpener uses this type of set up
    - Used in a number of 3D printers because of the compact space and better control of stepper motors
  - Worm gear
    - Cylindrical gears with spiral teeth (you will see this gears on a guitar or similar instrument)
    - Usually a worm gear is used in combination with an involute gear
    - Worm gears are used on guitars (and the like) to tune strings, for conveyor belts, and moving filter wheel in cryostat
    - Has very high gear ratios which makes in useful for many other applications



Epicyclic gearing from Wikipedia. Eric Pierce's Own work created in Blender and GIMP. CC BY-SA 3.0

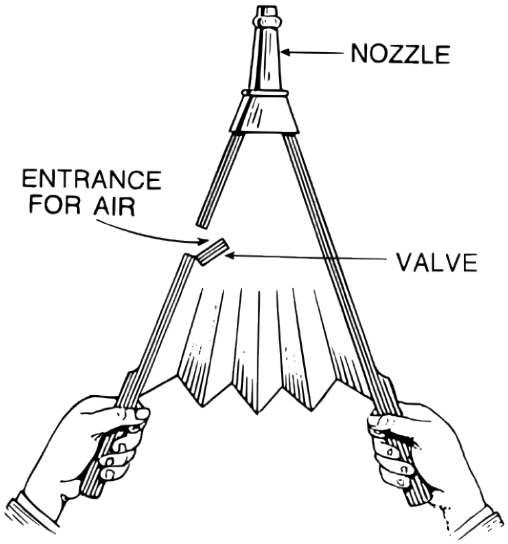


Worm gears for tuning a ma
tuners that is connected to
the wound wire on it.

Bellows

- Bellows
  - Leather bellows
    - Expandable and compressible
    - Basically a leather bag between to pieces of wood that are open and closed to send air into the fire
    - Used primarily for stoking fires to make them higher temperature for metallurgy
    - Fire pistons are similar systems developed in Asia
  - Metal bellows
    - Flexible metal structures that expand and compress with vacuum or pressure

- Many uses
  - Pressure gauges
  - Sealing (especially with toxic gases)
  - Coupling for stepper motors
  - Compressors
  - Vacuum pumps
- Musical pipe organs also have bellows



Old-fashion bellow that gives you the idea of the modern bellows. There are many web sites on bellows[4]. This image is public domain from Wikipedia by Pearson Scott Foresman.

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[1]This course is also know as strengths of materials or for short materials. For schools that have a materials science course this could lead to confusion. In this case schools refer to this course as solid mechanics which is a more general term which can include some of the additional topics that are not really strengths of materials.

[2]It is customary to use P for the force as it gives the idea of pressure, but it is fine if you want to use F as some do that as well.

[3]There are other stress like torsional stress (rotation), etc. but that will be reserved for the actual solid mechanics course.

[4]An instructive web site on the difference between metal bellows and non-meta bellows are at Bellowexpansionjoints.com

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14.12: Thermodynamics (Statistical Physics)

Thermodynamic Definitions

The previous section dealt with energy in the mechanical world, but not heat energy (or electric). Thermodynamics is the disciplines that deals with heat and its relation to other forms of energy. It is part of the overall discipline of statistical mechanics. To start off with this brief introduction to thermodynamics we will define some essential terms.

Property	In a system this is a quantity (measurable) that is to be described (like pressure or volume)
State	Properties at a given instance
Process	This is the path of successive states through the application of energy (heat, mechanical, etc.)
Thermodynamic Equilibrium	When the properties of a system have ceased to change AND have no tendency to change
Metastable	When the properties of the system are seemingly stable but have a tendency to change
iso's • isothermal - temperature • isobaric - pressure • isochoric - volume • isoentropic - entropy	When a property is constant You will normally see this terms used when discussing a P-V diagram or one of the other related diagrams
Adiabatic	Heat does not enter or leave a system
Phase	When the properties of a system are uniform
Phase transition	When a system goes from one uniform property system to another This is associated with a change in the degrees of freedom
Entropy	A measure of the disorder in a system
Enthalpy	A measure of the heat of a system
Free energy	Internal energy that is usable for work (hence it is "free" to use)
Helmholtz free energy	Entropy-based free energy which assumes constant temperature and volume Normally favored by Physicists
Gibbs free energy	Enthalpy-based free energy which assumes a constant pressure Normally favored by Chemists
Partition function	A function that describes statistically a thermodynamic equilibrium system in terms of thermodynamic properties Statistical mechanics

Statistical Mechanics

In physics thermodynamics is derived from the overall subject of statistical mechanics. Statistical mechanics uses statistical methods to describe systems of microscopic elements in terms of macro concepts like temperature and pressure. Temperature and pressure do not exist as far as single particles is concerned because they are statistical concepts. So in physics statistical is used to describe these concepts. For thermodynamics the most important concept is the partition function. The partition function is dimensionless entity that is express as

$$Z \equiv \sum_r e^{-\beta E_r} \tag{14.12.1}$$

where E_r is the total energy in each state r and β is the thermodynamic beta defined as $\frac{1}{kT}$ where k is the Boltzman's constant. The thermodynamic beta represents coldness. A little bit of heat added will increase entropy with is expressed by the "less" coldness of the thermodynamic beta. Thus the thermodynamic beta is directly related to entropy.

Most of the properties of thermodynamics can be derived from the partition function. Statistical mechanics is a graduated level physics course and as such we will not go any further on this subject, but instead move on to more concrete thermodynamics.

Entropy

Entropy is a measure of thermal energy not convertible into work. This is better understood as the disorder of a system as work is derivable from ordered state motion. The more disorder the higher the entropy and the less work available from thermal sources. We can express entropy as

$$S = k \ln \Omega \tag{14.12.2}$$

where Ω is the number of states in a system (for an ideal gas you would have $\Omega \propto V^N E^{3N/2}$ were V is volume, E is energy, and N is the number of molecules).

Laws of Thermodynamics

0^{th} Law	If $T_A = T_B$ and $T_B = T_C$ then $T_A = T_C$
1^{st} Law	$Q - W = \Delta U$ where Q is thermal energy (heat), W is work, and U is internal energy
2^{nd} Law	S_{total} cannot decrease (note however local entropy can decrease...)
3^{rd} Law	As $T \to 0K$ then $S \to 0 \Rightarrow \therefore$ can't reach absolute zero

Equations of some thermodynamic properties

There are many properties in thermodynamics but there are four properties that are generally very familiar: Volume, Pressure, Temperature, and mass density. Volume is mostly obvious from mathematics class. Pressure is $P = \frac{\vec{F}_n}{A}$ where F_n is the normal force on a surface and A is the cross-sectional area. However this is not in terms of a thermodynamic system because the force and area are a macro concept, not a micro concept, so we it might be better to define pressure as $P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ which states that pressure is the change of internal energy with respect to changing volume with entropy being held constant (and implicitly with the number of states not changing). Mass density is $\rho = \frac{m}{V}$ where m is mass and V is volume. Temperature[1] like pressure need to be expressed with a differential equation because of it is a micro concepts (atoms, molecules, particles). $T = \left(\frac{\partial U}{\partial S}\right)_{V,N}$ which states that temperature is the change of internal energy with respect to changing entropy with volume being held constant (and implicitly with the number of states not changing).

Ideal Gas Law (and less ideal gas law)

The ideal gas law is an equation that relates pressure, volume, and temperature with an assumption that the system is ideal (no interaction between the molecular particles). It is a fairly good approximate in a lot of situations. The equation is

$$PV = nRT \tag{14.12.3}$$

where P is pressure, V is volume, n is the number of moles, R is the universal gas constant, and T is temperature. This law is an accumulation of different laws like Boyle's Law (Mariotte's Law) and Charles's Law.

For a sort of non-ideal gas law we will start with the ideal gas law and modify it. The ideal gas law equation can also be expressed using a specific gas constant, $\overline{R}$ using mass (m).

$$PV = m\overline{R}T \tag{14.12.4}$$

If we use specific volume we can rewrite this equation as

$$P\nu = \overline{R}T \tag{14.12.5}$$

It is this equation where we can add a constant to that will allow

$$P\nu = Z\overline{R}T \tag{14.12.6}$$

where Z is a compressibility factor. For Z = 1 we have the ideal equation again and for $Z \neq 1$ we have an expression for a non-ideal situation. There is more to this story but in this brief introduction we will move on to the next subject which is a breakdown of the first law starting with work and its many forms.

The First Law

Work equations

In order to get a better understanding of work we list some differential work expressions with their "origin." This is not a complete list or any sort of extensive proof of concept. This is just for you all to get a handle on the idea of work after which we will discuss heat (flow) and then internal energy.

Differential work equation	Details	Type of work
dW = F ds	F is force, s is distance	General work (due to force) - kinetic energy
dW = P dV	P is pressure, V is volume	Expansion work
dW = mg dh	m is mass, g is gravity, h is height	Gravitational work - potential energy
dW = V dq	V is voltage, q is charge	Electrical work
dW = μ dn	μ is chemical potential, n is number of moles	Chemical work
dW = γ dA	γ is surface tension, A is surface area	"Surface tension" work
dW = k dx	k is spring constant, x is "stretching" of spring (or springy thing)	Spring work (or stretching work)

Heat (flow)

Generally when we talk about heat we start with the heat equation which is a parabolic partial differential equation which is rather complex at this stage of your academic career. To discuss heat we will instead go to the historical and still valid ideas of Newton and Fourier.[2]

So in the following table we present the equations on heat flow.

Law	Uses	Equation
Newton's Law	Convection (fluid)	$\dot{Q} = hA\Delta T = \frac{A\Delta T}{r}$

Law	Uses	Equation
Fourier's Law (1D)	Conduction (molecules/materials)	$\dot{Q} = \frac{kA\Delta T}{L} = \frac{A\Delta T}{r}$
Fourier's Law (3D)	Conduction/*heat flux*	$\vec{\phi}_q = -k\nabla T$
Stefan-Boltzmann Law	Radiation	$\dot{Q} = \sigma\epsilon A(T_1^4 - T_2^4)F_{1-2}$

where h is heat transfer coefficient, A is surface area where heat transfer is going through, r is resistivity factor, k is thermal conductivity, $\vec{\phi}_q$ is heat flux which using the heat equation can be reformulated in to heat flow which is done for one dimension above, σ is Stefan-Boltzmann[3] constant $\sigma = 5.670374 \times 10^{-8} W \cdot m^{-2} \cdot K^{-4}$, ϵ is emissivity (one for a black body), and F_{1-2} is a shape factor.

Integrating the heat flow[4] over time will give the heat energy.

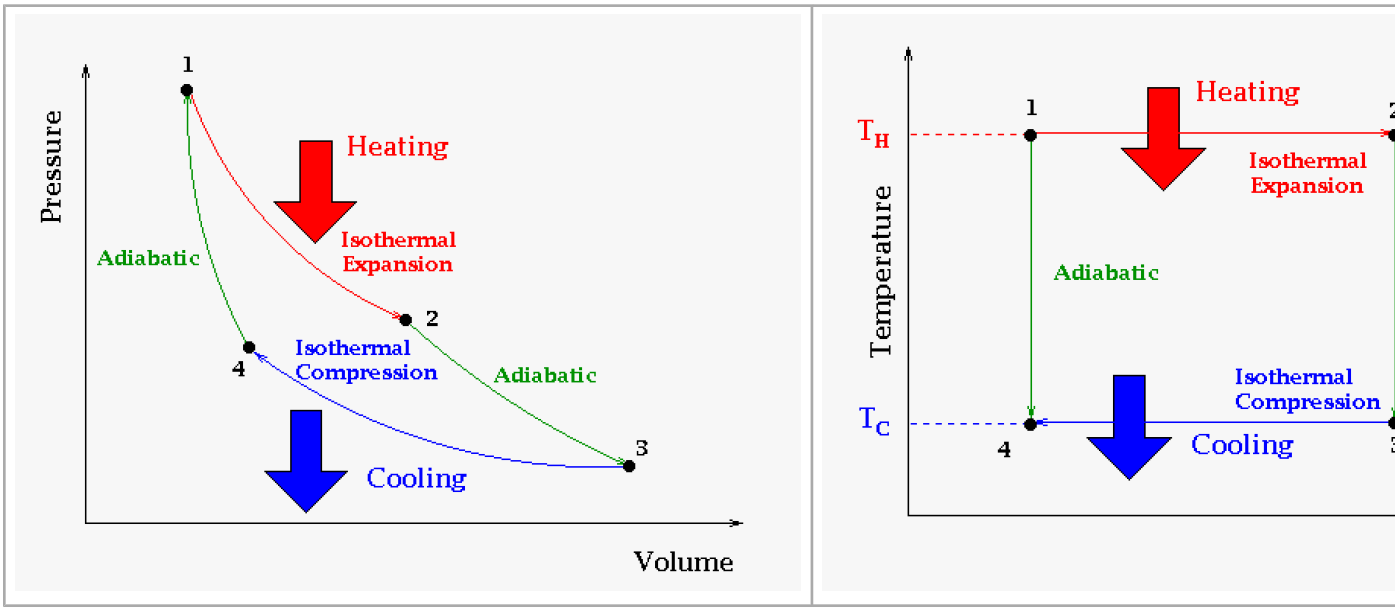
Internal Energy

Internal energy, U, is the energy that is transferred either through heat or mass internally in a system. It is not measurable except through other measurable energies. Internal energy is dependent on entropy, S, volume, V, and the particles. Internal energy does not depend on the kinetic and potential of the system as a whole, but you could view it as the kinetic and potential energy of the microscopic particles. That viewpoint will not work in the relativistic point of view however.

Carnot Cycle (Ideal Engine)

To complete this introduction to thermodynamics[5] we will briefly go over the Carnot cycle which describes the heat-work relationship particularly in engines or cryocoolers (reverse Carnot cycle). The Carnot cycles is the theoretical ideal thermal-work cycle (engine). Carnot cycle cannot be achieved but is the goal that all other cycles seek (Stirling, Otto, etc.).

Carnot cycles are normally described on a graph two different ways: Pressure versus Volume and Temperature versus Entropy.

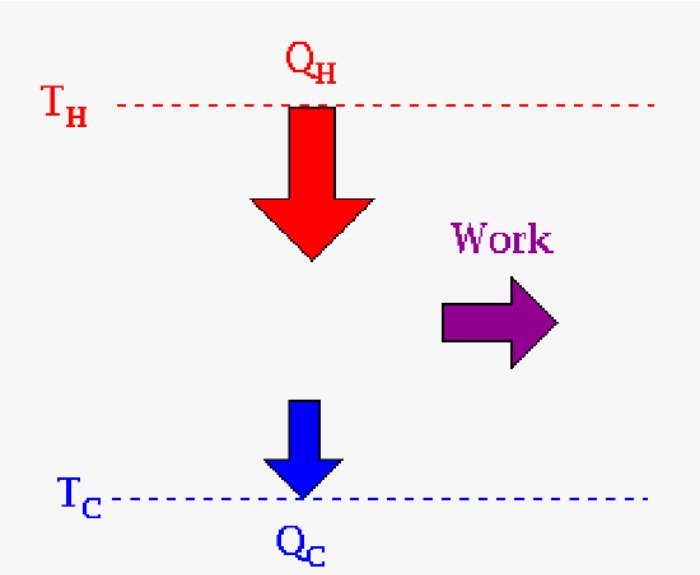
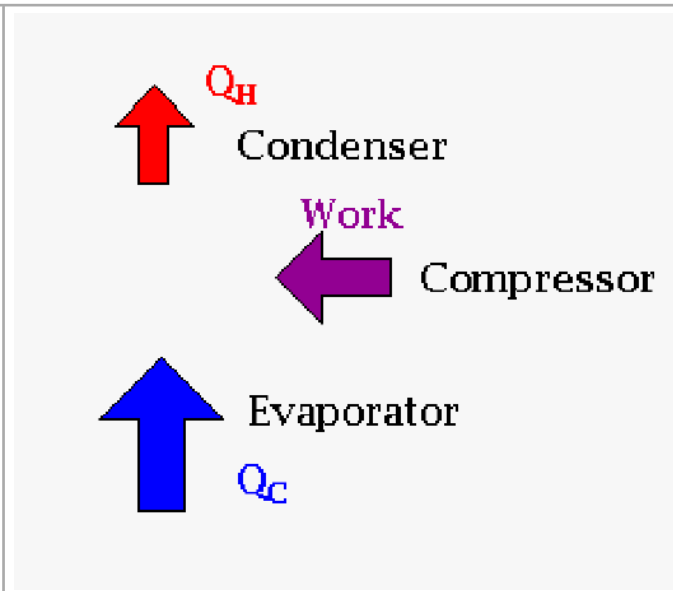


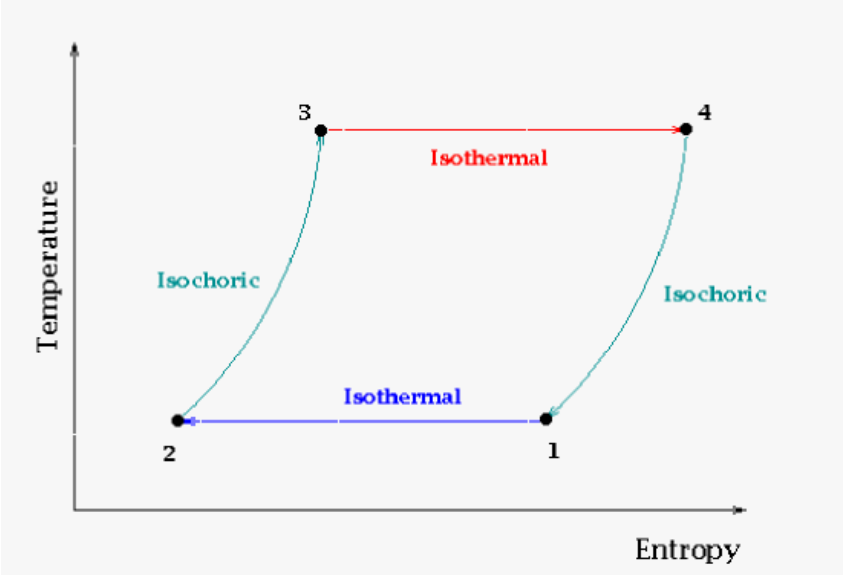
The pressure-volume description of the engine cycle is used most commonly. The above graph is of the Carnot cycle. When we apply a heat source at 1 to a system that is free to expand we go through an isothermal expansion as it absorbs heat and the volume increase as a result (pressure decreases). The heat source is removed at 2 and the system goes through an adiabatic expansion. Next we apply a cold source at 3 to the system and it goes through a isothermal compression and the volume decreases as a result. The cold source is removed at 4 and the system goes through an adiabatic compression. The work done is the area in the enclosed cycle.	The temperature-entropy description of the engine cy commonly. The main reason to use this is to see the Note that the ideal process (the Carnot cycle) does n entropy (universal). Basically the "entropy source" is e sink." Real world cycles like the Otto or Stirling cycle behavior (though they will get as close as they can). Basically the amount of useable heat to produce work producing work in the Carnot cycle.

Real world cycles will modify the Carnot cycle such as putting heat and work into the system (Otto cycle) to get more work out. Some cycles are isochoric so that clearly also will modify the Carnot cycle.

Reverse Carnot Cycle (Cryocoolers)

If we reverse the Carnot cycle we have a description of how refrigerators or cryocoolers work. The Stirling cycle is one of the most common cycles reversed to produce the Stirling cryocooler (as you might have noted when doing your laboratory design project the Stirling cyrocooler was efficient, but produced a lot of vibration).

	
The relationship between heat and work for an engine (Carnot cycle). The engineering result that a customer would want is work.	The relationship between heat and work for a cryocooler. Here work is done to produce a cold surface that you might place yo detector on to cool it. *Why?* If you remember from the laborato design it has something to do with noise on your detector.



Stirling cycle as presented using the temperature-entropy graph. Here you will note the volume is constant.

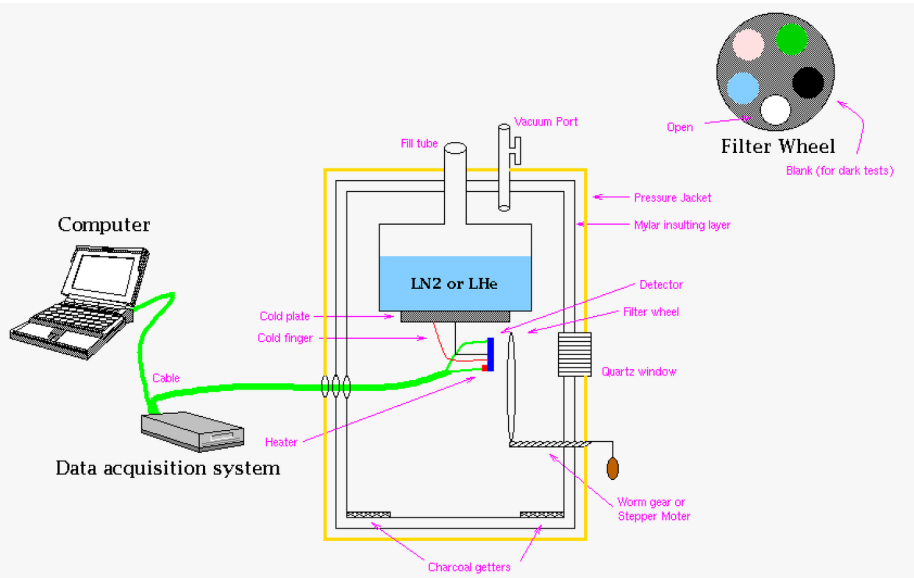
Going back to a previous section, the laboratory design, you should have investigated the idea of a cryocooler (and compared it to a cryostat (cryo-Dewar) and TECs). Here is a simple outline of cryocoolers with a description of a real world issue that is not discussed in thermodynamics: Vibration.

- Stirling Engine (Stirling Cryocooler)
  - Repeated cooling and heating of a sealed amount of working gas
    - Air, hydrogen, helium
    - Follows standard gas law behavior PV=nRT
    - Essentially same theoretical efficiency as Carnot engine
  - Stirling engines can be reversed to make a Stirling cryocooler
  - High vibration
- Gifford-McMahon Cryocoolers
  - Version of a Stirling cryocooler
  - Inexpensive
  - High vibration
- Vuilleumier Cryocoolers
  - Version of a Stirling cryocooler
  - Mechanically linked two displacers (different from Stirling)
    - The displacer is a special "piston" in certain Stirling-type cryocoolers whose purpose is to move the working gas back and forth between the hot and cold exchangers.
    - The displacer is not the main piston(s) that is normally attached to a flywheel.
    - The classic Stirling cryocooler does not have a displacer.
  - Low noise
  - Reliable (as are all Stirling-type cryocoolers)
- Joule-Thomson Cryocoolers
  - Throttling process
  - Moisture problem in the throttling process is a known problem. Mini-JT cryocoolers might suffer from this problem worse.
  - Low vibration
  - Low efficiency
  - Usually combined with other cryocooler technology
  - Can be operated without other cryocoolers, but need special compressor
- Sorption cryocooler
  - Type of Joule-Thomson cryocooler
  - Achieves the thermodynamics through a chemical means (adsorption).
  - Low efficiency

    - Very low vibration
- Brayton Engine (Brayton Cryocooler)
    - Turbine-based compressor-type engines
    - Can be reversed to produce a Brayton cryocooler
    - Highly efficient at low temperatures
    - HST's NICMOS uses a turbo-Brayton cryocooler uses balanced turbines to achieve vibration-free performance
- Flourescent cryocooler
- Solid state cryocooler
    - No moving parts as in Stirling-type cryocoolers
    - Low mass
    - New technology still being developed but could be promising.
- Pulse-Tube Cyrocoolers
    - Using a compressor on a regenerator connected to a sealed tube produces a cold end and a warm end
    - Efficiency can be increased by changing manipulating the mass oscillation (see Orifice Pulse Tube handout)
    - Benefits
        - No cold moving parts as in Stirling-type cryocoolers (which will have vibrational benefits, of course)
        - Low vibration (mostly caused by the compressor which can be isolated)
        - High efficiency
        - Inexpensive
    - Drawbacks
        - Compressor, usually very big and noisy
        - Cooling of compressor either by water or air
        - Large for space applications, but versions for space have been achieved
        - Miniature pulse-tubes might suffer from boundary condition effects with smaller and smaller tubes. This will decrease their efficiency.

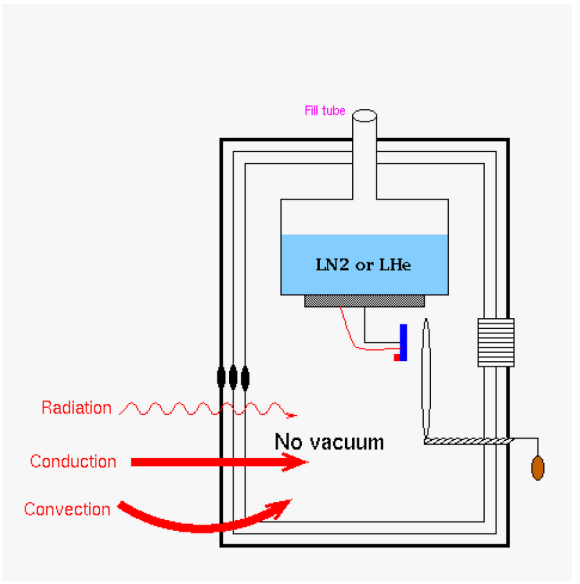
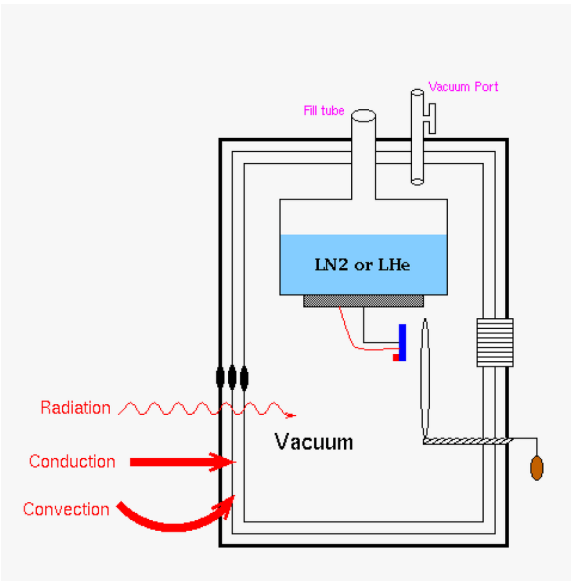
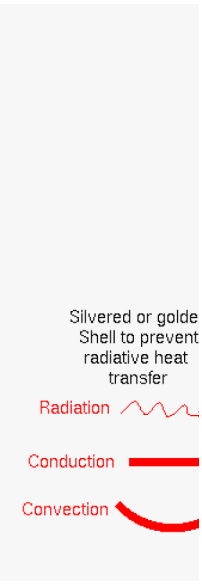
Liquid cryostat ("Dewars")

Cryocoolers (non-liquid cryostats) are to be compared to liquid cryostats (in the sense of "scientific instrument Dewars") and Thermoelectric Coolers (TECs). Cryocoolers main drawback is vibration where as Dewars and TECs have essentially zero vibration though TECs may have a thermal footprint as any electronic device might have especially in the scientific arena.



A cartoon of a Dewar (liquid cryostat) for the propose of cooling detectors. Pictured here are the items usually associated with running a detector such as a data acquisition system and computer. In the Dewar is the detector and a filter wheel that is moved by a worm gear (see gears in Solid Mechanics section or just do a quick search on the web). Because the detector is cooled there are getters to take up any residual moisture in the vacuum chamber. This image is exaggerated in order to label items within a Dewar system (note there are many many variations on this idea).

Using the Dewar figure above we could look at the various heat flow laws in relationship to preventing heat flow into the cooled chamber where the detector is mounted and tested in the system. This is instructive cyrogenic systems.

		
This is our Dewar with no vacuum and a non-silvered (or similar cladding) jacket. All mechanisms of heat transfer will cause the system to heat up. This is not a good design.	Pumping down the Dewar through the vacuum port allows for a vacuum chamber where the detector is mounted in the system. Conduction and convection heat flow is stymied, however radiation heat transfer is still getting into the system.	Putting a silvered (Dewar will prevent best configuration

That completes this brief journey into thermodynamics[6] (and applications) for this course. You will learn more about thermodynamics and the thermodynamic cycle in physics and possibly a thermodynamics course (should take if you are interested, of course).

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[1]A more intuitive way but not reliable way of looking at temperature is by using kinetic theory to used average molecular kinetic energy and the ideal gas law to derive a so-called kinetic temperature. While more intuitive it is not accurate in a number of situations.

[2]Joseph Fourier, the mathematician behind the Fourier series and Fourier transform, also was a physicist who worked on a theory of heat flow that was based of Newton's Law of Cooling. From these formulas comes a the more complex heat equation which we will not cover here. A side note, Fourier is the first to figure out the greenhouse effect.

[3]If on a test you were given this constant but not the equation you could use the units of this constant to help you remember that radiative heat flow should have T^4 rather then T in the equation. Hint, hint.

[4]We use heat flow here because heat transfer is implied in heat, Q, itself. $\dot{Q}$, which is the time derivative of heat, is really then the flow of heat transfer of time. However engineers call this quantity heat transfer which is like saying heat transfer transfer not unlike saying ATM machine or SAT test. We believe heat flow is more expressive.

[5]The ideas in this section of apply heating and cooling for the purpose of producing engines is where the term thermodynamics came from. This process is clearly dynamic and is the counter to the debate that thermodynamics is actually thermostatics.

[6]It should be noted that while in modern times thermodynamics is derived from statistical mechanics, historically thermodynamics was developed first. Statistical mechanics was developed to give thermodynamics a theoretical underpinning.

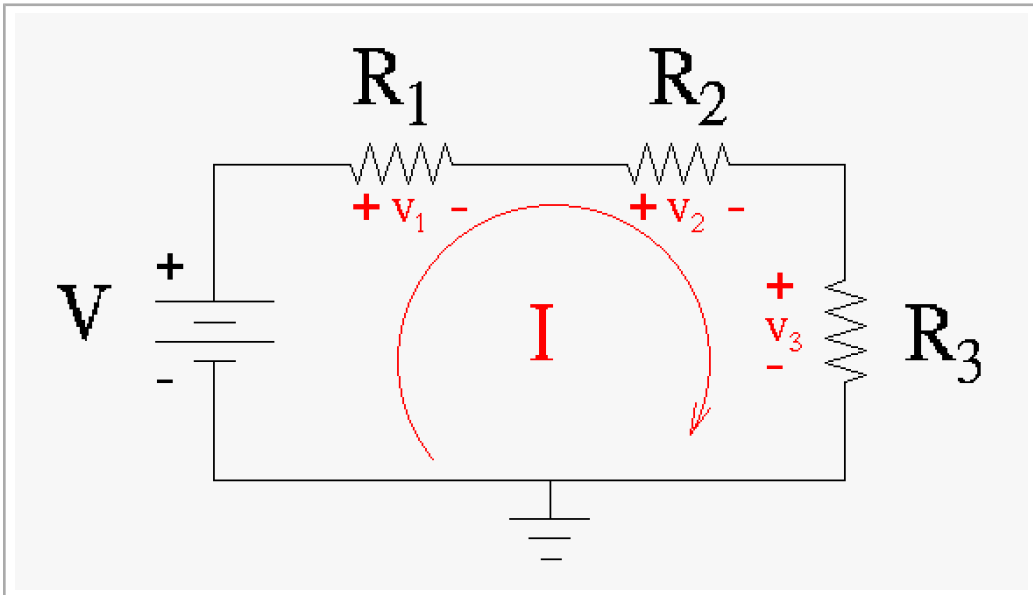
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14.13: Electrical Circuits

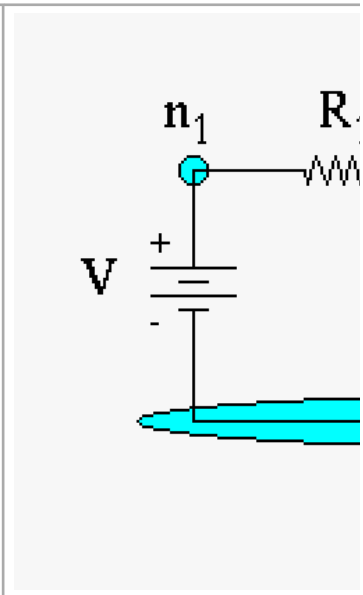
Electrical circuits are very important to all engineering disciplines either because there are electric circuits in those disciplines or because the underlying physical ideas are easily translated to other disciplines. The two most important laws in circuit analysis are the two ***Kirchhoff's Laws*** which are just another form of the *conservation laws of physics*. These laws are ***ALWAYS*** valid in every situation. This can get confusing sometimes as some techniques in circuit analysis might seem to violate these rules - they do not, when you encounter these situations remember that they are just shortcuts (approximations).

Before we continue we must define some ideas in electric circuits, for instance *what is a circuit?* A circuit follows from the regular definition of a circuit, a circular path, but with the addition of electrical elements. This is extended to have possible **branches** off from the original circuit. For each branch of a circuit typically we assign a **node** as can be seen in the figure below (note: nodes could be defined anywhere though generally you want it to be useful).

Description of an simple electric circuit



A simple electric circuit consisting of a voltage source and some resisters (example: incandescent light bulbs). If we follow the current loop this would be how **mesh analysis** is done. For this simple mesh using Kirchhoff's law (presented below) we would get the equation: $V = v_1 + v_2 + v_3 = IR_1 + IR_2 + IR_3$



A circuit with branches (the wire circuit is a little more advanced a analysis: mesh analysis which rev aforementioned nodes. Typically

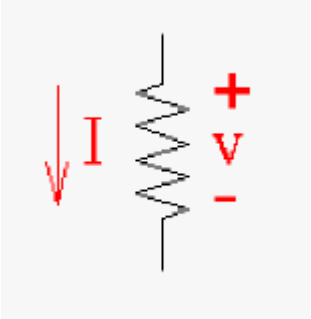
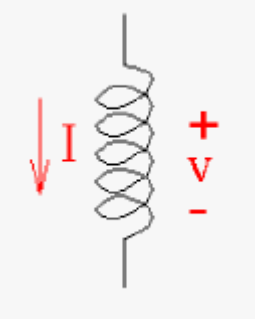
So what are electrical elements? The are the various building blocks of an electrical circuit that represents energy storage and release (as is the case for mechanics as well...see control systems). They are literally hundreds of electrical elements, so here we will just introduce the most important ones to give you a taste of circuits.

Electrical Elements

Passive electrical elements

Passive elements just store and release energy that is originally introduced into the circuit from an active element. The three most important passive elements are the resistor (releases energy into the air by heat), the capacitor (stores energy electrically), and the inductor (stores energy magnetically).

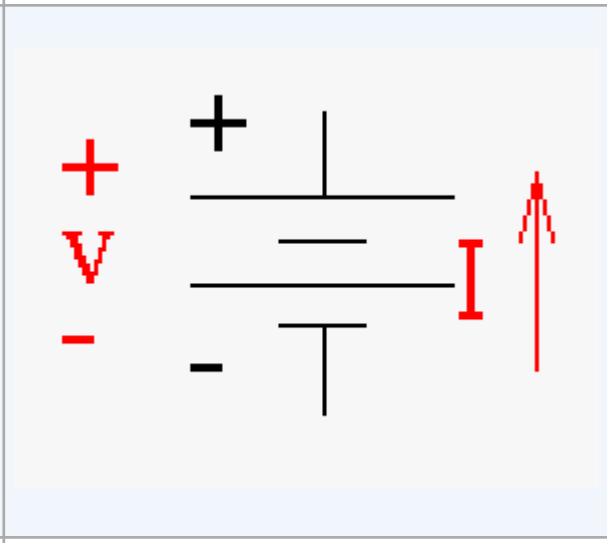
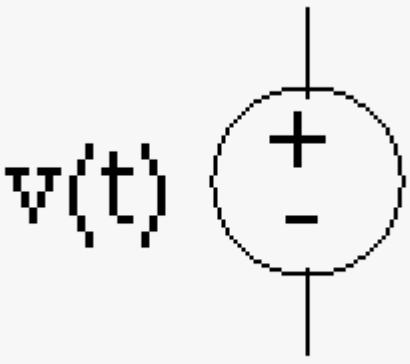
Element	Resistor	Inductor	Capacitor (Condenser)
Constant (Symbol)	R	L	C

Electronic Symbol (Schematic)			
Equation	v=Ri Ohm's Law	$v = L\frac{di}{dt}$ $i = \frac{1}{L}\int v(t)dt$	$i = C\frac{dv}{dt}$ $v = \frac{1}{C}\int i(t)dt$
Physics	Releases energy through heat	Stores energy by means of a magnetic field	Stores energy by means of an electric field
Units	Ohms	Henry	Farad

These elements are often used in control system equivalent circuits.

Active electrical elements

Active elements store and release energy in an active manner with either the redirection of energy or the introduction of energy through an external source (example: an electric plug) or a chemical process (example: a battery). The five most important active elements are the voltage source (example: a battery), current source, transistor (bipolar or FET), diode, and operational amplifier.

Element	Voltage source (DC)	Voltage source (AC)	
Electronic Symbol			
Description	Direct current voltage source	Alternating current voltage source (time varying)	Ty yc

This is definitely not a complete list of active elements but this is good for a quick introduction especially for engineers that might not be going into electrical engineering.

Kirchhoff's Laws

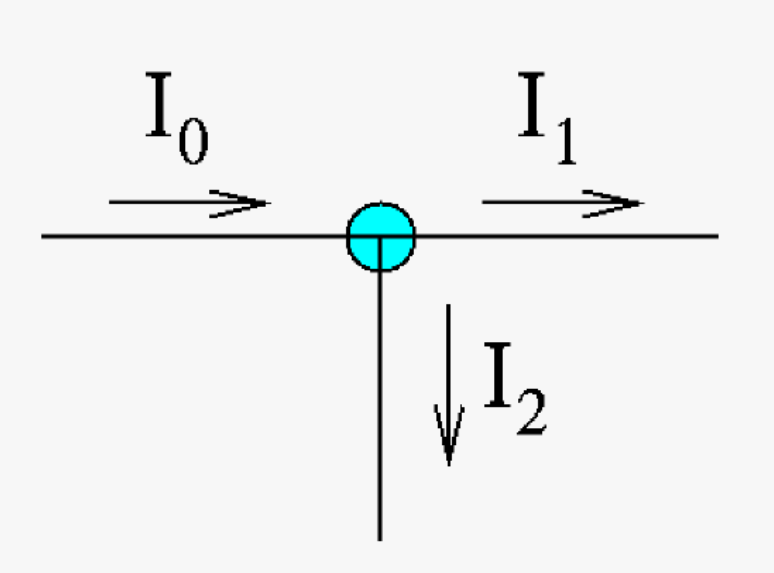
Kirchhoff's Voltage Law

The sum of the voltages in a **closed circuit** is equal to zero. $\sum V_i = 0$

Example: Using the simple circuit above we would have $V - v_1 - v_2 - v_3 = 0$

Kirchhoff's Current Law

The sum of the currents going into a **node** are equal to the sum of the currents going out of a node. $\sum I_{in} = \sum I_{out}$



"Current in" is equal to "current out" (or sum of current equals zero if you are consistent in signs). Here we have current I_0 entering a node and splitting into current I_1 and I_2 whose sum is equal to current I_0.

It should be noted that this is a simplification of the actual Kirchhoff's laws which are more complicated and involve partial differential equations (see Kirchhoff's original papers on the subject). These laws however are sufficient for the practitioner to solve circuit problems.

Equivalence Theorem

The equivalence theorem allows the electrical engineer to make an equivalent circuit for a circuit that is for all intensive purposes a black box (example: the plug in the wall - we don't need to know the full circuit; we can just make an equivalent circuit). Measurements can be made on any black box circuit to form an equivalence without knowing the details.

Thévenin's Theorem

Any linear network can be converted to an equivalent form that consists of a voltage source in series with a resistor (or an impedance so as to include other elements).

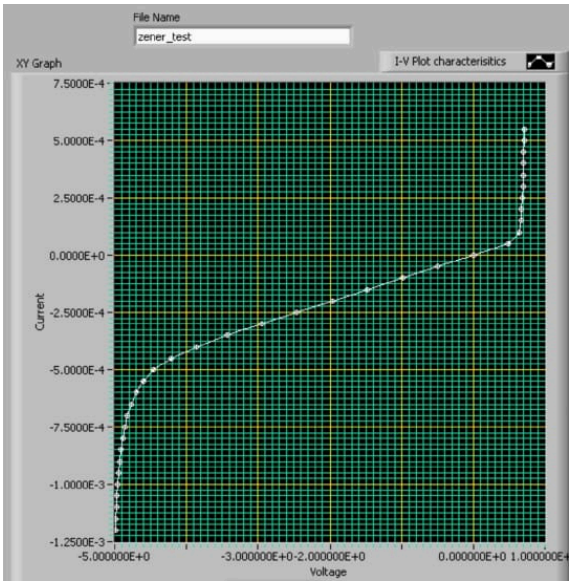
Norton's Theorem

Any linear network can be converted to an equivalent form that consists of a current source in parallel with a resistor (or an impedance so as to include other elements).

For circuits with a non-linear component the system can be divided into a non-linear portion and a linear portion where the equivalence theorems are still valid.

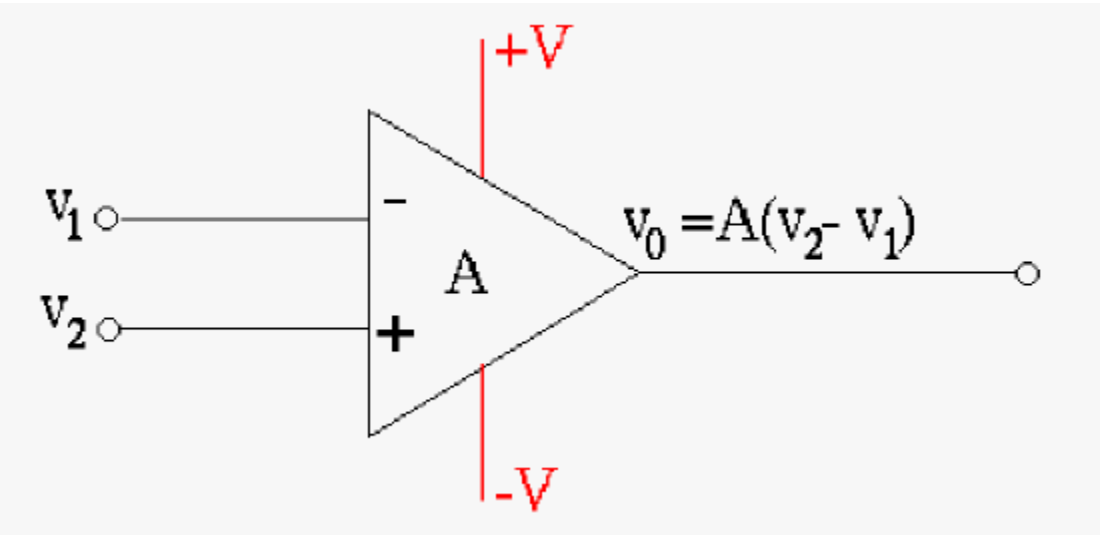
Non-linear circuits

Diodes, transistors, etc are non-linear and need to be expressed in terms of an I-V characteristic curve. In general the idea of creating a curve to describe elements is used in multiple disciplines of engineering (example: phase diagrams for materials; expansion curves used by civil engineers; etc.). There is a science to using these curves which will be taught in later classes; have fun!



I-V characteristic plot of a Zener diode. This image is an actual result displayed using LabVIEW which is a very useful electrical engineering software package.

Operational Amplifiers



The operational amplifier's function is shown above. The output is difference of the two voltage inputs (v_2 and v_1) amplified by a gain A. The +V and -V is the power supplied to the amplifier which also limits the output voltage. The output voltage cannot exceed V the supply voltage.

As a last topic we will review operational amplifiers (opamp) uses but not the details (that is for later classes). An opamp is a device that takes the difference of two inputs and amplifies that signal. This seems to make the opamp a one purpose device, but that would not be accurate. An opamp can be used as a summing amplifier, an inverting and non-inverting amplifier, a voltage follower (used as a buffer), a differential amplifier (used extensively), an ECG amplifier (in the past referred to as an EKG amplifier), instrumental amplifier (extension of an ECG amplifier), level shifter (for sensor calibration), lock-in amplifier, active filters (faster and much better slope then passive filters), differentiator (which takes an analog differential, just like in math), and integrator (which takes an analog integral, just like in math).

An ECG amplifier is the basis of an electrocardiogram though they can be used in a multitude of instruments (though in general an instrumental amplifier would be better). Given the importance of electrocardiogram it is clear how important opamps are to world.

Filters are useful for clearing up poor signals among other uses. Active filters are really much better filters but because they have limitations with voltage, current, and frequencies they do not totally replace passive filters. Passive filters are made with the passive elements (resistor, capacitor, and/or inductor) and while inferior to active filters they still have there uses in high voltage and high current applications.

A differentiator and integrator can be used in an analog computer or a PID controller. While you might think a differentiator and integrator is just doing calculus and question why you would need that, it turns out that differentiation and integration has useful engineering properties that can be used to achieve certain results. While the opamp method is an electrical engineering method there are mechanical ways to differentiate and integrate as well.

For more details enroll in an electric circuit analysis course.

Xfig was used for the images in this chapter. It is a free software which is very useful for various CAD purposes.

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14.14: Signals and Systems (Control systems)

Signals and Systems

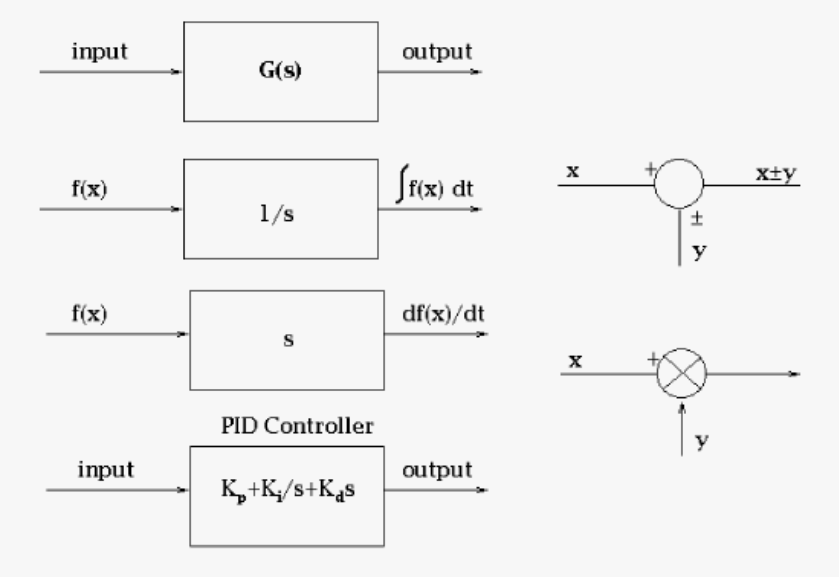
"Signals and systems" is the study of processes with signal input(s) and signal output(s). The methods of this is applicable to ***all*** engineering disciplines. Therefore all engineers should take a course in signals and systems. So *what is a signal?* and *what is a system?*

A **signal** is the representation of a physical "wave" expressed as a variable in time-space (example: x(t)). Signals may be voltage or current of a circuit, the force in a mechanical circuit, heat flow in a thermal circuit, or hydraulic flow in a fluid circuit. It is likely this is familiar to the student already so as a challenge to the student: find ten more examples of signals in engineering or science.

A **system** is a process that has an input a signal or signals and outputs a different signal or signals (usually). A system is generally expressed as a transfer function or an operator. An example of this would be an integrator that integrates the input signal over time as its output. Another more simple example might be an operator that just multiplies the signal by a constant and outputs that new signal.

The methods that are used for the systems are in general the same methods generally learned in **linear algebra**. The methods used to model the physical aspects of the "signals" are the same methods generally learned in **numerical methods**.

A system can be expressed mathematically as a transfer function or an operator. A simple example of an operator could be an integrator that takes a signal and integrates it over time or it could be as simple as adding a constant to a signal. Transfer functions are specific to engineering and have the notation of a Laplace transform, where as operators are more specific to physics (and other sciences) and have the notation of mathematics. In the following figure are some examples of how engineers express this symbolically.



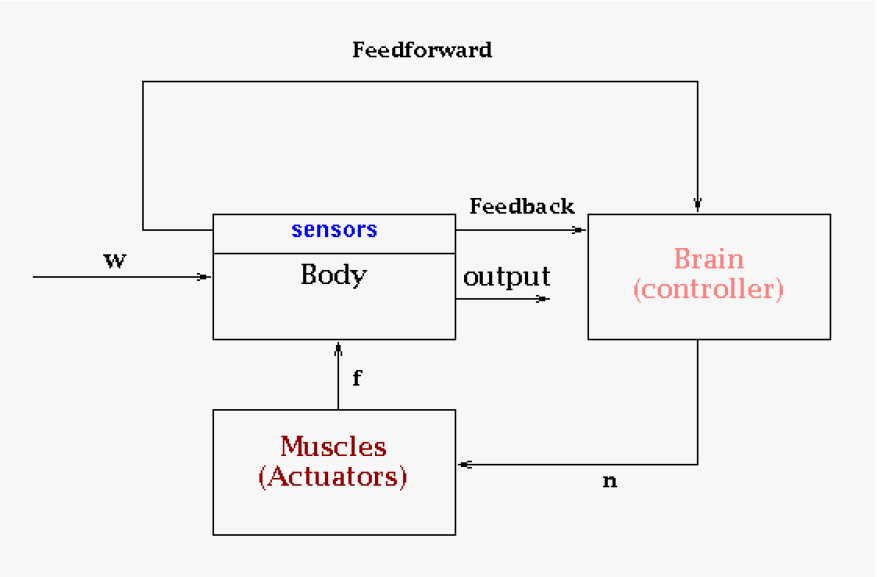
In this figure are examples of blocks that represent transfer functions. The signal is the input and the output and inside the block is the transfer function, represented by G(s) in the first example. The s-space notation is consistent with Laplace transformations. The blocks other than the first general block shows examples of how the blocks might be used in a diagram.

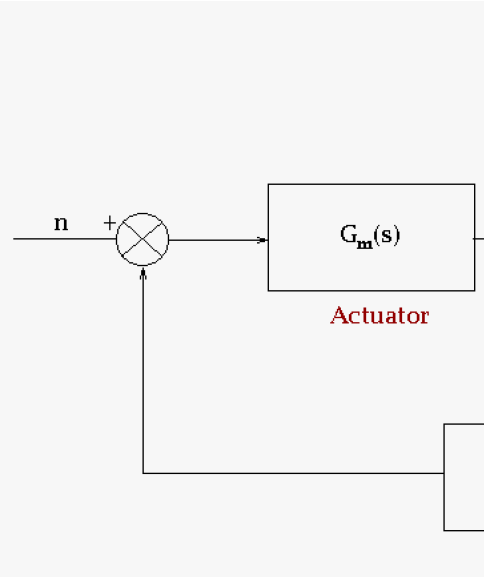
Simple Example: Signals and Systems

Let us imagine we are standing in a heavy wind and we are almost blown over. In this instance our brain, through our skin sensors

and our muscles, detect that the wind is putting us in a metastable state that could lead to us falling over. So the brain (our controller) sends a message to the muscles and tells them to adjust our position to lean into the wind saving us from falling. So let us look at this from the point of view of signals and systems. The wind is a signal as well as our skin sensor and our muscles communication to the brain; our systems are our body, our controller (brain), and our muscles (a muscle acts as a senor and actuator).

In the following table we look at a conceptual block diagram of the system described above and then look at the conversion of that to a transfer function block diagram that is consistent with control system concepts that engineers generally use.





In this conceptual block diagram the body is a system of muscles and skeleton that represent our structure (this could equally well be a building), the control system that takes feedback from sensors is the brain and the brain through a neural network instructs the muscles (actuators) what to do. The main input is the wind, w, but within the block diagram there are also inputs of force, f, and neural net input, n.

The sensors for the eyes, muscles, ears, and skin. The eyes can predict wind coming our way and correct our body (by maybe leaning it into the wind) as a preparation. The eyes give us feedforward information which for the transfer function block diagram we will ignore (feedforward is complicated is generally avoided the same reason as extrapolation is avoided in science). For our feedback system we have the skin as a pressure sensor, the ears as a balance sensor, and the muscles are strain sensors. Feedback is crucial in designing a control system.

This is a transfer function block diagram that
Transfer functions are operators and describe
an output. The transfer function is usually a v
differential equation that act as operators. Fo
equation will be a force equation $\mathbf{M_s\ddot{x} + C_s\dot{x}}$

The variables are bold here because they are
differential equations. A structure is complex
equations which would need to be solved wit
Laplace formalism would convert the differer
would allow for easier calculations, possibility

Note that the force equation could be an equ
the idea of dynamic analogies. See next sect

This is just a brief look at control systems and the use of transfer functions. More formal descriptions would happen in your Signals and Systems course (which hopefully would have some Numerical Methods discussion).

Bode Plots (Nyquist plots and Nichols plots)

Bode plots are frequency response plots of a system described by the transfer function. Bode plots consist of both a magnitude plot and a phase plot help to characterize the stability of the system through gain, gain margin, and phase margin. The are very important in system analysis.

Nyquist plots are similar to the Bode plots but in a different coordinate systems ("polar"). Nichols plot is a different form of a Bode plot. For a simple transfer function we will look at a mass, damper, and spring system: $G(s) = \frac{1}{ms^2+cs+k}$

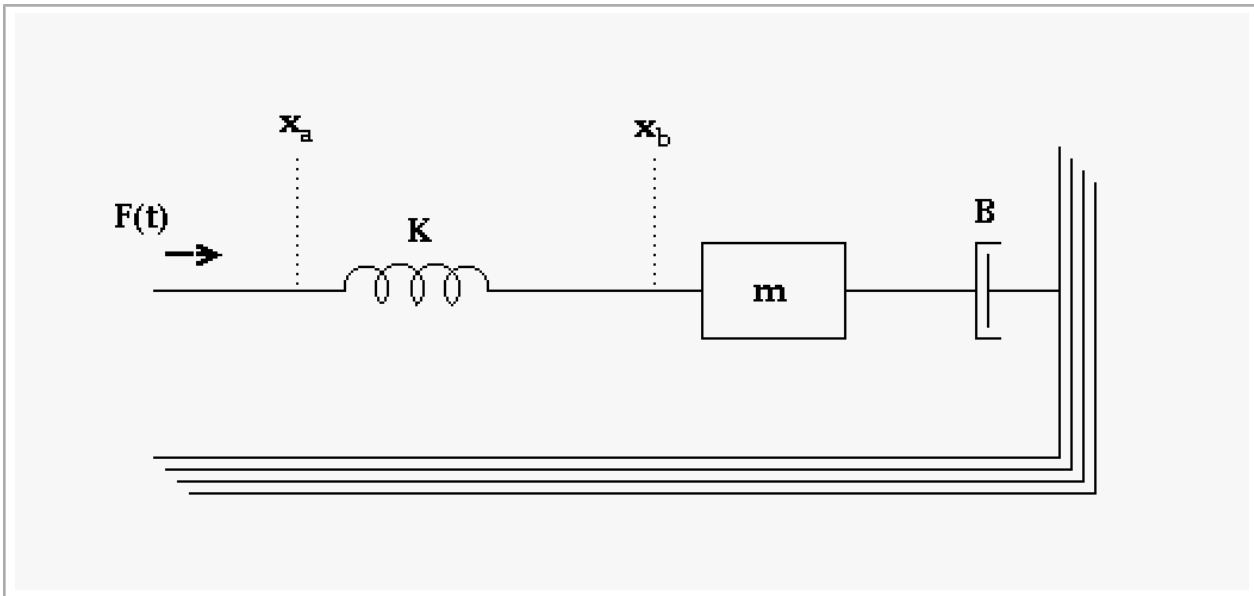
```
pkg load control
T = tf([1],[5 4 3])
bode(T)
figure(2)
nyquist(T)
```

The output of this simple systems shows an stable oscillatory system. Nothing very noteworthy, but for more complicated systems this can be very helpful. Again if you want more detail check out your Signals and Systems course.

Dynamic Analogies Discussion

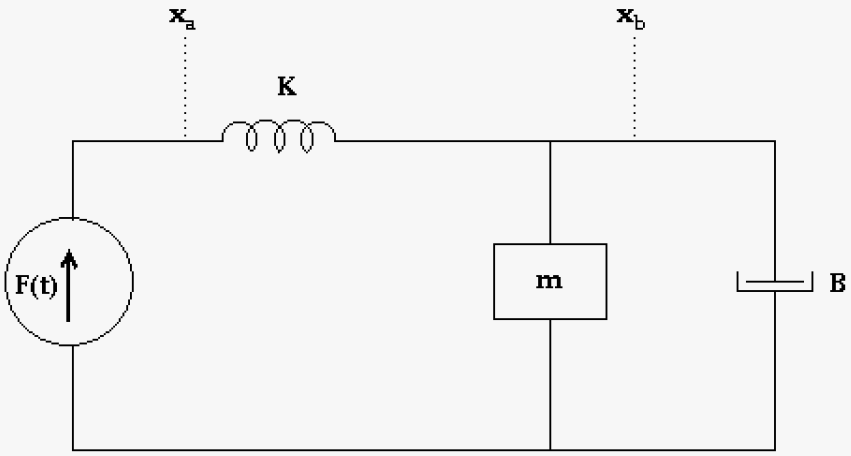
We use generic transfer function descriptions when modeling systems rather than discipline specific symbols because of the idea of dynamic analogies. The idea is that the systems of one discipline can be used in other disciplines. Below we give a simple example of this idea and then a incomplete table of analogies (enough to give an idea which should be pursued more fully in later classes).

This model is based off the comparison that $F = \frac{dp}{dt} = m\frac{dv}{dt}$ and $I = C\frac{dV}{dt}$ where force[1] is to be associated with current and velocity with voltage. That leads to the idea that mass is associated with capacitance. Similarly with the equation $I = \frac{1}{L}\int V dt$ the spring constant, K, is to be associated with 1/L and with the equation $I = \frac{1}{R}V$ the damping coefficient, B, is associated with 1/R.

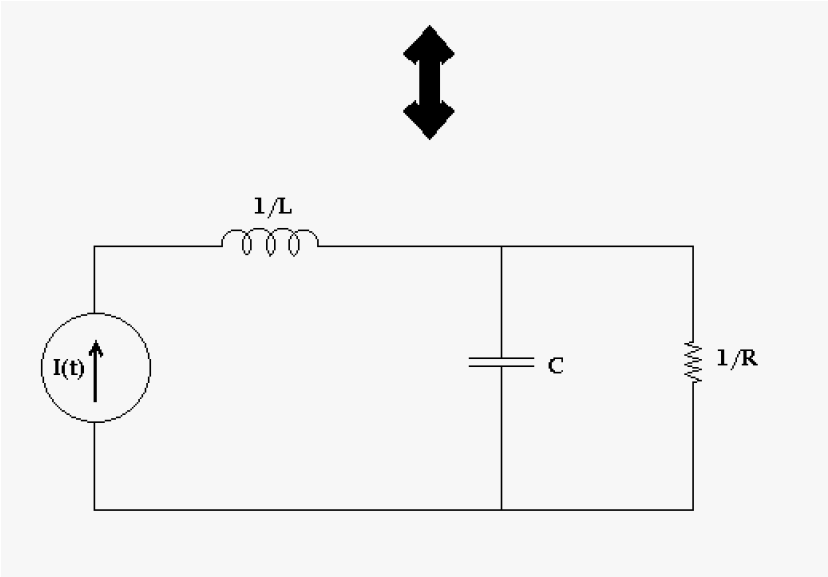


This is a system that includes a spring, a mass, and a damper. K is the spring constant, m is the mass and B is the damping coefficient. A force is being applied on this system F(t). This is the system we wish to use as a system model.

Many mechanical process can be described by these three elements.



This takes the above system and converts it to a mechanical circuit which is a convenience that will help us going to the next step.



This is the equivalent electrical circuit that would emulate the above mechanical circuit.

Mechanical circuits might describe systems that are very large, however converting them into an electric circuit would make a physical simulation easier. Alternatively this could be run in a Spice, which would then make the electronic simulator into an mechanical simulator.

The following table gives a list of some equivalencies to different physical system. Traditionally these are call dynamical analogies. Different ways to create models will change this list so this is not an all encompassing list. There are many missing analogies here as, again, this is not meant to be complete.

Mechanical	Electrical	Thermal	Hydraulic	Mechanical - rotational
Force, F	Current, I	Rate of heat flow, q	Volumetric flow rate, Q	Torque, τ
Velocity, v	Voltage, V	Temperature (θ)	Pressure, P	Angular velocity, ω
Mass, m	Capacitance, C	Thermal capacitance (C = mc)	Tank area, A_T	Moment of inertia, I
Spring constant[2], K	1/L	-	-	Torsion spring constant[2], κ
Damping coefficient, B	1/R	-	Flow conductivity, K	Rotational damping coefficient, D

Mechanical	Electrical	Thermal	Hydraulic	Mechanical - rotational
-	Capacitor voltage, V_c	-	Liquid height, h	-
-	Resistance, R	Thermal resistance, R	Flow resistance, R	-
Momentum, p	-	heat	-	Angular momentum, L
-	-	Specific heat, c	-	-

From an energy perspective these analogies are more obvious and it is one reason you will find Lagrangian and Hamiltonian methods used in physics.

Signals and Systems Concluded

Signals and systems really describe a large number of engineering and science process. The brief discussion here is to show you what this is all about, but is definitely not complete. For an introduction in engineering class this is more then enough so we will move on to the next topic.

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[1]Alternatively we could look at force as associated with voltage which would change the electrical analogs correspondingly. We believe that force and current are are better analogies then force and voltage. Voltage is a difference which is more like how velocity is defined with dx.

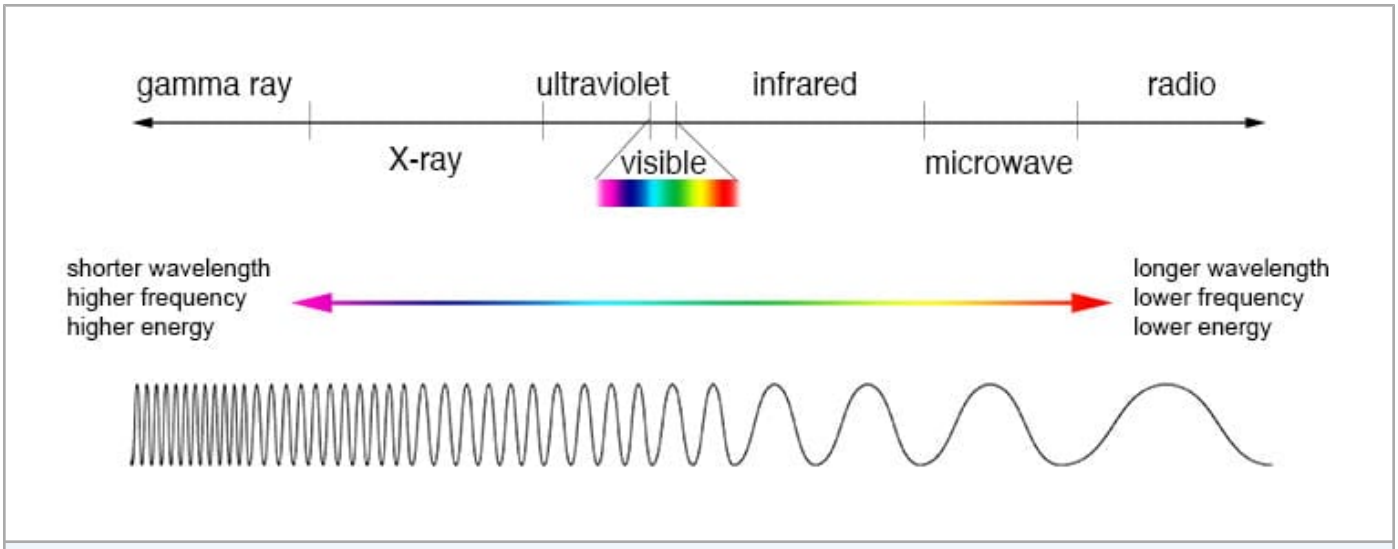
[2]From a dynamical analogy point of view whether this is a spring coefficient or stiffness coefficient is not material as it still would be represented by a spring. This is similar for the torsion spring coefficient or rotational stiffness coefficient.

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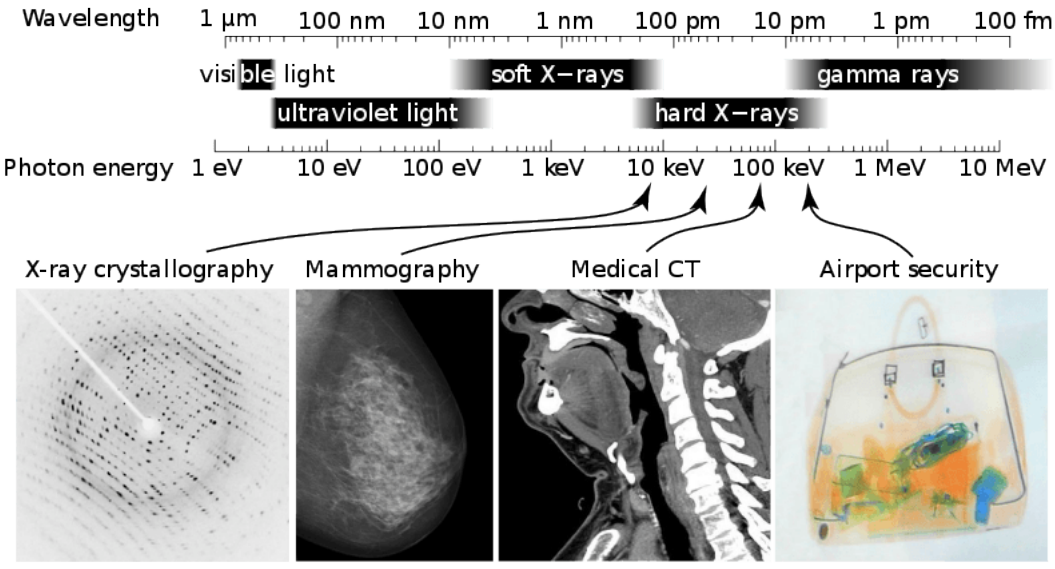
14.15: Optics

Light and Optics

Optics is technically the study of light and its properties when interacting with different materials. Note that the here light is rather broad and includes a large portion of the electromagnetic spectrum of which only a small part is visible.



A description of the electromagnetic spectrum using wavelength, frequency, and energy. Note that visible is just a very small part of the electromagnetic spectrum. Most wavelengths (not just visible) can be "bent" to work with a parabolic (or close to parabolic) "mirror" to produce telescopes. This image is from NASA's Image the Universe.



A different perspective on the electromagnetic spectrum with applications in the X-ray region of the electromagnetic spectrum (X-ray crystallography, mammography, computed tomography (CT or CAT) scan, and security). This image is from Wikipedia credited to Ulflund, Del45, Nevit Dilmen, ChumpusRex, and Mattes.

Here we will explore just a few aspects of optical engineering through the discussion of lenses and mirrors. You will do basic study of optics in your physics class, but in general only electrical engineers will delve into the topic more thoroughly (and if you want

to go really in-depth then you need to study graduate physicists). This topic can get intense, but not herein.

Snell's law

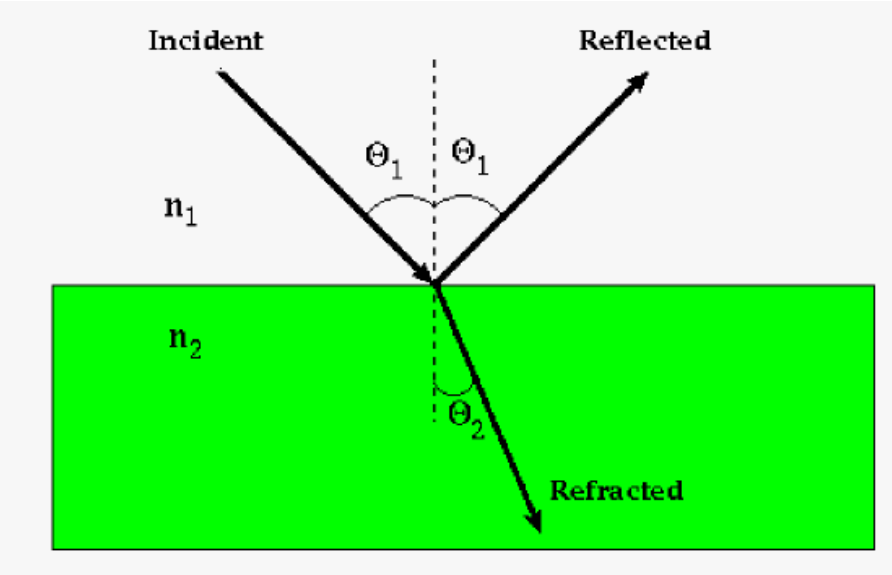
The first law to learn in optics and the most important is Snell's law[1] which describes refraction and reflection mathematically. From a more general wave front description refraction is described by a form of least time known as Fermat's principle[2].

Snell's law of reflection simple states that the angle of incident of a light ray is equal to the angle of reflection. Snell's law of refraction states that the incident ray will bend as a function of index of refraction (a property of optical materials which is wavelength dependent) and the sine of the angles.

$$n_1(\lambda)sin(\theta_1) = n_2(\lambda)sin(\theta_2) \tag{14.15.1}$$

Sometimes this is expressed with the assumption of a visible wavelength that corresponds close to yellow and the assumption that within a reasonable visible area the wavelength is constant (this is not a good assumption in the blue-violet range).

$$n_1 sin(\theta_1) = n_2 sin(\theta_2) \tag{14.15.2}$$



This is a cartoon describing the relationship between incident, reflected, and refracted waves. The arrows describe light in a manner we call ray optics. Real waves are described by wave fronts, just like waves from the ocean. Ray optics works fairly well and is easier to use for most cases but not all.

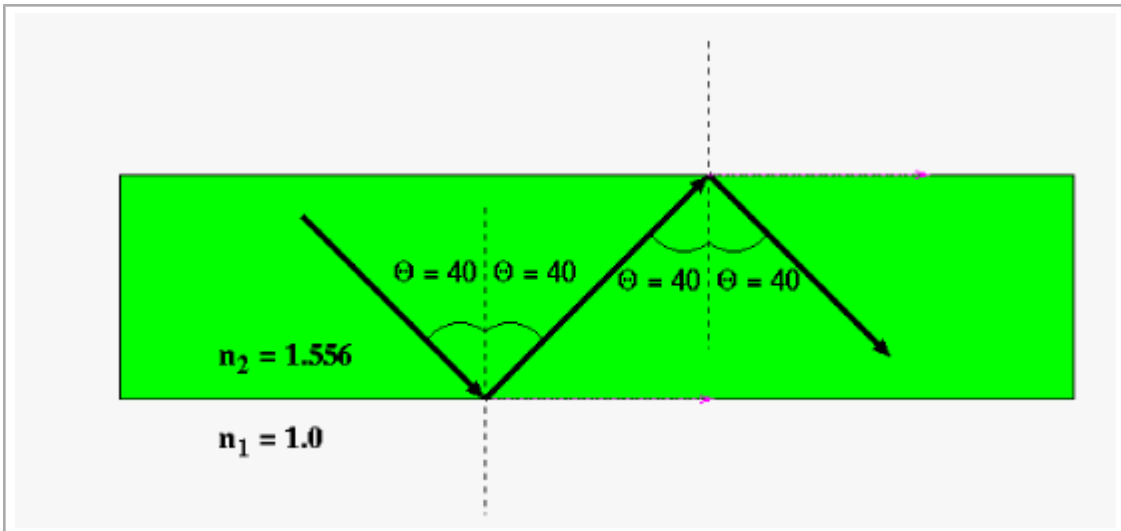
Here note the incident angle of the light ray is equal to the reflected angle of the light ray and the refracted ray is "bent" based of the Snell's law of refraction.

Important in this is the idea of the index of refraction of materials. The speed of light, c, is normally reported as the speed of light in a vacuum. However, technically, the speed of light is actually dependent on the material it travels in. This dependency is described by the index of refraction as follows:

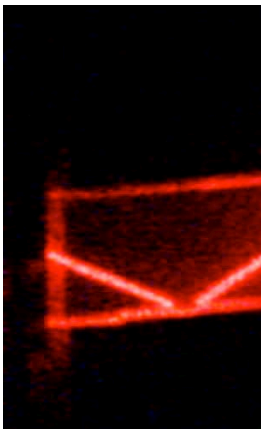
$$c_m = \frac{c_v}{n_m(\lambda)} \tag{14.15.3}$$

where c_m is the speed of light in a particular medium, c_v is the speed of light in a vacuum, and n_m is the index of refraction of material m. In this equation the index of refraction is positive. Traditionally there were only positive index of refractions, but with the advent of metamaterials, we now have negative index of refraction as a concept with *Snell's law still being valid*. We will not discuss that here as that is beyond the scope of this course (but it is really interesting and mind-blowing).

This property of refraction can lead to some interesting results such as internal reflection of light in a material. A careful calculation of the index of refraction and the angle of incident will lead to this effect. This is what is used in fiber optics for communications (though in generally it is a bit more complex as fibers are usually multi-clad). Light pipes (or light tubes) use similar principles but are usually more complex as they could involve prisms and mirrors as well.



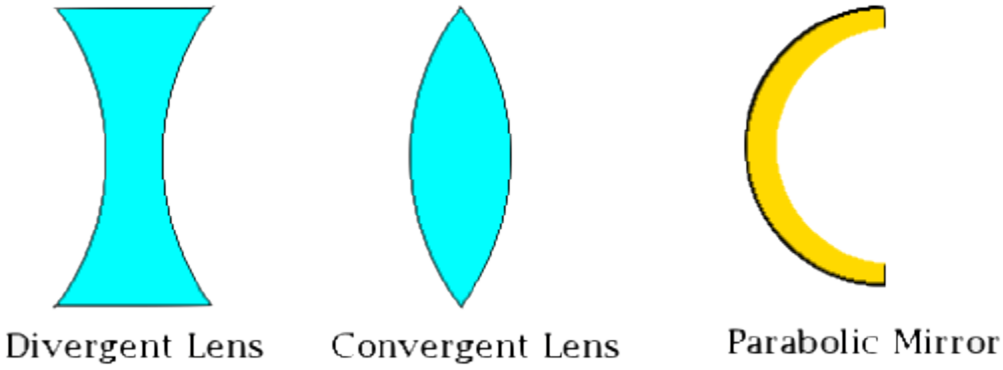
This is a cartoon describing internal reflection (note the "refracted" portion of the wave is along the surface in pink at 90 degrees). Here the indices of refraction and angle of reflection was determined to assure internal reflection. The critical angle will be defined by the index of refraction (note since this is wavelength dependent you would need to know the wavelength of light for any application). The equation for the critical angle: $\theta_c = arcsin\left(\frac{n_2}{n_1}\right)$



Example of internal refle
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design project) under the

Basic Optical Elements

The basic optical elements are lenses and mirrors. In this brief section we will just give a sketch on those elements.

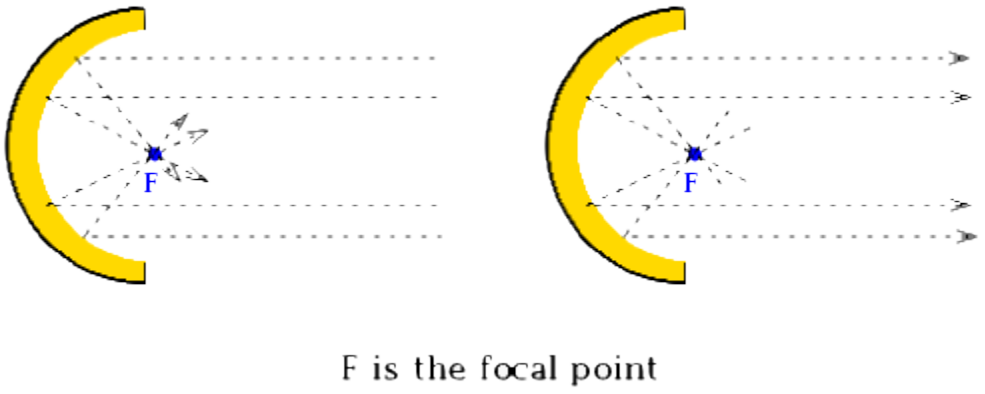


There are many optical elements but for a basic description the divergent lens, convergent lens, and parabolic mirror are the conceptually the most important. The convergent lens is as its name implies: it converges light. This lens is the most oft mentioned lens as it is used in basic descriptions of refractor telescopes and microscopes. The divergent lens is as its name implies: it diverges light. This lens is usually used in telescopes and cameras. Parabolic mirrors or almost parabolic mirrors (made more hyperbolic to reduce aberrations - at a cost however) are used in reflecting telescopes such as Hubble Space Telescope.

Mirrors

There are a number of type of mirrors that are used in science and engineering. The flat mirror is obvious, but parabolic or

close to parabolic mirrors are useful for reflecting telescopes. In particular a parabolic mirror will focus parallel light onto a focal point which can be used for astronomical observation (light from distance stars, etc. is essentially parallel). This can also be used in reverse where a light is used at the focal point and parallel light rays are projected out. This idea is used in flashlights and spotlights (theater lighting).

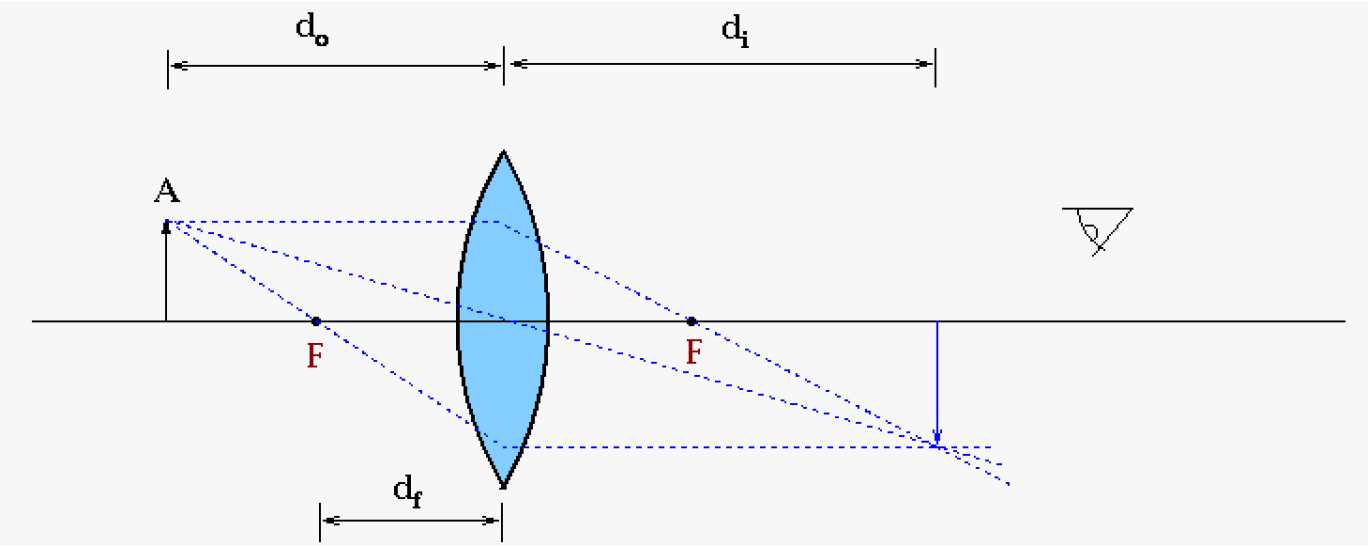


This cartoon of a parabolic mirror shows how parallel light (shown with ray tracing) reflects to a singular point called the focal point. This is particularly useful for astronomical observations (though two mirrors are usually used one concave and another convex normally). Likewise light from a focal point reflects off the mirror in a parallel direction. This is what makes flashlights and spotlights so effective. The parabolic mirror is a convergent optical element (as opposed to the convex mirror which is divergent - see below for a telescope example).

Lenses

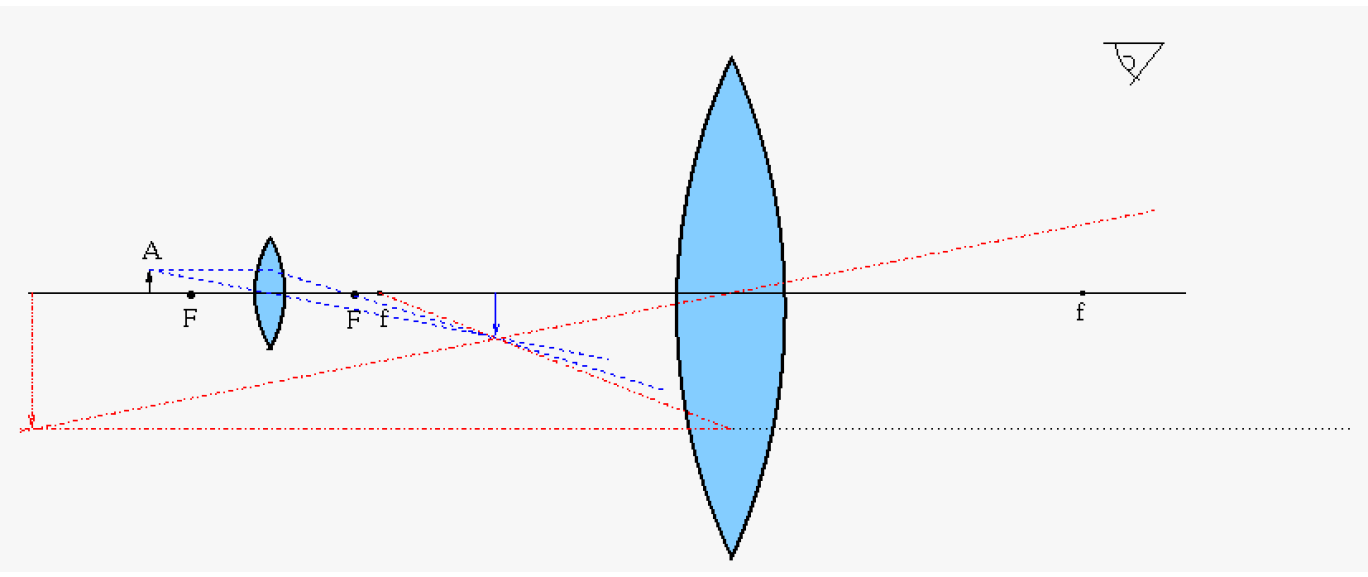
Lenses are used in an array of optical devices including camera, telescopes (refracting type telescope though some components in a reflecting telescope might have lenses as well), Raman spectrometers (with mirrors as well), microscopes, and a host of other optical devices. There are two main types of lenses: converging and diverging. The converging are the most widely known as that is the basis for magnifying glasses which almost everyone has used in their lifetime. Converging lenses are based-off convex lens (take note that convex mirrors are divergent, so be careful which element you are discussing) and can be plano-convex, bi-convex, and positive meniscus and diverging lenses are based-off concave lens and can be plano-concave, bi-concave, and positive meniscus.

Ray tracing is one method to describe how the path of waves (light in the case of optics, though for seismology they would use s and p waves) through a medium. Usually the wave front is not included in this description for simplification. Here we will describe ray tracing or a bi-convex and bi-concave lens assuming the lenses are fairly thin (thin lens equation: $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$ where d_o is the distance of the real object from the center of the lens, d_i is the distance of the image object from the center of the lens, and d_f is the focal distance from the center of the lens as seen in the figure below).



Example of ray tracing for a converging thin lens. The real object is A at a distance d_o from the center of the lens and is represented by a line with an arrow on top. The final image is the blue line at a distance d_i. Technically each part of the object line should be ray trace, but this is not necessary as each point can be determined by ray tracing the top and bottom of the object and then interpolation the rest; at least for an ideal lens. The bottom is trivial as it is the bottom of the image as well, so that just leaves the top to ray trace. The ray trace, as you can see in this cartoon, consists of three dotted lines (only two are really needed) and follow the rules that through the focal point the line bends at the lens in a "parallel" manner, a "parallel" line bends at the lens through the focal point and a ray that goes through the center of the lens does not bend.

For telescopes, microscopes, cameras, etc. many optical elements are combined to form a whole. For an example we will look at a fairly simple microscope consisting of two converging lenses.



This is a simple cartoon of a two lens microscope. In general a real microscope would have multiple optical elements some of which will be to help the non-ideal real world issues like aberrations.

Optical Aberrations

Above we discussed ideal mirrors and lenses but in the real world there are always aberrations because of how we need to make

the optical elements[3]. Here we list a few of the many issues with lenses and mirrors.

Optical Aberration	Issue	Possible resolution
Chromatic Aberration	Different wavelengths of light have different focal points leading to all but one wavelength slightly out of focus. Sometimes this is referred to as color fringing in cameras.	Mirrors do not have chromatic aberration, this is one of the benefits of using mirrors for telescopes.
Spherical Aberration	Different rays of light have different focal points in areas away from the center of the optical element; this aberration is due to the spherical nature of lenses and mirrors in the manufacturing process.	Parabolic mirrors[4] and aspheric[5] lenses are mostly immune to this effect. Aspheric lenses are difficult but not impossible to manufacture (mass production circa 1950s). It is possible to get the same effect as an aspheric lens with multiple "normal" lenses but it does cost in the number of optical elements and hence weight.
Coma	An off-axis effect that produces a tail for each optical point (looks like a comet).	Parabolic mirrors are susceptible to this when the light is not parallel. The idea of using a parabolic mirror is based off the idea that all light of the object will be parallel (theory) because the object is so far away. While this is a good assumption it is not perfect; some light will be not parallel. This will also be a problem with lenses that are not made perfectly (not as easy as it sounds). Usually have to post-process this.
Astigmatism	This is where there are different focal points in different planes and essentially is a blurred image. There are two types of astigmatism the first for symmetrical optical elements and the second for non-symmetrical optical elements.	Aspheric lenses help with astigmatism, but note some systems actually employ astigmatism for their processes. Astigmatism usually is caused by defects in manufacturing. Making optical elements is not an easy process so these defects happen all the time (even for Hubble Space Telescope).
Field curvature	Defocusing in the radial caused by the lack of flatness in optical systems	Usually needs to be corrected post processing
Fringing (not color fringing which is just chromatic aberration)	An interference pattern effect is caused by the edge of a lens, a small imperfection of the lens, temperature variations, vibration, an alignment issue, a particle on the lens, etc.	This generally has to be corrected post processing (obviously if cleanable you could always clean a particle off, but here we mean that the lens is not inaccessible). Optics isn't easy is it.

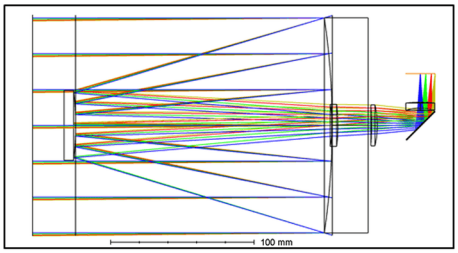
Note that aberrations can be caused by a multitude of factors such after manufacturing processes and even dust, plus aberrations can be introduced when assembling a number of optical elements into an optical system. An optical engineer works to reduce these effects. This is precision work that can take quite awhile. Not everything can be solved (for instance dust, etc. that might accumulate on a telescope after assembly - even in space). So after the system is perfected by an optical engineer we move on to post processing (i.e. we let a computer correct it). Finally note this brief introduction does not take into account many other aspects of optical engineering that needs to be considered (weight of overall system, light leaks, light transmission and reflectivity, masking issues, etc.).

There are many software techniques to correct optical problems after the system is already designed and in use. Generally they depend on images taken before the system is put into use (in say a detector/optical system laboratory). Using software however is a sort of last resort method and it is best to produce the best system possible before this action is taken.

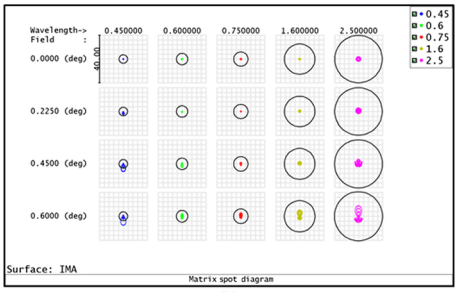
Spot Diagrams

There are many ways to characterize an optical systems. Characterizing a system would start with a spot diagram which is a very accurate ray tracing of the optical system. Aberrations can be identified in this characterization. Typically spot diagrams of standard telescopes are common place and easily found on the web. The point spread function (PSF) which is related through a Fourier transform to the optical transfer function (OTF) or real portion of the Modulation Transfer function (MTF) is an actual test

of the optical system that shows how a point spreads going through the optical system. A number of software packages (Zemax, COMSOL Multiphysics, etc.) will do spot diagrams or you could write your own program.



(a)

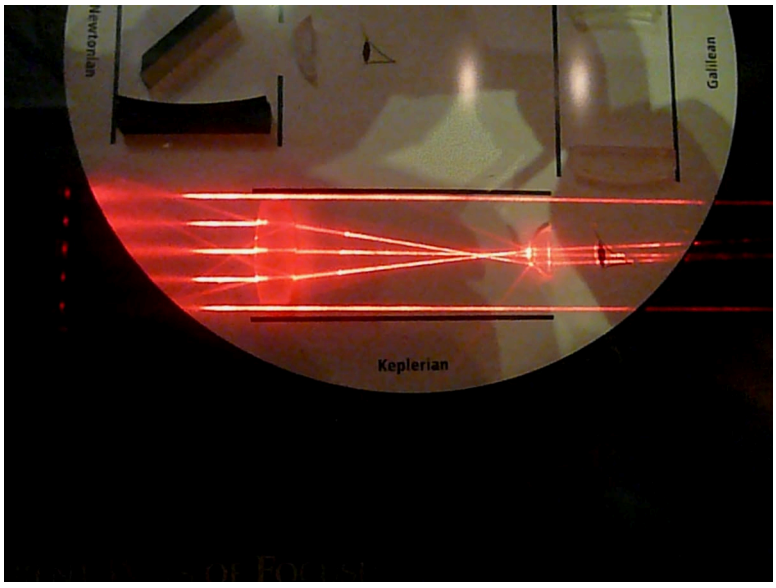
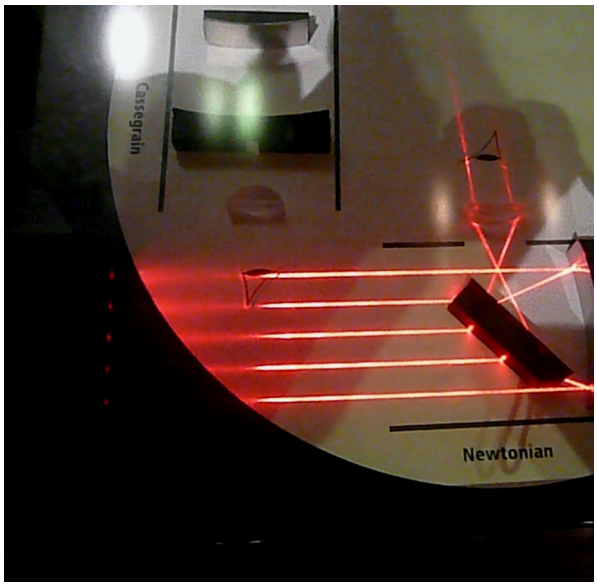


(b)

Example of ray tracing and the resulting spot diagrams. Paul C. T. Rees, Ian P. Baker, David A. Thomson, Dean Catterall, Martin Coleman, Martyn L. Jones, John B. Mitchell, "Development of a lightweight camera for high altitude platform systems," Opt. Eng. 59(10) 105104 (20 October 2020) https://doi.org/10.1117/1.OE.59.10.105104. SPIE, Optical Engineering, CC by 4.0. (Assignment: Read this paper)

Telescopes

There are many type of telescopes that have pros and cons. Engineers have to balance the pros and cons when choosing a telescope. As in other engineering systems any time you improve on aspect of your system (your telescope) you worsen another aspect of your system. Most major telescope systems are based off the Cassegrain-type reflecting telescopes. Keplerian is not favored mainly because of the weight of the lenses in large systems for these refractor telescopes.

Keplerian	Newtonian
	

Keplerian	Newtonian
Refractor telescope where the light comes in from the left and is viewed by your eye at the right. From this picture taken by authors at the Air & Space Museum in Washington DC you can see the parallel light from the left coming into a converging lens (where the magnified image will be formed) and going through the divergent lens . The refractor telescope's "inventor" is unknown.	Reflector telescope where the light comes in from the le incident on the primary mirror and reflects to a seconda reflects the image at 90 degrees to the detector (eye in Newtonian telescope is credited to Newton.

Refractor telescope major issues

- Chromatic aberration
- Distortion on the edges
- For large telescopes very heavy (try picking up a one meter piece of glass)
  - Solution
    - Use a reflector telescope as they are lighter than refractors for large telescopes (try picking up a one meter thin piece of aluminum)
- Works almost exclusively in the visual range only as most glasses do not transmit infrared (IR) and ultraviolet (UV), at least efficiently
  - Solution for IR and UV
    - Use mirrors which can be used for X-ray[6], UV, visible, IR, and radio[7].
    - Use quartz, though this might be too expensive and a bit overkill as mirrors would do
- Solutions for aberrations:
  - Use convex lenses (Kepler)
  - Use mirrors (reflecting)

Reflecting telescopes major issues

- Coma
- Spherical aberrations
- Central obstruction
- Sensitive to collimation

Types of Telescopes with notes

- Newtonian
- Cassegrain (classic version)
  - To reduce spherical aberrations
    - Parabolic primary
    - Slightly hyperbolic secondary
  - Compact
- Schmidt-Cassegrain
  - Very short, compact, and cheap
  - Corrector plate to make shorter
    - More coma and spherical aberration as a result
    - Inexpensive especially for laypeople
- Ritchey-Chrétien
  - To reduce coma
    - Make the primary more hyperbolic than the classic Cassegrain
    - Make the secondary more hyperbolic than the classic Cassegrain
  - Specialized Cassegrain
  - Hubble Space Telescope (HST)
    - Advantages
      - Very little coma (due to hyperbolic mirror)
      - Only two surfaces (so less light loss)
      - No corrector plate (refractive elements)

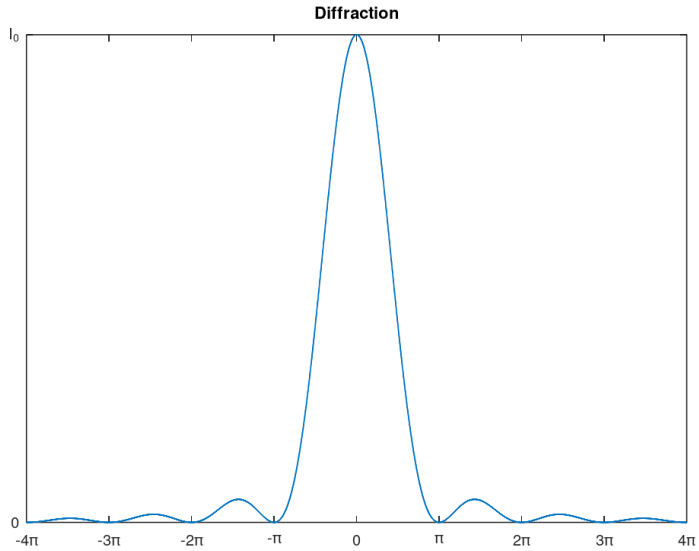
        - Glass, etc. reduce Infrared transmission which HST needs to view
        - Less light loss
    - Good flat field
- Disadvantages
    - Third-order coma
    - Severe large-angle astigmatism (the ***cost*** for reducing coma)
    - Severe field curvature
        - May not be an issue if you are not try to view large fields
        - Getting mostly rid of coma is more important
- Maksutov-Cassegrain
    - Better resolution than schmidt-cassegrain, however heavier
- Nasmyth
- Gregorian
    - Concave Mirror
    - Usually, Earth observing though some space versions
- Dall-Kirkham
- Herschelian
    - Off-axis telescope (therefore no central obstruction)
- Schiefspiegler
    - Off-axis telescope (Cassegrain variant)
- Yolo
    - Off-axis telescope
    - Toroidal reflectors

Diffraction

Diffraction is the effect of light spreading out do to a small slit, opening, or screen. A result of these is bending of light around objects that you might otherwise think is blocking the light. This is of particular concern to low light activities like scientific observation through a telescope. We will only discuss briefly some of the ideas behind diffraction without going over the physics. Your physics and materials science will give more through discussions of this topic.

Fraunhofer

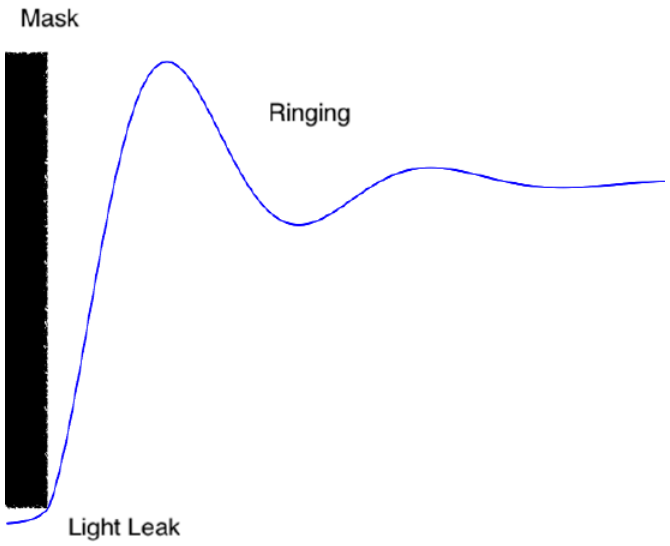
Fraunhofer diffraction happens when the light source is far[8] from the slit or opening. The wave fronts are therefore planar. We see Fraunhofer diffraction from apertures[9] either rectangular or circular. Circular apertures produce the well known Airy disk where the center is very bright and radially continue outwards interweaving between dark rings and every dimming light rings. The optical engineer would work to reduce this as much as possible.



The basic diffraction pattern is described by the square of the sinc function. As you will see there is at maximum initial intensity at the center and then goes from no intensity to an every small percentage of the initial intensity. Note: this graph could be viewed as a 2-D point spread function (3-D PSF would be better).

Fresnel

Fresnel diffraction[10] happens when the light source is near the slit, opening, or mask. The wave fronts are therefore spherical. Mask apertures[9] can act as a Fresnel diffraction generator where the source is the edge of the mask and the opening is the gap between the mask and the detector. While this cannot be avoided totally in real systems the optical engineer would work to reduce this as much as possible.



Cartoon (image) of light intensity as a result of a mask. The the black rectangle represents the mask and the cartoon shows how light leaks under the mask (not related to Frensnel, just an issue for optical engineers) and the light pattern of oscillation (rings) directly after the mask ends (ringing is a term used in electrical engineering and we employ it here as well). The "ringing" is due to Fresnel diffraction. This cartoon is more dramatic then it would be in a real system for the purpose of emphasis nevertheless light leak and ringing will be observed in any masked system.

The previous discussion is just a brief taste of some of the issues the an optical engineer may encounter with diffraction.

Polarization

Polarization of light[11] describes how their the electric field and magnetic field oscillate. Polarization can be linear or circular with circular (or elliptical) being subdivided into right and left circular polarization. Unpolarized light is a mix of all the various polarization. Results of polarization in certain materials (plastics) can be used to measure stress in objects which would be useful to engineers. For science polarized waves indicate directional[12] information that give further detail of an event or object.

Some materials index of refraction are dependent on polarization (among other dependencies) and these materials are call birefringent. A number of optical systems use birefringence.

The next topic is a brief discussion of chemistry which is mostly covered in the other Libretexts course associated with this course.

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[1]Also known as Descartes' law in France because Rene Descartes independently described refraction and reflection (though he assumed the speed of light was infinite). The description of reflection and refraction has a long history and includes Ptolemy, Ibn Sahl (who greatly extend the work of Ptolemy), and other polymaths. Assigning a name to these laws is just a convention and does not accurately acknowledge the work of many scientists and engineers.

[2]Principle of least action as described by the Hamilton's principle. Note that Fermat's proof did not assume an infinite speed of light but a finite speed of light.

[3]The most obvious issue would be with lenses that are not really infinitely thin, but have a thick part in the center and become thinner as they go to the ends.

[4]For very high end telescopes for astronomical observation the "fast" or low f ratio requires the parabolic mirror to be more hyperbolic (*how much?* well that is a complex design issue and is why we have optical engineers.).

[5]The aspheric lens design takes standard shape of a lens and reshapes it at the ends. This needs to take into account the index of refraction of the material and then design the surface to be more hyperbolic in looks.

[6]X-rays have to have mirrors designed to have the light graze on multiple mirrors as the energy is very strong and would travel straight through the mirror if the trajectory was head-on with the mirror. Gamma rays' energy is too powerful for mirrors and are usually capture through a crystal's reaction to the ray.

[7]Radio uses parabolic steel or mesh structures that acts exactly like a mirror so we believe you could still define that as a mirror.

[8]Technically it a function of distance, width of opening, and wavelength, but we will assume that some standard design standards are employed such that really only distance is an issue here (fully study of optics should include all factors).

[9]Apertures are used to reduce the amount of light coming into a system in order to prevent saturation of detectors. Sometimes an aperture is used as a mask unwanted light signals impinging on the detector (sometimes in places that are meant to be dark but can detect light themselves).

[10]Just like Fraunhofer, Fresnel is also a function of distance, width of opening, and wavelength, but we will only look at one aspect of Fresnel diffraction as we present just some tidbits for interest not an optics course. Technically diffraction for very near light sources is Kirchhoff diffraction defined by the Kirchhoff-Fresnel equations but that is too far off topic at this point of your academic career.

[11]Light waves are transverse waves as opposed to pressure or sound waves with are longitudinal waves.

[12]Symmetry (is something asymmetric or symmetric)

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14.16: Chemistry

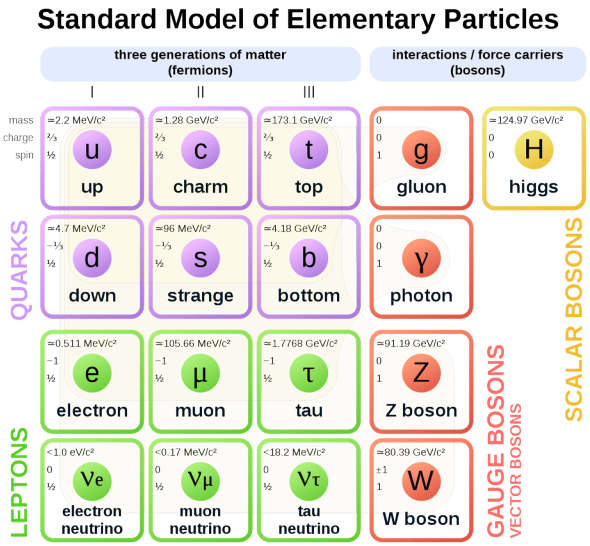
This is a small discussion of chemistry, for a true engineering for chemistry course go to the Chemistry for Engineers course for Prince Georges Community College (PGCC):

https://chem.libretexts.org/Courses/Prince_Georges_Community_College/Chemistry_2000%3A_Chemistry_for_Engineers_(Sinex)

The course above represents an atoms first approach which basically starting at the most fundamental particle and moving up to molecules and structures. For this brief introduction we will only focus on the basic atom first introduction and crystal structures alone.

Particles form other particles form other particles

The most basic particles are the elementary particles. These particles consist of quarks, electrons, and other particles. With the exception of the electron these particles really don't play a role in most of chemistry (except nuclear chemistry). Quarks make protons (two up quarks and one down quark) and neutrons (two down quarks and one up quark).



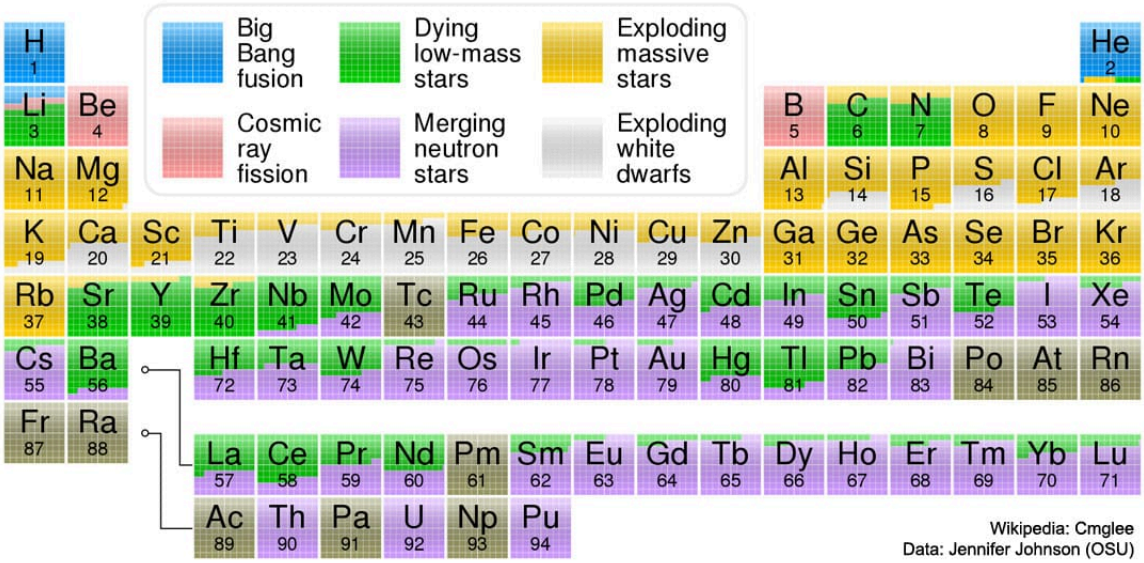
This is a table of the most common elementary particles and can be found on Wikipedia on the subject of the Elementary Particles. Credit: MissMJ, Cush - Own work by uploader, PBS NOVA [1], Fermilab, Office of Science, United States Department of Energy, Particle Data Group

Atoms

Atoms consist of protons, neutrons, and electrons with the protons (and neutrons) making up the nucleus which is surrounded by "orbiting" electron. There are less than 100 unique stable atoms which we call **elemental** atoms. The bulk of the mass of an atom is in the nucleus, but the volume is the inner portion of the "electron cloud." Electrons give rise to most of the electrical, mechanical, chemical, and thermal properties, or more precisely the orbital shells (or better yet cloud) determine the properties. Elements form bonds to other elements to form molecules.

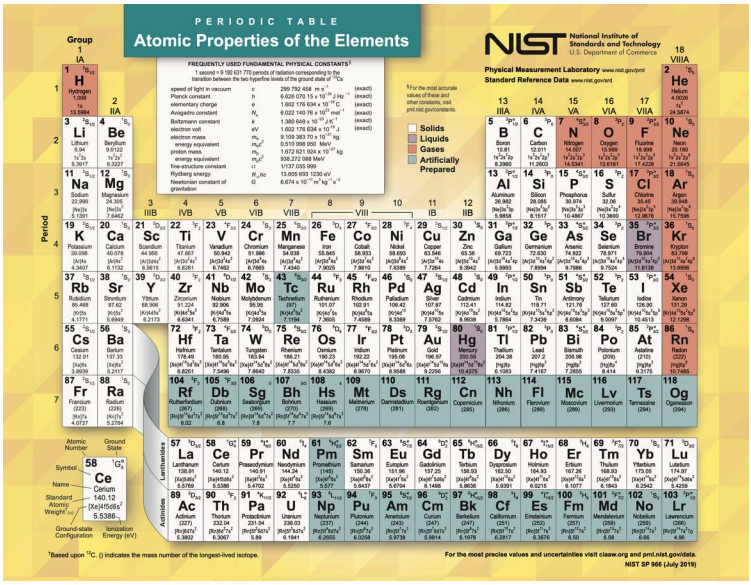
Periodic Tables

All atoms in nature that are relevant to chemistry were created by astronomical events.



Periodic table that explains the origins of the various elements (in nature only) through astronomical events. Note the elements not in this periodic table but in the periodic below and elements in this periodic table that do not have a color in the legend (like Np) are elements that are not natural but created by humankind. Credit: Wikipedia: Cmgless. CC BY-SA 3.0

For a more traditional periodic table we have atomic weights and ionization energies among other useful information.



Modern-day traditional periodic table from NIST. This has the artificially created elements which will not have any useful application in engineering. Credit: Alexander Kramida, Karen Olsen, and Yuri Ralchenko, NIST, Physical Measurement Laboratory

Molecules and Crystals

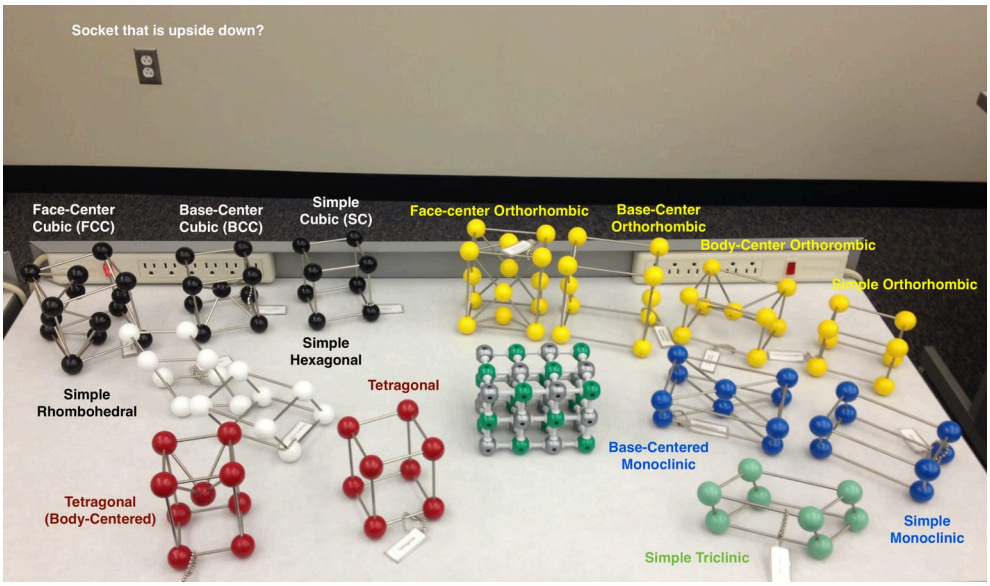
Atoms bond to other atoms with different bonding methods with the most common being ionic and covalent bonds. When atoms bond together in a stable structure that is a molecule. Gases, liquids, and solids are three of the most common phases of matter. Gases, liquids, and solids consist of numerous atoms or molecules or compounds. Gases and liquids have some applications in engineering, but primarily solids dominate the field. For this brief topic page, gases and liquids will not be discussed but rather saved for Chemistry. Solids can take different forms some of which are amorphous and some of which are crystalline. Crystals can be formed from elements or from molecules. For instance salt (NaCl) is formed by elements but not molecules whereas white

phosphorus is formed from molecules (other allotropes of phosphorus are not considered molecular crystals but macromolecules or amorphous polymers[1]). In general engineers refer to solids as materials as in solid materials (civil and mechanical engineering) and materials science (most disciplines of engineering).

Materials

There are three main type of materials: metallic materials, polymeric materials, and ceramic materials. Metals are crystals, polymers can be crystalline but are in general considered not crystals, and ceramics are a mix of metallic and non-metallic materials. Metals are ductile and strong (not brittle) and are also electrically and thermally conductive. Polymers are less easily characterized because ot the many different types of polymers. DNA, RNA, and proteins are polymers. Wood is a polymeric material that is used in engineering. Ceramics have a high degree of hardness and temperature resistance, are light weight, but are brittle and porous. Detailed discussions of polymers and ceramics will wait until after completion of chemistry for engineers.

Crystals are solids that have a repeated structure defined by seven different lattice systems[2] that have fourteen different "unit cells" which we call Bravais lattices. Most elemental metals are either body-centered cubic (BCC), face-center cubic (FCC), or hexagonal closed-back (HCP) because these structures are densely packed and, therefore, have lower energy and are more stable. Iron has a unit cell of BCC with a length of 0.287 nanometers at room temperature which is quite small. The lattice structure of a metal helps determine a number of properties of metals (ductility, electric properties, etc.). For instance, the Fourier transform of the Bravais lattice (which are Bravais lattices themselves) is a reciprocal lattice that is helps define bandgaps (determines where we have a conductor, insulator, or semiconductor) in materials.



Models of the fourteen possible basic crystal structures of solid elements. This consists of seven lattice systems that make up the fourteen Bravais lattices shown here. Most metals have either an FCC, BCC, or SC crystal structure.

Manipulatives

Below are two demonstration projects[3] from Wolfram (Mathematica) allowing you to manipulate (rotate and change): 1) the seven lattice systems (not crystal systems) and 2) the SC, FCC, BCC, and diamond (two FCCs merged) lattice structures for additional information. The demonstration projects at Wolfram have interesting examples of physics, chemistry, engineering, and other subjects that you might want to investigate on your own.

This concludes this discussion on chemistry with a dose of materials science which is very important to all disciplines of engineering (for example, materials for building structures like dwellings, laboratories, mining equipment, cars, pencils, and musical instruments, and materials for electronics and electrical devices). All students are advised to take chemistry for engineers and scientists and materials science for engineers and scientists for a complete engineering education.

[1]Polymers have a range of types from crystalline polymers to amorphous polymers. Sometimes the material is polymer-like but does not consist of monomers and as such is considered a macromolecule (not a polymer, crystalline, or ceramic), *though* there is some semantics debate on this. This subject would be explored in Materials Science for Engineers and Scientists.

[2]Crystal systems (geology) are not lattice system though they are very similar. Sometimes they are confused, however, there is an easy way to tell if lattice system should be used instead of crystal system. If one of the structures is rhombohedral (which is a trigonal crystal system) then we have a lattice system.

[3]Click on the menu symbol next to "Created with Wolfram Technology" to get general information. The attribution to the two demonstrations shown, respectively, are S. M. Blinder, "The Seven Crystal Classes", http://demonstrations.wolfram.com/TheSevenCrystalClasses/, Wolfram Demonstrations Project, Published: March 7 2011 and Bianca Eifert, "Cubic Crystal Lattices", http://demonstrations.wolfram.com/CubicCrystalLattices/, Wolfram Demonstrations Project, Published: March 7 2011

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15: LABORATORY PROJECT FOR INTRODUCTION TO ENGINEERING

For laboratory students will team up in 4 or 5 person teams. Each team would be expected to choose their own experiment (with instructor approval within a set of criteria that will ensure substantial sophistication). The main objective here is for the students to learn how real engineers work. It is very likely only prototypes will be developed by the end of this section, but that does not mean analysis, lab notebooks, and lab reports cannot be done even if the project is incomplete. Lab work usually takes many years to accomplish a significant result and it is important that students understand the process at this stage of their academic career.

Each student is expected to have a laboratory journal for taking detailed notes during the experiment. All experiments expect a laboratory journal of some sort (professional lab books are the best, but for academia a simple non-spiral bound composition notebook). Each student is expected to write their own laboratory report. While writing group laboratory reports might be acceptable in the "real world" in academia the purpose is to learn and everyone must be involved by writing their own laboratory report and by so doing learn some laboratory techniques.

For a modern laboratory project it is suggested that a control system with actuators and sensors be used to develop project that is capable of being tested in an experimental way. For this type of experiment there would need to be an graph of the data which would require an "independent" variable and a "dependent" variable (an x-y plot). For error bars you would need to perform an experiment at least five times (preferably more) for each independent variable. There are other types of graphs and analysis but they are less engineering-based (still you will need to learn them, but not here).

The intent is for a group to choose their own project and present it to the instructor for approval. This is so the student have a full experience.

Keys to all the labs:

- Labs must have one or more actuators
- Labs must have one or more sensors
- Actuators and sensors are to be controlled by a control system software (LabView, SciLab, or Octave) or just a standard programming language like Fortran or C++.
- All notes, drawings, data, etc. are to be entered into a lab book following lab journal rules given herein. Each student should have a lab book and it will be graded for completeness and format. This is different from the final laboratory report which each student also must turn in but will have a different sense of completeness.

To assist with ideas as small sample of ideas from other classes are presented here:

Example lab projects (with important details missing – for student to figure out if they go for that lab):

- Releasing different diameter rolled paper and measuring the speed from one point to another point. One actuator to drop the paper and two sensors to register time start and time end.
- Catapult distance given different release angles. While this can be theoretically figured out that does not matter, the point is the process not the experiment per se.
- Rotational speed of an AC generator connected to a light bulb (incandescent won't work with an LED light) versus the brightness.
- Length of string from fix point to bob versus period.
- Weight of load for a vehicle versus the time to travel a set distance.
- Coefficient of friction (of known surfaces) versus velocity of vehicle.
- Power of motor versus velocity of a vehicle. Note the equipment this group had could control the power of the motor by software.
- Cooling water by stirring and not stirring as a function of time (two graphs to compare).
- For a full tub of water the size of an opening versus rate of water flow. This had some technical problems that made the build of this very difficult but was doable.
- Water flow on a water wheel versus electrical output. Lots of water requires some work outside so don't do this type of experiment in the winter as you will get wet (even the professor might get wet during observation).

15.1: Laboratory Journal Format
15.2: Laboratory Report Format
15.3: Example of Laboratory Report
15.4: Sketch of how to present data in laboratory (i.e. graph with errors)

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15.1: Laboratory Journal Format

Laboratory Journal Format

General notes:

- *All laboratory work should be kept in a lab journal, which is a bounded notebook. Separate sheets combined do not constitute a lab journal. The lab journal can be a composition book, though specific books are sold for laboratory work.*
- *A pen (not a pencil) should be used in a laboratory journal.*
- *Mistakes are to be crossed out with single line. Under no circumstances are they to be erased or scribbled out. They should still be observable to the lab reviewer. In some instances trying to hide mistakes might be looked on as dishonesty. If you believe you made a really stupid mistake, cross it out and put a statement next to it (for example "opps, my brain stopped functioning for a bit."). Everyone makes mistakes even the smartest and brightest.*
- *The journal format will have the same feel as the laboratory report but will by its nature be a bit more haphazard because it is like a diary of your work but with more professionalism (don't let this scare you though, it does not need to be stodgy just more organized and through).*
- *Try to maintain order as best you can, but don't re-write or re-do pages of a lab journal. They are meant to stay intact as they were done. Your lab report WILL have order.*

On every page you should have:

- Dates, times, and page numbers. This is essential.

Each lab day should have:

- Basic condition for the day like temperature and humidity (or estimates). Do as best you can on this one.

Title Page

The title page gives all the information that is relevant for a reviewer in order to figure out who was involved with the experiment. It should contain the title of the experiment, the names of the participants, and ways to contact the participants if questions arise on the experiment and the results. It should also include dates, times, and locations of the experiment.

Overview

The overview contains observation that leads to a hypothesis, the hypothesis, and an experiment to test it along with the predicted outcome of the experiment. This section will include a summary of the reading notes used in determining the hypothesis. Experiments will in general be controlled experiments, and, to be generally accepted, will be replicated multiple times by independent experiments. This means that your notes and musings on this subject must be complete and thorough so that others may understand how you arrived at your hypothesis.

All experiments will involve some sort of hypothesis regardless of how mundane it might seem. If you can't figure out a hypothesis then you don't have a true experiment. Whether a design will succeed or fail is NOT an experiment. Hypothesis can fail. Failure is not a mistake, it is a result that can lead eventually to success. Trying to avoid failure is a mistake.

(Note: Some courses have laboratories that are not true laboratories but training on how to use a procedure or equipment. Learning how to titrate is not an experiment, it is learning a procedure. Still you might want to follow a laboratory journal format in this case more in the note taking sense then an experiment sense.).

Abstract

The abstract should be a concise description of the experiment, the results, and the conclusion. The main purpose of the abstract is for the reader to determine if she wishes to read further. The abstract differs depending if it is in a final report or a continual lab journal but should generally be consistent with the idea of informing a reader what is to follow.

Introduction

A description of the idea behind the experiment should be presented in this section. This will include a number of items enticing the reader to understand your experiment. This will describe the reasons behind your experiment. Work by others on the subject matter will usually be summarized here as they apply to your experiment.

Materials

This includes all the equipment, tools, and materials used in this experiment. Include model numbers and serial numbers for the equipment. Sketches of the equipment should suffice, however in some instances pictures of the equipment can be of use. This section along with procedure is the most important part of the laboratory journal.

Procedure

The procedure should be outlined in this section with detailed drawings of the equipment mentioned above and how they were used. The drawings do not have to be perspective drawings. Detailed sketches with important items pointed out should be sufficient for the reader to recreate the experiment. The procedure does not include data. Data is reserved for the result section.

Results

All the results should be included in this section. All data should be done in a permanent marker and all mistakes should be crossed out with one single line. Hiding mistakes is NOT appropriate and might call into question the veracity of the data. We all make mistakes; so don't overreact by erasing mistakes or hiding them in some fashion. If the mistake is due to equipment malfunction or some other explainable action make note of it in the lab book. Tables of data with the appropriate units, graphs with appropriate labels, images of the result with appropriate caption, sketches of some action, diagrams, and simple calculations should all be shown in this section or reference should be made to the appendix. All measure quantities have an error associated with them. These should be noted here. Give a brief summary of the various sources of error.

Appropriate use of significant figures should also be used when reporting results.

Discussion

This section should include a discussion of the results. Any unusual or note-worthy features should be described here. Any problems with the present experiment or methods to improve the experiment should be cited here. This is NOT a conclusion section.

Conclusion

This section includes the conclusions derived from the result. The results must support the conclusion and the reasons for the support should be cited here. The conclusion should be brief. Any lengthy discussion should be reserved for the discussion section of the lab report. In the lab book you might have an initial conclusion that you reject after analysis. That is fine don't try to hide this fact.

References

List all books, web sites, and articles used in the completion of the laboratory. Sometimes this is at the beginning of the lab journal and maybe in the middle as well.

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15.2: Laboratory Report Format

Laboratory Report Format

Title Page

The title page gives all the information that is relevant for a reviewer in order to figure out who was involved with the experiment. It should contain the title of the experiment, the names of the participants, and ways to contact the participants if questions arise on the experiment and the results. It should also include dates, times, and locations of the experiment.

In the spirit of saving precious resources, this "title page" maybe just a small portion of the front of the report rather than a whole page.

Abstract

The abstract should be a concise description of the experiment, the results, and the conclusion. The main purpose of the abstract is for the reader to determine if she wishes to read further. The abstract differs depending if it is in a final report or a continual lab journal but should generally be consistent with the idea of informing a reader what is to follow.

Introduction

A description of the idea behind the experiment (i.e. the overview -- in lab journal area or see next paragraph). This will include a number of items enticing the reader to understand your experiment. This will describe the reasons behind your experiment. Work by others on the subject matter will usually be summarized here as they apply to your experiment.

Also included in the introduction would be an overview of the experiment which contains observation that leads to a hypothesis, the hypothesis, and a suggested experiment to test the hypothesis, and the predicted outcome of the experiment. This section will include a summary of the reading notes used in determining the hypothesis. Experiments will in general be controlled experiments, and, to be generally accepted, will be replicated multiple times by independent experiments. This means that your notes and musings on this subject must be complete and thorough so that others may understand how you arrived at your hypothesis.

All experiments will involve some sort of hypothesis regardless of how mundane it might seem.

Materials

This includes all the equipment, tools, and materials used in this experiment. Include model numbers and serial numbers for the equipment. Sketches of the equipment should suffice, however in some instances pictures of the equipment can be of use.

Procedure

The procedure should be outlined in this section with detailed drawings of the equipment mentioned above and how they were used. The drawings do not have to be perspective drawings, detailed sketches should be sufficient. The procedure does not include data. Data is reserved for the result section or the appendix.

Results

All the results should be summarized in this section. Data should be graphed along with any analysis information like fits. Fit equations should be reported in this section. Raw data may be placed in the appendix and a citation should be made in this section. Tables of data with the appropriate units, graphs with appropriate labels, images of the result with appropriate caption, sketches of some action, diagrams, and simple calculations should all be shown in this section (if brief) or reference should be made to the appendix. All measure quantities have an error associated with them. These should be noted here. Give a brief summary of the various sources of error.

Appropriate use of significant figures should also be used when reporting results.

Discussion

This section should include a discussion of the results. Any unusual or note-worthy features should be described here. Any problems with the present experiment or methods to improve the experiment should be cited here. This is where you might state reasons your conclusion from the results might be not right. This is NOT a conclusion section.

Conclusion

This section includes the conclusions derived from the result NOT what you believe the conclusion should be. The results must support the conclusion and the reasons for the support should be cited here. The conclusion should be brief. Any lengthy discussion should be reserved for the discussion section of the lab report.

References

List all books, web sites, and articles used in the completion of the laboratory.

Appendices (optional)

This section, which can be many different sections, should include raw data or lengthy calculations that, while important, overshadow the general hypothesis, procedure, discussion, and conclusion. Data should be placed in tables that give the units and indicate whether it is a measured quantity or a derived quantity. Derivations should be as complete as possible if they are different from known work. In any paper for publication this section will likely be removed, so if there is any important information that is necessary to understand the experiment then it should be summarized in the text with a reference to the appendix.

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15.3: Example of Laboratory Report

This is an example of the laboratory report format. It is not meant to represent a real lab (not sufficient for the lab requirement in this course) but meant to highlight the format. It is purposely made very simple to not obfuscate the purpose of this example.

Linearity of a resistor
Anakin Skywalker
Ahsoka Tano
askywalker@jeditemple.edu
atano@jeditemple.edu
1:00 pm - 3:00 pm 10/21/1980
Laboratory Location: Jedi Temple, Coruscant, Room 210E

Abstract

This experiment seeks to determine if a 5 ohm resistor is linear to within 5% in the voltage range 0.160 to 2.50 volts. A simple serial circuit was devised and tests showed that the resistor was linear to well within 5%.

Introduction

Common resistors, like ACME off-the-shelf resistors, are usually carbon-composite resistors. An ideal resistor has the mathematical relationship V = IR, called Ohm's Law, where V is the instantaneous voltage across the resistor, I is the current, and R is a proportionality constant that is called resistance. Ohm's law holds in most circumstances however is not valid at extreme temperature ranges, voltage ranges, and high frequencies.

A non-ideal resistor will have a linear range which follows Ohm's Law closely and a non-linear range associated with it. For most applications it is desirable to operate the resistor in the linear range. Testing the linearity of circuits whether they are resistors or other circuit items is commonplace in industry.

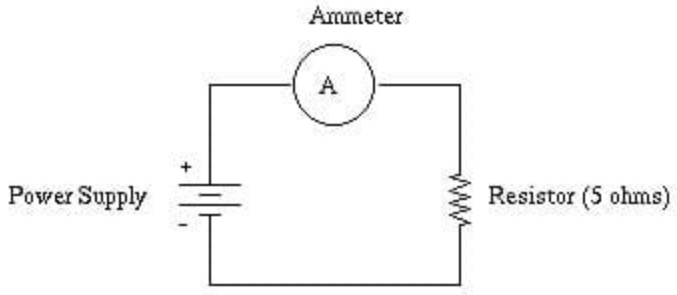
Several tests performed by a number of groups seem to indicate that resistors are linear with respect to voltage. However resistors are known to go non-linear in certain voltage ranges. Our group has an application that requires a 5 ohm resistor to be linear between the voltage ranges of 0.160 to 2.50 volts. We believe that an off the shelf resistor found an ACME should be linear to better than 5%. To test this we will measure the current through a resistor with a known applied voltage within the above voltage range.

Materials

Hewlett Packard 6253A DC Power Supply
Fluke T5-600 ammeter
Fluke 189 Multimeter
Connector Board
plug leads
ACME resistor [1 Watt 5 ohm 10% tolerance (silver band)]

Procedure

1. Using the connector board connect the resistor to the power supply (see diagram).
2. Connect the ammeter in series.
3. Change the voltage to the following 5 voltages: 0.160, 0.300,0.630,1.30, and 2.50 volts (in a cyclic order)
4. Repeat this sequences 5 times recording the current each time.

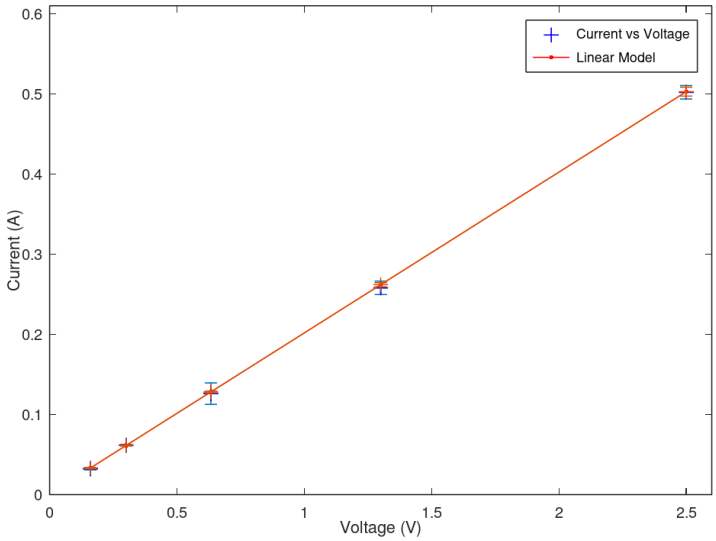


Results

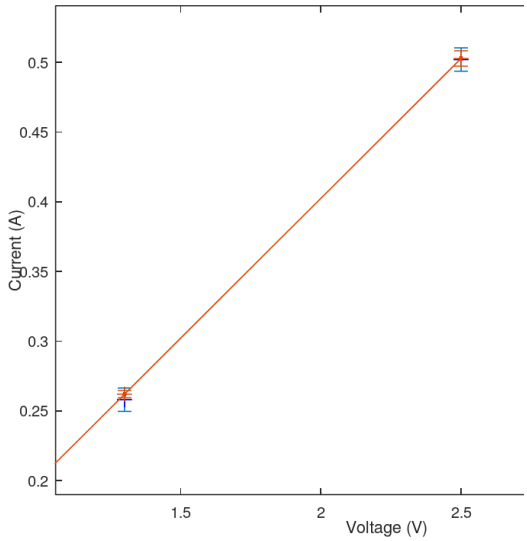
The results are tabulated in the table (table 1) in the appendix with current readings repeated a number of times to give us a reasonable error bar. The laboratory's temperature was 25 degrees C and had a steady relative humidity of 45% during the experiment.

A linear fit (see figure 1) gives us the following relationship between current and voltage:

I = (0.201 +/- 0.002)V + (0.00112 +/- 0.001)



Fit of current versus voltage based off this simple lab example. A linear fit is clearly evident with no significant deviation.



Close up of error bars to show that the model error b
from the data error bars (blue). These differences are
visual confidence in our data (statistical confidence v

Discussion

The relationship derived in the results is fairly close to the theoretical relationship of I = V/R with R in the range of 5 ohms. We anticipated an intercept of zero in this experiment, however, the linear fit suggests otherwise. This could be an unknown effect of the resistor or power supply however we believe that we did not take enough data to provide a ***perfect*** linear fit. It is our opinion that increasing the number of data points will produce a zero intercept. Further experiments should have more voltage points and more current readings.

While this experiment showed that the resistance is linear within this voltage range, it was very limited in scope. A better experiment would have tested multiple resistors at many voltage levels.

Conclusion

Within the voltage range of 0.160 to 2.50 volts we have determined that resistance is linear. The fit closely matches the theoretical equation I = V/R.

Reference

Electronic Components and Measurements, Bruce D. Wedlock and James K. Roberge, Prentice-Hall, Inc. Englewood Cliffs, NJ (1969).

Through a vision of the future, we have found a replacement for the above book that is now out of print:

Practical Electronics for Inventors, 4th edition, Paul Scherz and Simon Monk, McGraw-Hill (2016).

Appendix

Table 1: Current versus voltage to determine if the resistor provide is linear within a range of 0.160 to 2.50 volts. (We don't have a lot of data so we feel it is alright to place the data into the lab report. We understand that if we had 100s of data points we should just show the graph.)

Voltage (V)	$I_{measured}$ *(A)*		σ_I
0.160	0.0310	0.0322	0.00130
0.160	0.0330		
0.160	0.0310		
0.160	0.0340		
0.160	0.0320		
0.300	0.0610	0.0618	0.000837
0.300	0.0620		
0.300	0.0630		
0.300	0.0610		
0.300	0.0620		

Voltage (V)	$I_{measured}$ *(A)*		σ_I
0.630	0.140	0.126	0.0134
0.630	0.120		
0.630	0.110		
0.630	0.120		
0.630	0.140		
1.30	0.250	0.258	0.00837
1.30	0.270		
1.30	0.260		
1.30	0.260		
1.30	0.250		
2.50	0.500	0.502	0.00837
2.50	0.490		
2.50	0.510		
2.50	0.510		
2.50	0.500		

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15.4: Sketch of how to present data in laboratory (i.e. graph with errors)

This is notes (yes it is actual just a bunch of random notes) on how to present data for your laboratory report. This is just one example and the student is encouraged to investigate how to present data beyond this (there are many examples on the web).

The main idea of this lab is to take data at different intervals (of length, time, speed, or something else that can be done in intervals). Your could image that this would give you some sort of line or curve on an xy graph, but for an experiment it is desirable (and might even be demanded) to have error associated with your data.

So how do we get a numerical error? One possibility and the method normally used in academic exercises is to take data multiple times for each interval. From this you can calculate a mean and a standard deviation (square root of the variance). This method of calculating error is in general conservative and should suffice for simple lab experiments.

Here is an example (in octave style) for calculating the mean and standard deviation:

```
% Ok, let us say we have some measured data y at x.

%
% The data is x which is a distance (for example) that we will assume is errorless (not completely correct,
% but good enough here). The test data is y which is time (for example). We will repeat the experiment at
% the same distance at least five times in order to get a good representative
% value (the mean). Associated with this value will be an error (standard deviation).
%

%
% So at distance 1 feet we measured times of 2.200 sec, 2.000 sec, 2.400 sec, 1.600 sec, 1.065 sec,
% 1.800 sec,and 2.935 sec. %

y1 = [2.200,2.000,2.400,1.600,1.065,1.800,2.935];
mean(y1)
% ans = 2
std(y1)
% ans = 0.59839 -- which for clarity purposes I used as 0.6 for sy.

%
% So at distance 2 feet we measured times of 4.0500 sec, 4.000 sec, 4.1500 sec, 3.6900 sec, 3.8500 sec,
% 3.9500 and 4.3100 sec. %

y2 = [4.0500,4.0000,4.1500,3.6900,3.8500,3.9500,4.3100];
mean(y2)
% ans = 4
std(y2)
% ans = 0.200991 -- which for clarity purposes I used as 0.2 for sy.
%
% Repeat for all the other x values...how many x's you use will depend on what type of fit you wish
% to achieve. For a linear fit you need at least 3 values of x, for a quadratic fit
% you need at least 4 values of x, for a cubic fit you need at least 5 values of x, and for
% a quartic at least 6 values of x, and so on. Obviously more data points would always be preferable
% to get the best fit possible. This assumes polynomial fits, if it is an exponential fit (or sinusoidal)
% then you should have even more values of x.
%
% x = [1,2,{other values}];
% y = [2,4,{other values}];
% sy =[0.6,0.2,{other values}];
```

%

On paper you would write the first data point calculated above as $y_1 = 2 \pm 0.6$

Assuming we have already calculate the mean and standard deviation of a lab with five data points, here is the presentation of the data for you to run.

```
%
% Script
% Author: Dr. Scott D. Johnson
% Date: 8/2008
% The data is x,y, and sy ("error" -- that is the standard deviation)
%
x = [1,2,3,4,5,6];
y = [2,4,6,8,9,12];
sy =[0.6,0.2,0.4,0.3,0.5,0.4];
%
% Now to plot this data we should use the errorbar function (note this function is slightl
% in MATLAB, but it should work similarly).
%
h1 = errorbar(x,y,sy);
%
% The above statement is good for most older Octaves, but the recent version
% of Octaves have the annoying "feature" of connected the lines of the
% error bars (something that no researcher would ever want -- after all,
% then why have error bars?). The FMT should handle this, but errobar()
% doesn't use the FMT correctly, so the following two lines will fix this.
%
set(h1,"marker","+");
set(h1,"linestyle","none");

%
% Of course this doesn't work unless you returned handle h1
%
%
% wpolyfit will be used which is a special optional package (optim) for Octave.
% wpolyfit allows for the error to be used as a weight. There are better ways to do this,
% course, this is sufficient. The function wpolyfit is similar to polyfit where a vector i
% with the coefficients of the polynomial and a structure is returned that includes the Ch
% factors of the Vandermonde matrix plus other information that is used to create the erro
% (as opposed to the error on the data itself).
%
pkg load optim;
[p,s] = wpolyfit(x,y,sy,1);
%
% polyconf is used to get the values of y and sy for the MODEL (not the data).
% ci is confidence interval method which is
% appropriate for error bars in the physical sciences.
%
[yn,syn] = polyconf(p,x,s,'ci');
%
% Now we continue the graphing. Plotting the model over the data. And making the graph pre
% You can do different things here depending on YOUR data.
%
hold on;
plot(x,y,'b+;The data;');
plot(x,yn,'r.-;Linear model;');
xlabel('X-data (Size of the {\alpha} tiger)');
ylabel('Experiment data with error {\sigma}');
title('x versus y showing that y is linearly dependent on x');
%
% Return handle h2 so you can change the color of the model errorbars, to be different the
% the data errorbars
```

```
%
h2=errorbar(x,yn,syn);
set(h2,"color","red");
set(h2,"marker","+");
%
% Let's neaten up a bit
%
axis([0,7,0,15]);
```

Finally we have a graph with the data, the error on the data, the model of the data (linear fit in this case), and the error on the model data (note the error bar is different for different model data points just like for different points of the actual data).

You will note (if you run the octave program above) that the error bars on the model (the fitted line) have different values (which they should have though most algorithms do not do that). If you believe your model is a polynomial curve then you should change the 1 to the degree polynomial you believe would best fit your data in wpolyfit (which is from the optim package which if you have octave you may have to go get). In the help for wpolyfit is a description of how to use a chi-squared test to see which polynomial fit might be better.

From the data that wpolyfit and polyconf provide an equation for the line (or polynomial) with error can be formulated to be part of your laboratory report.

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16: Beyond the basics of computers

Review

Previously we discussed the concepts that will allow you to make any program of moderate complexity.

- Data structures, operators, and program control is all you really need
- Defined programs as programs or functions with either functions or subroutines nested in the programs
- Defined scripts
- Discussed how to write programs

In this chapter we will discuss concepts that are advanced and required for complex problems.

- These concepts are less used by the general engineer and/or scientist but are important to understand
  - Because these concepts are limited in importance to the general engineer, we will only briefly outline the concepts
  - A computer engineer would need to take a course to fully understand these concepts
- These concepts allow the user to do many useful things, but also take more effort to master

Service Routines

These are control concepts that allow the user to have more control than the basic program control of input/output, looping, and conditions.

- Interrupt
  - A variable that registers when some portion of the hardware has been physically interacted with, like for instance pressing a key on the keyboard (we would say this is a register that allows you to interact with the hardware, like for instance writing a letter from the keyboard onto the screen)
  - The idea basically is that you interrupt normal processing to do something
  - Mostly used in assembly language but some languages have access to this as well
  - Example: when cropping an image if you have selected the region you press a button to crop, an interrupt detects that pressing of a button and starts the routine to crop the image
  - Example: you are typing in your computer and the letters appear on the screen; an interrupt detects each letter you press and starts a routine to put the corresponding letter onto the screen
  - ```
    1   mov ah, 6
    2   mov dh, 255
    3   int 21h
    ```

    This snippet of an assembler program is an example of how to read a character through the use of an interrupt in the old 8080/8086 assembler language. This is not a complete program. There are online simulators of this assembly language (https://www.tutorialspoint.com/compile_assembly_online.php) and a working compiler for this assembly language (*nasm*). NB: Some basic processors (not Arduino, however, which uses C++) have assembler as their base language. NB_2: This is an old snippet and works with MS-DOS/Windows (16-bit), if you are going to use this in Linux you should use 80h, however the authors would advise you to use the simulator to learn the language before you use it in reality. Since Macs now have Intel, *nasm* should work on them as well, however, it will be different then what you will mostly find on the web so you really should take a course if you want to go into this detailed work.

- Stream
  - Data supplied continuously over time
  - Standard streams: Input, Output, Error
  - For a stream of data from a satellite it probably would be better to use assembler and the interrupt routine
    ```

Data Objects

These are simple constructs to help with data usage.

- Register
  - A small portion of memory in the CPU for fast access, kind of like a special variable
    - Interrupt registers are for interrupts
    - Program counter
    - Accumulator
    - Memory registers
    - etc.
  - In general this is used in assembler language
- Pointers
  - A variable that contains an address or reference to an actual value location
  - Useful in languages that have call by value and the all important link list
  - Can use to manipulate your computer memory in your own program (for experts)
    - Can have significant advantages
    - Can also lead to significant errors; problems; debugging time
- Data structures
  - A group of variables that are of the same type or different type
  - This is basically an expression of COBOL's DATA division

```
type refcollection
    character(len=100) :: title
    character(len=50) :: mainauthor
    character(len=150) :: otherauthors
    character(len=50) :: genres
    character(len=15) :: isbn
    character(len=10) :: type
    integer :: year
    integer :: edition
    integer :: volume
    integer :: vnumber
    real :: version
    real :: usefulscale
    logical :: available
end type refcollection
```

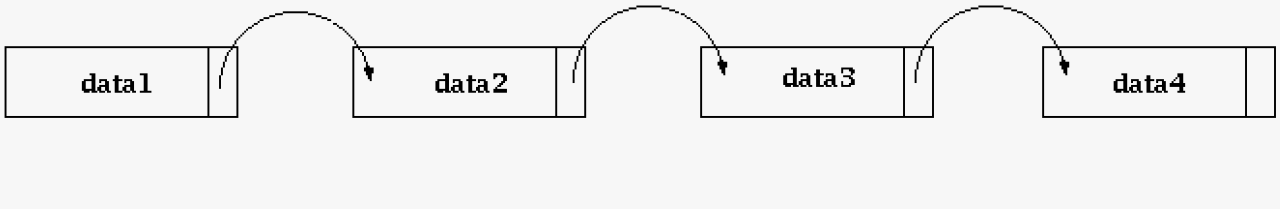
This is an example of a data structure with different data types using Fortran. This is the data structure for database of a collection of books, magazines, and papers for a rather large and esoteric "book" collection.

Advanced Data Structures

These are data constructs that utilize other data concepts. For the general engineer and scientists these are not necessary but might be useful in a limited way so a brief bread crumb is presented here. The student can study these on their own though this goes very much into computer science and less into solving problems in engineering and science.

- Linked Lists
  - A set of variables connected by pointers
  - Pointers can go one or two directions
  - For increased efficiency can be modified to include variable arrays and a host of other ideas
  - Very important concept for computers
    - It is very useful in general computer theory to increase speed by deleting things without really deleting them

- This is the reason things are recoverable after deleting them (can be very useful if you deleted a picture that was irreplaceably but could be not so good if a criminal is able to recover what you thought you deleted)
- Unfortunately this leaves a mess which eventually has to be cleaned up (part of garbage collection)
-



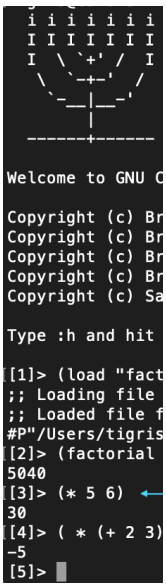
Example of linked list where the data (say words in a file) are linked together.

- Stack
  - Array that places each variable on a stack just like a stack of plates in a cafeteria
  - Used in stack-based programming languages (LISP) and calculators (old HP calculators)
  - Can be implemented in hardware making it useful in basic operations
- Queue
  - An ordered array
  - Useful as a buffer
  - Usually implemented with a linked list
- Hash table
  - Uses keys and values (think COBOL)
  - Useful for looking up data
- Graphs (as in data structures)
  - A set of data which consists of *nodes* (vertices) connected by *edges*
    - The node contains the data the edge is the link
  - Flexible data structure that goes beyond matrices
  - Requires a more standard data structure and an algorithm to implement it
    - List structure (like linked list)
      - Con: Slow
      - Pro: Low memory
    - Matrix structure (like array)
      - Pro: Fast
      - Con: Memory intensive
- Tree (as in data structures)
  - A set of *nodes* (*leafs*) connected by *branches* to other *leafs*
  - A special graph data structure
  - In typical format the root is up and the branches and leafs go down
  - A parent node is above the child node
  - Nodes can have internal nodes (children as well)
  - Used for searches, workflows, anything that is hierarchical
  - Can be pruned or grafted: this is what give this structure its advantage over other possible structures
  - Implementation of trees can either be with linked lists or a set of matrices (in languages that don't have pointers - which is very few now)
    - For Fortran (95 and above) you could define a type tree in a module to create a tree
    - For C++, Objective-C, C#, or D define a class tree
    - MATLAB and Octave have the ability to define class tree as well
    - C and older versions of Fortran do not have the ability to define classes so instead a system of matrices would need to be defined

- A decision matrix (say T)
- Some leaf matrices (say L1,L2, etc.)

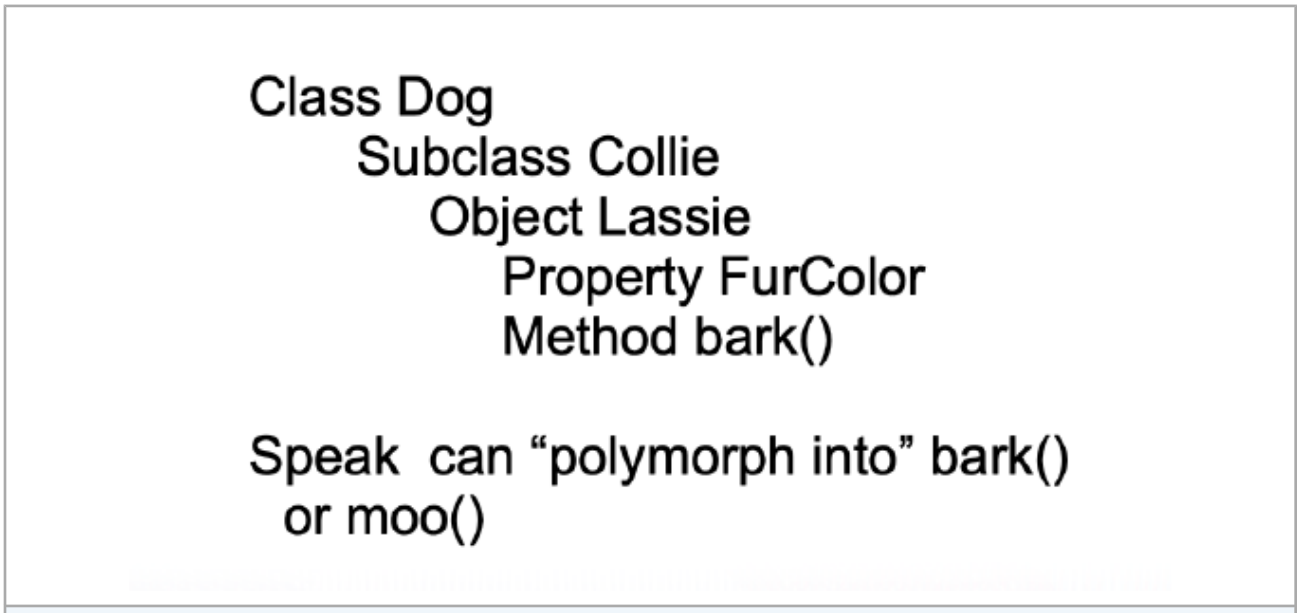
Advanced Programming Methods

These methods are very useful in engineering and science. They are a bit more complicated for a beginning computer course so we will only briefly introduce these concepts.

- Recursion
  - When a function call itself
  - In mathematics the perfect example is the factorial where $n! = n(n-1)! = n(n-1)(n-2)!...$
  - Used in AI languages like LISP
  - LISP natively uses recursion but most other *modern* programming languages are able to use recursion now as well
  - LISP Example: (defun factorial(n) (if (<= n 1) 1 (* n (factorial (- n 1)))))
    - Yes the example is just one line of code, LISP is know for its brevity
    - The operations here are an example of stack computing where <= n 1 means in the non-stack world n <= 1, etc.
    - Assignment: Figure out the rest of this small LISP program
  - <table>
<tr><td><pre>1 (defun factorial(n) (if (<= n 1) 1 (* n (factorial (- n 1)))))</pre></td><td></td></tr>
<tr><td>This is the LISP program described in the outline.</td><td>LISP is an inter
first run clisp (v
clisp program v
arrow). For a lit
some simple ca
if you understar
learning LISP.</td></tr>
</table>

- Vector processing
  - When a program divides its data to have the same operation done on multiple data sets
  - The idea is to speed up mathematical operations
  - This is SIMD (Single Instruction Multiple Data) as opposed to the usual SISD (Single Instruction Single Data) that you will find on almost all non-specialized computers
  - This type of processing use to be available only on a *Cray* (supercomputer) and *Convex* (mini-supercomputer) but now is available on a lot of CPUs
- Parallel processing
  - When a program splits its parts into independent parts that are run on different processors
  - So part of the program run at the same time
  - Very useful in Monte Carlo simulations
  - Technically this is MISD (Multiple Instruction Single Data) but almost all implementation include vector processing

which is MIMD (Multiple Instructions Multiple Data)
    - *mpirun* (or *mpiexec*) can be used to run programs in MIMD
        - This program and many others are installed when you install *openmpi* (https://www.open-mpi.org) or *mpich* (https://www.mpich.org)
        - There is also mpicxx and mpifort plus others that can be used to compile your parallel program C++ or Fortran programs
    - iPython has *ipcluster* to run a program in parallel
- Event Driven Language
    - A programming language whose control is dictated by the interrupts of various sensors or user actions
    - Sometimes confused with Object Oriented Programming, but it is distinct
    - EDL/EDX (extinct)from IBM Series/I computers a long time ago
    - Currently this ability is available in JavaScript, Java and Python (and obviously assembler) which are not directly event driven languages
- Object Oriented Programming (OOP)
    - "Routines" are preformed by objects
    - On the surface similar to functions and subroutines but distinctly different
    - Objects can directly interact with one another (different from subroutines)
        - Objects can have one or more methods (or "operations")
        - Uses message passing and inheritance
    - Data structure is more complicated
        - Operators themselves are part of the data structure
        - The operators are associated with the variables they operate on
    - Has a class structure
    - All "contained programs" are objects
    - Can have abstraction and polymorphism
    - C++ is object oriented, C is not
        - While C++ is object oriented it can also be run without any objects
        - Learning C++ is not learning object oriented programming you must learn the object oriented part of it...
    - Modern Fortran has the ability to be object oriented, but older versions are not
    - Most programming languages have the ability to do object oriented programming now



An non-computer human way of looking at an object. Here we have a class, subclass, object, property, and method. We also take about a method speak that polymorphs bark() into moo()

    - Object oriented programming has useful features but it is often not used in the engineering and science world for many "real world" practical reasons
        - Better for large projects like making a program to do scientific programming with all the bells and whistles (where the scientific program would not be object oriented)

        - Parallel processing does not work with object oriented programming
        - Recursion does not work with object oriented programming (though it can be emulated like they use to in Fortran before recursion was implemented
- Functional programming
    - This is the "new competitor" to object oriented programming using basically old ideas (which are still good ideas) with some new ideas and vocabulary
        - Technically each method of programming is better suited to different task so not a real competition, but that does stop people from online wars
        - Fortran and C++ can do either of these methods but some programming languages are better suited for one or the other
    - The idea behind functional programming is "pure" functions are used to create a program
    - Functional programming has useful features and is preferred by the engineering and science world when large projects are not an issue
    -

Functional programming	Object oriented programming
Function is primary method of getting things done	Data is the primary method of getting things done
Recursion and parallel programming is available	Recursion and parallel programming is **NOT** available

▪ Functions rule
  ▪ Pure functions
  ▪ With same inputs you always get same outputs
  ▪ No side effects (as you might find in OOP)
  ▪ Higher-order functions (functions that return functions)
▪ Type method is used for data (basically the original method used in most programming languages before OOP)
▪ Evaluation can be non-strict
▪ No overhead like in extensively nested classes in OOP (highly nested classes are for certain projects an advantage, but not all projects; hence a good criteria of when you should use OOP or not)
	▪ Classes for data structures
	  ▪ Object for data
	  ▪ Methods for "functions"
	▪ Abstraction
	▪ Inheritance
	▪ Polymorphism
	▪ Encapsulation (hides data from user that they shouldn't need)

Octave and Object Oriented Programming

Octave and its ilk are perceived by some to not have true object oriented programming and to some extend that is true but it is vastly over exaggerated.

- Problems with OOP in Octave
    - Untyped variables (this is a feature a lot of programmers actually like, however it is not good for OOP)
    - Weak polymorphism (because of weak typing)
- Advantages
    - Vector structures is useful even in OOP
    - Can be made to operate as fast as any other OOP language
- Using an OOP approach gives a measure of definition which allows Octave to produce very large projects
- Octave differs from other languages that use OOP in that it fits into a development role that deals with new problems that are not easily defined
    - Good for wicked problems
    - Many OOP languages are only good for "tamed problems"
- Wicked Problems (as opposed to the theoretical tame problems)
    - Characteristics
        - No final solution or true or false solution, just a better or worse solution (theoretically the tame problem has a true or false solution)
        - No final test to the solutions (can't prove the solution, can't stop development)
            - Of course you can do intermediate tests
            - ***TEST*** your program even if you don't have a final solution

        - Problems and solutions are in general unique (at least in the details)
    - Can't be solve by traditional method of adding more management; actually makes matters worse
    - Work for solution before all the data is in (very typical of engineering and science; or anything new)
    - Octave and its ilk are good for solving this type of problems
- Solution to wicked problems: Extreme Programming (XP)
    - Extreme programming is one of many methods that might be used to solve wicked problems
    - Characteristics
        - Simple design
        - Small sustainable working releases; adds functionality and "bug" fixes
        - Pair programming (pilot; co-pilot approach)
        - Collective ownership (requires programs to help development like *cvs* - the program not the store)
        - Documentation (actually all methods require this) - see comments in any programming language (all programming languages have the ability to comment)

Value versus reference languages

There are two type of languages those that pass arguments by value and those that pass arguments by reference. Some languages do both. This definition however is not strict as call by value and call by reference can mean different things even within the same categories. With this said...

- Normally pass by value means that a copy of the inputs are passed to a function; could make things slow for large structures
- Normally pass by reference means an address to a variable is passed to a function; error could lead to "crashed" program in certain programming languages
- Languages that pass function input/outputs by value
    - Advantage is in general speed but not in memory
    - C (in practice users of C pass a pointer to the variable they wish to change)
        - The pointer method can lead to errors if not done carefully
    - Pascal
    - GDL/IDL
    - Java (very different from C though)
    - Octave/MATLAB/Scilab
- Languages that pass in calls by reference
    - Efficient in time and space
    - Fortran (not pure however which is a copy of the reference; which is better)
    - Perl
- Languages that pass by both methods
    - C++
    - Some Fortran versions
    - Basic
    - LABView

So this is a quick brief bread crumb chapter to give hooks to other methods that an engineer/scientists may need, but likely not need for their career. We will end the chapter here. Next chapter will finish off some concepts on computers.

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17: Documentation and such

Documentation

Why have a chapter on documentation? Because documentation is needed for any program you write, however, it is also the most avoided essential. *Why?* Because of the oft used excuse that "I know what it does." But unfortunately you forget after awhile...so...***DOCUMENT!!!!***

Documentation is one of the most important parts of programming, it is also the toughest, and most poorly implemented in general. Here we will present some general rules for documenting any program (software project).

- Documentation for all programming projects
  - A traditional documents on paper (or pdf or html or some universal way to read the document)
    - Manual
    - One page quick guide
    - Paper (or papers) explaining the raison d'etre for your program and the theory and solutions
  - Documentation within the program itself
- **All** programming languages have some method of allowing the user to enter comments
  - Comments should be added at the beginning of the program
  - Comments should be throughout the program itself near *important* lines
  - Don't overdo comments as this distracts from the overall documentation
    - Example: we don't need a comment on a print statement that says it displays the string to the screen...we can figure this out without documentation
    - Example: you use a special numerical method that is not very common...yes please document this
  - Octave and its ilk note: the **help** in Octave give the front comments; **lookfor** give the first line
    - So if you don't have the front comments, help won't work...the professor will not be pleased...
    - Front comments can also be where you credit yourself (and your partners if applicable)
- Good coding techniques acts as a defacto documentation of the code
  - Use tabs for statements in loops and conditionals, etc.
    - Note you have to do this in Python (so you are force to have good coding techniques)
    - A lot of editors (for coding) do this for you as long as you have a template for that language
  - Separate functions that are used multiple times into function, subroutines, or the like
-

```
% Function add n planar vectors
%
% SYNTAX: [result_force,result_angle]=forceaddmulti(f1,theta1,...)
% PURPOSE: To add planar vectors for a class demonstration
% EXPLANATION: Uses Pythagoreans theorem to get the magnitude of the resulting
% force and trigonometry to get the resulting angle
% CALLING SEQUENCE EXAMPLE: [force,theta] = forceaddmulti(5,30,6,20,4,60)
% INPUTS: f1, theta1,... are the magnitude and the angle of the different vectors
% OUTPUTS: result_force, result_angle is the magnitude of the added vectors with
% their directions as defined by an angle from the reference direction (x)
% DEPENDENCIES: sqrt,cosd,sind,atand2d
% NOTES: Extends the first class example (forceadd) to allow for a variable number of
% arguments by using varargin
%
% Author: Scott D. Johnson as an example to understanding Octave/MATLAB  2006
% Revisions:
% 1. SDJ minor revisions 3/2008
% 2. SDJ minor revisions 3/2012
% 3. SDJ comment revision 3/2021 for better clarity
%
%
function[f,theta]=forceaddmulti(varargin)
%
%
fxtot = 0;
fytot = 0;
%
% Varargin uses the "cell" structure...
% To extract pieces of a cell use {} instead of ()
%
for k=1:2:length(varargin)-1
%
% cosd, sind, atand, atan2d are for degrees instead of radians
%
fx = varargin{k}*cosd(varargin{k+1});
fy = varargin{k}*sind(varargin{k+1});
fxtot = fxtot + fx;
fytot = fytot + fy;
end
f = sqrt(fxtot*fxtot+fytot*fytot);
%
% Should use atan2d, not atand. atand cannot take into account the direction of the vectors, atan2d can
% so atand(-10/10) = atand(10/-10) = -45 degrees while as
% so atan2d(-10,10) = -45 degrees but atan2d(10,-10) = 135 degrees
% could write code to handle this for atand, but since atan2d exists might as well use that...
%
%theta = atand(fytot/fxtot);
%
theta = atan2d(fytot,fxtot);
end
```

```
GNU Octave, version 6.2.0
Copyright (C) 2021 The Octave Project Developers.
This is free software; see the source code for copying conditions.
There is ABSOLUTELY NO WARRANTY; not even for MERCHANTABILITY or
FITNESS FOR A PARTICULAR PURPOSE.  For details, type 'warranty'.

Octave was configured for "x86_64-apple-darwin20.3.0".

Additional information about Octave is available at https://www.octave.org.

Please contribute if you find this software useful.
For more information, visit https://www.octave.org/get-involved.html

Read https://www.octave.org/bugs.html to learn how to submit bug reports.
For information about changes from previous versions, type 'news'.

octave:1> help forceaddmulti
'forceaddmulti' is a function from the file /Users/tigris/howtos/exampleprograms/octave/forceaddmulti.m

 Function add n planar vectors

 SYNTAX: [result_force,result_angle]=forceaddmulti(f1,theta1,...)
 PURPOSE: To add planar vectors for a class demonstration
 EXPLANATION: Uses Pythagoreans theorem to get the magnitude of the resulting
 force and trigonometry to get the resulting angle
 CALLING SEQUENCE EXAMPLE: [force,theta] = forceaddmulti(5,30,6,20,4,60)
 INPUTS: f1, theta1,... are the magnitude and the angle of the different vectors
 OUTPUTS: result_force, result_angle is the magnitude of the added vectors with
 their directions as defined by an angle from the reference direction (x)
 DEPENDENCIES: sqrt,cosd,sind,atand2d
 NOTES: Extends the first class example (forceadd) to allow for a variable number of
 arguments by using varargin

 Author: Scott D. Johnson as an example to understanding Octave/MATLAB  2006
 Revisions:
 1. SDJ minor revisions 3/2008
 2. SDJ minor revisions 3/2012
 3. SDJ comment revision 3/2021 for better clarity



Additional help for built-in functions and operators is
available in the online version of the manual.  Use the command
'doc <topic>' to search the manual index.

Help and information about Octave is also available on the WWW
at https://www.octave.org and via the help@octave.org
mailing list.
```

Comments from program are the official help!

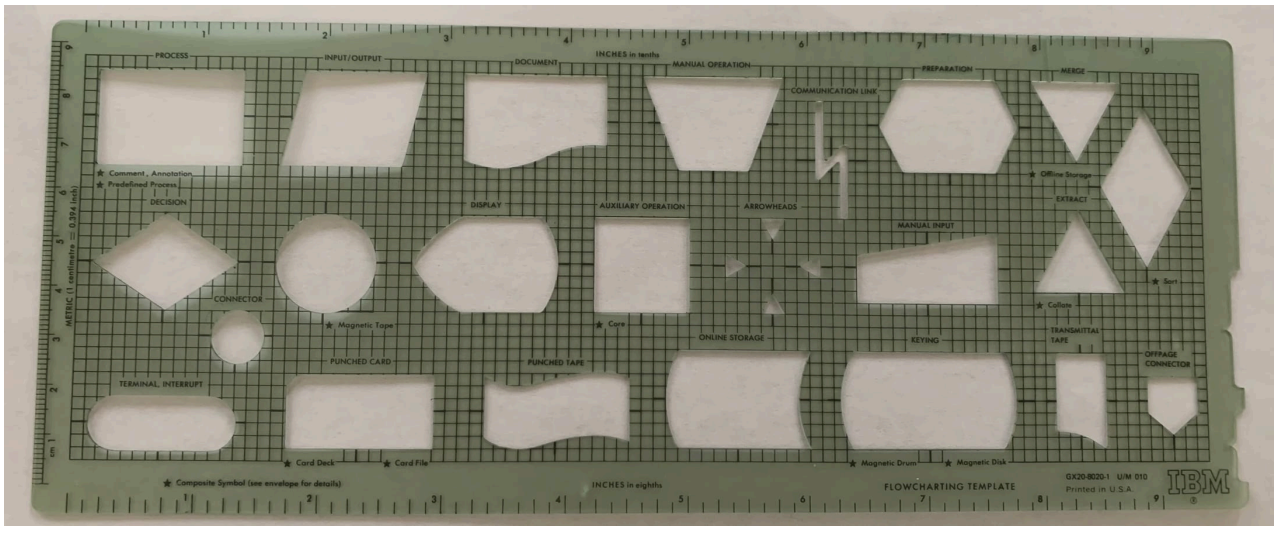
This is the program that we will ask for help on the right. Observe the comments written in this program. *NB: open the image in new tab or window if you are having trouble reading it.*	This is what help returns in Octave/MATLAB for the program on the left. You will see that the comments were important in order to get the help. This is true for all the functions in Octave/MATLAB, so do your comments!

- Documenting in a group project
  - Documenting in a group project requires some negotiation and sharing of responsibility
    - It is required of engineers to work in groups
    - It is required of instructors to grade you on group dynamics
  - Extreme Documentation
    - Characteristics
      - Document as project progresses (not after it is done)
      - Team ownership of project
      - Independent documentation from each programmer in the group (more eyes theory)
      - Compile documentation at the end in a traditional document and in the program itself

Writing Good Programs

To write good programs you should follow the procedures on documentation above, but *what about the actual code?* It is generally agreed that you should outline the procedure before you begin to actually code your program. There are two methods for outlining a program, flowcharts which has actually done before computer programming but adopt by programmers as well, and pseudocode where you code in a generic sense and to the specific syntax after you have gone over the pseudocode.

- Flowcharts
  - Set of graphic "blocks" with connection arrows that represents the programming flow in a visual way
  - There has been work on using set theory to actually prove if a program is valid using flowcharts
    - Just proves you get a result and it follows you procedure as you believe would work
    - Can't really prove if the result is right or not as that takes an expert opinion in the field that the programming is being written for
  - There are programs that help with flowcharting or even run programs that you have flowcharted (*raptor* or *flowgorithm*)
  - There are modern additions that allow for parallel processing in flowcharting
  - Flowcharts have official ANSI symbols
  - While an old technique it is still useful



Flowchart symbols on a template that was used before personal computers.

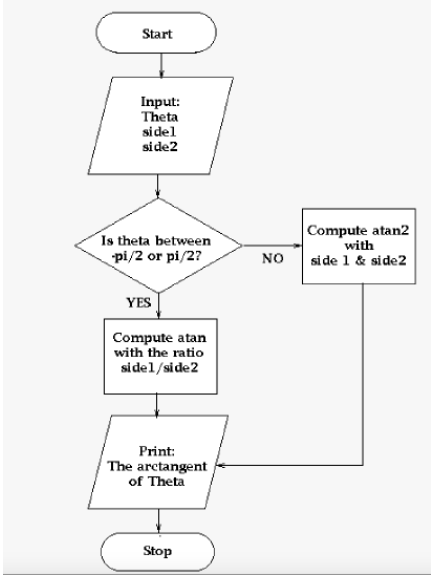
- Pseudocode
  - Programming in plain English (or any other language you wish) with a limited set of rules

- Takes away the burden of strange (in the sense of our normal English sensibility) variable names, structures, methods, etc. imposed by different computer languages
- Lets us write the way we "think"
- Some pseudocodes have been developed with hardware and software interaction in mind
  - Software may actually be involved in changing voltage levels (think a controller) which is hardware in nature but controlled by software
  - Pseudocode is a generally enough idea that it can be used things not necessary computer programming related - just like flowcharts
- While generally pseudocode is what you like to say in English it is generally agreed on that it should have words similar to programming languages
  - Might use read and print in pseudocode for a C++ program and those "pseudocode" words would be changed to cin and cout when you start to program
  - There is no official pseudocode

Comparison of pseudocode and flowchart with very simple routine

```
input theta,side1,side2
if (-pi.2 < theta < pi/2) then
    answer = atan(side1/side2)
else
    answer = atan2(side1,side2)
endif
print answer
return
end
```

Pseudocode of a simple routine to decide whether to use atan or atan2. Note there is input and output and a conditional.



Flowchart of a simple routine to decide whether to use atan or atan2. Note there is input and output and a conditional.

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18: ADVANCED PROGRAMMING CONCEPTS

18.1: Analysis Methods

18.2: Analysis Techniques for Laboratory

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18.1: Analysis Methods

In engineering and science, analysis is important for proper design. Most analysis and modeling techniques are covered in numerical methods which has, in recent years, been unfortunately sacrificed in a number of engineering programs in favor of a pure signals and systems class. Some engineering programs include numerical methods by including a section or two in signals and systems and various programming courses. It is important for a student to make sure they have some background in numerical methods. Here we will review some essentials ideas in numerical methods in computer science. This will not be a complete picture.

Signals and Systems and Numerical Methods

A system is a process that can accomplish a task or is a process that by nature accomplishes a task. The term system is so generic it can be used in the sciences, engineering, social, computer science, music, etc. In biology for instance we have a digestive system that takes in food and processes to output materials the body needs and waste, in computer system you have system calls that input a command you wish the computer system to do and outputs the action of the computers system (which is a system in itself), in engineering you may have a LED which you input a potential and the LED outputs light, and so on. Think of all the terms where system is used...a government system, a railroad system, a system of stars, a musical system (staves), natural system, social system, etc. This makes a consistent definition of system a little problematic, however we will try to define it to fit more of a science and engineering purpose.

A system is a **process** that has an *input* and has an *output*. You can see that in the examples above. In engineering we add another term to this complicated definition and that is signal. Therefore, a system is a **process** that has an *input signal or signals* and an *output signal or signals*. The output signal is usually different but in certain instances and be the same as the input. A subset of this is a *control system* that controls or regulates a system. This now leaves the question of *what is a signal?*

A signal is the representation of a physical "wave." That is normally expressed as a variable in time-space, for instance x(t). Examples of signals might be the voltage or current of an electrical circuit, the force in a mechanical circuit, heat flow in a thermal circuit, hydraulic flow in a fluid circuit, biological mass flow in a digestive circuit, etc. This definition gives us a basis to move to a more mathematical approach to systems and signals which can lead to computational methods.

A system can be expressed mathematically as a transfer function or an operator. A simple example of an operator could be an integrator that takes a signal and integrates it over time or it could be as simple as multiplying the signal by a constant. Transfer functions are specific to engineering and have the notation of a Laplace transform, where as operators are more specific to physics (and other sciences) and have the notation of mathematics.

How we express these ideas tangible is through different mathematical techniques or methods. The system methods that are used are generally the same methods that a student would learn in linear algebra. The signal methods that are used are generally the ones learned in *numerical methods*. This makes both methods important in engineering so we will give a quick bread crumb run down of the various methods starting with numerical methods.

Numerical Methods

Technically we already did one type of numerical methods in parachute person. This method is the Euler's method for numerically solving differential equations. There are better methods to numerical solving differential equation but it the most intuitive.

¬ What type of problems do engineers/scientists solve

- We solve problem of balance, like balancing forces or balancing voltages (Kirchhoff's voltage law)
  - Any balanced equation can be written such that the function equals zero
  - These are problems of finding a root or roots
  - *Root solving methods*
- We solve optimization problems like minimum energy, minimum volume, or maximized efficiency
  - Mathematically this requires taking a derivative and solving the resulting function when it is set to zero
  - Solving for a system set to zero is a root problem so root solving methods and optimization methods are related

    - Though related methods they are not the same because of slightly different goals
    - *Optimization methods*

¬ What other tasks do engineers/scientists perform on a routine basis

- Analysis for experimental data
    - Fitting experimental data to a curve (see sections in laboratory chapter)
    - Filtering experimental data or filtering signals (which produces the same results as fitting)
    - Fourier analysis (or other related analysis)
        - Fourier fast transform (FFT)
        - Wavelets
    - *Filter methods (or fitting methods)*
    - *Transform methods (operators)*
- Simulation
    - Circuit analysis
    - Finite difference analysis
    - Finite element analysis
    - Finite volume analysis
    - Monte Carlo analysis
    - "Molecular" dynamic analysis
    - *Simulation methods*

Conservation Laws

The idea of balanced equations comes to us from the conservation laws of physics. Here is a brief list of the laws with notes.

From a theorem by Emmy Noether[1] -- Symmetry implies conservation (breaking symmetry is a phase change)

- Conservation of Energy
    - Historically promoted by engineers over the objections of physicists
    - First Law of Thermodynamics – Chemical Engineering
    - Voltage Balance – Electrical Engineering
    - Invariant under time translation
- Conservation of linear momentum (or just momentum)
    - Force balance -- Civil and Mechanical Engineering
    - No external force is required
    - Invariant under translation
- Conservation of angular momentum
    - Angular analogy to linear momentum – Civil and Mechanical Engineering
    - No external torque is required
    - Invariant under rotations
- Conservation of electric charge
    - Current balance – Electrical Engineering
    - Charge in a note must eventually leave that node
- Conservation of mass
    - Mass balance equation – Chemical Engineering
    - Mass into a system must equal the mass out of a system plus any accumulated mass in the system

- Conservation fails in certain instances if the equivalence of energy and mass is not considered
  - Does not fail in reality...just in our understanding of what is going on

From these laws we get a number of the laws (Kirchhoff's law, etc.) that we have already discussed in this course/book. The equations produced here can then be programmed into a computer and solved for the problem you wish to solve. Of for that more advance languages some of the laws maybe already incorporated into the language itself (like say SPICE for circuits).

Finding the roots of an equation

- Bracketing methods
  - Converges but can go really slow in certain instances
  - Need to points to bound the root
  - Bisection method
    - Similar to what you do when someone says guess a number between 1 and 50
    - Just keep dividing by 2, right?
  - Regula Falsi method
    - Improves bisection method by speeding it up
    - The speed up can result in issues itself
- Open methods
  - Finds solutions fast but may not converge
  - Only need one point
  - Fixed-point iteration method
    - Isn't used by itself but maybe combined with another method
  - Newton-Raphson method (Newton method)
    - Most popular method
    - Need derivative of function
    - Uses Taylor's series expansion
  - Secant method
    - Similar to Newton method
    - Uses two points so it does not require a derivative like Newton's method does
- There are hundreds of rooting finding methods which attest to it's importance
- In general root finding methods in higher level languages such as Octave employ a combination of the previous methods (or similar methods)

Here is a bisection method example. Review it and see if you can understand what this does. This uses a feature of Octave called handles.

```
% Uses the bisection method to calculate a root of the function
%
% SYNTAX: root=bisect(function_to_find_root,bounds_of_roots,tolerance)
% PURPOSE: To get the root of a function using the bisection method
% EXPLANATION: Uses bisection method to solve for a root
% CALLING SEQUENCE EXAMPLE: root = bisect(@myfunc,[0,4],1e-6)
% INPUTS: The function, fin, the bounds to find the root, bounds, and the tolerance, tol
% OUTPUTS: The root of the function
% DEPENDENCIES: feval
% NOTES: Class example with limited error checks
%
% Author: Scott D. Johnson as an example to understanding Octave/MATLAB  2019
%
function root = bisect(fin,bounds,tol)
xl = bounds(1);
xu = bounds(2);
```

```
f1 = feval(fin,xl);
f2 = feval(fin,xu);
if (f1*f2 > 0)
   printf("Error in your bounds\n");
   return
else
   printf("Bounds are good let's proceed\n");
endif
xnew = (xl+xu)/2.;
while (abs(f1*f2) > tol)
   f1 = feval(fin,xl);
   f2 = feval(fin,xnew);
   if (f1*f2 < 0)
      xu = xnew;
   elseif (f1*f2 > 0)
      xl = xnew;
   endif
   if (abs(f1*f2) <= tol)
      root = xnew;
      printf("Root has been found\n");
   endif
   xnew = (xl+xu)/2.;
endwhile
return
end

% A function to use to test root finding methods
% This is an example of a function to be used in either MATLAB or octave.
% There is a difference between the two programs though they are generally
% alike.
%
function ansfunc = myfunc(x)
%
ansfunc = (x-4).^4-3.*x+2;
return
end

root = bisect(@myfunc,[0,4],1e-6)
```

Optimization methods

- Golden search method
  - Uses golden ratio
  - Just like in art
  - Like bracketing methods, converges but is slow
  - Unfortunately slow in optimization is not good given the likelihood of very complicated optimization problems
- Newton's method
- Non-gradient methods like Powell's method
  - Don't use these as they are slow and not very efficient
- Gradient methods
  - Levenberg-Marquardt method (one of the best)
  - Octave routine, *leasqr* in the package *optim*
- "Simulation" methods
  - Predictor-corrector methods
    - Idea extends what we did in parachute person
    - Predictor is what we did in parachute person, corrector is the improvement of what we did
  - Simulated Annealing
  - Neural Nets
  - Genetic Algorithms
- Linear programming
  - Simplex method

- There are hundreds of optimization methods which attest to it's importance

Fitting data

- Least square method
  - This method is typically available on most engineering/science oriented higher level languages
  - Octave, *polyfit* or *wpolyfit* (from *optim* package)...see discussion in laboratory pages
- Splines
  - This is a very useful method of fitting data
  - There are a number of different spline methods that are typically available on most engineering/science oriented higher level languages
- Digital filters (remember filtering is fitting and fitting is filtering)

Differential analysis methods

- Finite element analysis
  - Solve partial differential equations on a grid (many points) and interpolate rather than a continuous system (infinite points)
  - Use extensively in engineering
- Finite difference analysis
- Finite volume analysis

These methods and more are normally discussed in numerical methods or in signals and systems (assuming a section on numerical methods is included). This course is essential for engineering and required in many Universities (names might be slightly different depending on discipline).

While there are other topics to discuss here the intent is for you to see what is out there with this bread crumb chapter, not to actually teach the course. So with that let us move on to the next section.

---

[1]Emmy Noether was a famous German mathematician during the time period of World War II. She helped many scientists, in particular physicists, solve problems, including Albert Einstein.

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18.2: Analysis Techniques for Laboratory

Data Analysis (for introductory students)

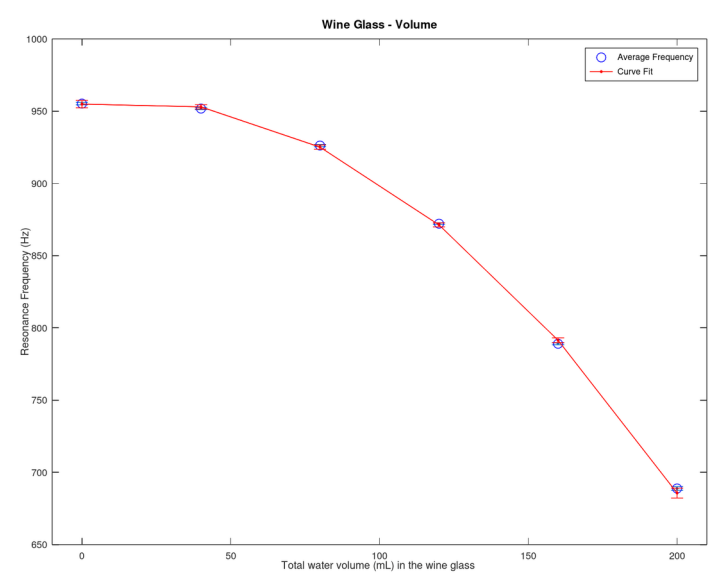
To continue on the previous section when we get data from experiments we wish to analysis it to be able to make reasonable conclusions. There are multiple methods of analysis for data, but for this introduction class we will look at just two: Fitting and Fast Fourier Transform (FFT).

In a typical experiment we would have one independent variable and one dependent variable (in the cases were there are a lot of independent variables producing different effects we would need to go to more sophisticated analysis techniques that are not the focus of this course - however from a computer point of view the functions to do these sophisticated analyzes are available in programs like Octave). Here we will show how one might use the fitting method for laboratory results and the FFT method to analyze data acquired in noisy circumstances (i.e. reality) and possibly pull out some useful data.

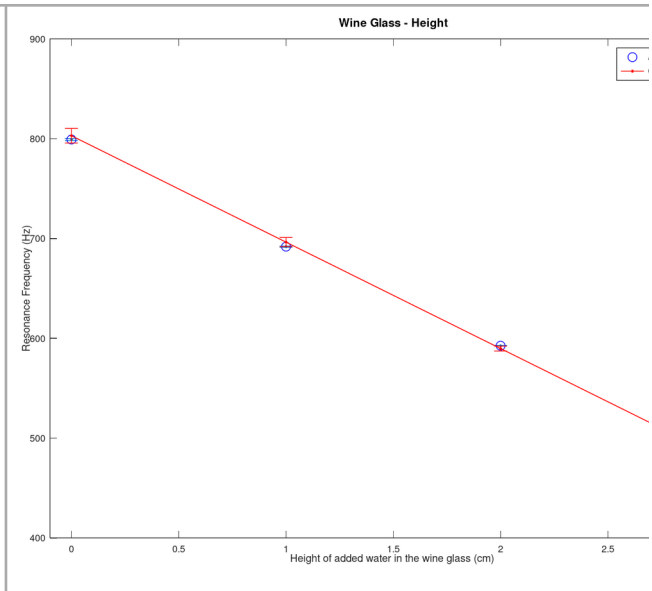
Fitting

During an experiment we will vary some variable to get results that, of course, we would like to draw some conclusions about. One of the most common techniques is to see if there is a relationship (such as a linear or quadratic) between the independent variable and dependent variable. To do that one must fit a model to the data. The most common fitting technique (or regression)[1] is the least square method. In Octave (and MATLAB/Scilab, Python, IDL, etc.) there are routines available to do this type of fit for polynomials and other functions. Here we will detail the Octave method of doing one of these fits with the additional component of including error. Another fitting technique, the spline, will be demonstrated in the next section on FFTs.

- Here we display the results of an experiment along with there model fit establishing a relationship (the data and the script are below)



This is a plot of the data from an experiment with errors and the model from the least square fit on the data with errors (different for each model point). To get more detail open the image in another window (or run the program below in Octave). The relationship between the resonance frequency and the volume in the wine glass is $-(8.1152 \times 10^{-3} \pm 2.6242 \times 10^{-4})x^2 + (0.277000 \pm 0.050156)x + (954.88 \pm 2.1531)$



This is a similar plot as on the left but with resonance freque versus height. The equation is $-(106.62 \pm 2.1685)x + (802.96$

- The code for the data and the fit with some notes in the comments

```
%
% Example of fitting from old experiment
%
% Empty wine glass
%
% Experiment 1 uses the volume of water which, because of
% the shape of the wine glass, was
% not linearly proportional to the height of the water from the stem
%
% load the wpolyfit routines which will be used to
% get errorbars (standard deviation) on the fit to the data
% NB: Each point in the fit model has a different error - as it should be
% NB2: If you run wpolyfit without getting the output, you will get a
% graph of the data with the fit and a confidence interval...we don't
% want to do that here, but you might feel the need to do it elsewhere
%
% The optim package has
%
pkg load optim
%
% Experimental values (with a number of trials)
%
Empty = [954, 956, 955, 955, 956];
w40mL = [952, 952.5, 951, 952.4, 951];
w80mL = [925.5, 926, 926, 927.5, 926];
w120mL = [871.5, 872.5, 871.5, 872.5, 872.5];
w160mL = [790, 789, 789, 789, 788];
w200mL = [690, 688, 687, 689.5, 689.75];
%
% Experiment 2 uses height of water from the stem (zero centimeters means
% just wetting the wine glass on bottom)
%
w0cm = [798.2, 800, 800.5, 798.5, 798.8];
w1cm = [691, 691.8, 692.4, 692.2,  691.6];
w2cm = [592.4, 592.8, 592.2, 592.6, 592.8];
w3cm = [482.4, 482, 482.2, 482.4, 482];
%
% Experiment 3 used Corn Syrup...
% No need to show results
% Conclusion: never use corn syrup in an experiment like this
%
% Mean of experimental value and standard deviation which will be used for error
%
% Experiment 1
%
mempty = mean(Empty);
smempty = std(Empty);
m40 = mean(w40mL);
sm40 = std(w40mL);
m80 = mean(w80mL);
sm80 = std(w80mL);
m120 = mean(w120mL);
sm120 = std(w120mL);
m160 = mean(w160mL);
sm160 = std(w160mL);
m200 = mean(w200mL);
sm200 = std(w200mL);
%
% Experiment 2
%
mw0cm = mean(w0cm);
sw0cm = std(w0cm);
```

```
mw1cm = mean(w1cm);
sw1cm = std(w1cm);
mw2cm = mean(w2cm);
sw2cm = std(w2cm);
mw3cm = mean(w3cm);
sw3cm = std(w3cm);
wvol = [mempty,m40,m80,m120,m160,m200];
wvolstd = [smempty,sm40,sm80,sm120,sm160,sm200];
volume = [0,40,80,120,160,200];
%
% Experiment 1 graph
% Figure 1 (does not need to be expressly activated)
%
h1 = errorbar(volume,wvol,wvolstd);
set(h1,"marker",".");
set(h1,"linestyle","none");
%
% It is obvious that this will need to be an order 2 fit
%
[p,s] = wpolyfit(volume,wvol,wvolstd,2);
[wvoln,wvolstdn]=polyconf(p,volume,s,'ci');
hold on
plot(volume,wvol,'bo;Average Frequency;');
plot(volume,wvoln,'r.-;Curve Fit;');
xlabel('Total water volume (mL) in the wine glass ');
ylabel('Resonance Frequency (Hz)');
axis([-10,210])
h2 = errorbar(volume,wvoln,wvolstdn);
set(h2,"color","red");
set(h2,"marker",".");
% axis([650,1000,-50,250])
title("Wine Glass - Volume","fontsize",12);
hold off
%
% The equation:
%
dp = sqrt(sumsq(inv(s.R'))'/s.df)*s.normr
equation = p
%
% So we have -(8.1152e-3 +- 2.6242e-4)*x^2 + (0.277000 +- 0.050156)*x + (954.88 +- 2.1531)
%
% Experiment 2 graph
% Figure 2 does need to be expressly activated
%
figure(2);
wcm = [mw0cm,mw1cm,mw2cm,mw3cm];
wcmstd = [sw0cm,sw1cm,sw2cm,sw3cm];
height = [0,1,2,3];
%
h1 = errorbar(height,wcm,wcmstd);
set(h1,"marker",".");
set(h1,"linestyle","none");
%
% Issue; is this linear or a polynomial?
% We will need to do a chi^2 test to see which is better...
% F distribution is used to test hypothesis (see help of wpolyfit)
%
% Running this
% wpolyfit(height,wcm,wcmstd,1)
% will give you a graph and the polynomial;
% but the method below gives you control...
%
[poly1,s1] = wpolyfit(height,wcm,wcmstd,1);
[poly2,s2] = wpolyfit(height,wcm,wcmstd,2);
```

```
F = (s1.normr^2 - s2.normr^2)/(s1.df-s2.df)/(s2.normr^2/s2.df);
prob = 1-fcdf(F,s1.df-s2.df,s2.df)
%
% prob =  0.42651  This is definitely not less than 0.01 so we cannot reject
% lower order polynomial...
% so have some confidence in saying this linear
%
% Degree 3 is automatically rejected since there are only 4 points
%
% This data will not get a useful chi square because it is too little data
%
[p,s] = wpolyfit(height,wcm,wcmstd,1);
%
[wcmn,wcmstdn]=polyconf(p,height,s,'ci');
hold on
plot(height,wcm,'bo;Average Frequency;');
plot(height,wcmn,'r.-;Curve Fit;');
xlabel('Height of added water in the wine glass (cm) ');
ylabel('Resonance Frequency (Hz)');
axis([-0.1,3.1]);
h2 = errorbar(height,wcmn,wcmstdn);
set(h2,"color","red");
set(h2,"marker",".");
% axis([-1,4,400,900]);
title("Wine Glass - Height","fontsize",12);
hold off
%
% The equation:
%
dp = sqrt(sumsq(inv(s.R'))'/s.df)*s.normr
equation = p
%
% So we have -(106.62+-2.1685)*x + (802.96+-5.5370)
%
```

Fitting as shown in this part is commonly done in laboratory along with other techniques.

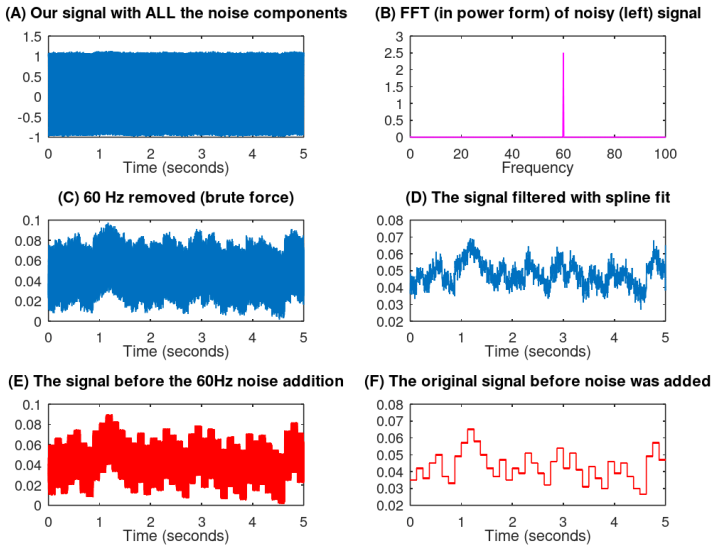
FFT

The Fast Fourier Transform (FFT) is the method of taking a Fourier transform in computers. The theory of these can wait until later courses, but we can still use the function as it is readily available in Octave (and MATLAB/Scilab, Python, IDL, etc.). As stated previously in other sections the Fourier transform transforms a time-space signal into a frequency-space signal and this transformation results in a different view of the data. For the purposes of this section, the FFT's main purpose for engineers is to extract hidden signals/data from time dependent signals and utilize that information for analysis or even cleaning up noise.

Example (FFT and some fitting as well)

For this example we create a program that generates some "perfect" data and then adds random noise to it plus a 60 Hz noise (which itself has random noise). With this very noise data we use the FFT and a spline fit to try and clean the data to see if we can find the original data. Note that while we know we added a 60 Hz noise we want to pretend that we do not have prior knowledge of what noise has corrupted our data. Instead we will see if there is any noise on our data with a specific frequency characteristic. *BTW, why 60 Hz?* This question is for you to answer.

- Here we display the results where (A) is the data we get from the instrument, what do we see in (A)?



- (B) is our data when we apply an FFT (actually a power density spectrum)
  - Here we can see that there is a 60 Hz signal which we did not want so we define that as noise - pretty obvious isn't it?
  - This "noise" was certainly not obvious in (A)
- (C) is our data with the 60 Hz removed just by zeroing out that component (we could do better but it obscures the topic)
  - Note we do this because the 60 Hz is noise, if it was not noise then the FFT analysis (B) above would be enough
  - Just zeroing out the component has as an undesirable side effect of zeroing out any actual signal we are interested in
- (D) is our spline fit that in this case acts as a filter
- With (D) we can actually see some data which we could not see in (A)
- (E) and (F) are the actually data that we generated (which in a real world situation we would not know)
  - (E) is the data with random noise added but not the 60 Hz noise
  - (C) and (E) compare favorably though (C) is still pretty noisy
  - (F) is the original pristine signal created with zero noise (which is NOT how reality works)
  - (D) and (F) compare favorably
- We have used some of our previously described techniques to take a very noisy signal (A) that seemingly is useless and through analysis (B) and a cleaning process (C and D) produce a reasonable signal (D)
  - Caveat: Any cleaning procedure will lose some of your signal so the engineer or scientist has to be very gentle on the data
  - This is a little bit like art restoration where you have to look at each piece carefully and slowly clean it without destroying the art
- Example program is below...

```
# For those who saw the old page: This works now...yes!
# This is an example script to produce a signal with two types of noise on it: Random and
# This is not a function because it is a one time use example
#
# We choose a sample frequence of 8000 samples/second because this program was originally
# that octave uses for its record (for recording sound)
#
# So here note the sampling of the data and the time (5 secs) of the data
#
fsample = 8000;
time = 0:1/fsample:5-1/fsample;
#
# Here we produce a brute force data signal with noise
#
 x1 = 0.01*ones(1,1000);
 x2 = 0.017*ones(1,1000);
 x3 = 0.011*ones(1,1000);
 x4 = 0.02*ones(1,1000);
 x5 = 0.025*ones(1,1000);
```

```
 x6 = 0.012*ones(1,1000);
 x7 = 0.008*ones(1,1000);
 x8 = 0.024*ones(1,1000);
 x9 = 0.032*ones(1,1000);
 x10 = 0.04*ones(1,1000);
 x11 = 0.033*ones(1,1000);
 x12 = 0.025*ones(1,1000);
 x13 = 0.017*ones(1,1000);
 x14 = 0.024*ones(1,1000);
 x14 = 0.012*ones(1,1000);
 x15 = 0.022*ones(1,1000);
 x16 = 0.01*ones(1,1000);
 x17 = 0.017*ones(1,1000);
 x18 = 0.014*ones(1,1000);
 x19 = 0.026*ones(1,1000);
 x20 = 0.02*ones(1,1000);
 x21 = 0.014*ones(1,1000);
 x22 = 0.007*ones(1,1000);
 x23 = 0.021*ones(1,1000);
 x24 = 0.029*ones(1,1000);
 x25 = 0.017*ones(1,1000);
 x26 = 0.026*ones(1,1000);
 x27 = 0.016*ones(1,1000);
 x28 = 0.006*ones(1,1000);
 x29 = 0.018*ones(1,1000);
 x30 = 0.011*ones(1,1000);
 x31 = 0.005*ones(1,1000);
 x32 = 0.021*ones(1,1000);
 x33 = 0.014*ones(1,1000);
 x34 = 0.02*ones(1,1000);
 x35 = 0.012*ones(1,1000);
 x36 = 0.005*ones(1,1000);
 x37 = 0.0015*ones(1,1000);
 x38 = 0.024*ones(1,1000);
 x39 = 0.032*ones(1,1000);
 x40 = 0.022*ones(1,1000);
 xcomp =[x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12,x13,x14,x15,x16,x17,x18,x19,x20,x21,x22,x2
 #
 # We will add noise with a random function, note it is only plus not minus so we have
 # actually changed the signal itself as well. Since this is a demo we will just assume
 # the we have added a component to the signal so the "true" signal is xcomp +0.025
 x = xcomp + 0.05*rand(1,40000);
 truexcomp = xcomp + 0.025*ones(1,40000);
#
# Now we produce the 60 Hz noise interferance
#
elec_noise = cos(2*pi*60*time) + 0.05*rand(1,40000);
sig = x + elec_noise;
#
subplot(321)
plot(time,sig)
title("(A) Our signal with ALL the noise components")
xlabel("Time (seconds)")
nlength = length(sig);
fftsig = fft(sig);
#
# Rids us of the two-sided
# portion of the fft
#
fftsig2 = fftsig(1:nlength/2+1);
#
# The first part is a normalization (though not quite)
#
psdsig = (1/(fsample*nlength)).*fftsig2.*conj(fftsig2);
```

```
#
# Multipling by two to ensure conservation of total power; plus pull out DC
# component
#
psdsig(2:end-1)=2*psdsig(2:end-1);
freq = 0:fsample/nlength:fsample/2;
#
# Test plot
#
subplot(322)
plot(freq,psdsig,"color","magenta")
title("(B) FFT (in power form) of noisy (left) signal")
xlabel("Frequency")
axis([0,100])
#
# From the power density function we can see there is 60 Hz noise (which we knew,
# but in real analysis we would not know)
#
# Let us see if we can clear this noice out to get our signal back again by
# zeroing out the signal out large signal in the fft (not the psd...so we will have to
# do both sides).
# This is a brute force method for demostration purposes.
#
pos1 =  (60/fsample)*nlength+1;
pos2 = (nlength - (60/fsample)*nlength) + 1;
fftsigsub = fftsig;
fftsigsub(pos1:pos1) = 0.0;
fftsigsub(pos2:pos2) = 0.0;
#
# Inverse the FFT and correct for power
#
timesig = ifft(fftsigsub)/sqrt(2);
#
subplot(323)
plot(time,timesig)
title("(C) 60 Hz removed (brute force)")
xlabel('Time (seconds)')
#
# Ok this recovers the original signal but it is really noisy; lets filter it
# Or in others words fit it with a spline
#
# For a spline we need to decide a good knot number; this is quite difficult
#
# knots = linspace(0,5,400)
# knots = linspace(0,5,800)
# The previous knots removed more noise, but also removed some signal; this is
# a delicate balance
#
knots = linspace(0,5,1600);
thespline = splinefit(time,timesig,knots);
timesigfilt = ppval(thespline,time);
subplot(324)
plot(time,timesigfilt)
title("(D) The signal filtered with spline fit")
xlabel('Time (seconds)')
#
subplot(325)
plot(time,x,"color","red")
title("(E) The signal before the 60Hz noise addition")
xlabel('Time (seconds)')
#
# This is the made-up signal with no noise whatsoever (you will never find
# this is real life)
#
```

```
subplot(326)
plot(time,truexcomp,"color","red")
title("(F) The original signal before noise was added")
xlabel('Time (seconds)')
```

There are many different applications of FFTs and there are other useful transformations as well (like the Radon Transform) so this is just a brief introduction of all the possible analysis techniques and data recovery techniques. Now we will look at an FFT-based idea that is primarily for analysis.

Spectrogram (FFT-based function)

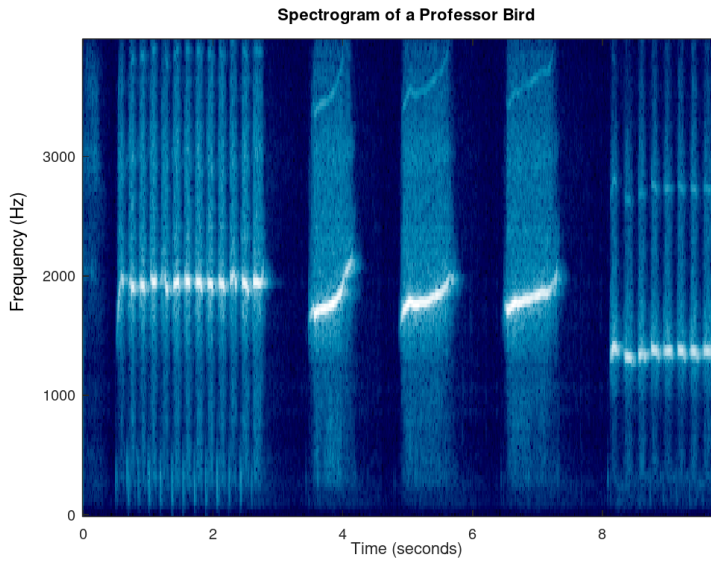
For analysis in a number of fields a spectrogram is a useful tool. Spectrograms can be used with any signal (wave) such as sound or electromagnetic waves. It is an image of what someone is hearing or seeing and is a convenient presentation that allows a engineer or scientist to see signals with a different view point.

This analysis method is used in many different fields such as speech analysis, bird identification[2], seismology, vibration analysis for materials, audio engineering, astronomy, etc. For a classic demonstration of the spectrogram we will analysis a bird call made by a professor.

Example

Because record sound requires different machine configuration this Octave program will have to be run, modified, and experimented with on your personal computer.

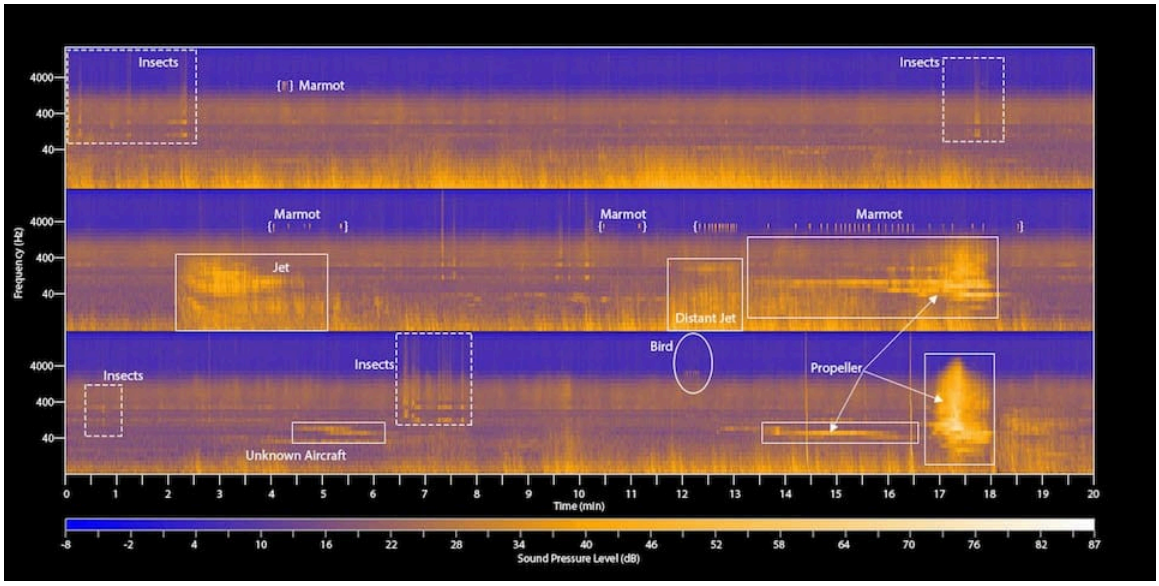
```
%
% Just a script for a demonstration of spectrograms
% Spectrograms are at their core an fft
%
% This should not be run as a script but be done
% line by line becuase you need to know when it
% is recording the sound..
% sound(profbird) will play the sound if wish to
% test the recording
%
colormap(ocean)
pkg load signal
profbird = record(10)
%
% Record in octave has a sampling rate of 8000
%
fsample = 8000;
[s,f,t] = specgram(profbird,256,fsample);
imagesc(t,f,log(abs(s)));
set(gca,"ydir","normal");
title("Spectrogram of a Professor Bird");
xlabel("Time (seconds)");
%
% Normally should use the ylabel, but it is intefering with the numbers
% So do a work around with the text function; ylabel does have a position
% option but it is as effective as text
%
%ylabel("Frequency (Hz)");
%
text(-1,1700,"Frequency (Hz)","rotation",90,"fontsize",12);
```



This scrip demonstrates how to make a spectrogram of sound for use in identifying birds (or other things). Because the *record* function is depending on machine configuration this Octave program will have to be run, modified, and experimented with on your own computer which shouldn't be difficult since Octave is free.

Spectrogram of the sound coming from the rare professor bird. A hun hear best between 1000 Hz and 5000 Hz[3] so you would anticipate th whistle would be in that range (which it is). Also notice the quick chirp followed by longer chirping.

- This is an example of a public domain image from the National Park Service of a spectrogram on their Soundscapes of Mount Rainier web site.



Spectrogram taken at Mount Rainier by the National Park Service. This spectrogram has been marked up with known sounds to help the general public understand some uses of spectrograms. There are more spectrograms at the web site linked above.

Final thoughts

This concludes are discussion of various techniques that engineers and scientists use computers for in their efforts to aid the world through knowledge and useful products. This is a small smattering of all the techniques used by engineers and scientists. A student should take a numerical methods course and signals and systems course (or a combination of them in the same course) to learn more and especially other important methods not discussed herein.

---

[1]Regression and curve fitting can, to some, mean the same thing and to others mean a different thing. Fitting as described here is not drawing a line from data point to data point, but more a regression to produce the best model (curve) of the data. If we wish to delve into the debates of the two subjects, please feel free to use you favorite web search to wade into the thorny issues of definitions.

[2]In ornithology the spectrogram is also called the sonogram (which can be confused with the medical sonogram which is not the same idea). Ornithology has been using this technique for decades and there are many examples available including apps that describe the sound of a bird usually accompanied with a spectrogram.

[3]The full range of human hearing is from about 20 Hz to 20000 Hz though this varies depending on genetics and age.

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19: USING COMPUTERS FOR ENGINEERING AND SCIENCE

19.1: Engineering and Science Languages
19.2: Programming Languages for Hardware

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19.1: Engineering and Science Languages

Here we present some of the programming languages that are useful for engineering and science.

IDL

This is a science language primarily used by astronomers but can be used by engineers. It's design centers around image processing for science. With that in mind it has a matrix-style ability similar to Octave. We will introduce IDL here and give an example using one of the routines from NASA/GSFC.

IDL is primarily used at NASA/GSFC, NIST, and NIH. At NASA/GSFC, it mostly supplanted the previous scientific image processing language, IRAF, though there is still some people who still used IRAF instead of IDL. The language has a host of routines available which we will detail later. GDL is a open source programming language that is highly compatible with IDL and can be used as an alternative to IDL especially in academic areas where financing is an issue. While GDL is not as powerful as IDL it offers a number of features that IDL offers in a similar syntax very much like Octave is to MATLAB.

- IDL/GDL is matrix-based (which is how image processing is done) like Octave/MATLAB/Scilab
- PV-WAVE comes from IDL but has evolved from it but it still has the looks of IDL
- In comparison IDL and GDL are almost alike like Octave and MATLAB, PV-WAVE diverges enough to be distinct but very similar like Scilab does

Routines for science and engineering

For IDL to be used effectively you would need routines that have been developed previously. This is no different then Python or Octave except that IDL does not have one general place to get routines but many web sites. Here is an incomplete list.

- David Fanning's IDL site - this is one of the top IDL web sites, however it is not as active as it once one
- https://people.ast.cam.ac.uk/~vasily/idl.htm - general site that offers some information and links to other web sites
- https://michaelgalloy.com/wp-content/uploads/2006/10/operators.pdf - Useful pdf of the operators in IDL
- NIH
  - Unfortunately most of NIH software is protected and needs a password unlike NASA/GSFC which is more open
  - NIH uses IDL but also uses MATLAB and Mathematica
- NIST
  - DAVE is an analysis environment developed on top of IDL
- NASA and NASA-related sites
  - The IDL Astronomy User's Library (also know as IDLASTRO - can download routines from github - Wayne Landsman's area)
  - DanDL - Science and Astronomy
  - WMAP software - actually a mixture of Fortran and IDL
  - RVLIN - exoplanet software
  - NICMOSlook - this is pretty old IDL code and program, but it can be used as an example (and if you want to analyze NICMOS data)
  - "SGP-IDL" program - the IDL program is on the bottom of the page (MATLAB programs are on the top)
  - JHU-APL library page - useful library of IDL routines that is heavily used by NASA

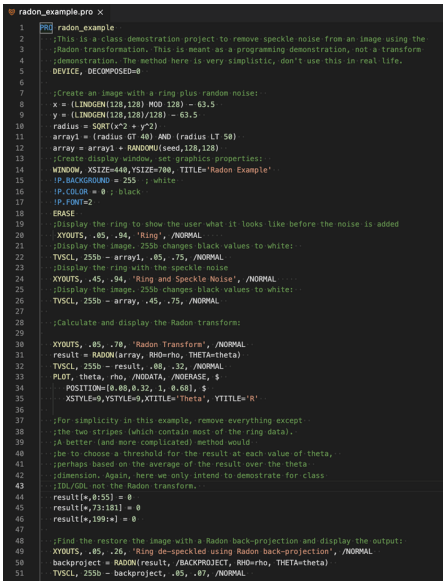
And now an example for IDL, but first we need to introduce a transform, Radon Transform. The Radon transform is an integrating transform that is typically used in tomography (imaging by sections) but is also used in barcode scanners, seismology, and in removing certain types of noise in imaging which includes astronomy. This transform is related to the Fourier transform which as you know is one of the most important transforms in engineering and science (remember FFT). In simple terms it reconstructs an image from scattered data pieces of a volume (hence tomography). Similar transforms include the Minkowski-Funk transform (Funk), the Hough transform (which is good for line detection unless your data is noisy with speckles then you should use the Radon transform), and the Penrose transform (which is just a complex version of the Radon transform).

All this might sound a bit technical at this point but as time goes on and you learn more you will be amazed how this becomes a sort of second nature. Below is a GDL script that utilizes the Radon transform from IDLASTRO. But before that let us review what

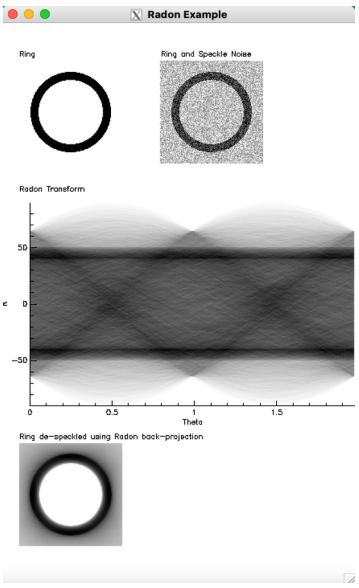
we are going to do.

Example

1. Generate a ring with speckle noise - this will represent our noisy data
2. Take the Radon transform (RADON) of the data and display that (just to show you)
3. Remove everything in the transform except the stripes (this is not the best approach - better approaches would obscure the message though)
4. Using the Randon back-projection reconstruct the image de-speckled
5. This is run in gdl so you can try it to if you get the open source program!



GDL programming demonstrating some image processing as might be found in science/engineering. Examine this program thoroughly and try to understand each command (use web sources to help if you wish - that is what would be done in the "real world").



This is the result of the GDL program on the speckled noise test algorithm. As you can see the speckle noise is removed, however the background seems to have become noisy on the edges (though not the center). Improvement of this process can be used to de-speckle noises in astronomy images or medical images without the noise that is still in the result here.

Python

Python is technically not a scientific programming language, however, through packages that can be obtained at the Python Package Index (pypi not to be confused with pipi which is another type of Python) you can use it as a scientific language. Octave/MATLAB/Scilab also have packages for additional scientific abilities though they have a lot more built-in then Python. These packages are a **must** if you want to do scientific/engineering work with Python.

Packages

Python has "modules" which contain pre-defined functions that our collected together to make package. These packages are essential for useful work with Python. For Python and in particular for science you would need the following packages at a minimum:

- Matplotlib - Python plotting package that you really need to do any plotting
- NumPy - Python package for computation especially with matrices
- SciPy - Python's scientific library (note SciPy has an "ecosystem" and has it's own web site: SciPy.org)
- Check out the pypi and scipy for more packages

Examples

- This is simple examples of some of NumPy's abilities.

```
import numpy as np
a = np.array([1,2,3],float)
b = np.array([0,1,1],float)
adotb = np.dot(a,b)               # This is a dot product of arrays a and b
#
a = np.array([[0,1],[2,3]],float)
b = np.array([[1,1],[4,0]],float)
astarb = np.dot(a,b)               # The dot product is also matrix multiplication
#
a = [1,2,3]
b = [4,5,6]
acrossb = np.cross(a,b)             # There also is a cross product command
#
# System of equation solver
# 3x + y = 9 and x + 2y = 8
#
a = np.array([[3,1],[1,2]])
b = np.array([9,8])
xy = np.linalg.solve(a,b)          # Solving a system of equations
print("The dot product is ",adotb)
print("Matrix multiplication is ",astarb)
print("The cross product is ",acrossb)
print("The solution to the equations are ",xy)
```

- A simple example using NumPy and Matplotlib.

```
import numpy as np
import matplotlib.pyplot as plt
t = np.arange(256)
sn = np.sin(t)                 # Sine for a range of numbers
sp = np.fft.fft(np.sin(t))     # Fast Fourier Transform (FFT) of Sine
freq = np.fft.fftfreq(t.shape[-1])
psd = sp*sp.conjugate()
plt.plot(freq,psd.real,marker='.',linestyle='none')
plt.title('PSD of Sine')
plt.xlabel('Frequency')
plt.ylabel('PSD')
plt.show()                     # If you just plot it doesn't show unless you do this!
```

- A simple example using SciPy. Here we have our data, a linear interpolation, and fit using a cubic spline. This complements the previous discussions by showing an example of fitting to a curve. The spline is very close to the data (which is a function that can be graphed if you want to see what it should look like...your assignment).

```
# 1-D interpolation using SciPy
#
import numpy as np
import matplotlib.pyplot as plt
from scipy.interpolate import interp1d       # note we don't import all of spipy
x = np.linspace(0,10,num=11,endpoint=True)   # Octave/MATLAB uses linspace as well
y = np.cos(-x**2/9.0)
f = interp1d(x,y)                            # Linear interpolation
f2 = interp1d(x,y,kind='cubic')              # Cubic spline
xnew = np.linspace(0,10,num=41,endpoint=True)
plt.plot(x,y,'o',xnew,f(xnew),'-',xnew,f2(xnew),'--')
plt.title('Our data and the cubic fit to our data')
plt.legend(['our data','linear interpolation','cubic spline'],loc='best')
plt.show()
```

Octave

Octave, MATLAB, and Scilab are science languages primarily used by engineers but can be used by scientists. These languages are designed to easily do matrix mathematics which is very important in engineering. Octave and MATLAB are almost completely compatible whereas Scilab is mostly compatible but does have some differences. MATLAB as a paid project is more functionality then either Octave or Scilab, but Octave or Scilab is excellent for those on a budget.

While a lot of science features are included in the core program for these languages so there is less need to go to packages (or toolboxes as MATLAB and Scilab calls them). However for some special functions these programs do have packages.

Octave Forge Packages

Octave packages can be found in many locations, but the best list of packages is at Octave Forge. For MATLAB and Scilab it is best just to go to the website and search for toolboxes. Note you have to pay for the toolboxes as well as the product for MATLAB. Scilab like Octave is open source so that is not an issue with that software.

Key Packages:

- octave-optim — non-linear optimization tools, but in particular wpolyfit which is used in laboratory herein
- octave-control — control system design functions; transfer function creation; **bode plot**
- octave-signals — signal generation and signal analysis tools includes IIR filter design routines; this is an absolute must
- octave-splines — splines are an extremely important tool; this package is a must
- octave-linear-algebra — linear algebra is need for control systems
- octave-mechanics — functions useful for mechanics included in this is some molecular dynamics routines (which is not exactly mechanics)
  - octave-quaternion — quaternion procedures that were found in mechanics originally but have since split off and expanded
- octave-symbolic — this gives octave the ability to manipulate symbols like MATLAB using Python's SymPy
  - This is not a must; this is just if you want to do symbolic mathematics
  - The authors believe that maxima would be better for this though; use wxMaxima/maxima instead
    - wxMaxima/maxima is like Mathematica however it is older then Mathematica
    - maxima's origins are from the 1960s (Macsyma:MIT:Lisp-based)
      - Interestingly because Macsyma required computers that were more expensive than the general user could afford a competitor to it was developed in the 1980s called Maple (University of Waterloo:B programming language)
      - As you might expect B was before the C programming language
      - Now however maxima can be run on laptops; while Maple still exists its original motivation is no longer relevant

Example

Example of octave has already been done extensively throughout this course/book. See the previous examples. May add one later here, but not for now.

Fortran (and C and C++)

Fortran is a programming language designed for science but it depends on libraries extensively to provide the best science functions (this is true of C and C++ as well). These libraries are like the packages of the interpretive science languages like Octave/MATLAB/Scilab or IDL/GDL. Frequently, the compiled libraries of Fortran and C/C++ are the basis of the packages in the interpretive science languages.

If these libraries are the source of most packages out there why not just go straight to the source? Because for most people these libraries are more complicated (less user friendly) and take extra effort to learn (which is valuable however). Some advantages of the libraries is that by editing and compiling these routines you can, with learned skill, speed up your programs and even add special cases that are not generally represented in routines that you might need. For the interpretive science languages you might be stuck with what you get. Still most people don't have major special cases and the ease of use makes the interpretive science languages' packages attractive.

This is where we would go back to numerical methods because most of the libraries/packages are based off the concepts in numerical methods.

Numerical Methods aspect of libraries

Numerical methods boil down to methods of approximating physical problems that are intractable using analytic techniques. Key ideas:

- Physical problems are represented by mathematical equations
  - Parachute person was an simplistic example of this
  - For real world problems you are likely to have multiple equations and following the systems example in the fundamental portion of this course we can see that almost any real world problem can be represented by this
    - Or to put it another way n unknowns and n equations
    - In order to solve n unknowns and n equations we would resort to matrices (hence the popularity of Octave/MATLAB)
    - In real world problems these n equations are actually differential equations, but the same matrices principles apply
- A number of different approximation techniques are available for the same physical expressions and are used depending on the conditions
- These techniques do have methods of assigning error
  - Assigning error can be a difficult task
  - This is an area where you might want to modify a program to fit the error analysis you believe is appropriate (when you are an expert!)
- Any of the routines found in the packages can be designed and written from scratch, however, these have existed and been regularly corrected for a long time so unless you truly have some evidence that there is a problem, best just use the libraries

Sample of numerical methods packages

Most of the numerical methods packages commonly used can be found on the NETLIB web site (either on the site or a link to another site - usually NIST). These libraries are written in Fortran, C, and C++. This does not mean that only Fortran libraries can be read by Fortran. C can read Fortran and C++ libraries, Fortran can read C and C++ libraries, and C++ can read Fortran and C libraries. You may need to take some extra care doing this, but most modern compilers make this rather easy.

Originally most of these libraries were written in Fortran but now most have been convert to C (and C++). If you want to investigate these programs it is probably better to look at the descriptive list at http://www.netlib.org/master/expanded_liblist.html.

Key libraries:

- EISPACK — Eigenvalue and eigenvector library (Fortran, note LAPACK has most of these routines now)
- LINPACK — Linear algebra pack (Fortran, note LAPACK has most of these routines now, with improvements)
- LAPACK — Modern linear algebra pack
- MINPACK — Non-linear problems
- NAPACK — Numerical linear algebra and optimization pack (Fortran)
- ODEPACK — Differential equations package
- QUADPACK — Integration package - generally incorporated into interpretive programs now
- FFTPACK — FFTs, of course
- SLAP — Sparse Linear Algebra Package
- BLAS — Basic Linear Algebra subprograms

GNU has rewritten a lot of these packages (in C and C++; but given binding abilities of modern languages they can be used in the language of your choice) and more to form the GNU scientific library (gsl). If you look at the reference manual for gsl you will see references to NETLIB. The GNU scientific library is extensive.

Examples

- Fortran using a routine from NETLIB

```
exampleprograms % gfortran -c ei.f
exampleprograms % gfortran ei.o -o ei
exampleprograms % ./ei
 EI(X)     14.505416450711849
 EONE(X)     6.4719102533039976E-003
 EXPEI(X)    0.41251699584804258
tigris@Carols-MBP exampleprograms %
```

This is a Fortran example program using a NETLIB subroutine to do some engineering/science calculation. Note the compiling and linking of Fortran to produce the program *ei*, before we run it (./ei). We did not use a library here, but if we did we would need either a static library (*.a) or a dynamic library (*.so). Fortran includes a lot of science functions by default so the need to link in a library may not be needed in most instances. These default functions are referred to as **intrinsic procedures**.

- C using gnu scientific library (gsl)

```
exampleprograms % gcc -c -I /opt/local/include bessel_test.c
exampleprograms % gcc -L /opt/local/lib bessel_test.o -lgsl -o bessel_test
exampleprograms % ./bessel_test
J0(1) = 7.651976865579666054e-01
J0(2) = 2.238907791412356185e-01
J0(3) = -2.600519549019334464e-01
J0(4) = -3.971498098638474028e-01
J0(5) = -1.775967713143382642e-01
J0(6) = 1.506452572509969778e-01
J0(7) = 3.000792705195555188e-01
J0(8) = 1.716508071375539013e-01
J0(9) = -9.033611187611125e-02
J0(10) = -2.459357644513483765e-01
J0(11) = -1.711903004071960999e-01
J0(12) = 4.768931079683355612e-02
J0(13) = 2.069261023770677665e-01
J0(14) = 1.710734761104586721e-01
J0(15) = -1.422447282678075166e-02
J0(16) = -1.748990739836291941e-01
J0(17) = -1.698542521511835490e-01
J0(18) = -1.335580572198409935e-02
J0(19) = 1.466294396596511951e-01
tigris@Carols-MBP exampleprograms %
```

```
#include <std
#include <gsl
int
main (void) {
    double p;
    double pp;
    double y;
    int i;
    p = 1.0;
    y = gsl_sf
    for(i = 1;
        {
            pp =
            y =
            prin
        }
    return 0;
}
```

This is the run on the C program on the left. Before that is the compiling of the program and the linking of the program. The first line is compiling of the program requires the author to tell the compiler where the gsl include file was (-I /opt/local/include). The second line is the linking of the program. The gsl library was linked by option -lgsl and again the author had to tell the compiler where the gsl library was (-L /opt/local/lib). Note the compiling and linking can be combined, but the author likes to check his syntax first before he links (otherwise it is harder to distinguish between syntax errors and linking errors).	This is a simple function at diffe only minor modi to use C++. Not scientific library we get the Bess need to connect through the com

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19.2: Programming Languages for Hardware

This section deals with software that is used for hardware design or access. Within this discussion there is a small discussion on html to help demonstrate a point about another language. In no way is learning html part of this course, it is just a means to an end herein.

SPICE and digital design languages

SPICE is the language of circuit design. It is the basis for all analog circuit simulators. There are both paid for products and open source products all based off the original Spice program. SPICE stands for Simulation Program for Integrated Circuit Emphasis. Here we will give a historical outline of SPICE following along the lines of our other history for programs.

- Created at UC-Berkely around 1975
  - Originally written in Fortran
  - Last version (SPICE3) was written in C
- Derived from CANCER (Computer Analysis of Non-linear Circuits, Excluding Radiation)
  - Radiation was excluded because of laws that would make anything dealing with radiation a domain of the Department of Defense (DOD)
    - Think 1960s and flower children
  - From a 1971 paper by Ronald A. Rohrer
- Uses a modified nodal analysis
  - The modified method allowed for circuits without an ideal source
  - Linear simultaneous equations (*How would we solve that in say Octave?* You answer.)
- Uses Newton-Raphson for non-linear components
- Transient analysis uses trapezoidal (or *Gear* if chosen) integration
  - Numerical methods course should be taken so you can understand these concepts...(hint)

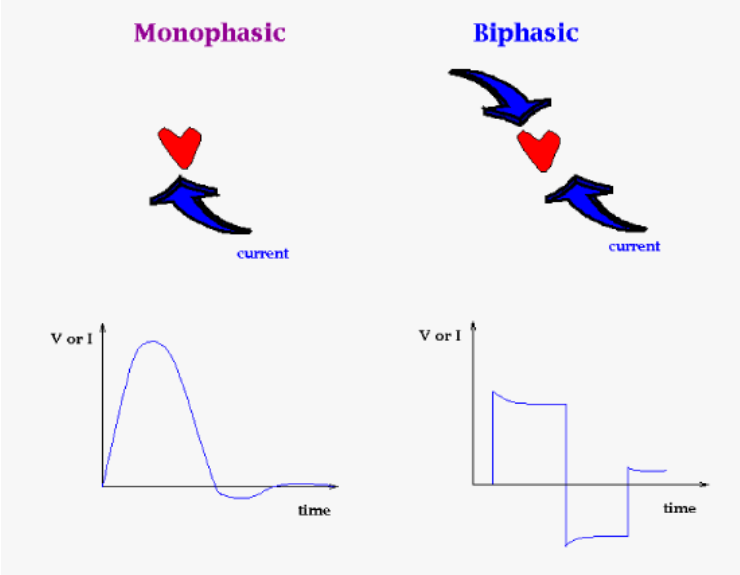
Spice simulators

This is a small selection of SPICE simulators available for "free" or for a price.

- SPICE — Spice3 from the original source
- ngspice — a part of the gEDA (GPL Electronic Design Automation tools)
- LTSpice
  - While a product of a business this program is free
  - Lots of preloaded elements are from Analog Devices but they offer it for free, so...
- TINA-TI
  - Two versions, one free the other paid for, but the free version is pretty loaded so you may not need the paid for one
  - From Texas Instruments
- CIRCUITMOD
- MacSpice 3
  - Version of SPICE 3 for Macs
- Circuit Construction Kit
  - This is for kids
  - Simulation through PhET at University of Colorado
    - PhET has lots of useful educational simulations
    - PhET simulations are for all ages (the one here is for kids but others are more advanced)
- MultiSim
  - National Instrument's circuit analysis program
  - Not free
- PSpice (OrCAD)
  - This is one of the more popular SPICE products
  - Not free

Example

Defibrillators are an important medical device that resuscitates a stopped heart (in certain conditions). Scientists and engineers worked on various versions of the defibrillator in the early 1900s but the origins of the device we use today comes from an electrical engineering in the 1930s. The device works by sending electric shocks to the heart, but why this works is still a matter of research. Since the simple defibrillator can be modeled rather simply by an electrical circuit we will use that as an example for Spice. Today there are two types of defibrillators in use: a monophasic and a biphasic defibrillator. The biphasic is considered to be more advanced but the monophasic is still used in situations where it is difficult to use the biphasic.

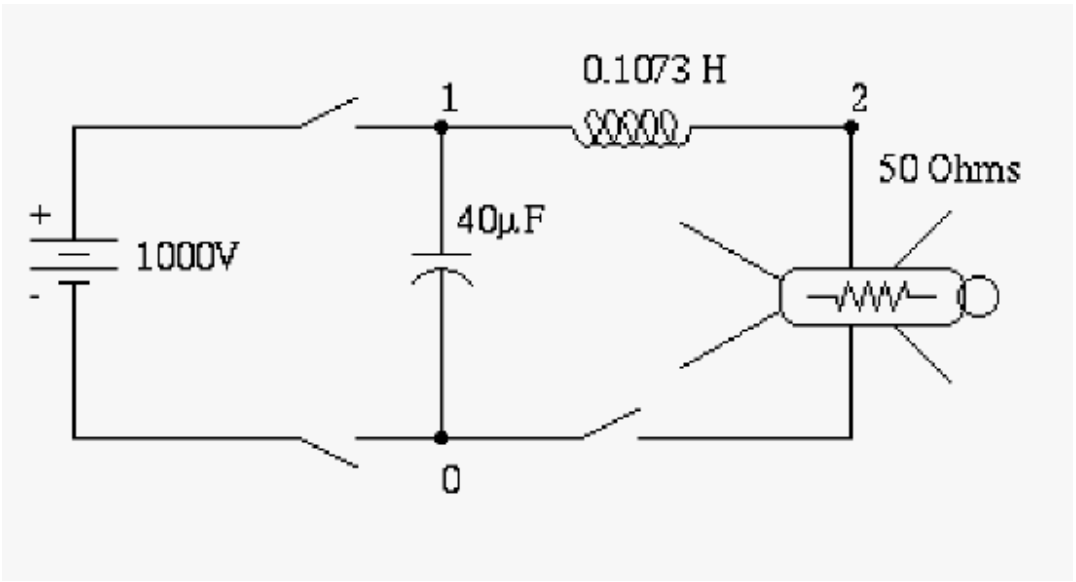


A monophasic defibrillator just sends current one way through the paddles that are on different sides of the heart whereas the biphasic sends current both ways. A sketch of the wave forms are show above. In our simulation we will try and produce a wave form that emulates the defibrillator. Since the somewhat square wave form of the biphasics will need additional circuitry that will obscure the programming message we will produce a more monophasic circuit and simulation (though it is not quite monophasic either).

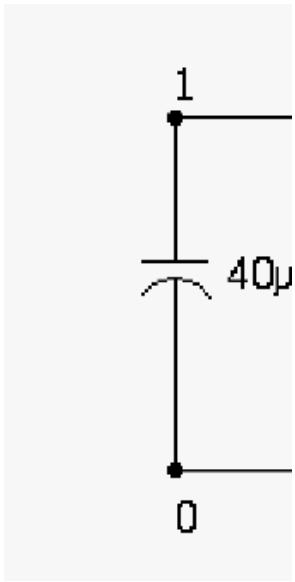
Since a defibrillator would be used on a human, we will need to simulate the human as a resistor. Outlined below are some real world considerations.

- Human resistance (or chest impedance)
  - A dry human is about a 100,000 Ω resistor
    - Too large a resistance
    - In order to make the defibrillator work we would burn the patient (in direct skin contact) at this resistance
    - Of course, we could open the patient and apply the paddles directly to the heart which was done in the past, but let's not go back to the past...the future is brighter if we want it to be
  - A wet human is about a 1000 Ω resistor
    - Better but still pretty large and this sounds dangerous
  - Old defibrillator concept inserted the lead into the human to reduce the resistance
    - A punctured human is about a 500 Ω resistor
    - Hmmm...still a large resistance and we are trying to avoid the past
  - A special conductive liquid could be put on the paddles to reduce the resistance
    - This can achieve a resistance of less than 100 Ω
    - This is the way
      - You might have seen in TV and movie medical dramas medical personal applying a wet gel on the paddles (though now there are gel pads and other advances)
      - With this we can simulate our human to be about a 50 Ω resistor

Let us now present a circuit that we believe will be good for our simple defibrillator.



A circuit for a simple defibrillator. The battery charges the capacitor which then acts as the power source (a capacitor will discharge fast). The human (cartoon) is represented by a 50 Ω resistor above.

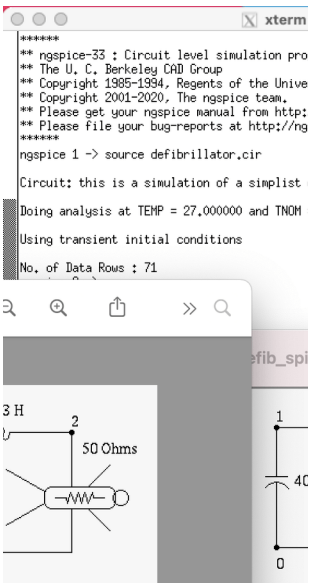


This is a circuit design for t
everyone who takes a seriou
That will be the voltage inpu
ic=1000V means that the c
avoids the switches and the
simulate those.

And now for the actual program and its execution.

```
defibrillator.cir
1   This is a simulation of a simplist deflibrillator
2   c1 1 0 40uf ic=1000
3   l1 1 2 107.3mh
4   r1 2 0 50
5   .tran 0.0005 0.03 uic
6   .control
7       run
8       plot V(2)
9   .endc
10  .end
11
```

This is the Spice program that represents the circuit discussed above. As you see it has a capacitor, inductor, and resistor (c1,l1, and r1).



This is a run of the program t
the defibrillator discussed be
discussed in your electrical c

Final thoughts for Spice

You might have noted that we simulated a human by a resistor. This idea which is reminiscence of the control system discussions in the Fundamentals section where physical objects can be represented by elements of a circuit. Given this it would be possible to use Spice to simulate any engineering system assuming we can represent it by elements of a circuit which BTW we can do

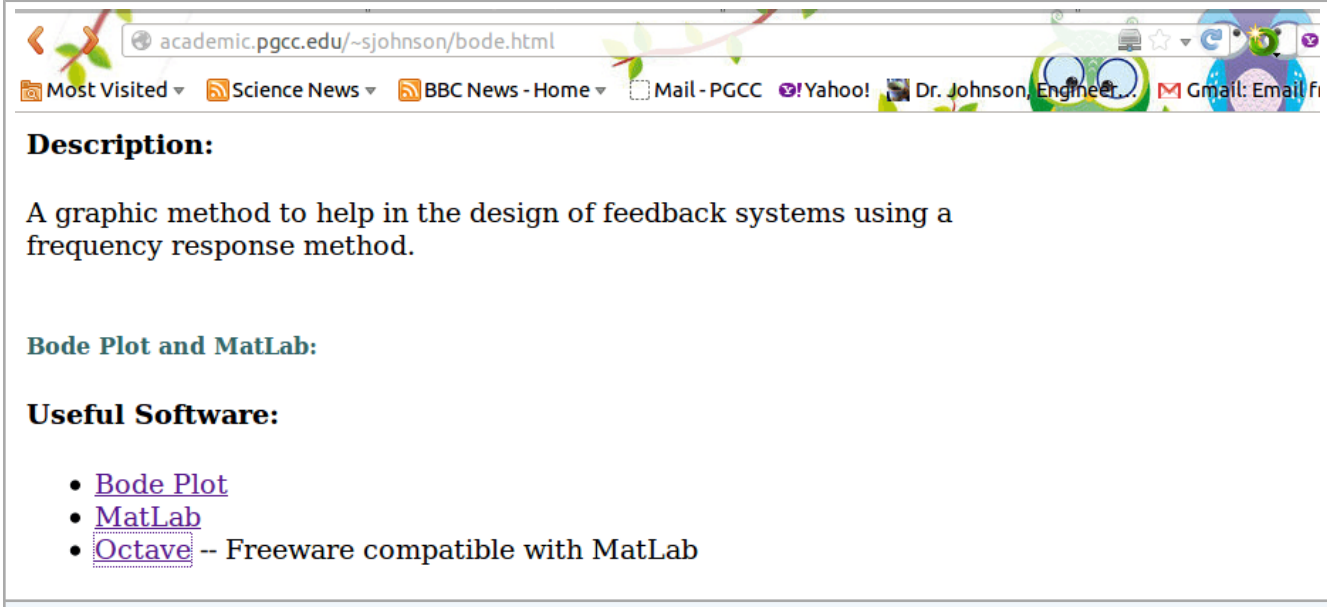
for most physical objects. Of course, Spice is not the only simulator for electrical circuits. Spice is for the analog circuits but todays circuits have a lot of digital components, so Hardware Description Languages (HDLs) were developed for digital circuit simulation. For complete circuit simulators the HDL engine is combine with the Spice engine.

Hardware Description Languages (HDLs)

HTML (as a prelude to HDLs)

HTML (HyperText Markup Language) has nothing to do with science and engineering, but it is a useful idea to examine in anticipation of a discussion on Hardware Description Languages (HDL). HTML is basically just text surrounded by tags that tell a software program (a browser) how a web page should be displayed on the screen. This is a lot like HDLs that describe the hardware for a program to simulate. That is text that has a specific syntax structure to emulate hardware (hence hardware description).

- Example of HTML



A simple browser page that is produce by the HTML code on the right. More complicated browser pages would have css templates and maybe javascript.

HDLs are used for modeling primarily digital circuits. Most programs meant to work with HDLs can also simulate the digital circuits. Analog components are generally simulated with Spice that can usually be linked to these programs. One program that used HDLs is the program Electric which is an open source program for use with VLSI and digital logic.

Let's take a look at the two most popular HDLs (note there are many HDL languages).

VHDL

VHDL is one a very popular HDL for modeling circuits (in particular digital circuits). Here is an outline of this language.

- VHSIC Hardware Description Language (VHDL)
  - Very High-Speed Integrated Circuits (VHSIC)
    - Useful for VLSI (Very Large Scale Integration)
    - Useful for ASIC (Application Specific Integrated Circuit)
  - A DOD standard originally to describe ASIC circuits (kind of like a hardware psuedo-code)
  - Borrows from the DOD product, Ada programming language
    - Heavy typing (can be difficult)

- Case insensitive

VHDL is the default language of Electric.

Verilog

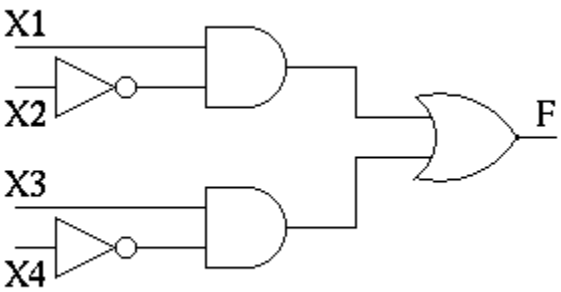
Verilog is a popular HDL because of its ease of use. Here is an outline of this language.

- Verilog is an HDL that was developed at about the same time as VHDL. Verilog is used to model electronic systems and to do verification of the circuit.
  - Portmanteau[1] of verification and logic
  - Borrows from C
  - Case sensitive

Verilog can be easily used by Electric, but is not default. Other HDLs are much harder to use with Electric.

VHDL example

Here we will give a simple example of the VHDL language. We will do this with a very simple logic circuit which is commonly used in digital circuit analysis.



This is a simple logic circuit that would be could be the basis of a digital circuit. It has two inverters (0 to 1 or 1 to 0) for inputs X2 and X4, two ANDS for X1 and ~X2 plus X3 and ~X4 and an OR for the results of the AND. This circuit could also be used in a logic class as well, btw. You can learn more about this in your digital circuits class if you chose to take that course.

This simple logic can be written in VHDL in many ways.

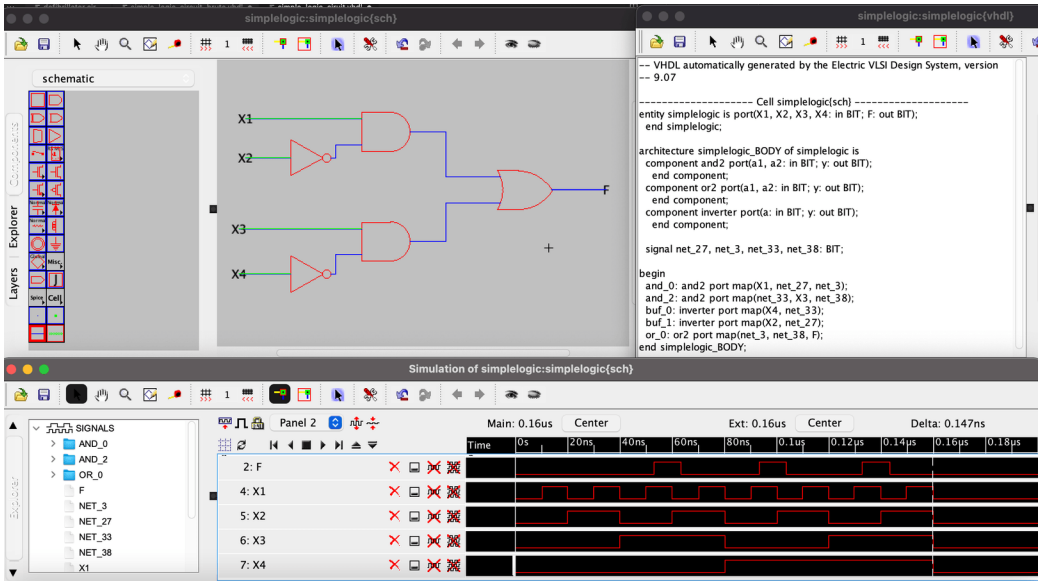
```vhdl
-- VHDL design of four port logic with two inverted ports; AND then OR
-- This method uses brute force; and, or, and not
-- This is does not include the use of this architecture
--
library IEEE;
use IEEE.std_logic_1164.all;
entity simple_cir is port (X1,X2,X3,X4: in std_logic; F: out std_logic);
    end simple_cir;
architecture logic_circuit of simple_cir is begin
    F <= (X1 and (NOT X2)) or (X3 and (NOT X4));
end logic_circuit
```

VHDL of simple logic circuit presented as is...brute force if you wish. The NOT, AND, and OR are incorporated in the equation. The output is F. This is very consistent with the ADA programming language.

```vhdl
-- VHDL desig
-- This metho
-- This is do
--
library IEEE;
use IEEE.std_
entity simple
    end simple
architecture
    F <= '1' w
         '1' w
         '0' ;
end logic_cir
```

VHDL of simple logic cir

Still another way of expressing the VHDL is the generated VHDL from the following electric example of this simple logic circuit.



This is the simple logic circuit created in the program Electric (linked to previously). The waveforms for of the circuit are presented here as well for completeness. The VHDL generated by Electric is shown in the upper right corner. It is more "wordy" then the hand written VHDL presented before. This is usually the case with program generated code. Electric is a program that can be used to learn VHDL by example.

LABVIEW (versus Simulink or XCOS)

National Instrument's LabVIEW is a IDE for general controllers generally in a professional setting. LabVIEW takes a graphical approach to programming (technically it is a Visual Programming Language (VPL) like Logo). LabVIEW and the G programming language (not to be confused with G-code which is very old command control language) are synonymous. LabVIEW is a *preferred language of engineers* for interacting wiht controllers and measurement devices through the General Purpose Interface Bus (GPIB or IEEE-488) because of its ease of use and extensive options. While LabVIEW is not the only programming language that can

do this as you could directly use C/C++, Fortran, or use a package from Octave/MATLAB (Instrument Control) or Python (pyVISA), it is the easiest.

Simulink and XCOS are for modeling control systems graphically (VPL) rather than directly interacting with controllers. However using these modeling system with MATLAB or Scilab and then using their separate packages for instrument control may be desirable for some. Note in Octave you can do all this but it will not be graphical, but it is free.

Example LabVIEW

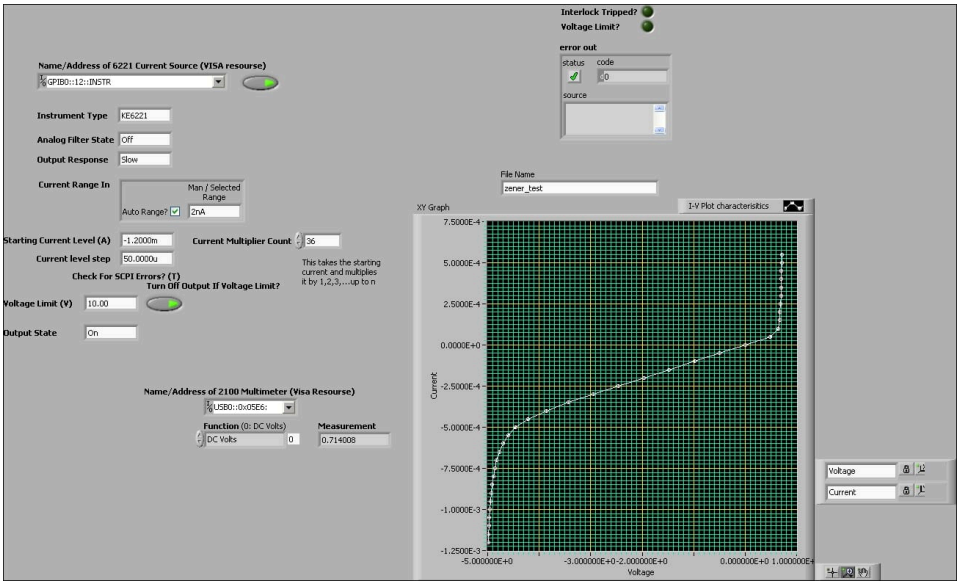
This example is an example of controlling the input into a diode and reading the output.

- Before showing the program we will show the set-up for a testing of a Zener diode using instruments controlled by the LabVIEW program.



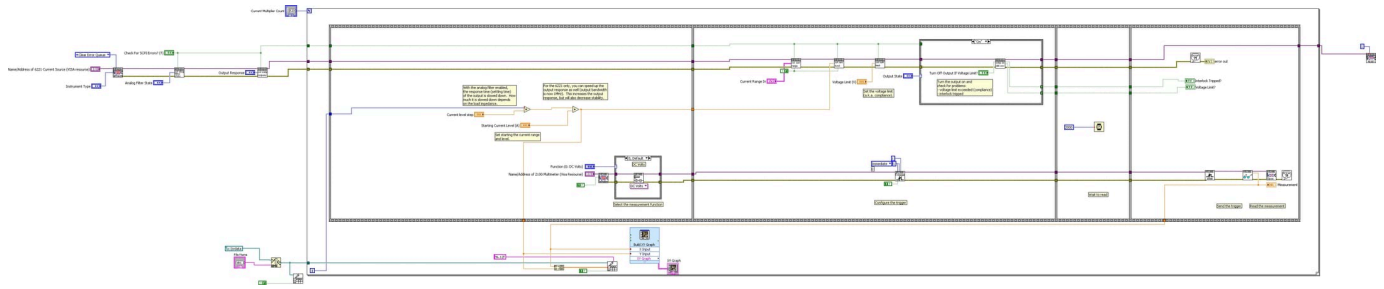
Here is the equipment that the LabVIEW program controlled for the simple Zener diode test. The actual circuit is behind the computer on a breadboard. *Note: These instruments are overkill for this type of test as this test was meant to ensure operation, not the actual work.*

- LabVIEW offers the ability to design your own GUI for your control panel where you set different parameters.



The instrument panel designed on LabVIEW (not programming, just a pretty control panel). The results (I-V curve) of a successful test are displayed here.

- And now the program which is too large to see unless you click on it and examine it with zoom. For convenience a small interesting piece of it is shown to the right.



The LabVIEW program which you can see is graphical (VPL). In this graphical program there is input/output, conditionals, and loops. The thin lined box represents a loop in the above image and the thicker lined boxes are either conditionals (case statements or while loops) or sequence structures.

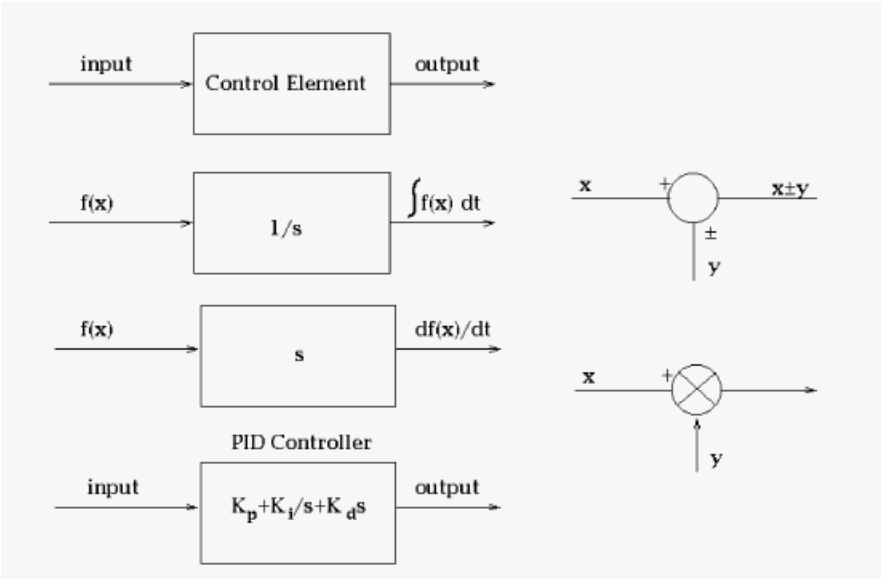
If you want to learn more about the LabVIEW programming language the LabVIEW Wiki is a good source of information and easy to navigate as well. There are really no open source alternatives to LabVIEW, a German product mostly in German and Spanish called MyOpenLab is the closest at present (2021). Of course you could use traditional languages to do what LabVIEW does but you will sacrifice ease of use.

Scilab Xcos

Scilab's Xcos (open source) is a control simulation programming language that is open source very much like MATLAB's Simulink.

This are graphical languages that have blocks that represent control actions like summing, integrating, differentiating, and other actions. This might best be describes as the blocks represent transfer function.

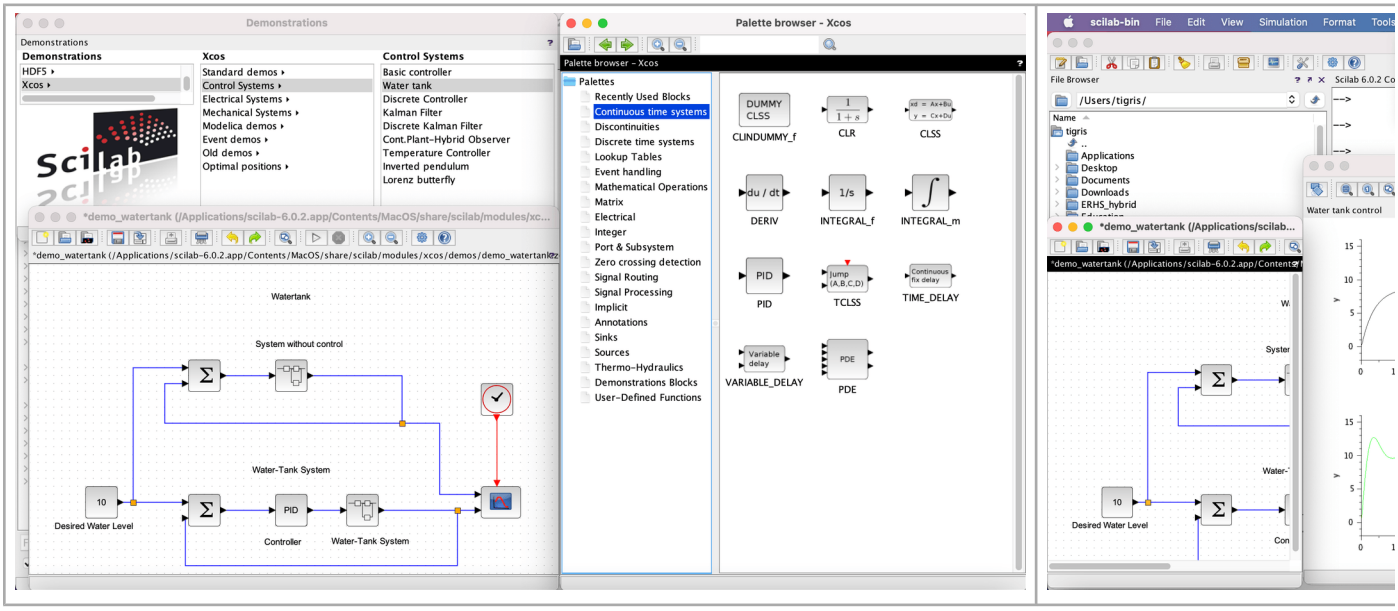
- In the following figure is a cartoon of how this blocks might look like in these types of control simulation programming languages.



Different control simulation languages have slightly different versions of the control block elements sketched above. Note that these blocks are just like they would have been done by hand before computers.

Scilab's Xcos has many examples which can be access in the Scilab program (or if you wish not to download Scilab they have some of those examples on the web page itself; this is true of MATLAB's Simulink as well).

- Xcos' examples are called demonstrations and are accessible through the Xcos' help



The snapshot above shows one of the demonstrations in Scilab's Xcos. It is a good example of a PID controller. Also in the picture are some of the blocks in Scilab. Of course since this product is open source you can go check this out for yourself.	The results of the demonstration *interesting engineering/control s this simulation?* In this artistic s demonstrations is through the he which you can "play" with to learn

VPLs are a great aid in programming more complex ideas and functions, but keep in mind that you can do the same thing in traditional languages. *Why use VPSs?* Well ease of use of course.

Arduino and "micro-computers"

Arduino is not the first microcontrollers/microprocessors but it's popularity has increased the awareness of these type of microprocessor. The uses for these microcontrollers range from drones to wearable light shows controlled by this miniature devices. A comparison of some of the more popular microprocessors show a range of operating systems (or lack thereof), languages, and usage.

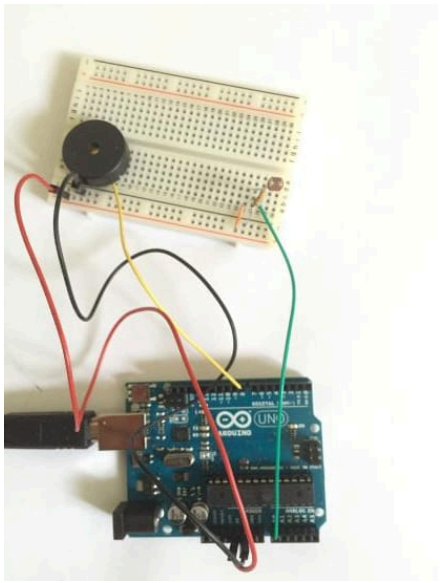
	Arduino	Raspberry Pi	Basic Stamp Processor	Beagleboard
Computer or not	Not a full-fledge computer	Full-fledged computer	Not a full-fledged computer	Full-fledged computer
Operating system	None	Linux OS (Debian - Raspberry Pi OS)	None	Linux OS (Ubuntu if you wish)
Primary Language	C++	Python (and others)	Unique (Basic Stamp 2) kind of like an Assembler language but more advanced (1.5 level?)	Python (and others)
Types	Many different types; many different sizes; more than 20 (not including retired products that are still widely available)	Two types	One module thought there are four different evolutions of it	Many different types; many different sizes
Website	https://www.arduino.cc	https://www.raspberrypi.org	Parallax	https://beagleboard.org

	Arduino	Raspberry Pi	Basic Stamp Processor	Beagleboard
Additional Info	• The most popular of these type of controllers/ processors • Good for beginners • Good for small projects • Lots of choices • No ports • Idea for this product came from the BASIC Stamp microcontroller	• Direct competitor to Arduino though technically they are really in different markets, so not really • More computer like • Lots of ports (Ethernet port; Video port, etc.) • *Note: Rasberry Pi Pico has no ports and is more like Arduino*	The BASIC Stamp microcontroller (or now the BASIC Stamp 2) was the most popular before Arduino. This has evolved into the Propeller microcontroller. *Arduino (circa 2005) is more like this product (circa 1992).*	• Open source • Not well known • Original product ◦ More expensive ◦ More features ◦ Kind of like a merge of Arduino and Raspberry Pi ◦ Ports (Ethernet port; etc.), • *Note: PocketBeagle has no ports is more like Arduino plus now they have more Arduino (Or is it BASIC Stamp?) like products*

These microcontrollers/microprocessors have add a new dimension to engineering and hence programming as well. With that we will give a simple example of Arduino from the starter kit.

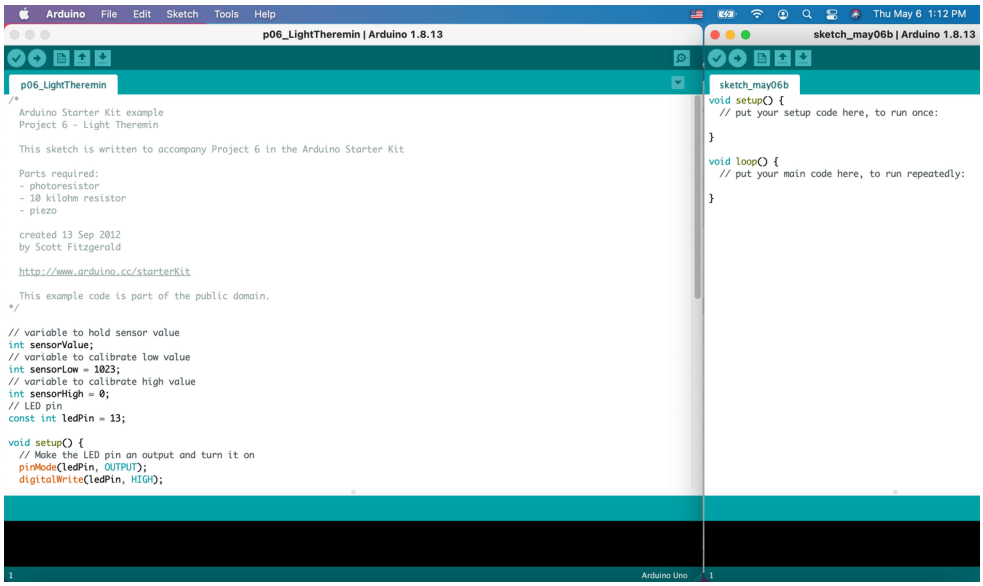
Example

The Arduino is meant to be plugged into your computer for you to program it. Here is an example of one of the projects in the Starter Kit, The Light Theremin[2].



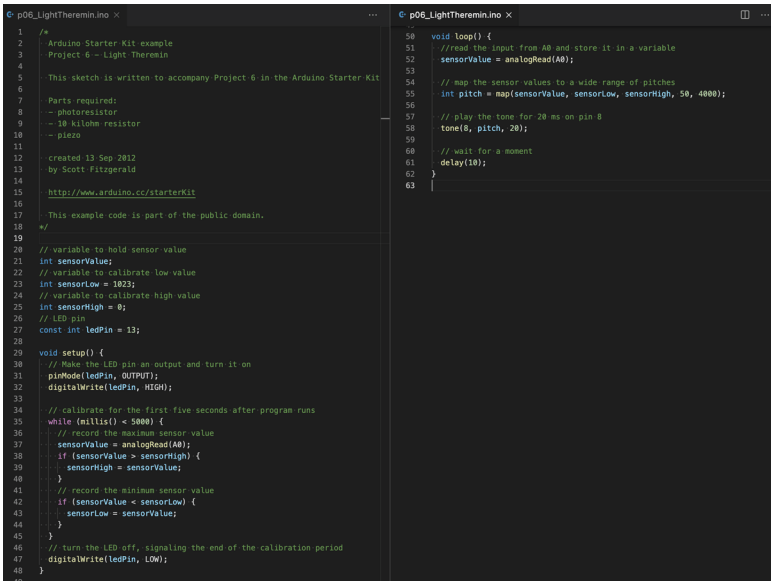
The simple light Theremin that is described in the Arduino Starter Kit. It is a take on the actual Theremin which does not use light. It is very simple, fun, and a great way to start your adventures with Arduino (or others microcontrollers as this is very simple and can easily be migrated to practically any electronic kit). The cable going out of the picture is attached to one of the authors' laptops.

The hardware is programming using the Arduino IDE software as shown here with the public domain Thermin program.



The software which is open source to program the hardware for the Arduino. The Light Theremin example is public domain and can be modified to improve the sound or even expand it's abilities by adding additional hardware and reprogramming. Just search the web and you will find many Theremin examples (light or otherwise).

Lastly we show the full code of the the Light Theremin project as our programming example.



This is the Light Theremin program used in the Arduino project. This is clearly a C/C++ program however you will note the more hardware like aspects to it.

This completes the discussion of computer programming, now it is time to produce a team project. The next section will detail that assignment. This will be a group project.

[1]Portmanteau in English is a traveling bag that was generally made of stiff leather and generally opens into two compartments. The word evolved into a "portmanteau word" which is a blended word believed to be first used by Lewis Carroll in the *Jabberwocky* poem of *Alice in Wonderland: Through the Looking-Glass*. Of course, *Alice in Wonderland* is one of the greatest works of "Englit."

[2]The Theremin is an invention of the modern world and is played without actually touching the equipment. The professional instrument is an example of heterodyning in radio engineering. It is basically an RLC circuit with the hand as ground. Sound from the Theremin can be heard in such great movies as *The Day The Earth Stood Still* (1951 Version) and *Spellbound*. For an utterly delightful theme song using a Theremin then you must watch *Midsomer Murders*. Though not originally done by a Theremin the theme song of the original *Star Trek* was so close to the sound of a Theremin that many people believed it actually was done by a Theremin. Today there are many Theremin versions of the *Star Trek* theme song on YouTube.

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20: Program Design Project

This section details the design project in the computer portion of the class (classes). This will represent the computer laboratory. Like the general engineering laboratory, teams will consist of 4 to 5 students (group work is requirement in most engineering courses, so no you can't do it alone).

Here you will write a computer program that is substantial in a team of 4 to 5 students (if circumstances make it less then this then instructor will make adjustments). The main objective of this laboratory is to learn how to write a program as a team that is more "real world" than academic. A report (with test examples) would accompany the program along with mini-manual. For academic purposes all individuals in a team will write their own report and mini-manual. In the "real world" a report could be written by just one member of the team, but academically that does not teach anything to the other members of the team. Hence the requirement herein.

The program must be able to run. Since there needs to be tests a non-functioning program would not meet the criteria. Reality means that in order to meet the functionality criteria a team may have to modify what they output and comment out sections that don't work. A report would include what your team actually had working and what your team failed to get working. There is no shame in failing on part of the assignment as the point of this is to teach and getting something is better than getting nothing.

So the laboratory will test:

- Your understanding of engineering (or scientific) programming
- General programming, documentation of programming, and testing of programming
- Teamwork
- Report writing (which is to be done by each individual of the team)

Assignment:

- Write a program (in a language of your choice) that investigates or models an engineering or science principle and/or project. The project is the team's decision with approval by the instructor. The whole group needs to participate (pop "visits" will be asking each team member to show what they are doing). As a guideline each student should write approximately 5 pages of code (this is not a hard and fast rule).
- Make sure the program is functional.
- Comment the program in appropriate places (not every where).
- TEST the program (each member of the team should do this separately - this assures everyone understands the program).
- Write a report that includes introduction, body with results and discussion, and conclusion. Each student should write their own report. A report should be about 3 pages (with figures) or longer, but not thesis size, please.
- Write a user manual that describes how to run the program. Discuss both input and outputs. Each student should write there own user manual.

Examples of programs done in the past:

Please note that in many cases simplifications had to be made to meet the time constraints in the academic world or to fit with the current mathematical and scientific understanding of the students (team). Simplifications were done by an experienced professor, not the students (it is not the job of the student to do this; though the students should as a group be following along and *asking questions*, if necessary). However sometimes no simplifications were made because the students wanted more of a challenge.

- Motion of car simulation using simplified Navier-Stokes Equations (simplified by instructor)
- Simulation of four stage rocket – *different rocket simulations have been done multiple times*
- Simulation of simple car crashing
- Simulation of car considering tires on various road surfaces
- Simulation of stopping distance for different breaking systems (disk, magnetic, drum, etc.)
- Simulation of flow in a various pipes
- Simulation of a wind tunnel (or particle simulation with obstructions, if you wish)

- Simulation of an ECG of the heart through Fourier series
- Simulation of an ECG of the heart through Daubechies wavelets
- Simulation of greenhouse warming through radiative forcing *(tk/tcl was used for graphic widgets in this project)*
- Simulation of a multibody problem (simplified by instructor)
- Simulation of the heat on the nose cone of the shuttle using finite elements *(NOT Simplified! Awesome work)*
- Simulation of Earthquake (p and s waves)
- Simulation of ground water flow (Darcy's Law) – *this has been done multiple times; each different; one went on to be used at a job site*
- Simulation of baseball pitches (including curve, sinkers, etc.) - *this has been done multiple times always with different ideas*
- Simulation of parachute dude – *a major extension of the program presented herein*
- Simulation of a hydrodynamic system: a real reservoir (*not simplified*) that is energy efficient and produces electricity in an environmentally friendly way

The choice of program is the team's choice. Examples are only given to guide your thinking, not to be chosen from. Many other project ideas were done that are not listed here.

To further help with the project idea, a pseudocode of a parachute extension is present below.

Parachute Dude Extension (pseudocode):

```
function (time,velocity) = paraperson(type_of_parachute, object_type,initial_time, final_time,

double pararect, paracirc,parahole
double saila,sailb
double generic double rocket, probot,pload
class parachute_descriptor ::length,width,radius,out_radius,in_radius,material_type
class mass_descriptor ::vararg(mass_pieces),vararg(distributed_force_pieces)

switch type_of_parachute
    case {“rectangular”}
        drag = pararect(length,width,material_type)
    case {“circular”}
        drag = paracirc(radius,material_type)
    case {“circularwhole”}
        drag = parahole(out_radius,in_radius,material_type)
    case {“parasail_a”}
        drag = saila(out_radius,in_radius,material_type)
    case {“parasail_b”}
        drag = sailb(out_radius,in_radius,material_type)
endswitch

switch(object_type)
    case {“generic human”}
        [center_of_mass,distribution_force_vector]=generic(mass_descriptor)
    case {“rocket”}
        [center_of_mass,distribution_force_vector]=rocket(mass_descriptor)
    case {“payload_robot”}
        [center_of_mass,distribution_force_vector]=probot(mass_descriptor
    case {“payload_drop”}
        [center_of_mass,distribution_force_vector]=pdrop(mass_descriptor)
endswitch

[time,velocity]=parap(initial_time,final_time,initial_velocity,drag,center_of_mass,distributio
end paraperson
```

Example of pseudocode shell to get drag coefficient given choices made when running program….

```
function drag_local = pararect(length,width,material_type)
   % Equations to get drag for this rectangular parachute
end pararect
```

```
....other drag functions...
```

Example of pseudocode shell to get center of mass of your object and the distribution of forces

```
function [center_of_mass,distribution_force_vector]=rocket(mass_descriptor)
    %Equations to get center of mass and your distribution of forces
end rocket

...other center of mass and force distribution functions...
```

Example of how you might want to run this in octave.

```
octave:1> [time,velocity] = paraperson(type_of_parachute, object_type,initial_time, final_time
octave:2> prettyplot(time,velocity)
```

Close message

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21: ETHICS AND GROUP DYNAMICS

Ethics is consider a requirement for "noble professions." Basically any profession that can cause substantial harm to people needs to have an ethical basis, however in reality every profession should have an ethical basis regardless of substantial harm or not. The ethics for engineers is not that much different from ethics for everyone else. Nevertheless we will have a brief discussion here.

Group dynamics is a theory that working in groups should have a purpose and a method. Here we discuss this.

21.1: Ethics

21.2: Group Dynamics

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21.1: Ethics

Ethics is a very difficult subject to explain in a brief presentation as there are nuances to ethics especially with regards to the law and employee and employer relationship. *What do you do if an employer asks you to do something wrong?* Report it, *but how do you keep your job?* It is true you most likely will lose your job in this case, *so what do you do? What is the right recourse?* As you see this is not easy.

It is your responsibility as a citizen of the world to be ethical and report ethical violations. *Will it be hard?* Yes. It is possible you will be unemployed but there is a possibility that an ethical business will "come to your rescue."

So what is ethics at any rate? First let us look at the engineering code of ethics presented by the National Society of Professional Engineers. While this is stated to be an **engineering** code of ethics it really can apply to **any** profession. This basically boils down to what you should have learned when you were a child: *Don't lie, don't cheat, don't blame it on the other person, and take responsibility for your actions.* It is pretty simple at a basic level, but the nuances are the things that go beyond the child. Still follow the rules you should have learned as a child.

Two statements in the fundamental canon needs some additional discussion.

- Engineers, in the fulfillment of their professional duties, shall **perform services only in areas of their competence.**

This is absolutely true, but note it says PERFORM, not discuss or talk about. If you see something in a presentation that is incorrect by, say, an optical engineer and you are a mechanical engineer, by all means TELL the person even if you are technically not competent in his field. As a matter a fact, in real world meetings a lot of people not competent in a particular field ask questions and suggest things. They could be wrong, but that presents the optical engineer a chance to educate them, or they could have a point and the optical engineer can say "thank you, I will revisit my calculations and get a better solution, I appreciate you actively listening to me so we can get a solution to our common problem."

Please ask questions! It is not unethical to do this ever!

- Engineers, in the fulfillment of their professional duties, shall **act for each employer or client as faithful agents or trustees.**

This is technically not a true ethics issue but it is the right thing to do at least up to a point. Looking for another job could be construed as being unfaithful to your employer, but this is NOT unethical. Ethically it would be akin to some form of indentured servitude if you did not have freedom to look for a job. That would be unethical. Obviously the authors did not intend this to be restrictive like that, but this is why there are nuances to ethical issues and why it really takes a complete course on ethics at your junior or senior year. Therefore this is all that will be said here. Do read the Code of Ethics as it is the best presentation of ethics for engineers, but remember that there will be nuances; and never do something you know is wrong.

For a kids prospective it is recommended that you watch the TV series Wow! Wow! Wubbzy! where Wubbzy seems like a businessman with an unclear definition of ethics; Widget, clearly, is an engineer, and Walden is a scientist or science-oriented creature, who both have a more clear definition of ethics. Ethics are best learned as a child, not as an adult because what we learn as a child influences us for life...it is very difficult to change an adult's ethics (not impossible though).

Engineering disasters have been related to unethical practices by either engineers or managers but not all engineering disasters are due to unethical practices. Some disasters are just bad management models, hence the next section: group dynamics.

Most professional engineering sites have a Code of Ethics paper or section (see Wikipedia's section for links) which you should read especially for your discipline. Reading other disciplines as well as yours would be even better. Nuances should be discussed in a course with an engineer who is also a lawyer as these subjects are intermingled and a good understanding from a true professional is necessary. It is suggested that this be done in person (or over web meeting software).

Ok that's it for ethics for this course.

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21.2: Group Dynamics

Group Dynamics

In engineering groups of engineers and other interested parties are assembled to solve problems. Most significant engineering and science problems require multiple inputs from a variety of people. Group dynamics are generally reflected in periodic meetings. In this context, properly run meetings are a very useful technique to help work through problems. Unfortunately it is possible for meetings to be poorly organized for a variety of reasons.

- Some reasons for poorly organized meetings
  - Different motives with each individual
  - Conflicting motives for each individual

The premise here is that effective group dynamics is the best way to solve complicated problems. This is borne out by experience by the authors at least.

- Effective group dynamics requires a level of professionalism
  - Members of a group must take responsibility and share of the work
  - Members must be dedicated and engaged
- Effective group dynamics requires a "schedule of things that should be done"
  - Define the problem (this is not as easy as it sounds)
    - There must be a clear and specific definition that is able to be solved, not a vague definition such as "build a bridge"
    - A purpose needs to be considered as part of the definition of the problem (what is the purpose of the bridge?)
    - Conditions must also be part of the definition of the problem (we need to build a bridge in a swamp, say)
    - Material limitations would be part of the conditions
  - Explore solutions to the problem
    - Research previous solutions or similar problems that have been solved or partially solved
    - Read peer reviewed papers that related to the problem
  - Plan how to implement the solution
    - Design based off previous works with maybe some innovative changes
    - Innovative changes are always welcomed BUT will delay projects due to the need to prototype before implementation
    - Note sometimes innovative changes are previously tested and investigated in other projects that eventually rejected them for time or money; so don't hesitate to do research...
  - Do the work
  - Did the project work as designed by your group?
    - Evaluate periodically and write reports (these can be used for future projects)
    - It is quite possible a project will work perfectly for many years but then a unforeseen flaw will be discovered...documenting is important for future projects
    - Documenting is important in all aspects of engineering from "hardware" to "software"
- Effective group dynamics requires full participation
  - Each member must define, explore, plan, and evaluate through research and discussion
  - Depending on the situation each member should be involved in the actual hands-on work, BUT only to the level of their expertise
  - Discussion is done before work is started
    - Get everything straight on paper in theory first
    - Don't rush to hands-on work (think first or to put it in a more classical way "measure twice THEN cut")
    - Prototype if necessary
  - Document (ALWAYS!!!)
- Effective group dynamics requires proper leadership and defined roles
  - Each member should have a role (some roles can have more than one person)
    - Coordinator (leader)
    - Checker (the thinker?)
    - The recorder
    - The skeptic (asks questions; doesn't accept - "we all know this")

        - Various "experts" or coordinator of experts (vice-leaders?)
    - Key to this is a leader - a leader must not stifle the though process and discussions however he must make sure the discussions do not get "out of hand"
        - Keep the meetings flowing
        - Don't allow one subject to dominate for months
            - Leader A: Ok this discussion has gone long enough, lets call it with the solution most are agreed upon
                - Good, but must not do this immediately - make sure every important detail is fleshed out
            - Leader B: Oh I don't want to make a decision, let's just keep on talking until we all agree years from now
                - Bad, nice that the leader tries to get everyone to agree, but at some point things must be decided

Let us take a closer look at the various roles:

- Leader
    - Organizes assignments and assigns tasks
    - Keeps group on the task at hand
- Checker
    - Monitors the groups comprehension of the subject at hand
    - Monitors the solution process
    - Checks "facts"
- Recorder
    - Monitors consensus on solutions
    - Writes solution (or edits and annotates)
    - In charge of documentation but does not write all the documentation (all members do that)
- Skeptic
    - Plays "Devil's Advocate"
    - Makes sure the group does not reach a conclusion without a full understanding
    - Suggests alternative solutions
        - The idea is that the skeptic would say that he might disagree but also presents an alternative solution
        - Does NOT just say "why?" or "prove it" but says "why do you think this bridge concept will not sink due to its enormous weight?"

Sometimes it is suggested that roles be rotated though there are problems with this concept.

- Roles should rotate but doesn't necessarily have to rotate if one member is "gifted" at the process
    - Some people just can't lead, though most should be able to do this role
    - Skeptics are generally skeptics by personality so rotating this seems to be counterproductive

For meetings there is a delicate balance of getting work done and going to meetings. If your day is filled with meetings it will be impossible to get any of your detailed expert work done. Meeting should have a gradient of scheduling that represents how far the group have progressed.

- Group meetings
    - Group meetings should be targeted and not general
        - With general meeting many people just should not be attending as it is not relevant for them
        - Large meetings are difficult to control and tend to be dominated by the loudest voice rather than be expert opinion
        - Group members can be in more than one group meeting
    - Scheduling of meetings depends on where in the solution or completion process you are at
        - Initially meetings should be one to two times a week
            - This is when the problem needs to be defined clearly and initially solutions need to be worked out
            - Requires more meeting to keep everyone on track
        - As time moves on meetings should spread out to one meeting a month
            - Need time for details to be worked out which usually take longer then initial solutions
        - After a solution or solutions have been implemented meetings should still continue with a meeting every three or four months
            - Monitor the final project for future projects
            - Note problems in design which need to be documented so future projects won't repeat these problems

Finally accountability is very important.

- Each individual is accountable for participation
  - Member should be informed early if it is believed they are not contributing
  - Each member is expected to contribute even if it is outside their realm of expertise: you should have sufficient academic training (physics, chemistry, system and signals, computers, and mathematics) to discuss any engineering discipline knowledgeable (but not as an expert unless, of course, the discussion is in your actual discipline)
  - If a member expects to just sit in on a meeting but not contribute they should be removed from the group (after given a warning to reform their ways)

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22: Storage of tests of Libretext's ability

Hi LibreBot,

Ok the style sheets on this system are incredibly inconsistent so sometimes you need to do things one way and sometimes another way. Just going to have to figure it out by trial and error.

Note: /html control does not really give you /html control...it is normally corrected by the style sheets again, but not always...it requires hours of trial and error to figure out what works and what doesn't...but R says it is the same on her NASA sheets, so maybe this is an overall problem with todays way of doing things.

Help in one end (css style sheets) makes it difficult in the other end (multifunction editors) ...ah! Another learning lesson that I will not give the students until they are ready to be masters...:)

In this particular list I use dl class="EXPAND_COLLAPSE" but in others this dl will fail (maybe in sections?) so you need to do just

. Symptoms of failure is that the list will expand then collapse immediately...that is when you should just use
...a bit strange...need to understand this better.

What is an engineer? Discuss in class or with friends. Note there is not just one answer as with anything in the real world definitions can be complex with multiple subtleties, but there is one specific word that is a must.

¬ A person who applies science and mathematics to produce a product?

¬ A person who fixes things?

```
version()
printf('Hello world!')
pkg load optim
pkg list
```

So this is an attempt just to use dl here...it does not work. This could be because this is a "chapter" not a "section"...section rules seem to be different then chapter rules. Need to be careful.

¬ A person who applies science and mathematics to produce a product?

This could describe an engineer, but doesn't it also describe artists and scientists (among others)? A professional artist might mix elements together to produce a specific effect ("chemistry of paint") and use mathematics to produce a pleasing image ("divine proportions").

Testing having another paragraph in this list....did it work? Yes

¬ A person who fixes things?

No this is not an engineer. This is more of a technician or even just a repair person. An engineer ***may*** fix things, but that does not define an engineer, just like you fixing the plumbing in your house does not make you a plumber.

Some equations tests:

$$\frac{dx}{dy} = \lim_{y \to -\infty} \frac{\Delta x}{\Delta y} \tag{22.1}$$

$$\cancel{\frac{\partial x}{\partial y}} \tag{22.2}$$

Above was wrong (***opps***, I know better than that...mental burp...) but it does tell me how to do infty...

Below is right.

$$\frac{dx}{dy} = \lim_{y \to 0} \frac{\Delta x}{\Delta y} \tag{22.3}$$

$$\frac{\partial x}{\partial y} \tag{22.4}$$

Something useful...how to do matrix in latex-like language here...

column:

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{22.5}$$

or row

$$\vec{x} = \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} i & j & k \end{pmatrix} \tag{22.6}$$

Another cool thing is doing inline equations where we need to use a "\ and a (together" and then a ") and a \ together" instead of the "$$". Here is our tensor inline, $\overleftrightarrow{I}$.

10 ⅔ meters per seconds (mps)...for "vulgar" fractions (which just means common fractions) use "⅔"

Let us say a car is moving at 10 ⅔ meters per second (mps), how long does it take for it to go 5 ⅞ meters? Try this problem in class and discuss the answer.

¬The answer is...

How to do sums and integrals and tabs (\quad,\qquad, and \hspace) for a space just use "\ "

$$\sum F_s = 0 \quad \sum F_x = 0 \quad \sum F_y = 0 \qquad \sum F_z = 0 \qquad \int f(x)dx \tag{22.11}$$

Integrals are \int....the vertical line from a solution of an integral (using \vert or \Biggr|, etc...):

$$A = \theta|_0^{2\pi} \tag{22.12}$$

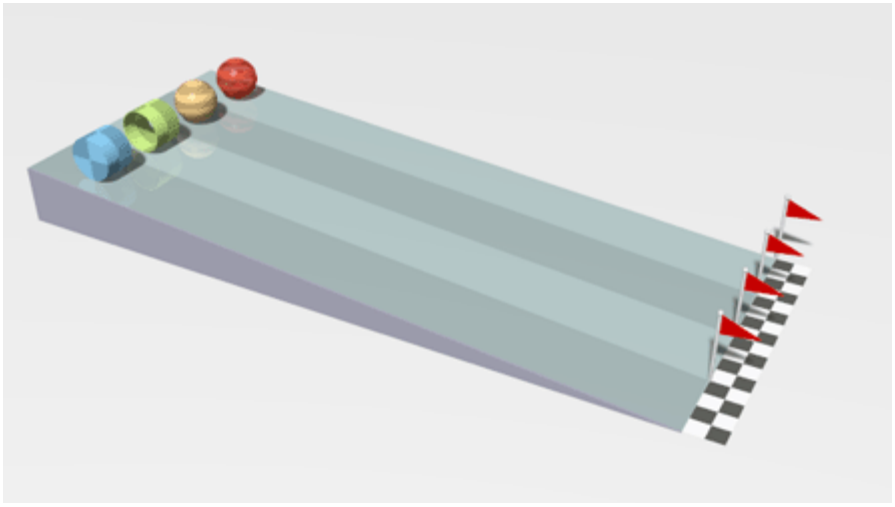
or

$$A = \theta|_0^{2\pi} \tag{22.13}$$

or

$$A = \theta\Big|_0^{2\pi} \tag{22.14}$$

Gif animations work...This is a simulation from Wikipedia (public domain) of moment of inertia...



What can I do in a math function (you know the "$$" stuff or the "\(" stuff). We can do text and arrows...watch...(rightarrow, leftarrow, uparrow, downarrow and \text, \textbf, etc...Note \textnormal does not seem to work here...)

$$\text{apples} = \textbf{apples} = \textit{apples} \rightarrow \text{oranges} \uparrow\downarrow \textbf{fruit} \leftarrow \text{What?} \tag{22.15}$$

Note here we used textbf but for math you would want to use \mathbf

$$\ddot{\mathbf{x}} \tag{22.16}$$

Testing a function...this works...now more complicated....

```
sombrero()
```

Oh my gosh! This works...awesome...I can do more then what I thought I could...

Ok now we are going to test writing a file...this would require some place to write this, so we will look for error actually. Ok THIS WORKS in the binder; surprise!!!

```
%
% Test of reading and writing files in libretext, very small file, for real work
% Use your own computer and octave (free!)
%
function nul = readwritefile()
%
%
num1 = [1,2,3];
num2 = [];
sznum1 = size(num1);
%
% Note there are many commands to write a file in octave including
% fprintf (which we use here), fputs, fwrite, dlmwrite, etc...
% this does not include octaves own method of
% saving matricies (for a much faster access) of just "save"...the better way to go
% mathematical work...
%
fidw = fopen("testfile","w");
for i = 1:sznum1(2)
    fprintf(fidw,"%d\n",num1(i))
    disp("Wrote in testfile"), disp(num1(i))
end %for
fclose(fidw);
disp("File written and closed")
disp("Now we will read it and print the result")
%
```

```
% Note there are many commands to read a file in octave including
% fgets (which we use here), fread, fscanf, dlmread, importdata, etc....
% this does not include ocaves own method of
% reading its matricies (for much faster access) of just "load"
%
fidr = fopen("testfile","r");
test = fgets(fidr);
do
    rnumber = str2num(test);
    num2 = [num2,rnumber];
    test = fgets(fidr);
until(test == -1)
for i = 1:sznum1(2)
    disp(num2(i))
end %for
disp("This is the end - JM")
return
end

readwritefile()
```

Ok some comments on the align feature. In general it is not something that should be used all the time, but it has it uses in situations like this.

$$\begin{aligned} 3 &= A + B \\ \underline{0 &= -A + 2B} \\ 3 &= 0 + 3B \\ 1 &= B \end{aligned}$$

Note there is a * in these aligns, that seems to shut off the ability to reference labels, so don't do * if you want to reference an equation.

$$\not{a} \tag{22.17}$$

$$\not{a} \tag{22.18}$$

$$\not{a} \tag{22.19}$$

Don't delete this template.Showorg() thing....just don't...no...no....no...no...no

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14.4.1: Circles
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19.2: Programming Languages for Hardware
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14.3.2: Trigonometric Functions
CNT
12: Case Study on Nanotechnology
code of ethics, engineers
21.1: Ethics
Codex on the Flight of Birds (da Vinci)
11.1: First Flight
cog train
11.3: Trains, trains, trains
Cohen's Kappa distribution
14.9: Statistics and Probability
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14.12: Thermodynamics (Statistical Physics)
cold plate
14.12: Thermodynamics (Statistical Physics)
column matrix
14.2.3.1: Matrices and Matrix Operations
combinations
14.2.4: Combinations and Permutations
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9: Programming in any language
compiled languages
5: What is a computer?
complex numbers
14.2.5: Complex Numbers
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19.1: Engineering and Science Languages
compressible
14.11.3: Hydraulics
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5: What is a computer?
computer operators
9: Programming in any language
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5: What is a computer?
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5: What is a computer?
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7: Operating Systems with Brief History
COMSOL Multiphysics
14.15: Optics
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14.7.4: Derivatives and the Shape of a Graph
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14.7.4: Derivatives and the Shape of a Graph
Concavity
14.7.4: Derivatives and the Shape of a Graph
concavity test
14.7.4: Derivatives and the Shape of a Graph
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9: Programming in any language
conduction
14.12: Thermodynamics (Statistical Physics)
confidence contour
14.9: Statistics and Probability
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18.1: Analysis Methods
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18.1: Analysis Methods
conservation of energy
14.11.3.2: Hydrodynamics
18.1: Analysis Methods
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18.1: Analysis Methods
conservation of mass
14.11.3.2: Hydrodynamics
18.1: Analysis Methods
conservation of physics
14.13: Electrical Circuits
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14.11.3.2: Hydrodynamics
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14.11.4: Solid Mechanics
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9: Programming in any language
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14.14: Signals and Systems (Control systems)
14.2.7.6: The Laplace Transform
18.2: Analysis Techniques for Laboratory
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14.12: Thermodynamics (Statistical Physics)
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14.2.7.2: Power Series and Functions
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14.2.6.3: Infinite Series
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14.2.6.2: Sequences
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14.2.7.6: The Laplace Transform
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14.4.3: The Hyperbola
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14.4.6: Coordinate Systems
coordinate systems, different types
14.4.6: Coordinate Systems
Coriolis force
14.11.2.1: Kinematics
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14.3.1: Basic Trigonometry
14.3.2: Trigonometric Functions
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14.3.1: Basic Trigonometry
14.3.2: Trigonometric Functions
cosine and exponentials
14.2.5: Complex Numbers
cotangent
14.3.1: Basic Trigonometry
14.3.2: Trigonometric Functions
counting boards
14.6: Calculus
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5: What is a computer?
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14.11.2.1: Kinematics
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14.7.3: Maxima and Minima
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14.10.2: Second-order Differential Equations

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14.5: Scalars, vectors, and tensors
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14.12: Thermodynamics (Statistical Physics)
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13.2: Information for example case study involving detector characterization
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14.16: Chemistry
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14.16: Chemistry
Cube root function
14.2.1.3: Roots and Radicals
cubic crystal structures, Wolfram demo
14.16: Chemistry
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11.2: The pyramids of Egypt
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11.2: The pyramids of Egypt
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14.9: Statistics and Probability
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14.5: Scalars, vectors, and tensors
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14.4.6: Coordinate Systems

D

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12: Case Study on Nanotechnology
damping coefficient
14.10: Differential equations
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13.1: Example case study involving detectors characterization
data structures
16: Beyond the basics of computers
Day The Earth Stood Still
19.2: Programming Languages for Hardware
DCL
13.1: Example case study involving detectors characterization
DCL discussion to aid student assignment
13.2: Information for example case study involving detector characterization
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14.2.6.2: Sequences
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14.2.5: Complex Numbers
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19.2: Programming Languages for Hardware
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11.1: First Flight
density and mass
14.8.3.4: Physical Applications of Integration
density function
14.8.3.4: Physical Applications of Integration
derivative table
14.7.1: Table of Derivatives
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14.4.5: Hyperbolic Functions
Descartes' law
14.15: Optics
desktop publishing
8: Brief History of Popular Programs
detector characterization
13.1: Example case study involving detectors characterization
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13.1: Example case study involving detectors characterization
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13.2: Information for example case study involving detector characterization
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14.12: Thermodynamics (Statistical Physics)
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12: Case Study on Nanotechnology
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14.10: Differential equations
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14.3.3: Trigonometric Identities
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14.2.3.1: Matrices and Matrix Operations
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14.6: Calculus
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14.10: Differential equations
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14.2.7.4: Working with Taylor Series
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14.15: Optics
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12: Case Study on Nanotechnology
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19.2: Programming Languages for Hardware
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18.1: Analysis Methods
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14.5: Scalars, vectors, and tensors
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14.5: Scalars, vectors, and tensors
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11.1: First Flight
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14.2.7.5: Fourier Series and Transform
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14.11.1: Statics
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14.2.1.1: Algebraic Expressions and Formulas
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14.5: Scalars, vectors, and tensors
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14.2.6.3: Infinite Series
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14.11.3.1: Hydrostatics
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14.11.3.1: Hydrostatics
divergent sequence
14.2.6.2: Sequences
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14.2.6.2: Sequences
DNA
12: Case Study on Nanotechnology
DNA faces
12: Case Study on Nanotechnology
don't panic
14: Fundamentals of Engineering
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14.6: Calculus
dot product
14.5: Scalars, vectors, and tensors
double angle formula, trigonometric identity
14.3.2: Trigonometric Functions
Douglas McCurdy
11.1: First Flight
drag
10: Parachute Person
Drax power station
14.4.3: The Hyperbola
Dumont's 14_bis
11.1: First Flight
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14.5: Scalars, vectors, and tensors
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14.14: Signals and Systems (Control systems)
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14.11: Mechanics
14.11.2: Dynamics
14.11.3: Hydraulics
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14.11.3.2: Hydrodynamics

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14.9: Statistics and Probability
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16: Beyond the basics of computers
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11.2: The pyramids of Egypt
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12: Case Study on Nanotechnology
14.6: Calculus
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11.1: First Flight
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18.1: Analysis Methods
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14.13: Electrical Circuits
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14.13: Electrical Circuits
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14.15: Optics
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12: Case Study on Nanotechnology
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5: What is a computer?
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14.4.4: The Ellipse
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14.2.7.4: Working with Taylor Series
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14.4.4: The Ellipse
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14.4.4: The Ellipse
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18.1: Analysis Methods

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4: What is engineering? Who are engineers?
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4: What is engineering? Who are engineers?
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4: What is engineering? Who are engineers?
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4: What is engineering? Who are engineers?
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7: Operating Systems with Brief History
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14.12: Thermodynamics (Statistical Physics)
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14.12: Thermodynamics (Statistical Physics)
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12: Case Study on Nanotechnology
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14.11.4: Solid Mechanics
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14.13: Electrical Circuits
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14.14: Signals and Systems (Control systems)
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15.4: Sketch of how to present data in laboratory (i.e. graph with errors)
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11.2: The pyramids of Egypt
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10: Parachute Person
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16: Beyond the basics of computers
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12: Case Study on Nanotechnology
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14.2.1.2: Rules of Exponents and Scientific Notation
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14.2.1.2: Rules of Exponents and Scientific Notation
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14.9: Statistics and Probability
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14.7.5: L'Hôpital's Rule
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14.8.2: Substitution
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14.7.2: Derivatives of Exponential and Logarithmic Functions
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14.8.1: Table of Integrals
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14.2.1.3: Roots and Radicals
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17: Documentation and such
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14.7.3: Maxima and Minima

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14.9: Statistics and Probability
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14.2.4: Combinations and Permutations
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14.9: Statistics and Probability
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11: Historical case studies in Engineering
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14.2.7.5: Fourier Series and Transform
14.2.7.6: The Laplace Transform
18.2: Analysis Techniques for Laboratory
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14.6: Calculus
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14.15: Optics
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18.2: Analysis Techniques for Laboratory
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14.2.6.2: Sequences
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14.11.2.1: Kinematics
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14.5: Scalars, vectors, and tensors
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18.1: Analysis Methods
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7: Operating Systems with Brief History
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14.7.4: Derivatives and the Shape of a Graph
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14.8.3.5: Moments and Centers of Mass
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18.2: Analysis Techniques for Laboratory
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18.1: Analysis Methods
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18.1: Analysis Methods
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17: Documentation and such
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14.11.3: Hydraulics
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14.6: Calculus
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14.4.2: The Parabola
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14.11.1: Statics
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10: Parachute Person
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14.14: Signals and Systems (Control systems)
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9: Programming in any language
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14.2.7.5: Fourier Series and Transform
18.2: Analysis Techniques for Laboratory
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14.12: Thermodynamics (Statistical Physics)
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14.12: Thermodynamics (Statistical Physics)
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14.2.6.1: Prelude to Sequence and Series
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14.2.6.1: Prelude to Sequence and Series
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14.2.1.3: Roots and Radicals
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11.1: First Flight
Fraunhofer diffration
14.15: Optics
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14.15: Optics
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14.8.3.3: Arc Length of a Curve and Surface Area
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14.8.3.5: Moments and Centers of Mass
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5: What is a computer?
9: Programming in any language
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14.2.7.2: Power Series and Functions
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14.2.7.3: Taylor and Maclaurin Series
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14.6: Calculus
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14: Fundamentals of Engineering

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14.15: Optics
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14.9: Statistics and Probability
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19.1: Engineering and Science Languages
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9: Programming in any language
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14.11.4: Solid Mechanics
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14.2.6.2: Sequences
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14.2.6.3: Infinite Series
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14.3: Geometry
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11.1: First Flight
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14.12: Thermodynamics (Statistical Physics)
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14.12: Thermodynamics (Statistical Physics)

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12: Case Study on Nanotechnology
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11.1: First Flight
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7: Operating Systems with Brief History
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19.1: Engineering and Science Languages
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18.1: Analysis Methods
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14.5: Scalars, vectors, and tensors
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14.11.3.1: Hydrostatics
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16: Beyond the basics of computers
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16: Beyond the basics of computers
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14.4.3: The Hyperbola
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14.7.4: Derivatives and the Shape of a Graph
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12: Case Study on Nanotechnology
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14.4.3: The Hyperbola
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21.2: Group Dynamics
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21.2: Group Dynamics
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14.7.5: L'Hôpital's Rule
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19.1: Engineering and Science Languages
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11.2: The pyramids of Egypt

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12: Case Study on Nanotechnology
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14.15: Optics
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18.1: Analysis Methods
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12: Case Study on Nanotechnology
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19.2: Programming Languages for Hardware
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19: Using Computers for Engineering and Science
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19.2: Programming Languages for Hardware
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14.2.6.3: Infinite Series
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12: Case Study on Nanotechnology
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16: Beyond the basics of computers
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14.16: Chemistry
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19.2: Programming Languages for Hardware
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14.12: Thermodynamics (Statistical Physics)
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14.2.2: Partial Fractions
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11.1: First Flight
Helium balloon
11.1: First Flight
Helmholtz free energy
14.12: Thermodynamics (Statistical Physics)
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11.1: First Flight
Henry Petroski
11: Historical case studies in Engineering
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5: What is a computer?
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14.8.3.4: Physical Applications of Integration
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14.8.3.1: Prelude to Applications of Integration
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14.4.4: The Ellipse
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19.2: Programming Languages for Hardware
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14.11: Mechanics
14.11.3: Hydraulics
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14.11.3: Hydraulics
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14.11.3.2: Hydrodynamics
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12: Case Study on Nanotechnology
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12: Case Study on Nanotechnology
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14.8.3.4: Physical Applications of Integration
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14.8.3.1: Prelude to Applications of Integration
14.8.3.4: Physical Applications of Integration
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14.8.3.4: Physical Applications of Integration
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14.8.3.4: Physical Applications of Integration
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14.11.3.1: Hydrostatics
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14.4.2: The Parabola
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14.4.3: The Hyperbola
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14.4.5: Hyperbolic Functions
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14.8.1: Table of Integrals
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14.3.1: Basic Trigonometry

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14.15: Optics
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14.11.2.2: Kinetics
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14.12: Thermodynamics (Statistical Physics)
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14.2.3.2: System of equations and Matrices
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19.1: Engineering and Science Languages
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19.1: Engineering and Science Languages
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9: Programming in any language
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11.2: The pyramids of Egypt
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9: Programming in any language
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14.2.1.3: Roots and Radicals
14.2.5: Complex Numbers
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14.11.2.2: Kinetics
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14.11.2.2: Kinetics
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14.11.2.2: Kinetics
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14.11.3: Hydraulics
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14.7.5: L'Hôpital's Rule
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14.15: Optics
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14.2.6.2: Sequences
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14.2.6.1: Prelude to Sequence and Series
14.2.7.1: Prelude to Power Series
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14.2.6: Sequences and Series
14.2.6.3: Infinite Series
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14.6: Calculus
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14.7: Infinitesimal calculus for derivatives
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14.8: Infinitesimal Calculus for integration
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14.7.4: Derivatives and the Shape of a Graph
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9: Programming in any language
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14.10: Differential equations
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14.8.1: Table of Integrals
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14.4.5: Hyperbolic Functions

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14.6: Calculus
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14.8.1: Table of Integrals
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14.8.2: Substitution
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14.12: Thermodynamics (Statistical Physics)
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14.15: Optics
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5: What is a computer?
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16: Beyond the basics of computers
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14.4.5: Hyperbolic Functions
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14.12: Thermodynamics (Statistical Physics)
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14.7.6: Newton's Method

J

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19.2: Programming Languages for Hardware
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11.1: First Flight
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14.11.2.1: Kinematics
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14.11.2.1: Kinematics
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11.1: First Flight

K

Kappa distribution
14.9: Statistics and Probability
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14.2.6.1: Prelude to Sequence and Series
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14.11.2.1: Kinematics
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14.11.2: Dynamics
14.11.2.1: Kinematics
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14.11.2: Dynamics
14.11.2.2: Kinetics
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14.13: Electrical Circuits
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14.13: Electrical Circuits
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14.2.6.1: Prelude to Sequence and Series
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14.2.6.1: Prelude to Sequence and Series
14.2.6.3: Infinite Series
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14.7.5: L'Hôpital's Rule
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15.1: Laboratory Journal Format
laboratory project for engineering
15: Laboratory Project for Introduction to Engineering
laboratory report format
15.2: Laboratory Report Format
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15.3: Example of Laboratory Report
LabVIEW
14.13: Electrical Circuits
19.2: Programming Languages for Hardware
5: What is a computer?
Lake Mead
14.8.3.1: Prelude to Applications of Integration
lamina
14.8.3.5: Moments and Centers of Mass
Laplace transform
14.2.7.6: The Laplace Transform
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14.2.7.6: The Laplace Transform
Laplace transform properties
14.2.7.6: The Laplace Transform
Laplace transform table
14.2.7.6: The Laplace Transform
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14.5: Scalars, vectors, and tensors
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14.8.3.3: Arc Length of a Curve and Surface Area
Latus rectum
14.4.2: The Parabola
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14.12: Thermodynamics (Statistical Physics)
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21.2: Group Dynamics
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18.1: Analysis Methods
Leibniz, Gottfried
14.6: Calculus
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14.15: Optics
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11.1: First Flight
14.2.6.2: Sequences
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11.4: What are the lessons learned?
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14.5: Scalars, vectors, and tensors
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14.5: Scalars, vectors, and tensors
LibreOffice
8: Brief History of Popular Programs
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11.2: The pyramids of Egypt
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14.2.6.2: Sequences
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14.11.2.2: Kinetics
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18.2: Analysis Techniques for Laboratory
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14.11.2.1: Kinematics
14.11.2.2: Kinetics
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16: Beyond the basics of computers
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19.1: Engineering and Science Languages
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7: Operating Systems with Brief History
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14.12: Thermodynamics (Statistical Physics)
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16: Beyond the basics of computers
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14.4.4: The Ellipse
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14.7.3: Maxima and Minima
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14.8.1: Table of Integrals
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9: Programming in any language
lookfor (MATLAB/Octave)
17: Documentation and such
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9: Programming in any language
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14.2.6.2: Sequences

M

Maclaurin polynomial
14.2.7.3: Taylor and Maclaurin Series
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14.2.7.3: Taylor and Maclaurin Series
MacOS
7: Operating Systems with Brief History
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14.16: Chemistry
magazines of interest (vetted/engineering)
6: Understanding (how to investigate on your own)
magazines of interest (vetted/science)
6: Understanding (how to investigate on your own)
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14.7.6: Newton's Method
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12: Case Study on Nanotechnology
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5: What is a computer?
9: Programming in any language
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19.1: Engineering and Science Languages
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14.2.3: Matrices and how to use them
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14.2.3.2: System of equations and Matrices
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14.2.3: Matrices and how to use them
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14.2.3.1: Matrices and Matrix Operations
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14.2.3.1: Matrices and Matrix Operations
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14.5: Scalars, vectors, and tensors
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14.9: Statistics and Probability
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14.6: Calculus
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14.11: Mechanics
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14.11.3: Hydraulics
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14.11: Mechanics
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12: Case Study on Nanotechnology
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14.13: Electrical Circuits
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14.8.3.2: Volumes of Revolution - Cylindrical Shells
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14.15: Optics
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19.2: Programming Languages for Hardware
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12: Case Study on Nanotechnology
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14.15: Optics
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7: Operating Systems with Brief History
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19.2: Programming Languages for Hardware
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14.16: Chemistry
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14.11.1: Statics
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14.11.2.2: Kinetics
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14.8.3.5: Moments and Centers of Mass
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14.11.2.2: Kinetics
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14.11.3.2: Hydrodynamics
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14.11.3.2: Hydrodynamics
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14.11.2.2: Kinetics
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14.11.2.1: Kinematics
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14.2.4: Combinations and Permutations
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14.2.4: Combinations and Permutations
Museum of Science and Industry in Chicago
14.4.4: The Ellipse

N

nabla
14.5: Scalars, vectors, and tensors
nanocrystalline metals
12: Case Study on Nanotechnology
nanometer
12: Case Study on Nanotechnology
nanosize hydroxyapatite
12: Case Study on Nanotechnology
nanotechnology
12: Case Study on Nanotechnology
nanotube
12: Case Study on Nanotechnology
netlib
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GLOSSARY

A

Abacus | Calculator based on stones moving on rods or in grooves

Aberration | A departure from the normal operation of a device or an optical element

Accuracy | Proximity to true value. Accuracy is a consistent error from true value, but it is not necessarily a good or precise error.

Acoustic | Science of sound

Adjacent side | In a right triangle, the side between a given angle and the right angle

AFM | Atomic force microscope is a microscope that uses a cantilever with a nanometer-size tip to "feel" the surface of the specimen. AFMs can "see" in the nanometer realm and has many different interchangable probes to investigate many different properties.

Ailerons | The small flaps on the back of wings on airplanes

Allotrope | Different forms of the same chemical element. Some allotropes of carbon are graphite, graphene, diamond, and buckyballs.

Angle of depression | The angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

Angle of elevation | The angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

ANOVA | Analysis of variance (ANOVA) is statistical models for estimations

Antiderivative | The opposite of a derivative which leads to a indefinite integral. Through the fundamental theorem of calculus an antiderivative can be made into a definite integral.

Apatite | A mineral group that consists of calcium phosophate minerals

Arc length | The arc length of a curve can be thought of as the distance a person would travel along the path of the curve

Arithmetic sequence | A sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence.

Aspheric | Slightly off from spherical (usually used to correct spherical aberration)

B

Bandgap | The gap in insulators and semiconductors between the valance band (the nucleus energy area) and the conduction bands (the electron shell energy area). Note: This is actually more complex than this.

Bellows | A device that has a bag and two flats plates that is used to squeeze air out at a fast pace for various purposes. Also, flexible structures that expand and compress for various purposes.

Binomial series | A series given by $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots$ for $|x| < 1$; from the Maclaurin series for $f(x) = (1+x)^r$

Boat keel | The long structure directly on the bottom of a boat that is used to support the hull. Can have hydrodynamics purpose or counterbalance purpose as well as support purpose.

Bounded above (sequences) | A sequence a_n is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

Bounded below (sequences) | A sequence a_n is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

Bounded sequence | A sequence a_n is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

C

Catenary | A curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary. Catenary comes from the Latin word meaning "chain" as a catenary is often observed with free hanging chains. In engineering a catenary arch is known to be a strong self sustaining structure.

Center of mass | The point at which the total mass of the system could be concentrated without changing the moment

Centroid | The centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

Ceramic | Hard brittle material made from inorganic substances and metals. Typically non-conductive, heat resistant, chemical resistant, and

corrosion resistant

Change of variables | The process in mathematics of substituting variables with another expression to simplify a problem.

Cipher | Encoded words to make a message secret. Used is "spycraft". Evidence of using ciphers first appear in antiquity (Ancient Egypt, Ancient Greece, Ancient Rome)

Codex | A manuscript usually hand written and on material other then paper

Cold finger | A strip of metal intended to transfer temperature from a cold plate to a device like a detector

Confidence Contour | A contour that represents the confidence of a true value that is a function of two or more variables (kinda like a 3D error bar)

Confidence Interval | A range of values that represents the confidence of a true value that is a function of one variable (kinda like an error bar)

Contour | Contour lines (isoline or isoarithm) which represent the path of a constant value

Convergence of a series | A series converges if the sequence of partial sums for that series converges.

Convergent sequence | A convergent sequence is a sequence a_n for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

Cryocooler | A mechanical cooling system that goes to cryogenic temperatures

Cryogenic temperature | Temperatures below 124 K, though in practical terms it is usually 77K and below

Cryostat | Cooling system that goes to cryogenic temperatures by means of a cryogenic liquid (LH or LHe)

Cubit | A length unit used in antiquity

D

Demoiselle | An ultralight aircraft in the context of this book. Young lady in general terms.

Density function (mass density) | A density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume.

Diffusion (physics) | Movement of particles to a high concentration area to a low concentration area

Dirigible | A rigid lighter-than-air aircraft that is steerable and powered by an engine. The Hindenburg was a dirigible. In contrast a Zeppelin is semi-rigid and a blimp is not rigid at all. So the Goodyear blimp is not a dirigible or Zeppelin. The dirigible was first demonstrated by a Brazilian in 1709. In 1785 a dirigible with bird-like steering and hand-powered propellors crossed the English channel with French balloonist Jean-Pierre-François Blanchard and American John Jeffries

Divergence of a series | A series diverges if the sequence of partial sums for that series diverges.

Divine Proportion | The Golden Ratio. This was detailed in the book illustrated by Leonardo da Vinci called "De Divina Proportione" by Luca Pacioli circa 1509.

Ductile | The ease of which metal can be drawn out into a wire. Also, the degree to which a material can deform until failure.

Dynamics | The study of objects in motion. This by its very nature suggests a study that is in the time domain. This differs from statics where time is not relevant. For fluids a hydro is attached to dynamics (hydrodynamics) but the idea is still the same.

E

eBird | Citizen science site for bird observation.

Electron microscope | Electron microscope is a microscope that uses electrons rather than light. Due to the wavelength of electrons, electron microscopes can "see" in the nanometer realm.

Ethics | A philosophy that is based off human morals that attempts to define right and wrong or good and evil

F

Frustum | A portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

G

Geometric Sequence | A sequence a_n in which the ratio a_{n+1}/a_n is the same for all positive integers, n, is called a geometric sequence.

Geometric series | A geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

Getter | Substance that absorbs unwanted residuals (like water) in a vacuum

Glassy carbon | A carbon solid that has a glassy appearance that is also known as vitreous carbon

Golden Ratio | A golden ratio is a ratio of two quantities, say a and b, that are equal to the ratio of the larger quantity and the sum of the quantities. It is said that this ratio defines a unique natural structure and hence it is sometimes called the divine proportion. Interestingly this quantity is used in numerical methods.

H

Handle (computer science) | Abstract reference to a resources (similar to a pointer, but not). A pointer is an address to a resource, normally data, where as a handle references functions.

Harmonic series | A geometric series is a series that can be written in the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$. This series is interesting in that it diverges albeit rather slowly.

Hydrophilic | Tends to mix, dissolve, or be wetted in water

Hydrophobic | Tends to repel water

Hydrostatic pressure | The pressure exerted by water on a submerged object

Hydroxyapatite | A phosphate material that forms bones and teeth but is rarely found in rocks

Hypotenuse | The side of a right triangle opposite the right angle

I

Incompressible (fluid) | A fluid that does not diverge but is the same density throughout the system

Indefinite integral | A primitive integral or an integral without an upper and lower limit. Sort of like a definition of a specific integral.

Indeterminate forms | When evaluating a limit, the forms $\frac{0}{0}$, $\infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

Index of refraction | A unitless number that describes the speed of light in a material.

Infinite series | An infinite series is an expression of the form $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Inflection point | If f(x) is continuous at c and f(x) changes concavity at c, the point (c, f(c)) is an inflection point of f

Integral | Integral has a number of meanings in mathematics, but they all relate to the standard definition of integral: the sum of the necessary parts that creates a whole. This leads to the most common use of integral in mathematics which is the sum of infinitesimal areas within a range.

Integration by substitution | A technique for integration that allows integration of functions that are the result of a chain-rule derivative

Interval of convergence | The set of real numbers, x, for which a power series converges

Iso | Equal as in equal temperature in *isothermal*, etc.

Iterative process | Process in which a list of numbers x_0, x_1, x_2, x_3, ... is generated by starting with a number x_0 and defining $x_n=F(x_{n-1})$ for $n\geq 1$

J

Jerk | Change of acceleration with respect to time. The names implies jerk as the feeling of being thrust suddenly in one direction. This is also sometimes called jolt (also aptly named).

K

Kinetics | In mechanics the effects of force on a body and in chemistry the rate of reaction. Kinetics implies a dynamic situation (time varying).

Koch curve | A fractal curve

L

Lamina | A thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

Limestone | Soft sedimentary rock made of calcium carbonate. Densities can vary from rather hard limestone to chalk like limestone.

M

Mastaba | A tomb that has slanted sides and a flat roof; stone bench

Metal | Crystal that conducts electricity and heat well. Ductile and malleable. Made of the metallic and semi-metallic elements.

Method of cylindrical shells | A method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this

method is different from the methods of disks or washers in that we integrate with respect to the opposite variable.

N

Nabla | An ancient stringed harp

Newton's method | Method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Nonelementary integral | An integral for which the antiderivative of the integrand cannot be expressed as an elementary function

O

Ostracon | A piece of pottery or stone that has writings scratch on it.

P

Partial sum | The k^{th} partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum $S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$

Periodic function | A function is periodic if it has a repeating pattern as the values of x move from left to right

Phase change | Change for one state of matter to another state of matter through "symmetry breaking" (except in some iso instances)

Polymer | Materials made of large molecules that are made of many subunits like DNA. Organic. Usually flexible.

Portmanteau | A combination word

Power series | A series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

Precision | A consistent result with a small error, but not necessarily anywhere near a true value

Principle of least action | A variation principle that is used to derive the equation of motions. It has nothing to do with the general accepted meaning of action or least action.

Probability Distribution Function (PDF) | This is a general term that describe the probability distribution of a system or experiment. There are three types of probability distribution functions:

- Probability Mass Function (PMF) for discrete systems
- Probability Density Function (PDF) for continuous systems
- Cumulative Distribution Function (CDF) is the cumulation of probabilities for discrete and continuous systems

Pseudocode | A plain language text of an algorithm to be translated into a computer program with appropriate syntax. Useful for thinking out ideas before implementing them.

Q

QED | "Quad erat demonstrandum." Sometimes hollow square ▫ is used for this. Sometimes a filled in square is used ■. These squares are sometimes referred to as a tombstone.

Quadrature | Integration

Quarks | Elementary particle that make up protons and neutrons

R

Radians | For a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s. This contrasts to degrees where a full circle is 360 degrees but 2π radians.

Radius of convergence | If there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for \(|x−a|R\), then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

Raman scattering | The basis of Raman spectroscopy which is the inelastic scattering of photons into a material.

Raman spectroscopy | Uses Raman scattering to determine the vibrational modes of a material. This will allow not only an identification of a substance but what allotrope it might be as well.

Read noise | The noise associated with reading an element of a detector. Includes all sources of noise but primarily due to ADC of a detector.

Recurrence relation | A recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

Recursion (computer) | A function that calls itself (typical example is a factorial)

Register (computer) | Quick memory usually associated with the CPU

Regression | Statistical model comparing dependent variables with independent variables usually using a mean value and the actual value in some fashion

Regula Falsi | False position

S

Sequence | An ordered list of numbers of the form a_1,a_2,a_3,...is a sequence

Small angle approximation | An approximation used extensively in engineering and physics. In this approximation when we have a small angle in our problem we can say that $\cos(\theta) = 1 - \frac{\theta^2}{2}$ and/or $\sin(\theta) = \theta$.

Surface area | The surface area of a solid is the total area of the "skin" of the object; for objects such as bricks, the surface area of the object is the sum of the areas of all of its faces; for a sphere is the sum of infinitesimal area slices of the surface of the sphere.

T

Telescoping series | A telescoping series is one in which most of the terms cancel in each of the partial sums

Theorem of Pappus for volume | This theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

Theremin | A modern electronic instrument that does not require physical contact

Tuples | Ordered set

U

Unix | An operating system philosophy of modularity; also the operating system itself

V

Variable | A representation of a changing value

Vitreous | Substance that resembles glass

W

Work | In science work is an energy transfer descriptor otherwise it means effort either physical or intellectually

X

x-ray | x-radiation which is high energy radiation

Y

Yeast | A fungus that people use to make bread, beer, wine, and root beer among other uses

Z

Zooniverse | Citizen science web portal. Citizen science goes back to antiquity. Recently with the advent of computers and clouds citizen science has become more popular and important to science.

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