STRENGTH OF MATERIALS



James K. Lord, Amy Richardson, Sneha Davison, and David A. Dillard
Virginia Tech



Strength of Materials

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James Lord is a Collegiate Associate Professor in the Department of Mechanical Engineering at Virginia Tech, where he has taught foundational mechanics courses since 2014, prior to which he taught a variety of courses at Cal Poly, SLO. Born and raised in England, he holds a master's degree in mechanical engineering and a PhD in Biomechanical Engineering from Newcastle University. He is active in the American Society for Engineering Education and has won numerous teaching awards. He is passionate about removing barriers to education and has previously contributed to other free mechanics resources, including the popular Engineering Statics: Open and Interactive. After a decade of watching students spend $100+ on a textbook for his course he thought “I could write one of those”. So he did.

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Amy Richardson is a research scientist in Engineering Education at Virginia Tech. Before joining Virginia Tech, she spent over 15 years as a community college faculty member, teaching mechanics courses every semester. Amy holds a BS and MS in Civil Engineering, as well as a PhD in Engineering Education. She is passionate about fostering equity in the classroom so that every student has the opportunity to succeed. Contributing to this OER textbook is her way of making foundational engineering mechanics more accessible to all—one small step toward removing barriers in engineering education.

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Sneha Davison is an advanced instructor in the Mechanical Engineering Department at Virginia Tech. She also completed her BS (1996), MS (1999), and PhD (2003), all in Engineering Science and Mechanics, from Virginia Tech. She has taught multiple mechanics courses every semester since the fall of 2016. These courses primarily include Statics, Dynamics, Deforms, and Thermodynamics. Beyond teaching in person, she feels that contributing to this OER textbook is the perfect way to reach more students and put the insight she has gained about how to communicate complicated concepts to students to good use.

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About the Accompanying Question Bank

This book has an accompanying set of practice questions that can be accessed by visiting the book's main landing page.

These questions are a mix of (1) new questions generated by the author team and (2) adapted questions from Kurt Gramoll's Eng Mechanics question bank (accessible via the Apple Store or Google Play).

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Introduction

What is Mechanics?

Mechanics can be broadly categorized in three parts, of which this text covers one. **Statics** explores the effects of *external loads* applied to *rigid bodies* in *equilibrium*. **Dynamics** explores the effects of *external loads* applied to *rigid bodies* that are not in *equilibrium*. In ***Strength of Materials***, we explore the mechanics of deformable bodies, learning about the effects of both *external* and *internal loads* on *nonrigid bodies* in *equilibrium*.

Since bodies in this text are in equilibrium, many of the principles from statics apply. However, because these bodies are nonrigid, a much deeper analysis is required. Objects discussed in this text will deform. When loads are applied, they will change their shapes. They will elongate and bend and twist and, if we're not careful, may even break. Real objects are deformable, and engineers must take great care to ensure that their designs do not break or deform so much that they are unfit for purpose.

Have you ever wondered why civil engineers tend to use certain materials for bridges and buildings, while automotive engineers use an entirely different set of materials in cars? Meanwhile, aerospace engineers don't make rockets out of the same materials that biomedical engineers use for prosthetics. There are many material properties that determine applications that a given material may, or may not, be suitable for. Two fundamental concepts explored in this text are stress, which helps us determine whether an object will break under a given load, and strain, which relates to the object's deformation.

Depending on how external loads are applied, there are a variety of potential internal loads on our object. You are likely already familiar with these loads and their effects, even if you haven't explicitly thought about them.

- **Axial forces** push or pull on a body. They cause elongation and compression. For example, imagine pulling on the ends of an elastic band. The band will stretch significantly. Objects may be pulled apart or crushed by this load. Imagine doing the same with a piece of chalk. There will be a small amount of deformation and the chalk will break.
- **Bending moments** act to bend an object. Imagine propping a long piece of plywood up on two supports and standing in the middle. The wood will bend. If you jump and land hard enough, it may even break.
- **Shear forces** are created when one body (or part of a body) slides relative to another. They commonly occur in conjunction with bending moments.
- **Torsional moments** act to twist an object. Imagine holding the ends of a cable and rotating your wrists so that the cable twists around itself along its length. You will see the cable rotate through ever larger angles. Twist it too far and it will break.

This text explores the effects of these loads and will expand your understanding of these effects. It aims to describe these effects in a conceptual way, building on your existing understanding. Once these effects are understood conceptually, you'll learn how to calculate the exact amount of deformation and the exact size of the stress created as well as how to make sure that your designs stay within appropriate limits.

Scope

This book is intended for use in introductory undergraduate mechanics of materials courses. Such courses are typically early in the curriculum, following statics and generally serving as a prerequisite to more advanced solid mechanics courses. As such, this book begins with a review of static equilibrium before introducing the concepts of stress and strain and the relationship between them, along with pertinent material properties. The following chapters explore stress and deformation under four loading conditions: axial loads, torsional moments, bending moments, and transverse loads. Methods for determining internal loads and relevant geometric properties are included.

Once stress and deformation under each loading condition are well understood, stress transformation is introduced. This is followed by a brief discussion of thin-walled pressure vessels before tying together many of the previous chapters in an exploration of combined loading scenarios. Finally, column buckling is examined as a potential failure mode.

Target Audience

This textbook is intended for a wide range of engineering undergraduate students. The content is fundamental to several disciplines, including mechanical, civil, aerospace, and materials engineering. This text covers fundamental skills and concepts that students will rely on throughout their studies and their careers.

Motivation

Existing textbooks covering this material are expensive for students and tightly controlled by their publishers. There are some scattered open resources to aid student learning, but this book provides a cohesive resource that covers a typical mechanics of materials course from start to finish. It is free to use, and instructors are free to adapt the book to their specific needs.

Our experience teaching this class is that students report rarely using their textbooks for anything beyond assigned homework problems. They often find the books dense and impenetrable and struggle in a course they perceive to contain many disparate topics. This text attempts to more clearly demonstrate the links between the various topics and help students realize that the course is really about the application of just a handful of concepts. In our experience, students who make these connections find the course much easier than those who don't.

Approach and Features

A cohesive narrative across the book. Each chapter begins with a paragraph explaining the purpose of the chapter and where it ties into the course. Relevant topics are then introduced with a reason for studying the topic, the underlying concepts are explored, and any relevant equations are provided with derivations where appropriate. Applications are demonstrated, and each chapter ends with a summary of the key takeaways and equations.

Conceptual understanding first, followed by numerical problem solving. Conceptual understanding is key to performing well not only in this course but also in subsequent courses and as an engineer. Topics are approached conceptually first, with realistic applications used as a tool to aid understanding. Extensive worked examples are used liberally to demonstrate application of the concepts to numerical problems. Worked examples include thorough explanations of what is being done to solve the problem and why.

Diverse problem sets. Practice makes perfect, and so it is vital that students have diverse and extensive problem sets to practice applying the material. A large number of problems are included with each chapter, ranging from traditional idealized problems to problems with real-world applications in a variety of engineering disciplines.

This course is often high-enrollment and instructors often teach multiple sections. With so many students, grading can be a challenge. Consequently, the problem sets in this book have been recreated online with an interface that can intelligently provide feedback to students. Timely feedback is so important to student understanding. More than just letting students know whether they got the right or wrong numerical answer, this feedback helps students identify and understand their mistakes.

Color coding. Each chapter begins with defined learning objectives. Lists of learning objectives are blue.

Learning Objectives

- These blocks include leaning objectives for the chapter.
- These are short statements describing what you should be able to achieve upon completing the chapter.

Chapters are organized into short sections. Sections include explanatory text and worked examples for practice. Worked examples are green. They include a problem statement and a worked solution. Students may choose to access the solution at any time or may attempt the problem themselves before checking their solution against the worked example.

Worked Example

Here you'll find the problem statement and any relevant diagrams. The solution is in the next block, allowing you to attempt the problem before viewing the solution, if desired.

Solution

Here you'll find a full worked solution and final answer. The solution includes the final answer, all calculations, and an explanation of why each calculation is used.

Most sections end with step-by-step instructions for solving problems related to that section. These are orange.

Step-by-Step

1. These boxes contain a numbered list of steps for approaching problems related to each section.
2. These are helpful for reminding yourself how to approach problems.
3. However, they lack the conceptual detail that is vital for understanding.

Each chapter ends with a summary section containing key takeaways and key equations. Like the learning objectives, these are blue.

Organization

This book begins with a review of static equilibrium for single members, trusses, and frames. Internal loads are then introduced.

Chapters 2 and 3 contain an overview of stress and strain, and Chapter 4 discusses the relationship between them. Important material properties such as yield stress, elastic modulus, and Poisson's ratio are introduced, and Hooke's law is applied in one and three dimensions. Thermal strain and factor of safety are also introduced here.

The following chapters cover stress and strain under various loading conditions. Chapter 5 covers axial stress, deformation, stress concentrations, and thermal deformation. Statically indeterminate problems are also introduced here and repeated several times throughout the text. Chapter 6 covers torsional stress, deformation, power transmission, and statically indeterminate problems again.

Before considering bending and shear loads, Chapter 7 reviews several methods for determining internal shear force and bending moment. Equilibrium methods, integration, and graphical methods are all described. Chapter 8 provides methods for determining geometric properties such as centroid and second moment of area. Chapters 9 and 10 then discuss bending stress and shear stress respectively. These equations necessarily rely on the material from Chapters 7 and 8. Chapter 11 then covers beam deflection by integration and by superposition, as well as the application of these methods to statically indeterminate problems.

Once students have a thorough understanding of the effects of these four different loads, stress transformation is introduced in Chapter 12. General stress transformation in 2D and determination of the principal stresses and maximum in-plane shear stress are discussed first. Mohr's circle is then discussed as an alternative method and is expanded to cover out-of-plane shear stress.

Chapter 13 covers spherical and cylindrical thin-walled pressure vessels. Chapter 14 brings much of the earlier content together and introduces combined load problems. Eccentric axial loads are discussed first as a specific subset of combined load problems. The principles are then expanded to general combined load problems that include any of or all the previously discussed loads.

Finally, buckling of columns is considered as a potential failure mode. The course wraps up with a reminder that both stress and deformation must be considered and that anything students work on must satisfactorily resist both.

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Chapter 1: Statics Review

Learning Objectives

- Define external and internal force and moment reactions
- Calculate external reactions
- Determine internal pin and two-force member reactions for structures composed of connected members
- Determine internal reactions in continuous bodies
- Calculate reactions in three dimensions

All bodies and structures discussed in this text are assumed to be in static equilibrium, which means they experience no acceleration (the sum of all forces and moments are equal to zero).

Any body that is not moving or that is moving at constant velocity is in static equilibrium. However, unlike in a standard statics course—where all bodies are assumed to be rigid—in this text bodies are expected to be deformable. This means they may stretch, contract, twist, bend, break, buckle, and so on.

To determine how an applied loading situation affects a given body or structure—and potentially causes it to deform—we must first apply statics to establish the distribution of forces and moments within the body. This is the first step of many problems.

This chapter reviews those aspects of statics that are prevalent throughout this course.

- Section 1.1 reviews the concept of external loads and their representation on free body diagrams.
- Section 1.2 reviews equilibrium of structures in two dimensions, including analysis of two-force members and multiforce members.
- Section 1.3 addresses internal loads, including normal force, shear force, and bending moment.
- Section 1.4 reviews equilibrium in three dimensions.

1.1 External Reactions and Free Body Diagrams

External forces and moments are those that act on the boundaries of a system. They include loads applied to the main structure (weight, wind, pressure, etc.) and the reactions these loads induce in supporting elements (pins, rollers, welds, etc.).

In the case of external reactions, the term *reaction* refers to the forces and/or moments exerted by the supports on the body in response to the applied loading to maintain equilibrium.

Finding the external reactions is the first step in many of the problem types covered in this text. This process begins with drawing a free body diagram (FBD) of the body.

An FBD is a sketch of the body “freed” from its supports, showing all the applied and reaction forces and moments to be considered in the analysis. In cases where some loads are insignificant compared to others (e.g., weight sometimes falls into this category), those loads are omitted from the FBD.

The importance of drawing FBDs as a first step in calculating internal and external loads and reactions cannot be overstated. Creating these diagrams establishes which loads and reactions are being included in the analysis, as well as their assumed directions. With the diagrams prepared, you can maintain consistency when summing forces and moments to apply equilibrium equations, helping to ensure accurate results.

Some commonly encountered supports, along with their corresponding reactions and FBD representations, include the following:

- **Pin supports**: Pin supports resist any lateral movement of the body relative to the pin at its location. The direction of the overall reaction force (F) is generally unknown, so it is represented as a set of perpendicular force components (F_x and F_y) in the figure below but may also be represented as any set of perpendicular components.

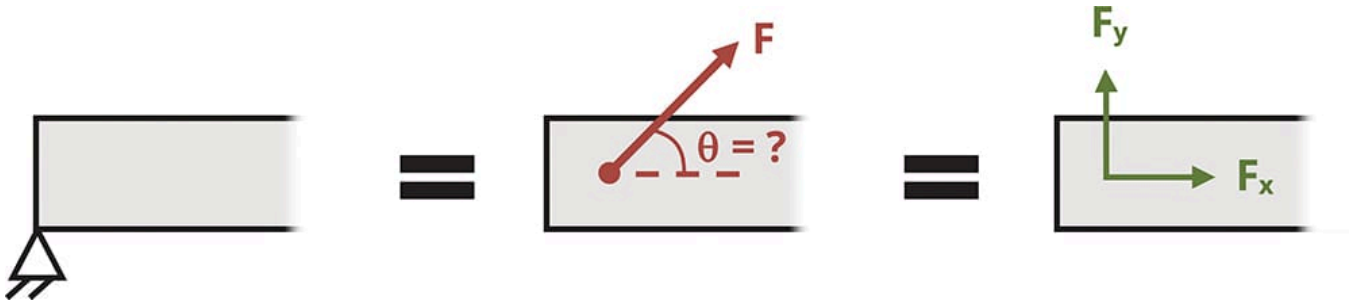


Figure 1.1: Pin supports

- **Normal supports**: Normal supports generally consist of surfaces on which the body simply rests (like a beam on a roller) and/or supports that are not fixed to any other surface (like a rocker). Similar to pin supports, normal supports prevent movement of the body relative to the support at its location. However, unlike pins, normal supports prevent only movement in the direction normal to the support site. Consequently, the reaction force (F) acts strictly perpendicular to the supporting surface.

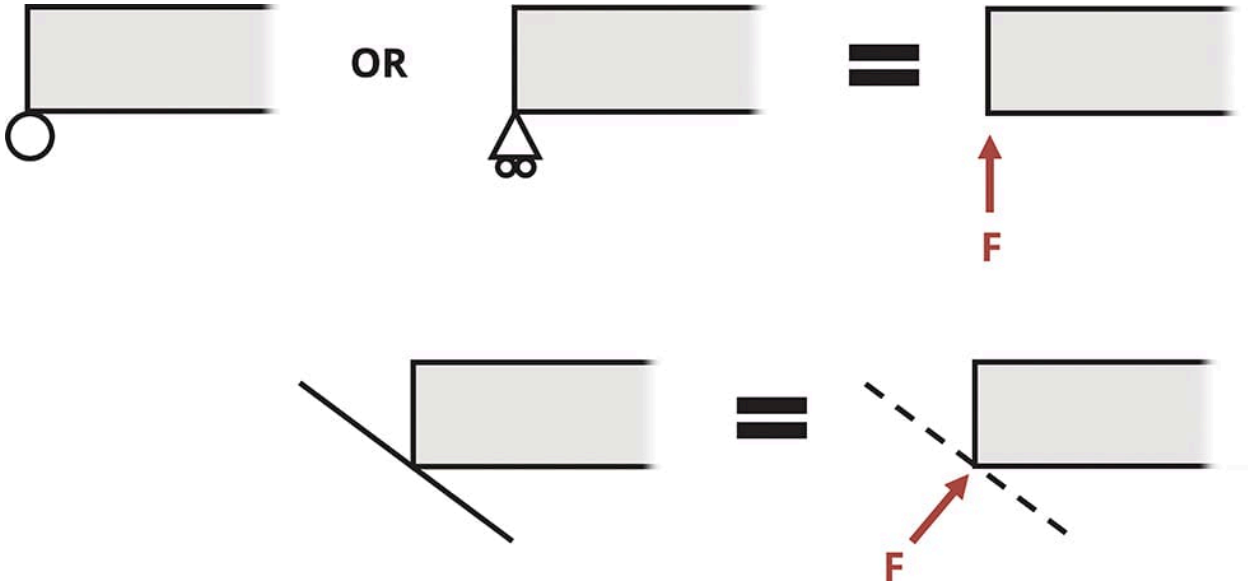


Figure 1.2: Normal supports

- **Cable**: A cable provides a tensile force reaction (T). It pulls on the body but never pushes. The reaction force acts in a direction along the path of the cable, extending away from the body and toward the external attachment point.



Figure 1.3: Cables

- **Fixed support**: A fixed support is capable of resisting both lateral movement and rotation, so its reactions consist of a reaction force (F) and a reaction moment couple (M). As with the pinned support, the direction of the overall reaction force is not necessarily known and is therefore represented by a set of perpendicular components.

  It is important to remember that the reaction moment couple to be solved for is an unknown, just like the reaction force components, and must be included in the moment equation about any point on the body.

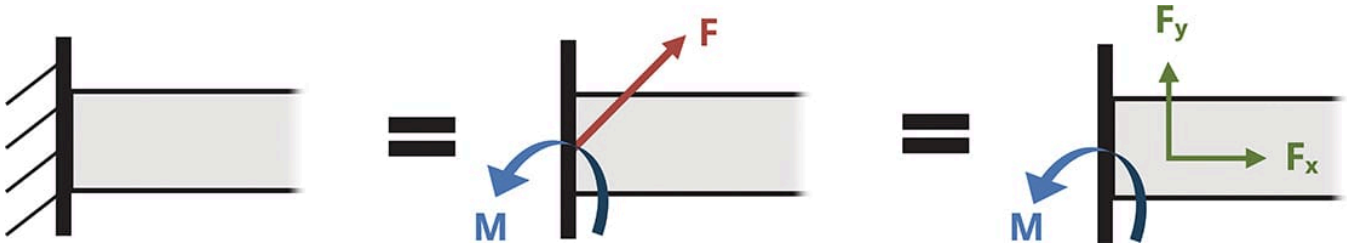


Figure 1.4: Fixed supports

Note that when components of the reaction force are found individually, as with the pin support and fixed support, the overall magnitude of the force (F) can be calculated as

$$F = \sqrt{F_x^2 + F_y^2}.$$

Its direction (i.e., angle measured from the horizontal axis) can be calculated as

$$\tan\theta_x = \frac{F_y}{F_x}.$$

Note how the FBDs are drawn in the examples provided throughout this chapter. Geometrical details of a body—such as lengths and angles—are often included on FBDs.

However, in this text these features are frequently excluded because they are provided in the problem statement, and the focus here is on representing the loading. You should keep in mind that in some courses and with some instructors these geometrical details may be considered required features of FBDs.

1.2 Equilibrium in two Dimensions

Once the FBD is drawn, the next step is to apply the equilibrium equations. In two dimensions (x-y plane) these are

$$\boxed{\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_{any\,point} = 0}.$$

(1.1)

A review of finding moments is given in Section 1.4.

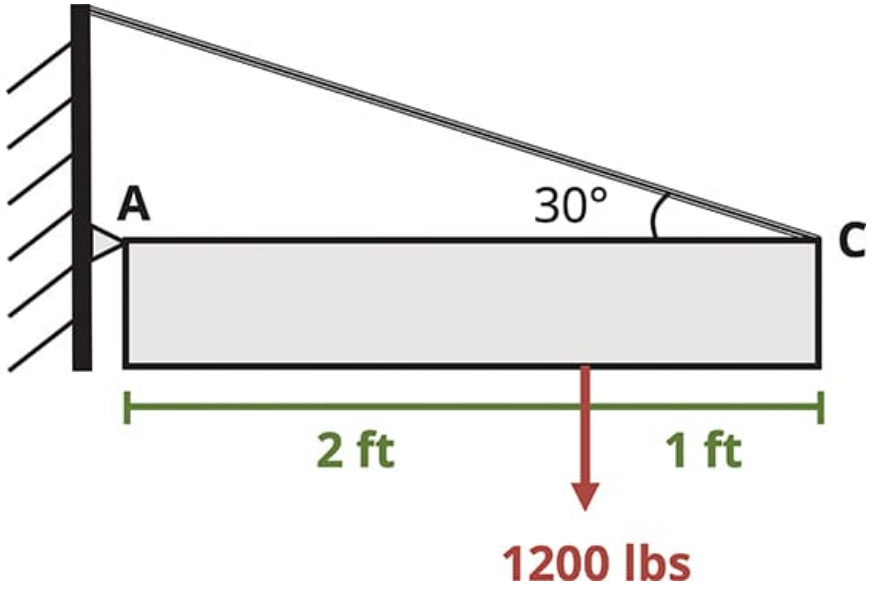
Since there are three equations, a statically determinate problem should have no more than three unknowns.

Example 1.1 illustrates the process of finding external reactions.

Example 1.1

A 3 ft beam is supported by a pin connection at the wall at point A and a cable at point C as shown. A load is applied 2 ft from point A.

Find the force in pin A as well as the tensile force in the cable.

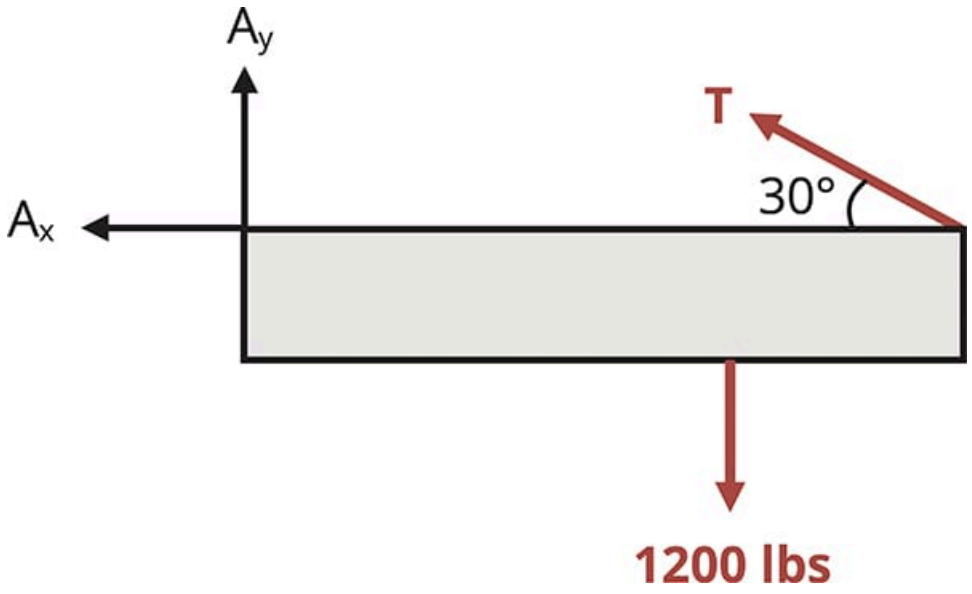


Solution

Step 1: Draw the FBD.

The beam is supported with a pin at A, so the support is represented on the FBD with force components A_x and A_y. Note that an assumption must be made about whether A_x points right or left and whether A_y points up or down. The actual directions of A_x and A_y are determined by whether the calculated values are positive or negative. A positive value means the assumed direction was correct, and a negative value means the force acts in the opposite direction.

Cables always exert a force that pulls away from the body. In the FBD below, the cable is represented by force T.



Step 2: Apply equilibrium equations.

If we start with the moment about A equilibrium equation, we can solve for the tensile force T since A_x and A_y do not contribute to the moment about A. Then substituting T into the two-force equilibrium equations allows us to find A_x and A_y.

$$\sum M_A = T\sin(30^\circ) * 3\ ft - 1,200\ lb * 2\ ft = 0$$
$$T = 1,600\ lb$$

$$\sum F_x = -A_x - T\cos(30^\circ) = 0$$
$$-A_x - 1,600\ lb\ \cos(30^\circ) = 0$$
$$A_x = -1,385.6\ lb$$

$$\sum F_y = A_y + T\sin(30^\circ) - 1,200\ lb = 0$$
$$A_y + 1,600\ lb\ \sin(30^\circ) - 1,200\ lb = 0$$
$$A_y = 400\ lb$$

Since A_x is negative, it acts in the opposite direction from the assumed direction on the FBD. Thus, A_x acts in the positive x direction.

The total force in pin A is

$$F_A = \sqrt{A_x^2 + A_y^2} = 1,442\ lb.$$

Answer:

- T = 1,600 lb
- F_A = 1,442 lb

1.2.1 Two-Force Members

One special type of pin connection for which the direction of the reaction force is known is one in which the pin is connected to a two-force member.

Contrary to its name, a two-force member is not necessarily a member on which only two forces act. Rather, it is a member where forces are applied at only two distinct locations regardless of how many forces there are.

A two-force member can have any shape, as is demonstrated in Figure 1.5. One easy way to recognize a two-force member is by noting the presence of only two pinned points and no other locations where a force or moment couple is applied.

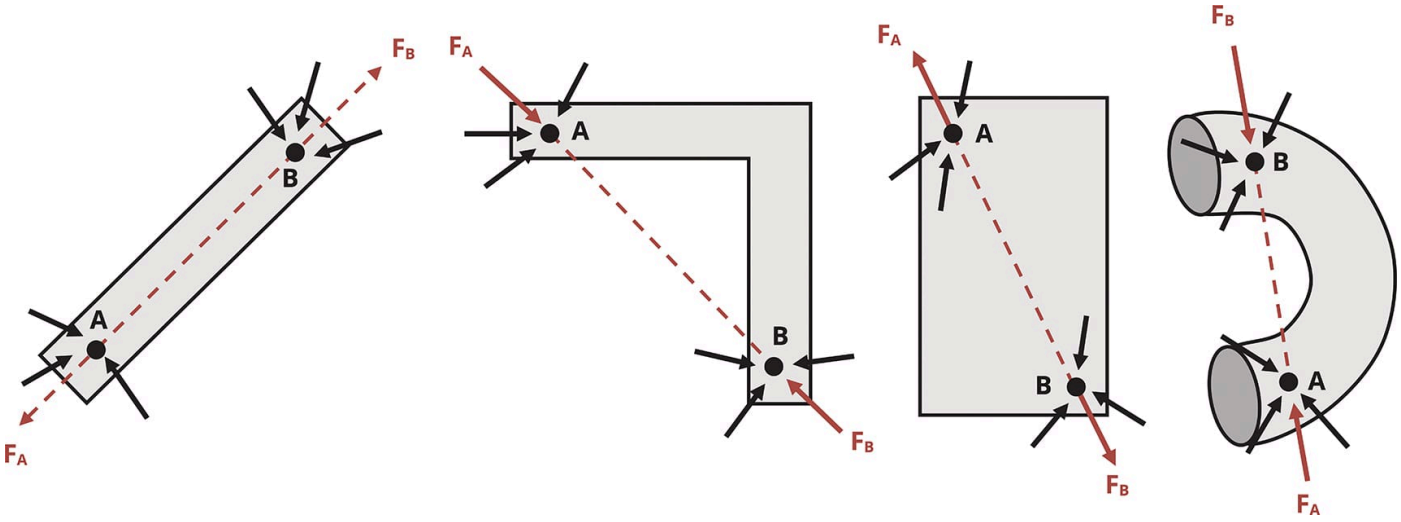


Figure 1.5: Illustrations of two-force members showing the line of action of the reaction force (red arrows) due to applied forces (black arrows) passing through the joints. Note that $F_A = F_B$.

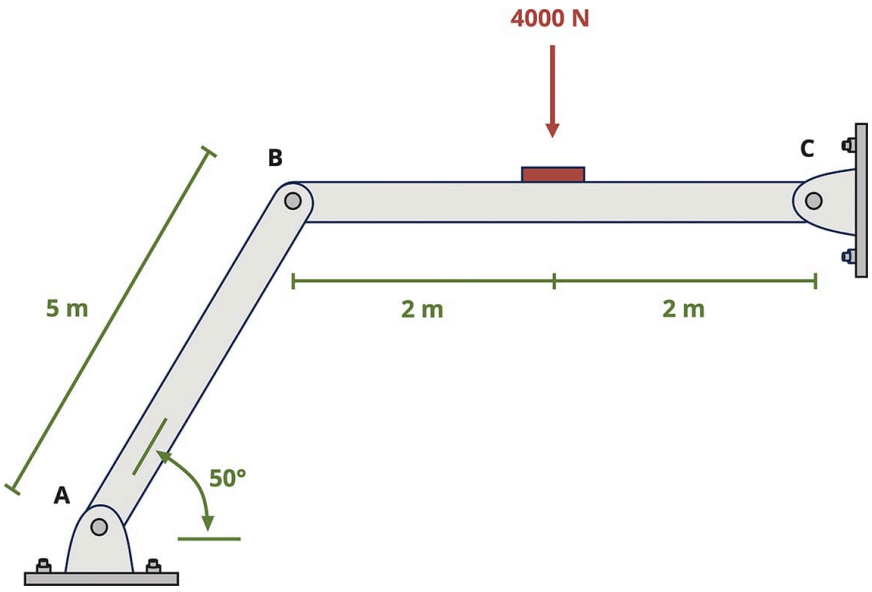
Once a member is identified as a two-force member, it can be concluded that the resultant force at both pins is equal in magnitude ($F_A = F_B$ in Figure 1.5), is opposite in direction, and follows a line of action that goes through the pins.

For a straight member, the internal force (discussed in Section 1.3) is equal to the reaction force in the pin.

The presence and recognition of two-force members simplifies equilibrium calculations by reducing the number of unknowns before any computations are performed. This is demonstrated in Example 1.2.

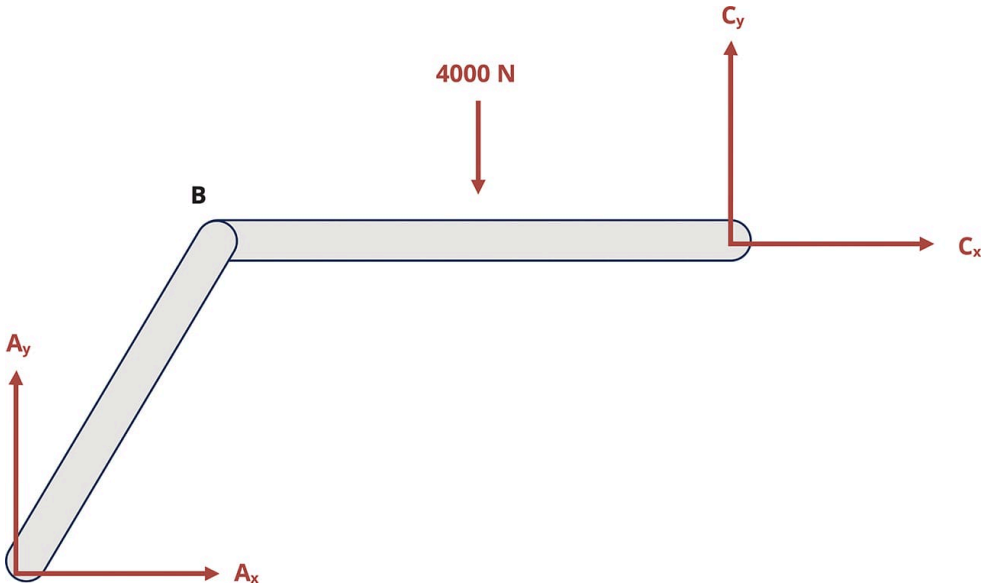
Example 1.2

Determine the force in pin C and in pin A.



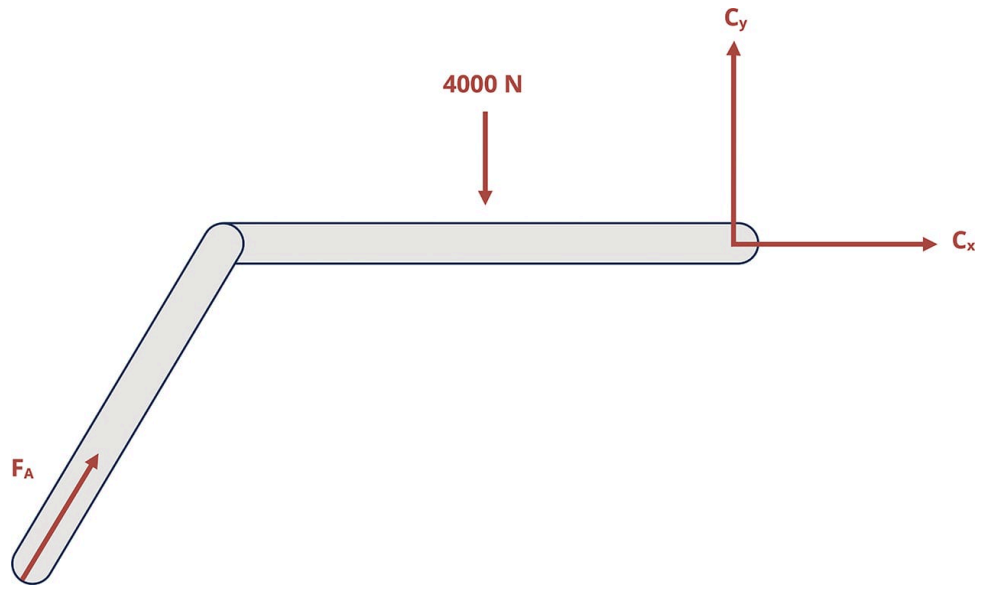
Solution

Step 1: Draw the FBD.



Given this FBD, it appears that there are four unknowns, so it cannot be solved with only three equilibrium equations.

However, recognizing that bar AB is a two-force member (since there is a pin at A and a pin at B but no other forces acting on that bar), we know that the line of action of the reaction force at A goes through points A and B. Thus, the FBD can be redrawn with only three unknowns.



Step 2: Apply equilibrium equations.

$$\sum M_C = F_A\cos(50^\circ) * (5\ m)\sin(50^\circ) - F_A\sin(50^\circ) * (4\ m + 5\ m(\cos(50^\circ)) + 4,000\ N * 2\ m = 0$$

$$\sum F_x = F_A \cos(50^\circ) + C_x = 0$$

$$\sum F_y = F_A \sin(50^\circ) - 4,000\ N + C_y = 0$$

Solving the first equation for F_A yields F_A = 2,611 N.

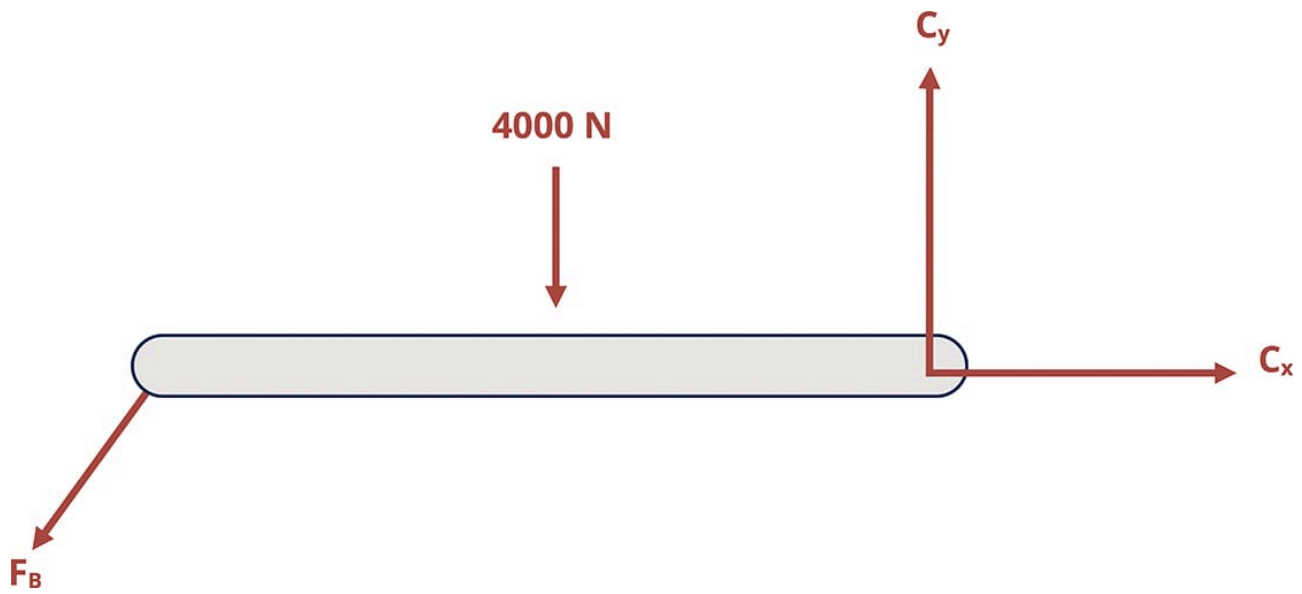
Substituting F_A into the force equations yields C_x = -1,678 N and C_y = 2,000 N.

Knowing C_x and C_y, we can find the overall pin force F_C.

$$F_C = \sqrt{C_x^2 + C_y^2} = 2,611\ N$$

Since C_x is negative, it points in the opposite direction from the assumed FBD. To avoid modifying the FBD, continue assuming the original direction but substitute in the negative value in further calculations.

As an alternative solution (explored in Section 1.3), bar AB and bar BC can be separated by removing the pin at point B and drawing the FBD of bar BC as shown. (Recall that because bar AB is a two-force member, F_B goes through points A and B and that $F_A = F_B$. We can solve for F_A by solving for F_B.)



$$\sum M_C = F_B \sin(50^\circ) * 4\ m + F_B \cos(50^\circ) * 0 + 4,000\ N * 2\ m = 0$$

This yields the same result: $\mathrm{F_A} = 2,611\ \mathrm{N},$ and the rest of the problem proceeds the same way.

Answer:

- F_A = 2,611 N
- F_C = 2,611 N

1.3 Internal Reactions

Internal reactions can be forces and moments at connection points between members (e.g., pins connecting multiple members of a frame, machine, or truss), as well as reactions at any point in a continuous body (e.g., a point in the middle of a beam). These reactions are the forces and/or moments necessary to hold a structure or a body together and are ultimately the loading aspect needed to determine if and how a body will deform or even break.

1.3.1 Internal Reactions at Pinned Connections

An FBD of an entire structure consisting of connected members omits the forces at the connecting pins because the connected members exert equal and opposite forces on each other, and these forces cancel each other in equilibrium equations. The pins

that connect the members nonetheless experience reaction forces you must determine to ensure the integrity of the structure. Do the following to find the reaction forces in these internal pins:

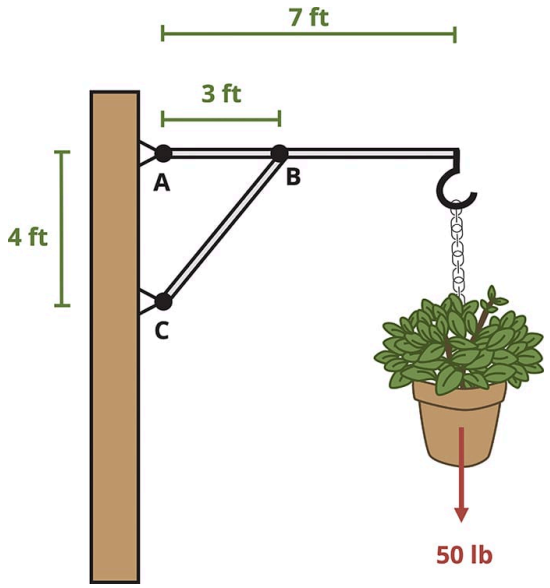
- Prepare the individual members' FBDs (imagine the pins pulled out and consider each member separately). With the members disconnected, the reaction forces at the connecting pins no longer cancel each other in equilibrium equations applied to the individual members. The reaction forces are drawn on the FBDs in the same way as for support pins, described in Section 1.1.
    - However, since the connected members do ultimately exert equal and opposite forces on each other, the reaction forces of pins common to two members are drawn in opposite directions on these members' FBDs.
- Given the FBDs, write and solve equilibrium equations for each body. Because three equilibrium equations can be determined for each FBD drawn, one could theoretically solve for three times the number of unknowns.
    - Note that the FBD of the whole structure may be useful to help solve for some unknowns.
    - In addition, as discussed, when one of the connected members is a two-force member, the reaction at the pin runs through the points where the forces act on the two-force member.

Example 1.3 and the alternative method used for Example 1.2 demonstrate these concepts.

Example 1.3

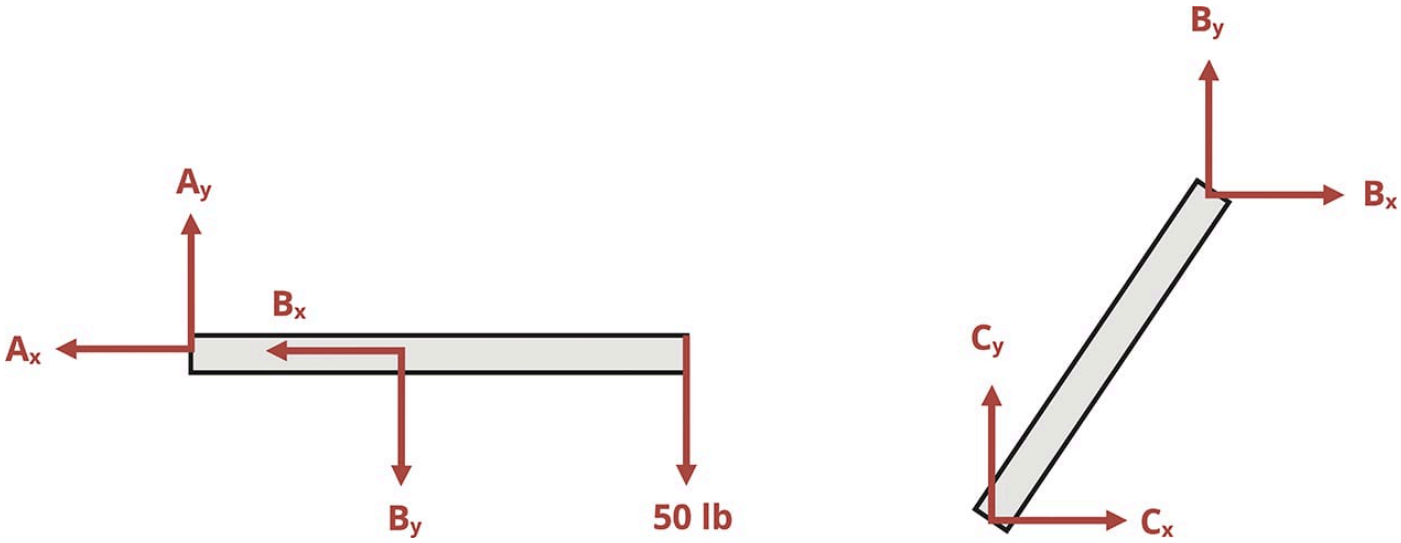
A plant hanger is secured to a wall with a pin and is supported by a pin brace connected to the hanger at B and to the wall at C.

Determine the external reactions at A and C as well as the reaction in the internal pin B.

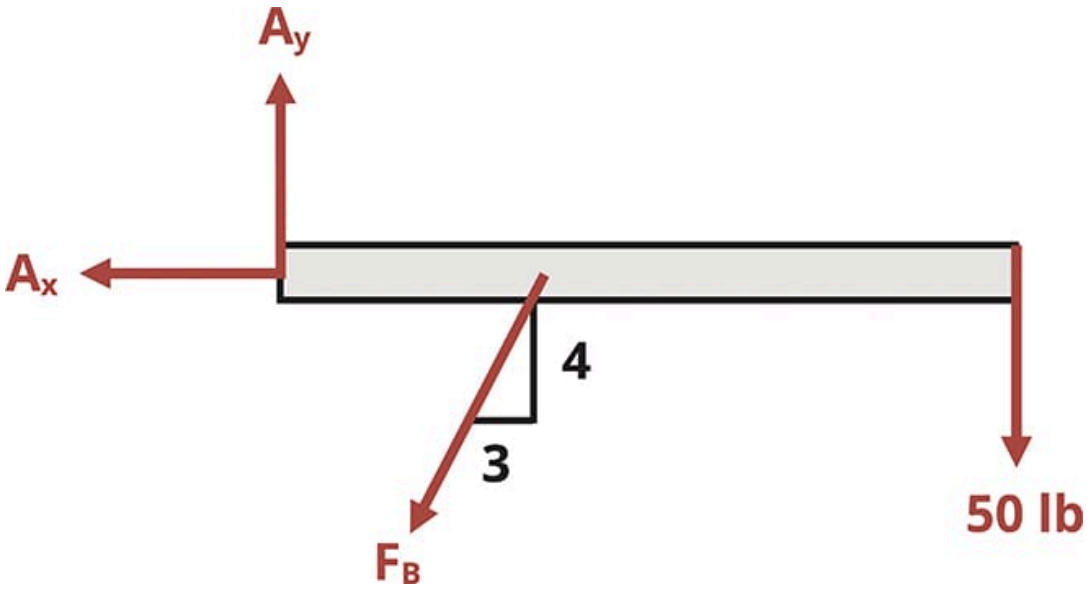


Solution

If BC were not a two-force member, we would approach the problem by separating the brace from the hanger and drawing an FBD of each part separately. Notice that B_x and B_y are drawn in opposite directions in the two different diagrams because the pin exerts an equal and opposite force on each bar.



Six unknowns and six equilibrium equations (three equations per body) are now available to use per bar, so the problem is technically solvable. However, since BC is a two-force member—there is a pin force at B and a pin force at C but no other forces at any other point on the bar—the reaction force at B follows a line of action that goes through B and C. We can also conclude that the force in pin C is equal to the force in pin B. The FBD of bar AB can be redrawn.



With the components B_x and B_y replaced with the resultant force F_B with known direction, the number of unknowns on bar AB is reduced to three. These unknowns can be solved for using the equilibrium equations.

$$\sum M_A = F_B\left(\tfrac{4}{5}\right) * 3\ ft - 50\ lb * 7\ ft = 0$$
$$\sum F_x = -A_x + F_B\left(\tfrac{3}{5}\right) = 0$$
$$\sum F_y = A_y + F_B\left(\tfrac{4}{5}\right) - 50\ lb = 0$$

Solving the equations yields F_B = 145.8 lb, A_x = 87.5 lb, and A_y = -66.7 lb. Therefore, the pin force in pin B is 145.8 lb and the pin force in A is $\mathrm{A} = \sqrt{\mathrm{A_x^2 + A_y^2}} = 110\,\mathrm{lb}$. Since BC is a two-force member, $F_C = F_B$.

Answer:

- F_A = 110 lb
- $F_B = F_C$ = 146.8 lb

1.3.2 Internal Reactions in Truss Structures

Truss structures are made up of only two-force members. The two main methods of determining internal reactions in planar (2D) truss structures are method of joints and method of sections.

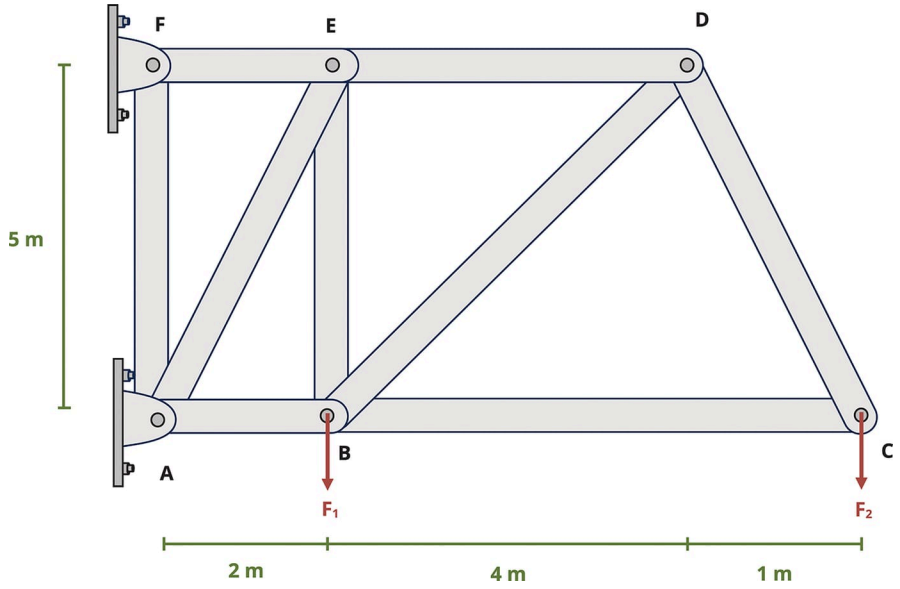
In using method of joints, we draw an FBD of the connecting pins (joints) within the truss. Since all the members connected at any given pin will be two-force members, the reactions can be drawn in known directions. However, because the forces all pass through the same point on the body, the moment equilibrium equation is not useful, so only the forces equilibrium equations can be used for each joint. This means that only two unknowns can be solved for at each joint. The method of joints is most useful when the forces of all the truss members are sought or when the only forces sought are attached to a joint with only two members.

To use method of sections, we cut through the truss structure and with the FBD analyze the intact part of the structure either to the left or to the right of the cut. The FBD of either side will show the applied forces and the reaction forces from the cut-through members. These reactions will be equal in magnitude but opposite in direction between the two sides of the cut. The side to examine is usually that which will not require finding external reactions (if there is a free end to the truss) and/or is least complicated to deal with in terms of geometry or applied loads. All three equilibrium equations can generally be effectively applied with method of sections, so three unknowns can be solved for with any given cut. Keep in mind that moments can be taken about points that lie outside the isolated section, which can lead to more efficient solutions.

Example 1.4 demonstrates the use of method of joints and method of sections to solve for forces in truss members.

Example 1.4

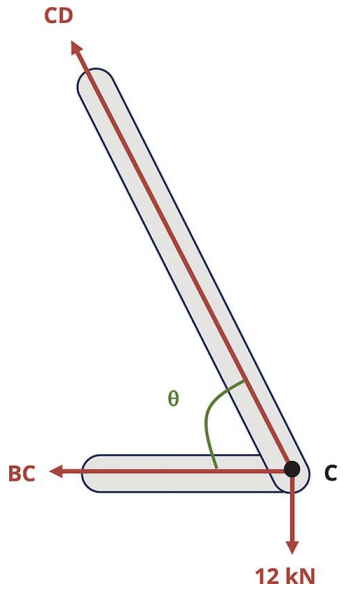
Determine the forces in members DE and BE. Let F_1 = 8 kN and F_2 = 12 kN.



Solution: Method of Joints

With method of joints, always work with a joint that has no more than two unknown forces acting on it. These unknown forces come either from members attached to the joint or from external support reactions. Joint C is a good place to start here, as there are only two unknown forces: BC and CD.

FBD Joint C



Angle θ can be found by setting up a right triangle at C with a base of 1 m and a height of 5 m. Once this angle is known, we can use equilibrium equations to find the forces in members BC and CD.

$$\theta = \tan^{-1}\left(\frac{5}{1}\right) = 78.7°$$

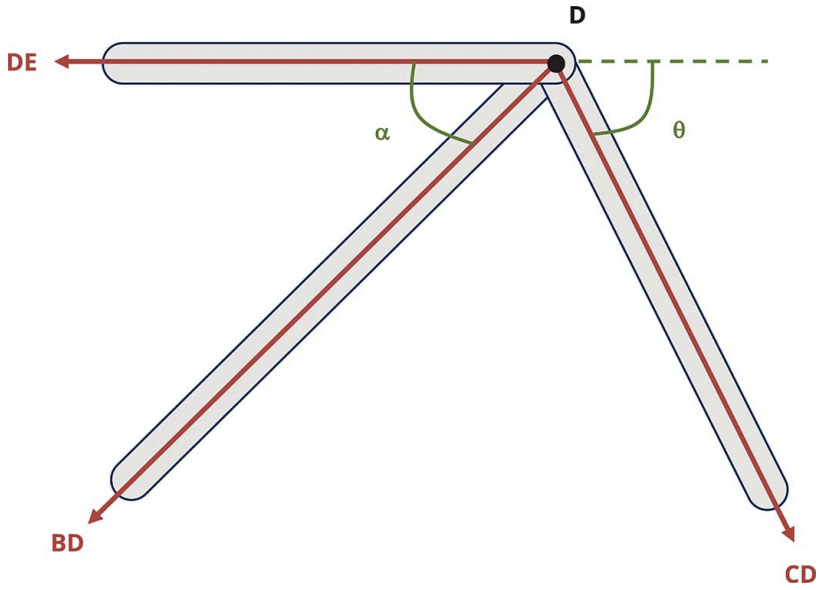
$$\sum F_y = CD\ \sin\ \theta - 12 = 0 \rightarrow CD = 12.2\ kN$$

$$\sum F_x = -BC - CD\ \cos\ \theta = 0 \rightarrow BC = -2.4\ kN$$

Since the forces in the FBD are shown to be tensile—the forces are pointed away from the joint—the negative answer indicates that force BC is actually compressive.

Now that force CD is known to be 12.2 kN in tension, two unknowns remain at joint D: BD and DE.

FBD Joint D



We know the 12.2 kN force acts at an angle of θ = 78.7° from the horizontal. Angle α can be found by considering right-angle triangle BDE, which has a base of 4 m and a height of 5 m. Once this angle is known, we can use equilibrium equations to find

the forces in members BD and DE.

$$\alpha = \tan^{-1}\left(\frac{5}{4}\right) = 51.3°$$

$$\sum F_y = -BC\ sin(\theta) - BD\ sin(\alpha) = 0$$

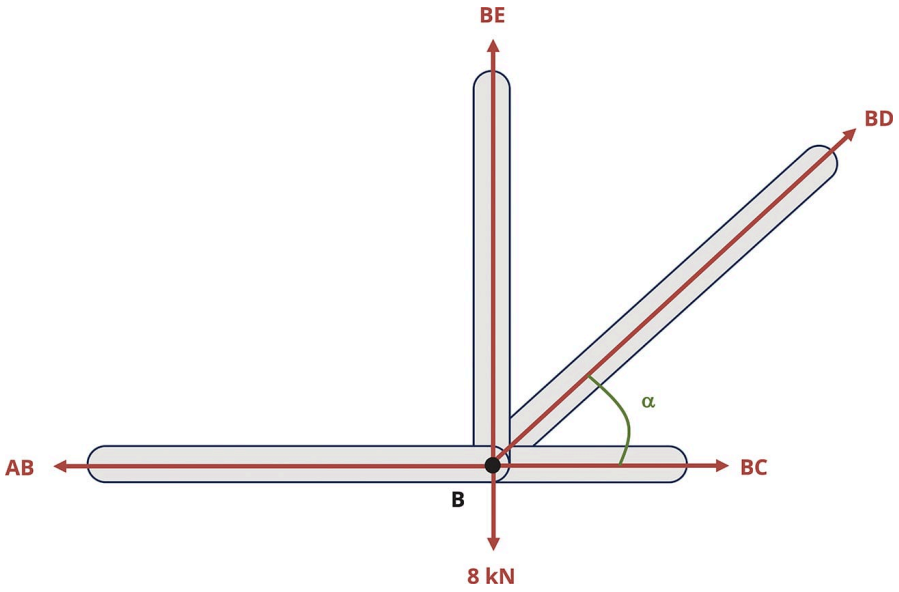
$$= -12.2\ \sin(78.7°) - BD\ \sin(51.3°) = 0 \rightarrow BD = -15.4\ kN$$

$$\sum F_x = -DE - BD\ cos(\alpha) + 12.2\ cos(\theta) = 0$$

$$= -DE - (-15.4\ kN)\cos(51.3°) + 12.2\ kN \cos(78.7°) = 0 \rightarrow DE = 12kN$$

Force BD is 15.4 kN in compression and force DE is 12 kN in tension. Now we need to find force BE. Two unknowns remain at joint B: AB and BE.

FBD Joint B



$$\sum F_y = BE - 8 + BD\ \sin(\alpha) = 0$$

$$= BE - 8 + (-15.4)\sin(51.3°) = 0 \rightarrow BE = 20\ kN$$

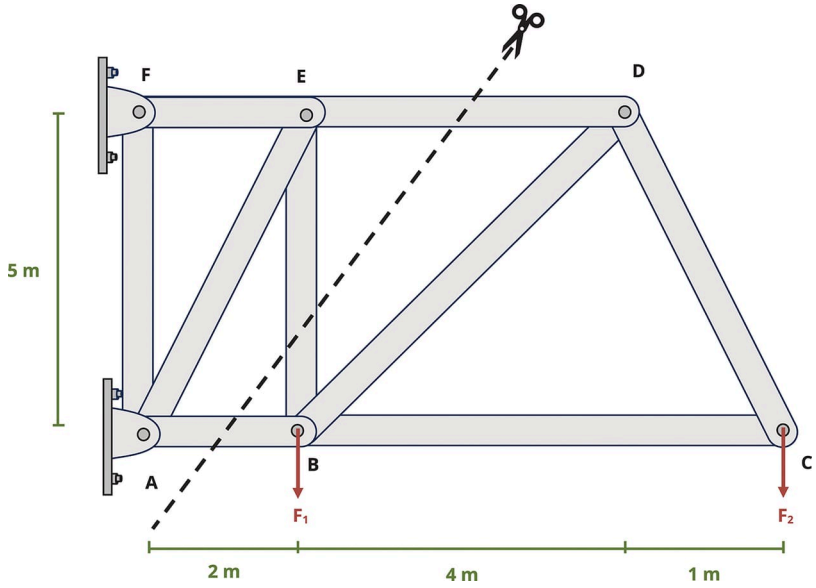
Answer:

- DE = 12 kN (Tensile)
- BE = 20 kN (Tensile)

Solution: Method of Sections

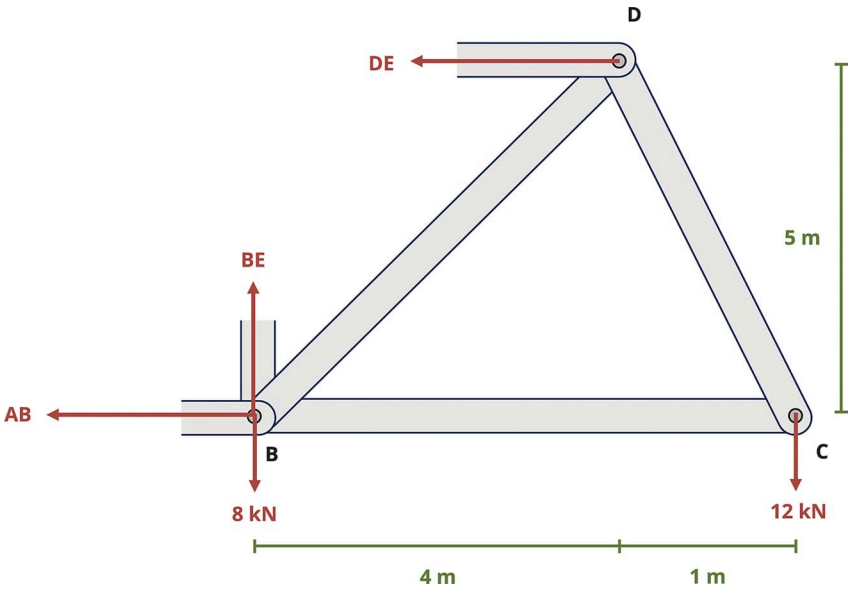
Although method of joints can be used here, it is an inefficient approach requiring us to draw and analyze FBDs of joints C, D, and B as well as to keep track of the forces at these joints and whether each force is in tension or compression.

As an alternative, we may use method of sections to find forces BE and DE. To apply this method, cut through no more than three unknown members and draw an FBD with no more than three total unknowns. For this problem, make a cut through the truss that passes through members AB, BE, and DE.



Once that cut is made, draw an FBD for the intact part of the truss either to the left or to the right of the cut. The external forces on the chosen side and the force of each cut-through member need to be shown on the FBD.

If we chose the left side, there would be unknowns AB, BE, and DE as well as unknown reactions A_x, F_x, and F_y. However, the right side would have no unknown support reactions, so the only unknowns are forces AB, BE, and DE.



$$\sum M_B = -(12 * 5) + (DE * 5) = 0 \rightarrow DE = 12\ kN$$
$$\sum F_y = BE - 8 - 12 = 0 \rightarrow BE = 20\ kN$$

Once again the forces are drawn on the FBD in the tensile direction. The positive answers confirm that the forces are indeed tensile.

Answer:

- DE = 12 kN (Tensile)
- BE = 20 kN (Tensile)

1.3.3 Internal Reactions in Continuous Bodies

Internal reactions also exist within a body or structure. These reactions are necessary to hold the body together and vary from point to point in a body depending on the distribution of external loading. As shown in Figure 1.6, the reactions at any given point can be examined by making a cut at the point of interest in the body. Think of any point within a body as acting as a fixed support for the rest of the body. That is, every point must potentially exert a force parallel to the cross-section where the cut is made, which is the shear force V; a force perpendicular to the cross-section where the cut is made, which is the normal force N; and a reaction moment where the cut is made, which is the bending moment M. Moreover, as discussed for internal pin reactions and the cut made for method of sections for trusses, the reactions at a cut will be equal and opposite on the two sides of the cut.

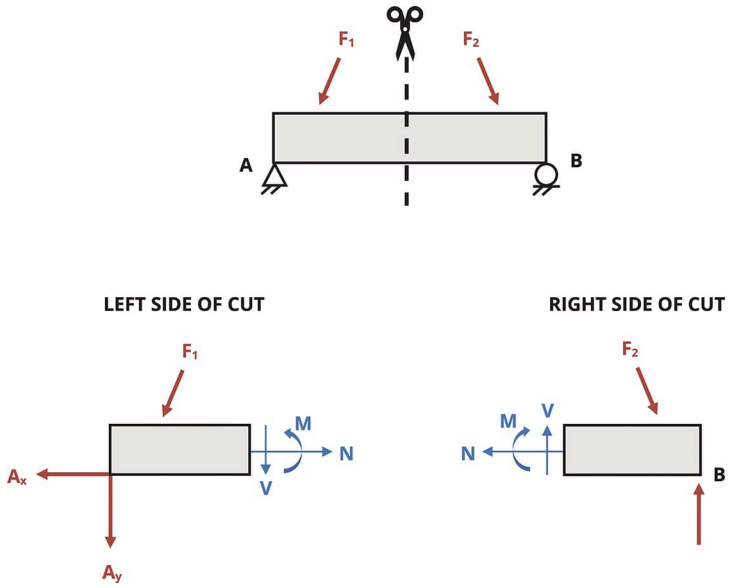


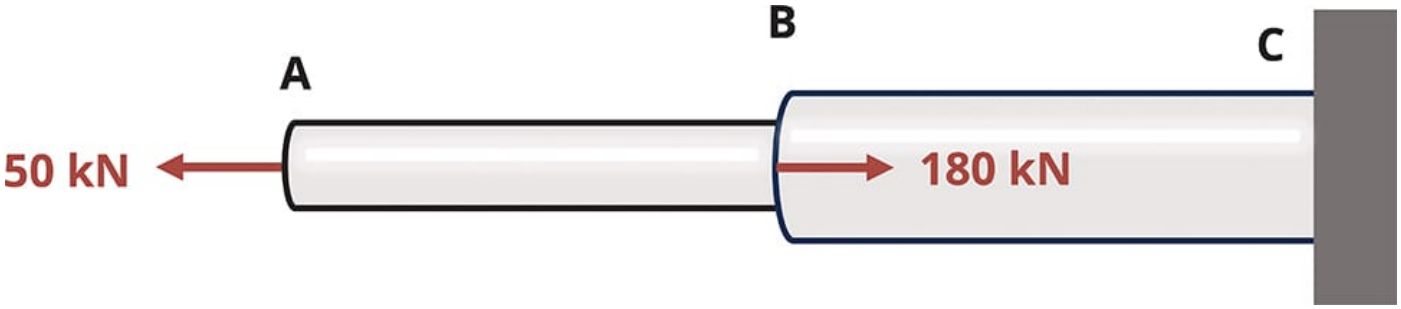
Figure 1.6: Cross-sections showing the internal loads on a continuous body

To determine the reactions, draw and use the FBD of the part of the beam either to the left or to the right of the cut. As with method of sections for trusses, choose which side of the cut to examine primarily by determining which side appears easiest and most efficient to analyze. Once the FBD of the cut section is drawn, the three equilibrium equations can be applied to determine the internal reactions. The determination of shear and bending moments in beams is reviewed in more detail in Chapter 7. The focus of Example 1.5 is on the determination of the normal force, as this is important in Chapter 2 and Chapter 3.

Example 1.5

Two solid bars make up the axial assembly loaded as shown.

Determine the normal force in each bar. State whether the force is tensile or compressive.



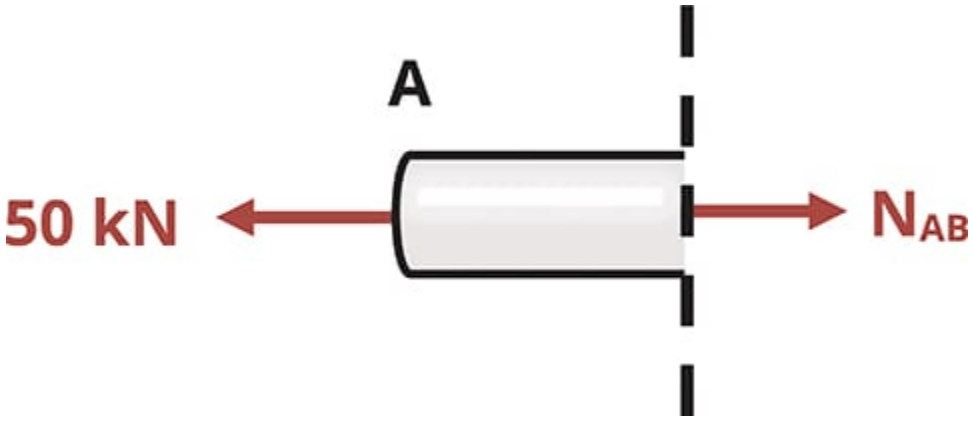
Solution

Though the assembly is not a beam, determining the internal reactions works in the same way. In this particular case, all the

forces are in the normal direction (no shear force). Furthermore, because of the central placement of the 50 kN and 180 kN forces, there is no bending moment. Consequently, only the normal reaction force is drawn on the FBDs.

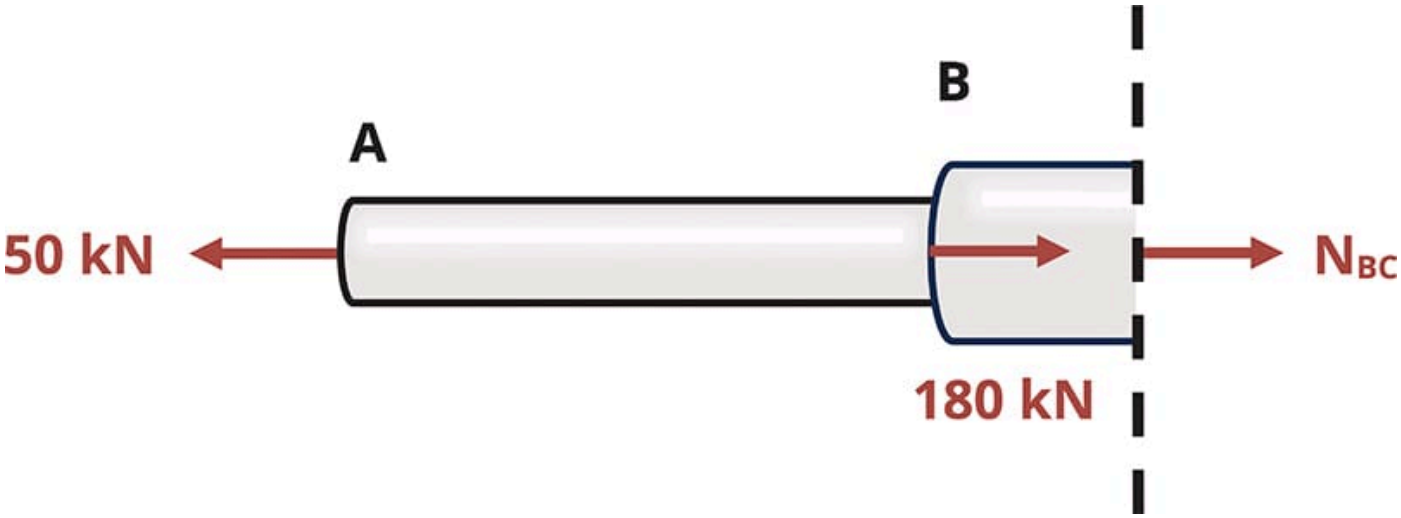
To find the internal load in a segment, cut a cross-section within that segment. Since the external loading (and therefore the internal loading) changes at point B, make one cut in segment AB and another in segment BC.

Making the cut in section AB and drawing the FBD allow us to determine the normal force in section AB. Drawing the FBD of the left section of the cut allows us to avoid needing to know the external reactions at wall C.



$$\sum F_x = -50\ kN + N_{AB} = 0 \rightarrow N_{AB} = 50\ kN$$

Making the cut in section BC and drawing the FBD allows us to determine the normal force in section BC. Once again draw the side to the left of the cut to avoid needing the reaction at the wall.



$$\sum F_x = -50\ kN + 180\ kN + N_{BC} = 0 \rightarrow N_{BC} = -130\ kN$$

For both AB and BC, the internal normal force was assumed tensile in the FBD and equilibrium equation. In the case of N_{AB}, the positive answer confirms that it is tensile. In the case of N_{BC}, the negative answer reveals that it is compressive.

Answer:

- N_{AB} = 50 kN (Tensile)
- N_{BC} = 130 kN (Compressive)

1.4 Equilibrium and Reactions in Three Dimensions

While we can often model structures as two dimensional, as in the previous examples, many real-life structures are subjected to forces and moments in all directions, necessitating the consideration of three-dimensional forces and three-dimensional moments. The process of finding external and internal reactions is the same in 3D as in 2D, but there are three additional equilibrium equations and three additional reactions.

For 3D systems there are six total scalar equilibrium equations as shown.

$$\begin{array}{|ll|}\hline \sum F_x = 0 & \sum M_x = 0 \\ \sum F_y = 0 & \sum M_y = 0 \\ \sum F_z = 0 & \sum M_z = 0 \\ \hline \end{array}$$

(1.2)

Recall that the moment vector describes the axis around which the body tends to rotate. Each individual component represents the tendency of the body to rotate around the specified axis.

In 3D there are three internal forces and three internal moments (Figure 1.7). One normal force (N_x) is perpendicular to the cross-section, and two shear forces (V_y and V_z) are parallel to the cross-section. Two bending moments (M_y and M_z) act around the axes parallel to the cross-section, and one torsional moment (T_x) acts around the axis perpendicular to the cross-section. This textbook examines each of these loads.

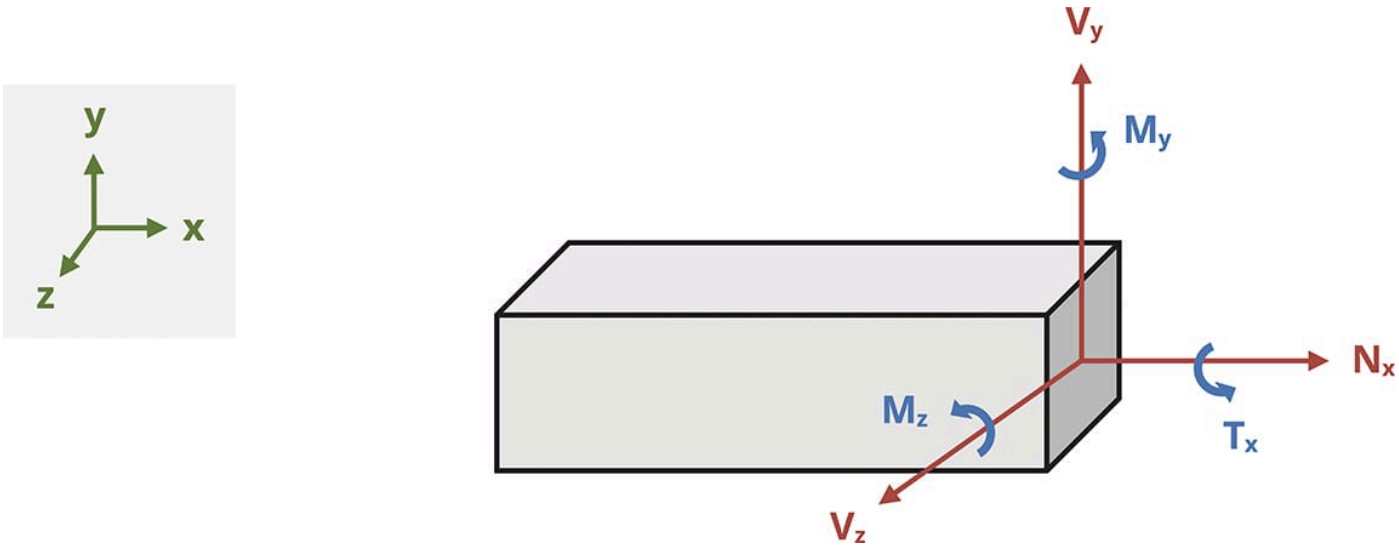


Figure 1.7: In 3D there are three internal forces (normal force N_x and two shear forces V_y and V_z) and three internal moments (torsional moment T_x and two bending moments M_y and M_z).

Note that here the choice of coordinate system is arbitrary (as long as it is a right-handed coordinate system). Loads are defined as normal, shear, bending, or torsional according to the effect they have on the body. For example, the normal force always acts in the direction perpendicular to the cross-section of the body and has the effect of extending or compressing the body. In the case of Figure 1.7, the x-axis is the axis perpendicular to the cross-section, so the force in the x direction is the normal force. If the axes or the body were oriented differently, the x component of the force could be a shear force instead of a normal force.

Whereas summing reaction forces in 3D is a straightforward process of adding forces in each direction, summing moments in 3D can be more complicated. To sum moments, generally two options are available. One option is to use the cross-product to calculate moments: $\overrightarrow{\mathrm{M}} = \overrightarrow{\mathrm{r}} \times \overrightarrow{\mathrm{F}}$.

1.4.1 Calculating Moments

Any position vector that goes from the point the moment is about to any point on the line of action of the force (F) will work for $\vec{\mathrm{r}}$ in the cross-product. Considering Figure 1.8, we could say

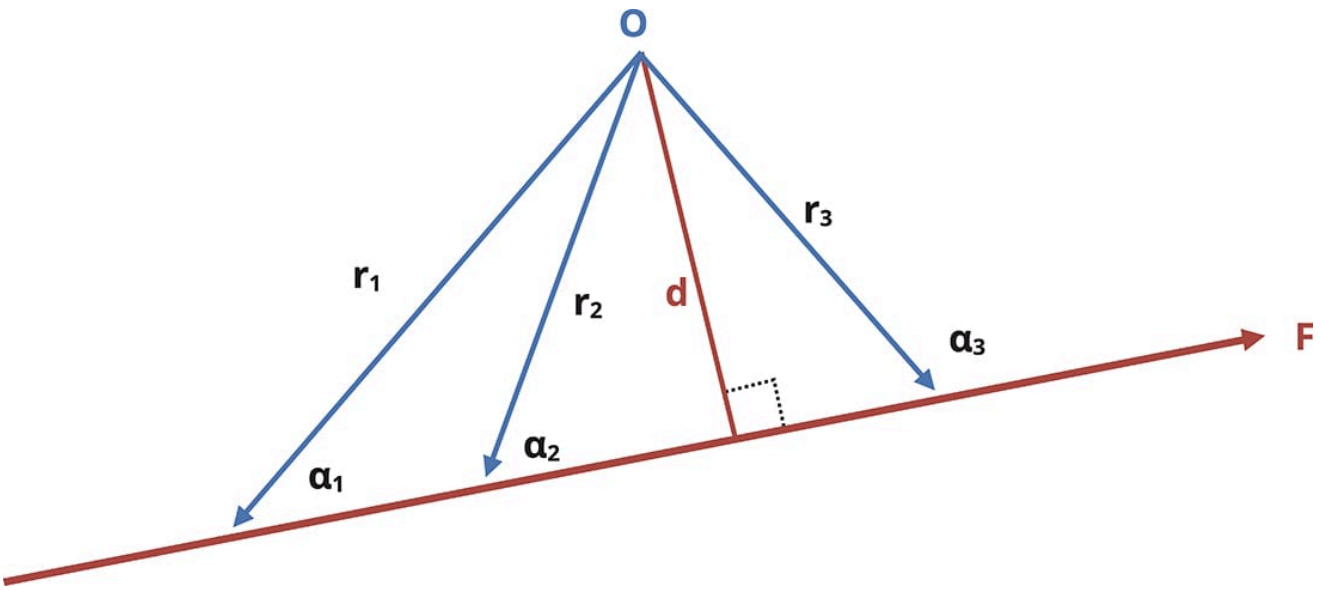


Figure 1.8: The moment around O due to the force (F) can be calculated using $\mathrm{M} = \mathrm{r}\ \sin(\alpha)$, M = Fd, or $\vec{\mathrm{M}} = \vec{\mathrm{r}} \times \vec{\mathrm{F}}$

$$\begin{aligned}\overrightarrow{M_O} &= \vec{r_1} \times \vec{F} \\ &= \vec{r_2} \times \vec{F} \\ &= \vec{r_3} \times \vec{F}.\end{aligned}$$

Using the cross-product to calculate moment will result in a vector expression for the moment equation that renders at one time all three components with the correct signs to indicate clockwise (negative) or counterclockwise (positive) rotation. The direction of rotation depends on the perspective, here taken from the positive end of the axis looking toward the negative.

The second option to calculate moments is to perform scalar calculations in which the sum of the moments about the x-, y-, and z-axis is calculated individually. To use this option, it might be helpful to recall several concepts:

1. The general scalar equation for moment is M = Fd, where d is the perpendicular distance between the point the moment is being taken about and the line of action of the force (see Figure 1.8). You can also use

   $$M = F\, r\, sin(\alpha),$$

   where

   - *r = any straight-line distance from the point the moment is being taken about to any point on the force vector* $\vec{F}$
   - α *=angle between* $\vec{F}$ *and the position vector* $\vec{r}$ *that corresponds to the distance r*

     Figure 1.8 shows various *α* and r pairs. Note that either *α* or its complement angle can be used because the sine of the angle is the same either way.

2. Forces do not cause moments about points they traverse or axes they act through.
3. Forces do not cause moments about axes they are parallel to—for example, F_x wouldn't cause a moment around an x-axis no matter where the point is.

4. When taking the moment about a point, move the origin of the coordinate axes to that point to determine the distance between the axis and the force.

Given all the foregoing reminders, you can apply the following equations:

- $\sum \mathrm{M_x} = \pm \mathrm{F_y} * \mathrm{z} \pm \mathrm{F_z} * \mathrm{y}$, where z and y are the distances from the x-axis at the point in question to F_y and F_z respectively.
- $\sum \mathrm{M_y} = \pm \mathrm{F_x} * \mathrm{z} \pm \mathrm{F_z} * \mathrm{x}$, where z and x are the distances from the y-axis at the point in question to F_x and F_z respectively.
- $\sum \mathrm{M_z} = \pm \mathrm{F_x} * \mathrm{y} \pm \mathrm{F_y} * \mathrm{x}$, where y and x are the distances from the z-axis at the point in question to the F_x and F_y respectively.

The ± is decided by the right-hand rule or visual inspection. As mentioned, when using visual inspection, judge the direction of rotation by looking from the positive end of the axis toward the negative end. A counterclockwise rotation is considered positive, and a clockwise rotation is considered negative.

Take these steps to apply the right-hand rule:

1. Orient your right hand so that the palm is at the moment point and the fingers are aligned with the moment arm, extended to the force. The moment arm is the axis along which the perpendicular distance to the axis would be determined. For example, if finding M_x due to F_y, the moment arm is in the z direction.
2. The thumb is aligned with the axis of rotation—that is, with the axis around which the moment is being calculated.
3. Curl your fingers in the direction of the force. If to perform this action your thumb is pointed in the positive rotation axis direction, the moment is counterclockwise. If your thumb is pointed in the negative rotation axis direction, the moment is clockwise. Typical convention designates counterclockwise rotation to be positive and clockwise to be negative.

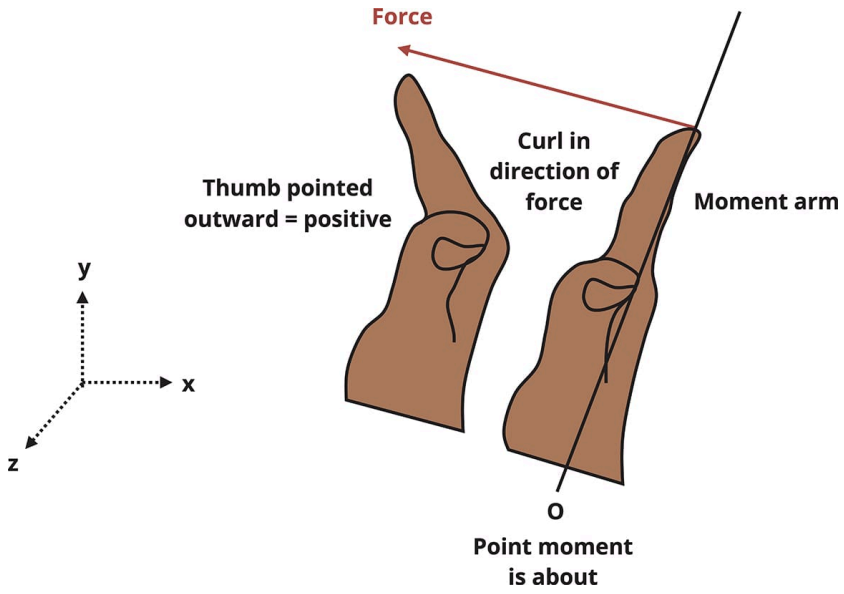
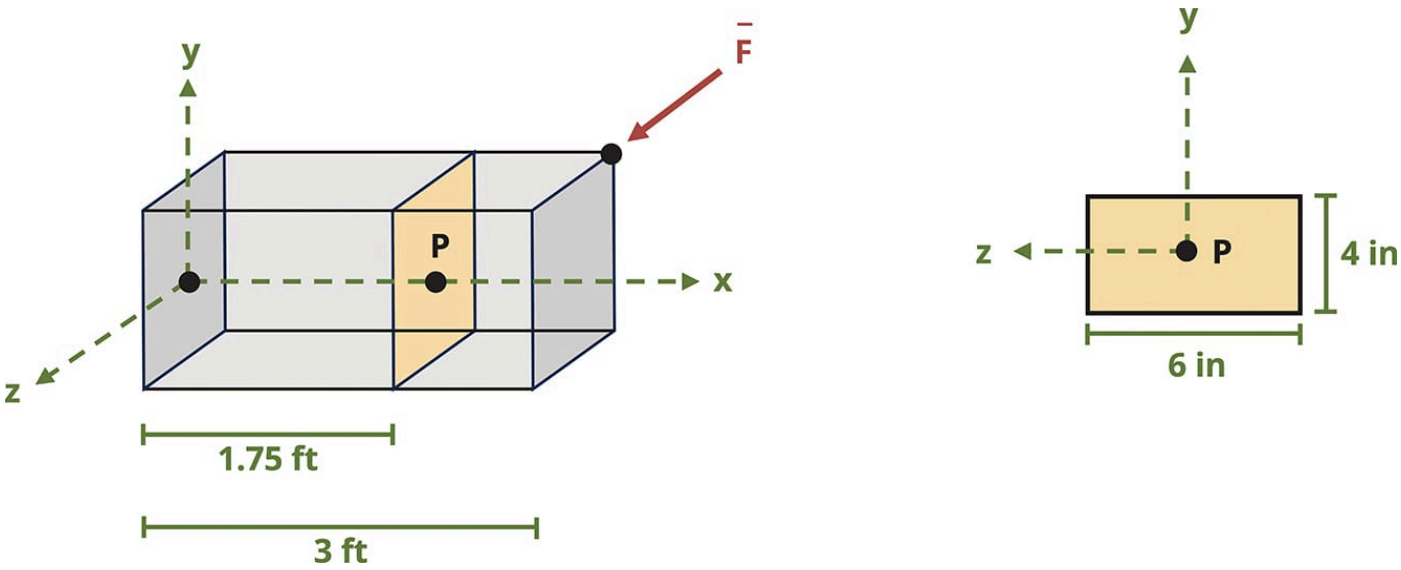


Figure 1.9: Illustration of applying right-hand rule

These concepts are further reviewed in Example 1.6.

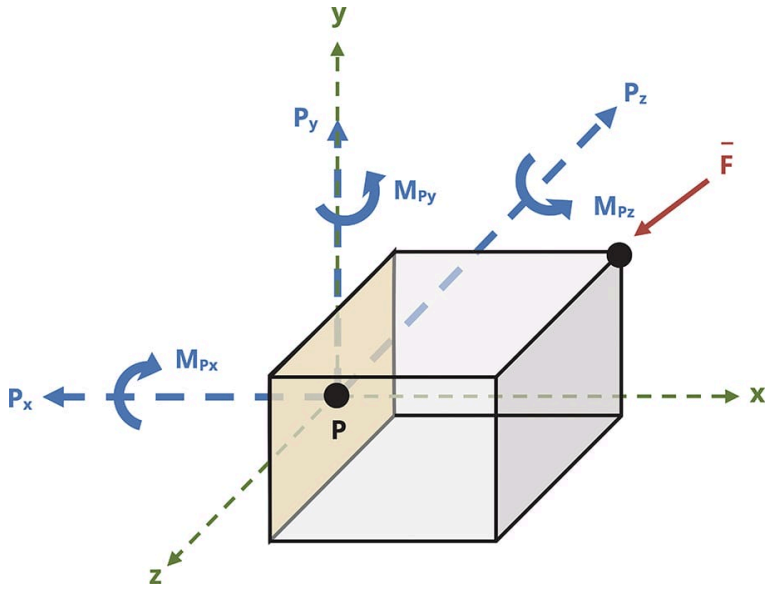
Example 1.6

Determine the internal reactions at point P, located at the center of the cross-section of the rectangular bar and 1.75 ft from the fixed support at the origin. Let F = -150i - 225j + 300k lb.



Solution

To determine the internal reactions at P, make a cut at P that is parallel to the cross-section. The section to the left of the cut includes the wall, but the section to the right of the cut has no support. Draw the FBD of the section to the right of the cut to avoid needing to know the reaction at the wall.



The x direction is normal to the cut section, so P_x is the internal normal force reaction. P_y and P_z are the internal shear force reactions.

The force reactions P_x, P_y, and P_z at P can be found by applying the force equilibrium equations.

$$\begin{aligned} \sum F_x &= -P_x + F_x = 0 \\ &= -P_x - 150\,lb = 0 \rightarrow P_x = -150\,lb \end{aligned}$$

P_x was assumed tensile (pointed away from the cut section), so the negative answer here means the force is compressive.

$$\begin{aligned}\sum F_y &= P_y + F_y = 0 \\ &= P_y - 225\ lb = 0 \rightarrow P_y = 225\ lb \\ \sum F_z &= -P_z + F_z = 0 \\ &= -P_z + 300\ lb = 0 \rightarrow P_z = 300\ lb\end{aligned}$$

Note that shear direction forces are not designated as either tensile or compressive because shear forces do not extend or compress a body. Signs on shear forces are discussed in Section 14.1.

The internal moment reactions M_{Px}, M_{Py}, and M_{Pz} can be found by taking the moments about point P. The sign on the individual multiplicative terms in each equation are determined by the right-hand rule or visualization.

The moment about the x-axis at point P is

$$\sum M_{Px} = \pm F_y z \pm F_z y = 0 = -\left[(225\ lb)\,(3\ in.)\left(\tfrac{1ft}{12\ in.}\right)\right] + \left[(300\ lb)\,(2\ in.)\left(\tfrac{1ft}{12\ in.}\right)\right] - M_{Px} = 0$$
$$M_{Px} = -6.25\ lb \cdot ft.$$

The reaction M_{Px} is subtracted because it is drawn on the FBD as counterclockwise around the negative x-axis, which means it is assumed clockwise around the positive x-axis. The negative answer means the reaction moment is opposite of the assumed direction, so it is actually counterclockwise around the positive x-axis.

The moment about the y-axis at point P is

$$\sum M_{Py} = \pm F_x z \pm F_z x = 0 = (150\ lb)\,(3\ in.)\left(\tfrac{1ft}{12\ in.}\right) - [(300\ lb)\,(1.25ft)] + M_{Py} = 0$$
$$M_{Py} = 337.5\ lb \cdot ft.$$

The reaction moment M_{Py} is added because it is drawn counterclockwise around the positive y-axis. The positive answer means it is actually counterclockwise.

The moment about the z-axis at point P is

$$\sum M_{Pz} = \pm F_x y \pm F_y x = 0 = \left[(150\ lb)\,(2\ in.)\left(\tfrac{1ft}{12\ in.}\right)\right] - \left[(225\ lb)\,(1.25ft)\right] - M_{Pz} = 0$$
$$M_{Pz} = -256.25\ lb \cdot ft.$$

Like M_{Px}, the reaction moment M_{Pz} is subtracted because it is drawn on the FBD as counterclockwise around the negative z-axis. The negative answer means the reaction is actually counterclockwise around the positive z-axis.

Answer:

- P_x = 150 lb (compressive)
- P_y = 225 lb
- P_z = 300 lb
- M_{Px} = 6.25 lb·ft (counterclockwise)
- M_{Py} = 337.5 lb·ft (counterclockwise)
- M_{Pz} = 256.25 lb·ft (counterclockwise)

Summary

Key Takeaways

Bodies in this text are in static equilibrium and subjected to external forces and moments that can cause deformations. To

determine the type and extent of deformation, we need to understand how to calculate external and internal reactions to loading.

External reactions can be found by drawing FBDs of the whole structure and applying equilibrium equations.

Internal reactions can be found by separating parts (for structures made of connected members) or making cuts (for solid bodies or truss structures when method of sections is used) and applying equilibrium to the separated parts or sections.

In the case of connected members, it can be helpful to the process to recognize two-force members and understand how they simplify analysis.

The effects of these internal loads on deformable bodies is the focus of this text.

Key Equations

Static equilibrium:

$$\begin{array}{ll} \sum F_x = 0 & \sum M_x = 0 \\ \sum F_y = 0 & \sum M_y = 0 \\ \sum F_z = 0 & \sum M_z = 0 \end{array}$$

2D scalar moment equations:

$$M = (F)(d)$$
$$M = Fr\sin(\alpha)$$

3D scalar moment equations:

$$\sum M_x = \pm F_z \times y \pm F_y \times z$$
$$\sum M_y = \pm F_z \times x \pm F_x \times z$$
$$\sum M_z = \pm F_x \times y \pm F_y \times x$$

Vector moment equation:

$$\vec{M} = \vec{r} \times \vec{F}$$

Force magnitude:

$$F = \sqrt{F_x^2 + F_y^2}$$

References

Figures

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Chapter 2: Stress

Learning Objectives

- Use internal loads to calculate average normal stress
- Use internal loads to calculate average shear stress
- Use the loads between surfaces to calculate bearing stress
- Calculate stresses on inclined planes

Having reviewed static equilibrium in Chapter 1, we know how to find the internal loads in a body using equilibrium. When determining whether a body can resist the loads applied to it, keep in mind that the internal load is only part of the solution. The dimensions of the body and the inherent properties of the material it is made from are also important.

In this chapter we explore the concept of stress, which can help us determine whether an object will physically break when subjected to a load. We cover average normal stress (stress perpendicular to the cross-section) in Section 2.1 and average shear stress (stress parallel to the cross-section) in Section 2.2. We then discuss bearing stress (the stress between two bodies in contact with each other) in Section 2.3 and finish with the average normal and shear stresses on an inclined cross-section in Section 2.4.

2.1 Average Normal Stress

Average normal stress is defined as the internal normal force divided by the cross-sectional area of the body. By convention, it is represented by the Greek letter sigma (σ). Stress increases as the force increases or as the cross-sectional area decreases.

$$\boxed{\sigma = \frac{N}{A}},$$

(2.1)

where

- σ = *Average normal stress [Pa, psi]*
- N = *Internal normal force [N, lb]*
- A = *Cross-sectional area [m^2, $in.^2$]*

As such the SI units of stress are N/m^2, more commonly referred to as the Pascal (Pa), where 1 Pa = 1 N/m^2. The US customary units for stress are $lb/in.^2$, commonly written as psi, which is short for pounds per square inch. Since stresses can get very large, it is common to use prefixes such as kilo (k) and mega (M) to represent 10^3 and 10^6 respectively. A stress of 15,000,000 Pa is written as 15 MPa. A stress of 35,000 psi is written as 35 ksi.

Normal stress occurs perpendicular to the cross-section and so is associated with either a pulling or pushing motion (Figure 2.1). Normal forces that pull on a cross-section are known as tensile forces and create a tensile normal stress. Normal forces that push on a cross-section are known as compressive forces and create a compressive normal stress. By convention, tensile normal forces and stresses are positive, while compressive normal forces and stresses are negative.

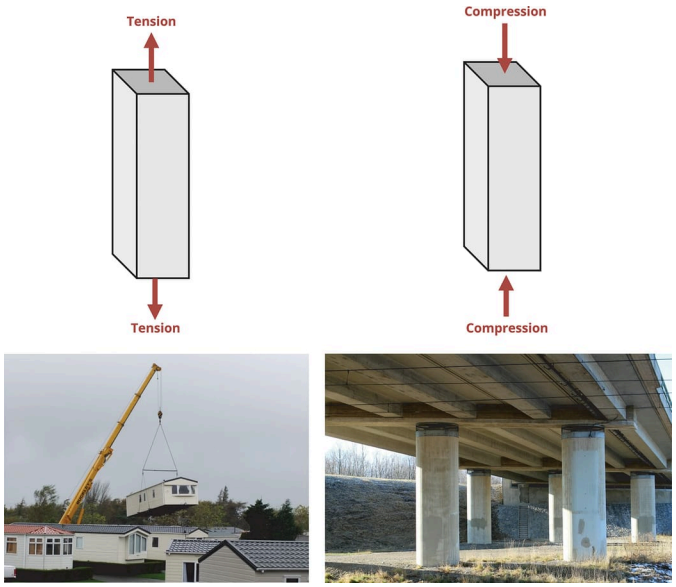
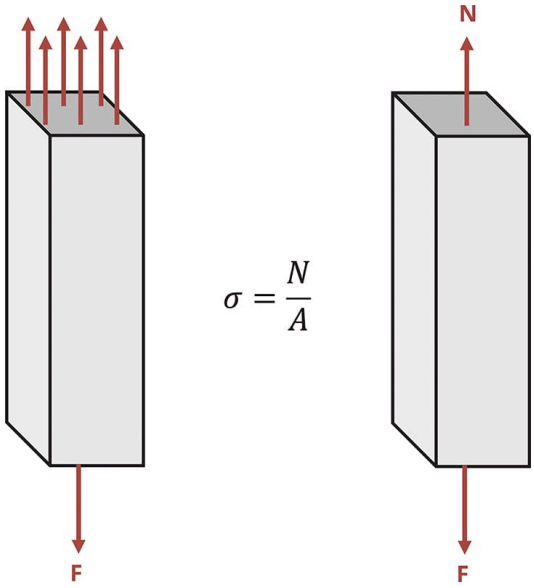


Figure 2.1: Forces pulling on a cross-section cause tension, while forces pushing on a cross-section cause compression. Both are very common. Left: A crane cable during a lift experiences tension. Right: The support columns under a bridge experience compression.

The normal stress calculated above is really an average normal stress because the force is distributed over the cross-section, but it can generally be simplified as a concentrated force that creates the same normal stress at every point on the cross-section (Figure 2.2).



$$\sigma = \frac{N}{A}$$

Figure 2.2: The internal load is equally distributed over the cross-sectional area but may be represented as a single concentrated load acting at the center (N in this case). The stress is therefore an average normal stress.

The internal normal force in a body can be found through the method of equilibrium as reviewed in Section 1.2. See Example 2.1 for a demonstration.

Example 2.1

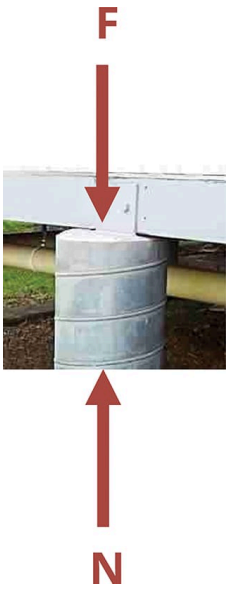
The support column will be subjected to a compressive force F = 65 kips.



1. The diameter of the column is 4 in. Determine the average normal stress in the column.
2. The column is to be made of concrete with an allowable compressive stress of 4 ksi. For the same force F = 65 kips, determine the required diameter of the column so that the average normal stress does not exceed 4 ksi.

Solution

1. Cut a cross-section through the column and draw a free body diagram (FBD). Although it is clear in this case that the internal load will be 65 kips, it is best to get in the habit of writing out equilibrium equations.



$$\sum F_y = N - F = 0$$
$$\sum F_y = N - 65\ kips = 0$$
$$N = 65\ kips$$

Recall that 65 kips = 65,000 lb. It is acceptable to keep the force in kips and calculate the stress in units of $\frac{\text{kips}}{\text{in}^2} = \text{ksi}$.

The column has a circular cross-section of area $\mathrm{A} = \pi \mathrm{r}^2 = \pi(2)^2 = 4\pi\ \mathrm{in.}^2$

The average normal stress can now be found from

$$\sigma = \frac{N}{A} = \frac{65\ kips}{4\pi\ in.^2} = 5.17\ ksi.$$

2. Use the average normal stress equation again, but this time the stress is known to be 4 ksi. The loading has not changed, so the internal normal force will continue to be 65 kips.

$$\sigma = \frac{N}{A} \rightarrow A = \frac{N}{\sigma} = \frac{65\ kips}{4\ ksi} = 16.25\ in.^2$$

Since $\mathrm{A} = \pi \mathrm{r}^2$, we can find $\mathrm{r} = \sqrt{\frac{\mathrm{A}}{\pi}} = \sqrt{\frac{16.25\ \mathrm{in.}^2}{\pi}} = 2.27\ \mathrm{in.}$

Then $\mathrm{d} = 2\mathrm{r} = 2 * 2.27\ \mathrm{in.} = 4.55\ \mathrm{in.}$

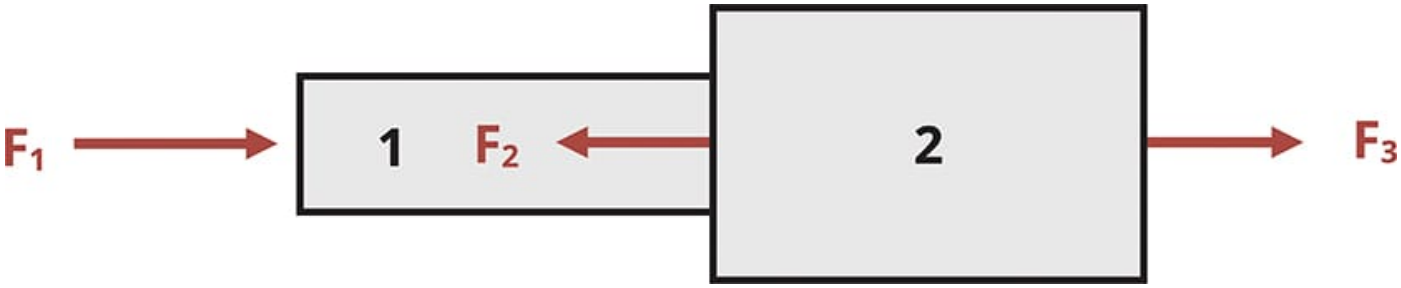
Note that this is the minimum required diameter to ensure the average normal stress doesn't exceed 4 ksi. If the diameter is any smaller than this, the stress will exceed the 4 ksi limit.

Sometimes the loading or cross-sectional area will not be the same at different points in the body, resulting in different stresses. In these cases the stress can be calculated separately in each segment of the body by first finding the internal load in each segment and then dividing the internal load by the cross-sectional area of the respective segment. Different parts of the body may experience different stresses. Generally, the highest stress is of most importance, as that is the stress that typically determines whether the body will break. Because we don't know in advance where the largest stress is, however, it is typically necessary to calculate the stress at each cross-section to determine the highest stress. See Example 2.2 for a demonstration.

Example 2.2

Two hollow pipes are welded together as shown. Pipe 1 has an outer diameter of 70 mm and an inner diameter of 40 mm, while pipe 2 has an outer diameter of 110 mm and an inner diameter of 60 mm. Forces are applied at the end of each pipe and at the weld, where F_1 = 40 kN, F_2 = 70 kN, and F_3 = 30 kN.

Determine the average normal stress in each pipe.

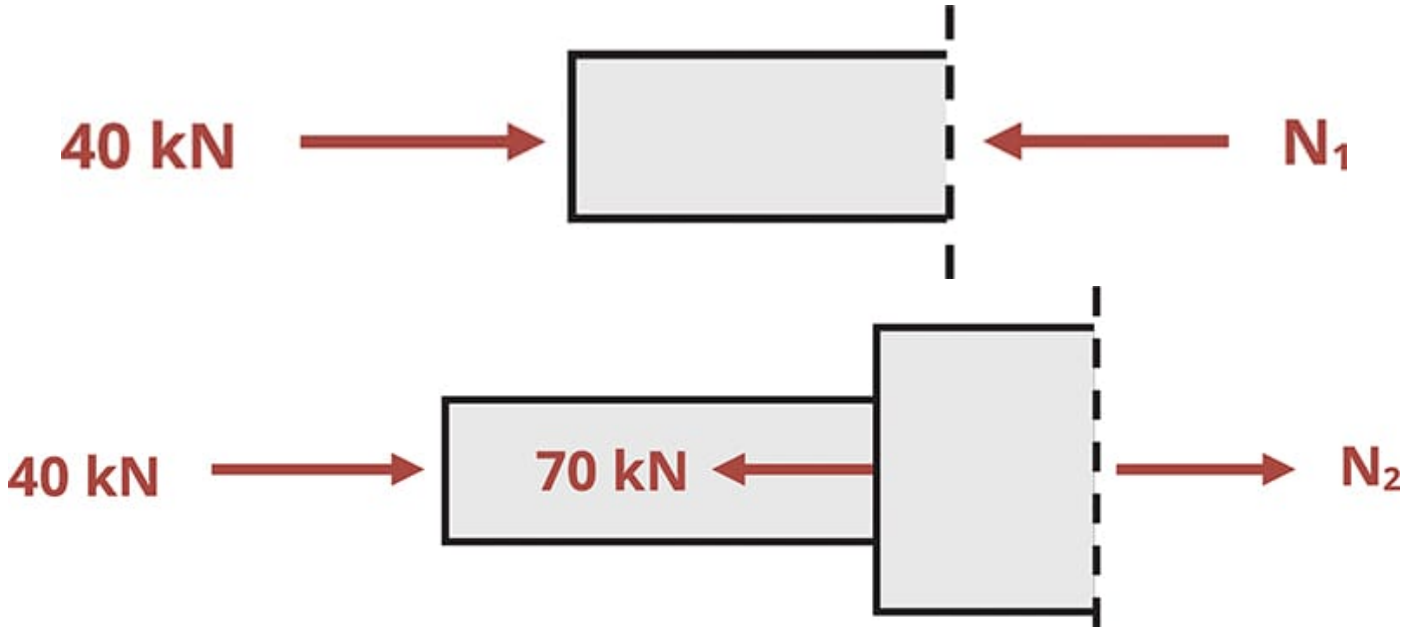


Solution

We can calculate average normal stress using $\sigma = \frac{\mathrm{N}}{\mathrm{A}}$, so we need to find the area of each pipe and the normal force in each pipe. This can be completed in any order. Starting with the areas, calculate as follows:

$$A_1 = \pi(0.035^2 - 0.02^2) = 0.00259\ m^2$$
$$A_2 = \pi(0.055^2 - 0.03^2) = 0.00668\ m^2.$$

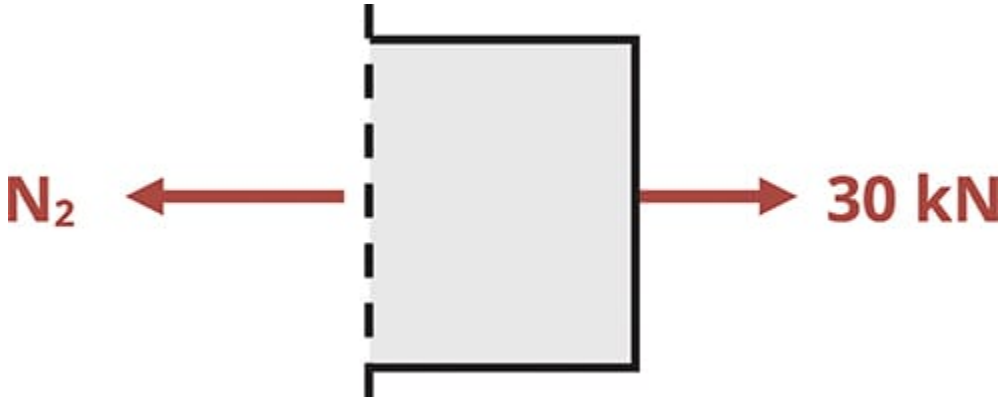
The internal loads can be found by cutting a cross-section through each pipe, drawing a free body diagram, and writing an equilibrium equation.



$$\sum F_x = 40\ kN - N_1 = 0 \ \rightarrow \ N_1 = 40\ kN$$
$$\sum F_x = 40\ kN - 70\ kN + N_2 = 0 \ \rightarrow \ N_2 = 30\ kN$$

Note that we may choose to draw the internal load in either tension or compression. The answer must be compared to the FBD. A positive answer from the equilibrium equation indicates that the direction drawn on the FDB is correct. Although both answers here are positive, the FBD shows that N_1 is drawn in compression and N_2 is drawn in tension. These positive answers indicate that the drawings are correct. N_1 is 40 kN in compression and N_2 is 30 kN in tension.

Note also that it is acceptable to draw an FBD of either side of the cross-section. This doesn't change the result. Perhaps drawing the right-hand side of section 2 would have been an easier approach.



$$\sum F_x = 30\ kN + N_2 = 0 \quad \rightarrow \quad N_2 = 30\ kN$$

We again find that N_2 is 30 kN in tension. You may always choose to draw either one side of a cross-section or the other. Make sure to include everything on the chosen side of your cut, all the way to the end of the structure (e.g., don't stop at the weld).

Now that the areas and internal loads are known, we can calculate the internal normal stresses.

$$\sigma_1 = \frac{N_1}{A_1} = \frac{-40{,}000\ N}{0.00259\ m^2} = -15.4\ MPa$$
$$\sigma_2 = \frac{N_2}{A_2} = \frac{30{,}000\ N}{0.00668\ m^2} = 4.49\ MPa$$

Step-by-Step: Average Normal Stress

1. Determine reaction loads: Use equilibrium equations to determine reaction loads at any supports.
2. Draw the FBD: Cut a cross-section through the member at the point where you want to determine the internal normal stress, and prepare an FBD of the member on either side of the cut. Be sure to include everything on the chosen side, including external loads, support reactions, and the internal normal force.
3. Calculate internal force: Use equilibrium equations to determine the internal normal force (N) in the member.
4. Calculate average stress: Determine average normal stress using $\sigma = \frac{\mathrm{N}}{\mathrm{A}}$.

2.2 Average Shear Stress

Average shear stress is defined as the internal shear force divided by the cross-sectional area of the body. While normal forces (and therefore normal stresses) occur when a body is under tension or compression, shear forces (and therefore shear stresses) occur when a force is tangentially applied parallel to the cross-section, resulting in a sliding motion (Figure 2.3).

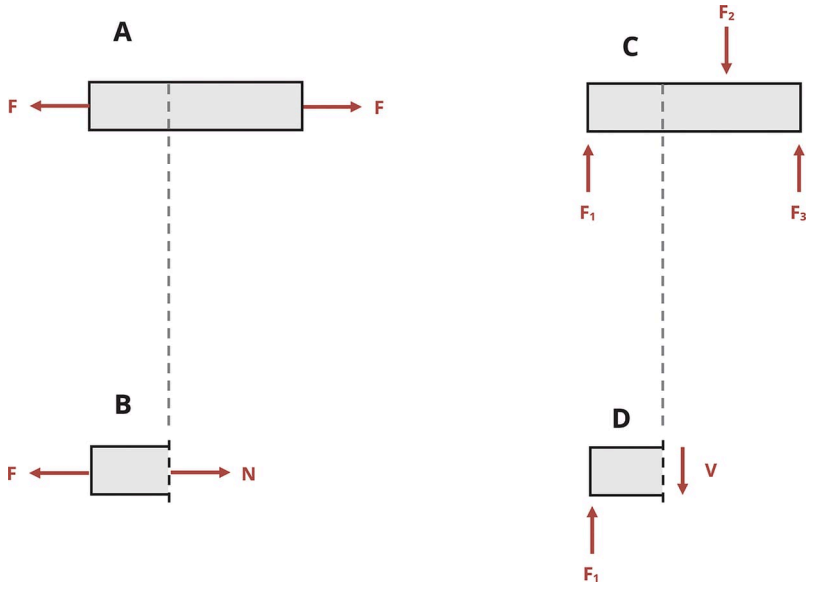


Figure 2.3: (A) External loads F creating (B), an internal normal force (N) perpendicular to the cross-section. (C) External loads F_1, F_2, and F_3 creating (D), an internal shear force (V) parallel to the cross-section. Note that for diagram (D) to be in equilibrium, there would also need to be an internal bending moment, which has been omitted here.

Although the basic definition of average normal stress and average shear stress is the same—force divided by area—there are some differences. To differentiate which stress we're talking about, we denote shear stress by the Greek letter tau (τ).

$$\boxed{\tau = \frac{V}{A}},$$

(2.2)

where

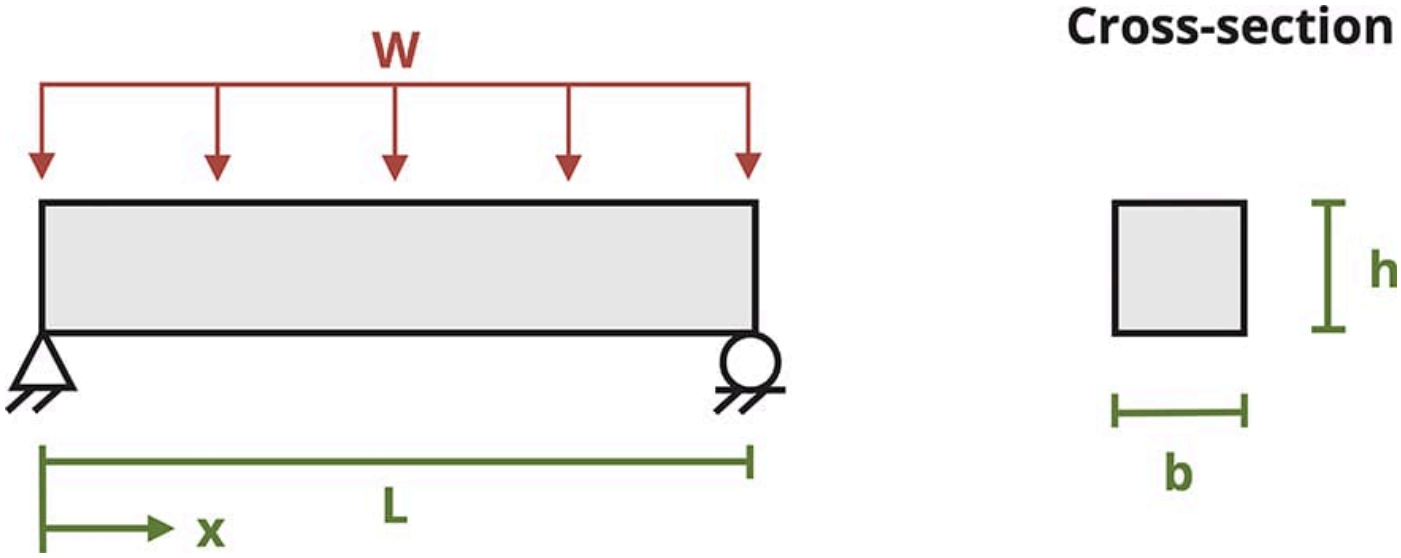
- τ = *Average shear stress [Pa, psi]*
- V = *Internal shear force [N, lb]*
- A = *Cross-sectional area [m^2, $in.^2$]*

Like normal force, internal shear force can be found by cutting a cross-section through the body, drawing an FBD, and applying equilibrium equations. See Example 2.3 for a demonstration.

Example 2.3

A bridge spans a gap of L = 150 ft. The roadway may be considered simply supported and has a rectangular cross-section of base b = 10 in. and height h = 6 in. It is subjected to a uniform distributed load of w = 200 lb/ft.

Determine the magnitude of the average shear stress in the cross-section at x = 30 ft and x = 80 ft.



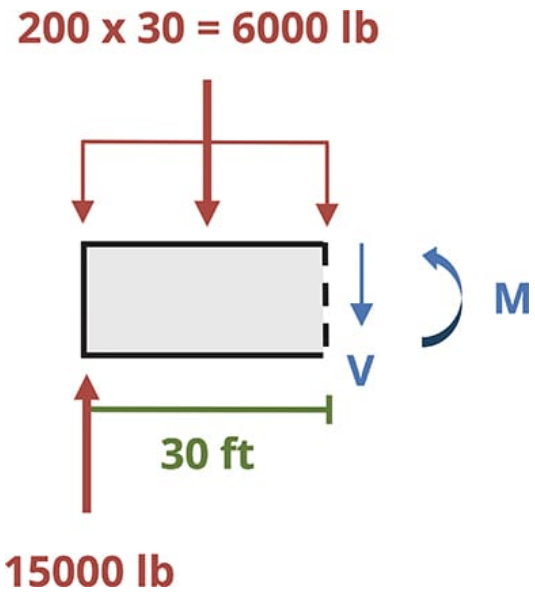
Solution

Average shear stress can be calculated from $\tau = \frac{V}{A}$. The cross-sectional area is $A = 10\text{ in.} * 6\text{ in.} = 60\text{ in.}^2$

The reactions at the support can be found by converting the distributed load into a statically equivalent concentrated load. This load will be equal to $F = 200\text{ lb/ft} * 150\text{ ft} = 30,000\text{ lb}$ and will be located at the center of the bridge. Since the loading is symmetric, each support will support half this load, or 15,000 lb each.

The internal shear force can be found by cutting a cross-section through the point of interest, drawing an FBD, and applying equilibrium equations. Remember to include the internal loads V and M and to again convert the distributed load into a statically equivalent concentrated load over this portion of the bridge.

At x = 30 ft:



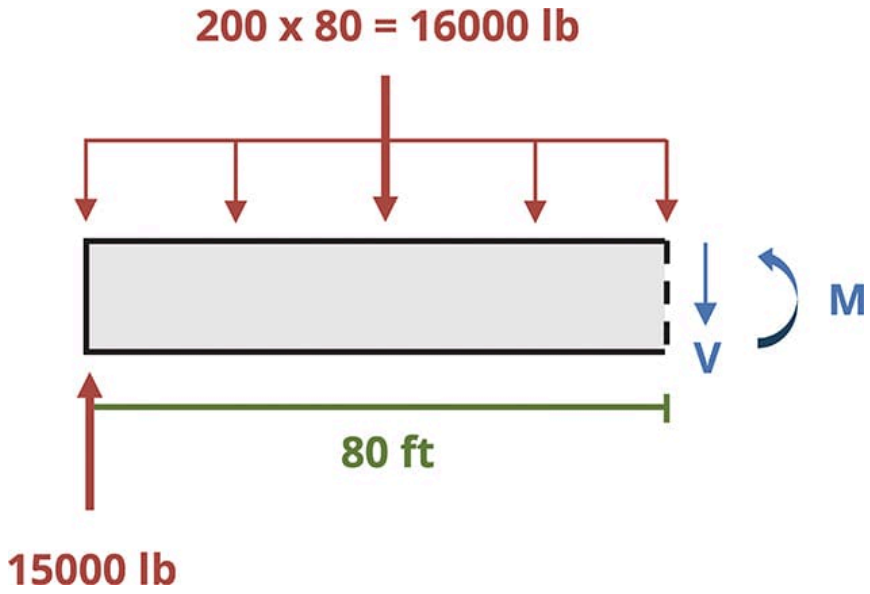
$$\sum F_y = 15,000\ lb - 6,000\ lb - V = 0 \quad \rightarrow \quad V = 9,000\ lb.$$

Note that an internal bending moment is also necessary to maintain equilibrium. We could find M by summing moments about any point, but the step is not required to solve this problem.

The shear stress can now be calculated.

$$\tau = \frac{V}{A} = \frac{9,000\ lb}{60\ in.^2} = 150\ psi$$

At x = 80 ft:



$$\sum F_y = 15,000\ lb - 16,000\ lb - V = 0 \quad \rightarrow \quad V = -1,000\ lb.$$

The negative sign here simply indicates that V is acting in the opposite direction of that drawn in the FBD. It can be discarded here because we are calculating the magnitude of the average shear stress at each point.

$$\tau = \frac{V}{A} = \frac{1,000\ lb}{60\ in.^2} = 16.7\ psi$$

Shear stresses are common in both members of a structure and also in the fasteners between members. Depending on how these connections are formed, we may see multiple shear planes. It is very important to correctly identify the internal force in the body. Figure 2.4 demonstrates the difference between single and double shear configurations for a pinned joint and the effects on internal shear force. Notice that in the case of double shear, the internal shear force is half that of the applied force, F_1. See Example 2.4 for an example of double shear.

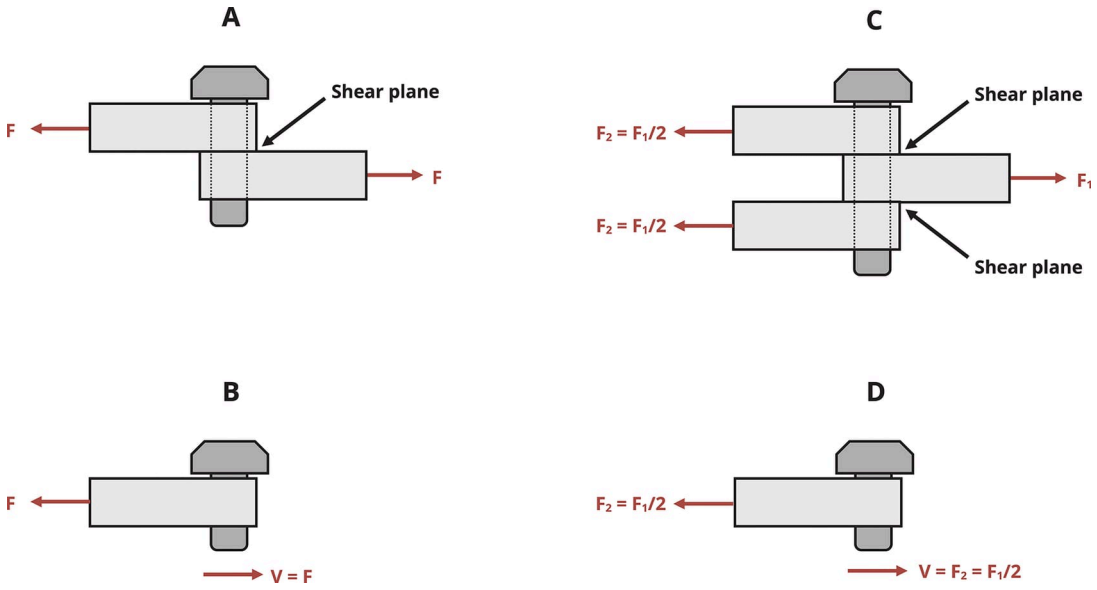
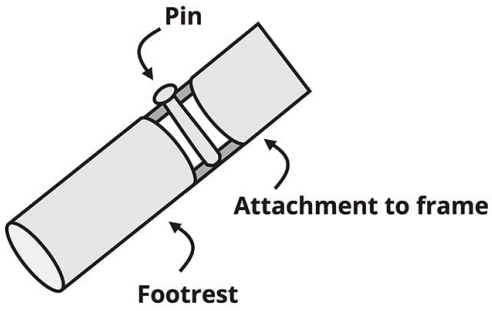


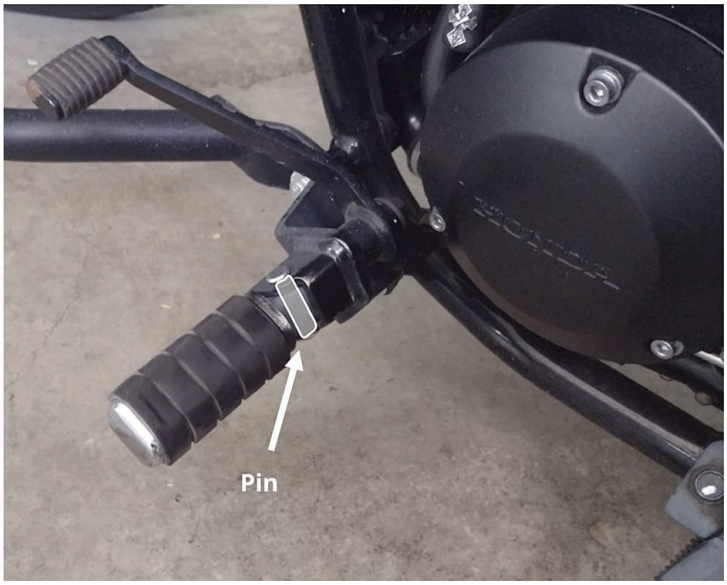
Figure 2.4: (A) Pin experiencing single shear and (B) the resulting internal shear force is equal to applied force F. (C) Pin experiencing double shear and (D) the resulting internal shear force is equal to half that of the applied force F_1.

Example 2.4

A pin made of a tin alloy with an allowable shear stress of 3 ksi is used to connect a footrest to the frame of a motorcycle. When in motion, the footrest supports a load of 200 lb, which is transferred to the pin.

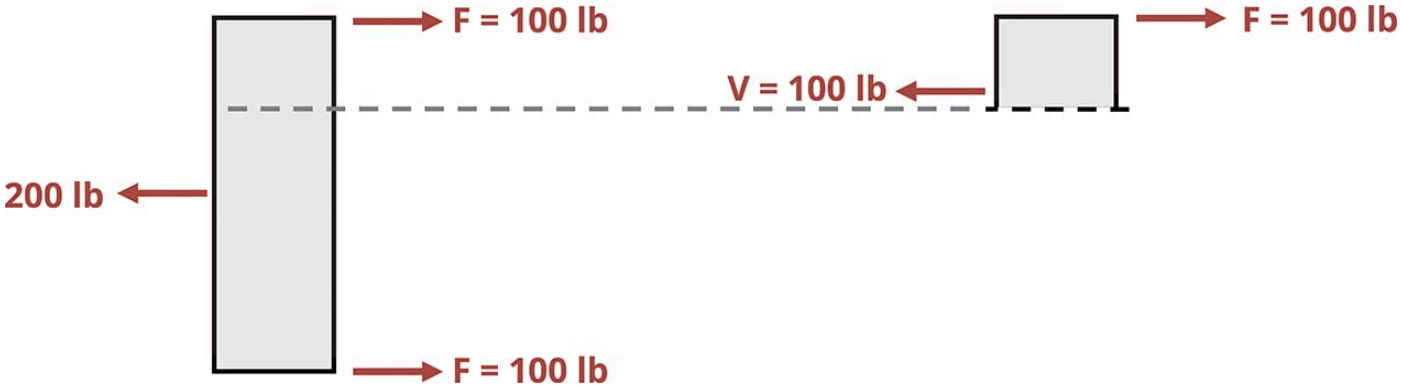
Determine the required diameter of the pin such that the stress will not exceed 3 ksi.





Solution

Average shear stress can be calculated from $\tau = \frac{\mathrm{V}}{\mathrm{A}}$. Rearrange this equation to $\mathrm{A} = \frac{\mathrm{V}}{\tau}$. Start with an FBD of the pin to determine the internal shear force.



From the diagram it should be apparent that the pin is in double shear and the maximum internal shear force in the pin is 100 lb. Thus the required cross-sectional area $\mathrm{A} = \frac{100\ \mathrm{lb}}{3{,}000\ \mathrm{psi}} = 0.0333\ \mathrm{in.}^2$

Since $\mathrm{A} = \frac{\pi}{4}\mathrm{d}^2$, the required diameter is $\mathrm{d} = \sqrt{\frac{0.0333\ \mathrm{in.}^2 * 4}{\pi}} = 0.206\ \mathrm{in.}$

Step-by-Step: Average Shear Stress

1. Determine reaction loads: Use equilibrium equations to determine reaction loads at any supports.
2. Draw the FBD: Cut a cross-section through the member at the point where you want to determine the internal shear stress and prepare an FBD of the member, selecting either side of the cut. Be sure to include everything on the chosen side, including external loads, support reactions, and the internal shear force.
3. Calculate internal force: Use equilibrium equations to determine the internal shear force (V) in the member.
4. Calculate average stress: Determine average normal stress using $\tau = \frac{\mathrm{V}}{\mathrm{A}}$.

2.3 Bearing Stress

Bearing stress is similar to normal stress, except that it occurs at the contact area between two bodies instead of within a single body. Bearing stress is calculated with the average normal stress equation, where the cross-sectional area is the contact area between the two bodies (Figure 2.5).



Figure 2.5: Circular column sitting upon a rectangular foundation. Contact areas between a circular and rectangular surface, and between two rectangular surfaces, are shown.

One common situation is contact between two curved edges, such as the bolt in Figure 2.6. In this situation it is common to use the projected contact area between the curved surfaces, which forms a rectangle of base d and height t.

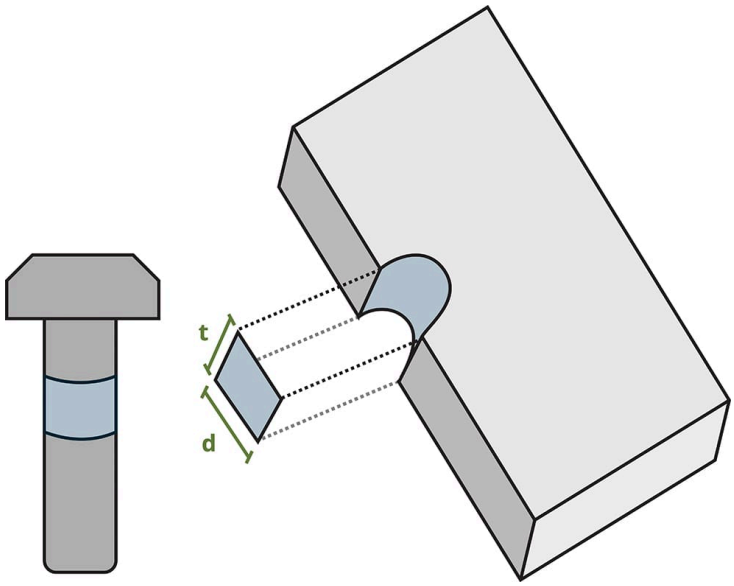


Figure 2.6: When calculating the bearing stress between curved surfaces, project a rectangle of area A = dt.

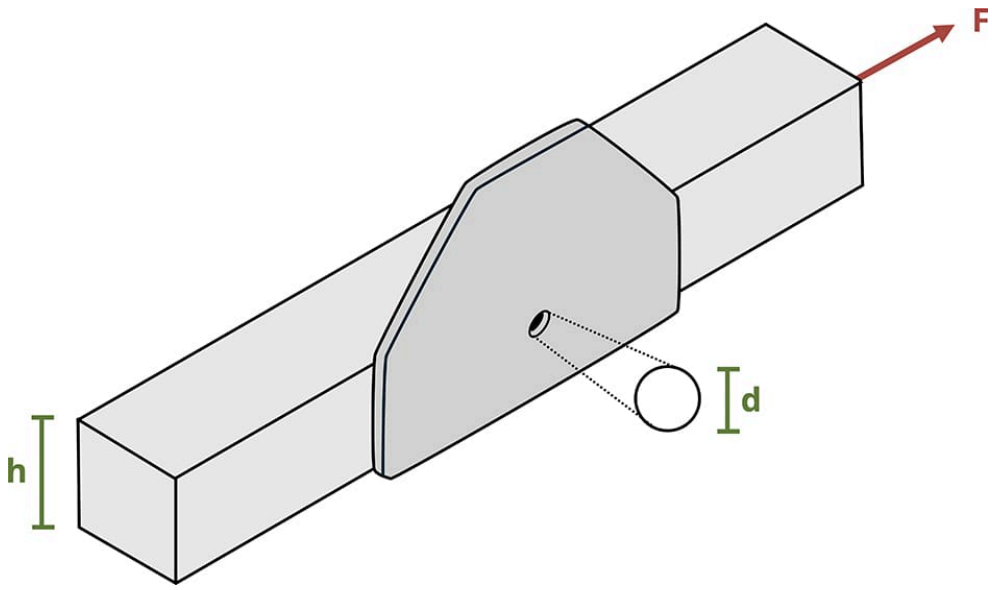
$$A = d * t$$

This greatly simplifies the calculation but again represents an average value for the contact or bearing stress. See Example 2.5 for a problem involving the bearing stress between a bolt and a plate.

Example 2.5

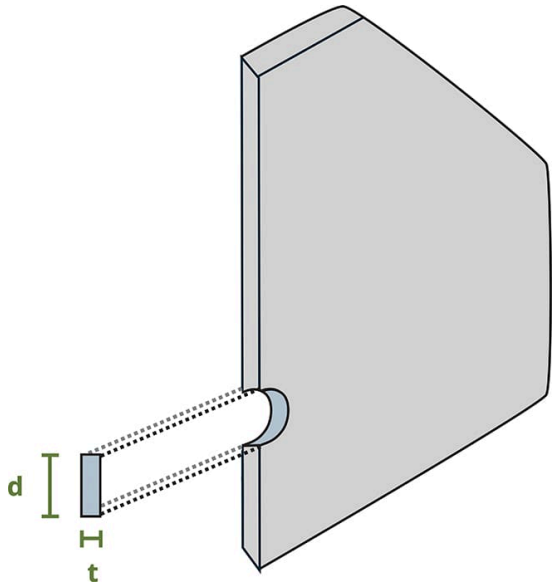
A bolt of diameter d = 80 mm is used to attach a steel bar to a gusset plate. The steel bar dimensions are height h = 200 mm and thickness t = 30 mm. It is subjected to a tensile force F = 50 kN.

Determine the bearing stress created in the steel bar at the pin.



Solution

Bearing stress can be calculated from $\sigma = \frac{N}{A}$, where N is the normal force between the objects and A is the projected contact area.



The force between the pin and the bar will be F = 50 kN = 50,000 N.

The projected contact area between the pin and the plate can be found from the diameter of the pin (80 mm =0.08 m) multiplied by the thickness of the plate (30 mm =0.03 m).

$$A = d * t = 0.08 * 0.03 = 0.0024\, m^2$$

Now we can calculate the bearing stress.

$$\sigma = \frac{50,000\, N}{0.0024\, m^2} = 20.8 * 10^6\, Pa = 20.8\, MPa$$

Step-by-Step: Bearing Stress

1. Determine reaction loads: Use equilibrium equations to determine reaction loads at any supports.
2. Determine area: Determine the contact area between the two components (A).
3. Calculate bearing force: Use equilibrium equations to determine the normal force (N) between the two components.
4. Calculate average stress: Determine average bearing stress using $\sigma = \frac{N}{A}$.

2.4 Stress on an Inclined Plane

We have so far been cutting cross-sections perpendicular to the external force or parallel to it. However, no rule says we have to do this. It is perfectly acceptable to cut cross-sections at an angle, and in many situations it may make sense to do so (Figure 2.7).

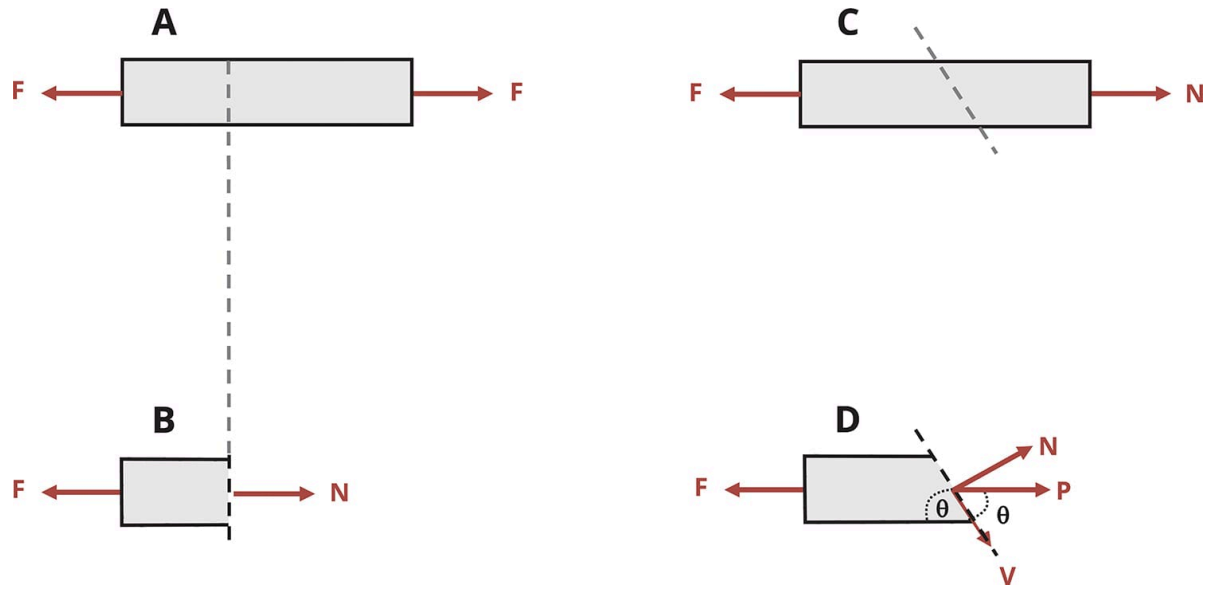


Figure 2.7: (A) External loads, F, will create (B) an internal normal force, N, when a cross-section is cut vertically. (C) If the cross-section is cut along an inclined plane, the internal force, P, will be neither a normal force nor a shear force. (D) The internal force should be broken into normal (N) and shear (V) components perpendicular and parallel to the plane respectively.

In this scenario, to maintain equilibrium the internal load must still be equal and opposite to the external load, but because the cross-section is cut at an angle, this internal load is neither parallel nor perpendicular to the cross-section. Therefore, it is not entirely a normal force or a shear force. However, the internal force may be broken into components that are perpendicular and parallel to the cross-section.

$$N = P \sin(\theta)$$
$$V = P \cos(\theta)$$

The area used to calculate the average normal and shear stresses must be the area of the inclined plane (Figure 2.8). Setting up a right-angled triangle enables us to find the area of the plane, A_p.

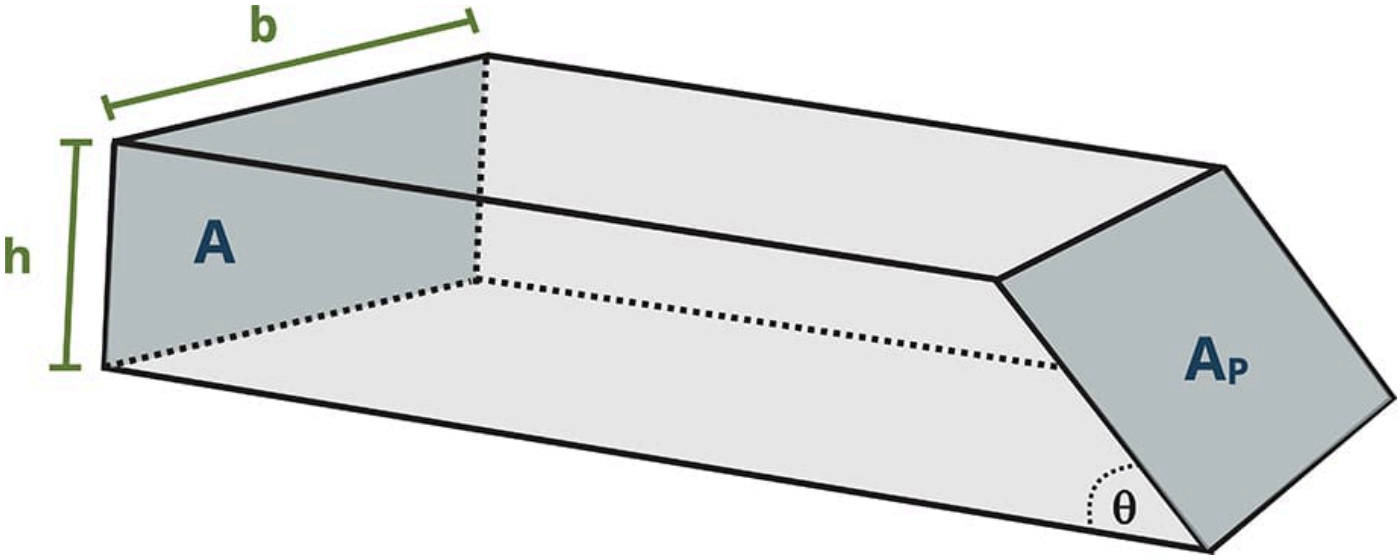


Figure 2.8: When calculating the stresses on an inclined plane, it is important to use the area of the plane, A_p, rather than the area of the cross-sectional plane, A.

$$A_p = \frac{A}{\sin(\theta)}$$

We can then calculate average normal stress.

$$\boxed{\sigma = \frac{N}{A_p} = \frac{P\sin(\theta)}{\frac{A}{\sin(\theta)}} = \frac{P\sin^2(\theta)}{A}}$$

(2.3)

We can calculate average shear stress from

$$\boxed{\tau = \frac{V}{A_p} = \frac{P\cos(\theta)}{\frac{A}{\sin(\theta)}} = \frac{P\sin(\theta)\cos(\theta)}{A}},$$

(2.4)

where

- *σ = Average normal stress on the inclined plane [Pa, psi]*
- *τ = Average shear stress on the inclined plane [Pa, psi]*
- *N = Internal normal force perpendicular to the inclined plane [N, lb]*
- *V = Internal shear force parallel to the inclined plane [N, lb]*
- *A_p = Area of the inclined plane [m^2, $in.^2$]*
- *A = Cross-sectional area [m^2, $in.^2$]*
- *P = Internal load perpendicular to cross-sectional area A [N, lb]*
- *θ = Angle between the inclined plane and the axis perpendicular to cross-sectional area A [°]*

These equations assume the angle (θ) is measured from the axis perpendicular to area A. For a horizontal beam, area A is in the

vertical plane, so angle θ is measured from the horizontal axis.

Even if the external load remains constant, we can obtain different values for the internal normal and shear forces (and therefore different values for the stresses) by changing the angle at which we cut the cross-section. Chapter 12 explores the implications. For now, a demonstration of calculating the stresses on an inclined plane is given in Example 2.6.

Example 2.6

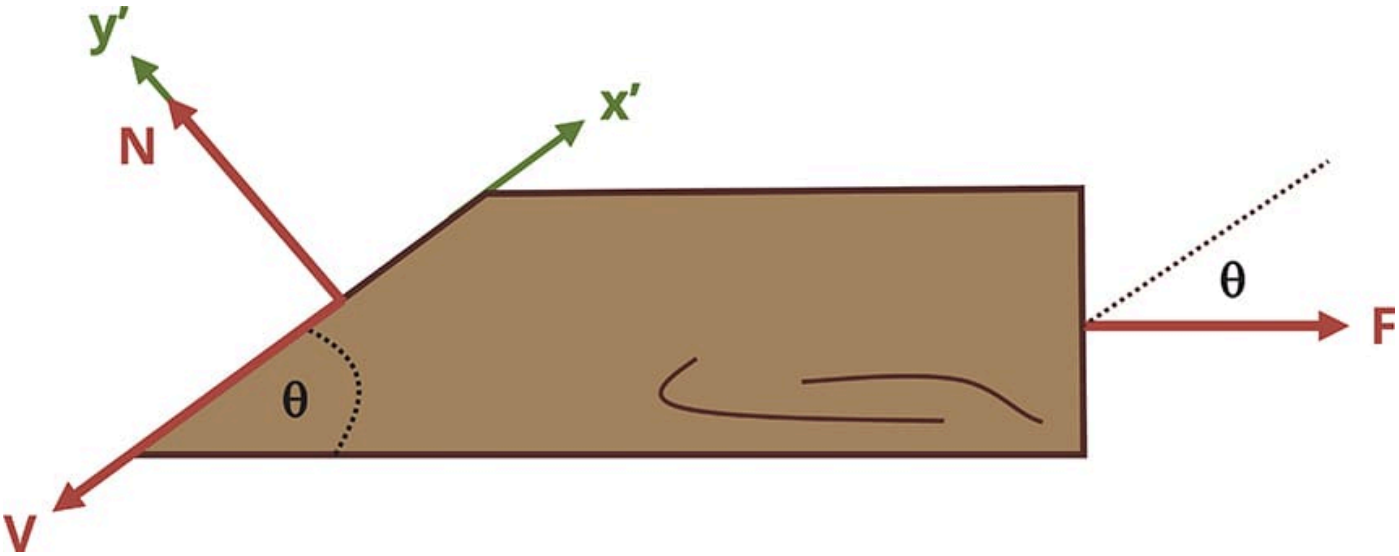
A beam is formed of two structural wooden members glued together along an inclined plane at angle θ = 40° and subjected to a tensile force of F = 30 kN. The height of the beam is 50 mm and its thickness is 20 mm.

Determine the normal and shear stresses created along the inclined plane.



Solution

Begin by cutting a cross-section along the inclined plane and drawing an FBD. Remember to include the internal normal and shear forces perpendicular and parallel to the cross-section.



Use equilibrium equations to determine the internal forces. It will be easiest to define axes parallel and perpendicular to the inclined plane.

$$\sum F_{x'} = F\ \cos(\theta) - V = 0$$
$$\sum F_{y'} = N - F\ \sin(\theta) = 0$$

With F = 30 kN and θ = 40°, these can be solved for.

$$\sum F_{x'} = 30\ kN\ \cos(40^\circ) - V = 0$$
$$V = 23.0\ kN$$
$$\sum F_{y'} = N - 30\ kN \sin(40^\circ) = 0$$
$$N = 19.3\ kN$$
$$\mathrm{N} = 19.3\ kN \text{ and } \mathrm{V} = 23.0\ kN.$$

Next, determine the cross-sectional area of the inclined plane.

$$A_p = \frac{A}{\sin(\theta)}$$

With $A = 0.05\ \mathrm{m} * 0.02\ \mathrm{m} = 0.001\ \mathrm{m}^2$ and $\theta = 40°$, $\mathrm{A_p} = 0.00156\ \mathrm{m}^2$.

Finally, determine the average normal stress and the average shear stress.

$$\begin{aligned} \sigma &= \frac{N}{A_p} = \frac{19{,}300\ N}{0.00156\ m^2} = 12.4 \times 10^6 \frac{N}{m^2} = 12.4\ MPa \\ \tau &= \frac{V}{A_p} = \frac{23{,}000\ N}{0.00156\ m^2} = 14.8 \times 10^6 \frac{\mathrm{N}}{m^2} = 14.8\ MPa \end{aligned}$$

Step-by-Step: Stresses on Inclined Planes

1. Determine reaction loads: Use equilibrium equations to determine reaction loads at any supports.
2. Draw an FBD: Cut a cross-section along the inclined plane and determine the angle (θ) of the plane with respect to the axis perpendicular to cross-sectional area A.
3. Calculate internal force: Use equilibrium equations to determine the internal load (F).
4. Calculate average stress:
    - Determine average normal stress using $\sigma = \frac{\mathrm{F}\ \sin^2(\theta)}{\mathrm{A}}$.
    - Determine average shear stress using $\tau = \frac{\mathrm{F}\ \sin(\theta)\cos(\theta)}{\mathrm{A}}$.

Summary

Key Takeaways

Objects under load experience stress. It is the stress level (rather than just the load by itself) that determines whether an object will break under load. We can calculate the average normal stress and average shear stress acting on a body, though a complete understanding of the stress field within a complex object often requires more advanced concepts.

Stress depends on the internal load in the body, the cross-sectional area, and the orientation of the plane. Normal stress occurs when there is an internal normal force and may be tensile or compressive. Shear stress occurs when there is an internal shear force. Members may experience multiple shear planes, in which case the internal shear force in the member is equal to the load applied to the member divided by the number of shear planes.

If multiple loads are acting on a body, or if the body has different cross-sectional areas at different points, the parts may experience different stresses. In such cases, we may calculate the stress in each part of the body by finding the internal load and cross-sectional area in that part of the body. This chapter has covered average stress. Subsequent chapters cover further details on stress distributions.

Bearing stress is similar to normal stress, but it occurs between two bodies.

We may calculate stresses along inclined planes by cutting a cross-section at an angle. Determining the stresses at different angles is important later in our studies.

Key Equations

Average normal stress:

$$\sigma = \frac{N}{A}$$

Average shear stress:

$$\tau = \frac{V}{A}$$

Bearing stress:

$$\sigma = \frac{N}{A}$$

Average stresses on an inclined plane:

$$\begin{aligned} \sigma &= \frac{P\sin^2(\theta)}{A} \\ \tau &= \frac{P\sin(\theta)\cos(\theta)}{A} \end{aligned}$$

References

Figures

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- Figure 2.1: Forces pulling on a cross-section cause tension, while forces pushing on a cross-section cause compression. Left image: Orangeaurochs. 2012. CC BY 2.0. https://flic.kr/p/dqmXdM. Right image: Unknown author. 2017. Public domain. https://pxhere.com/en/photo/993074. Diagrams: Kindred Grey. 2024. CC BY.
- Example 2.1: Jbarta. 2010. CC BY-SA 3.0. https://en.wikipedia.org/wiki/File:Permanent-column-form.jpg.
- Example 2.4: Image: James Lord. CC BY.
- Figure 2.5: Circular column sitting upon a rectangular foundation. Jiří Sedláček - Frettie. 2010. CC BY-SA 3.0. https://commons.wikimedia.org/wiki/File:Footing_of_column_in_collections_of_Muzeum_Vyso%C4%8Diny_in_T%C5%99eb%C3%AD%C4%8D,_T%C5%99eb%C

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Chapter 3: Strain

Learning Objectives

- Define strain
- Explain the difference between normal strain and shear strain
- Calculate normal and shear strains due to applied loads

Chapter 2 showed how to calculate normal and shear stresses with respect to a plane of interest, such as a cross-sectional area or an inclined plane. If stresses increase too much, the object will break, so it is important that engineers design structures and machines such that the stresses stay within certain acceptable limits. It is also important that objects do not deform so much that they are no longer fit for their original purpose. Strain is a measure of the intensity of a deformation—that is, the deformation per unit length.

Imagine two points on a body, A and B, separated by some distance L (Figure 3.1). In statics (and dynamics), it is assumed that objects are rigid and do not deform when subjected to forces. In reality, forces will cause an object to deform and consequently its dimensions will change. As points A and B move with the object, the distance between them will change to some new distance L'.

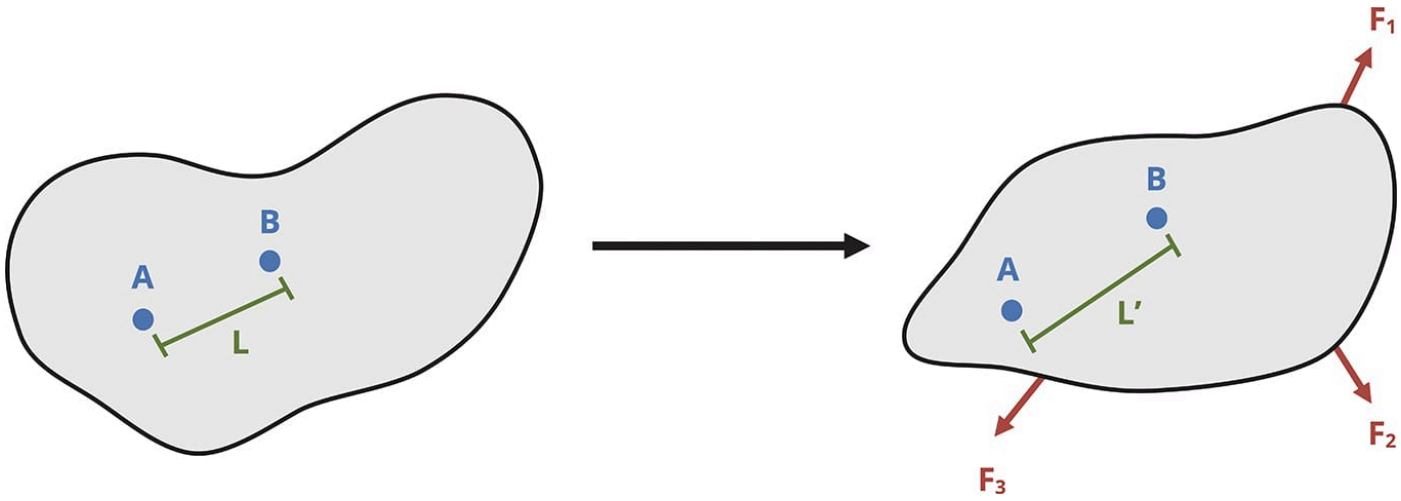


Figure 3.1: Nonrigid bodies will deform under load, and the distance between any two points A and B will change as the body deforms.

These deformations may be very large relative to the size of the object (e.g., a rubber band), or they may be relatively small (e.g., structural members), but there will always be some amount of deformation because no material is infinitely stiff, as we will learn in Chapter 4. Engineers must design structures and machines such that these deformations are not excessively large for the intended application.

This chapter introduces two types of strain. Section 3.1 covers normal strain, which like normal stress occurs when objects are subjected to axial loads. Section 3.2 covers shear strain, which like shear stress occurs when objects are subjected to shear loads.

3.1 Normal Strain

A simple example of normal strain is the deformation of a bar subjected to an axial load. Consider a bar of length L subjected to a tensile axial load P as shown in Figure 3.2. The load creates normal stress in the rod and also causes the rod to elongate by an amount ΔL.

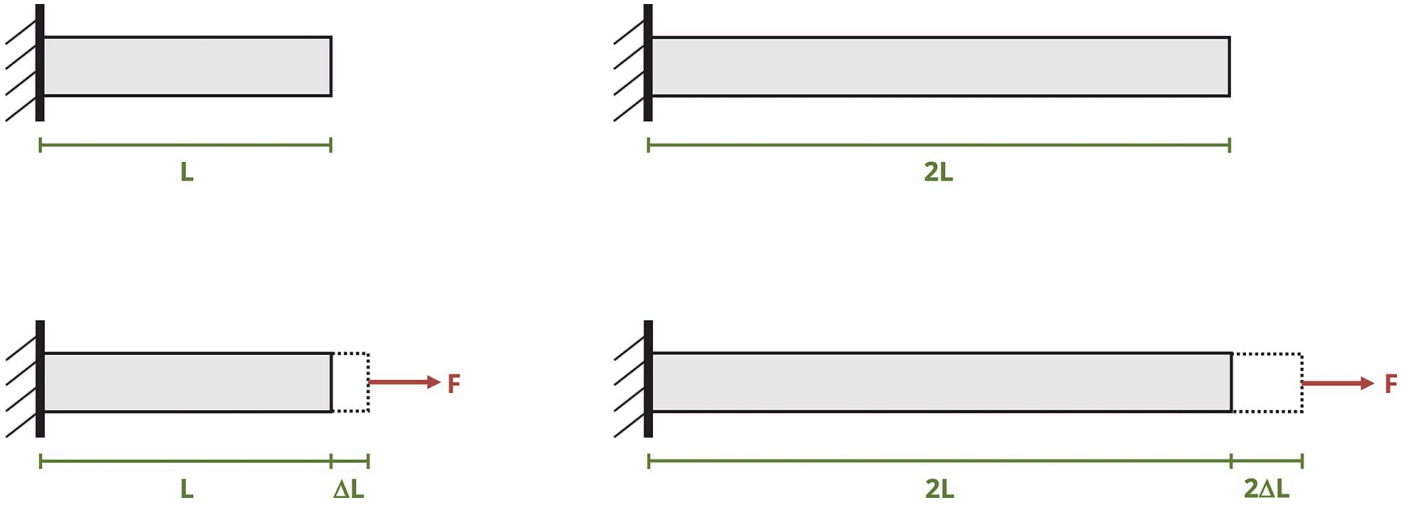


Figure 3.2: Axial loads cause a change in length that depends partly on the original length of the object.

If the same load were applied to a longer rod of length 2L with the same cross-sectional area, the longer rod would elongate by an amount 2ΔL. The deformation depends on the original length of the rod in this case, but the strain doesn't. Strain is a measure that normalizes the change in length by the original length. Normal strain is represented by the Greek letter epsilon (*e*) and is defined as

$$\boxed{\varepsilon = \frac{\Delta L}{L}},$$

(3.1)

where

$$\gamma = Shear\ strain [rad]$$
$$\tfrac{\pi}{2} = Original\ angle\ [rad]$$
$$\theta' = New\ angle\ [rad]$$

This is a normal strain because it occurs under axial load and is associated with a change in the object's length. Since normal strain is defined by dividing one length by another, it is a dimensionless quantity—that is, it has no units. It is nonetheless fairly common to include units of mm/mm or in./in.

For example, if the rod was initially 5 m long and it elongated by 12 mm, the normal strain can be calculated as

$$\varepsilon = \frac{\Delta L}{L} = \frac{12}{5,000} = 0.0024\ mm/mm = 0.0024.$$

Note that the units used for ΔL and L must be the same, but whether they are expressed in mm or m doesn't matter. Using meters instead yields the same answer.

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.012}{5} = 0.0024\ m/m = 0.0024$$

In fact, because the normal strain is dimensionless, its magnitude is the same in any system of units.

Because strains tend to be quite small in many engineering applications, they are sometimes expressed with a prefix to eliminate the leading zeros. For example, a strain of 0.0024 may be expressed as 2.4 mm/m or 2,400 μm/m or simply 2,400 μ. Strains may also be expressed as a percentage, which can be found simply by multiplying the strain by 100 percent. The following strains are all equivalent:

$$\varepsilon = 0.0024\ mm/mm = 0.0024\ m/m = 0.0024 = 2.4\ mm/m = 2,400\ \mu m/m = 2,400\ \mu = 0.24\%.$$

Here we will use the sign convention that we defined for normal stress. Tensile forces and stresses, associated with elongation

of the bar and tensile normal strain, are positive. Compressive forces and stresses, associated with shortening of the bar and compressive normal strain, are negative (Figure 3.3).

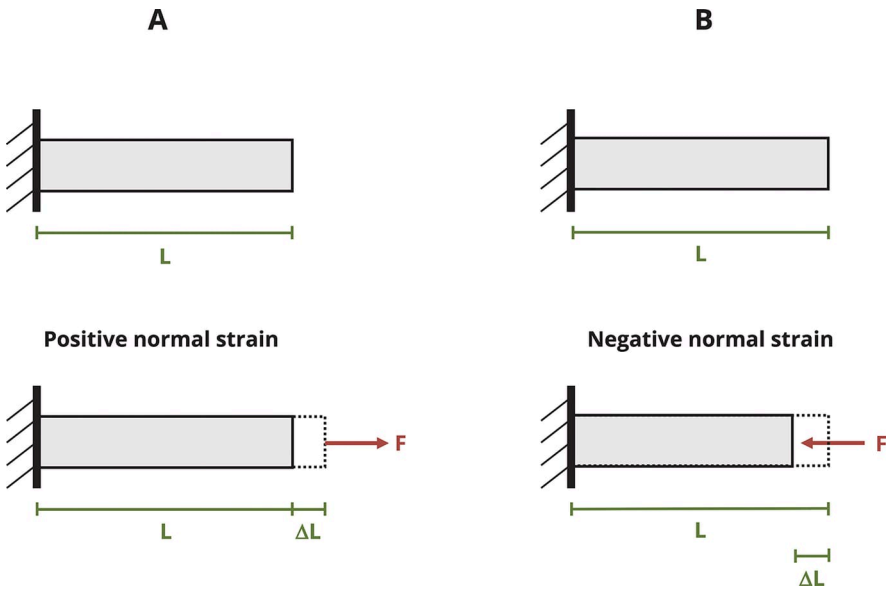


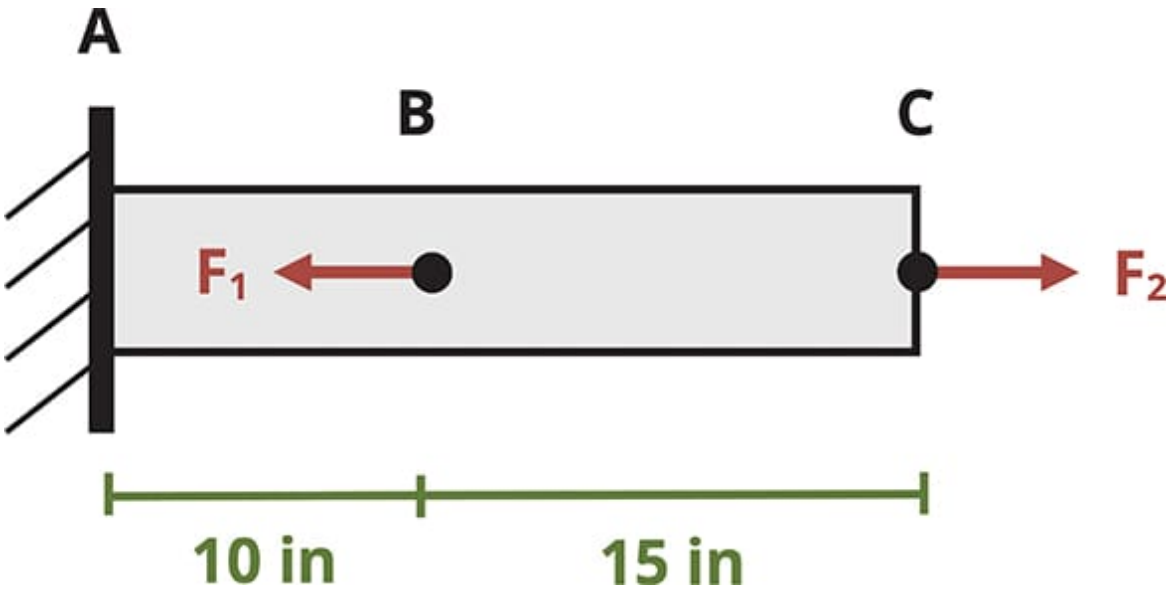
Figure 3.3: (A) Objects in tension will elongate. By convention, tension and elongation are positive. (B) Objects in compression will shorten. By convention, compression and shortening are negative.

See Example 3.1 for a problem involving a bar experiencing both tension and compression. Example 3.2 involves strain in two parallel cables.

Example 3.1

The bar is subjected to forces F_1 = 30 kips and F_2 = 10 kips as shown. Segment AB is originally 10 in. in length and segment BC is originally 15 in. in length. As a result of the applied forces, the normal strain in segment AB is -0.03 in./in. and the strain in segment BC is 0.045 in./in.

1. Determine the change in length of each segment.
2. Determine the average strain for the entire bar.



Solution

Using $\varepsilon = \frac{\Delta L}{L}$ for each segment, rearrange to calculate the change in length of each segment.

$$\begin{array}{ll} \Delta L & \varepsilon * L \\ \Delta L_{AB} & \varepsilon_{AB} L_{AB} = -0.03 * 10\ in. = -0.3\ in. \\ \Delta L_{BC} & \varepsilon_{BC} L_{BC} = 0.045 * 15\ in. = 0.675\ in. \end{array}$$

To find the average strain for the entire bar, first find the overall change in length for the entire bar.

$$\Delta L = -0.3\ in. + 0.675\ in. = 0.375\ in.$$

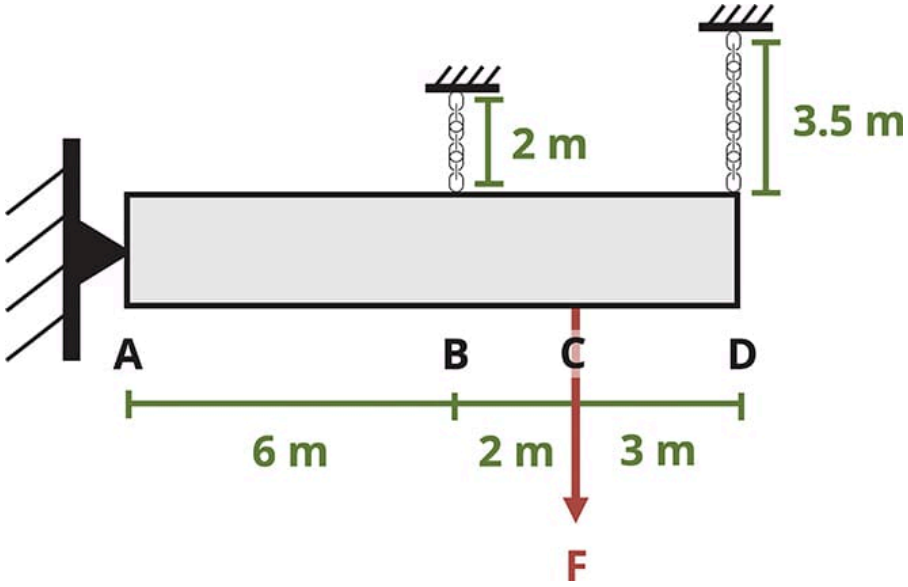
Overall the bar elongates by 0.375 in. The original length of the bar was 25 in. We can now determine the strain for the entire bar.

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.375\ in.}{25\ in.} = 0.015\ in./in.$$

Example 3.2

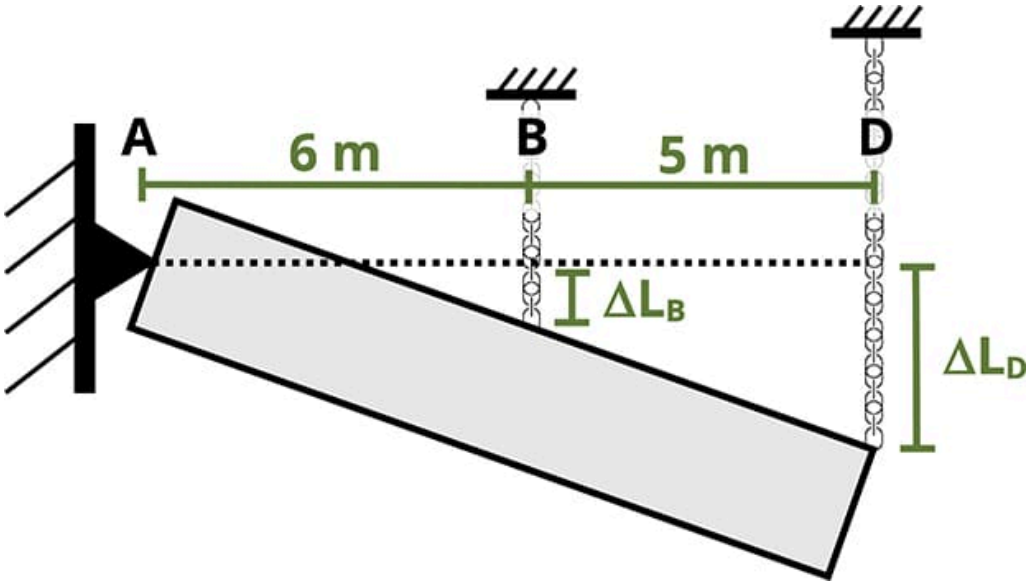
Rigid bar ABCD is supported by a pin at A and cables at B and D. After load F is applied, the strain in cable D is 2,300 μm/m.

Determine the strain in cable B.



Solution

Consider the deflected position of rigid bar ABCD. Now calculate the change in length of cable D.



$$\varepsilon = 2,300\frac{\mu m}{m} \times \frac{1m}{10^6 \mu m} = 2,300 \times 10^6 \frac{m}{m}$$

$$\varepsilon = \frac{\Delta L}{L}, \Delta L = \varepsilon \times L$$
$$\Delta L_D = \varepsilon L = 2,300 \times 10^{-6} \times 3.5m = 0.00805m = 8.05mm$$

Point D will deflect downward by 8.05 mm.

Use similar triangles to find the deflection of point B.

$$\frac{\Delta L_B}{6\,m} = \frac{8.05\,mm}{11\,m}$$
$$\Delta L_B = \frac{6\,m * 8.05\,mm}{11\,m} = 4.39\,mm$$

Cable B elongates by 4.39 mm. Use this to determine the strain in cable B. Be careful to be consistent with units. Use either ΔL = 4.39 mm and L = 2,000 mm or ΔL = 0.00439 m and L = 2 m.

$$\varepsilon = \frac{\Delta L}{L} = \frac{4.39\,mm}{2,000\,mm} = 0.002195\,m/m = 2,195\,\mu m/m$$

Step-by-Step: Normal Strain

1. Determine the change in length of the object.
2. Calculate the normal strain, $\varepsilon = \frac{\Delta L}{L}$.

3.2 Shear Strain

Consider a small square element that may be part of a body such as that shown in Figure 3.1. Just as the distance between points changes as the body deforms, so the shape of this square element also changes. The corners of the square element initially form right angles. As the body deforms and the points move with the object, the element's shape changes and the corners no longer form right angles (Figure 3.4). There is a small horizontal displacement of amount Δx.

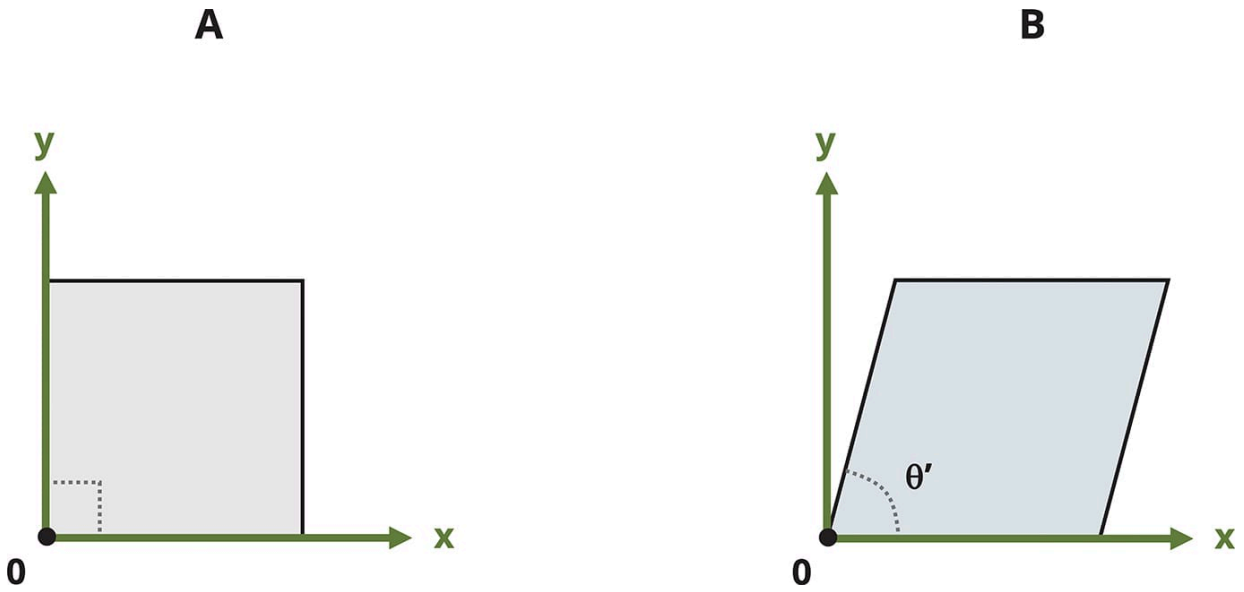


Figure 3.4: (A) The undeformed element is initially square, with right-angled corners. (B) As the body deforms, the square element skews and the corners are no longer right angles.

Shear strain is denoted by the Greek letter gamma (γ) and, like normal strain, is a dimensionless quantity. Recall that normal strain was defined in Equation 3.1 as the change in length of the object divided by its original length. Shear strain is similarly defined, but the deformation is perpendicular to the side length instead of parallel to it (Figure 3.5). If the element has side length L and displacement Δx, the shear strain is defined as

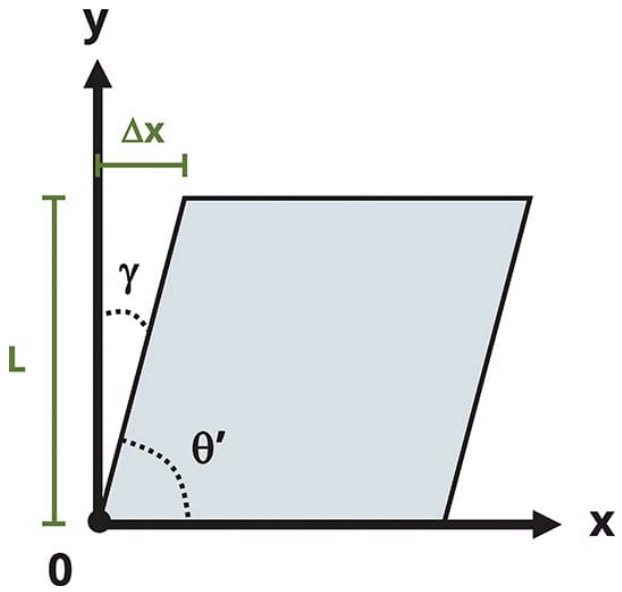


Figure 3.5: As the element deforms under shear, the shear strain at corner O is defined as $\gamma = \frac{\Delta x}{L}$.

$$\boxed{\gamma = \frac{\Delta x}{L}},$$

(3.2)

where

$$\gamma = Shear\ strain$$
$$\Delta x = Horizontal\ displacement[m, in.]$$
$$L = Vertical\ side\ length[m, in.]$$

For small deformations, the shear strain is commonly approximated as a change in angle. Specifically, provided we start with right-angled corners, shear strain can be approximated as the original angle (before deformation) minus the new angle (after deformation). Regardless of the shape of the body, a square element can always be defined such that the original angle is always a right angle. After deformation, the new angle between these three points is θ'. By convention, shear strain is represented in radians. Thus the original right angle is represented as $\frac{\pi}{2}$ radians and the shear strain as

$$\boxed{\gamma = \frac{\pi}{2} - \theta'},$$

where

- γ = *Shear strain [rad]*
- $\frac{\pi}{2}$ = *Original angle [rad]*
- *θ' = New angle [rad]*

Finding angle γ directly, without first finding angle θ', is often an easier approach. Consider the deformed element in Figure 3.5. If the element has side length L and horizontal displacement Δx, the angle γ can be found from $\gamma = \tan^{-1}\left(\frac{\Delta x}{L}\right)$. For small angles where $\tan(\gamma) \approx \gamma$ (when calculated in radians), this is approximately $\gamma = \frac{\Delta x}{L}$, as defined earlier.

Note that because we always subtract the new angle from the original angle, if the angle increases, the result is a negative shear strain, and if the angle decreases, it is a positive shear strain (Figure 3.6).

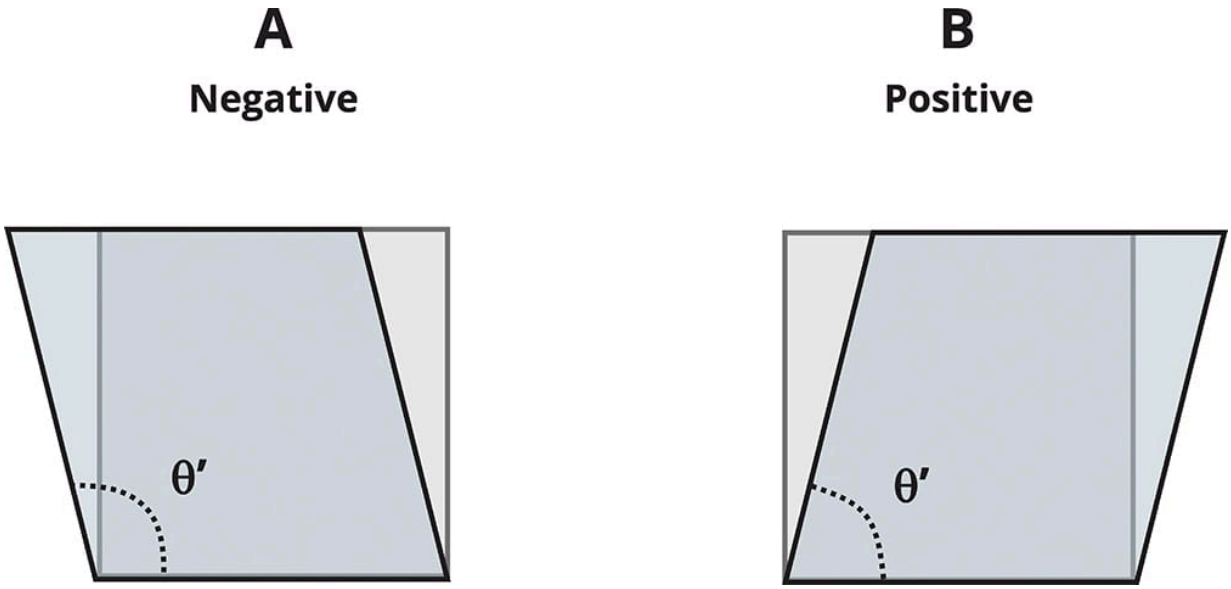


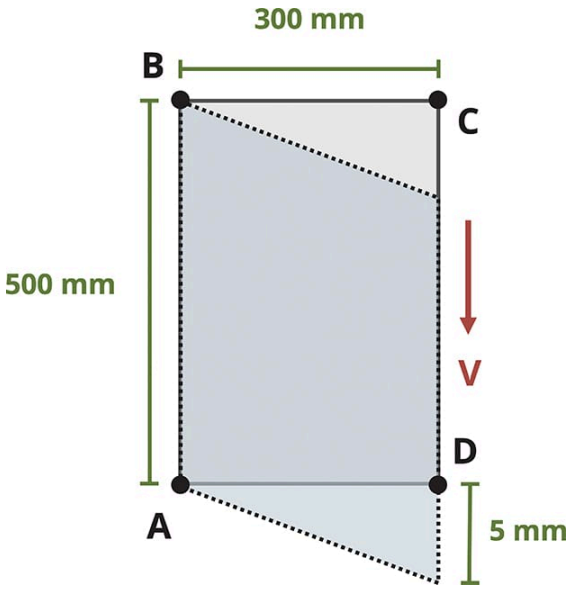
Figure 3.6: (A) If the new angle θ' is larger than $\frac{\pi}{2}$ radians, the shear strain is negative. (B) If the new angle θ' is smaller than $\frac{\pi}{2}$ radians, the shear strain is positive.

See Example 3.3 and Example 3.4 to practice calculating shear strain.

Example 3.3

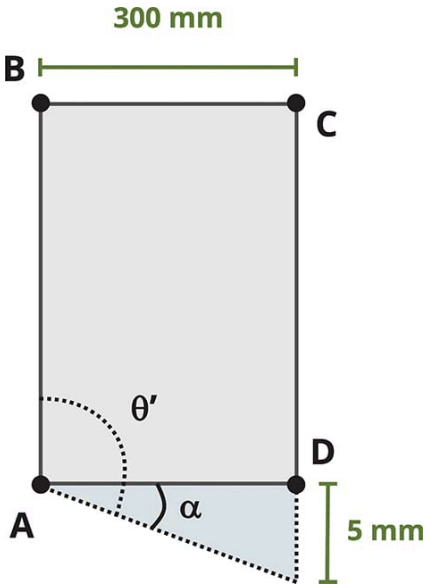
A thin rectangular plate has a base of 300 mm and a height of 500 mm. Initially corners A, B, C, and D are all right angles. The plate is subjected to a shear force V, which deforms the plate into the dotted lines shown. Corner D moves straight downward by 5 mm.

Determine the shear strain at corner A.



Solution

We could calculate the new angle at corner A as $\theta' = \frac{\pi}{2} + \alpha$.



Angle α can be found, since there is a triangle of base 300 mm and height 5 mm.

$$\alpha = \tan^{-1}\left(\frac{5}{300}\right) = 0.01667\, rad$$

Then

$$\theta' = \frac{\pi}{2} + 0.01667\, rad = 1.5875\, rad$$

Now the shear strain can be calculated.

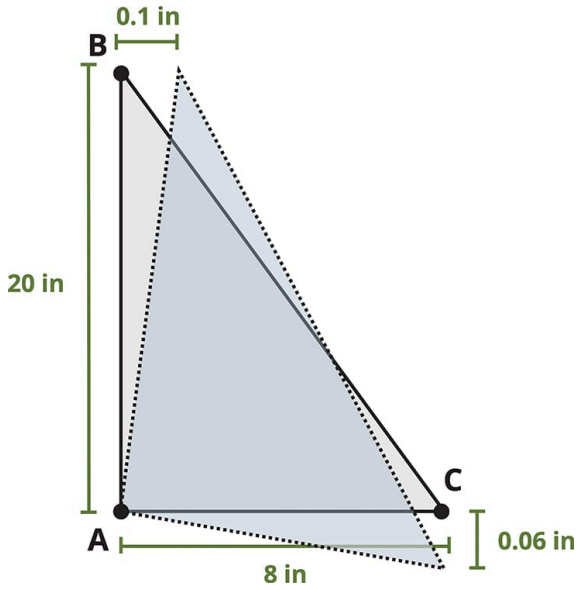
$$\gamma = \frac{\pi}{2} - \theta' = \frac{\pi}{2} - 1.5875\, rad = -0.01667\, rad$$

Notice that this is just the negative of angle a. Since the shear strain is the change in angle and α represents the increase of the angle, we could simply have said that angle α is the shear strain. As defined in Section 3.2, the shear strain is negative because the angle is increasing.

Example 3.4

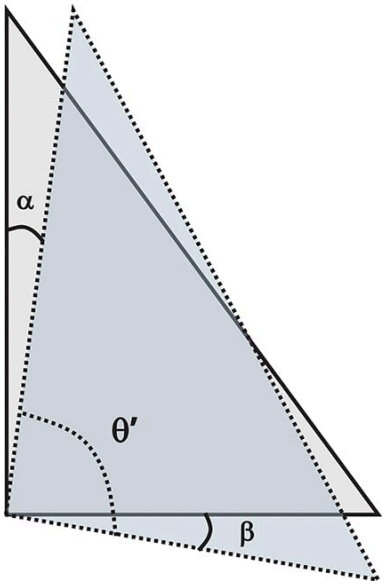
A thin triangular plate ABC has a base of 8 in. and a height of 20 in. Corner A is initially a right angle. When subjected to a shear load, the plate deforms as shown. Corner B moves 0.1 in. to the right and corner C moves 0.06 in. downward.

Determine the shear strain at corner A.



Solution

The angle at A will decrease by angle α and increase by angle β. Start by finding these angles.



$$\alpha = \tan^{-1}\left(\frac{0.1}{20}\right) = 0.005\, rad$$
$$\beta = \tan^{-1}\left(\frac{0.06}{8}\right) = 0.0075\, rad$$

The new angle at A is

$$\theta' = \frac{\pi}{2} - \alpha + \beta = 1.5708 - 0.005 + 0.0075 = 1.5733\, rad.$$

With the new angle known, we can calculate the shear strain.

$$\gamma = \frac{\pi}{2} - 1.5733\, rad = -0.0025\, rad$$

Note that the shear strain can be calculated more directly. Given that shear strain is just the change in angle, we could have said that the angle at A decreases by angle α and increases by angle β.

Recall that shear strain is positive if the angle decreases and negative if the angle increases. The change in angle at A, and thus the shear strain, could be calculated as

$$\gamma = \alpha - \beta = 0.005\, rad - 0.0075\, rad = -0.0025\, rad.$$

Step-by-Step: Shear Strain

1. Begin with a corner that forms a right angle ($\frac{\pi}{2}$ rads) prior to deformation.
2. After deformation, determine the new angle (θ') at this corner.
3. Determine the shear strain, $\gamma = \frac{\pi}{2} - \theta'$.

Summary

Key Takeaways

Objects may experience a change in length. If, for example, the axial load is tensile, the length will increase. If the load is compressive, the length will decrease.

Normal strain is defined as the change in length divided by the original length. Although strain has no units, it is commonly written with units of mm/mm or in./in. or similar. Normal strain is positive if the length increases and negative if the length decreases.

Objects subjected to shear loads experience a change in angle. Shear strain is defined as this change in angle, expressed as original angle minus new angle, where the original angle is always a right angle. Shear strains are positive if the angle decreases and negative if the angle increases.

Shear strains are expressed in radians, not degrees, and so the original angle is always defined as $\frac{\pi}{2}$ radians.

Key Equations

Normal strain:

$$\varepsilon = \frac{\Delta L}{L}$$

Shear strain:

$$\gamma = \frac{\Delta x}{L}$$
$$\gamma \approx \frac{\pi}{2} - \theta'$$

References

Figures

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Chapter 4: Mechanical Properties of Materials

Learning Objectives

- Explain important mechanical properties of engineering materials, including representative behavior within an initial linear elastic region where stresses and strains are proportional to one another
- Describe how these material properties are determined, relating behavior on an actual specimen (a simple structure) to what the material experiences (local stresses and strains)
- Recognize different behavior of brittle and ductile materials and understand the important concept of yield strength
- Explore relationships among various moduli and Poisson's ratio for linear elastic, isotropic materials
- Analyze linear elastic response of materials subjected to simple stress states, as well as the more complex multiaxial stresses imposed on real engineering products
- Apply a factor of safety as one simple means to begin to accommodate unexpected loading conditions, material variations, and other possible uncertainties

To meet design requirements of load-bearing structures typically involves determining the geometries and sizes of the components involved. For instance, a structure may have to be capable of carrying anticipated loads without failure (e.g., an aircraft wing), be stiff enough to deflect by only a limited amount (e.g., a building girder), or be material that returns to its original configuration when the loading is removed (e.g., a spring). To meet these design requirements we must understand the behavior of the material or materials being used within the structure. Exploring several of these basic mechanical properties is the focus of this chapter. An understanding of material behavior and these mechanical properties will enable us to relate stresses to strains, relate temperature change to strains, and explore the coupling that exists when subjecting materials to an arbitrary stress state.

The mechanical properties described in this chapter vary among materials. These properties can be measured and recorded for reference in engineering handbooks. Some useful properties for common engineering materials are recorded in Appendix C.

For the purposes of this book as we learn the principles, we assume that mechanical properties are constant—that is, independent of temperature, rate of loading, and other factors. This assumption works well for many engineering materials such as steel and aluminum at moderate temperatures, but not necessarily at very high temperatures. In addition, the mechanical properties of polymers, including plastics and rubber, are known for their temperature, time, and rate dependence. That is, their properties change as these factors vary, often at common use temperatures. This makes presenting their properties in a simple table challenging. Practicing engineers often need to recognize and address such complications as they work with such materials.

Of particular importance is that materials in this text are assumed to be isotropic, which means they have the same properties in all directions. Many metals, polymers, concrete, and ceramics, including as discussed in this book, are often considered to be isotropic. Note that anisotropic materials, which display different properties depending on the direction of loading, are quite common but are beyond the scope of this text. Examples include wood, fiber-reinforced plastics such as fishing rods or airplane wing skins, and biological tissues such as tendon.

Section 4.1 discusses a common method of determining mechanical properties, known as the tensile test. The output of the tensile test—the stress-strain diagram—is analyzed in Section 4.2.

Important mechanical relationships, including Hooke's law, which relates stress to strain, and Poisson's ratio, which relates normal strain in the axial and lateral directions, are covered in Section 4.3 and Section 4.4 respectively.

Section 4.5 considers the relationship between shear stress and shear strain, and Section 4.6 introduces the concept of thermal strain. Section 4.7 expands Hooke's law from Section 4.3 to the general three-dimensional case.

Finally, Section 4.8 focuses on applying a factor of safety to our designs to ensure they don't fail.

4.1 Tensile Test

A common method of measuring mechanical properties of materials is the tensile test, which involves placing a cylinder or bar of material in a load frame (such as shown in Figure 4.1 (A)) and subjecting the specimen to tensile loading. A specimen (such as shown in Figure 4.1 (B)) is mounted in grips that are then separated at a prescribed displacement rate. Because of the likelihood of failures at the grip due to concentrations of stress, specimens are typically “waisted” to form dog bone or dumbbell shapes. Large fillets are used to reduce stress concentrations associated with the different areas (see Section 5.2), and failure typically occurs within the narrowed region of the specimen because of the smaller cross-sectional area. This region is often referred to as the gauge length, as this becomes the area of interest for the specimen.

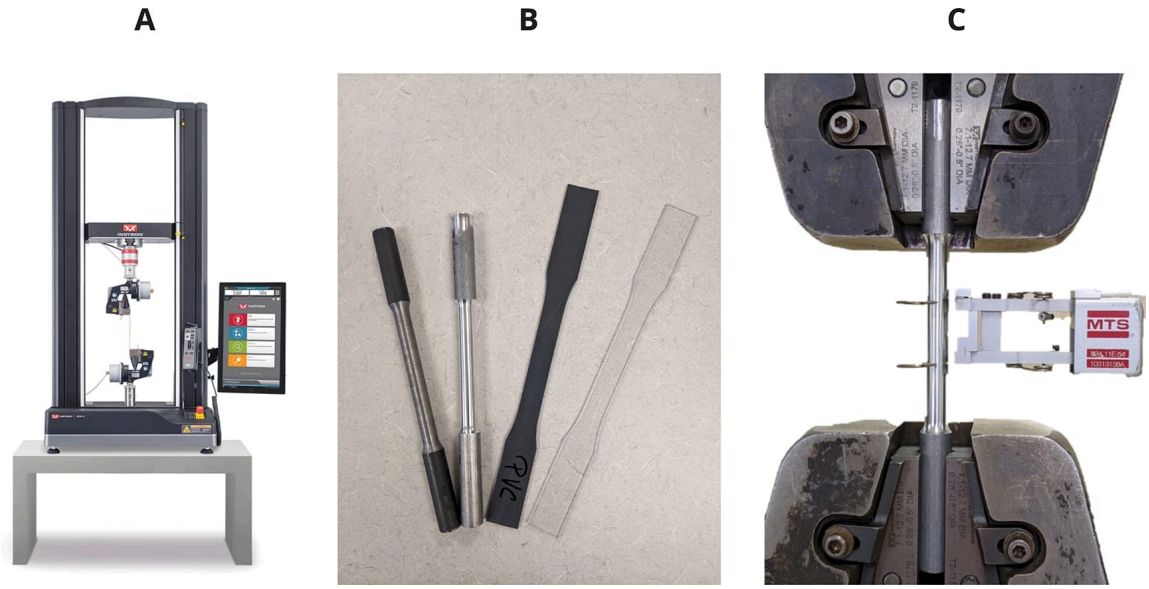


Figure 4.1: Tensile tests are a common means to measure mechanic properties of materials. (A) Illustration of a tensile test frame. (B) Left to right: Examples of steel, aluminum, poly(vinyl chloride), and poly(methyl methacrylate) tensile specimens. (C) Image of a circular steel specimen mounted in grips and instrumented with a clip-on extensometer.

The applied tensile load is steadily increased and along with the deformation of the specimen is measured and recorded at regular intervals. To determine deformation, a device such as a mechanical extensometer—see Figure 4.1 (C)—is mounted on the specimen. The specimen produces a uniform uniaxial stress state within its gauge region. For each data point, the load is used to calculate the average normal stress from Equation 2.1.

$$\sigma = \frac{N}{A}$$

The deformation is used to calculate the average normal strain from Equation 3.1, where DL is the deformation measured by the strain gauge over the active length of the extensometer, L.

$$\varepsilon = \frac{\Delta L}{L}$$

These quantities are essential in engineering design because properties from appropriate tests on laboratory specimens may be used to predict the behavior of much more complex engineering components.

4.2 Stress-Strain Diagram

The goal in conducting tests of materials, such as the tensile test described above, is to learn qualitatively how the specimen responds to the applied loads and to quantitatively determine a material's important mechanical properties.

As discussed in the introduction to this chapter, this text assumes these material properties to be constant and so, once

determined from a test specimen, can be used to design and analyze components made of that same material. Appendix C contains material properties for a selection of these materials. Again, caution is needed as these properties can vary, sometimes substantially, depending on temperature, loading rate, prior loading history, and other factors.

The stress and strain data collected from a tensile test can be plotted to create a stress-strain diagram. Figure 4.2 illustrates an exaggerated stress-strain curve for a steel specimen. Let's discuss the shape of the curve, the behavior of the material, and the material properties that can be determined from this plot.

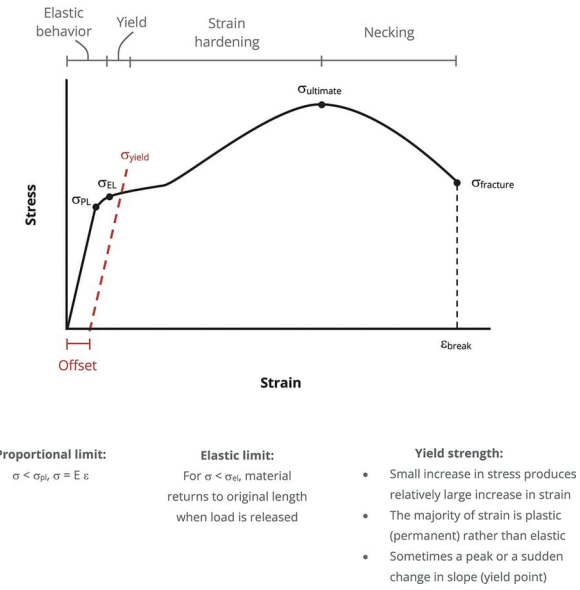


Figure 4.2: Idealized stress-strain curve for a steel specimen, indicating the proportional limit (σ_{PL}), elastic limit (σ_{EL}), yield stress (σ_{yield}), ultimate tensile strength ($\sigma_{ultimate}$), and fracture stress ($\sigma_{fracture}$).

4.2.1 Yield Stress

Initially a linear relationship exists between stress and strain, represented as a straight line on the stress-strain diagram. Following this is a plateau where a relatively small increase in stress results in a large increase in strain. The transition between the linear region of the stress-strain curve and the subsequent behavior marks an important engineering property of the material known as the yield stress, σ_{yield}.

There are actually three values worth mentioning here:

1. The point where the relationship between stress and strain stops being linear is known as the proportional limit, σ_{PL}.

2. Following the proportional limit is the elastic limit, σ_{EL}. If a material is loaded so that the stress does not exceed the elastic limit and then the load is removed, the object will return to its original dimensions. The elastic limit marks the point beyond which materials do not return to their original state.

3. The yield point is typically defined using the 0.2 percent offset method where a straight line beginning at a strain of 0.002 (0.2%) and running parallel to the linear portion of the diagram is drawn. The point where this line intersects the stress-strain curve is the yield point.

Since these three points for a material often occur within a fairly narrow range of one another, this text refers to these three points collectively as yielding. We use the yield stress because this is the easiest of the three points to consistently identify when working with real data.

4.2.2 Elastic and Plastic Deformation

Prior to yielding, the material undergoes elastic deformation. If the load is removed during elastic deformation, the object will

return to its original dimensions. This makes the yield strength of the material a highly important design parameter, as many engineering components are designed to operate within the linear region of material response and hence remain elastic.

When loaded past the yield point, the material continues to experience increasing stress and strain, though it is no longer a linear relationship. Deformation beyond the yield point is permanent. This is known as plastic deformation. If an object is loaded beyond its yield point and the load is then removed, any elastic deformation is recovered but any plastic deformation is not. The object's dimensions have been permanently changed.

You likely have experienced material yielding if you have ever deformed a coat hanger or paperclip such that it doesn't return to the original state. Such behavior is undesirable for many engineering components designed to continue operating in their original state, so engineers commonly consider the yield stress to be the maximum acceptable stress in a component.

However, plastic deformation following yielding can be very important in the fabrication process. For example, sheets of steel or aluminum can be placed between molds and compressed under high pressure to deform them into the shape of an automobile panel as seen in Figure 4.3 (A). Plastic deformation can also be useful in absorbing energy, such as seen in the concertina crushing of a structural component for an automobile, shown in Figure 4.3 (B). This book, however, focuses on the pre-yield, linear elastic region.

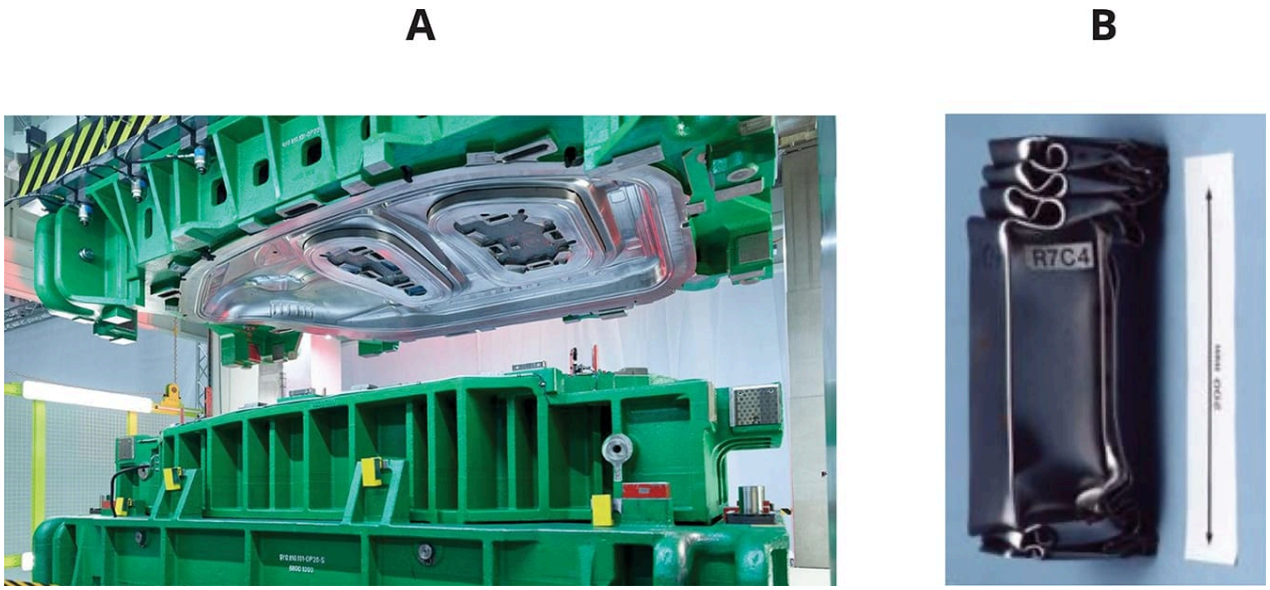


Figure 4.3: Engineering materials are often loaded beyond yield to permanently deform the material, as in (A) the stamping process for an automotive panel or in (B) the crushing of a weld-bonded hat section used to absorb energy in an automobile accident.

If we continue to load the specimen into the plastic region, the stress and strain continue to increase until some maximum stress known as the ultimate tensile stress, σ_{UTS}. Past this point the material undergoes necking—a thinning of the cross-section—and so the load it can withstand actually decreases. This continues until the fracture point, $\sigma_{fracture}$.

4.2.3 Ductile and Brittle Materials

Figure 4.2 shows an idealized stress-strain diagram for a steel specimen. The diagrams for other materials will look a little different because their mechanical properties are all different. Figure 4.4 (A) illustrates representative stress-strain behavior for several engineering materials. These are drawn to scale, and the popout shows an enlarged view of the elastic region.

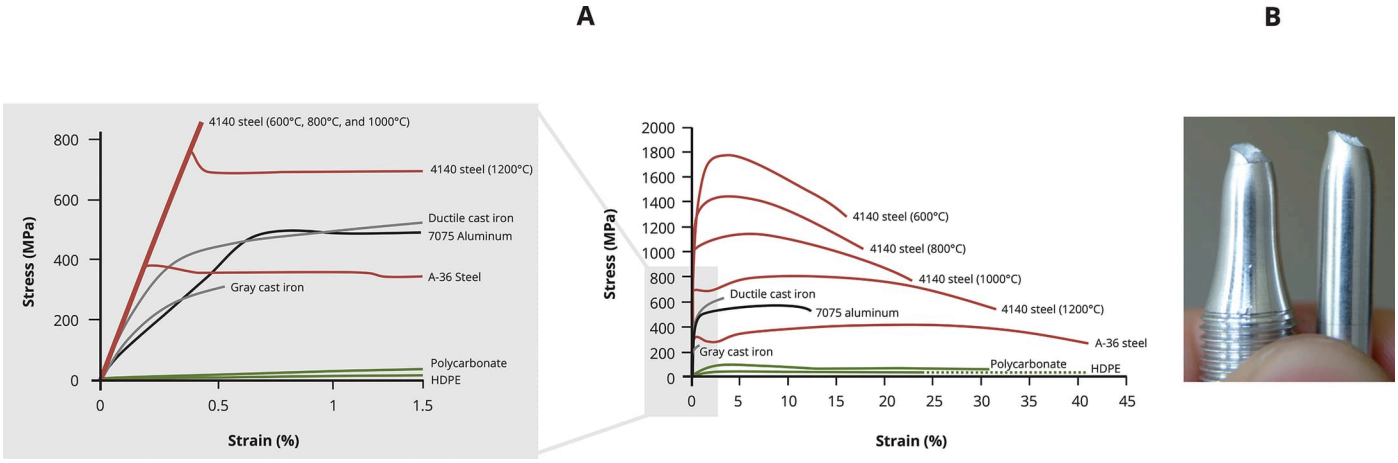


Figure 4.4: Results obtained from tensile tests. (A, left) Illustration of typical stress-strain behavior for some common materials. Temperatures denote anneal temperatures. (A, right) Same data plotted up to 5 percent strain. (B) Tensile test of an AlMgSi alloy. This is a ductile fracture type, as seen by the local necking and the cup and cone fracture surfaces.

Each material initially experiences a linear relationship between stress and strain, but the yield stress is significantly higher for some materials than others. Materials with a high yield stress are of course more appropriate for high-stress applications.

Consider the plastic deformation for the materials shown in Figure 4.4. Some materials, such as cast iron, fail at relatively small strains, whereas other materials, such as high-density polyethylene (HDPE) often used for milk jugs, reach very large strains well beyond the axis limit.

Brittle materials are those with short plastic regions, indicating they break without experiencing much plastic deformation. The behavior remains nearly linear until failure.

Ductile materials are those with long plastic regions, indicating they experience significant plastic deformation before breaking. Figure 4.4 (B) shows the failure region of an aluminum alloy specimen tested to failure, illustrating necking and ductile failure. This can be useful in many engineering applications where the material will deform significantly if the yield stress is accidentally exceeded and can thereby allow time for the component to be replaced before it breaks. Consider the difference between dropping a ceramic mug on a hard floor compared to a mug made of aluminum or tin. The ceramic mug is likely to shatter, whereas the tin mug may end up with a permanent dent (indicating that its yield stress was exceeded and it experienced plastic deformation) but will still be functional.

4.3 Hooke's Law

The slope of the linear region of this stress-strain diagram represents an important material property known as the elastic modulus (also known as Young's modulus or the modulus of elasticity), so designated by E. Thomas Young (1773–1829), a British scientist who noted the proportionality between applied force and extension, which he recorded as *ut tensio sic vis*, meaning “as is the extension, so is the force.”

The elastic modulus can be thought of as a measure of stiffness of the material. A high elastic modulus for stiff materials is represented by a steep slope—that is, a large increase in stress with only a small increase in strain. A low elastic modulus for soft materials is represented by a shallow slope, indicating a small increase in stress but a large increase in strain.

The linear elastic region of material response is of enormous importance for most engineering structures. Loading significantly beyond this proportional limit can result in pronounced yielding and permanent deformation that is often unacceptable, unless one is using this deformation to absorb energy (as in an automobile frame involved in collision). The elastic modulus is illustrated in Figure 4.5, which plots a portion of the normal stress versus normal strain, for example, from a tensile test.

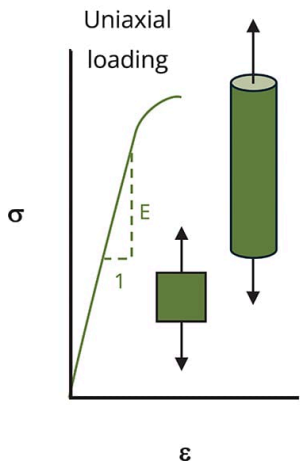


Figure 4.5: Illustration of the initial linear elastic portions of the stress-strain curve and slope corresponding to elastic modulus for uniaxial tension.

$$E = \frac{\sigma}{\varepsilon}$$

Since strain is unitless, elastic modulus will have the same units as stress (Pa or psi).

This relationship is more commonly called Hooke's law and written as

$$\boxed{\sigma = E\varepsilon},$$

(4.1)

where

$$\sigma = Average\ normal\ stress\ [Pa, psi]$$
$$E = Elastic \bmod ulus\ [Pa, psi]$$
$$\varepsilon = Normal\ strain$$

This equation is appropriate for uniaxial stress (i.e., when the load is applied and the strain measured in only the same single direction). Section 4.7 expands this relationship to cases where normal stresses occur in more than one direction.

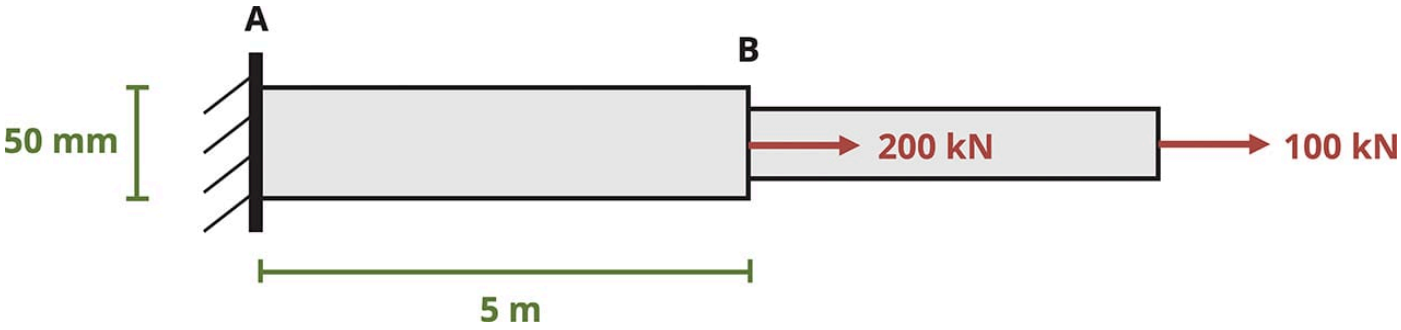
Especially important to note is that this relationship is applicable only within the linear elastic region of the material's response. This relationship, applicable at the material level for stress and strain, also extends to structural elements (e.g., bars in tension or compression, beam deflections, or helical springs in which the force and deflection are related by the spring constant) and many other engineering components and structures, as covered in the following chapters of the book.

Example 4.1 uses Hooke's law to determine the elastic modulus of a material where the applied load and deformation are known.

Example 4.1

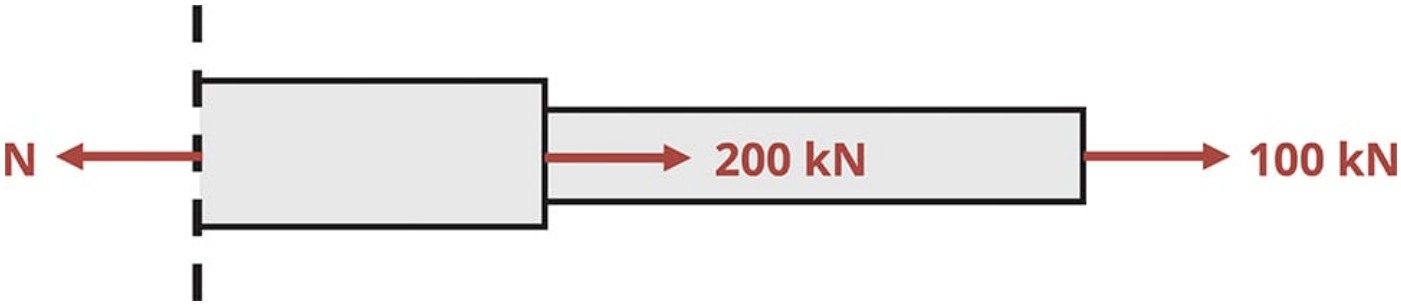
Two bars are welded together and fixed to a wall at A. Two loads are applied as shown. Bar AB is 5 m long and has a diameter of 50 mm.

If the change in length of bar AB is 4 mm, determine the elastic modulus of this bar.



Solution

Cut a cross-section through bar AB and determine the internal load.



$$\sum F_x = 100\ kN + 200\ kN - N = 0 \rightarrow N = 300\ kN$$

Calculate the normal stress.

$$\sigma = \frac{N}{A} = \frac{300,000\ kN}{\pi(0.025)^2\ m^2} = 152.8\ MPa$$

Calculate the normal strain.

$$\varepsilon = \frac{\Delta L}{L} = \frac{4\ mm}{5,000\ mm} = 0.0008$$

Use Hooke's law to calculate the elastic modulus.

$$E = \frac{\sigma}{\varepsilon} = \frac{152.8 \times 10^6\ Pa}{0.0008} = 191\ GPa$$

Step-by-Step: Elastic Modulus

1. Determine the internal load in the member by equilibrium.
2. Determine the normal stress in the member by $\sigma = \frac{\mathrm{N}}{\mathrm{A}}$.
3. Determine the normal strain in the member by $\varepsilon = \frac{\Delta \mathrm{L}}{\mathrm{L}}$.
4. Use Hooke's law to calculate the elastic modulus, $\sigma = \mathrm{E}\varepsilon$.

4.4 Poisson's Ratio

When you stretch a rubber band, you'll notice that the band gets longer in the direction of the applied force but the width of the band decreases. That is, a normal strain occurs in the axial (length) direction and also in the lateral (width or thickness) direction. These normal strains couple in perpendicular directions.

Poisson's ratio is the material property that describes the proportional relationship between axial and lateral strains. Named after Siméon Denis Poisson (1781–1840), the French mathematician and physicist, Poisson's ratio is defined as the negative of the lateral strain divided by the axial strain for a uniaxially loaded material. It may be expressed as

$$\boxed{\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}}},$$

(4.2)

where

$$v = Poisson's\ ratio$$
$$\varepsilon_{lateral} = Lateral\ normal\ strain$$
$$\varepsilon_{axial} = Axial\ normal\ strain$$

Recall that strains are positive if the dimension lengthens and negative if it shortens. For most common engineering materials, applying a tensile force causes the object to become longer and thinner, while applying a compressive force causes it to become shorter and wider. This is shown in Figure 4.6. Thus, for these materials, either the axial or lateral strain will be positive and the other will be negative.

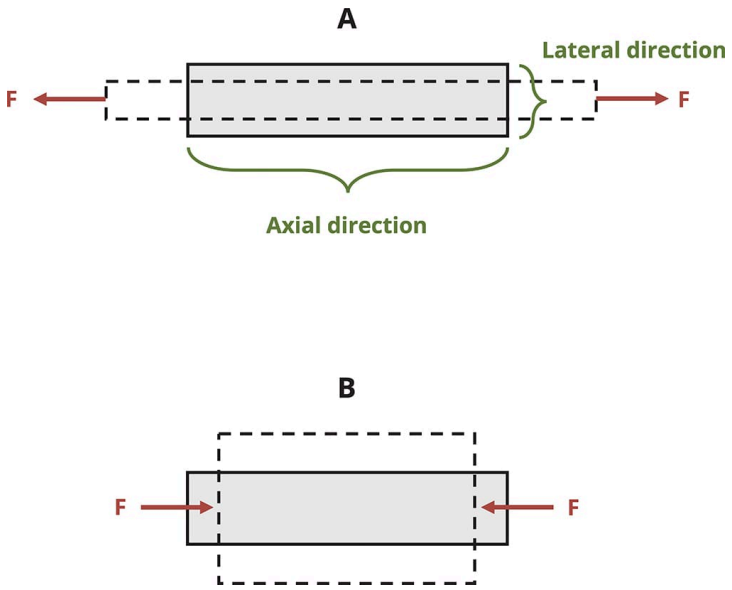


Figure 4.6: Illustration of axial and lateral strains for (A) Tensile loading and (B) Compressive loading. Solid lines show original (undeformed) shape, and dotted lines show deformed shape. Note that one strain is positive (elongating) and the other is negative (shortening) in both cases.

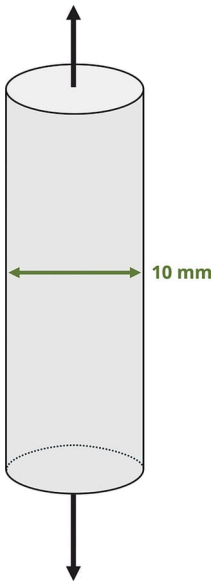
Note that this relationship is appropriate for a uniaxially loaded specimen. Section 4.7 expands this relationship to cases of normal stresses in more than one direction.

Poisson's ratio depends on the material as well as its structure. Theoretically, Poisson's ratio can range from –1 to +0.5 for isotropic materials but usually ranges from near zero for foams to almost exactly 0.5 for soft materials such as elastomers and gels. Most metals fall in the range from 0.2 to 0.35. A positive value of Poisson's ratio means that as the material is extended in one direction, a contraction occurs in lateral directions. See Example 4.2 for a simple application of Poisson's ratio.

Example 4.2

A circular polymeric rod with a diameter of 10 mm is subjected to a uniaxial force, resulting in an axial strain of 0.5%. A corresponding reduction in diameter of 0.02 mm is measured.

Determine the Poisson's ratio of this material.



Solution

The longitudinal strain is given as $\varepsilon_{\text{longitudinal}} = 0.005$.

The lateral strain is determined from the reduction in diameter divided by the original diameter and is negative because it is contracting.

$$\varepsilon_{lateral} = -\frac{0.02\,mm}{10\,mm} = -0.002$$

Poisson's ratio is then easily determined as

$$\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{axial}} = -\frac{-0.002}{0.005} = 0.4$$

Step-by-Step: Poisson's Ratio

1. Determine the axial strain, given the change in length (L): $\varepsilon_{\text{axial}} = \frac{\Delta \text{L}}{\text{L}}$.
2. Determine the lateral strain, given the change in diameter (D), height (H), or width (W): $\varepsilon_{\text{lateral}} = \frac{\Delta \text{D}}{\text{D}} = \frac{\Delta \text{H}}{\text{H}} = \frac{\Delta \text{W}}{\text{W}}$.
3. Calculate Poisson's ratio: $\nu = -\frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{axial}}}$.

4.5 Shear Stress and Strain

In addition to considering normal stresses and strains, engineers require an understanding of the relationship between shear stresses and shear strains. Like the tensile test for normal stress and strain, a shear test can be performed to produce a diagram of shear stress against shear strain. The slope of a shear stress versus shear strain diagram is referred to as the shear modulus or modulus of rigidity and is designated by G. The corresponding slope of a shear stress versus shear strain plot is referred to as the shear modulus or modulus of rigidity, as shown in Figure 4.7 (B).

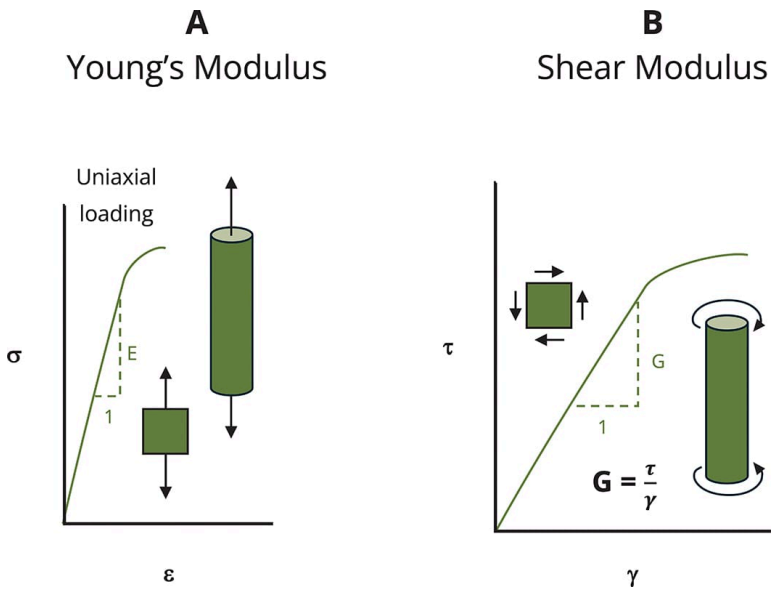


Figure 4.7: Illustration of the initial linear elastic portions of the stress-strain curve and slopes corresponding to (a) Young's modulus for uniaxial tension; (b) shear modulus for pure shear loading

$$\boxed{\tau = G\gamma},$$

(4.3)

where

$$\tau = Shear\ stress\,[Pa, psi]$$
$$G = Shear \bmod ulus\ or \bmod ulu\ s\ of\ rigidity\,[Pa, psi]$$
$$\gamma = Shear\ strain$$

Note that the elastic modulus, the shear modulus, and Poisson's ratio are not independent of each other. For isotropic materials, only two are independent, linear elastic properties; knowing any two allows us to calculate the others. It can be shown that

$$G = \frac{E}{2(1+v)},$$

(4.4)

where

- *G = Shear modulus or modulus of rigidity [Pa, psi]*
- *E = Elastic modulus [Pa, psi]*
- *v = Poisson's ratio*

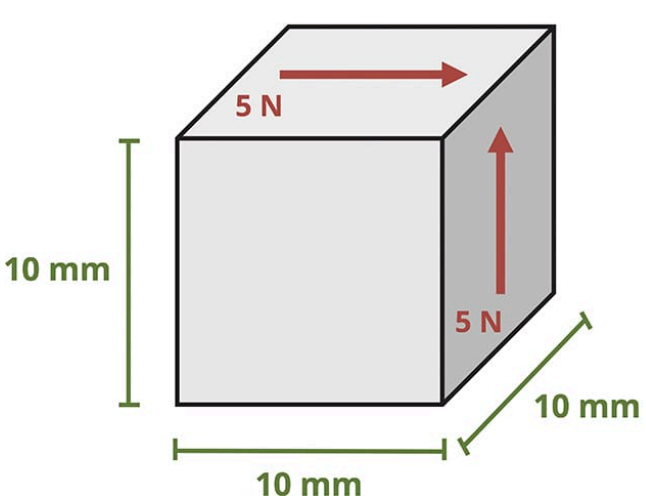
See Example 4.3 for a problem involving the shear modulus.

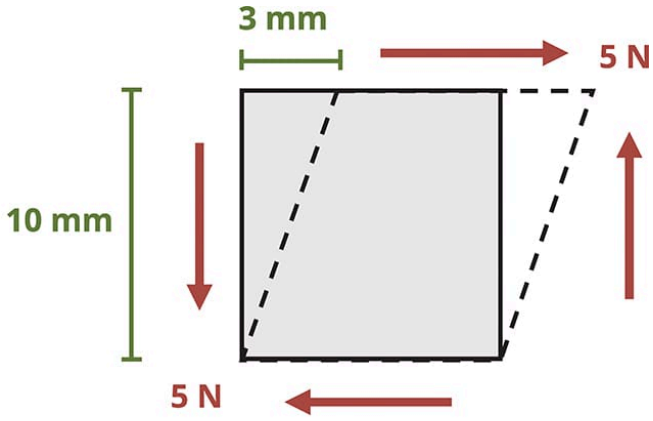
Example 4.3

Soft contact lenses often consist of a hydrogel, an elastomeric network that is imbibed with water, expanding its volume and making the lens more soft and compliant.

Suppose a 10 mm cube of the dry elastomer were subjected to a shearing action as shown. Shearing forces of 5 N each are applied to the four faces of the cube, resulting in a displacement of 0.3 mm.

What is the shear modulus of this elastomeric block?





Solution

First determine the shear stress.

$$A = L^2 = (0.01m)^2$$
$$\tau = \frac{V}{A}$$
$$\tau = \frac{5\ N}{(0.01\ m)^2} = 50\ kPa$$

Then determine the shear strain.

$$\gamma = \frac{\Delta x}{L}$$
$$\gamma = \frac{0.3\ mm}{10\ mm} = 0.03$$

Finally, calculate the shear modulus.

$$G = \frac{\tau}{\gamma} = 50\ kPa/0.03 = 1.667\ MPa$$

Step-by-Step: Shear Modulus

1. Determine the internal load in the member by equilibrium.
2. Determine the shear stress in the member by $\tau = \frac{\mathrm{V}}{\mathrm{A}}$.
3. Determine the shear strain in the member by $\gamma = \frac{\Delta \mathrm{x}}{\mathrm{L}}$.
4. Use Hooke's law to calculate the shear modulus, $\tau = \mathrm{G}\gamma$.

4.6 Thermal Strain

Other factors in addition to relationships arising between load, stress, and strain can affect dimensional changes of materials as well as changes in the stresses present in a structure. Most materials will expand, for example, when their temperature is increased. Many engineering structures will experience significant temperature variations during their life, beginning with thermal processes involved in their manufacture, cyclic temperature changes (e.g., day/night or summer/winter), or because of their usage, such as an engine block heating up or cooling down. Although dimensional changes from temperature often appear to be quite small, they can lead to significant changes in stress state, and in fact numerous failures result not from applied mechanical loading but from thermal exposure issues. Though beyond the current scope, dimensional changes can also result from other environmental factors, such as the expansion of a material as water is absorbed, or from the application of electric fields for materials that exhibit piezoelectric properties.

Vibrational energy of atoms increases as temperature increases, typically leading to slightly greater atomic separation distances.

This expansion leads to dimensional changes in the material, and the behavior is characterized by the material's coefficient of thermal expansion (CTE). The linear CTE is often designated by α and is defined as the change in strain per unit change in temperature. As with the other material properties discussed in this chapter, α is assumed to be constant for any given material.

The CTE is independent of direction in isotropic materials. We can determine the CTE by heating or cooling a specimen and measuring the resulting strain (ε) or dimensional change and recording the change in temperature (DT).

$$\alpha = \frac{\varepsilon}{\Delta T} = \frac{\frac{\Delta L}{L}}{\Delta T},$$

where $\Delta \mathrm{T} = \mathrm{T}_{\mathrm{final}} - \mathrm{T}_{\mathrm{initial}}$. Since strain is unitless, CTE has units of $\frac{1}{^{\circ}\mathrm{C}}$ or $\frac{1}{^{\circ}F}$. Knowing the CTE enables us to easily determine the strain expected when a material is subjected to a temperature change of DT.

$$\boxed{\varepsilon_T = \alpha \cdot \Delta T}$$

(4.5)

Or dimensional changes can be determined as

$$\boxed{\Delta L = \alpha \cdot \Delta T \cdot L},$$

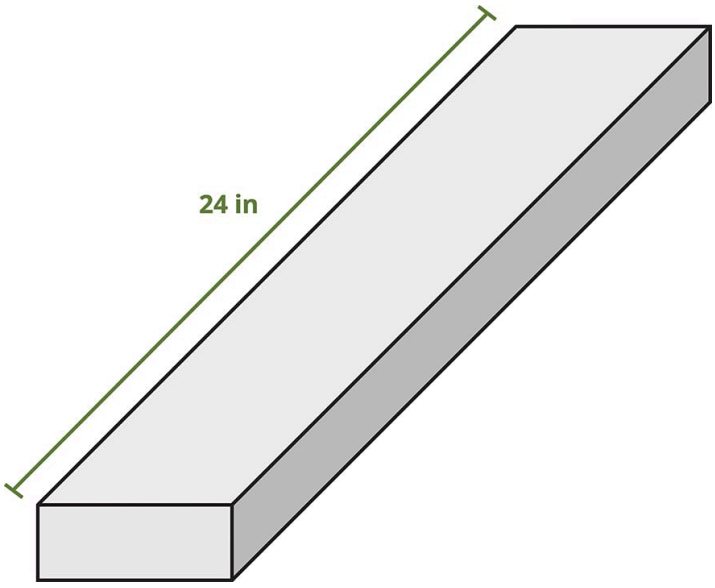
(4.6)

where

$\varepsilon_T = Thermal\ strain$
$\alpha = Coefficient\ of\ thermal\ expansion\ \left[\frac{1}{^{\circ}C}, \frac{1}{^{\circ}F}\right]$
$\Delta T = Change\ in\ temperature\ [^{\circ}C, ^{\circ}F]$
$\Delta T = Change in length [m, in.]$
$L = Original length [m, in.]$

See Example 4.4 for a problem involving a temperature change resulting in a thermal strain.

Example 4.4



A 24 in. long aluminum bar with a coefficient of thermal expansion of 13 × 10^{-6}/°F and Young's modulus of 10,000 ksi is cooled from 100°F to 25°F.

1. What change in strain results from this temperature change if the bar is free to contract?

2. What change in length results from this temperature change if the bar is free to contract?

Solution

1. Thermal strain can be calculated from $\varepsilon_{\mathrm{T}}=\alpha \Delta \mathrm{T}$.

$$\varepsilon_T = \alpha \Delta T$$
$$\varepsilon_T = 13 \times 10^{-6}/°F \times (-75°F)$$
$$\varepsilon_T = -0.00098$$

2. Change in length can be determined from $\varepsilon=\frac{\Delta \mathrm{L}}{\mathrm{L}}$.

$$\varepsilon = \frac{\Delta L}{L} \rightarrow \Delta L = \varepsilon L$$
$$\Delta L = -0.00098 \times 24\, in.$$
$$\Delta L = -0.0235\, in.$$

Step-by-Step: Thermal Strain

1. Determine the change in temperature, $\Delta \mathrm{T}=\mathrm{T}_{\text{final}}-\mathrm{T}_{\text{initial}}$.
2. Look up the coefficient of thermal expansion (α) for the material.
3. Calculate the thermal strain, $\varepsilon_{\mathrm{T}}=\alpha \Delta \mathrm{T}$.
4. Calculate the change in length from $\varepsilon=\frac{\Delta \mathrm{L}}{\mathrm{L}} \rightarrow \Delta \mathrm{L}=\varepsilon \mathrm{L}$.

4.7 Multiaxial or Generalized Hooke's Law

Materials are often tested under idealized conditions (e.g., uniaxial tension or shearing resulting from torsional loading). These tests are important because they provide material properties necessary for engineering design. Such simple stress states are less common in actual engineered components, where more complex, multiaxial stress states are present, as shown in Figure 4.8.

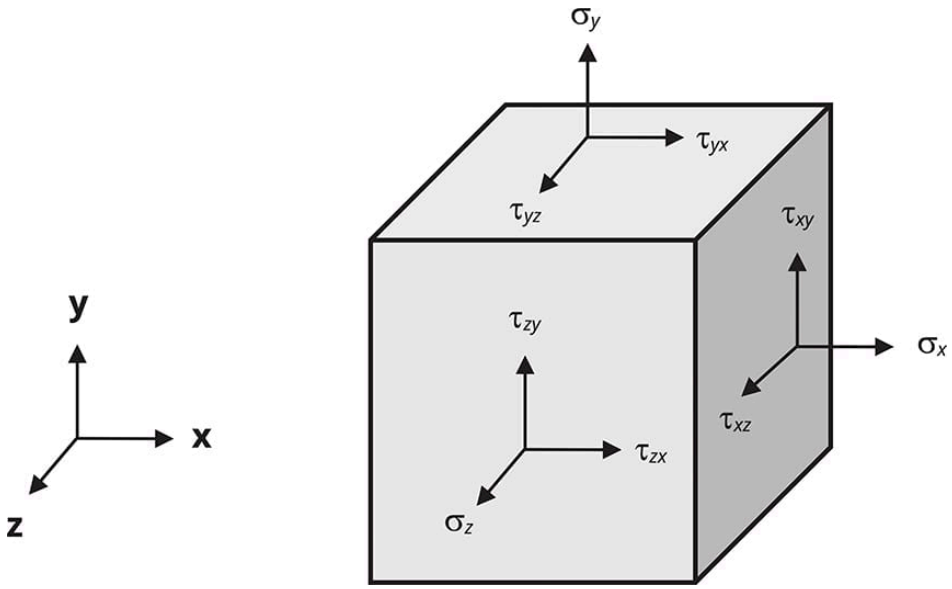


Figure 4.8: Illustration of a general, multiaxial stress state, showing normal and shear stresses acting on the positive x-, y-, and z-planes. Note that equal and opposite stresses exist on the hidden faces of the element.

Here there are three normal stresses, referred to with a single subscript indicating their direction (σ_x, σ_y, and σ_z), though some texts use repeated subscripts for these (σ_{xx}, σ_{yy}, and σ_{zz}). There are six shear stresses, designated by τ. These are referred to with

two subscripts, one indicating the plane or “face” the stress acts on and the other indicating the direction it acts in. For example, on the y-z plane (the x-face) are the normal stress τ_x and two shear stresses, τ_{xy} and τ_{xz}. The first subscript indicates that these shear stresses act on the x-face, and the second subscript indicates that they point in the y and z directions respectively.

Consider the x-y plane of the stress element in Figure 4.8. This two-dimensional plane is shown in Figure 4.9. It experiences two normal stresses (σ_x and σ_y) and two shear stresses (τ_{xy} and τ_{yx}). Of these, τ_{xy} will cause the element to rotate counterclockwise, whereas τ_{yx} will cause the element to rotate clockwise. For an element in equilibrium, these two stresses must be equal for these rotation effects to balance. Expanding this to the other planes, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, and $\tau_{yz} = \tau_{zy}$.

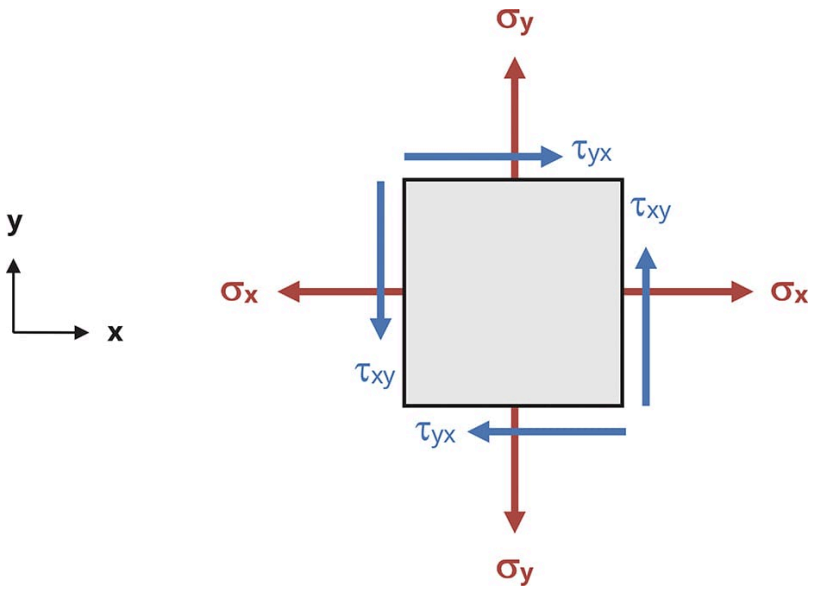


Figure 4.9: 2D view of the x-y plane of a 3D stress element.

Given the concept of Poisson's ratio, recognize that a normal stress in one direction creates normal strains in all three directions (x, y, and z). We have reviewed this in responding to the application of a uniaxial stress state—a normal stress applied in only one direction. For example, a normal stress in the x direction means there will be a normal strain in the x direction and also in the y and z directions.

$$\varepsilon_x = \frac{\sigma_x}{E}$$
$$\varepsilon_y = -\upsilon\varepsilon_x = -\frac{\nu\sigma_x}{E}$$
$$\varepsilon_z = -\nu\varepsilon_x = -\frac{\nu\sigma_x}{E}$$

Most engineering structures, however, involve more complex stress states, in which stresses may be applied in the x, y, and z directions. We refer to these as multiaxial stress states. We can define similar equations for a normal stress applied in the y direction.

$$\varepsilon_y = \frac{\sigma_y}{E}$$
$$\varepsilon_x = -\nu\varepsilon_y = -\frac{\nu\sigma_y}{E}$$
$$\varepsilon_z = -\nu\varepsilon_y = -\frac{\nu\sigma_y}{E}$$

And we can do likewise for a normal stress applied in the z direction.

$$\varepsilon_z = \frac{\sigma_z}{E}$$
$$\varepsilon_x = -\nu\varepsilon_z = -\frac{\nu\sigma_z}{E}$$
$$\varepsilon_y = -\nu\varepsilon_z = -\frac{\nu\sigma_z}{E}$$

If loads are applied in all three directions simultaneously, we can determine the overall strain in each direction by simply adding together the above components.

$$\boxed{\begin{aligned}\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E} = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]\\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]\\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]\end{aligned}},$$

(4.7)

where

- ε_x, ε_y, ε_z = *Normal strain in the x, y, and z directions*
- σ_x, σ_y, σ_z = *Normal stress in the x, y, and z directions [Pa, psi]*
- *E = Elastic modulus [Pa, psi]*
- *ν = Poisson's ratio*

This is the generalized Hooke's law for the three-dimensional state of stress. Example 4.5 demonstrates the application of the generalized Hooke's law to a problem with stresses acting in the x, y, and z directions.

Similarly, normal strains are written with a single subscript as ε_x, ε_y, and ε_z, whereas shear strains are written with two subscripts as $\gamma_{xy} = \gamma_{yx}$, $\gamma_{xz} = \gamma_{zx}$, and $\gamma_{yz} = \gamma_{zy}$.

The relationships between shear stresses and strains do not change in isotropic materials subjected to additional other stress components, so

$$\boxed{\begin{aligned}\gamma_{xy} &= \frac{\tau_{xy}}{G}\\ \gamma_{yz} &= \frac{\tau_{yz}}{G}\\ \gamma_{xz} &= \frac{\tau_{xz}}{G}\end{aligned}},$$

(4.8)

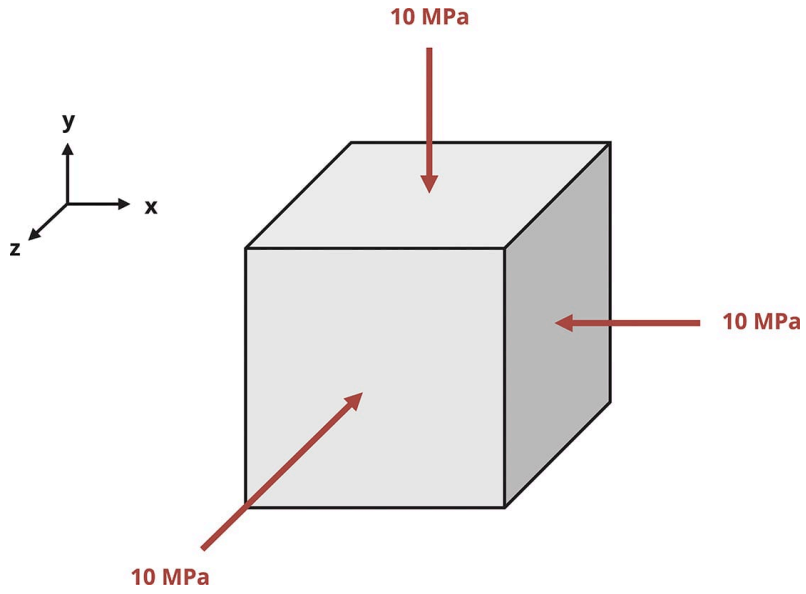
where

- γ_{xy}, γ_{yz}, γ_{xz} = *Shear strains*
- τ_{xy}, τ_{yz}, τ_{xz} = *Shear stresses [Pa, psi]*
- *G = Shear modulus [Pa, psi]*

Example 4.5

A solid, 50 mm polycarbonate cube is subjected to a compressive stress of $\sigma = 10\,\mathrm{MPa}$ on all sides.

Assuming the material has a Young's modulus of 3 GPa and a Poisson's ratio of 0.38, what is the change in length of one of the edges?



Solution

First note that a compressive stress acts in all directions, so $\sigma_x = \sigma_y = \sigma_z = -10\ \mathrm{MPa}$. We can choose any one of the generalized Hooke's law equations to determine the strain in the corresponding direction, but note that all normal strains will be the same.

$$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)]$$
$$\varepsilon = \frac{-\sigma}{E}[1 - 2v]$$
$$\varepsilon = \frac{-10\ MPa}{3\ GPa}[1 - 2 * 0.38] = -0.0008 = -0.08\%$$
$$\varepsilon = \frac{\Delta L}{L}$$
$$\Delta L = \varepsilon L = -0.0008 * 50\ mm = -0.04\ mm$$

If we used the simple uniaxial Hooke's law, $\sigma = \mathrm{E}\varepsilon$, we would erroneously obtain a strain of $\varepsilon = \frac{\sigma}{\mathrm{E}} = \frac{-10\ \mathrm{MPa}}{3\ \mathrm{GPa}} = -0.033 = -0.33\%$ and a change in length of –0.167 mm, more than four times the correct answer!

Step-by-Step: Multiaxial Hooke's Law

1. Determine the stress, α, acting on each side of the stress element.
2. Use these stresses to calculate the strain, ε, on each side of the stress element, using the generalized Hooke's law.
3. Calculate the change in length of each side, using $\varepsilon = \frac{\Delta \mathrm{L}}{\mathrm{L}}$.

4.8 Allowable Stress Safety Factor

Safety is of primary importance in engineering practice, and various engineering principles and codes are built on the necessity of considering the safety of products we design. Using the methods in this book, we can predict the stresses in a component. Rather than allowing the maximum anticipated stress to equal the failure stress, however, we apply a factor of safety (FS) such that the maximum allowable stress in the component is only a fraction of the failure stress. The factor of safety varies depending on the application, consequences of failure, uncertainty around loading, industry standards, and many other factors.

$$\boxed{\sigma_{allow} = \frac{\sigma_{fail}}{FS}},$$

(4.9)

where

- σ_{allow} = *Allowable stress [Pa, psi]*

- σ_{fail} = *Failure stress [Pa, psi]*
- *FS = Factor of safety*

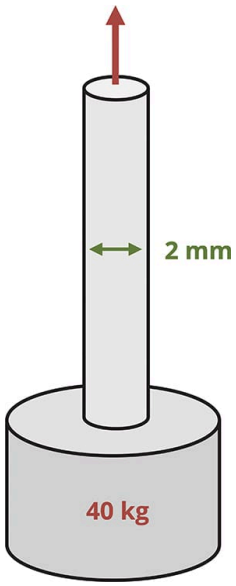
A larger factor of safety makes the structure less likely to fail but may add weight or cost or may limit functionality. If the factor of safety value is too small, the margin of safety for the design may become unacceptable. Unexpected overload situations, engineering design and assumption errors, and general wear and tear of a material over time, including in challenging environments, could lead to failure and consequences ranging from negative publicity to loss of life. The factor of safety must always be greater than 1. Values of slightly over 1 up to 10+ are common.

Example 4.6 demonstrates a simple calculation of the factor of safety given the failure and allowable stresses in a component.

Example 4.6

A 2 mm diameter steel wire is used to suspend a mass of 40 kg.

Using the yield strength of 200 MPa as the failure stress, what is the factor of safety of the wire?



Solution

First determine the normal force in the wire caused by the mass.

$$N = 40\ kg \times 9.81\frac{m}{s^2} = 392\frac{kg \cdot m}{s^2} = 392\ N$$

The stress in the wire can be calculated from

$$\sigma = \frac{N}{A} = \frac{392\ N}{\pi * (0.001\ m)^2} = 124.8\ MPa.$$

Since the failure stress in the wire is 200 MPa, the factor of safety can be determined from

$$FS = \frac{\sigma_{fail}}{\sigma_{allow}} = \frac{200\ MPa}{124.8\ MPa} = 1.60.$$

In practice working the other way is also common. Given a material with a known failure stress and a desired factor of safety, we can determine the allowable stress in the component and then design the component to meet this requirement. This is demonstrated in Example 4.7.

Example 4.7

A square timber column is used to support a deck. The timber has a failure stress in compression of 1,200 psi.

Using a factor of safety of 2.5, determine the required dimensions of the square cross-section if the column experiences a compressive load of 3,000 lb.

Solution

Use the failure stress and factor of safety to determine the allowable stress in the column.

$$\sigma_{allow} = \frac{\sigma_{fail}}{FS} = \frac{1,200\ psi}{2.5} = 480\ psi$$

Next, determine the required cross-sectional area of the column so that the stress does not exceed 480 psi.

$$\sigma = \frac{N}{A}$$
$$A = \frac{N}{\sigma}$$
$$= \frac{3,000\ lb}{480\ psi}$$
$$= 6.25\ in.^2$$

Finally, determine the dimension of each side of the square cross-section.

$$A = L^2$$
$$L = \sqrt{A} = \sqrt{6.25} = 2.5\ in.$$

The column cross-section must be at least 2.5 in. on each side so that the stress doesn't exceed the allowable stress of 480 psi.

We may safely make the dimension larger than 2.5 in., which reduces the stress further but also increases weight and cost. We may not make the dimension any smaller than 2.5 in., as this will result in a stress greater than 480 psi.

Step-by-Step: Factor of Safety

Determine the failure stress of the member. Depending on the application, this may be the yield stress or the ultimate stress.

Determine the allowable stress in the member from $\sigma_{\text{allow}} = \frac{\sigma_{\text{fail}}}{\text{FS}}$.

Use the allowable stress to calculate the allowable force in the member or the required cross-sectional area, using $\sigma_{\text{allow}} = \frac{\text{F}}{\text{A}}$.

Summary

Key Takeaways

The mechanical properties of materials need to be known for engineering design purposes. For many applications, the linear elastic properties are the first step in characterizing a material and in determining stresses resulting from imposed displacements or strains and deflections caused by imposed loading scenarios.

Tensile tests of material samples are a common means to measure material behavior, both in the linear elastic region and beyond.

Tensile tests can indicate the normal stress-strain behavior, and the Young's modulus (or modulus of elasticity) is defined as the slope of the linear portion of this curve.

The proportional limit, elastic limit, and yield point of some engineering materials are quite similar, and yield strength is used throughout the book to denote this region of material response.

Because safe designs often involve using materials below their yield strength, the linear elastic properties—elastic modulus, shear modulus, and Poisson's ratio—are important parameters for engineering design.

Poisson's ratio characterizes the coupling between normal stresses in one direction and strains in lateral directions through the generalized Hooke's law.

The shear modulus (or modulus of rigidity) relates shear stresses and shear strains in the linear elastic region.

The factor of safety is one simple means to quantify how conservative a design may be, an important parameter in engineering design.

Key Equations

Hooke's law:

$$\sigma = E\varepsilon_{long}$$

Poisson's ratio:

$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}}$$

Shear stress and strain:

$$\tau = G\gamma$$

Relationship between E, G, and ν:

$$G = \frac{E}{2(1+\nu)}$$

Thermal strain:

$$\varepsilon = \alpha \Delta T$$
$$\Delta L = \alpha \Delta T L$$

Generalized Hooke's law:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - v(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$
$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

Factor of safety:

$$FS = \frac{\sigma_{fail}}{\sigma_{allow}}$$

References

Figures

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- Figure 4.1: Tensile tests are a common means to measure mechanic properties of materials. A: Credit Instron ®. All rights reserved. Permission granted for this use case. B: David Dillard. 2024. CC BY. C: Mac McCord. 2024. CC BY.
- Figure 4.3: Engineering materials are often loaded beyond yield to permanently deform the material, as in (A) the stamping process for an automotive panel or in (B) the crushing of a weld-bonded hat section used to absorb energy in an automobile accident. A: © Porsche USA. Reproduced under fair use. Retrieved from https://www.autonews.com/suppliers/porsche-deal-lets-tool-supplier-shine. B: © Ford Motor Company. Released under CC BY-NC-SA 4.0.

- Figure 4.4: Results obtained from tensile tests. A: Kindred Grey. 2024. CC BY. B: Sigmund. 2011. CC BY-SA 3.0. https://commons.wikimedia.org/wiki/File:Al_tensile_test.jpg.



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Chapter 5: Axial Loading

Learning Objectives

- Explain stress concentrations and use geometry to calculate maximum stress
- Calculate axial deformation in a bar subjected to axial load
- Determine deformation in a series of parallel bars
- Define statically indeterminate problems and solve them using knowledge of deformation
- Calculate thermal deformation and solve statically indeterminate problems involving changes in temperature

This chapter takes a closer look at the effects of axial loading. Axial loads are those applied perpendicular to the cross-section of an object (Figure 5.1). Such loads affect the object in several ways.

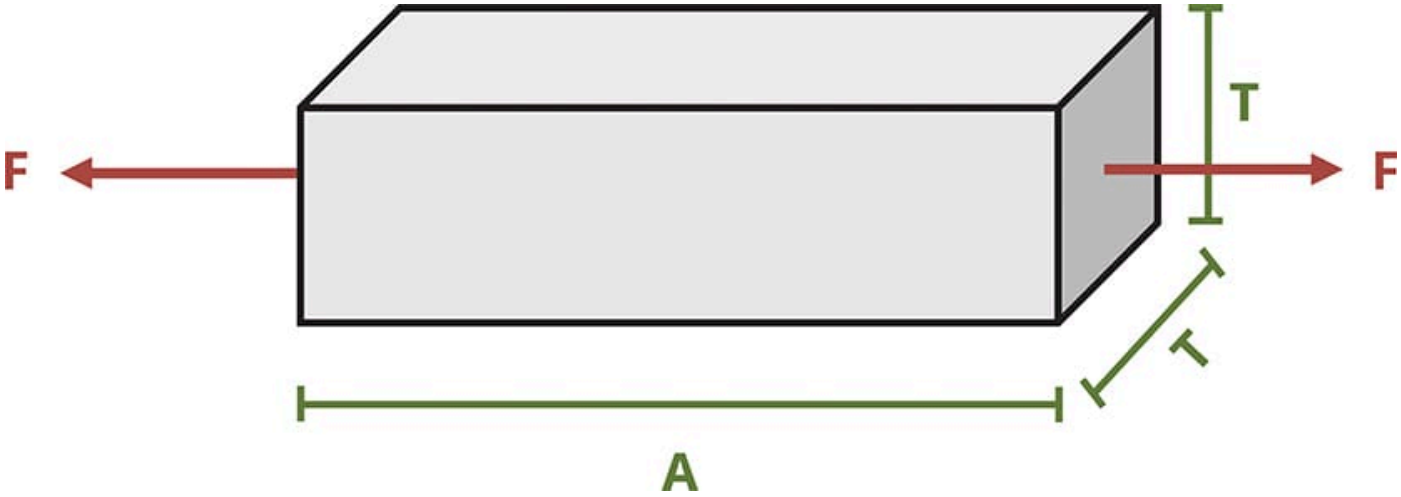


Figure 5.1: Axial loads act perpendicular to the cross-section. They cause normal stress on the cross-section and normal strains in both the axial (A) and transverse (T) directions.

Axial loads create axial stresses, as seen in Section 2.1 and revisited briefly in Section 5.1. We previously considered the average normal stress in a cross-section, but the geometry of the cross-section can cause large localized stresses much higher than the average. These stress concentrations are discussed in Section 5.2.

This text is concerned with both stress and deformation, so Section 5.3 explores the deformation caused by axial loads. This extends to a special case of axial deformation involving parallel bars in Section 5.4, before we apply our knowledge of axial deformation to solve statically indeterminate problems in Section 5.5. These are problems where the equilibrium equations are insufficient to determine the reaction and internal forces.

Finally we'll investigate the effects of temperature in Section 5.6. As the temperature of an object changes, the object expands or contracts. Thus deformation can occur even in the absence of any applied forces.

5.1 Axial Stress

Axial loads create axial stresses, also known as normal stresses. As discussed in Section 2.1, the average normal stress is calculated from

$$\boxed{\sigma = \frac{N}{A}},$$

where

- σ = *Average normal stress [Pa, psi]*
- N = *Internal normal force [N, lb]*
- A = *Cross-sectional area [m^2, $in.^2$]*

5.2 Stress Concentrations

5.2.1 Saint-Venant's Principle

The normal stress mentioned in the introductory section is averaged over the cross-section and assumes that stresses (and therefore strains) at the cross-section are uniform. In reality stresses and strains can vary across a cross-section, especially if the cross-section is close to an applied load, a support, or a change in geometry. At these points localized stress concentrations occur, leading to large, localized stresses and strains. The intensity of these concentrations depends on the type of loading, support, or geometry (Figure 5.2).

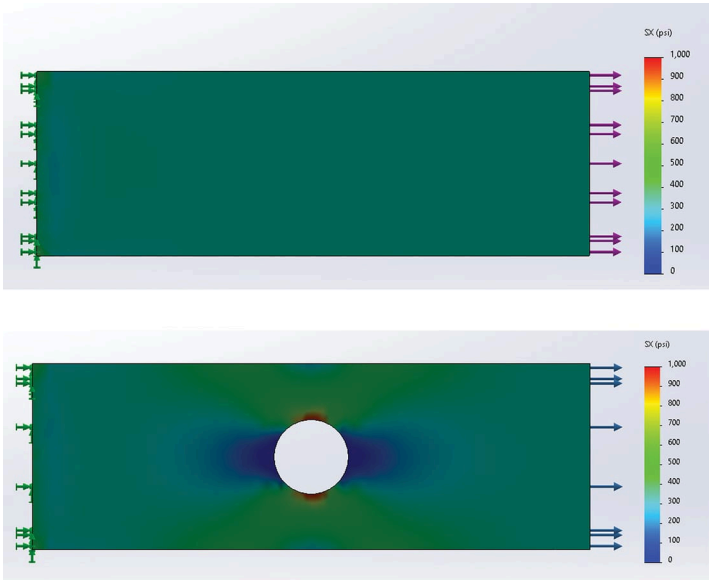


Figure 5.2: Each bar has a fixed support on the left, a cross-section of 30 $in.^2$, and is subjected to a force of 10,000 lb. The top bar is uniform and experiences a uniform average normal stress of $\sigma = \frac{10{,}000}{30} = 333$ psi at all points, except those close to the support and load. The bottom bar experiences the same stress at most points, but significantly higher stress concentrations close to the hole.

However, these effects disappear a certain distance away from the force, support, or local geometry, so it is acceptable to assume uniform stress and strain (as we have done so far) provided that we also assume our cross-section is a sufficient distance away from these points. This is known as Saint-Venant's principle, which can be formally stated thus: “The stresses and strains created at a point in a body by two statically equivalent loads are equivalent at points sufficiently far removed from the applied load” (Figure 5.3).

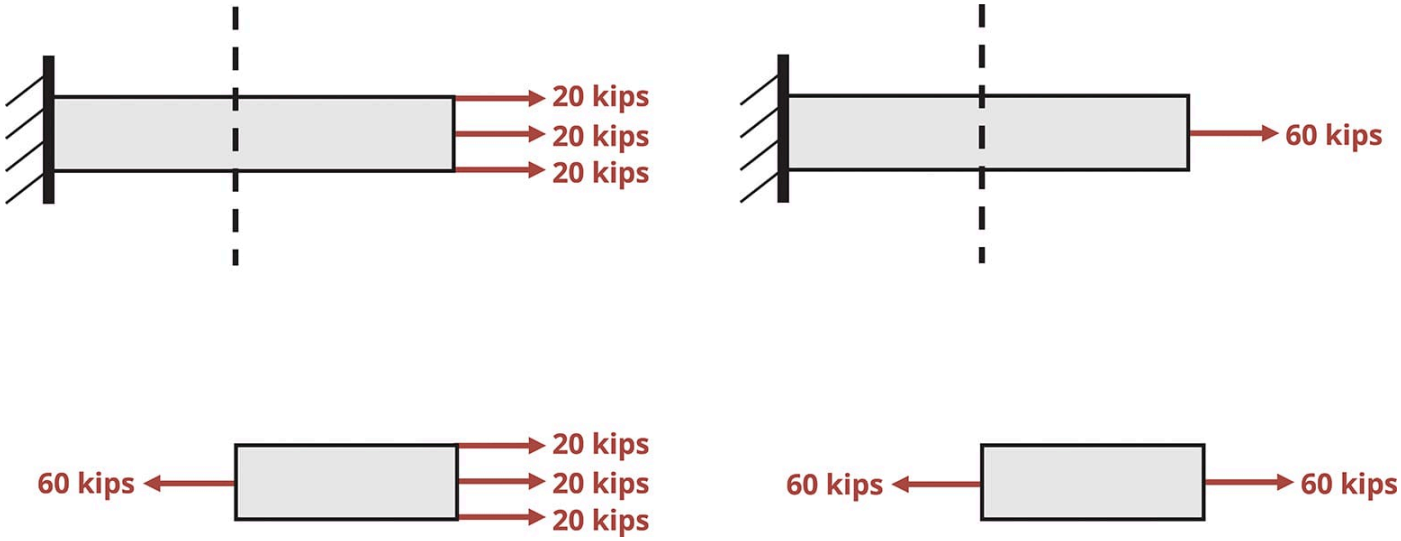


Figure 5.3: Two statically equivalent loads applied to a bar. Close to the point of application there will be localized stresses and deformations, but at a cross-section sufficiently far away from the applied loads the internal effects are equivalent.

5.2.2 Stress Concentrations Around Changes in Geometry

More advanced courses use the theory of elasticity to study the areas of variable stress and strain, but for now we'll continue to apply Saint-Venant's principle and assume that our cross-sections are sufficiently far away from the applied loads to eliminate the need to consider these stress concentrations. Here, however, the focus is on the question of stress concentrations around changes in geometry. We'll study two specific examples: holes and fillets. Figure 5.2 shows an example of the stress concentrations around a hole. Figure 5.4 shows an example of the stress concentrations around a fillet, which is a rounded corner used to help transition between two geometries.

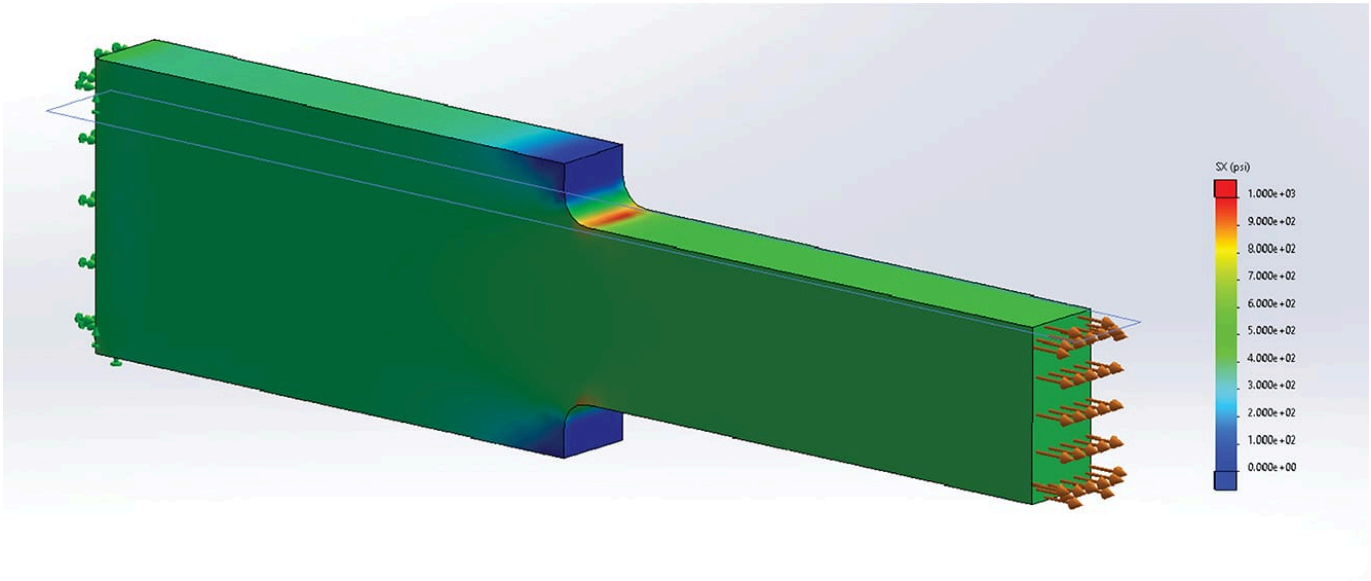


Figure 5.4: The fillet helps prevent a sharp corner but still causes stress concentrations. The bar has a thickness of 3 in. The height of the smaller section is 6 in. and the larger section is 10 in. The applied load is 10,000 kips, so the average normal stress in the smaller section is 556 psi and in the larger section is 333 psi. The maximum stress at the fillet, however, is 973 psi.

Fully modeling these stress concentrations is very complicated, but for our purposes it is sufficient to find only the maximum stress that occurs around these stress concentrations. These can be significantly larger than the average stress we have been

calculating and can cause localized failure even if the average stress is below the yield stress for the material. The maximum stress can be found simply by multiplying the average stress by a stress concentration factor, K.

$$\boxed{\sigma_{max} = K\sigma_{avg}},$$

(5.1)

where

- σ_{max} = *Maximum normal stress [Pa, psi]*
- *K = Stress concentration factor*
- σ_{avg} = *Average normal stress [Pa, psi]*

This stress concentration factor depends on the geometry at hand and can be found from curves in design handbooks. Figure 5.5 shows curves for holes and fillets. The average normal stress is calculated at whichever plane gives the largest value. For the case with a hole, this is the plane passing through the center of the hole. For the case with a fillet, this is a plane passing through the smaller segment of the member. See Example 5.1 and Example 5.2 to see how these curves can be used.

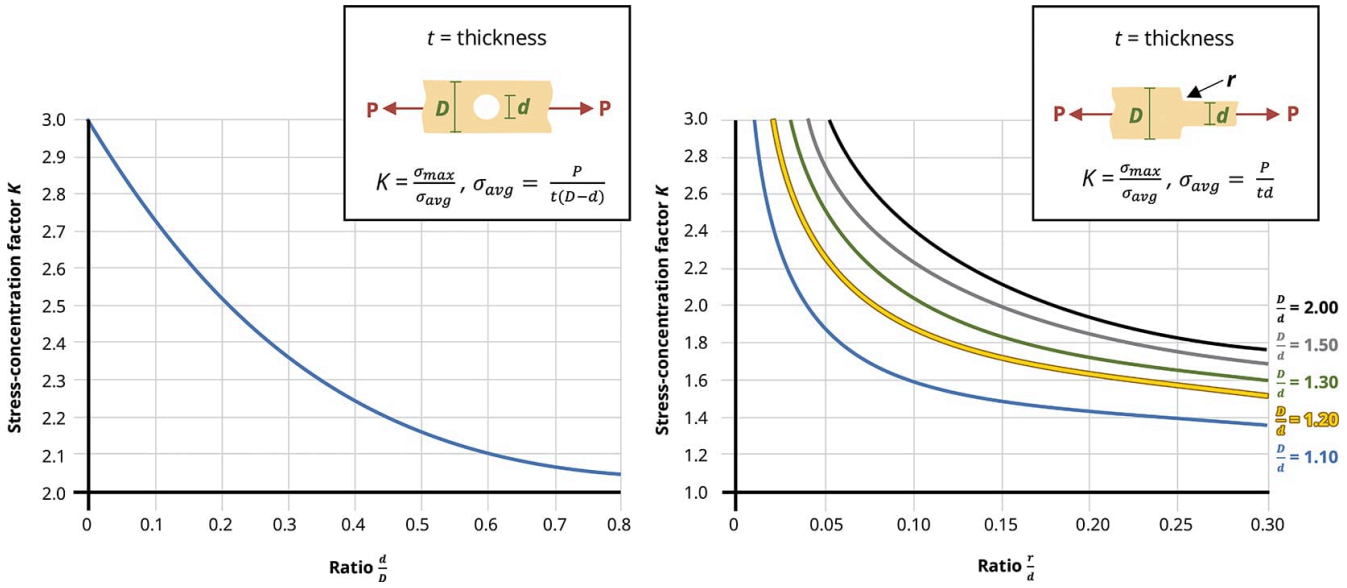
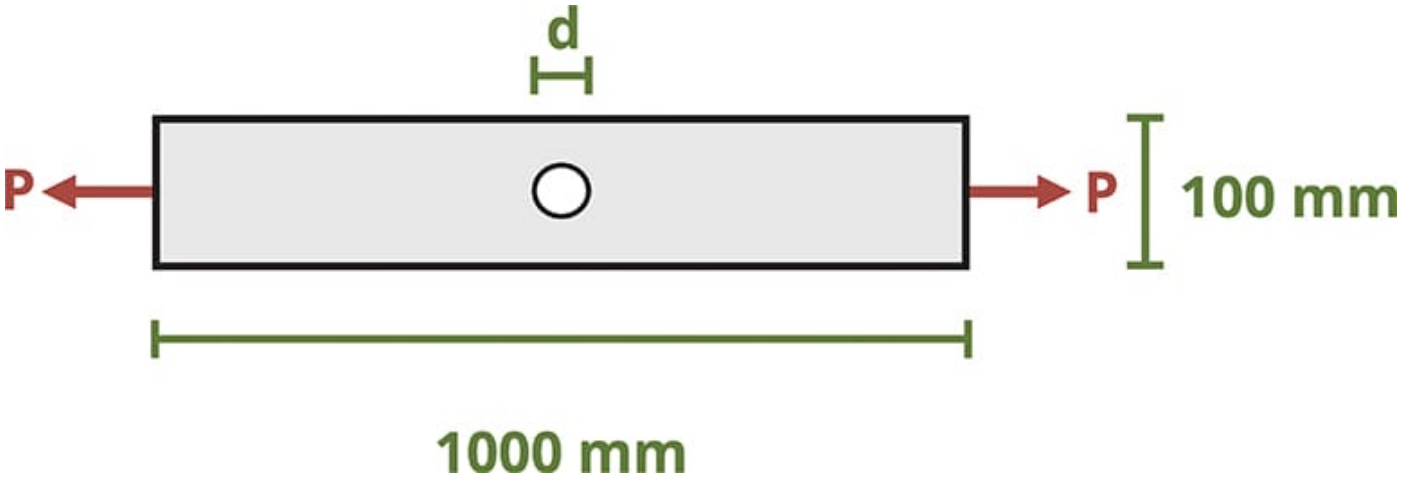


Figure 5.5: Graphs showing how the stress concentration factor, K, changes according to the geometry of the object.

Example 5.1

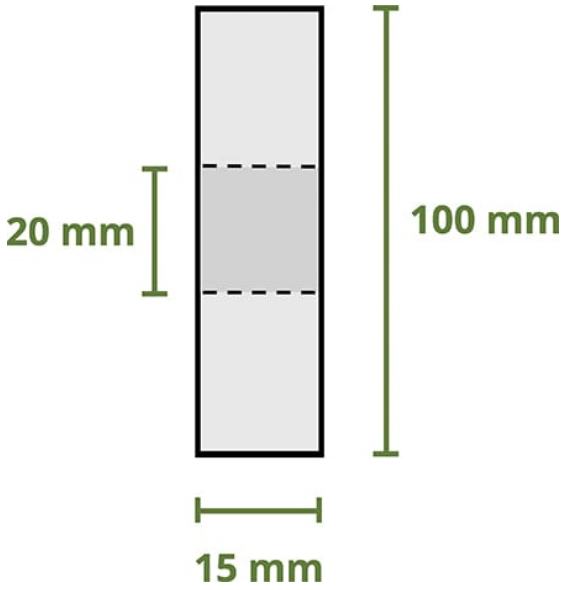
A hole is drilled through a steel plate to allow two components to be bolted together. The plate has the dimensions shown and is 15 mm thick. The plate is subjected to an axial load of P = 50 kN.

Determine the maximum stress in the plate if the hole diameter is d = 20 mm.



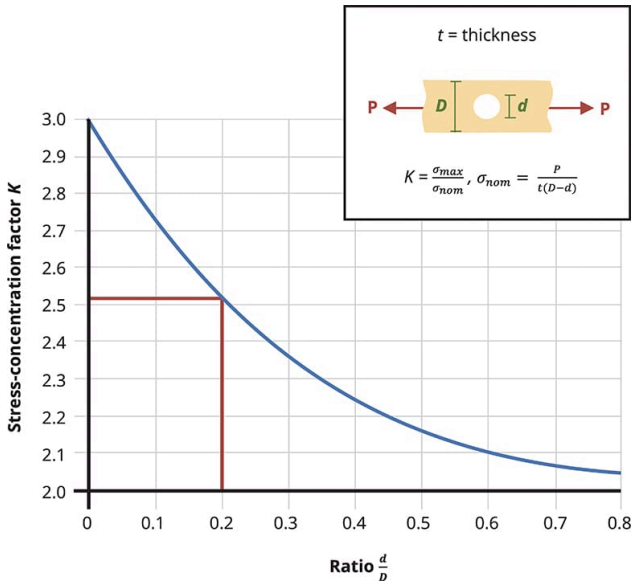
Solution

Start by calculating the average normal stress in the plate at the location of the hole. The cross-section is a rectangle with a base of 15 mm (0.015 m) and a height of 100 mm (0.1 m), with a 20 mm (0.02 m) diameter hole cut out.



$$\sigma_{avg} = \frac{N}{A} = \frac{50,000\ N}{0.015\ m * (0.1 - 0.02)\ m} = 41.7\ MPa$$

Determine the ratio $\frac{d}{D}$, where d = hole diameter and D = height of the cross-section. Use the appropriate stress concentration curve to read off the stress concentration factor K for this geometry.



$$\frac{d}{D} = \frac{20\,mm}{100\,mm} = 0.2$$
$$K = 2.525$$

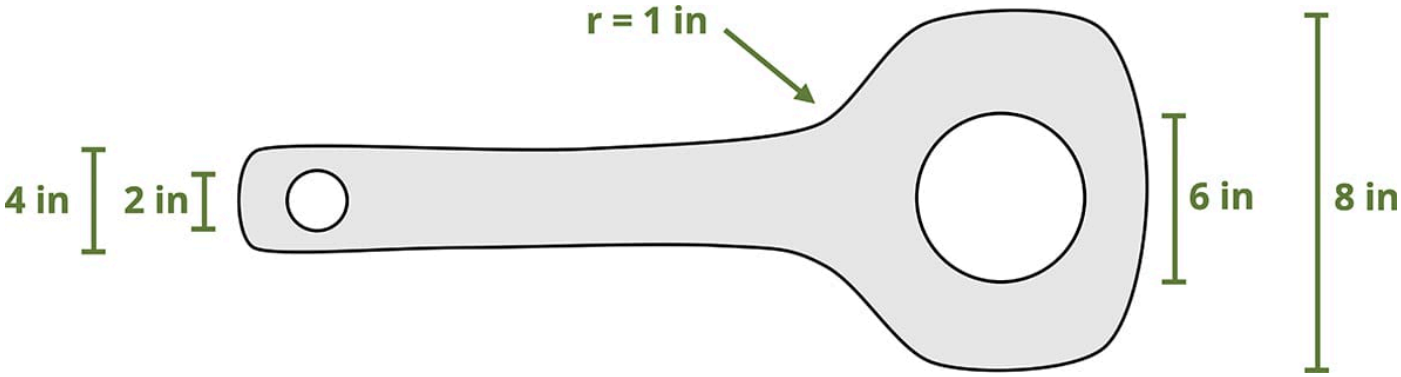
Calculate the maximum stress.

$$\sigma_{max} = K\sigma_{avg} = 2.525 * 41.7\,MPa = 105\,MPa$$

Example 5.2

A 1 in. thick connecting rod in an engine assembly has the dimensions shown. The maximum allowable stress in the rod is 40 ksi.

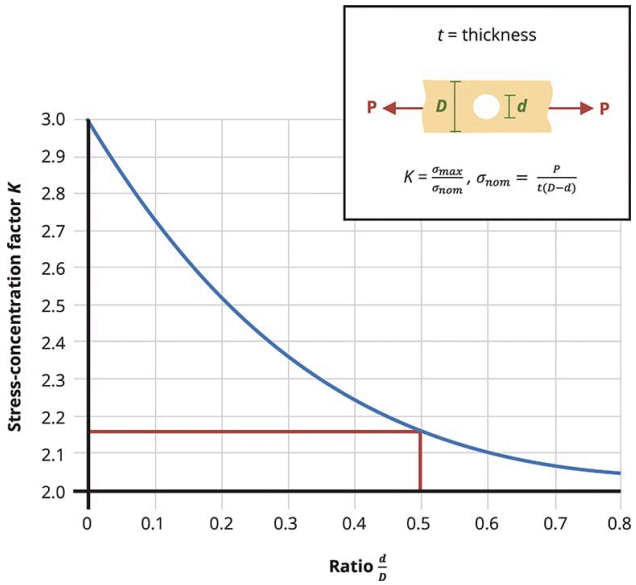
Determine the maximum axial load that may be applied to the rod.



Solution

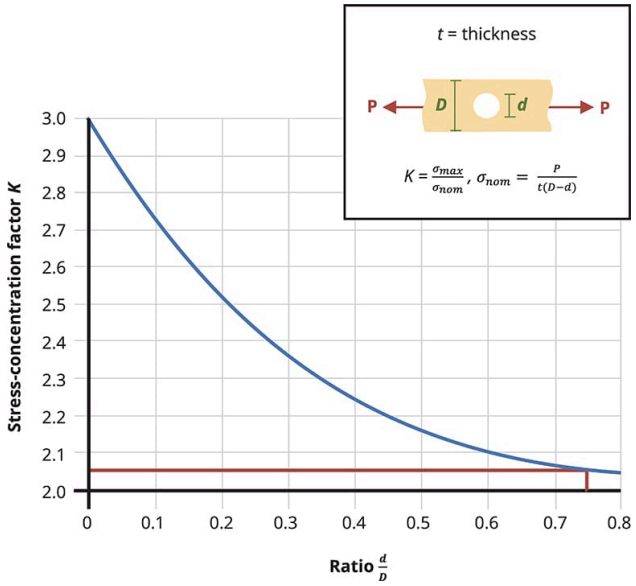
The connecting rod has two holes and a fillet. We can start by determining which of these geometries experiences the largest maximum stress, considering applied load P. For each geometry, calculate the average normal stress and then use the appropriate stress concentration curve to determine the maximum normal stress.

For the smaller circle:



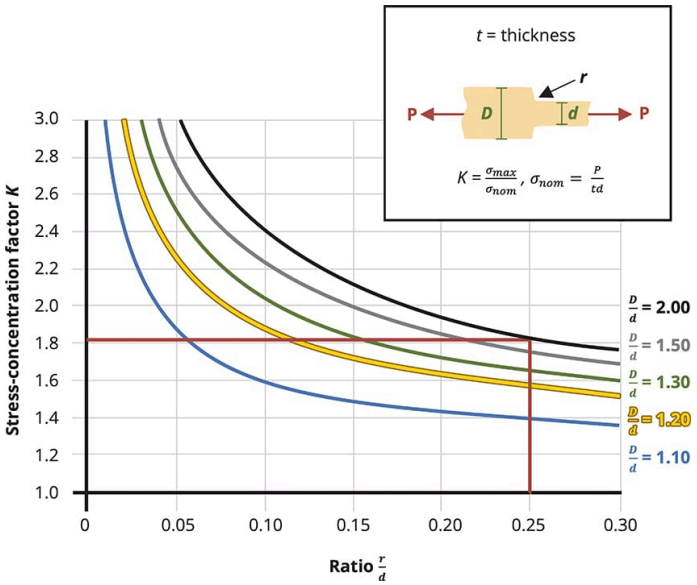
$$\sigma_{avg} = \frac{N}{A} = \frac{P}{1*(4-2)} = \frac{P}{2} = 0.5P$$
$$\frac{d}{D} = \frac{2\,in.}{4\,in.} = 0.5$$
$$K = 2.15$$
$$\sigma_{max} = K\sigma_{avg} = 2.15 * 0.5P = 1.075P$$

For the larger circle:



$$\sigma_{avg} = \frac{N}{A} = \frac{P}{1*(8-6)} = \frac{P}{2} = 0.5P$$
$$\frac{d}{D} = \frac{6\,in.}{8\,in.} = 0.75$$
$$K = 2.05$$
$$\sigma_{max} = K\sigma_{avg} = 2.05 * 0.5P = 1.025P$$

For the fillet:



$$\sigma_{avg} = \frac{N}{A} = \frac{P}{1*4} = \frac{P}{4} = 0.25P$$
$$\frac{D}{d} = \frac{8\,in.}{4\,in.} = 2$$
$$\frac{r}{d} = \frac{1\,in.}{4\,in.} = 0.25$$
$$K = 1.82$$
$$\sigma_{max} = K\sigma_{avg} = 1.82 * 0.25P = 0.455P$$

The largest stress is at the small hole, where s_{max} = 1.075P.

Since the maximum allowable stress is 40 ksi, $40\ \text{ksi} = 1.075\text{P} \rightarrow \text{P} = \frac{40\ \text{ksi}}{1.075} = 37.2\ \text{kips}.$

Apparent from these examples is that sharp corners and tight radii lead to high stress concentrations. When designing parts, it is best to use smooth, flowing transitions where possible and avoid sharp corners to minimize stress concentrations.

Step-by-Step: Stress Concentrations

1. Calculate the average normal stress using $\sigma_{\text{avg}} = \frac{\text{N}}{\text{A}}$. Do this at the hole or at the thinner section of the fillet.
2. Use the appropriate stress concentration curve to determine the stress concentration factor, K, for the given geometry.
3. Calculate the maximum stress from $\sigma_{\text{max}} = \text{K}\sigma_{\text{avg}}$.

5.3 Axial Deformation

Consider a simple bar of uniform cross-section subjected to an axial force (Figure 5.6).

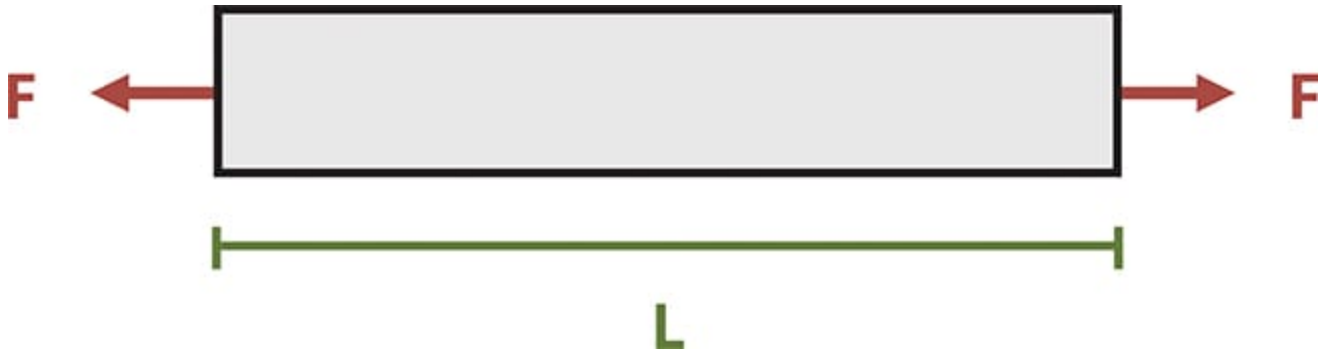


Figure 5.6: A bar of uniform cross-section, A, and length, L, subjected to axial load, F.

Assuming elastic behavior, we have equations for stress and strain, as well as Hooke's law, which relates the two equations.

$$\sigma = \frac{F}{A}$$
$$\varepsilon_{long} = \frac{\Delta L}{L}$$
$$\sigma = E\varepsilon_{long}$$

Replace the stress and strain terms in Hooke's law.

$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Rearrange the equation.

$$\Delta L = \frac{FL}{AE}$$

This last equation can be used to directly find the change in length of an object subjected to an axial load.

In practice multiple axial loads may be applied to the bar. The cross-sectional area may change at different points along the bar, or perhaps different materials with different elastic moduli are connected in series to form the bar. In any of these cases we can split the bar into sections where each section has a constant F, A, and E. We can calculate the change in length of each section separately and sum them to find the total. See Example 5.3 for deformation of a simple bar subjected to a single load and Example 5.4 for a problem involving multiple segments of a bar.

$$\boxed{\Delta L = \sum \frac{FL}{AE}}, \tag{5.2}$$

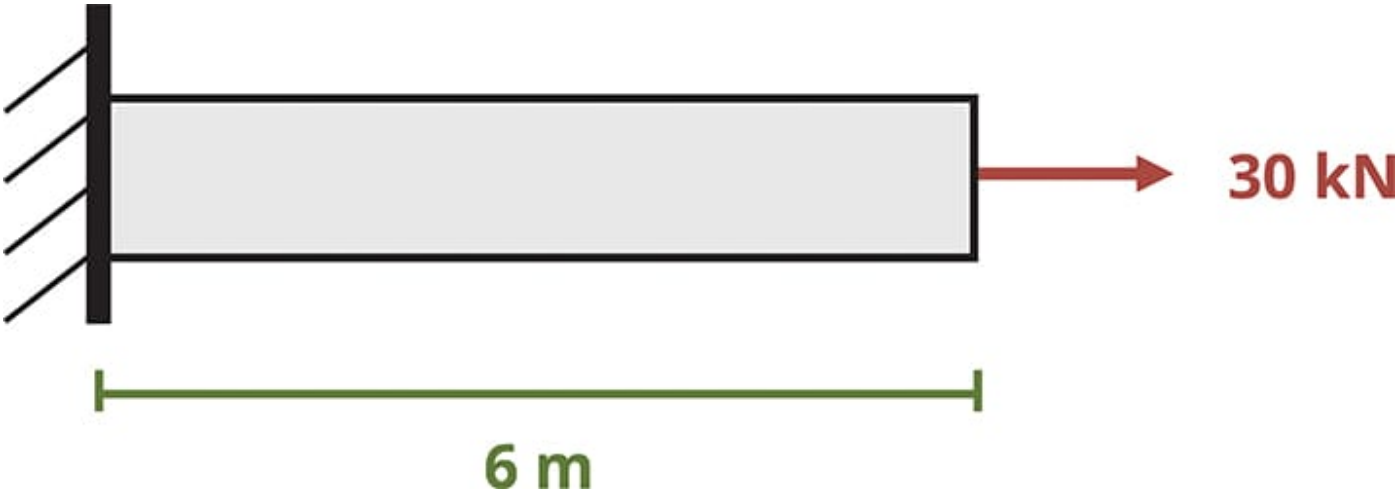
where

- ΔL = *Change in length [m, in.]*
- F = *Internal axial load [N, lb]*
- L = *Original length [m, in.]*
- A = *Cross-sectional area [m^2, $in.^2$]*
- E = *Elastic modulus [Pa, psi]*

Example 5.3

A wooden beam (E = 12 GPa) of length L = 6 m has a rectangular cross-section of base b = 50 mm and height h = 80 mm. The beam is subjected to a tensile load P = 30 kN as shown.

Determine the change in length of the beam.

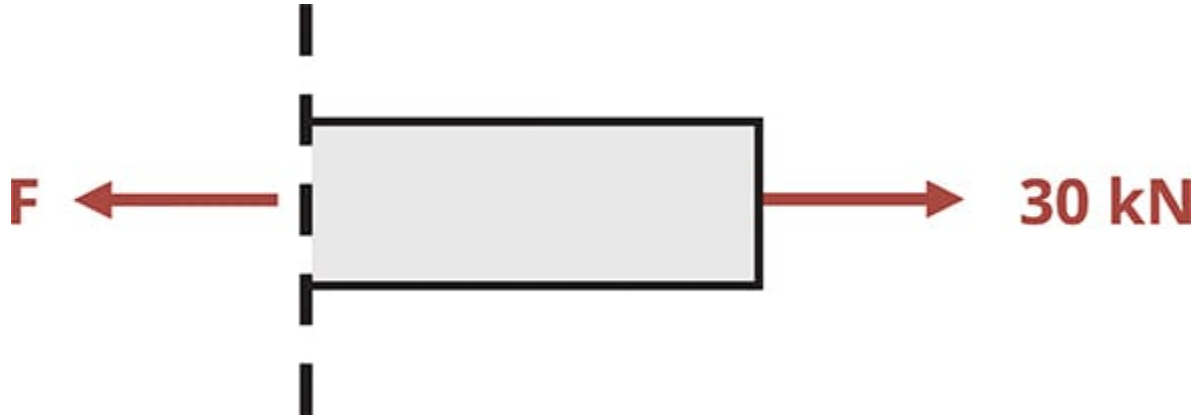


Solution

We can find the change in length of the beam using $\Delta \mathrm{L} = \Sigma \frac{\mathrm{FL}}{\mathrm{AE}}$.

Here only one load is applied and the beam is made of a single material with a uniform cross-section, so F, A, and E are constant.

We next find the internal load in the beam by cutting a cross-section and drawing a free body diagram (FBD).



$$\Sigma F_x = 30 - F = 0$$
$$F = 30\ kN$$

The beam has a rectangular cross-section so the area is given by

$$A = bh$$
$$= 0.05\ m * 0.08\ m$$
$$= 0.004\ m^2.$$

We are given L = 6 m and E = 12 GPa, so the change in length of the beam is given by

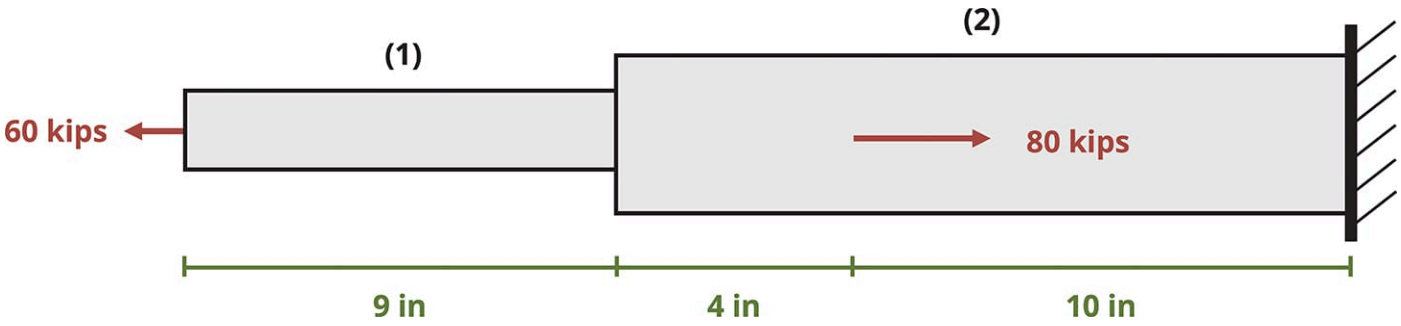
$$\Delta \mathrm{L} = \frac{\mathrm{FL}}{\mathrm{AE}}$$
$$= \frac{30{,}000\ \mathrm{N} * 6\ \mathrm{m}}{0.004\ \mathrm{m}^2 * 12 * 10^9\ \frac{\mathrm{N}}{\mathrm{m}^2}}$$
$$= 0.00375\ \mathrm{m}$$
$$= 3.75\ \mathrm{mm}.$$

The beam lengthens by 3.75 mm.

Example 5.4

A component is made by welding together two circular rods. Rod 1 is made of steel (E = 29 x 10^6 psi) and is hollow, with an outer diameter of 4 in. and an inner diameter of 2 in. Rod 2 is made of copper (E = 17 x 10^6 psi) and is solid, with a diameter of 5 in.

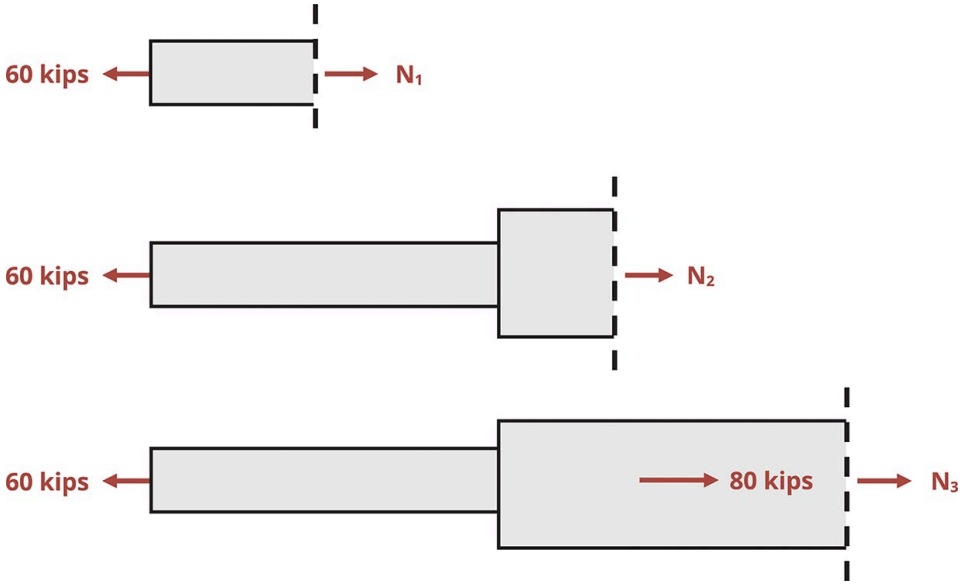
If the component is subjected to the axial loads shown, determine the total deformation of the component.



Solution

We can find the change in length of the component using $\Delta \mathrm{L} = \sum \frac{\mathrm{FL}}{\mathrm{AE}}$.

First, break the component into three segments and determine the internal load in each segment. The first segment covers the 9 in. steel rod. After the first 9 in. of the component, both the cross-sectional area and elastic modulus change, so a second segment is needed. This segment covers the next 4 in. At this point the loading changes, so a third segment is required to cover the final 10 in. of the component.



$$N_1 = N_2 = 60\ kips$$
$$N_3 = -20\ kips$$

Now calculate the deformation of each segment. Be sure to use the appropriate dimensions and material properties for each segment.

Cross-sectional areas:

$$A_1 = \pi * (2^2 - 1^2)\ in.^2 = 9.42\ in.^2$$
$$A_2 = \pi * (2.5^2) in.^2 = 19.6\ in.^2$$

Deformation of segment 1:

$$\Delta L_1 = \frac{60,000\ lb * 9\ in.}{9.42\ in.^2 * 29 \times 10^6 \frac{lb}{in^2}} = 0.00198\ in.$$

Deformation of segment 2:

$$\Delta L_2 = \frac{60,000\ lb * 4\ in.}{19.6\ in.^2 * 17 \times 10^6 \frac{lb}{in.^2}} = 0.000719\ in.$$

Deformation of segment 3:

$$\Delta L_3 = \frac{-20,000\ lb * 10\ in.}{19.6\ in.^2 * 17 \times 10^6 \frac{lb}{in.^2}} = -0.000599\ in.$$

Note that segment 3 is in compression and therefore becomes shorter, while segments 1 and 2 are in tension and therefore become longer. The total deformation in the component is simply the sum of these three deformations.

$$\Delta L = 0.00198\ in. + \ 0.000719\ in. - \ 0.000599\ in. = 0.002\ in.$$

Step-by-Step: Axial Deformation

1. Split the bar into sections of constant F, A, and E. Any time any term changes, begin a new section.
2. Determine the length of each section.
3. Calculate the change in length of each section using $\Delta \mathrm{L} = \frac{\mathrm{FL}}{\mathrm{AE}}$.
4. Determine the total change in length by summing the change in length of each section. Remember that some sections may elongate while others contract.

5.4 Deformation in Series of Bars

Sometimes a structure has more than one axially loaded member. In such cases multiple bars will experience a change in length and the amount of deformation won't necessarily be the same in each member. The deformable members are generally connected by a rigid (nondeformable) beam. These problems generally involve a little geometry alongside the deformation equation (Figure 5.7).

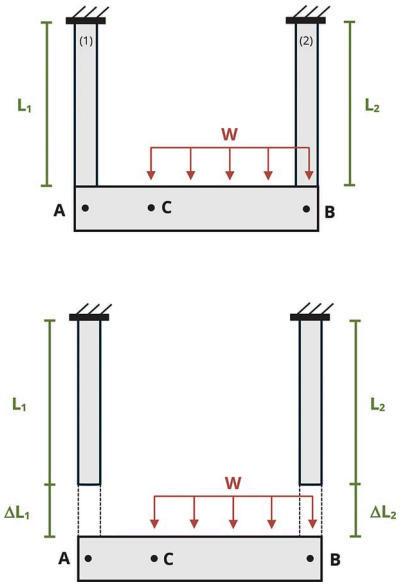


Figure 5.7: Rigid beam AB is attached to deformable poles 1 and 2. The poles may deform different amounts, so point A is displaced by amount DL_1, point B by amount DL_2, and point C by an amount in between these two values.

By identifying the force in each member through equilibrium, we may calculate the deformation of each member separately by using

$$\Delta L = \frac{FL}{AE}.$$

Once each member's change in length is known, use simple geometry to find the displacement at different points on the rigid beam (Figure 5.8). This process is demonstrated in Example 5.5.

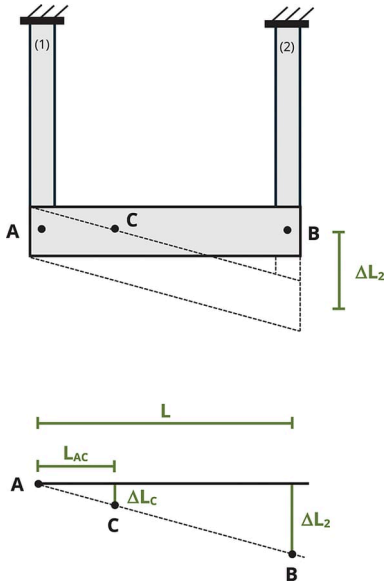


Figure 5.8: Beam deflection in the case where only bar 2 deforms. Point B will deflect downward by amount ΔL_2. The deflection at point C can be determined through geometry.

The deflection of point C will be somewhere between the deflection at A and the deflection at B. Assuming that there is no deflection at A and that the deflections (and thus the angle at A) are small, use similar triangles to find

$$\frac{\Delta L_C}{L_{AC}} = \frac{\Delta L_2}{L} \rightarrow \Delta L_C = \Delta L_2 \frac{L_{AC}}{L}.$$

If a deflection exists at point A (Figure 5.9), this simply becomes

$$\Delta L_C = \Delta L_1 + (\Delta L_2 - \Delta L_1)\frac{L_{AC}}{L}.$$

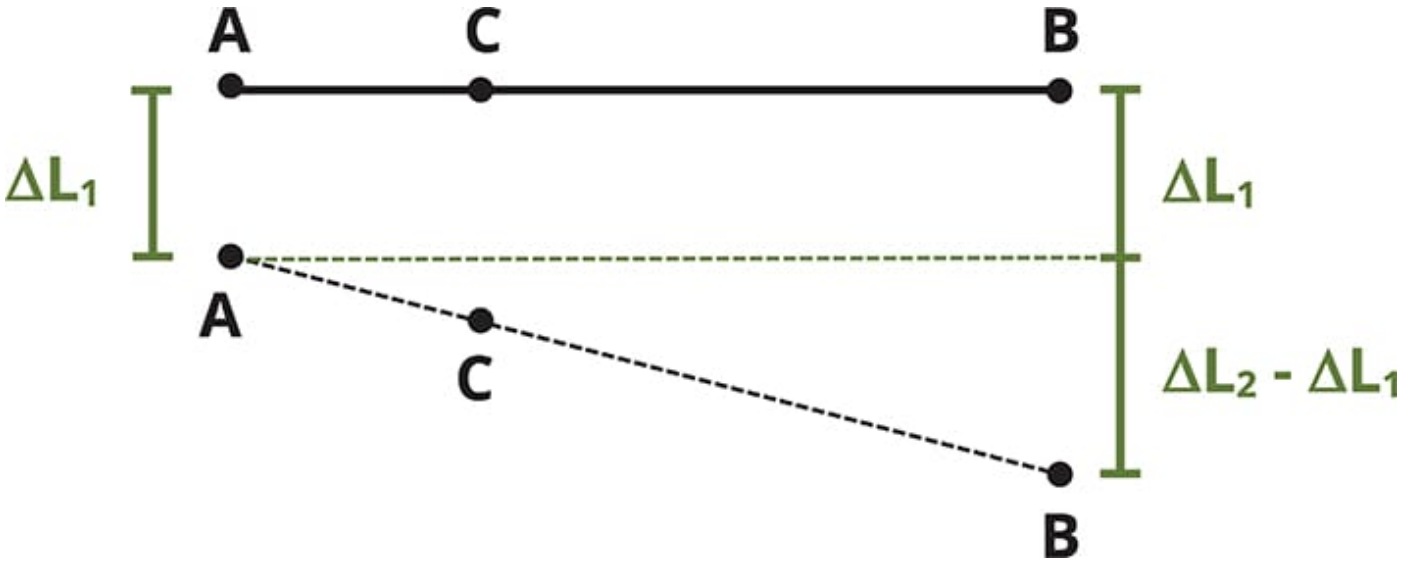
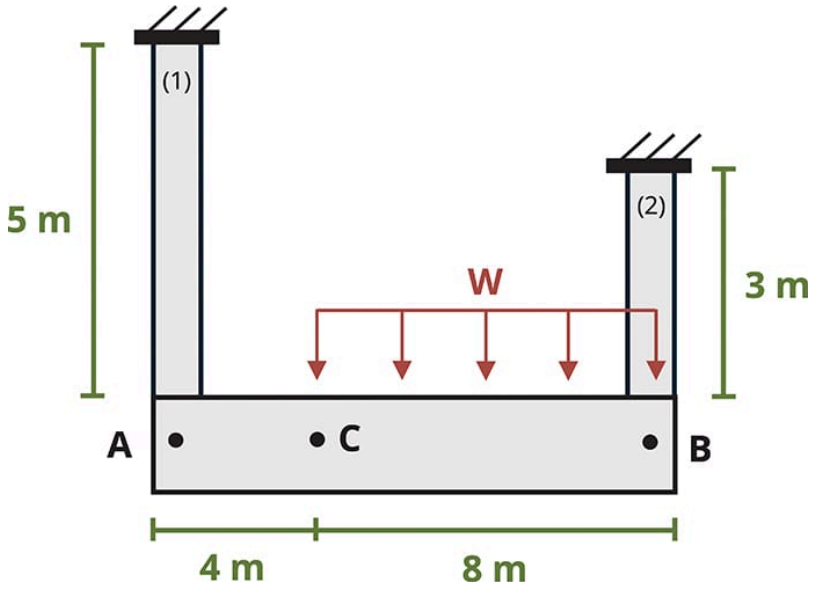


Figure 5.9: Similar triangles for calculating deflection at a point when both bars experience a change in length.

Example 5.5

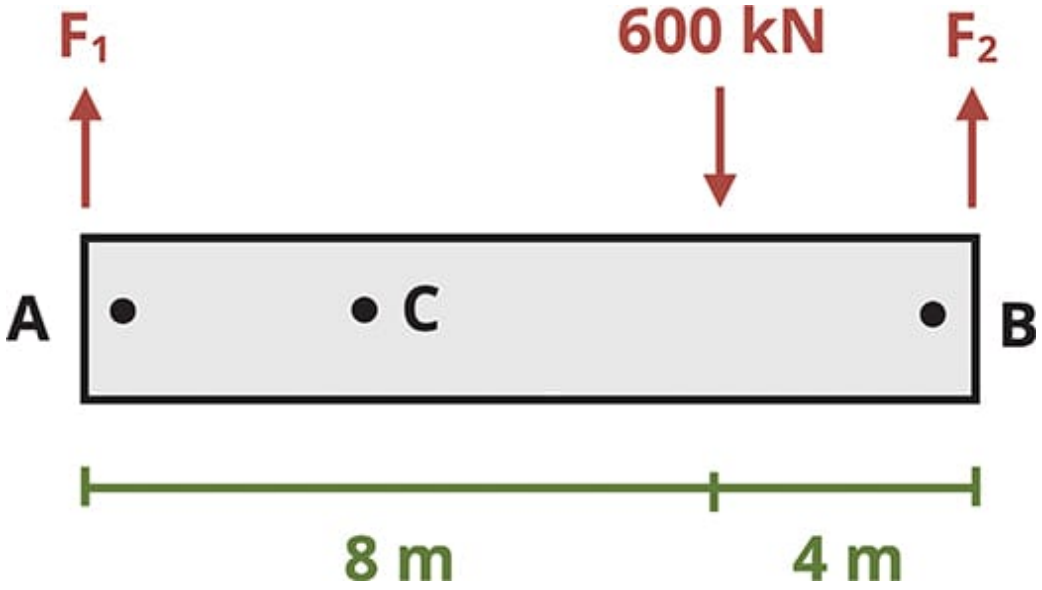
A rigid beam is supported by two nonrigid poles and subjected to a distributed load w = 75 kN/m. Pole 1 is made of steel (E = 200 GPa) and has a diameter of 30 mm. Pole 2 is made of cast iron (E = 70 GPa) and has a diameter of 50 mm.

Determine the deflection at point C of the rigid beam.



Solution

Although the beam is rigid, poles 1 and 2 will both elongate. We can find the force in each pole by drawing an FBD and applying equilibrium equations.

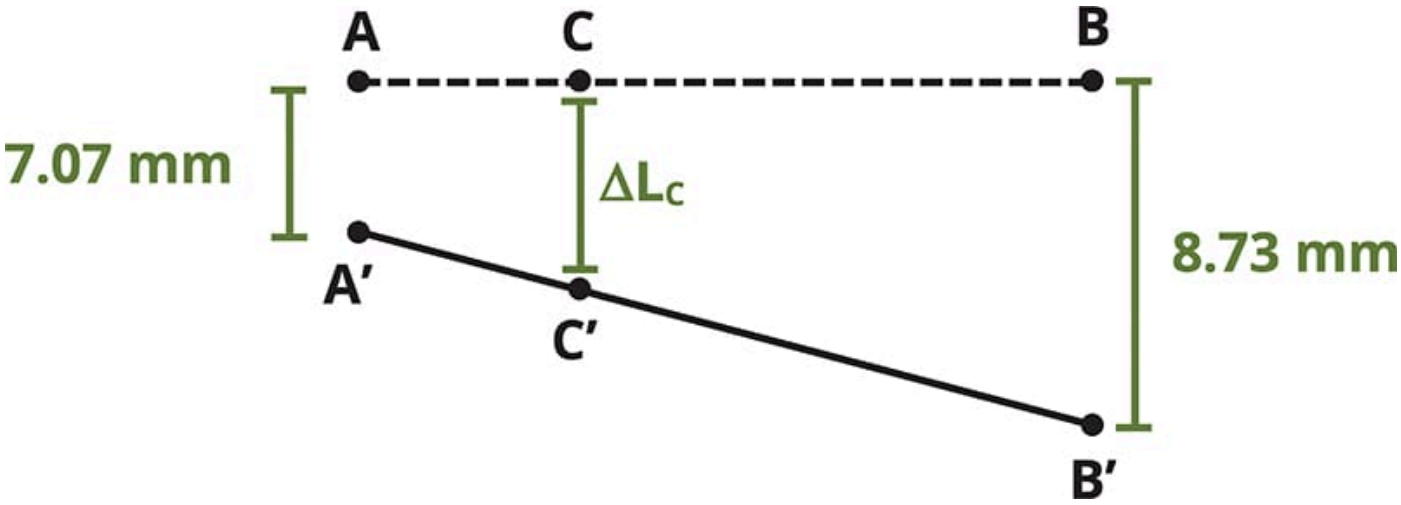


$$\sum M_A = -(600\ kN * 8\ m) + (F_2 * 12\ m) = 0\ \rightarrow\ F_2 = 400\ kN$$
$$\sum F_y = F_1 - 600\ kN + 400\ kN = 0\ \rightarrow\ F_1 = 200\ kN$$

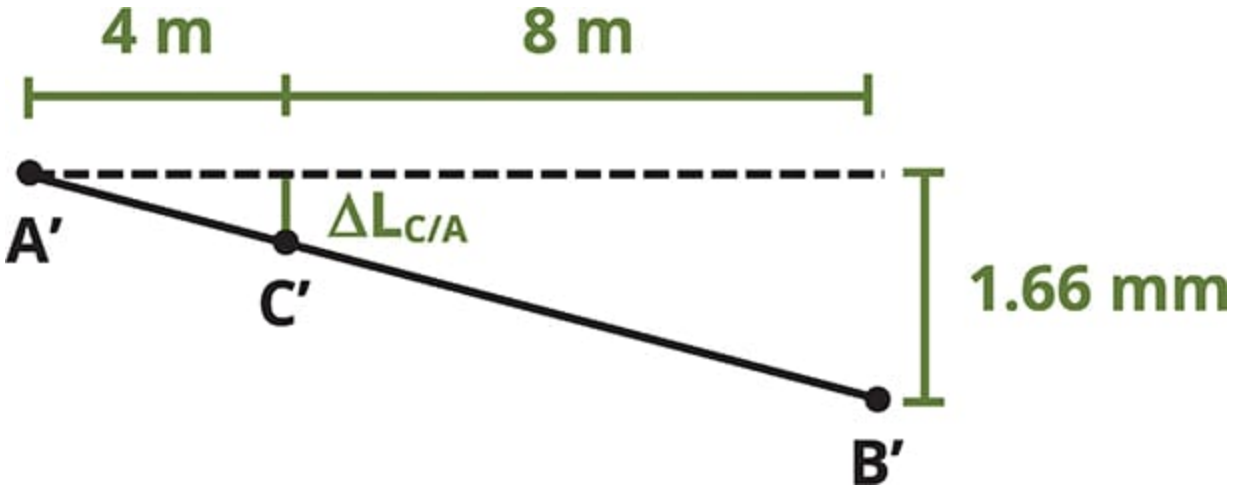
We can then calculate each pole's change in length.

$$\Delta L_1 = \frac{F_1 L_1}{A_1 E_1} = \frac{200{,}000\ N * 5\ m}{\pi * (0.015\ in.)^2 * 200 * 10^9 \frac{N}{m^2}} = 0.00707\ m = 7.07\ mm$$
$$\Delta L_2 = \frac{F_2 L_2}{A_2 E_2} = \frac{400{,}000\ N * 3\ m}{\pi * (0.025\ in.)^2 * 70 * 10^9 \frac{N}{m^2}} = 0.00873\ m = 8.73\ mm$$

To determine the deflection at point C, first determine that point B has deflected: (8.73 mm –7.07 mm) = 1.66 mm more than point A.



We can find how much more point C has deflected than point A by using similar triangles.



$$\frac{1.66\,mm}{12\,m} = \frac{\Delta L_{C/A}}{4\,m} \rightarrow \Delta L_{C/A} = 0.553\,mm$$

So point C deflects 0.553 mm more than point A, which is a total deflection at point C of

$$\Delta L_C = 7.07\,mm + 0.553\,mm = 7.62\,m.$$

Step-by-Step: Deformation in Series of Bars

1. Use equilibrium to determine the internal force in each bar.
2. Calculate the change in length of each bar using $\Delta \mathrm{L} = \frac{\mathrm{FL}}{\mathrm{AE}}$.
3. Use similar triangles to find the deflection at any point between the parallel bars.

5.5 Statically Indeterminate Problems

A statically indeterminate problem is one that has more unknowns than we have equilibrium equations to solve for them. This issue prevents us from finding the internal loads, and so we cannot calculate stress or deformation.

There are two types of statically indeterminate problems. The first type has additional supports beyond those needed to maintain equilibrium. These are known as redundant supports and are quite common in practice (Figure 5.10).



Figure 5.10: An example of redundancy on a suspension bridge. There are multiple supports and multiple cables at each support, such that if one cable fails the entire structure does not collapse.

In such problems, determining all the reaction forces using equilibrium alone isn't possible. However, we can use our knowledge of deformation to help. If a member is held between two supports, then its total deformation must be zero. Since the change in length depends on the internal force in the member, this constraint introduces an additional equation to use alongside the equilibrium equations. Two approaches are possible here.

Approach 1: To determine the reaction force at the redundant support, begin by removing the redundant support from the problem and determining the deformation that would occur without the support. Then replace the reaction force at the support, which will cause the member to deform in the opposite direction (Figure 5.11).

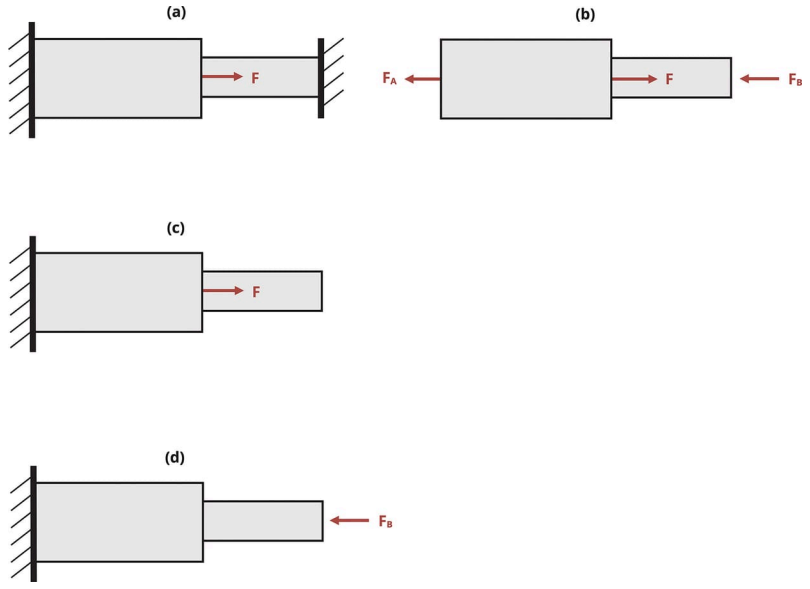


Figure 5.11: (a) A bar held between two rigid supports. (b) A free body diagram of the beam reveals that there are two unknowns but only one equilibrium equation. (c) Removing one of the supports allows the bar to elongate. (d) Replacing the support force causes the bar to contract.

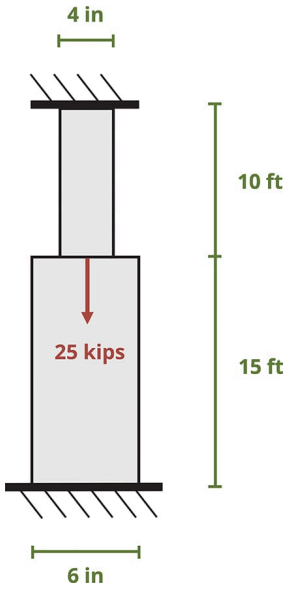
The sum of these two deformations must equal the member's actual deformation. If the member is held between two rigid

supports, the total deformation will be zero. If there is a small gap or the supports allow a certain amount of movement, the total deformation will be equal to the size of this gap. Example 5.6 shows this process applied to a bar made of two materials.

Example 5.6

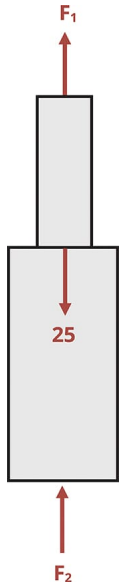
A 15 ft tall concrete (E = 4,000 ksi) column has a square cross-section of 6 in. × 6 in. A 10 ft tall copper (E = 17,000 ksi) cylinder with an outer diameter of 4 in. and inner diameter of 3 in. is attached to the top of the concrete. The structure is fixed between two supports. A force of 25 kips is applied as shown.

Determine the average normal stress in each material.



Solution

To determine the internal force in each material, first find the reaction forces at the supports. Usually we draw an FBD and apply equilibrium equations to find these forces. However, this won't work here because the problem is statically indeterminate.

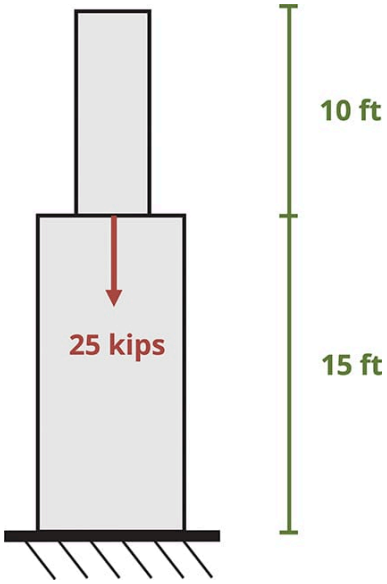


$$\sum F_y = F_1 + F_2 - 25\ kips = 0 \ \rightarrow \ F_1 + F_2 = 25\ kips$$

To solve this statically indeterminate problem, remove one of the supports and allow the structure to deform. Either support may

be removed.

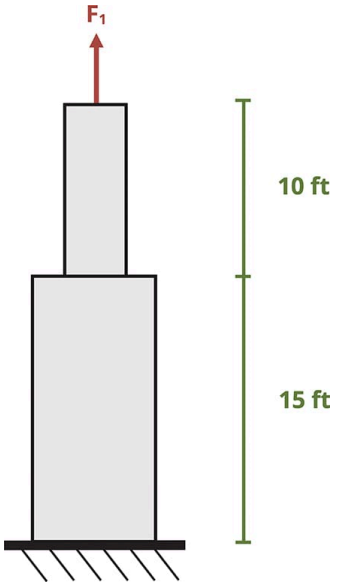
Remove the top support. In this scenario there is no load in the copper cylinder and there is a compressive load of 25 kips in the concrete column.



The total deformation of the structure in this scenario is

$$\Delta L = \sum \frac{FL}{AE} = 0 - \frac{(25\ kips) * (15 * 12)\ in.}{(6 * 6) in.^2 * (4,000)\ \frac{kips}{in.^2}} = -0.03125\ in.$$

Then replace force F_1 and calculate the deformation caused by this force. Because force is applied at the top of the structure, both the concrete and the copper elongate.



$$\Delta L = \sum \frac{FL}{AE}$$

$$\triangle L = \frac{F_1 \times (10 \times 12) in.}{\pi (2^2 - 1.5^2) in.^2 \times (17,000) \frac{kips}{in.^2}} + \frac{F_1 \times (15 \times 12) in.}{(6 \times 6) in.^2 \times (4,000) \frac{kips}{in.^2}}$$

$$\triangle L = 0.001284 F_1 + 0.001254 F_1$$

$$\triangle L = 0.002534 F_1$$

Since the structure is fixed at both ends, the actual total deformation must be zero.

$$-0.03125\ in. + \ 0.002534F_1 = 0 \rightarrow F_1 = 12.3\ kips$$

Returning to the equilibrium equation, also find force F_2.

$$F_1 + F_2 = 25\ kips$$
$$12.3\ kips + F_2 = 25\ kips$$
$$F_2 = 12.7\ kips$$

The internal force in the copper cylinder is 12.3 kips, and the internal force in the concrete column is 12.7 kips. Finally, calculate the stress in each material. Note from your original FBD that the copper cylinder is in tension and the concrete column is in compression. Introduce a negative sign here to indicate compressive stress in the concrete.

$$\sigma_{copper} = \frac{N_1}{A_1} = \frac{12.3\ kips}{\pi(2^2 - 1.5^2)in.^2} = 2.24\ ksi$$
$$\sigma_{concrete} = -\frac{N_2}{A_2} = -\frac{12.7\ kips}{(6*6)in.^2} = -0.353\ ksi$$

Approach 2: An alternative approach is to start by noting that the total deformation of the bar must be zero. In the case shown in Figure 5.12, the deformation of segment 1 plus the deformation of segment 2 must add up to zero.

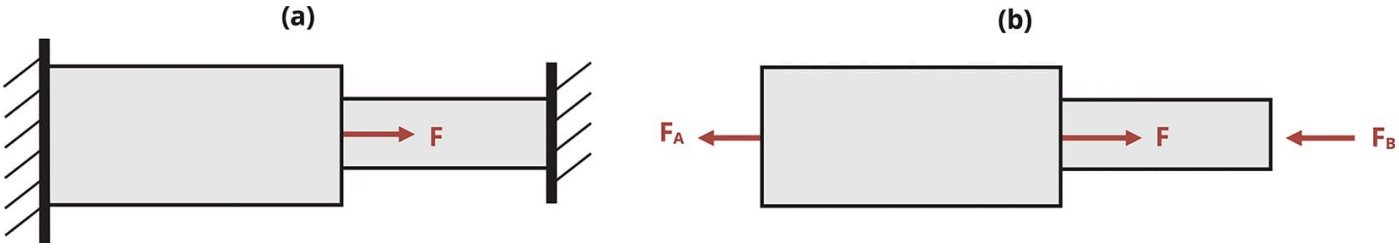


Figure 5.12: (a) A bar held between two rigid supports. (b) A free body diagram of the bar.

The internal load in segment 1 is F_A and the internal load in segment 2 is F_B, so there are currently two unknowns. We may use an equilibrium equation to solve for these two unknowns simultaneously.

$$\frac{F_A L_1}{A_1 E_1} + \frac{F_B L_2}{A_2 E_2} = 0$$

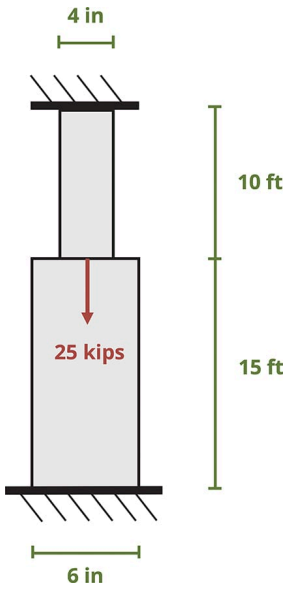
$$F - F_A - F_B = 0$$

Example 5.7 solves Example 5.6 using this method instead.

Example 5.7

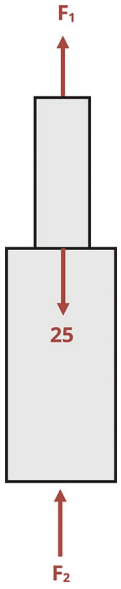
A 15 ft tall concrete (E = 4,000 ksi) column has a square cross-section of 6 in. x 6 in. A 10 ft tall copper (E = 17,000 ksi) cylinder with an outer diameter of 4 in. and inner diameter of 3 in. is attached to the top of the concrete. The structure is fixed between two supports. A force of 25 kips is applied as shown.

Determine the average normal stress in each material.



Solution

As before, begin with an equilibrium equation that relates the support loads to the applied load. In this equation we use the convention that forces pointing upward are positive and those pointing downward are negative.



$$\sum F_y = F_1 + F_2 - 25\ kips = 0$$
$$F_1 + F_2 = 25\ kips$$

This bar consists of two segments: the copper cylinder (segment 1) and the concrete column (segment 2). Since the bar is held between two rigid supports, the total deformation of these two segments must sum to zero. Note that in this diagram segment 1 is in tension and segment 2 is in compression. Remember to be consistent with the sign convention that tension is positive and compression is negative.

$$\frac{F_1 L_1}{A_1 E_1} - \frac{F_2 L_2}{A_2 E_2} = 0$$

Here F_1 and F_2 are the internal forces in segments 1 and 2 of the bar. These will be the same as the reaction loads at the supports that we are trying to solve for.

$$\frac{F_1 * (10 * 12)\, in.}{\pi(2^2 - 1.5^2) in.^2 * (17,000)\, \frac{kips}{in.^2}} - \frac{F_2 * (15 * 12)\, in.}{(6 * 6) in.^2 * (4,000)\, \frac{kips}{in.^2}} = 0$$

Rearrange the equilibrium equation and substitute into the deformation equation.

$$F_1 = 25\ kips - F_2$$
$$\frac{(25 - F_2) kips * (10 * 12) in.}{\pi(2^2 - 1.5^2) in.^2 * (17{,}000) \frac{kips}{in.^2}} - \frac{F_2 * (15 * 12) in.}{(6 * 6) in.^2 * (4{,}000) \frac{kips}{in.^2}} = 0$$

Simplify and solve for force F_2.

$$0.0321 - 0.001284 F_2 - 0.00125 F_2 = 0$$
$$0.0321 = 0.002534 F_2$$
$$F_2 = 12.7\ kips$$

Then use the equilibrium equation again to find F_1.

$$F_1 = 25\ kips - 12.7\ kips = 12.3\ kips$$

These are the same reactions found in Example 5.6 when we solved this problem using the other approach. From here find the stress in each material as before, noting again from the diagram that segment 2 is in compression.

$$\sigma_{copper} = \frac{N_1}{A_1} = \frac{12.3\ kips}{\pi(2^2 - 1.5^2) in.^2} = 2.24\ ksi$$
$$\sigma_{concrete} = -\frac{N_2}{A_2} = -\frac{12.7\ kips}{(6 * 6) in.^2} = -0.353\ ksi$$

The second type of indeterminate problem involves two materials bonded together in parallel. In these problems it is possible to find the reactions at the supports, but not possible to find the internal force in each material using only equilibrium (Figure 5.13).

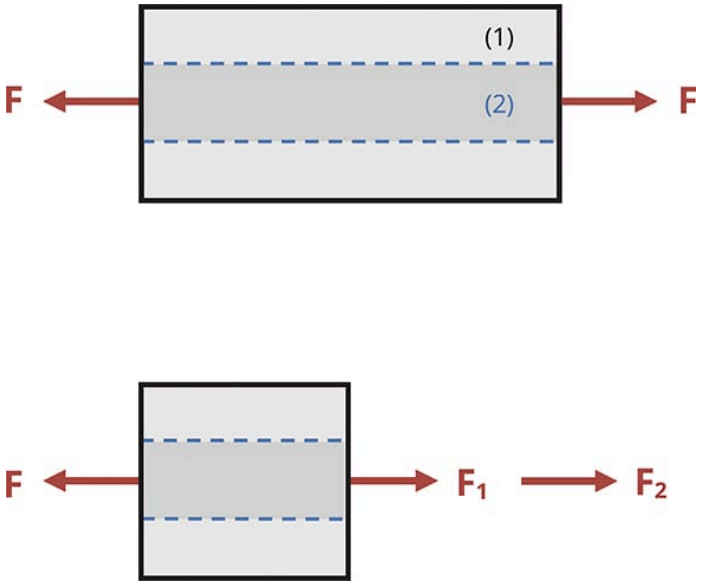


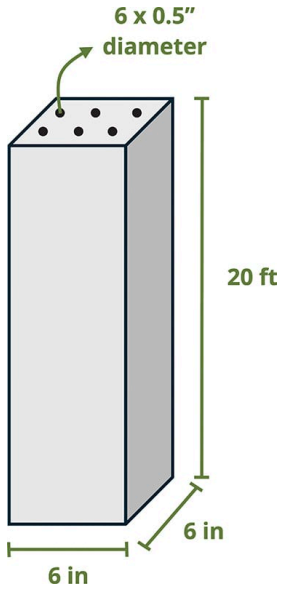
Figure 5.13: When two materials in parallel are subjected to force F, the force is split between the two materials (F_1 and F_2). By equilibrium $F_1 + F_2 = F$, but this is not sufficient to find the force in each material.

We have one equilibrium equation but two unknown internal forces. However, since the materials are bonded together, they must deform by the same amount. By setting the deformation the same for each material, we can define a second equation that involves the two internal forces and, combined with the equilibrium equation, now solve for both internal forces. See Example 5.8 for a demonstration.

Example 5.8

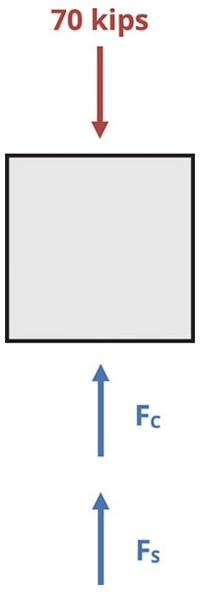
A 20 ft tall concrete (E = 4,000 ksi) column has a square cross-section 6 in. on each side. It is reinforced by six pieces of steel (E = 30,000 ksi) rebar that extend through the column's length. Each has a diameter of 0.5 in. The column is subjected to a compressive load of 70 kips.

Determine the stress in each material.



Solution

The 70 kips force will be split between the two materials. We need to find the force in each material before we can calculate the stress. Start by cutting a cross-section through the column, drawing an FBD, and writing an equilibrium equation.



$$\sum F_y = F_C + F_S = 70\ kips$$

This problem is statically indeterminate, since we have two unknowns and only one equilibrium equation. However, because the two materials are bonded together, they must both deform the same amount.

$$\Delta L_C = \Delta L_S$$

Since we know $\Delta L = \frac{FL}{AE}$,

$$\frac{F_C L_C}{A_C E_C} = \frac{F_S L_S}{A_S E_S}.$$

The length of both materials is 20 ft, so this term will cancel.

Although the steel comprises six individual pieces, it is fine to determine the total area of the steel and the total force in the steel. The total area of the steel is $\mathrm{A_S} = 6 * \pi * (0.25\ \mathrm{in.})^2 = 1.178\ \mathrm{in.}^2$.

For the area of the concrete, calculate the area of the square and then remove the area of the six rebar rods.

$$A_C = (6 * 6) in.^2 - 1.178\ in.^2 = 34.82\ in.^2$$

Substitute these into the deformation equation and rearrange.

$$\frac{F_C}{34.82\ in.^2 * 4{,}000\ ksi} = \frac{F_S}{1.178\ in.^2 * 30{,}000\ ksi}$$
$$F_C = F_S \left[\frac{34.82\ in.^2 * 4{,}000\ ksi}{1.178\ in.^2 * 30{,}000\ ksi}\right]$$
$$F_C = 3.941 F_S$$

Substitute this into the equilibrium equation.

$$F_C + F_S = 70\ kips$$
$$3.941 F_S + F_S = 70\ kips$$
$$4.941 F_S = 70\ kips$$
$$F_S = 14.2\ kips$$

Then since $\mathrm{F_C} = 3.941 \mathrm{F_S} \rightarrow \mathrm{F_C} = 3.941 * 14.2\ \mathrm{kips} = 55.8\ \mathrm{kips}$.

Now that we know the force in each material, calculate the stress in each material.

$$\sigma_S = \frac{14.2\ kips}{1.178\ in.^2} = 12.0\ ksi$$
$$\sigma_C = \frac{55.8\ kips}{34.82\ in.^2} = 1.60\ ksi$$

Step-by-Step: Statically Indeterminate Axial Problems

Problems with redundant supports

Approach 1:

1. Write out equilibrium equations. There will be too many unknowns, so put these to one side for now.
2. Remove one of the supports and calculate the amount of deformation that would occur if the support were not there, using $\Delta \mathrm{L}_1 = \sum \frac{\mathrm{FL}}{\mathrm{AE}}$.
3. Replace the force from the removed support and determine the deformation, $\Delta \mathrm{L}_2$, caused by this force in terms of the force itself.
4. Set $\Delta \mathrm{L}_1 + \Delta \mathrm{L}_2$ equal to the total allowed deformation for the bar and use this to solve for the unknown force at the redundant support.
5. Now that one force is known, use the equilibrium equations to determine the other unknown forces.

Approach 2:

1. Draw an FBD of the entire structure and write out the relevant equilibrium equations. For axial loads this will be just one sum of force equation, either horizontal or vertical depending on the applied loading.
2. Write an equation in which the total deformation in each segment of the bar must sum to the total deformation of the bar (zero if the bar is held between two rigid supports). A new segment must be made any time the internal load, cross-sectional

area, or material changes.

3. This deformation equation involves the same two unknowns as the equilibrium equation. Solve these equations simultaneously to find the reaction loads at the supports.

Problems with two materials in parallel

1. Cut a cross-section through the member and set up an equilibrium equation for the internal forces where the internal forces will sum to equal the external applied force.

2. Set the deformation of the two materials equal.

3. Solve the deformation equation and equilibrium equation simultaneously to determine the internal force in each material.

5.6 Thermal Deformation and Thermal Stresses

So far we've studied the effects of axial forces on an object and how they create stresses and deformations. Temperature changes will also cause an object to deform. As seen in Section 4.6, the strain due to temperature can be predicted by

$$\varepsilon_T = \alpha \Delta T.$$

As before, strain is dimensionless. The coefficient of thermal expansion is a material constant that can be looked up in handbooks or in Appendix C. Note that for a given change in temperature, the thermal strain will be the same in the axial and transverse directions.

Since strain is also defined as $\varepsilon = \frac{\Delta L}{L}$, we can also predict the deformation resulting from a change in temperature.

$$\varepsilon_T = \alpha \Delta T = \frac{\Delta L}{L} \rightarrow \Delta L = \alpha \Delta T L$$

That the object is free to expand or contract doesn't cause any issues and is easy to account for. Many real applications include a small gap to allow for changes in length due to temperature changes (Figure 5.14). It is even possible that an object is subjected to both a physical force and a temperature change and that the total change in length is simply the sum of these effects.



Figure 5.14: Expansion joint on a bridge, allowing for thermal expansion and contraction of the bridge as temperature changes through the year.

$$\boxed{\Delta L = \Delta L_F + \Delta L_T = \frac{FL}{AE} + \alpha \Delta T L},$$

(5.3)

where

- ΔL = *Change in length [m, in.]*
- ΔL_F = *Change in length due to applied load [m, in.]*
- ΔL_T = *Change in length due to temperature change [m, in.]*
- F = *Internal force [N, lb]*
- L = *Original length [m, in.]*
- A = *Original cross-sectional area [m^2, $in.^2$]*
- E = *Elastic modulus [Pa, psi]*
- α = *Coefficient of thermal expansion* $\left[\frac{1}{°C}, \frac{1}{°F}\right]$
- ΔT = *Change in temperature [°C, °F]*

However, if we do not design a gap, or if the gap isn't large enough, then the object is not free to expand or contract. As the member pushes or pulls on its supports, a physical force is created. This in turn creates a stress in the object, and these stresses can be very large. Such problems are statically indeterminate because the force the support applies on the member is unknown and cannot be found using only equilibrium.

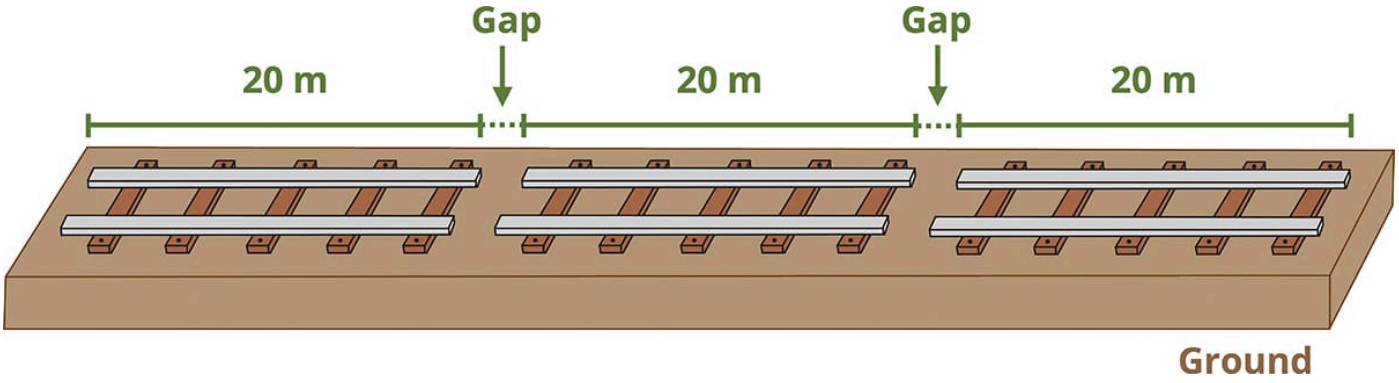
Solving these problems is very similar to solving the first type of statically indeterminate problem. First, remove a support and determine the amount of deformation that would occur as a result of the change in temperature if the object were free to deform. Then replace the force from the removed support, which will cause the object to deform in the other direction. The sum of these two deformations will equal the total deformation of the member as before. Example 5.9 works through a statically indeterminate thermal expansion problem.

Example 5.9

Steel (E = 200 GPa, α = 11.7 × 10^{-6}/°C) train rails are laid end-to-end. Each rail is 20 m long. A small section of this track is shown below. The rails are laid in winter when the temperature is 0°C. In summer, the maximum temperature is 40°C.

Determine the compressive stress in the rail as a result of the temperature change if

1. In a particular section of track, the rails are laid with no gap between them.
2. In another section, the rails have a 5 mm gap between them.



Solution

1. First determine the deformation of each rail that would occur if it were free to expand.

$$\Delta L_T = \alpha \Delta T L = 11.7 \times 10^{-6}/^\circ C * 40^\circ C * 20\ m = 0.00936\ m$$

Since each rail contacts the rail next to it as it tries to expand, each rail will experience a compressive force. This force causes compression in the rail.

$$\Delta L = -\frac{FL}{AE}$$

The actual deformation of each rail must be zero, given that the rails are in contact end-to-end. Thus these deformations must sum to zero.

$$0.00936\ m - \frac{F * 20\ m}{A * 200 * 10^9\ Pa} = 0$$

Although we don't know the cross-sectional area of the rail, we can replace $\sigma = \frac{\mathrm{F}}{\mathrm{A}}$.

$$0.00936\ m - \sigma * \frac{20\ m}{200 * 10^9\ Pa} = 0$$
$$\sigma = \frac{0.00936\ m * 200 * 10^9\ Pa}{20\ m} = 93.6\ MPa$$

2. In this case again begin by determining the deformation of each rail that would occur if it were free to expand.

$$\Delta L_T = \alpha \Delta T L = 11.7 \times 10^{-6}/^\circ C * 40^\circ C * 20\ m = 0.00936\ m$$

Each rail would expand 9.36 mm, but the gap between rails is only 5 mm, so the rails close this gap and make contact, which means there will again be some compressive stress. Set up the deformation equation as in part (a), but this time the deformation is 5 mm (0.005 m) instead of zero.

$$0.00936\ m - \frac{F * 20\ m}{A * 200 * 10^9\ Pa} = 0.005\ m$$

Replace $\sigma = \frac{F}{A}$ as before and determine the stress.

$$0.00936\ m - \sigma * \frac{20\ m}{200 * 10^9\ Pa} = 0.005\ m$$
$$0.00936\ m - 0.005\ m = \sigma * \frac{20\ m}{200 * 10^9\ Pa}$$
$$\sigma = \frac{0.00436\ m * 200 * 10^9\ Pa}{20\ m} = 43.6\ MPa$$

We can see that installing the rails with a small gap has significantly reduced the thermal stress. Accurately predicting the amount of deformation and leaving a suitably sized gap would reduce the stress even further.

Step-by-Step: Thermal Deformation

1. Remove one of the supports and calculate the total deformation that would occur if the support were not there, using $\Delta \mathrm{L_T} = \alpha \Delta \mathrm{TL}$ for thermal deformation and $\Delta \mathrm{L} = \frac{\mathrm{FL}}{\mathrm{AE}}$ for any applied axial loads.

2. If the total deformation causes the member to contact a support that prevents some of the deformation, replace the force from the removed support and determine the deformation caused by this force in terms of the force itself, using $\Delta \mathrm{L} = \frac{\mathrm{FL}}{\mathrm{AE}}$.

3. Sum these deformations and set them equal to the total allowed deformation. Use this equation to solve for the force at the redundant support.

4. Use equilibrium to determine the force at any other supports.

Summary

Key Takeaways

Axial loads cause normal stresses. While we generally calculate the average normal stress, stress concentrations occur close to applied loads, supports, and changes in geometry.

These stress concentrations can be large and cause localized failure even if the average stress is within acceptable limits. Modeling stress concentration is beyond the scope of this course, but we can determine maximum stresses for different geometries using stress concentration curves.

It is often desirable to limit the total change in length of a bar subjected to axial loads. Change in length can be predicted directly.

Leaving a gap to allow for change in length is useful in practice. If a change in length is prevented from happening, this creates a stress in the bar.

Changes in temperature also cause axial deformation of a bar, known as thermal deformation. Thermal deformation can lead to a physical stress in the bar if a support prevents the deformation from occurring.

Key Equations

Stress concentrations:

$$\sigma_{avg} = \frac{N}{A}$$
$$\sigma_{max} = K\sigma_{avg}$$

Axial deformation:

$$\Delta L = \frac{FL}{AE}$$

Thermal strain:

$$\varepsilon_T = \alpha \Delta T$$

Thermal deformation:

$$\Delta L = \alpha \Delta T L$$

References

Figures

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- Figure 5.2: Each bar has a fixed support on the left, a cross-section of 30 in.^{2}, and is subjected to a force of 10,000 lb. James Lord. 2024. CC BY.
- Figure 5.4: The fillet helps prevent a sharp corner but still causes stress concentrations. James Lord. 2024. CC BY.
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- Example 5.1: First and second image: Kindred Grey. 2024. CC BY. Third image: Adapted under fair use by Kindred Grey from Figures 5.14 and 5.15 in Philpot, T.A. (2020) Mechanics of Materials, 4th Edition, Wiley.
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- Figure 5.10: An example of redundancy on a suspension bridge. Unknown author. 2017. Public domain. https://pxhere.com/en/photo/759379.
- Figure 5.14: Expansion joint on a bridge, allowing for thermal expansion and contraction of the bridge as temperature

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Chapter 6: Torsion

Learning Objectives

- Calculate internal reaction torques to use as inputs for determining shear stress and angle of twist at any given point in a body subjected to pure torsion
- Understand and apply the shear stress equation for torque to determine shear stress at any given point in a body subjected to pure torsion
- Understand and apply the torque-twist equation to determine angle of twist at any given point in a body subjected to pure torsion
- Use kinematic constraints in combination with equilibrium to solve statically indeterminate problems in which the body is subjected to pure torsion
- Apply torsional concepts to shafts used in power transmission assemblies

Moments can produce multiple effects on a body or structure, depending on the axes around which they act. Whereas later sections of the text discuss the effect of moments that cause bending (see Chapter 7, Chapter 9, Chapter 10, and Chapter 11), this chapter concerns the effects of moments that cause torsion.

Torsion occurs when a moment is applied around the longitudinal axis of a body. This type of moment is known as torque, and the effect of torsion is a twisting action. Torsion is found in many common situations. Some examples of applications in which torsion is a primary form of loading include these:

1. In machinery, rotating shafts, such as transmission shafts that transmit power, are subjected to torque.
2. When one uses a wrench to tight or loosen a bolt, the bolt is subjected to torque.
3. Hinge pins are subjected to torque when the rotating elements they are attached to (e.g., doors) are rotated.
4. Vertical poles from which road signs and traffic lights are mounted are subjected to torque when forces are applied to the horizontal elements of the structure. For example, in Figure 6.1 wind force applied to the horizontal poles or traffic lights causes the vertical pole to twist around the vertical axis.

   Note that in this particular example the vertical pole would also be subject to bending (from moments around the cross-sectional axes of the pole). Problems in which this type of combined torsion and bending effects occur are discussed in Chapter 14.

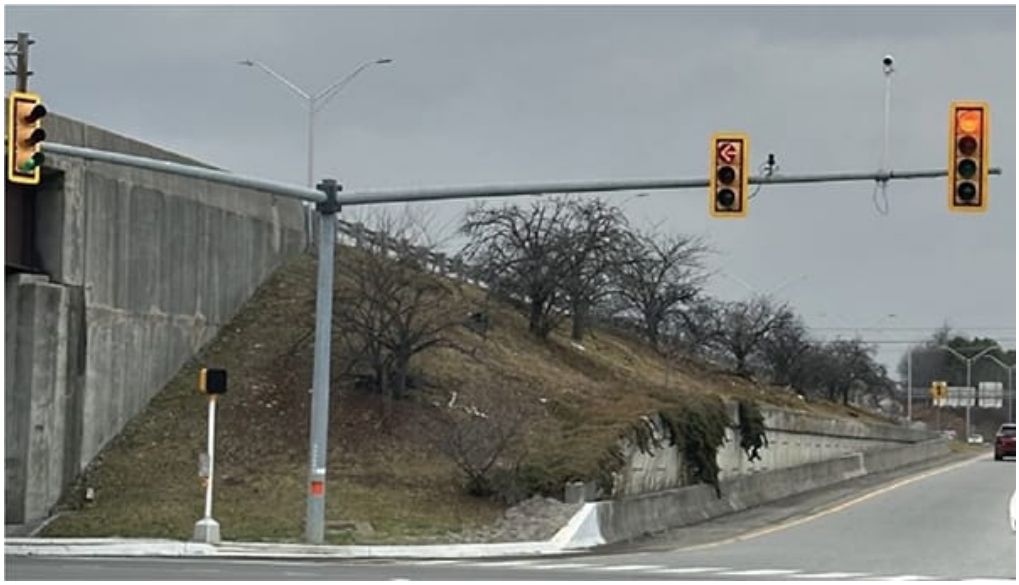


Figure 6.1: Horizontal wind blowing on the traffic lights will create a torsional moment in the vertical pole that results in twisting.

This chapter focuses on the effects of pure torsion (no axial or bending loads) on bodies.

Section 6.1 describes the general effects of torsion. Section 6.2 details the process to determine internal torques as a preliminary step to stress and deformation calculations. Section 6.3 and Section 6.4 offer derivation equations for calculating the stress and deformation caused by torsional loads. Section 6.5 examines how these concepts can be applied to solving statically indeterminate torsion problems, which is a similar process as axial loading statically indeterminate problems described in Section 5.5. Finally, Section 6.6 considers how the concept of torque applies in power transmission applications.

6.1 The Effect of Torsion

As mentioned, a torsional moment causes the body to twist, and this action results in shear stress.

Consider the pleated tube shown in Figure 6.2. Figure 6.2 (A) shows the tube in an unloaded state with some of the pleats colored for identification. Figure 6.2 (B) shows that after a torque is applied, the pleats move vertically but not horizontally. We can say that all points remain in their original plane even as they twist to new locations around the perimeter of the cross-section. In other words, the shape of the cross-section is unchanged despite the twisting action.

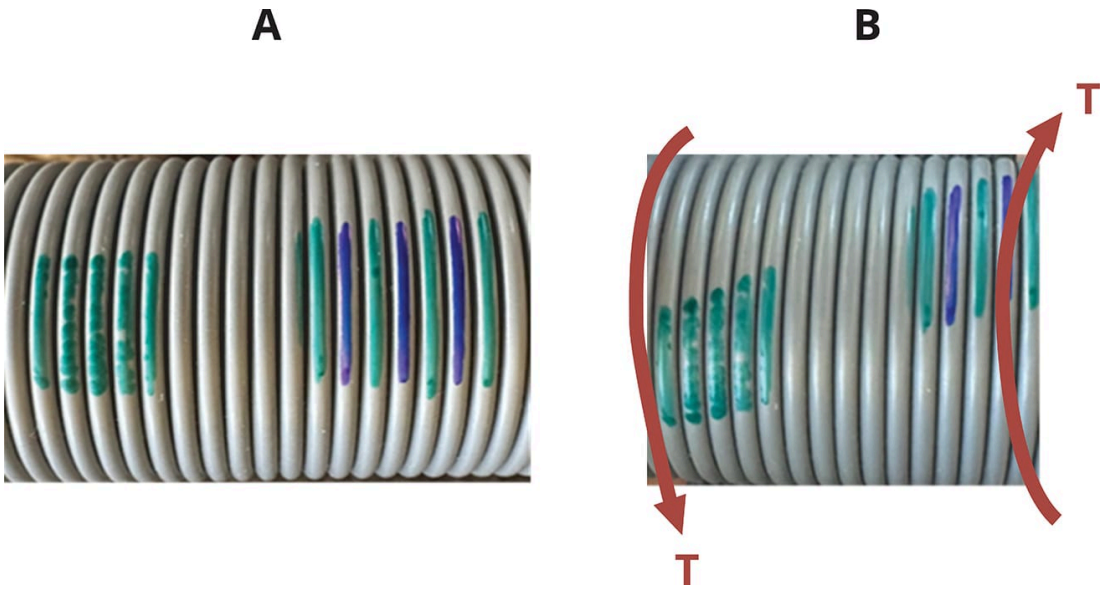


Figure 6.2: Unloaded tube (A) and tube with torque applied (B).

Now consider the I-beam subjected to torsion in Figure 6.3. In this case the shape of the cross-section changes and the points do not remain in the same vertical plane (there is horizontal displacement as well as vertical). This shape change is referred to as warping. In this text we consider only circular cross-sections with no warping.

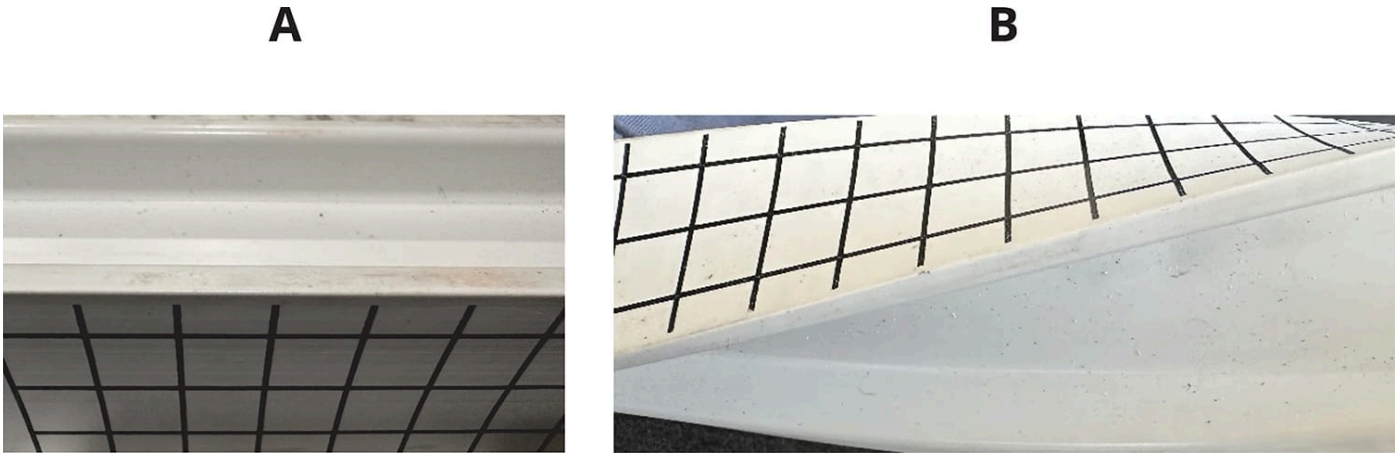


Figure 6.3: I-beam subjected to torsion unloaded (A) and with applied torque (B).

Returning to Figure 6.2, let's consider the shape formed by the nondisplaced colored lines in Figure 6.2 (A) versus the displaced lines in Figure 6.2 (B). These images are shown again in Figure 6.4. In the nondisplaced case, a rectangle can be drawn around the green lines on the left with 90° corners. In the displaced case, the corners are no longer at 90°. From Section 3.2 we know that this change in angle is the definition of shear strain. The evidence of shear strain indicates that the applied torque results in shear stress.

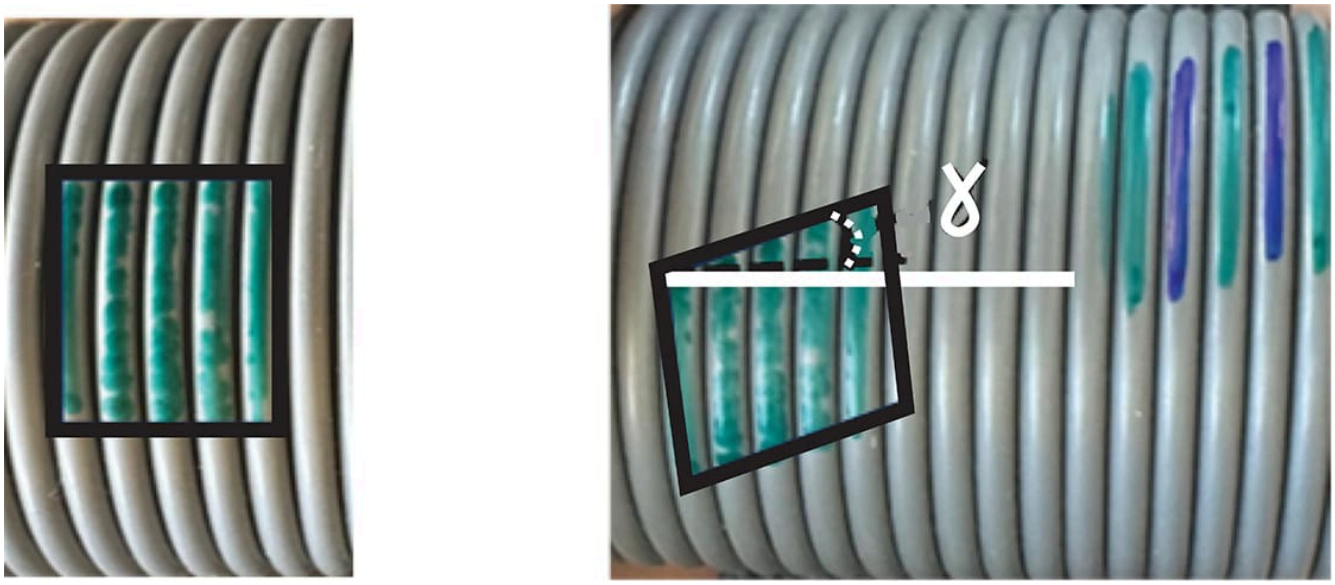


Figure 6.4: Right angles in unloaded tube (left) change when torque is applied (right), giving of shear strain.

6.2 Determination of Internal Torques

As with axial loading, stress and deformations from torsion vary from point to point within a body and are dependent on the internal torques at the individual points and geometric section properties. Internal torques can be found by sectioning bodies and applying equilibrium equations, just as we did when finding internal forces in axial loading problems (see examples 1.5, 2.1, and 2.2). In the case of internal torques, moment equilibrium around the longitudinal axis is applied instead of force summation.

Before we examine the process of finding internal torque reactions, it would be beneficial to establish the sign convention and ways of representing torque used in this text.

6.2.1 Sign Convention and Representation of Torque

In this text counterclockwise torques are considered positive and clockwise torques negative. However, the apparent direction of a torque depends on the perspective from which the rotation is observed. In Figure 6.5, the applied torque would appear clockwise if viewed from the left end of the bar toward the right and counterclockwise if viewed from the right end of the bar toward the left. As long as you are consistent, which perspective is used isn't usually important. However, a typical convention (and the one followed in this text) is to look from the free end of the shaft (if there is one) toward the fixed end, or from the positive end of the longitudinal toward the negative.

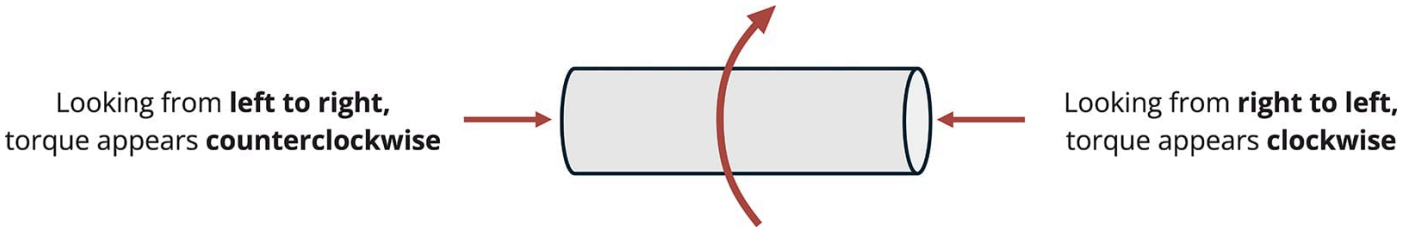


Figure 6.5: Torque applied to a segment of shaft. The viewing direction affects the perception of clockwise or counterclockwise rotation.

To visually determine whether a torque appears clockwise or counterclockwise can sometimes be difficult, so one helpful tool that can be used is the right-hand rule: Curl the fingers of your right hand in the direction of the rotation represented by the torque arrow. If in doing this your thumb points in the direction of the positive longitudinal axis (the x-axis in Figure 6.6), a positive

torque is indicated, and if it points away, a negative torque.

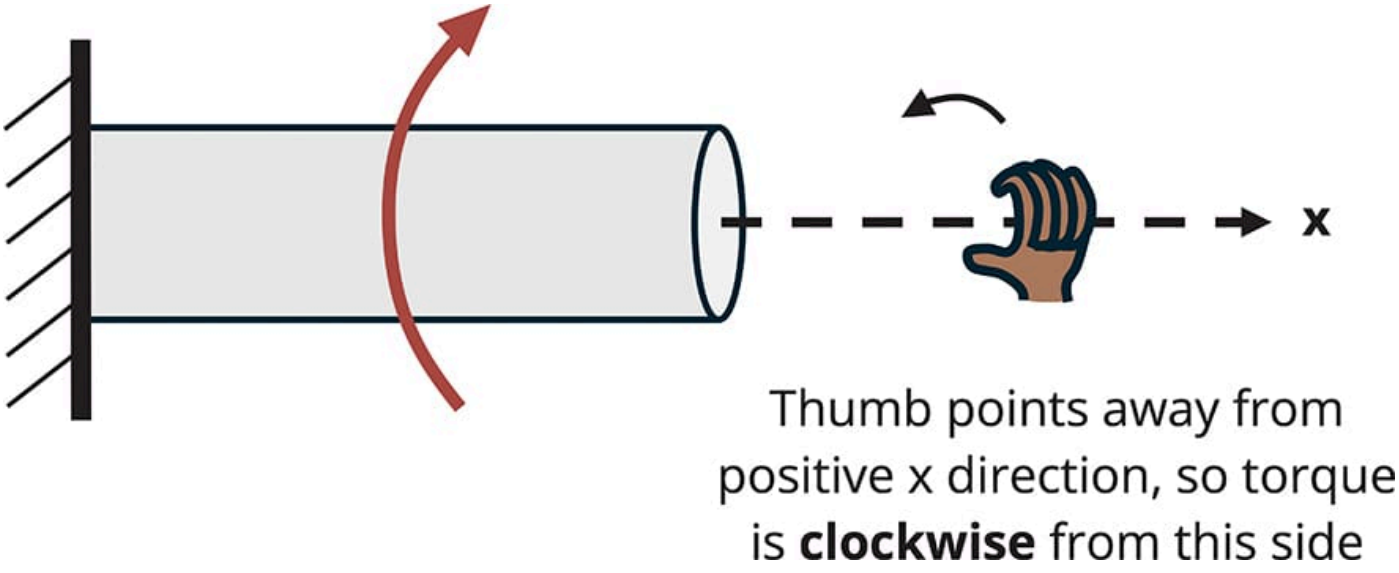


Figure 6.6: Using the right-hand rule to determine sign for torque

For bodies that are sectioned to find internal torques, consider the direction of the torsional moment from the perspective of looking from the cut edge towards the other end of the cut section. Imagine your right hand to be at the cut edge and curl your fingers in the direction of the torque. If your thumb points away from the cut edge, the internal torque is positive, and if your thumb points toward the cut edge, the internal torque is negative. This is illustrated in Figure 6.7.

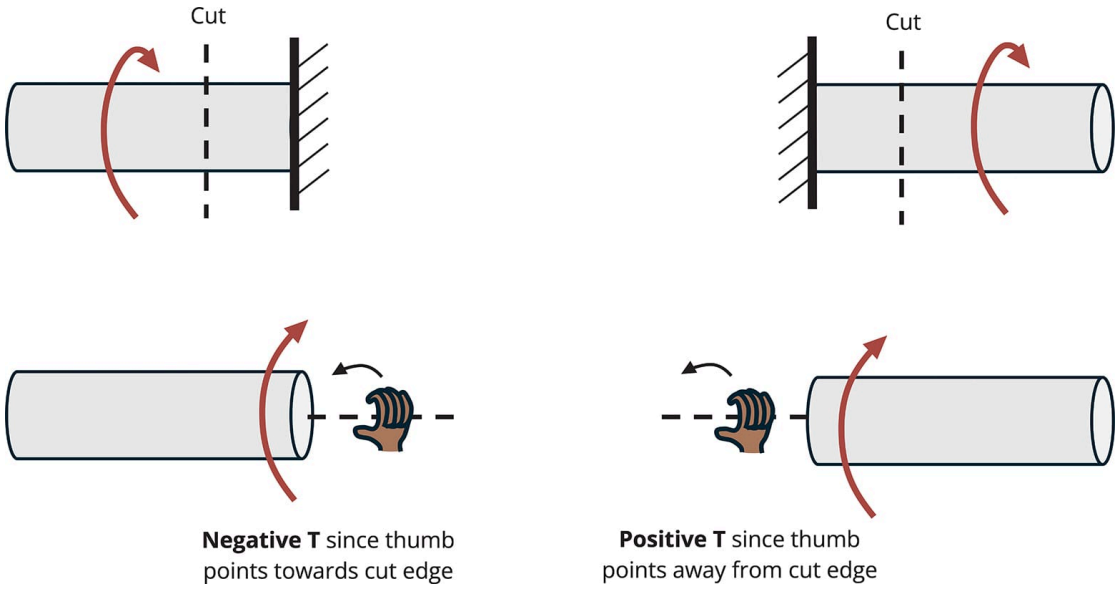


Figure 6.7: Tool to determine sign of internal torque

Finally, since circular arrows can be difficult to draw in a way that clearly conveys direction, double-headed line arrows are frequently used to represent torque. Note that conventions can vary between texts and other sources. This text uses the following convention:

1. On a whole body FBD used to determine external reactions, arrows facing in the direction of the positive longitudinal axis indicate positive torques (counterclockwise) and those facing the opposite direction negative torques.

2. When applying equilibrium, the positive or negative sign from step 1 is used.

3. For FBDs of cut sections used to determine internal torques, positive torques are indicated by arrows pointed away from the section's cut edge and negative torques by arrows pointed toward the cut edge. However, when summing moments, the regular convention of adding right arrows and subtracting left arrows is used.

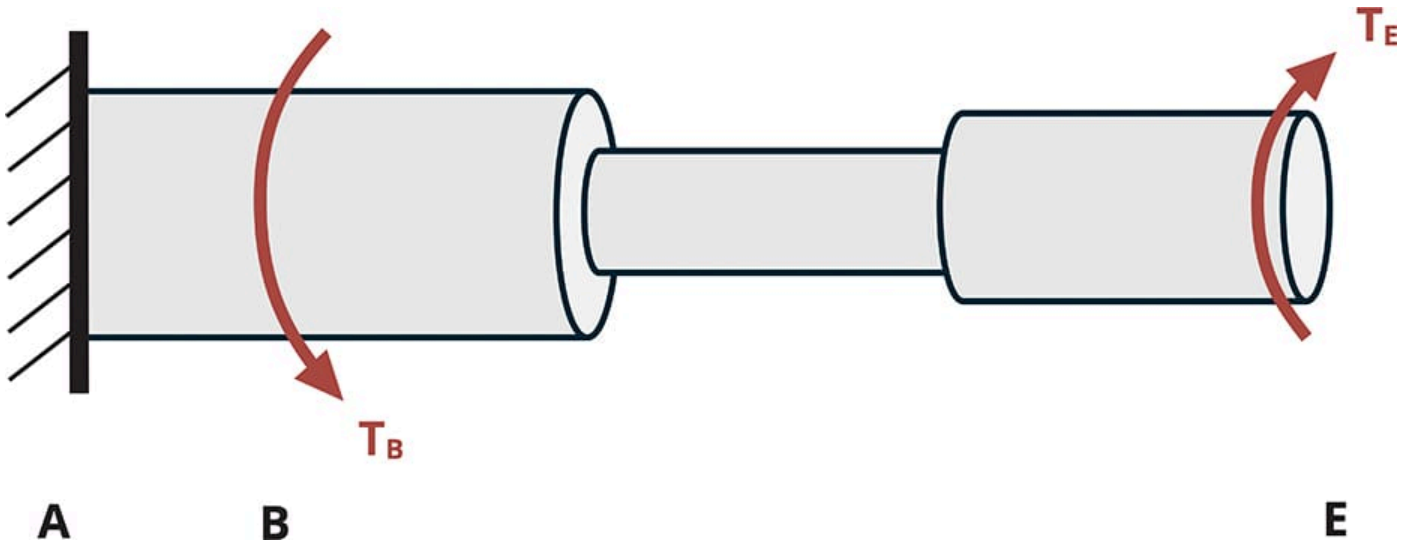
The direction of the internal torque is unimportant in calculating shear stress (the sign on the stress requires concepts discussed in Chapter 12) but is important when calculating angle of twist, as evident in Section 6.2.

Example 6.1 demonstrates how to determine internal torques in a multisection body as well as how to apply the described sign conventions.

Example 6.1

A multisection steel bar is supported by a fixed support at A and is subjected to torque $T_B = 70$ N·mm and $T_E = 30$ N·mm.

Determine the internal torque at all points in the shaft.

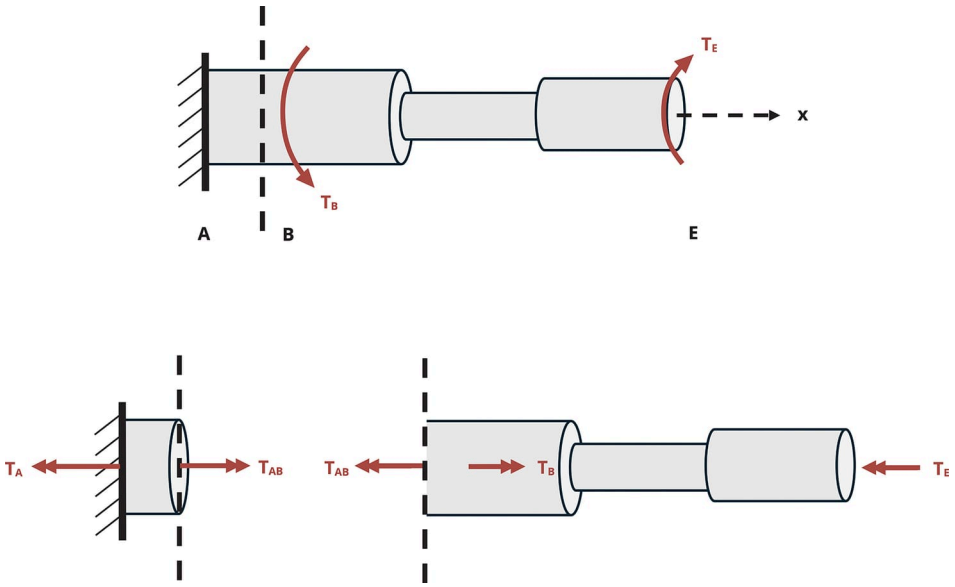


Solution

With torques applied only at points B and E (there is also a reaction moment from the wall at A), section AB and section BE are sections of constant internal torque. The reaction torque at A could be obtained by applying equilibrium to the whole shaft, but this example shows how to avoid that.

Section AB:

To determine the internal torque in section AB, we can cut the shaft anywhere between points A and B and draw an FBD of the section to the left of the cut or the section to the right of the cut.



Note that in the following illustration of the cut and the resulting FBDs, the internal torque T_{AB} is drawn in opposite directions on the two sections. This must be the case if the bar is in equilibrium.

T_{AB} is drawn as a positive internal torque on both the left and the right section diagrams since in both instances it points away from the cut. We assume that T_{AB} looks counterclockwise if we view the left section from the cut toward A or the right section from the cut toward E.

For the applied loads, since T_A is drawn in the direction opposite of T_{AB} on the section to the left of the cut, it is assumed to appear clockwise from the cut toward A.

On the right side diagram, T_B is drawn in the direction opposite of T_{AB} since it appears clockwise from the cut toward E. Applied torque T_E is drawn in the same direction as T_{AB} since it appears counterclockwise from the cut toward E.

In writing the equilibrium equation to find T_{AB}, we use either the left section FBD or the right section FBD and obtain the same result. To avoid needing to find T_A, we will choose to use the right section FBD.

Recall that torques are moments around the longitudinal axis, which is the x-axis in this case. Applying the x-axis moment equilibrium equation to the right section gives

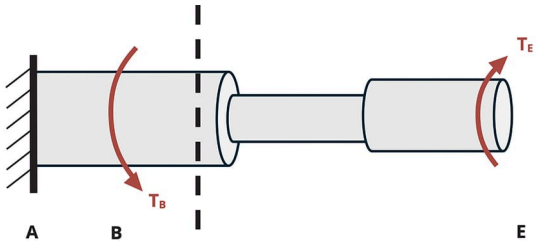
$$\begin{aligned} \sum M_x &= -T_{AB} + T_B - T_E = 0 \\ \sum M_x &= -T_{AB} + 70\ N \cdot mm - 30\ N \cdot mm = 0 \\ T_{AB} &= 40\ N \cdot mm. \end{aligned}$$

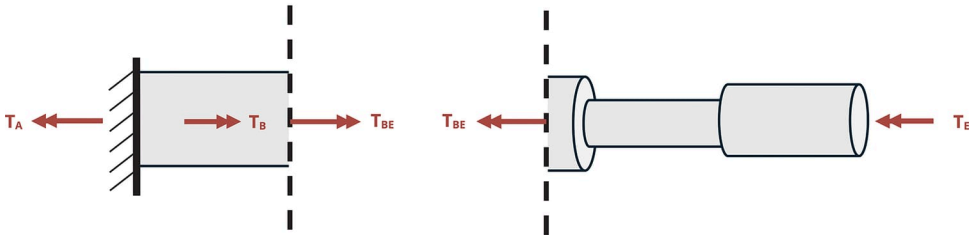
Note that in performing the summation, right directed arrows were added and vice versa. The opposite convention can also be used with the same end result.

The positive result for T_{AB} in this case confirms that the assumed direction for T_{AB} is the correct one, so it is a counterclockwise internal torque.

Section BE:

Similarly, the internal torque in all parts of the shaft between B and E can be determined by cutting the shaft at a nonspecific point between B and E and applying equilibrium to either the section of shaft to the left of the cut or the section to the right of the cut, as shown in the next two FBDs.





Once again the internal torque (T_{BE} in this case) is drawn in the direction away from the cut on both sections, so it is assumed to be counterclockwise on both sections.

Applying the moment equilibrium equation on the right section results in

$$\begin{aligned} \sum M_x &= -T_{BE} - T_E = 0 \\ \sum M_x &= -T_{BE} - 30\ N \cdot mm = 0 \\ T_{BE} &= -30\ N \cdot mm\ (or\ 30\ N \cdot mm\ clockwise). \end{aligned}$$

The negative result indicates that T_{BE} goes in the opposite direction than what was assumed on the FBD, so it is clockwise.

Answer:

Internal torque for all points between A and B is 40 N·mm counterclockwise.

Internal torque for all points between B and E is 30 N·mm clockwise.

6.3 Calculation of Shear Stress Due to Pure Torsion

This section explains the derivation of the shear stress equation that results from torsion of a circular shaft. Understanding the derivation lends clarity to the inputs of the equation and the assumptions necessary for use. If you wish to skip the derivation, go directly to Equation 6.1.

Consider a circular shaft fixed on one end and subjected to torque on the free end, as shown in Figure 6.8. The shaft has an outer radius, c, as shown in Figure 6.9 (B). For the given loading, the internal torque at any x location along the length of the shaft would also be T. We will examine a small sliver of the shaft, marked as AB.

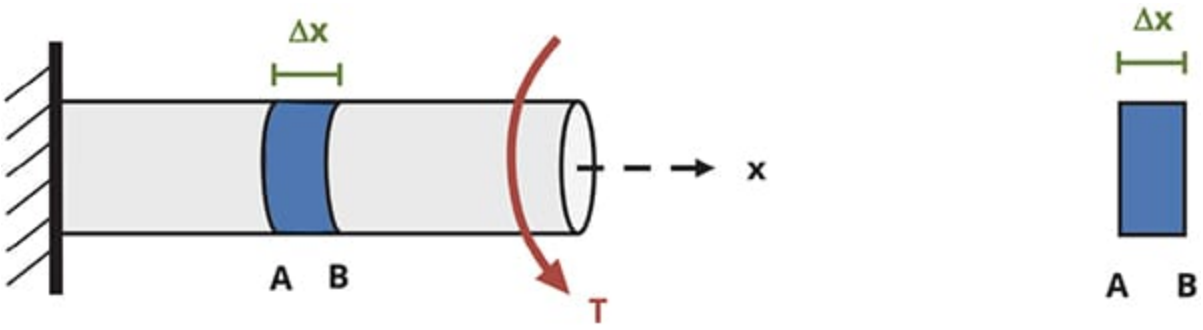


Figure 6.8: Illustration of fixed end shaft subjected to torque and isolated section AB

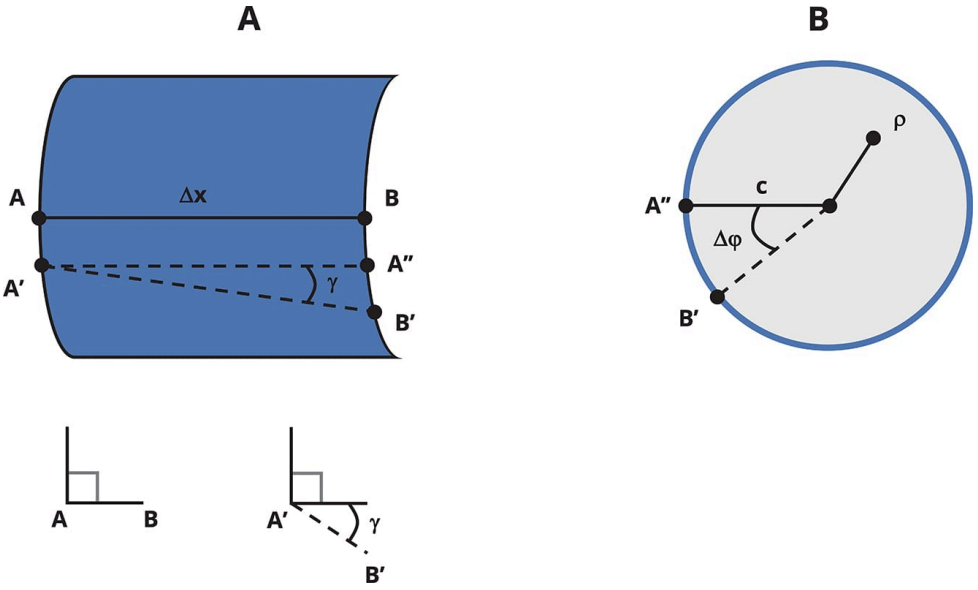


Figure 6.9: Illustration of the displacement of points A and B on the shaft (A) and the cross-section of the shaft (B)

Before the torque, is applied, a horizontal line segment drawn across AB has a length of Dx. Now let's consider the same shaft segment after the torque has been applied and points A and B move to A' and B' respectively. We can approximate that A'B' is linear for a very narrow section of shaft (infinitely small Δx). An illustration of the resulting geometry can be seen in Figure 6.9 (A). Since the shaft is fixed on the left end, point B on the shaft will rotate more than point A. Point A″ represents the relative location of A' on the cross-section of B and B'.

Recalling that shear strain is the change in angle between two lines that are originally perpendicular, we can see that the angle of tilt of A'B' relative to the horizontal line A'A″ is the shear strain, γ, that results from the torque. For an infinitely small Δx, the arc segment from A″ to B' can be approximated as vertical so that the shear strain γ can be expressed using trigonometry.

$$\tan(\gamma) = \frac{A''B'}{\Delta x}$$

The small-angle approximation (given an infinitely small Δx) results in

$$\tan(\gamma) \approx \gamma = \frac{A''B'}{\Delta x}.$$

Now consider the cross-section of the shaft where points B and B' are located, shown in Figure 6.9 (B). The angle between the radial lines leading from the center to points A″ and B' respectively is the shaft segment's angle of twist. Since we are examining a very small section of the shaft, the angle is represented as $\Delta\phi$ (as opposed to the absolute angle of twist, ϕ, as measured from the fixed end). If the shaft's outer radius is r = c, then the length of A″B' is given by the arc length, so

$$A''B' = c(\Delta\phi)$$

and

$$\gamma = \frac{c(\Delta\phi)}{\Delta x}.$$

Generalizing this result for any point on the cross-section that is located at an arbitrary radial distance ρ from the center yields

$$\gamma = \frac{\rho(\Delta\phi)}{\Delta x}.$$

The resulting expression of γ shows that the shear strain varies linearly with ρ from the center of the cross-section to the outer radius, as illustrated in Figure 6.10. The shear strain is 0 at the center, and the maximum shear strain occurs at the max value of ρ which is ρ = c.

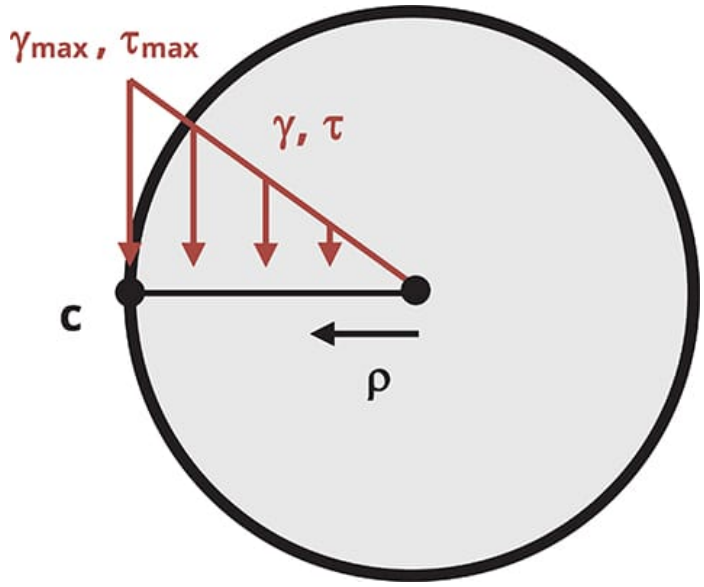


Figure 6.10: Shear strain and shear stress vary linearly from the center to the outer edge of the circular cross-section.

$$\gamma_{max} = \frac{c(\Delta\phi)}{\Delta x}$$

Substituting the general and max expressions for shear strain into Hooke's law for shear stress, $\tau = G\gamma$, reveals that shear stress also varies linearly from 0 to t_{max} in going from the center of the cross-section to the outer radius.

$$\tau = G\gamma = G\frac{\rho(\Delta\phi)}{\Delta x}, \tau_{max} = G\frac{c(\Delta\phi)}{\Delta x}$$

Figure 6.10 shows that the slope of the linear variation is $\frac{\tau_{max}}{c}$ and that the shear stress can also be expressed as

$$\tau = \frac{\tau_{\max}}{c}\rho.$$

The last step in developing the shear stress relationship to torque involves considering the shaft's equilibrium. Refer to Figure 6.11, in which the square is a representative point (i) subjected to an infinitesimal force dF_i corresponding to the shear stress discussed previously. If each point (i) on the cross-section has an area of dA_i then dF_i can be calculated as

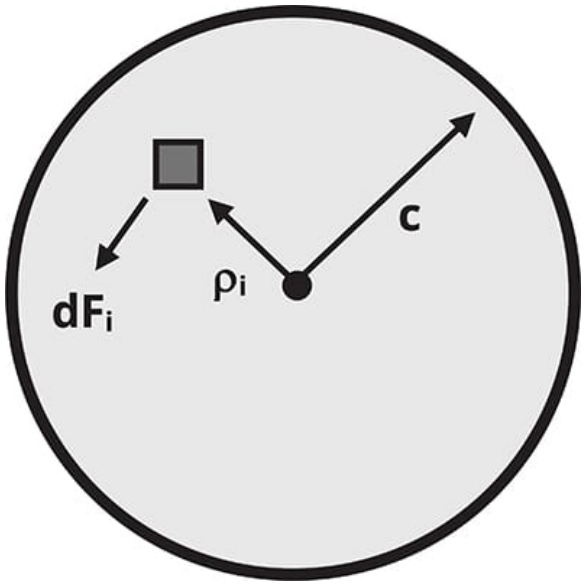


Figure 6.11: Illustration of an infinitesimal force on the shaft cross-section

$$\mathrm{dF}_i = \tau_i \mathrm{dA}_i = \frac{\tau_{max}}{c}\rho_i d.$$

Recalling that the internal torque everywhere on the shaft is T, note that equilibrium dictates that the sum of the moments about the center of the cross-section should be equal to T. For an infinite number of points on the cross-section, this summation can be performed as an integral.

$$T = \int dF_i \rho_i = \int \frac{\tau_{\max}}{c}\rho_i^2 dA$$

Since $\tau_{\max}$ and c are constant, they can be pulled outside the integral, leaving the integral portion to be $\int \rho_{\mathrm{i}}^2 \mathrm{dA}$, which is the definition of the polar second moment of area (also known as moment of inertia and area moment of inertia). This is the moment of inertia around the out-of-plane axis, which is the x-axis in this case. The polar second moment of area is denoted with the letter J in this text.

For a solid circular cross-section of outer radius r and outer diameter d:

$$J_{solid} = \frac{\pi}{2}r^4 = \frac{\pi}{32}d^4.$$

For a tube or hollow circular cross-section with outer radius and diameter r_o and d_o respectively and inner radius and diameter r_i and d_i respectively:

$$J_{hollow} = \frac{\pi}{2}\left(r_o^4 - r_i^4\right) = \frac{\pi}{32}\left(d_o^4 - d_i^4\right).$$

Replacing $\mathrm{J} = \int \rho_{\mathrm{i}}^2 \mathrm{dA}$ in our previous equation results in

$$T = \frac{\tau_{max} J}{c}.$$

This can be rearranged to

$$\tau_{max} = \frac{Tc}{J}.$$

This equation yields the maximum stress on a given circular cross-section, which will occur on the outer edge of the shaft (where ρ=c). A more general form of the equation that can be used to find the stress at any point on the cross-section is

$$\boxed{\tau = \frac{T\rho}{J}},$$

(6.1)

where

- τ = *Shear stress due to torsion [Pa, psi]*
- *T = Internal torque [N·m, lb·in.]*
- *ρ = Radial distance from center of cross-section to point of interest on the cross-section [m, in.]*
- *J = Polar second moment of area [m^4, $in.^4$]*

Here linear elastic behavior and small deformations of a circular cross-section have been assumed.

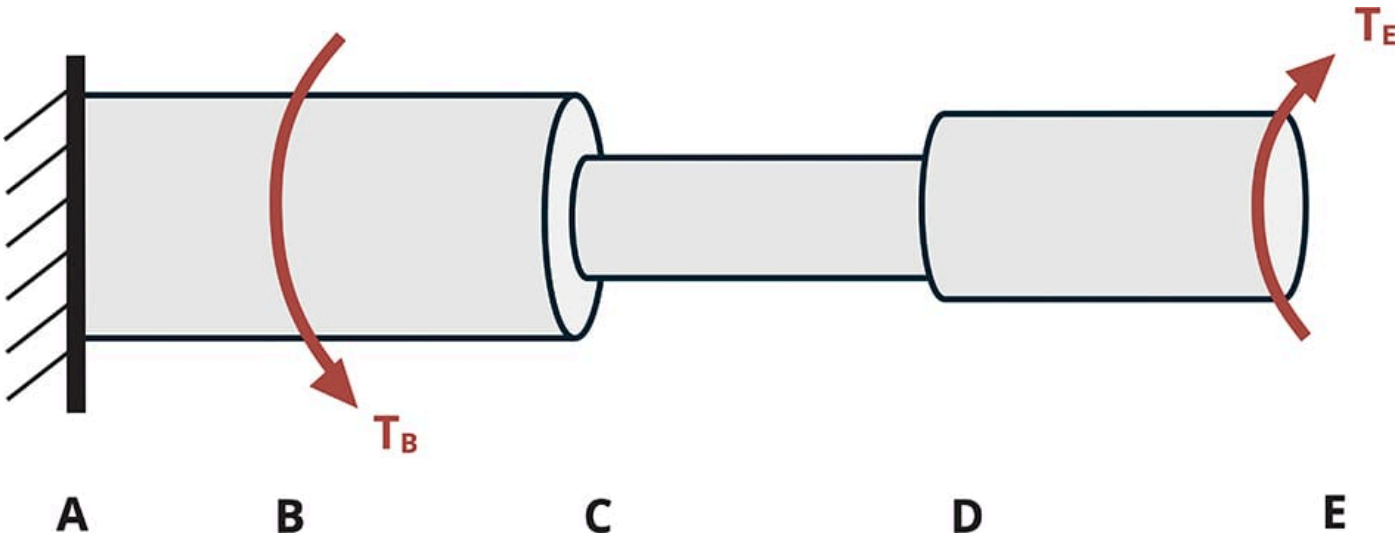
Note that for now only the magnitude of the shear stress (without sign) will be determined, since the sign for shear stress is not based purely on the sign of the torque. The sign on shear stress is discussed in Chapter 12.

Example 6.2 demonstrates the use of the above equations to determine shear stress in the same circular bar assembly used in Example 6.1.

Example 6.2

The multisection solid steel bar of Example 6.1 is repeated here with the same loading (T_B = 70 N·mm and T_E = 30 N·mm).

Also given the diameters d_{AC} = 50 mm, d_{CD} = 20 mm, and d_{DE} = 35 mm, determine the magnitude and location of the maximum shear stress in the bar assembly.



Solution

Equation 6.5 shows that the maximum shear stress depends on the internal torque and the shaft's geometrical properties. So, to find the maximum shear stress, we must calculate the shear stress at the outer radius for each distinct section of the shaft for which the internal torque and/or cross-sectional dimensions change.

Recall from the previous example that all points between A and B have constant internal torque T_{AB} = 40 N·mm and all points between B and E have constant internal torque T_{BE} = 30 N·mm. This example has four distinct sections for which we need to calculate shear stress:

1. Section AB with internal torque T_{AB} and diameter d_{AC}

2. Section BC with internal torque $T_{BC} = T_{BE}$ and diameter d_{AC}
3. Section CD with internal torque $T_{CD} = T_{BE}$ and diameter d_{CD}
4. Section DE with internal torque $T_{DE} = T_{BE}$ and diameter d_{DE}

Recall that only the magnitude of shear stress will be calculated (sign is discussed in Chapter 12).

Because the bar has a solid cross-section with the diameters given, we use $\mathrm{J}_{\text{solid}} = \frac{\pi}{32}\mathrm{d}^4$ to calculate the polar moment of inertia at each section.

Now we can calculate the maximum stress in each section.

$$\tau_{AB} = \frac{T_{AB} * c_{AB}}{J_{AB}} = \frac{0.04\ N{\cdot}m * \frac{\frac{0.05}{2}\ m}{}}{\frac{\pi}{32}(0.05\ m)^4} = 1,629.7\ Pa = 1.630\ kPa$$

$$\tau_{BC} = \frac{T_{BC} * c_{BC}}{J_{BC}} = \frac{0.03\ N{\cdot}m * \frac{\frac{0.05}{2}\ m}{}}{\frac{\pi}{32}(0.05\ m)^4} = 1,222.3\ Pa = 1.222\ kPa$$

$$\tau_{CD} = \frac{T_{CD} * c_{CD}}{J_{CD}} = \frac{0.03\ N{\cdot}m * \frac{\frac{0.02}{2}\ m}{}}{\frac{\pi}{32}(0.02\ m)^4} = 19,098\ Pa = 19.10\ kPa$$

$$\tau_{DE} = \frac{T_{DE} * c_{DE}}{J_{DE}} = \frac{0.03\ N{\cdot}m * \frac{\frac{0.035}{2}\ m}{}}{\frac{\pi}{32}(0.035\ m)^4} = 3,563.6\ Pa = 3.564\ kPa$$

Answer:

Maximum shear stress in the bar is 19.1 kPa and occurs in section CD.

Step-by-Step: Torsional Stress

1. Use equilibrium to determine reaction torques at any supports (if necessary).
2. Cut a section through the member at a point of interest and draw an FBD of everything either to the left or to the right of the cut.
3. Using the FBD from the previous step, write the moment equilibrium equation around the longitudinal axis to determine the internal torque in the chosen section.
4. Determine the torsional stress using $\tau = \frac{\mathrm{T}\rho}{\mathrm{J}}$.
5. Repeat steps 2 through 4 as necessary for other points of interest.

6.4 Calculation of Angle of Twist

The deformation of a shaft that arises from applying torsion—assuming deformation is restricted to the elastic deformation—is quantified by the angle of twist, ϕ. As discussed in Section 6.3 and illustrated here again in Figure 6.12, the angle of twist is the angle formed between the radial lines that correspond to the original location of a point on the cross-section and the location it moves to as a result of applied torque. The equation to determine angle of twist at various points along the shaft follows. If you wish to skip the derivation, go directly to Equation 6.2.

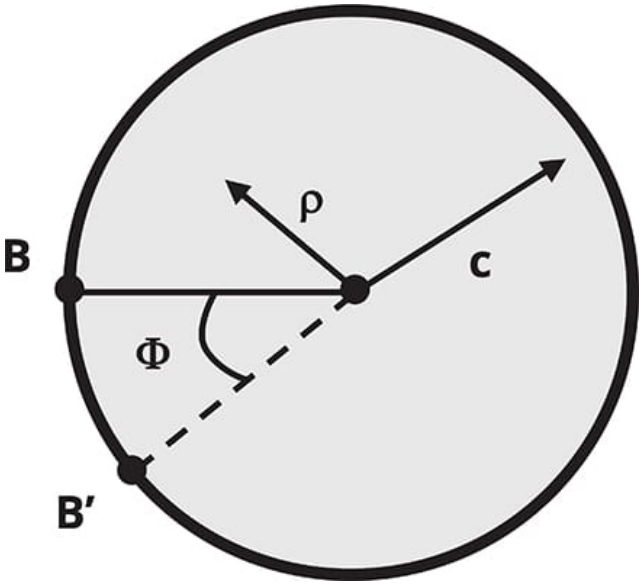


Figure 6.12: The angle between B (original location) and B' (where B moves to after torque is applied to the shaft) is the angle of twist, ϕ.

Recall from Section 6.3 that the shear strain can be expressed as $\gamma = \frac{\rho(\Delta\phi)}{\Delta x}$. Also recall that Dx represents the length of a small section of shaft and $\triangle\phi$ is the change in angle of twist that occurs across that length. For an infinitely small section of shaft, the shear strain equation can be expressed in terms of differentials: $\gamma = \frac{\rho(\mathrm{d}\phi)}{\mathrm{dx}}$.

Applying Hooke's law ($\tau = \mathrm{G}\gamma$) and the equation for shear stress (Equation 6.1) yields

$$\frac{\rho(d\phi)}{dx} = \frac{\tau}{G} = \frac{T\rho}{JG}.$$

Solving for $\mathrm{d}\phi$ yields

$$d\phi = \frac{Tdx}{JG}.$$

Integrating both sides of the equation gives the total angle of twist for an entire shaft or section of shaft.

$$\phi = \int \frac{Tdx}{JG}.$$

For a section (i) of shaft of length L_i for which torque (T_i), material (G_i), and geometry (J_i) stay constant, this becomes

$$\phi_i = \frac{T_i L_i}{J_i G_i}.$$

For multisection bodies where each individual section has constant T, J, and G, the integral can be expressed in the form of a sum where the angle of twist of each distinct section is calculated individually and then all are added together.

$$\boxed{\phi = \phi_i = \sum \frac{T_i L_i}{J_i G_i}},$$

(6.2)

where

- ϕ = *Change in angle [rad]*
- *T = Internal torque [N·m, lb·in.]*

- *L = Length [m, in.]*
- *J = Polar moment of inertia [m^4, $in.^4$]*
- *G = Shear modulus [Pa, psi]*

In these calculations the sign of internal torque is important as it indicates the direction of the twist. If a section twists in one direction and the adjoining section twists in the opposite direction, the angles will subtract from each other, resulting in a smaller overall angle of twist. If, however, two adjoining sections twist in the same direction, the effect will be additive. According to the assumed sign convention discussed in Section 6.2, an overall positive angle of twist indicates that the shaft twists counterclockwise when looking from the positive end of the longitudinal axis to the negative and clockwise when looking from the negative to the positive end.

6.4.1 Units

If values for T, L, J, and G are used in Equation 6.2 such that the units are all consistent with each other, the overall unit works out to be unitless, as is expected for an angle calculation. Since the calculation is for an angle, the actual unit is in radians. To obtain an answer in degrees, convert the result of the equation by multiplying it by $\frac{180^\circ}{\pi}$.

Because the angle of twist often works out to be very small, you may want to express the value in terms of μrad, which is 10^{-6} radians. To convert from rad to μrad, multiply the number of radians by 10^6.

6.4.2 Notation and Relative Angle of Twist

When finding the angle of twist, we encounter the angle at any given point along the length of the shaft relative to some reference point. Often the reference point is a fixed point (e.g., at a fixed support), so the relative angle is an absolute angle. In this case, the symbol ϕ is subscripted with the point along the shaft where the angle was calculated. For example, given the shaft assembly in Figure 6.13, the notation ϕ_B denotes the angle of twist at point B relative to the fixed point A.

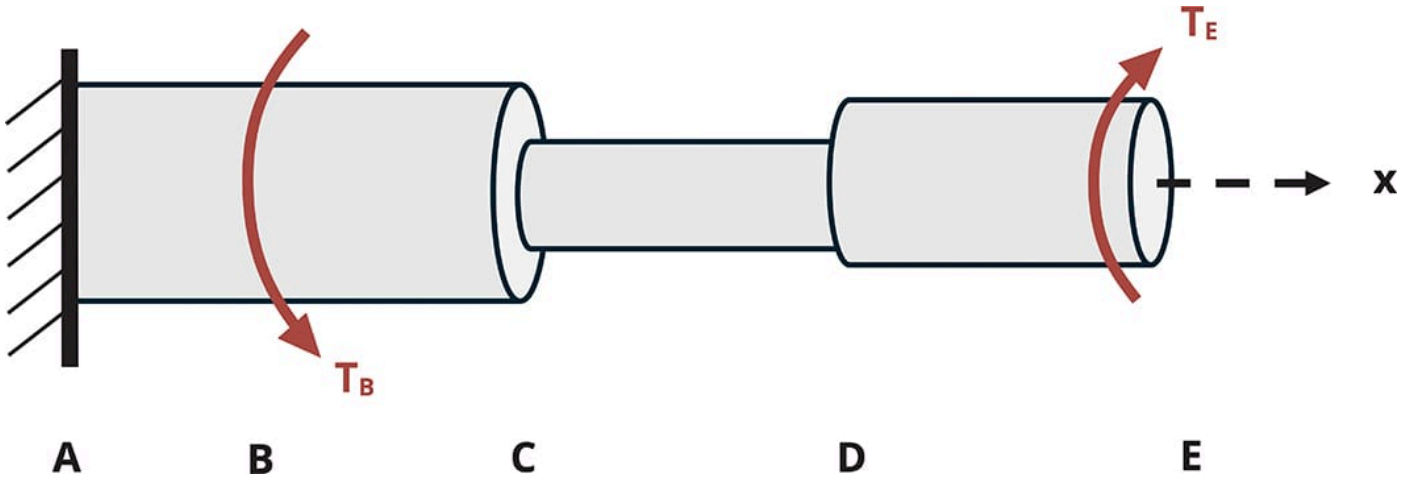


Figure 6.13: A multisection body with fixed end A and constant torque in each section

When finding the relative angle of twist between two nonfixed points on the same shaft section (same T, J, and G), we notate the angle of one point relative to another on the same section by using two subscripts specifying the two points. For example, the angle of twist of D relative to C in Figure 6.13 is notated as ϕ_{CD}. In this case we perform the calculation for the angle of twist at D as if point C is fixed.

$$\phi_{CD} = \frac{T_{CD}L_{CD}}{J_{CD}G_{CD}}$$

To find the relative angle of twist between two nonfixed points (let's say B and D in Figure 6.13) on different shaft sections (i.e.,

one or more sections change between them due to changing T, J, or G), sum the angles of twist of each section between them. In this case, the notation would be $\phi_{D/B}$, which is the angle of twist of D relative to B. If point C is where the section change occurs, the calculation would be

$$\phi_{D/B} = \phi_{BC} + \phi_{CD}.$$

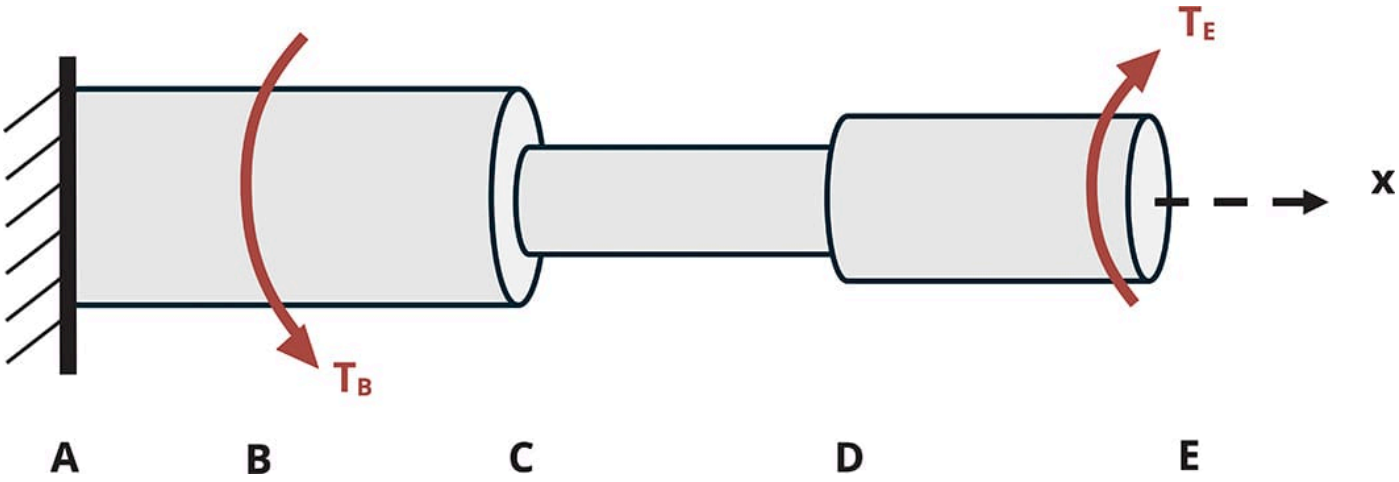
These concepts are illustrated in Example 6.3, which continues with the same bar assembly and loading used in Example 6.1 and Example 6.2.

Example 6.3

The multisection steel bar of Example 6.1 and Example 6.2 is repeated here with the same loading (T_B = 70 N·mm and T_E= 30 N·mm). The diameters are given as d_{AC} = 50 mm, d_{CD} = 20 mm, and d_{DE} = 35 mm, and section lengths as L_{AB} = 300 mm, L_{BC} = 200 mm, L_{CD} = 400 mm, and L_{DE} = 700 mm. All sections have a shear modulus of G = 40 GPa.

Determine the angle of twist of points C, D, and E, all relative to fixed point A.

A negative result indicates a clockwise twist, and a positive result indicates a counterclockwise twist.



Solution

As was the case when calculating shear stress in Example 6.2, the changes in torque and geometry result in four distinct sections for which we need to calculate angle of twist: AB, BC, CD, and DE. Example 6.1 showed that the internal torque in section AB is counterclockwise and equal to 40 N·mm. It also showed that the internal torque in all sections between B and E is clockwise and equal to 30 N·mm. The directions in this case are from the positive end of the x-axis to the negative (i.e., point E to point A). This means the angle of twist in section AB is positive while the angles of twist in BC, CD, and DE are negative.

Since point A is a fixed point, $\phi_{B/A} = \phi_B = \phi_{AB}$.

$$\phi_B = \phi_{AB} = \frac{T_{AB}L_{AB}}{J_{AB}G_{AB}} = \frac{0.04\ N \cdot m * (0.300\ m)}{\frac{\pi}{32}(0.05\ m)^4\left(40 \times 10^9\ \frac{N}{m^2}\right)} = 4.889 \times 10^{-7}\ rad$$

To find the angle of twist of point C relative to A, $\phi_{C/A}$, add the angle of twist of section BC to that of section AB.

$$\begin{aligned}\phi_{C/A} &= \phi_C = \phi_{AB} + \phi_{BC} = 4.889 \times 10^{-7}\ rad + \frac{T_{BC}L_{BC}}{J_{BC}G_{BC}} \\ &= 4.889 \times 10^{-7}\ rad + \frac{-0.03\ N{\cdot}m{*}(0.200\ m)}{\frac{\pi}{32}(0.05\ m)^4(40{\times}10^9\ \frac{N}{m^2})} \\ &= 2.445 \times 10^{-7}\ rad\end{aligned}$$

Note that T_{BC} was substituted as a negative value because it is clockwise.

To find the angle of twist of point D relative to A, $\phi_{D/A}$, the angle of twist of section CD is added to that of AB and BC.

$$\begin{aligned}\phi_{D/A} &= \phi_D = \phi_{AB} + \phi_{BC} + \phi_{CD} = 2.445 \times 10^{-7}\ rad + \frac{T_{CD}L_{CD}}{J_{CD}G_{CD}} \\ &= 2.445 \times 10^{-7}\ rad + \frac{-0.03\ N{\cdot}m{*}(0.400\ m)}{\frac{\pi}{32}(0.02\ m)^4(40{\times}10^9\ \frac{N}{m^2})} \\ &= -1.885 \times 10^{-5}\ rad\end{aligned}$$

Finally, to find the angle of twist of E relative to A, $\phi_{\mathrm{E/A}}$, we add the angle of twist of section DE to that of AB, BC, and CD.

$$\begin{aligned}\phi_{E/A} &= \phi_E = \phi_{AB} + \phi_{BC} + \phi_{CD} + \phi_{DE} = -1.885 \times 10^{-5}\ rad + \frac{T_{DE}L_{DE}}{J_{DE}G_{DE}} \\ &= -1.885 \times 10^{-5}\ rad + \frac{-0.03\ N{\cdot}m{*}(0.700\ m)}{\frac{\pi}{32}(0.035\ m)^4(40{\times}10^9\ \frac{N}{m^2})} \\ &= -2.242 \times 10^{-5}\ rad\end{aligned}$$

Answer:

- $\phi_{\mathrm{B}} = 4.89 \times 10^{-7}\mathrm{rad} = 0.489$ μrad (counterclockwise)
- $\phi_{\mathrm{C}} = 2.45 \times 10^{-7}\mathrm{rad} = 0.245$ μrad (counterclockwise)
- $\phi_{\mathrm{D}} = 1.89 \times 10^{-5}\mathrm{rad} = 18.9$ μrad (clockwise)
- $\phi_{\mathrm{E}} = 2.24 \times 10^{-5}\mathrm{rad} = 22.4$ μrad (clockwise**)**

Step-by-Step: Angle of Twist

1. Use equilibrium to determine reaction torques at any supports (if necessary).
2. Cut a section through the body at the point of interest and draw an FBD of everything either to the left or to the right of the cut.
3. Using the FBD from the previous step, write the moment equilibrium equation around the longitudinal axis to determine the internal torque in the chosen section.
4. Determine the segment's angle of twist using $\phi = \frac{\mathrm{TL}}{\mathrm{JG}}$.
5. The total angle of twist between two points can be determined by summing the angle of twist for each segment between those points. Be careful with signs, as some segments may rotate clockwise, while others rotate counterclockwise.

6.5 Statically Indeterminate Torsion

As discussed in Section 5.5 for axial loading, we may sometimes encounter loading and constraint conditions that are statically indeterminate. In these situations, though a structure may be in static equilibrium, equilibrium equations do not provide enough equations to solve for all the unknown reactions. In many cases, the known constraints on deformation can be used to provide the additional necessary equations. Let's consider the same two categories of statically indeterminate loading:

1. **Redundant supports**: These are cases where each end has a fixed support. In this configuration, the shaft is constrained on both ends such that the overall angle of twist must be 0 if both ends are truly fixed. As a variation, we could also consider the case where the angle of twist is limited to a specified amount if there is a fixed degree of freedom on one end (for example if one side involves a loose connection).

   Example 6.4 demonstrates how to solve a statically indeterminate torsion problem that falls into this category. The example considers both the case where the total angle of twist is 0 and the case where there is a fixed degree of freedom.

   Example 6.5 re-examines part (a) of Example 6.4 to demonstrate a different method of solving the problem. This method is known as the method of superposition and is outlined below.

2. **Coaxial or parallel shafts**: Multiple shafts are bonded concentrically (one within another) so that they are constrained to twist as a unit by the same amount. In this case, the extra equation needed in addition to equilibrium comes from setting each shaft's angle of twist equal to the other's.

Example 6.6 demonstrates how to solve a statically indeterminate torsion problem that falls into this category of loading.

6.5.1 Method of Superposition

For the case of redundant supports, an alternative is to use the method of superposition. The method of superposition is based on the idea that the effects of individual loads on the deformation of a body can be calculated individually and then added together to obtain the total effect. To apply this method do the following:

1. Draw the FBD of the whole body and write out the relevant equilibrium equations.
2. Choose one of the supports to consider redundant, remove the corresponding reaction torque, and find the total angle of twist from the remaining loads.
3. Put the reaction torque from the redundant support back, remove the applied loads, and then find the total angle of twist due to the replaced reaction torque in terms of the corresponding variable.
4. Sum the angle of twist from step 2 and step 4 and set the total equal to zero (or a given angle as appropriate).
5. Solve the equation resulting from step 4 for the reaction at the redundant support and then use the whole body equilibrium to solve for the reaction at the other support(s).

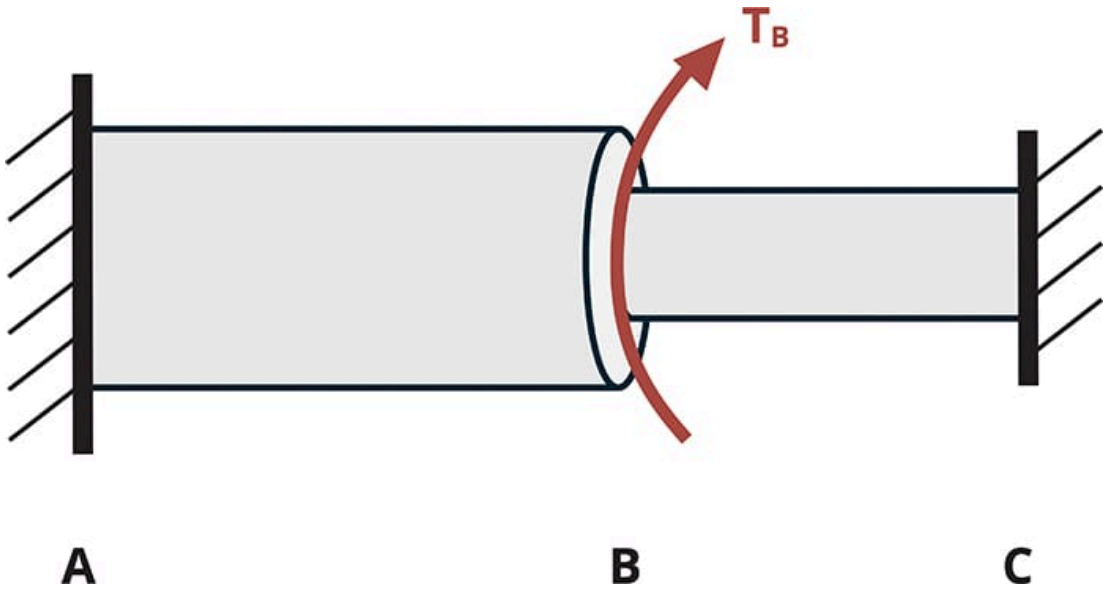
Example 6.5 demonstrates the use of superposition to solve part a of Example 6.4.

Example 6.4

A shaft assembly consists of a 12 in. long steel section AB (d = 10 in., G = 11,600 ksi) bonded rigidly to a 9 in. long solid brass section BC (d = 7 in., G = 5,800 ksi). The assembly is to be bolted to walls A and C.

If a torque T_B of magnitude 2,000 kip·in. is applied at B, determine the shear stress in the steel section and the brass section after the ends are bolted for each of the following end conditions:

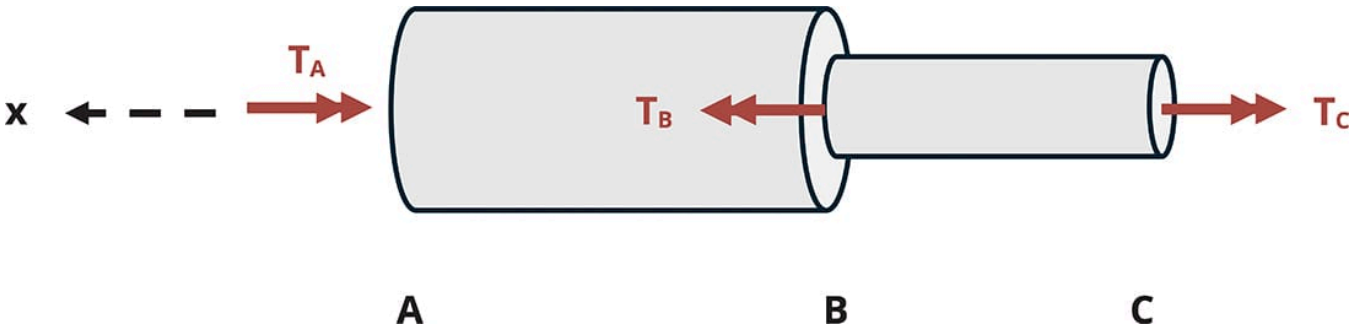
1. Both ends A and C are tightly connected immediately at installation.
2. A slight misalignment exists at end A that necessitates turning that end by 0.5° to lock in.



Solution

To calculate stresses, we need to know the internal torque in each section. To find the internal torque, we need to know at least one of the reaction torques exerted by the walls at A and C. The steps below can be used to find the reaction torques.

- **Step 1: Draw the FBD of the whole body and write the relevant equilibrium equation(s).**

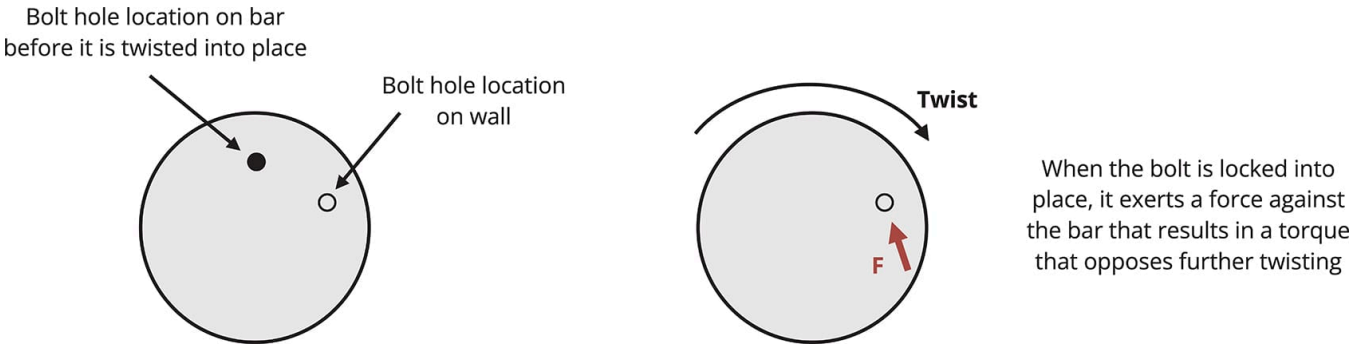


In this case the only helpful equilibrium equation is $\sum \mathrm{M_x} = 0.$

Note that the assumed directions for T_A and T_C (the reaction torques exerted by the bolts) would usually be completely assumed, with the signs of the answers indicating if the assumed directions are correct. For case (a), this is still true.

For case (b) we must be more careful. The direction of the 0.5° turn is not specified, but it must be in the direction of T_B, since that is the only applied load. The direction of T_A, which is the reaction torque exerted by the bolt at A, should be in the direction opposite to the way the shaft turns since the bolt is there to fix the assembly in place.

The assumed direction of T_C can be in either direction for both case (a) and case (b).



The positive x-axis goes from right to left in the whole body FBD. As long as the choice of signs applied to torques for any whole body equation is based on the same assumption, it does not matter which direction is chosen to be the positive x direction. In this case, the signs on the torques will be based on the way the torques are perceived when looking in the direction from A towards C.

Since the direction of T_B is counterclockwise from that perspective, it is drawn in the positive x direction. T_A is drawn in the negative x direction. T_C is arbitrarily assumed to be the same direction as T_A. Write the moment equilibrium equation.

$$\sum M_x = -T_A + T_B - T_C = 0$$
$$T_C + T_A = T_B$$

Because there are two unknowns (T_A and T_C) but only one equilibrium equation, the problem is statically indeterminate.

- **Step 2: Apply kinematic constraint.**

Applying the kinematic constraint will provide the second equation needed to solve for T_A and T_C. For case (a), both ends are fixed at the time of installation so the total angle of twist should be zero. Thus

$$\phi_A = \sum \frac{TL}{JG} = \phi_{AB} + \phi_{BC} = 0.$$

For case (b), the misalignment means end A must be turned by 0.5° before it gets locked in, so the magnitude of the total angle of twist is also 0.5°. In step 1, it was discussed that the twist will be in the same direction as T_B, which is

counterclockwise looking from A toward C.

So for case (b), the kinematic equation is

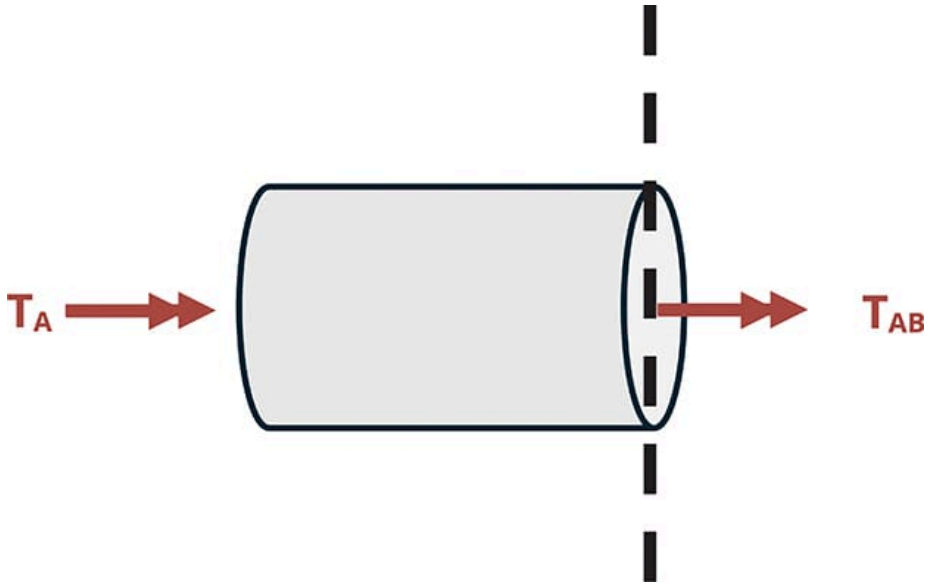
$$\phi_A = \sum \frac{TL}{JG} = \phi_{AB} + \phi_{BC} = 0.5^\circ * \frac{\pi}{180^\circ}.$$

- **Step 3: Write the compatibility equation.**

  Write the compatibility equation by substituting the appropriate torque-twist equations for the angles of twist for the angles of twist in the kinematic constraint equation.

  To implement this step, establish the internal torque (and sign) for each section by cutting sections and applying equilibrium.

  The FBD for a section made by cutting through section AB is as shown.



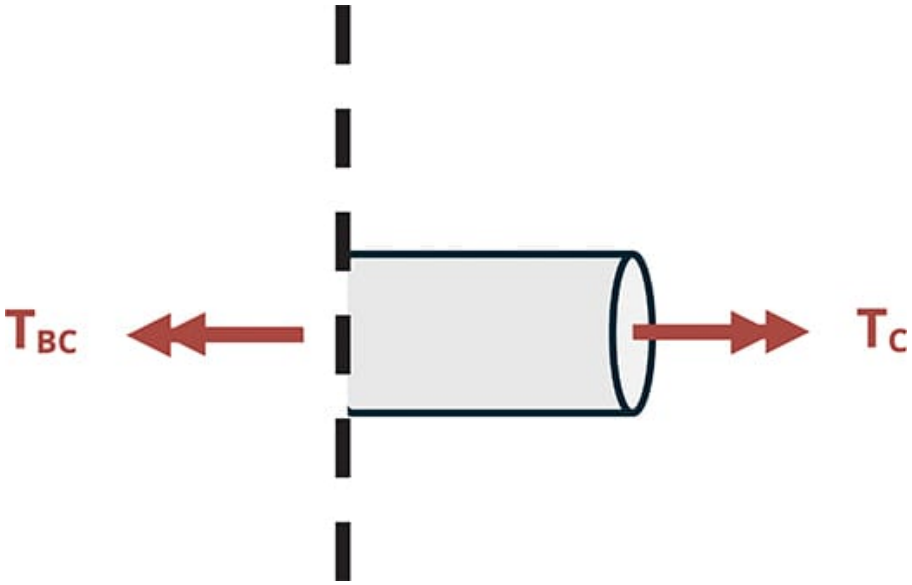
The equilibrium equation for this section is

$$\sum M_x = T_A + T_{AB} = 0$$
$$T_{AB} = -T_A.$$

Thus, the angle of twist in section A_B is

$$\phi_{AB} = -\frac{T_{AB}L_{AB}}{J_{AB}G_{AB}} = \frac{-T_A * (12\,in.)}{\frac{\pi}{32}(10 * in.)^4 * (11,600\,ksi)}.$$

The FBD for a section made by cutting through section BC is as shown.



The equilibrium equation for this section is

$$\sum M_x = -T_{BC} + T_C = 0$$
$$T_{BC} = T_C.$$

The angle of twist in section BC is

$$\phi_{BC} = +\frac{T_{BC} L_{BC}}{J_{BC} G_{BC}} = \frac{T_C * (9\ in.)}{\frac{\pi}{32}(7\ in.)^4 * (5,800\ ksi)}.$$

Substituting the expressions for ϕ_{AB} and ϕ_{BC} into the kinematic constraint equation for case (a) gives the compatibility equation

$$\frac{-T_A * (12\ in.)}{\frac{\pi}{32}(10\ in.)^4 * (11,600\ ksi)} + \frac{T_C * (9\ in.)}{\frac{\pi}{32}(7\ in.)^4 * (5,800\ ksi)} = 0.$$

Case (b) gives the compatibility equation

$$\frac{-T_A * (12\ in.)}{\frac{\pi}{32}(10\ in.)^4 * (11,600\ ksi)} + \frac{T_C * (9\ in.)}{\frac{\pi}{32}(7\ in.)^4 * (5,800\ ksi)} = 0.5^\circ * \frac{\pi}{180^\circ}.$$

- **Step 4: Solve the equilibrium and compatibility equations** simultaneously for T_A and T_C.
  - Case (a):

$$T_A = T_C \left[\frac{9\ in. * \frac{\pi}{32}(10\ in.)^4 * (11,600\ ksi)}{12\ in. * \frac{\pi}{32}(7\ in.)^4 * (5,800\ ksi)} \right]$$
$$T_A = T_C[6.247]$$
$$\text{Substitute into } T_A + T_C = 2,000\ kip \cdot in.$$
$$6.247 T_C + T_C = 2,000\ kip \cdot in.$$

$$T_C = 276\ kip \cdot in.$$
$$T_A = 1,724\ kip \cdot in.$$

  - Case (b):

$$T_A = \left(\frac{T_C * (9\ in.)}{\frac{\pi}{32}(7\ in.)^4 * (5{,}800\ ksi)} - 0.5^\circ * \frac{\pi}{180^\circ} \right) * \left(\frac{\frac{\pi}{32}(10\ in.)^4 * 11{,}600\ ksi}{12\ in.} \right)$$
$$T_A = T_C\,[6.247] - 8,282\ kip \cdot in.$$
$$\text{Substitute into } T_A + T_C = 2,000\ kip \cdot in.$$
$$6.247T_C + T_C - 8,282\ kip \cdot in. = 2,000\ kip \cdot in.$$
$$T_C = 1,419\ kip \cdot in.$$
$$T_A = 581\ kip \cdot in.$$

The results are positive for both cases, so the assumed directions are the actual directions.

- **Step 5: Calculate stresses.**

  Now that the reaction torques are known and it was established in step 3 that $T_{AB} = T_A$ and $T_{BC} = T_C$ in magnitude, the shear stresses in the two sections can be calculated using Equation 6.1.

  - Case (a):

$$\tau_{AB} = \frac{T_{AB} * c_{AB}}{J_{AB}} = \frac{1{,}724\ kip{\cdot}in. * 5\ in.}{\frac{\pi}{32}(10\ in.)^4} = 8.78\ ksi$$
$$\tau_{BC} = \frac{T_{BC} * c_{BC}}{J_{BC}} = \frac{276\ kip{\cdot}in. * 3.5\ in.}{\frac{\pi}{32}(7\ in.)^4} = 4.10\ ksi.$$

  - Case (b):

$$\tau_{AB} = \frac{T_{AB} * c_{AB}}{J_{AB}} = \frac{581\ kip{\cdot}in. * 5\ in.}{\frac{\pi}{32}(10\ in.)^4} = 2.96\ ksi$$
$$\tau_{BC} = \frac{T_{BC} * c_{BC}}{J_{BC}} = \frac{1{,}419\ kip{\cdot}in. * 3.5\ in.}{\frac{\pi}{32}(7\ in.)^4} = 21.1\ ksi.$$

Answer:

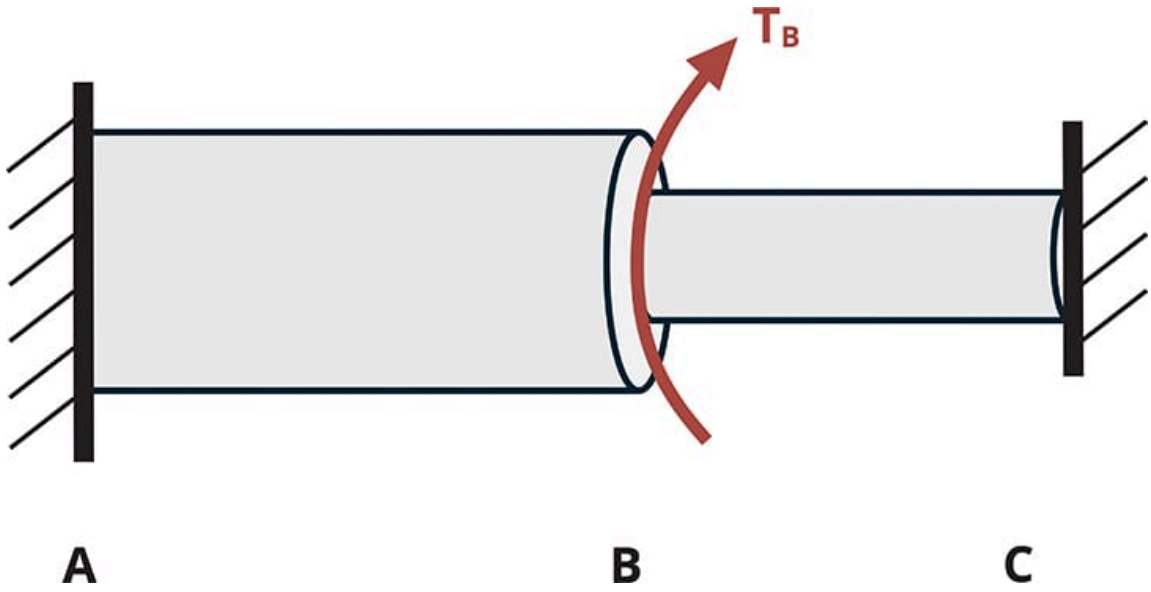
- Case (a): $\tau_{\mathrm{AB}} = 8.78$ ksi, $\tau_{\mathrm{BC}} = 4.10$ ksi
- Case (b): $\tau_{\mathrm{AB}} = 2.96$ ksi, $\tau_{\mathrm{BC}} = 21.1$ ksi

Example 6.5

Solve part (a) of Example 6.4 using the method of superposition. The problem is repeated here for convenience.

A shaft assembly consisting of a 12 in. long steel section AB (d = 10 in., G = 11,600 ksi) is bonded rigidly to a 9 in. long solid brass section BC (d = 7 in., G = 5,800 ksi). The assembly is bolted to walls A and C.

If both ends A and C are tightly connected to the walls and a torque T_B of magnitude 2,000 kip·in. is applied at B, determine the shear stress in the steel section and the brass section.

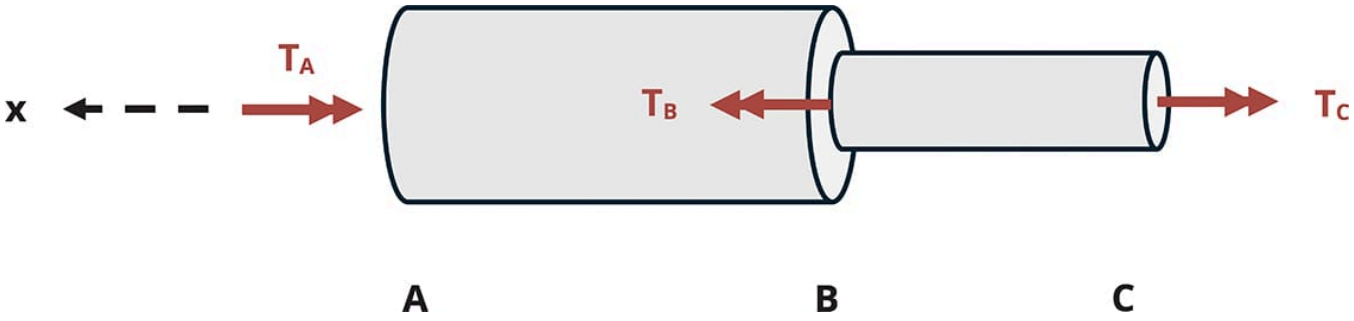


Solution

The first step to solving (global equilibrium) remains the same. The method of superposition is applied in step 2 and step 3.

- **Step 1: Draw the FBD of the whole body and write out the relevant equilibrium equations.**

  Draw the whole body FBD and write out the relevant equilibrium equations. Shown on the FBD are the assumed directions for T_A and T_C as well as the direction used for the positive x direction. Recall that the direction of the torque should be determined by looking from the positive x-axis toward the negative (A to C).

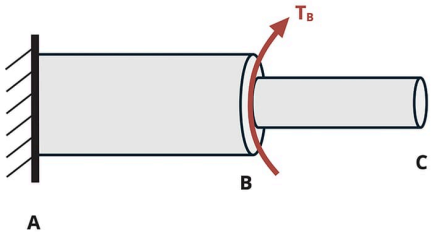


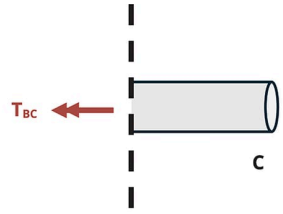
  Applying equilibrium in this case reduces to the sum of the moments about the x-axis as the only nontrivial equation.

$$\sum M_x = -T_A + T_B - T_C = 0$$
$$T_C + T_A = T_B$$

  Because there are two unknowns (T_A and T_C) but only one equilibrium equation, the problem is statically indeterminate.

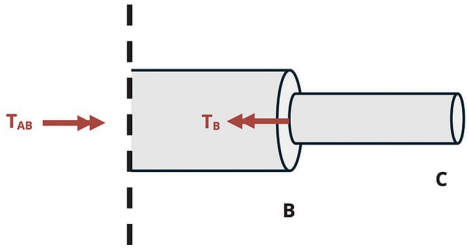
- **Step 2: Choose one of the supports to consider redundant, remove the corresponding reaction torque, and then find the total angle of twist from the remaining loads.**





$$\sum M_x = -T_{BC} + 0 = 0$$

$$T_{BC} = 0$$



$$\sum M_x = T_{AB} - T_B = 0$$

$$T_{AB} = T_B$$

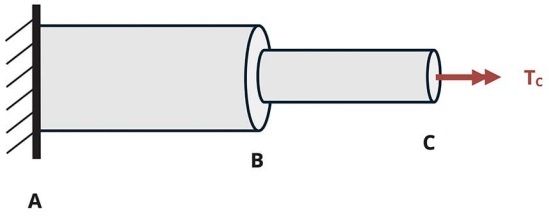
Let's consider the support at C to be redundant. Note that the support at A could alternatively be considered redundant with the same results.

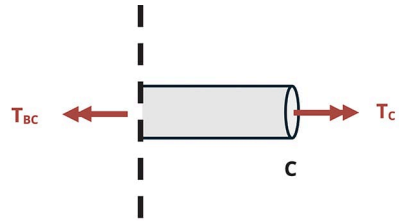
With no support at C, a section cut within BC shows that the internal torque T_{BC} must be 0 for equilibrium.

A section cut through AB shows that the internal torque $T_{AB} = T_B$ with the direction of T_{AB} opposite of T_B. Viewed from the cut towards C, T_B is counterclockwise, so T_{AB} is clockwise. Therefore, the angle of twist is negative and equal to

$$\phi_{C/A} = \sum \frac{TL}{JG} = \frac{T_{AB}L_{AB}}{J_{AB}G_{AB}} = \frac{-(2,000\ kip \cdot in.) * (12\ in.)}{\frac{\pi}{32}(10in.)^4 * (11,600\ ksi)} = -2.107 \times 10^{-3}\ rad.$$

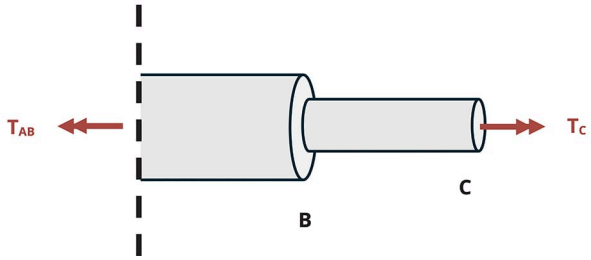
- **Step 3: Put the reaction torque from the redundant support back, remove the applied loads, and then find the total angle of twist due to the replaced support torque in terms of the corresponding variable.**





$$\sum M_z = -T_{BC} + T_C = 0$$

$$T_{BC} = T_C$$



The section FBDs in this case reveals that both T_{AB} and T_{BC} are equal in magnitude and opposite in direction from T_C. Since both are pointed away from the cut, the direction is taken to be counterclockwise, or positive. So the angle of twist with this loading is

$$\begin{aligned}\phi_{C/A} &= \sum \frac{TL}{JG} = \frac{T_{AB}L_{AB}}{J_{AB}G_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}G_{BC}} \\ &= \frac{T_C*(12\ in.)}{\frac{\pi}{32}(10\ in.)^4*(11{,}600\ ksi)} + \frac{T_C*(9\ in.)}{\frac{\pi}{32}(7\ in.)^4*(5{,}800\ ksi)} \\ &= 7.637 \times 10^{-6}\ T_C.\end{aligned}$$

- **Step 4: Set the sum of the angles of twist equal to 0 (or other fixed angle when applicable) and solve for the unknown reaction torque.**

  The total angle of twist should be equal to zero when the ends are tightly bolted to the walls without extra twisting.

  $$2.107 \times 10^{-3}\ rad + 7.637 \times 10^{-6}\ Tc = 0$$

  Solving for T_C gives T_C = 276 kip·in which is the same result found in Example 6.4.

- **Step 5: Use the whole body equilibrium equation from step 1 to solve for the other reaction torque.**

  Substituting the above result for T_C into the whole body equilibrium equation from step 1 gives T_A = 1,724 kip in as was found in Example 6.4. Once again the corresponding shear stresses are

  $$\tau_{AB} = \frac{T_{AB}*c_{AB}}{J_{AB}} = \frac{1{,}724\ kip{\cdot}in.*5\ in.}{\frac{\pi}{32}(10\ in.)^4} = 8.78\ ksi$$
  $$\tau_{BC} = \frac{T_{BC}*c_{BC}}{J_{BC}} = \frac{276\ kip{\cdot}in.*3.5\ in.}{\frac{\pi}{32}(7\ in.)^4} = 4.10\ ksi.$$

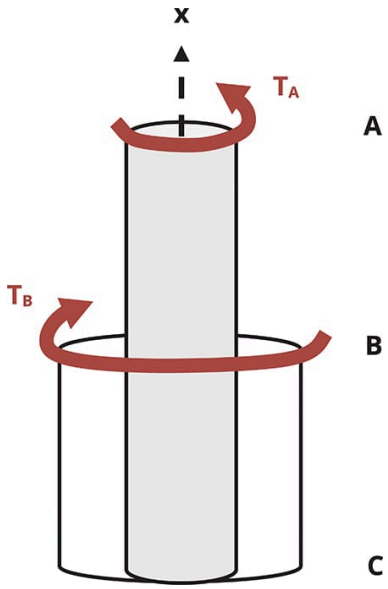
Answer:

- τ_{AB} = 8.78 ksi
- τ_{BC} = 4.10 ksi

Example 6.6

The vertical shaft assembly is fixed to the bottom surface and consists of a solid brass (G = 40 GPa) circular shaft completely bonded in section BC to a steel (G = 80 GPa) reinforcing tube. The diameter of the brass is 25 mm along the whole length, and the outer diameter of the steel is 50 mm. The assembly is subjected to torques T_B = 800 N·m and T_A = 300 N·m in the directions shown.

Determine the magnitude of the shear stress in the brass in sections AB and BC.

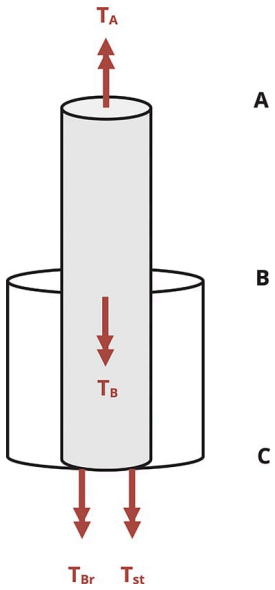


Solution

First we'll find the shear stress in the brass in section BC. To begin, we determine how much of the internal torque within that section is supported by the brass versus how much is supported by the steel.

- **Step 1: Draw the FBD of a section cut through BC to examine the internal torque in BC.**

  Since the reaction at C is not strictly necessary to solve this problem, the whole body FBD is not needed here. Start by cutting a section through BC and drawing the FBD. Note that instead of representing the internal reaction at the cut as T_{BC}, it is split into the components that make up T_{BC}: T_{Br} and T_{St}.



  With only one nontrivial equilibrium equation and two unknowns, the problem is statically indeterminate.

$$\sum M_x = T_A - T_B - T_{Br} - T_{St} = 0$$

- **Step 2: Apply the kinematic constraint to obtain a second equation.**

  In this case, the materials are fully bonded in section BC, so the materials are constrained to twist as a unit in that section. In BC, the kinematic equation is

$$\phi_{Br} = \phi_{st}.$$

- **Step 3: Substitute the torque-twist equation into the kinematic equation to get the compatibility equation.**

  Substituting Equation 6.2 into the kinematic constraint equation gives us a second equation that relates T_{Br} and T_{St}.

$$\frac{T_{Br}L_{Br}}{J_{Br}G_{Br}} = \frac{T_{St}L_{St}}{J_{St}G_{St}}$$

  The lengths are equal and therefore cancel each other. Rearrange this equation to get an expression for T_{St} in terms of T_{Br}.

$$\begin{aligned} T_{St} &= \tfrac{T_{Br}J_{St}G_{St}}{J_{Br}G_{Br}} \\ &= T_{Br}\left[\frac{\frac{\pi}{32}\left((0.05\ m)^4-(0.025\ m)^4\right)\left(80\times10^9\ \frac{N}{m^2}\right)}{\frac{\pi}{32}(0.025\ m)^4\left(40\times10^9\ \frac{N}{m^2}\right)}\right] \\ &= 30.0\ T_{Br} \end{aligned}$$

- **Step 4: Use the equilibrium equation and the simplified compatibility relationship to solve for T_{Br}.**

  Substite T_{st} = 30 T_{Br} into the equilibrium equation.

$$300\ N \cdot m - 800\ N \cdot m - T_{Br} - 30\ T_{Br} = 0$$
$$T_{Br} = -16.1\ N \cdot m$$

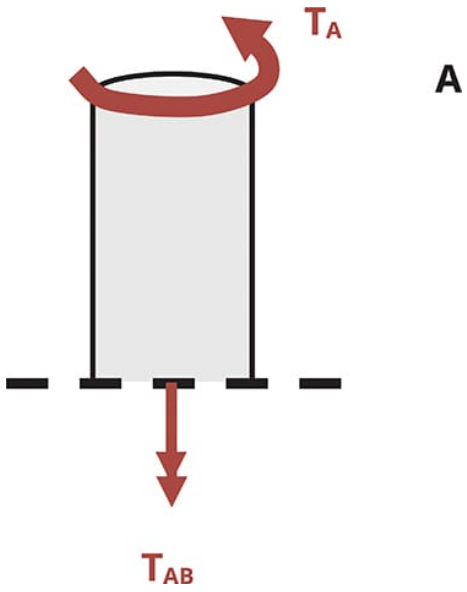
- **Step 5: Apply the torque-shear stress equation to find the shear stress in the brass for sections AB and BC.**

  Note that because only the magnitude of the stress is requested, we substitute torque values in as positive values.

  Applying Equation 6.1 in section BC yields

$$\tau_{Br(BC)} = \frac{T_{Br} * c_{Br}}{J_{Br}} = \frac{16.1\ N \cdot m * (0.0125\ m)}{\frac{\pi}{32}(0.025\ m)^4} = 5.26\ MPa.$$

  In section AB, which consists only of brass, all the internal torque is carried by the brass. Making a cut within that section and applying equilibrium shows that the magnitude of the internal torque $T_{AB} = T_A = 300$ N·m.



  Applying the shear stress equation yields

$$\tau_{Br(AB)} = \frac{T_{Br} * c_{Br}}{J_{Br}} = \frac{300\ N \cdot m * (0.0125\ m)}{\frac{\pi}{32}(0.025\ m)^4} = 97.8\ MPa.$$

Answer:

- τ_{Br} in section BC is 5.26 MPa
- τ_{Br} in section AB is 97.8 MPa

Step-by-Step: Statically Indeterminate Structures

1. Draw the whole body FBD and apply whichever equilibrium equations are relevant given the loading and problem geometry. For this chapter's pure torsion problems, only the moment equilibrium about the longitudinal axis is useful. All other equilibrium equations are identically 0.
2. Write the equation of kinematic constraint or the equation that expresses the restriction on the deformation. These equations would generally be
   - $\sum \phi = 0$ if a shaft assembly is tightly supported by fixed supports on the ends.
   - $\sum \phi = \text{angle}$ if shaft assembly is supported by fixed supports on both ends but one is loose, misaligned, and so on such that the end to twist has some degree of freedom.

- $\phi_1 = \phi_2$, where 1 and 2 represent two different shafts bonded together co-axially.

3. Write the compatibility equation by substituting the appropriate torque-twist relationship ($\phi = \frac{TL}{JG}$) into the kinematic constraint equation.

4. You should now have the same number of equations as unknowns. Solve the equations.

6.6 Power Transmission

As mentioned in the introduction to torsion, one common application where we encounter circular shafts subjected to torsion is in power transmission shafts. In these types of assemblies, shafts are connected and apply torque to one another through gears and belts. If a shaft somewhere in the assembly is connected to a motor that causes it to rotate, that rotation is then transmitted to other connected shafts, which can in turn transmit rotation to even more shafts. Note that such systems are considered to be in equilibrium even with the rotation as long as the angular velocity remains constant.

Below is a brief derivation of the equation that relates torque to power, followed by a discussion of how torque and angle of twist are affected by the gear or belt connections between shafts.

6.6.1 Derivation of Power-Torque Relationship

Recall from other subjects of study that power is defined to be the time rate of change of work. The definition of work is force times distance. So power can be calculated as

$$P = \frac{W}{t} = \frac{F \times d}{t}.$$

Within the context of rotating shafts, the force is applied to the shaft's outer edge resulting in torque, and the distance is the arc length of a point on the shaft's edge that travels around the circumference in a given amount of time, t. Figure 6.14 illustrates this point.

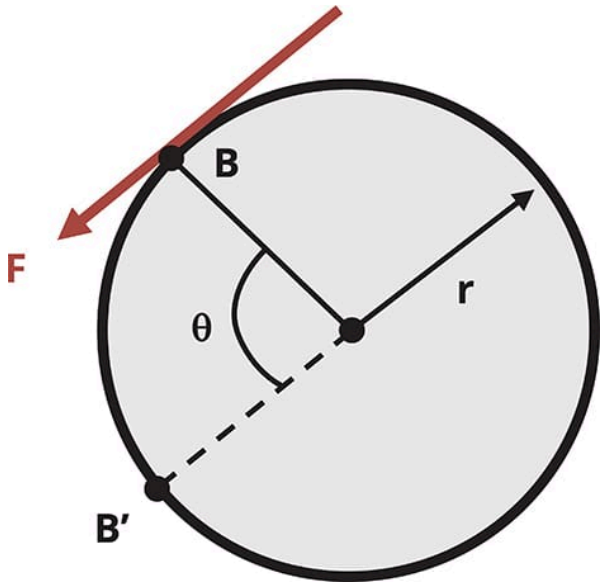


Figure 6.14: Force applied to rotation shaft results in work.

If the arc length over which point B travels is $\mathrm{BB}' = \mathrm{r}\theta \text{ then } \mathrm{W} = \mathrm{F}(\mathrm{r}\theta)$. Since torque is equal to (F)(r), we can see that W = Tθ. Substituting this expression for work into the power equation above, we then have $\mathrm{P} = \frac{\mathrm{T}\theta}{\mathrm{t}}.\mathrm{The}\ \frac{\theta}{\mathrm{t}}$ term is angular velocity, which is denoted here with the Greek letter ω. So the power equation can finally be expressed as

$$\boxed{P = T\omega},$$

(6.3)

where

- *P = Power [Watts, in.-lb/sec, horsepower]*
- *T = Torque [N·m, lb·in.]*
- *ω = Angular velocity [rad/s, rad/sec]*

6.6.2 Units

In SI the standard unit for power is the Watt (W), which is (N·m)/s. The N·m unit is also referred to as a Joule, so a Watt is a Joule/s.

In US customary units, power is commonly expressed as ft·lb/s or in.·lb/s. However, it is also common to express power in terms of horsepower, where 1 hp = 550 ft·lb/s. Note that there is a conversion difference between US horsepower and British horsepower. In this text, horsepower refers to the US version.

In applying Equation 6.3, the angular velocity w must be expressed in radians per second. Sometimes angular velocity is given in Hz, where 1 Hz = 1 revolution/s. In this case, one must use the fact that 1 revolution = 2p radians to convert from Hz to rad/s. If angular velocity is given in revolutions per minute (rpm), then we must multiply by 2π radians and also divide by 60 to convert 1/min to 1/s. This reduces to an overall multiplication factor of π /30 to convert from rpm to rad/s.

A summary of the power related units and conversions is provided in Table 6.1 and Table 6.2.

Table 6.1 Power units

SI (watt)	$\mathrm{W} = \frac{\mathrm{Nm}}{\mathrm{s}} = \frac{\mathrm{J}}{\mathrm{s}}$
US	$\left(\frac{\mathrm{ft\,lb}}{\mathrm{s}}\right) \text{ or } \left(\frac{\mathrm{in.\,lb}}{\mathrm{s}}\right)$
US (horsepower, hp)	$1\ \mathrm{hp} = \frac{550 \mathrm{ft\,lb}}{\mathrm{s}}$

Table 6.2 Angular velocity units

Hertz (Hz)	$1\ \mathrm{Hz} = 1 \frac{\mathrm{rev}}{\mathrm{s}}$
Angular velocity given Hz (rad/s)	$1\ \frac{\mathrm{rev}}{\mathrm{sec}} \left(2\pi \frac{\mathrm{rad}}{\mathrm{rev}}\right) = \frac{2\pi \frac{\mathrm{rad}}{\mathrm{s}}}{\mathrm{Hz}}$
Angular velocity given rpm (rad/s)	$1 \frac{\mathrm{rev}}{\mathrm{min}} \left(\frac{2\pi\ \mathrm{rad}}{1\ \mathrm{rev}}\right) \left(\frac{1\ \mathrm{min}}{60\ \mathrm{s}}\right) = \frac{\frac{\pi}{30} \frac{\mathrm{rad}}{\mathrm{s}}}{\mathrm{rpm}}$

6.6.3 Torque Transfer Between Connected Gears (Gear Ratio)

Consider two connected gears as shown in Figure 6.15. The interaction of the teeth results in a force applied to each gear from one gear to the other. For equilibrium, this force must be equal in magnitude but opposite in direction. The force results in a torque that can be expressed as T = Fr.

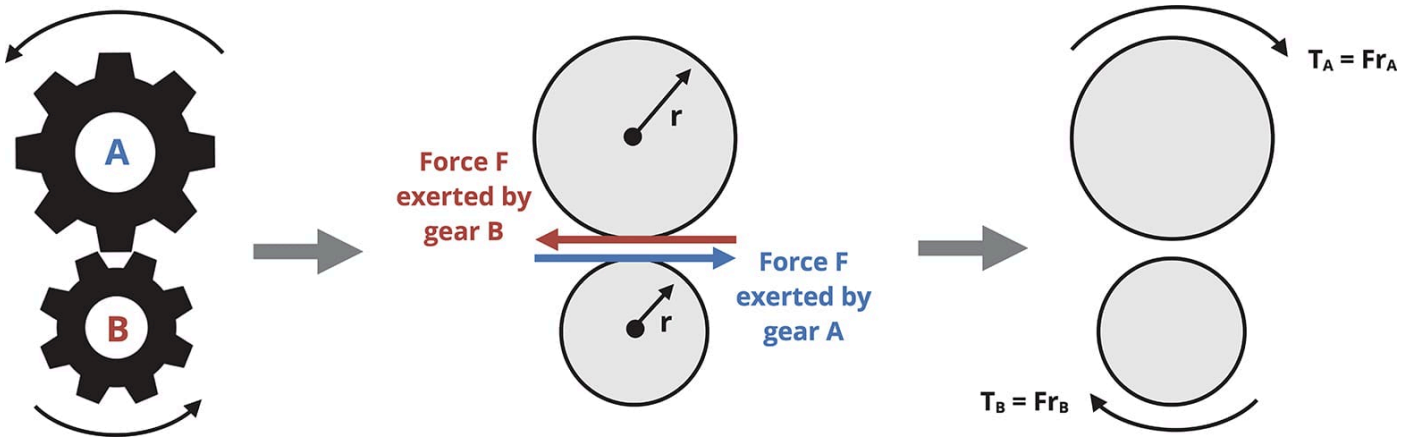


Figure 6.15: Gear interaction results in the application of force and torque.

Given that the force on each gear is equal in magnitude to the force on the other, we can write

$$F = \frac{T_A}{r_A} = \frac{T_B}{r_B},$$

which then allows us to express the torque exerted by one gear in terms of the torque exerted by the other as well as the ratio of radii.

$$\boxed{T_A = \frac{r_A}{r_B} T_B}$$

(6.4)

The ratio of radii is known as the gear ratio. It can be expressed as the ratio of gear diameters or the number of teeth on each gear.

While the forces that result in torque are opposite in direction for each gear of the connected pair, the torques themselves are in the same direction. In Figure 6.15, the rightwards force gear A exerts on gear B results in a clockwise torque on gear B. Likewise, the leftward force gear B exerts on gear A results in a clockwise torque on gear A.

6.6.4 Angular Velocity Ratio of Connected Shafts

When two shafts are connected by a pair of gears, conservation of energy dictates that the power transferred into one shaft equals the power transferring out of the other (power in = power out, assuming losses are negligible).

For two connected gears A and B, if $P_A = T_A\omega_A$ and $P_B = T_B\omega_B$ and $P_A = P_B$, then $\omega_A = \frac{T_B}{T_A}\omega_B$.

Substituting in Equation 6.4 gives us

$$\boxed{\omega_A = \frac{r_B}{r_A}\omega_B}.$$

(6.5)

Note that the gear ratio in this case is inverse of the one used to relate torques.

6.6.5 Angle of Twist

When two connected gears, each attached to a separate shaft, rotate they travel the same distance for a given amount of shaft rotation, or angle of twist. The distance of travel is equal to the arc length $r \cdot \phi$. Given that the gears exert forces in opposite directions (as illustrated in Figure 6.15), the direction of travel of each gear will also be opposite of the other, as depicted in

Figure 6.16. For connected gears A and B, attached to shafts A and B respectively this can be expressed as

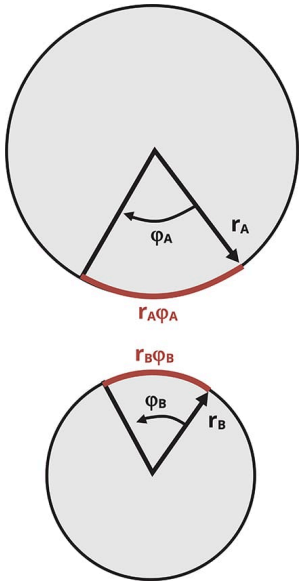


Figure 6.16: Connected gears rotate the same distance in opposite directions.

$$r_A\phi_A = -r_B\phi_B.$$

where ϕ_A is the angle of twist of shaft A and ϕ_A is the angle of twist of shaft B. Rearranging this relationship gives

$$\boxed{\phi_A = -\frac{r_B}{r_A}\phi_B}.$$

(6.6)

As with the angular velocities, the gear ratio is the inverse of that used to relate torques.

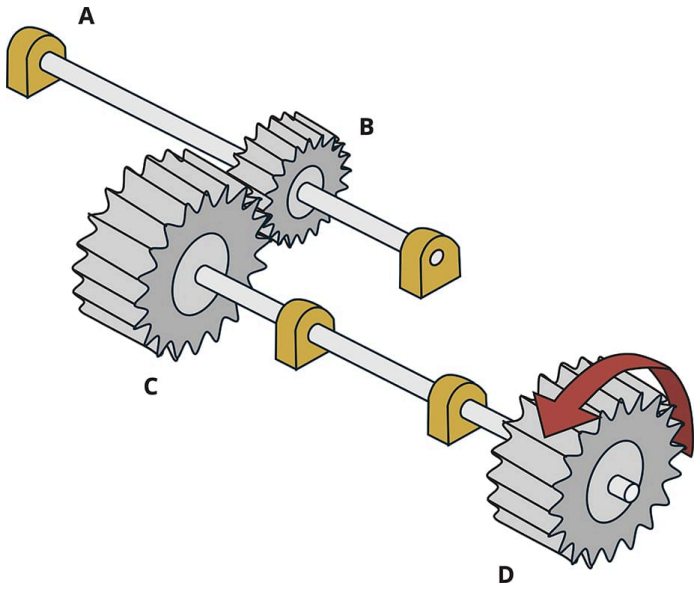
Example 6.7 presents a problem of power transmission with connected gears that utilizes the above concepts.

Example 6.7

Power of 60 kW is transmitted at 33 Hz to gear D of the assembly. Shaft CD is allowed to turn freely within its bearings in the direction shown at D. End A of shaft AB is attached rigidly to the wall.

Determine the angle of twist of point D relative to A.

Assume gear C has 35 teeth (N_C = 25) around its circumference and gear B has 50 teeth (N_B= 50). In addition, shaft AB (d = 40 mm) is 450 mm long and shaft CD (d = 30 mm) is 700 mm long. Both shafts are made of steel (G = 80 GPa).



Solution

In general, think of the angle of twist at D as equal to whatever the angle of twist of point C is plus the angle of twist of shaft CD.

$$\phi_{D/A} = \phi_C + \phi_{CD}$$

With the connection of gear C to gear B, we have

$$\phi_C = -\frac{r_B}{r_A}\phi_B = -\frac{N_B}{N_A}\phi_B,$$

so

$$\phi_{D/A} = -\frac{N_B}{N_A}\phi_B + \phi_{CD}.$$

With N_B and N_A given, we need to find ϕ_{B} and ϕ_{CD}.

1. Given the power information, calculate the torque being applied at D. When considering units, recall that a Watt is an N m/s and a Hz is a revolution per second.

$$T_D = \frac{P_D}{\omega} = \frac{60,000\,W}{33\,Hz\left(\frac{2\pi\,\mathrm{rad}}{Hz}\right)} = 289.4\,N\cdot m$$

2. Given the torque at D, calculate the torque at C by applying equilibrium to shaft CD.



$$\sum M_x = T_D - \mathrm{T}_C = 0$$
$$\mathrm{T}_C = T_D$$

Looking from D toward C, note that the given direction of T_D is counterclockwise, so T_C = 289.4 N·m (clockwise).

3. Use the gear ratio to relate the torque exerted on gear C to the torque exerted on gear B. Recall from section 6.6.3 that T_B will be the same direction as T_C.

$$T_B = \frac{N_B}{N_C} T_c = \frac{50}{35}(289.4\ N \cdot m) = 413.4\ N \cdot m\ \text{(clockwise)}$$

4. Calculate $\phi_B = \phi_{AB}$ (since point A is fixed).

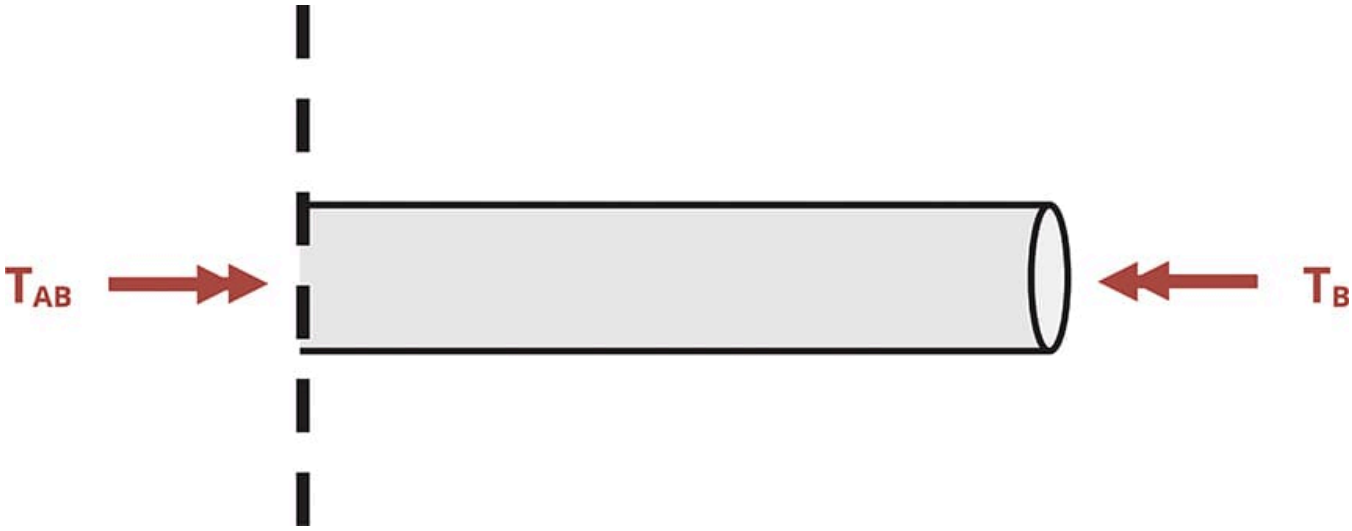
We can see from applying equilibrium to the FBD of a section cut through shaft AB that the internal torque $T_{AB} = T_B$ (clockwise). This means that T_{AB} results in a clockwise (negative) angle of twist ϕ_{AB}.



$$\phi_B = \frac{T_{AB} L_{AB}}{J_{AB} G_{AB}} = -\frac{413.4\ N \cdot m * (0.450\ m)}{\frac{\pi}{32}(0.04\ m)^4 \left(80 \times 10^9\ \frac{N}{m^2}\right)} = -0.009252\ rad$$

5. Determine and use the internal torque T_{CD} to determine ϕ_{CD}.

   Cutting a section through shaft CD and drawing the FBD shows that $T_{CD} = T_D$ (counterclockwise), so



$$\phi_{CD} = \frac{T_{CD} L_{CD}}{J_{CD} G_{CD}} = \frac{289.4\ N \cdot m * (0.700\ m)}{\frac{\pi}{32}(0.03\ m)^4 \left(80 \times 10^9\ \frac{N}{m^2}\right)} = 0.03184\ rad.$$

6. Calculate $\phi_{D/A} = -\frac{N_B}{N_A}\phi_B + \phi_{CD}$.

$$\phi_{D/A} = -\frac{50}{35}(-0.009252\ rad) + 0.03184\ rad = 0.04506\ rad$$

Answer:

- $\phi_{D/A}$ = 0.04506 rad (counterclockwise)

Step-by-Step: Power Transmission

1. Use P = Tw to find power, torque, or angular velocity depending on the specific problem.
2. Find internal torques by cutting sections and applying equilibrium.
3. If the problem involves connected gears (A and B), and the torque or angle of twist of one shaft is already known, determine the torque and/or angle of twist for the other shaft by using $T_A = \frac{r_A}{r_B} T_B$ and $\phi_A = -\frac{r_B}{r_A}\phi_B$.

4. Calculate the total angle twist for assemblies with multiple shafts by summing angles of twist for each shaft individually, accounting for the effect of connected gears using Equation 6.7, and then sum the results together.

Summary

Key Takeaways

Torque is a moment applied around the longitudinal axis of a body. The corresponding twisting deformation is quantified by the angle of twist. In a circular cross-section, the cross-section remains plane after deformation.

Torsional loading results in shear stress. Both stress and strain vary from 0 at the center of a circular cross-section to a maximum value on the outer edge of the cross-section.

Internal torques can be determined by cutting a section through the member, drawing an FBD of the cut section, and applying equilibrium equations.

Internal torques are considered positive if they act counterclockwise when viewed from the positive end of a section toward the negative. For cut sections, the direction away from the cut edge of a body is considered the positive direction.

Sign for shear stress has not been discussed in this text yet (this will come later), but the sign of the angle of twist is important. Many problems involve multiple segments experiencing different stresses and deformations. The stress and angle of twist may be determined for each segment independently, but we need to sum deformations across all the segments to determine the total angle of twist. Therefore the correct signs must be maintained.

One common application where torques are present is power transmission. We usually know the amount of power and the angular velocity but need to determine the torque. Problems involving power transmission often involve shafts with connected gears. These gears alter the torque and angle of twist according to the ratio of their radii. If we know the torque and angle of twist at one gear, we can determine the torque and angle of twist of the connected gear and attached shaft.

Key Equations

Torsional stress:

$$\tau = \frac{T\rho}{J}$$

$$\tau_{max} = \frac{Tc}{J}$$

Angle of twist:

$$\phi = \sum \frac{TL}{JG}$$

Relationship between power and torque:

$$P = T\omega$$

Transfer of torque, angular velocity, and angle of twist between connected gears:

$$T_A = \frac{r_A}{r_B} T_B$$

$$\omega_A = \frac{r_B}{r_A} \omega_B$$

$$\phi_A = -\frac{r_B}{r_A} \phi_B$$

References

Figures

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- Figure 6.1: Horizontal wind blowing on the traffic lights will create a torsional moment in the vertical pole. James Lord. 2024. CC BY.
- Figure 6.2: Unloaded tube (left) and tube with torque applied (right). James Lord. 2024. CC BY.
- Figure 6.3: I-beam subjected to torsion unloaded (a) and with applied torque (b). James Lord. 2024. CC BY.
- Figure 6.4: Evidence of shear strain. James Lord. 2024. CC BY.



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Chapter 7: Beams

Learning Objectives

- Apply equilibrium methods to determine internal shear forces and bending moments equations, and specific values in beams
- Draw the shear and bending moment diagrams using integration and graphical methods

Beams are structural members that support loads along their length. Typically these loads are perpendicular to the axis of the beam and cause only shear and bending moments. Figure 7.1 shows some examples.



Figure 7.1: Examples of structural beams

As evident in previous chapters, internal forces and moments are crucial to calculating stresses and deflections. The same is true for finding stresses and deflections in beams. Due to the nature of the loadings and constraints of beams, finding internal forces requires new methods in statics. This chapter describes these methods for finding a beam's internal shear force and bending moments.

Section 7.1 reviews how to find the internal shear force and bending moments at a specific point in a beam using equilibrium. Section 7.2 discusses relationships between the applied load, the internal shear force, and the internal bending moment.

Then follows a discussion of a more general approach to visualizing the internal shear force and bending moment at different points in a beam using diagrams of these. Section 7.3 explores how to draw these diagrams using an equilibrium method. Then Section 7.4 provides an alternative graphical method for drawing these diagrams.

Before presenting methods for finding internal shear and moments of beams, we need to establish a sign convention to provide consistency and act as a guide to interpreting our results. Note that anytime you start a new project or subject, you should ensure that a sign convention is established.

For the purposes of this book, the shear is positive when the external forces shear off as depicted in Figure 7.2. Another way to think about it is that a positive shear is when the internal shear forces cause a clockwise rotation of the beam segment.

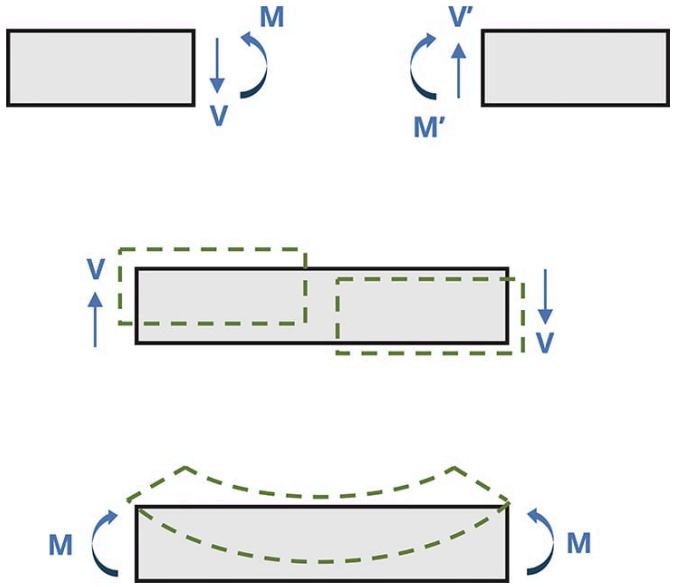


Figure 7.2: Positive sign convention for internal shear force (V) and bending moment (M)

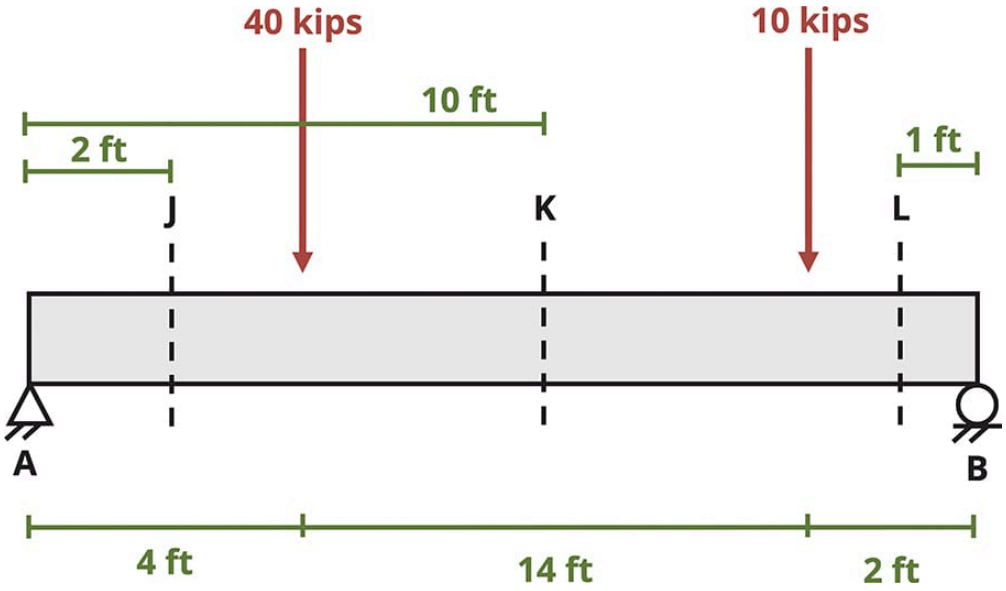
The bending moment is positive when the external forces bend the beam in a concave up shape as indicated in Figure 7.2. This causes the top fibers of the beam to be in compression while the bottom fibers are in tension.

7.1 Internal Shear Force and Bending Moment by Equilibrium

One way to find internal shear and bending forces is to cut sections and analyze the free body diagrams (FBDs), as we did in previous chapters of this book. The first step for a statically determinate beam—one where all forces can be found using only equilibrium equations—is to find the external reactions. To determine the internal forces at any point along the beam, we cut the section at that point and draw the FBD from that point to one end of the beam. We'll include the internal shear force, V, and internal bending moment, M, at the cut section and then use equilibrium equations to solve for those internal loads. This process is demonstrated in Example 7.1.

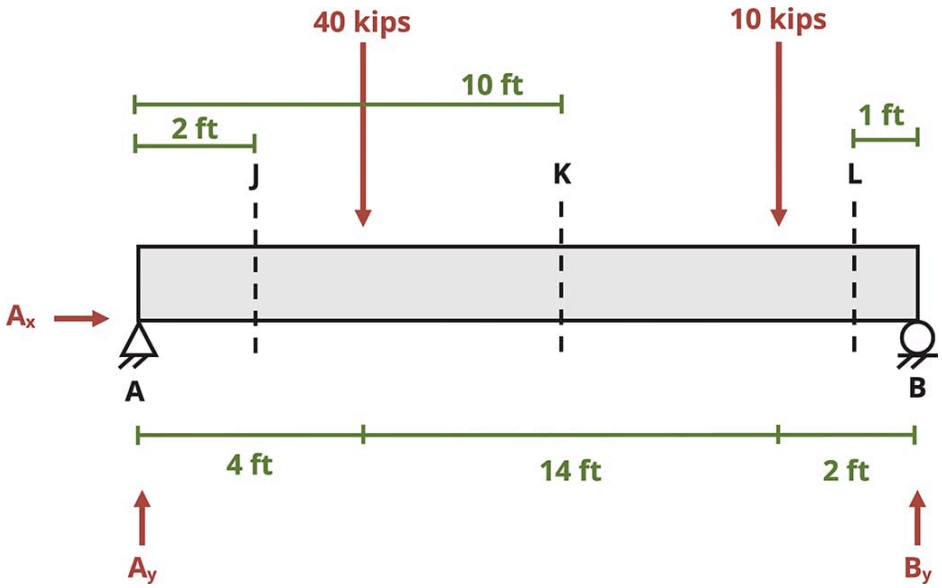
Example 7.1

Find the internal shear and bending moment at points J, K, and L in the pictured beam.



Solution

The first step is to find the external reactions at the supports A and B.

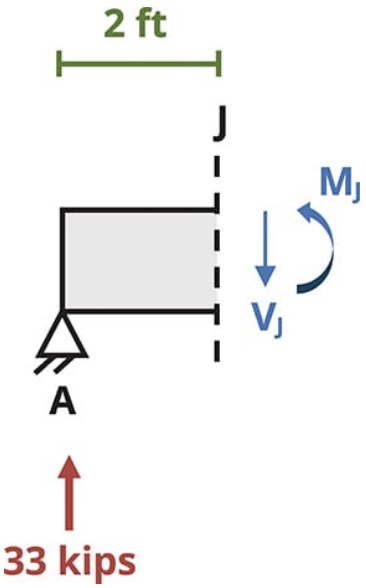


$$\sum F_x = A_x = 0$$
$$\sum M_A = -40\ kips * 4\ ft - 10\ kips * 18\ ft + B_y * 20\ ft = 0\ \rightarrow\ B_y = 17\ kips$$
$$\sum F_y = A_y - 40 - 10 + 17 = 0\ \rightarrow\ A_y = 33\ kips$$

Next we draw an FBD of the right end of the beam from point A to J by cutting the beam at section J. The internal forces V_J and M_J are placed at the point of the cut using the positive sign convention. Note that you can assume the direction of these forces since the statics will work out the correct direction.

Section J

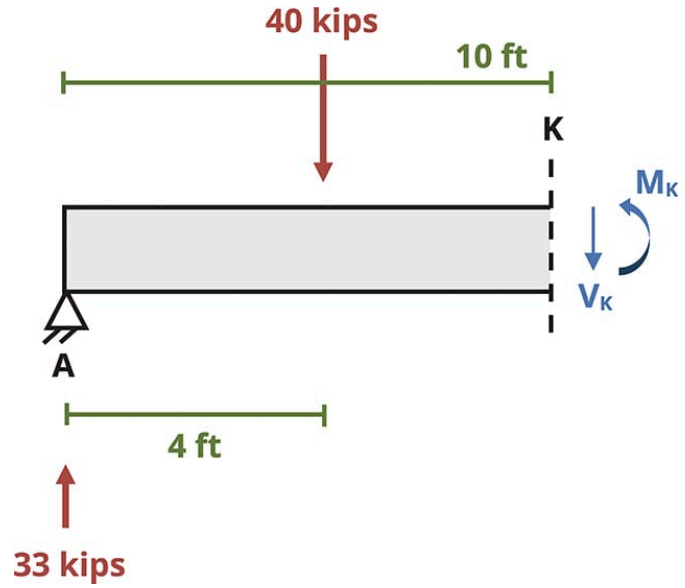
To find the internal shear, sum forces in the y direction. To find the internal moment, sum moments at point J.



$$\sum F_y = 33\ kips - V_J = 0\ \rightarrow\ V_J = 33\ kips$$
$$\sum M_J = -33\ kips * 2\ ft + M_J = 0\ \rightarrow\ M_J = 66\ kip \cdot ft$$

Section K

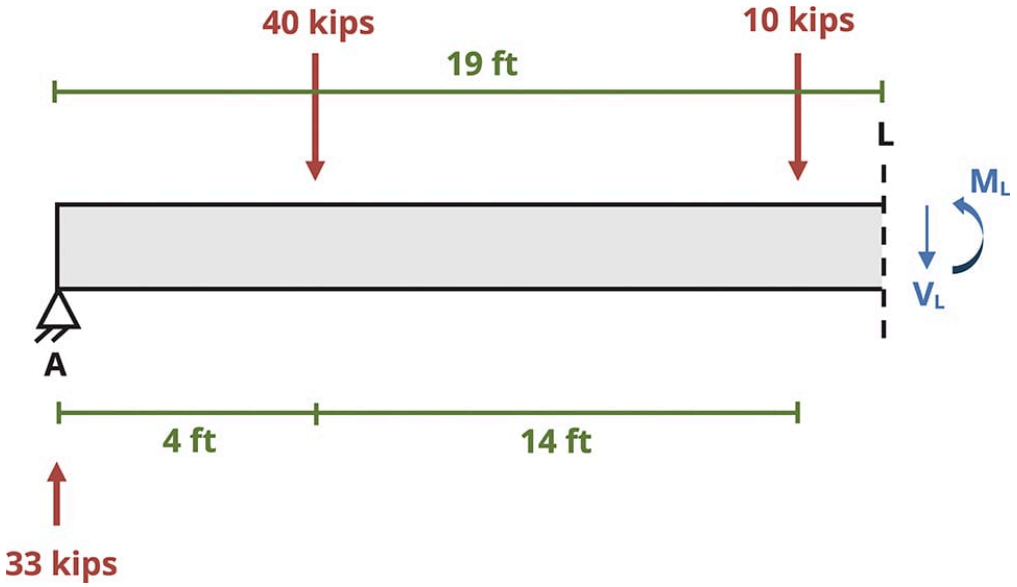
To find the internal shear, sum forces in the y direction. To find the internal moment, sum moments at point K.



$$\sum F_y = 33\ kips - 40\ kips - V_k = 0\ \rightarrow\ V_k = -7\ kips$$
$$\sum M_k = -(33\ kips * 10\ ft) + (40\ kips * 6ft) + M_k\ \rightarrow\ M_k = 90\ kip \cdot ft$$

Section L

To find the internal shear, sum forces in the y direction. To find the internal moment, sum moments at point L.



$$\sum F_y = 33\ kips - 40\ kips - 10\ kips - V_L = 0\ \rightarrow\ V_L = -17\ kips$$
$$\sum M_L = -(33\ kips * 19\ ft) + (40\ kips * 15\ ft) + (10\ kips * 1ft) + M_L = 0\ \rightarrow\ M_L = 17\ kip \cdot ft$$

Observe that the internal forces vary at different points along the length of the beam. This method of finding the internal stresses is convenient when the loading is simple or when you know a specific point along the length of the beam. However, as the loading becomes more complex, consider one of the methods outlined in the following sections.

7.2 Relationship Between Load, Shear, and Moment

There is a relationship between loading, shear, and moment—and as shown later in this text, slope and deflection—in a beam. Consider the beam in Figure 7.3, which is subjected to a distributed load, w per unit length.

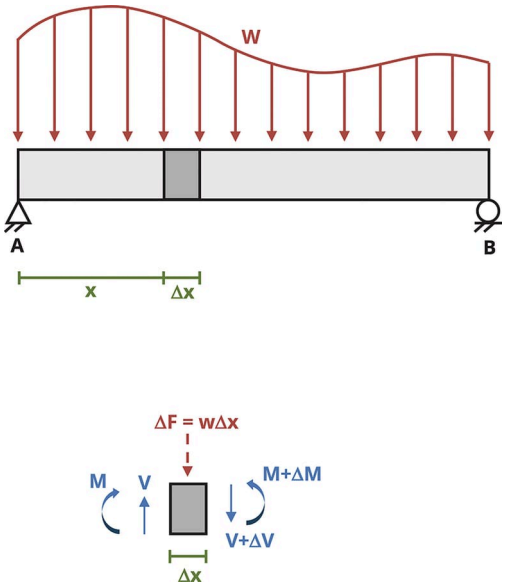


Figure 7.3: Small segment of a beam subjected to an arbitrary distributed load

Now look at a small section from that beam that has a width of Δx, shown below. The distributed load has been replaced with a resultant force ΔF (this is indicated by a dashed arrow in the FBD).

Since both sides of the beam have been cut, replace each cut with internal shear and moment forces assuming the positive sign convention established earlier in this chapter.

7.2.1 Relationship Between Load and Shear

This FBD is in static equilibrium, so we can use the equilibrium equation to sum forces in the y direction and set it equal to zero.

$$\sum F_y = V - \Delta \mathrm{F} - (\mathrm{V} + \Delta \mathrm{V}) = 0$$

$$\boxed{\Delta V = \Delta F}$$

(7.1)

Or

$$\Delta V = w \Delta x$$

Dividing both sides by Δx and then letting Δx approach zero produces

$$\frac{\Delta V}{\Delta x} = \frac{dV}{dx} = w.$$

We then rearrange this by multiplying both sides of the equation by dx.

$$dV = w dx$$

Now we can integrate between any two points A and B on the beam.

$$\underbrace{\Delta V}_{\text{change in shear}} = \int \underbrace{w dx}_{\text{area under loading curve}}$$

(7.2)

This equation is valid for distributed loads, but not when there is a discontinuity in the shear diagram that is caused by concentrated loads. This relationship should be used only between concentrated loads.

7.2.2 Relationship Between Shear and Moment

Let's return to the FBD of the small section of the beam subjected to a distributed load. This segment is shown in Figure 7.4. We can apply another static equilibrium equation, summing moments about point C.

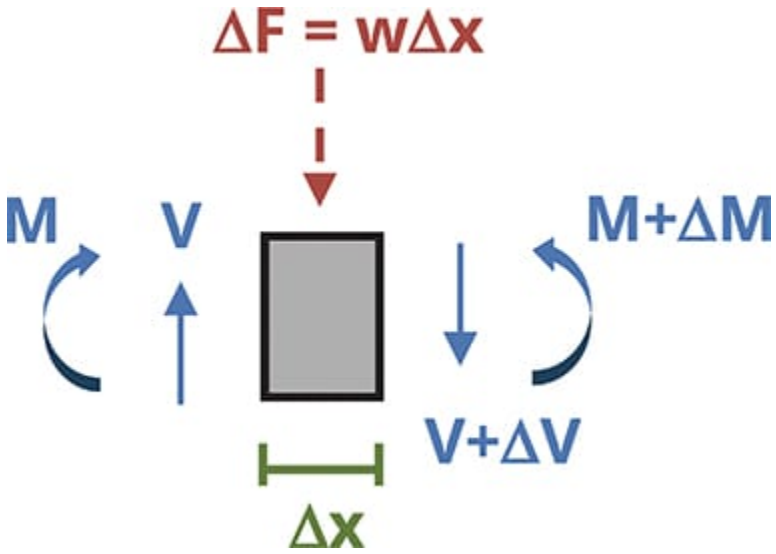


Figure 7.4: Free body diagram of a small segment of a beam subjected to an arbitrary distributed load

$$\sum M_c = 0 = -M - V(\Delta x) + \Delta F\left(\tfrac{1}{2}\Delta x\right) + (M + \Delta M)$$
$$\Delta M = M + V\Delta x - (w\Delta x)\left(\tfrac{1}{2}\Delta x\right) - M$$
$$\Delta M = V\Delta x - \tfrac{1}{2}w\Delta x^2$$

Dividing both sides by Δx and then letting Δx approach zero yields

$$\underbrace{\frac{\Delta M}{\Delta x}}_{\text{Slope of moment diagram}} = \underbrace{V}_{\text{shear}}$$

(7.3)

Then we rearrange this by multiplying both sides of the equation by dx.

$$dM = V dx$$

Now it is possible to integrate between any two points A and B on the beam.

$$\underbrace{\Delta M}_{\text{Change in moment}} = \underbrace{\int V dx}_{\text{Area under shear diagram}}$$

(7.4)

This equation is invalid at points where a concentrated force or concentrated moment occurs. These concentrated loads cause discontinuities in the moment diagram.

7.2.3 Using Calculus Knowledge to Build Shear and Moment Diagrams

In a basic sense, the relationship between load, shear, and moment can be described as in Figure 7.5.

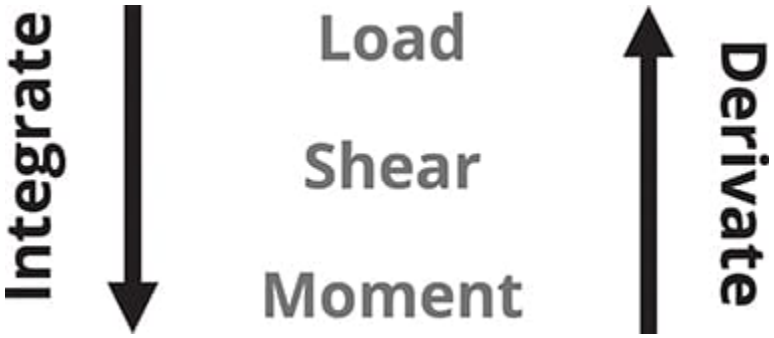


Figure 7.5: Relationship between load, shear, and moment

Combine these relationships with what you know about derivatives and integrals from your calculus course. Here are some items of note:

- The slope of the shear diagram is equal to the sign and magnitude of the distributed load. If there is no load on a section of the beam, then the slope of the shear diagram is zero.
- Similarly, the slope of the moment diagram at any point is the derivative of the moment function evaluated at that point. This is equal to the sign and magnitude of the shear at that point.
- The area under the distributed load between two points is equal to the change in the shear between those same two points.
- The area under the shear diagram between two points is equal to the change in the moment between those same two points.
- The maximum moment occurs where its derivative (shear) is equal to zero.
- At the points of concentrated forces, the shear diagram jumps up or down depending on the direction of the force. If the force is up, then the shear diagram will jump up, and if the force is down, the shear diagram will drop down.
- At the points of concentrated moments, the bending moment diagram jumps up or down depending on the direction of the rotation. It is a common mistake to assume that counterclockwise moments cause a jump upward because counterclockwise is generally considered positive in equilibrium equations. However, recall that we are plotting the internal bending moment on the bending moment diagram, not the externally applied moment, and the sign convention established at the start of this chapter for internal bending moments is the one we must follow. Therefore, if the concentrated moment is clockwise, the moment diagram will jump up, and if counterclockwise, it will drop down.
- If the load can be represented with a polynomial, then we can easily predict the degree of the subsequent shear and moment functions. For example, if the load is a uniformly distributed load (constant), the shear will be a linear function (n + 1) and the moment will be a parabola (n + 2).

Use all you know about calculus and the relationships between functions when you derive and integrate to build, check, and analyze shear and moment diagrams.

7.3 Determining Equations by Equilibrium

The relationship between load, shear, and moment allows us to build complete shear and moment diagrams using a few different methods. We'll explore two of them: an equilibrium method and a graphical method.

In Section 7.1 we found the internal shear force and bending moment at specific points in the beam by cutting a cross-section at a given point and using equilibrium. The method in this section is similar, but instead of cutting a cross-section at a specific point, we'll cut a cross-section at distance x from the left end of the beam. We'll draw an FBD and determine the internal shear force (V) and bending moment (M) as before, but this time the resulting equations for V and M will be functions of distance x.

To draw the diagrams, we can solve this equation at two values of x. It is often simplest to solve when x = 0 and when x = L, the length of the beam. These are two points we can then connect on our diagram. The diagram's shape depends on the highest power of x that appears in the equation.

- For x^0 we obtain a flat horizontal line.
- For x^1 we obtain a straight diagonal line.
- For x^2 we obtain a curved line.
- For x^3 we obtain a cubic curve.

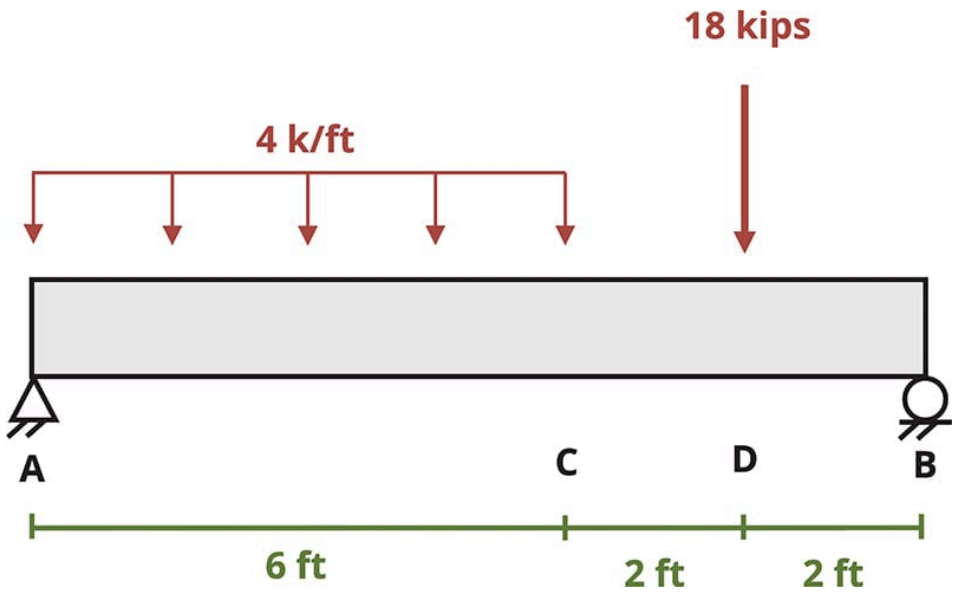
If two points are insufficient to determine the exact shape of the curve, simply solve the equations at a third point (e.g., where x = L/2). We can solve the equations at as many points as necessary, but two or three points, along with knowledge of the general shape of the line, are typically sufficient. The equations found using this method are valid as long as the external loading doesn't change.

Cases where the external loading does change require us to determine a new set of equations each time the loading changes. This includes any time a distributed load begins or ends, and both sides of a concentrated force or couple. In each region of load we will cut a cross-section at distance x and determine equations for V and M that are valid for that region. We then solve each equation at values of x at the start and end of their valid region and build up the complete shear force and bending moment diagrams.

Example 7.2 illustrates this method for a simply supported beam, and Example 7.3 for a cantilever beam.

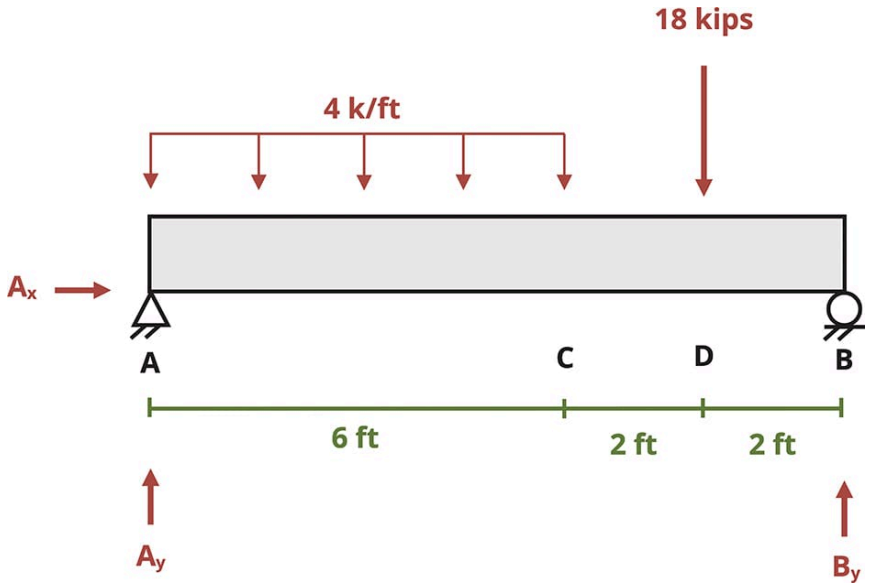
Example 7.2

Draw the shear force and bending moment diagrams for the beam shown.



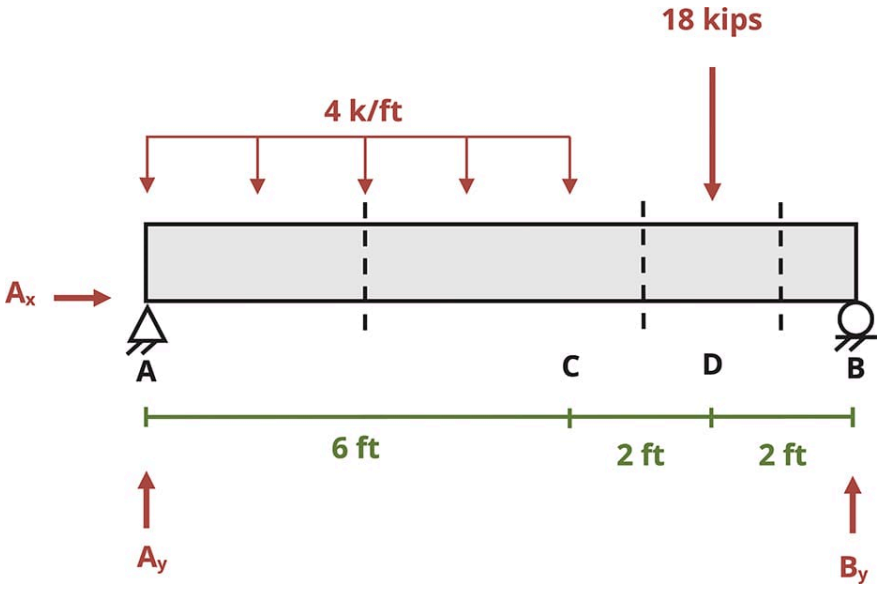
Solution

The first step is to draw an FBD of the beam, being sure to change the supports to the correct external reaction forces as shown.



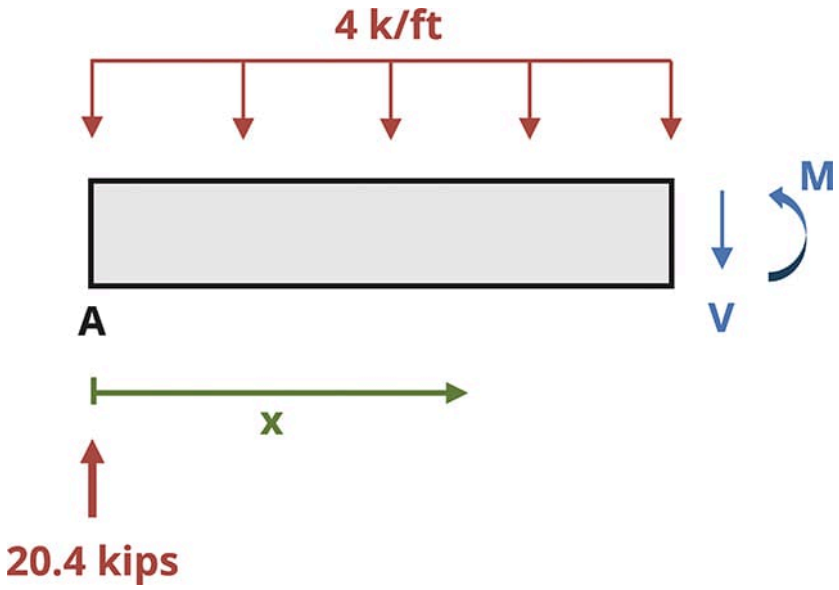
We then use static equilibrium equations to solve for the magnitude of the support reactions.

$$\sum F_x = A_x = 0$$
$$\sum M_A = -4\frac{kips}{ft}*6\ ft*3\ ft - 18\ kips*8\ ft + B_y*10\ ft = 0 \rightarrow B_y = 21.6\ kips$$
$$\sum F_y = A_y - 4\frac{kips}{ft}*6\ ft - 18\ kips + 21.6\ kips = 0 \rightarrow A_y = 20.4\ kips$$



Next we cut a cross-section within the loading region concerned. When choosing sections, be sure to cut within the uniform load and between loads and reactions. For this example, here we must cut three sections to create the complete shear and moment diagram. These sections are depicted with the green lines. Then the equilibrium equations are used to find the internal shear force as a function of x, V(x).

Section between A and C

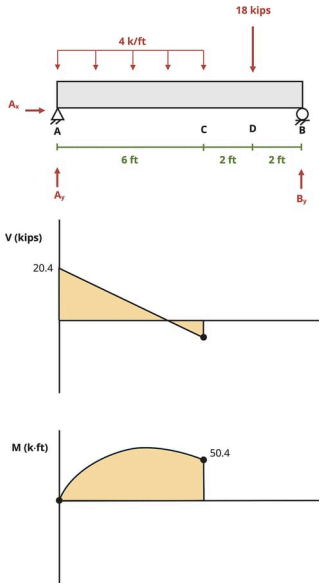


$$\sum F_y = 20.4 - 4(x) - V = 0 \rightarrow V(x) = -4x + 20.4$$
$$\sum M_0 = -20.4(x) + 4(x)(\tfrac{x}{2}) + M = 0 \rightarrow M(x) = -2x^2 + 20.4x$$

Note that you can easily check the statics because V(x) is the derivative of M(x). For polynomials, this is a quick test to see if the equations make sense.

We plot the V(x) and M(x) equations on an axis from x = 0 to x = 6 ft. Solving the equations at these points yields

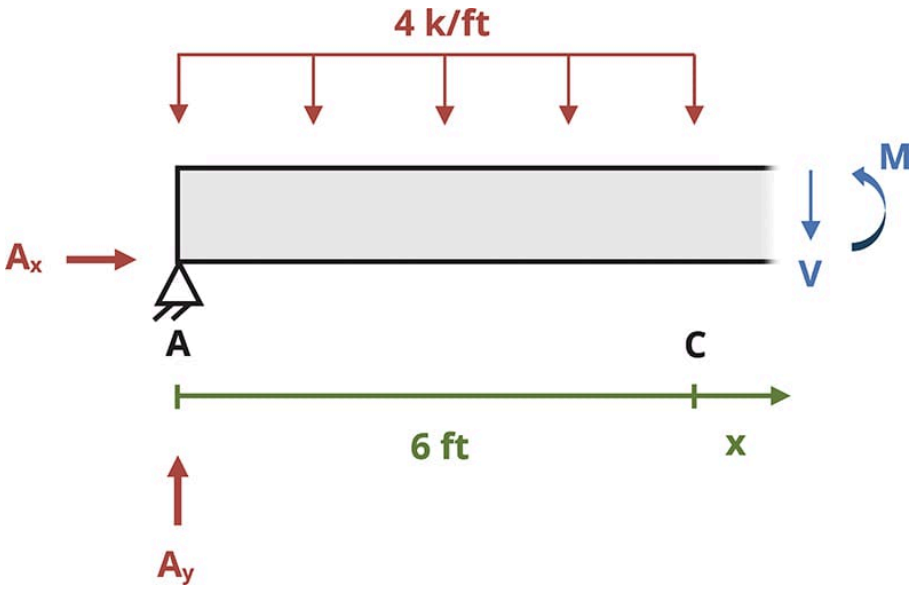
$$V(0) = -4(0) + 20.4 = 20.4\ kips$$
$$V(6) = -4(6) + 20.4 = -3.6\ kips$$
$$M(0) = -2(^0{+}20.4(0) = 0$$
$$M(6) = -2(^6{+}20.4(6) = 50.4\ kip \cdot ft.$$



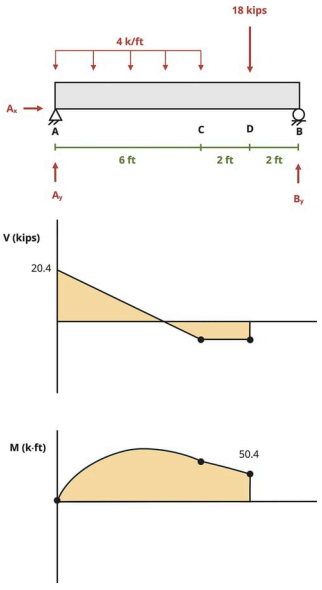
Note that the shear diagram is drawn directly below the beam, and the moment diagram directly below the shear. These three—load, shear, and moment—share the same x-axis, whereas the vertical axes for the shear (V) and moment (M) diagrams have different units and possibly different scales.

Drawing the shear and moment diagrams directly below the beam is good practice so that you can get a complete picture of what is going on along the length of the beam.

Section between C and D



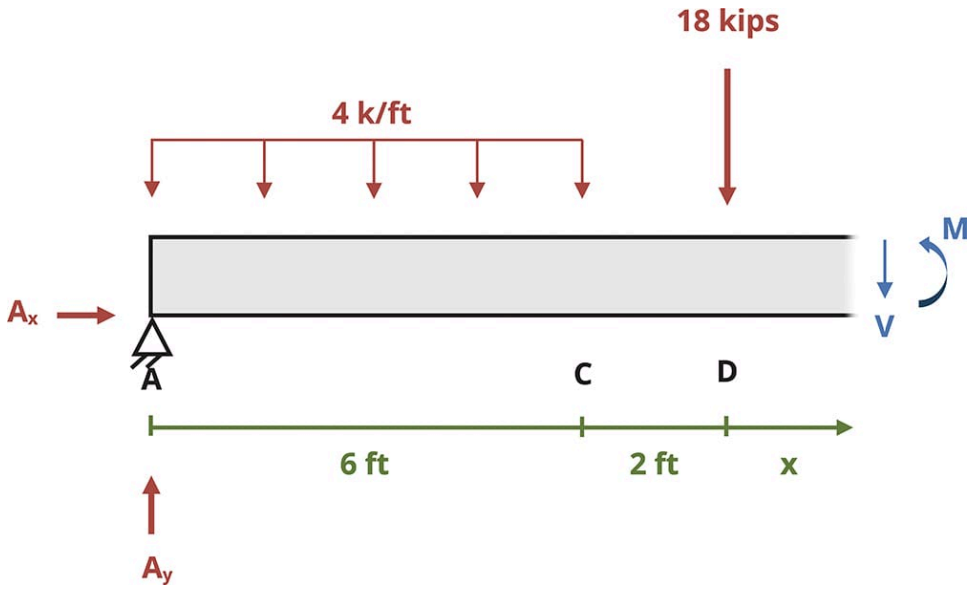
$$\sum F_y = 20.4 - 4(6) - V = 0 \rightarrow V(x) = -3.6$$
$$\sum M_0 = -20.4(x) + 4(6)(x-3) + M = 0$$
$$M(x) = +20.4x - 24x + 72$$
$$M(x) = -3.6x + 72$$



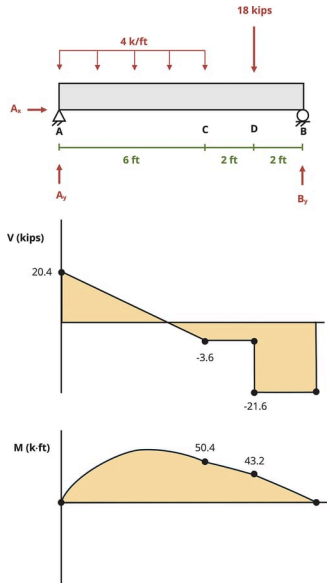
We then add the section from C to D using the V(x) and M(x) equations. Notice that since there are no applied loads on the beam from C to D, the shear diagram remains constant. This means the moment diagram is linear between x = 6 and x = 8 ft.

$$V(6) = -3.6\ kips$$
$$M(6) = -3.6(6) + 72 = 50.4\ kip \cdot ft$$
$$V(8) = -3.6\ kips$$
$$M(8) = -3.6(8) + 72 = 43.2\ kip \cdot ft$$

Section between D and B



$$\sum F_y = 20.4 - 4\,(6) - 18 - V = 0 \;\rightarrow\; V\,(x) = -21.6$$
$$\sum M_0 = -20.4\,(x) + 4\,(6)\,(x-3) + 18\,(x-8) + M = 0$$
$$M\,(x) = 20.4x - 24x + 72 - 18x + 144$$
$$M\,(x) = -21.6x + 216$$

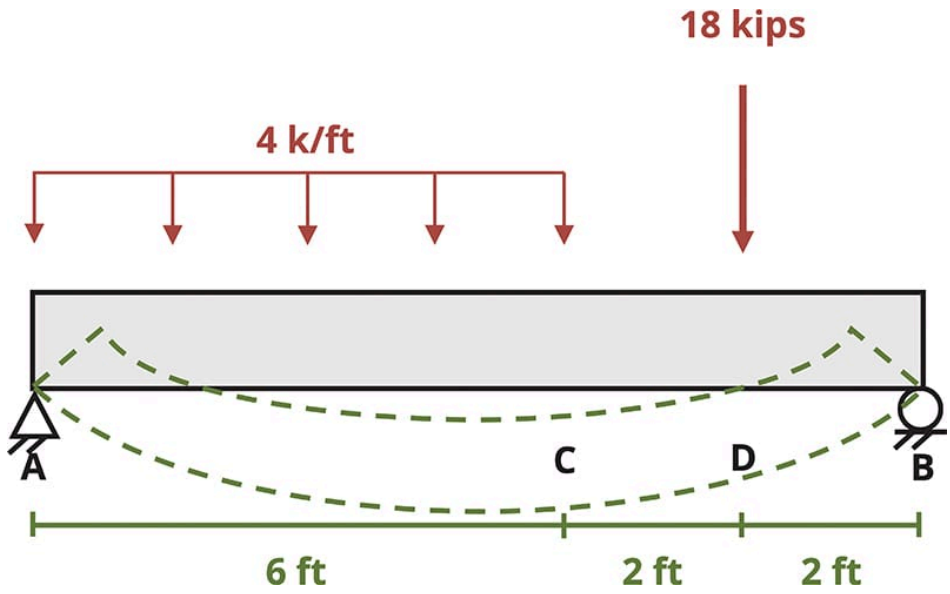


Finally, we graph the V(x) and M(x) equations obtained from point D to point B (x = 8 to x = 10 ft).

$$V\,(8) = -21.6\;kips$$
$$M\,(8) = -21.6\,(8) + 216 = 43.2\;kip \cdot ft$$
$$V\,(10) = -21.6\;kips$$
$$M\,(10) = -21.6\,(10) + 216 = 0$$

You now have complete shear and moment diagrams. Be sure to label the axes, including units, and all pertinent values.

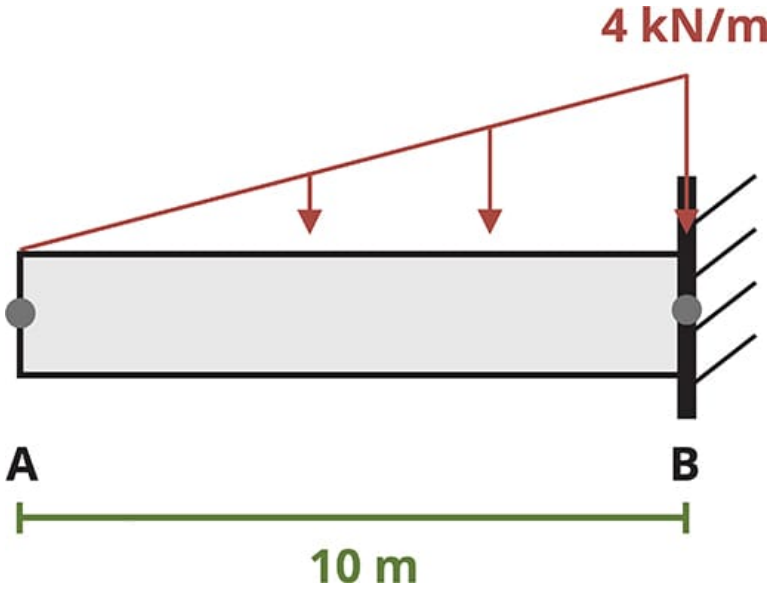
These diagrams are used in the design of the beam and its components.



Note that in this example the entire length of the beam is subjected to positive moment. According to the sign convention at the beginning of this chapter, remember, a positive moment indicates a concave up behavior. Given the beam and the external loads, this makes sense, for the beam will want to bend concave up between supports, as the figure shows.

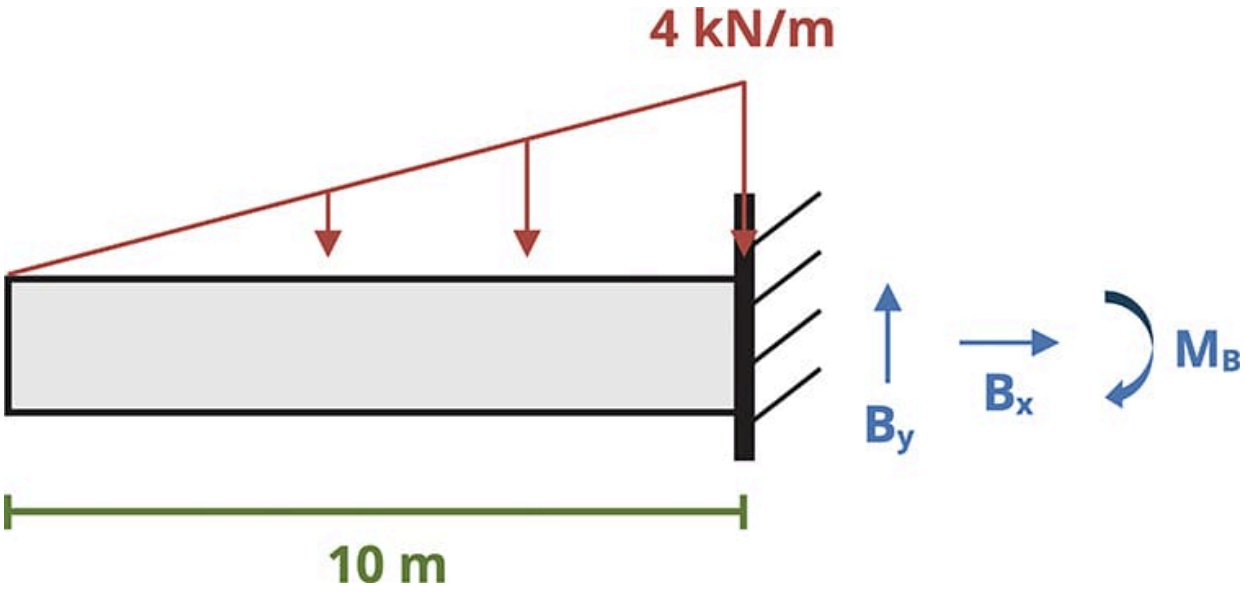
Example 7.3

Draw the shear force and bending moment diagrams for the beam shown.



Solution

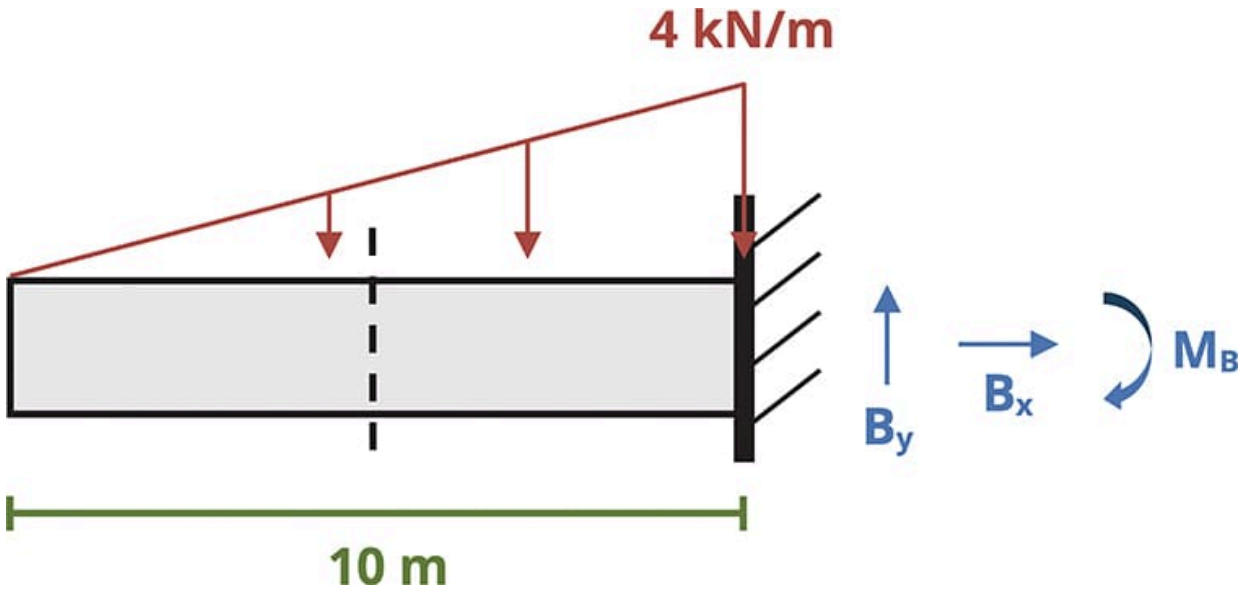
The first step is to draw an FBD of the beam, being sure to change the supports to the correct external reaction forces.



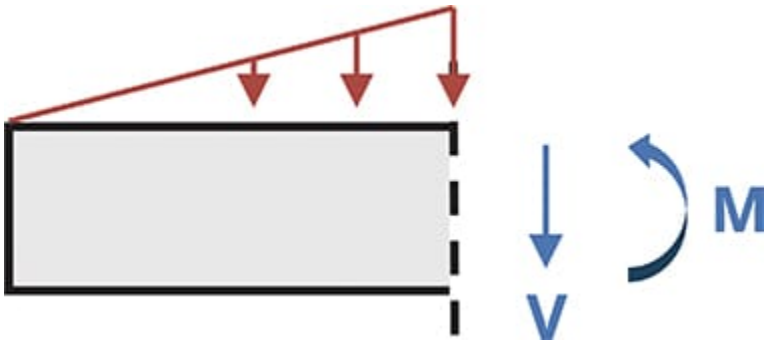
We then use static equilibrium equations to solve for the magnitude of the support reactions.

$$\Sigma F_x = B_x = 0$$
$$\Sigma F_y = -\tfrac{1}{2}(10\ m)\left(4\ \tfrac{kN}{m}\right) + B_y = 0\ \rightarrow\ B_y = 20\ kN$$
$$\Sigma M_B = \tfrac{1}{2}(10\ m)\left(4\ \tfrac{kN}{m}\right)\left(\tfrac{10\ m}{3}\right) - M_B = 0\ \rightarrow\ M_B = 66.7 kN \cdot m$$

Next we cut a cross-section within the loading region concerned. For this example, we cut one section between A and B to create the complete shear force and bending moment diagram. This section is depicted with the dashed line. Then equilibrium equations are used to find the internal shear force as a function of x, V(x).



Section between A and B

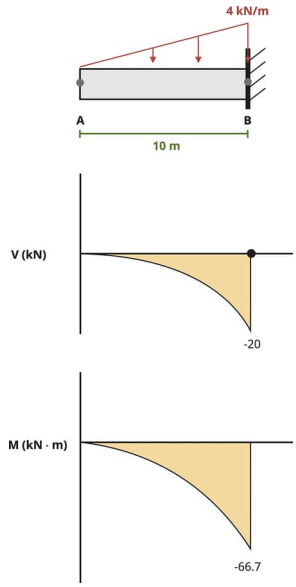


$$\Sigma F_y = -V - \tfrac{1}{2}(x)(\tfrac{4}{10}x) = 0$$
$$V(x) = -\tfrac{1}{5}x^2$$
$$\sum M = \tfrac{1}{2}(x)(\tfrac{4}{10})(\tfrac{1}{3}x) + M = 0$$
$$M(x) = -\tfrac{1}{15}x^3$$

Note that you can easily check the statics because V(x) is the derivative of M(x). For polynomials, this is a quick test to see if the equations make sense.

We plot the V(x) and M(x) equations on an axis from x = 0 to x = 10 m. Notice how the shear diagram is drawn directly below the beam, and the moment diagram directly below the shear. These three—load, shear, and moment—share the same x-axis, whereas the vertical axes for the shear (V) and moment (M) diagrams have different units and possibly different scales.

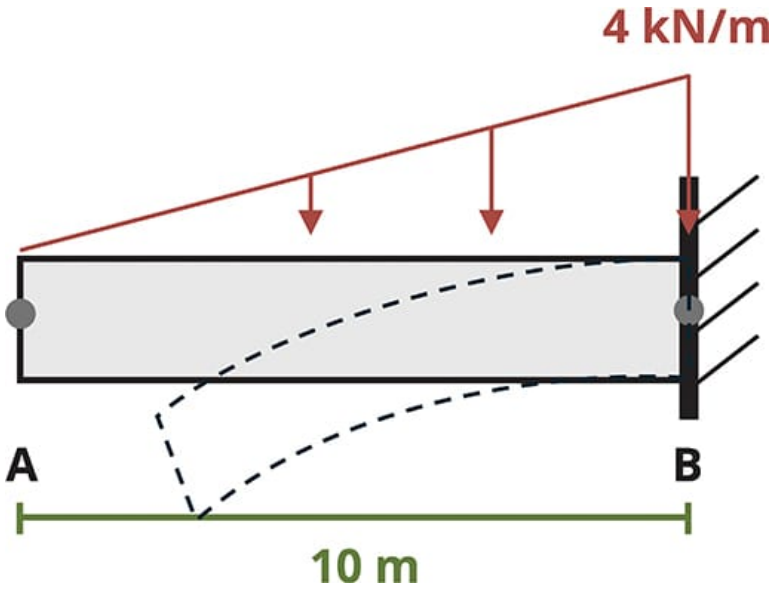
Drawing the shear and moment diagrams directly below the beam is good practice so that you can get a complete picture of what is going on along the length of the beam.



See that the shear diagram starts at zero, then at x = 10 m, V(10) = –20 kN. The reaction calculated was 20 kN going up, so adding that reaction closes the shear diagram.

Similarly, the moment diagram is the graph of the cubic equation from x = 0 to x = 10. M(10) = –66.7 kN·m is the same as the reaction calculated to close the moment diagram.

Note that in this example the entire length of the beam is subjected to negative moment. According to the sign convention presented at the beginning of this chapter, remember, a negative moment indicates a concave down behavior. Given the beam and the external loads, this makes sense because the beam will want to bend concave down, as shown in the figure.



Step-by-Step: Shear Force and Bending Moment Diagrams (Equilibrium Method)

1. Use equilibrium to determine the reaction loads at any supports.
2. Identify the distinct regions of external loading and cut a cross-section within each region.
3. Draw an FBD of the left side of the first cut, which should be made at distance x_1. Use equilibrium to determine equations for the internal shear force (V) and bending moment (M) as functions of x_1.
4. Repeat step 3 for each loading region and develop equations for V and M that are valid in each region.
5. Plot each set of equations on axes of V(x) and M(x). Each equation is valid only in the region where it was developed. Solve each equation at the start and end points of its region and then connect the points with an appropriate line, according to the highest power of x that appears in the equation.

7.4 Graphical Method Shear Force and Bending Moment Diagrams

An alternative, often faster method for drawing shear force and bending moment diagrams is the graphical method. This approach leverages the relationships between load, shear force, and bending moment, allowing us to derive one diagram from another.

Specifically, Section 7.2 showed that $\frac{\mathrm{dV}}{\mathrm{dx}} = \mathrm{w}$ and $\frac{\mathrm{dM}}{\mathrm{dx}} = \mathrm{V}$. That is, the slope of the shear force diagram at any point equals the applied distributed load at that point, and the slope of the bending moment diagram at any point equals the shear force at that point.

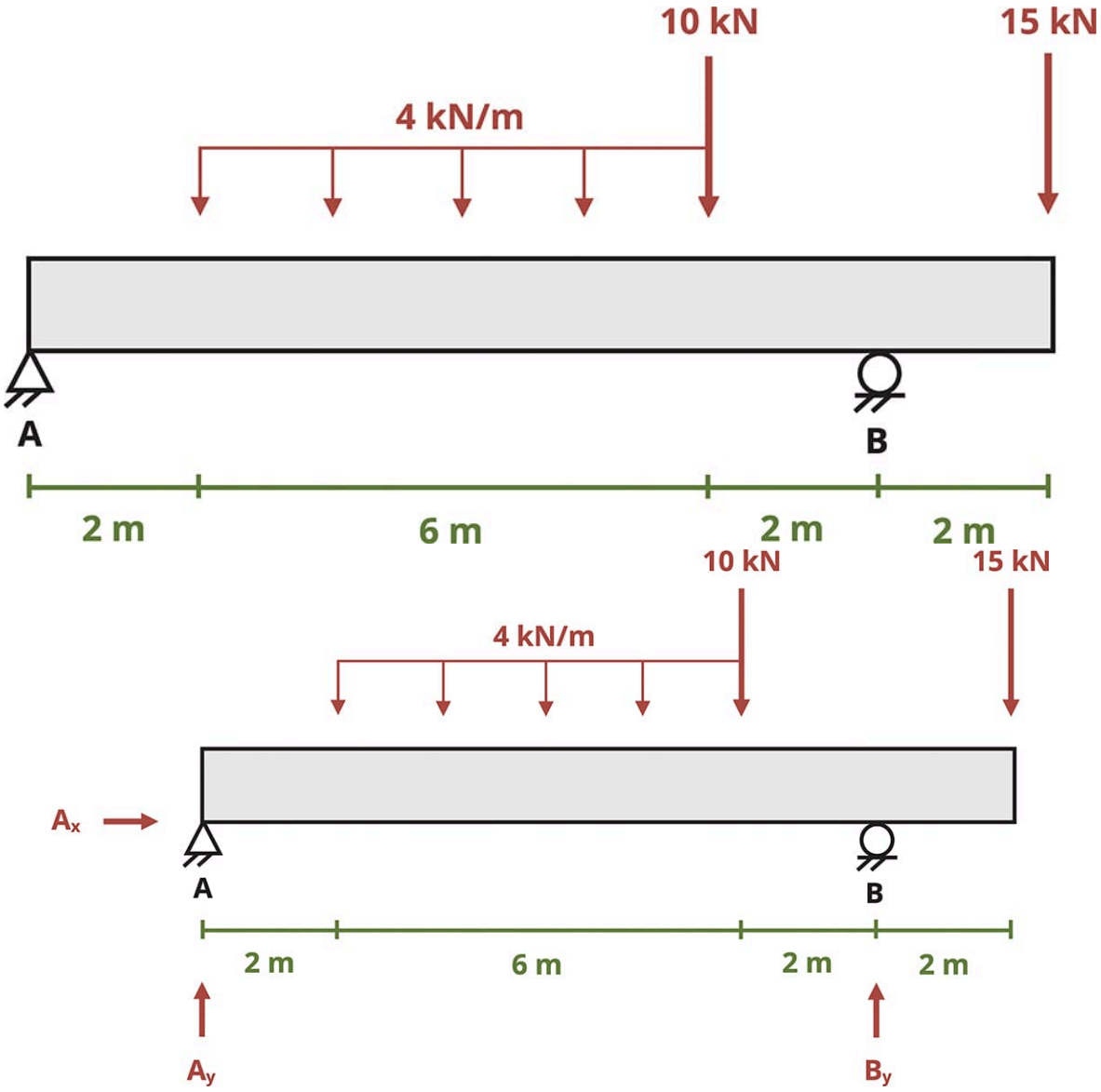
Further, any applied concentrated force causes a jump in the shear diagram (e.g., an upward force results in an upward jump). Any applied couples causes a jump in the bending moment diagram (e.g., a clockwise moment causes an upward jump). Thus the shape of the shear force diagram is based on the applied loads, and the shape of the bending moment diagram is based on the shear force diagram.

Finally, Equation 7.2 also showed that the change in shear force between any two points is equal to the area under the distributed load between those same two points. Evident from Equation 7.4 is that the change in bending moment between any two points is equal to the area underneath the shear force diagram between those two points. This allows us to add numbers to the diagrams and determine the value of the shear force and bending moment at key points.

When the loading is relatively simple, consisting of concentrated forces, concentrated moments, and uniformly distributed loads, we can use geometry to find the area of the load and shear diagrams since the shapes are simple. Example 7.4 and Example 7.5 illustrate this method.

Example 7.4

Draw the shear force and bending moment diagrams for the beam shown.



Solution

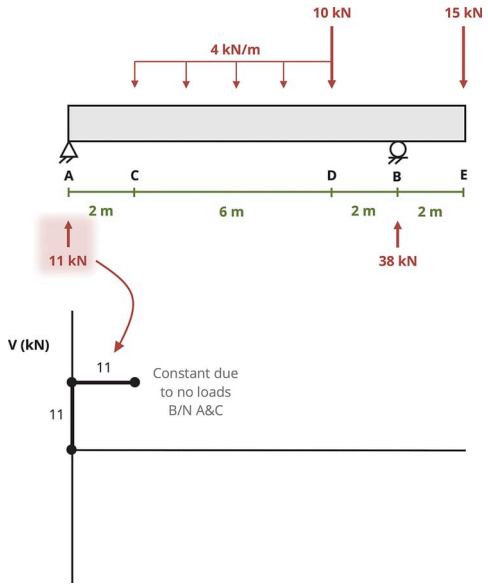
The first step is to draw an FBD of the beam, being sure to change the supports to the correct external reaction forces, as shown below.

Then use static equilibrium equations to solve for the magnitude of the support reactions.

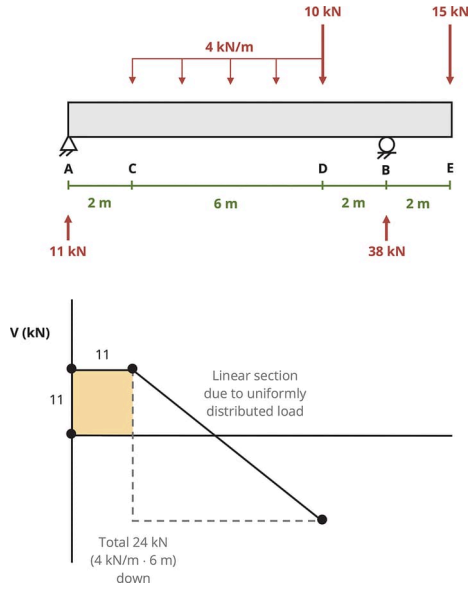
$$\sum F_x = A_x = 0$$
$$\sum M_A = -4\tfrac{kN}{m}(6\ m)\,(5\ m) - 10\ kN\,(8\ m) + B_y\,(10\ m) - 15\ kN\,(12\ m) = 0\ \rightarrow\ B_y = 38\ kN$$
$$\sum F_y = A_y - 4\tfrac{kN}{m}(6\ m) - 10\ kN - 15\ kN + 38\ kN\ \rightarrow\ A_y = 11\ kN$$

Now that the external forces are known, build the shear diagram. Start at the leftmost point of the beam, point A.

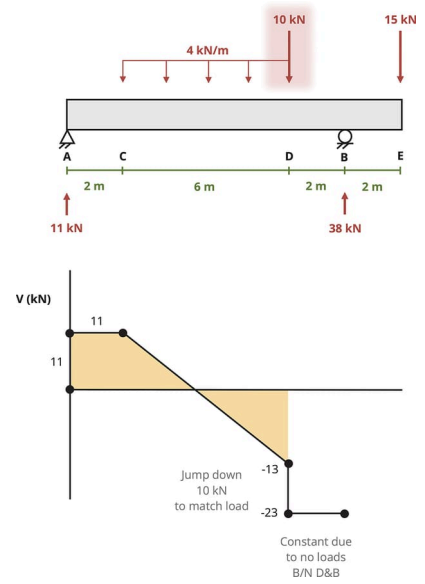
The first force encountered is at point A, the reaction force of 11 kN. The force is going up, so do that same thing on the shear diagram.



From that point look at the beam and note that no forces are acting between points A and C. This indicates the shear diagram will remain constant at 11 kN.

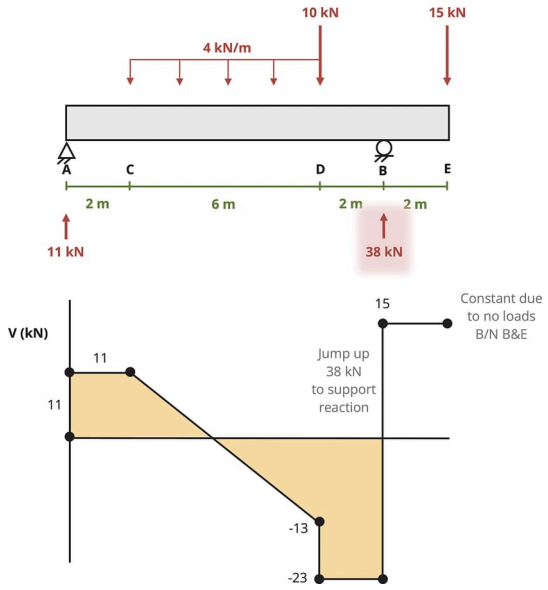


The 4 kN/m uniformly distributed load starts at point C and goes for 6 m until point D. The force is going down for a total of 24 kN (4 kN/m·6 m). Since the load is constant from point C to D, the shear will be linear between those points. The slope of the shear diagram will be -4 and over those 6 m will decrease the total of 24 kN.



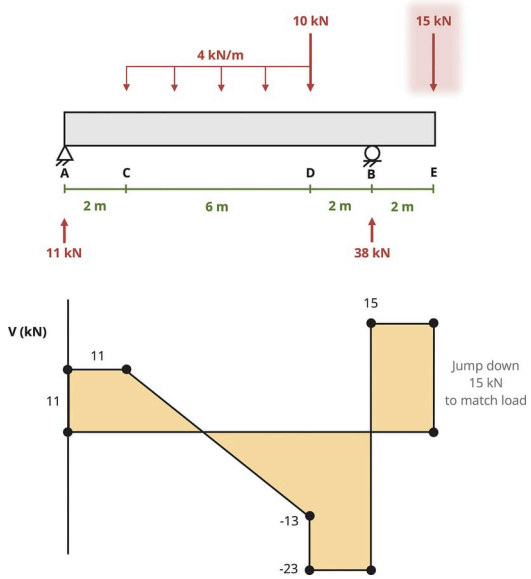
At point D there is a concentrated 10 kN load going down. On the shear diagram, this load will be represented by a discontinuity, jumping down by 10 kN to –23 kN.

From that point look at the beam and note that no forces are acting between points D and B. This indicates the shear diagram will remain constant at –23 kN.



At point B, the roller support, there is an external reaction of 38 kN going up. This concentrated force will cause a discontinuity in the shear diagram. From –23 kN add 38 kN to end at 15 kN.

From that point look at the beam and note that no forces are acting between points B and E. This indicates the shear diagram will remain constant at 15 kN.



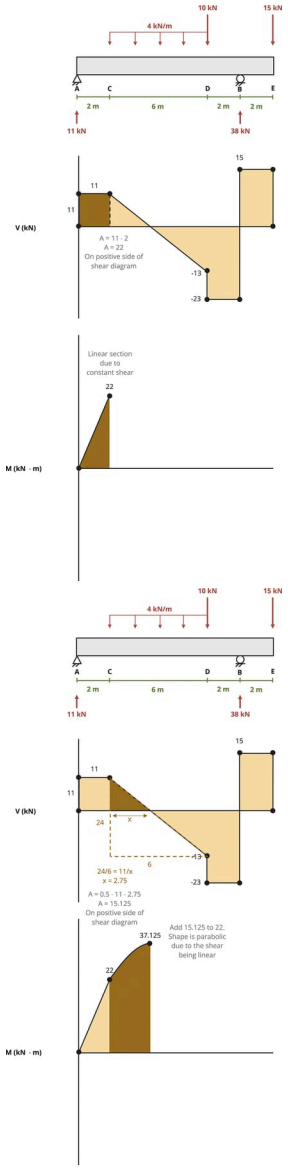
At point E see a 15 kN concentrated force going down. This concentrated force will cause a discontinuity in the shear diagram. From 15 kN subtract 15 kN to end back at zero.

The shear diagram should start and end at zero. At the end of the beam the shear diagram “closes,” which means that it ends back at zero. This is a good check that you are on the right track. However, rounding reactions might lead to a slightly off result at the end of your shear diagram.

Now having the shear diagram, you can build the moment diagram. Remember from the previous section that the internal moment is the area under the shear diagram. Because the shear diagram consists of basic shapes (rectangles and triangles), you can use geometry to find these areas.

Just as for the shear diagram, start at zero at the leftmost point of the beam, point A. The first section from A to C is a rectangle. Calculate the area under the shear curve to find the change in the internal moment between A and C. Keep in mind the following three things:

- **Magnitude of the change**: Magnitude of the change is the area under the curve. The height of the rectangle is 11 kN, and the width is 2 m, so the area is 22 kN·m.
- **Direction of the change**: Direction of the change is on the positive side of the shear diagram, which indicates that it will go up from A to C.
- **Shape of the segment**: The shear diagram is constant, so the moment diagram will be linear with a slope of 11.

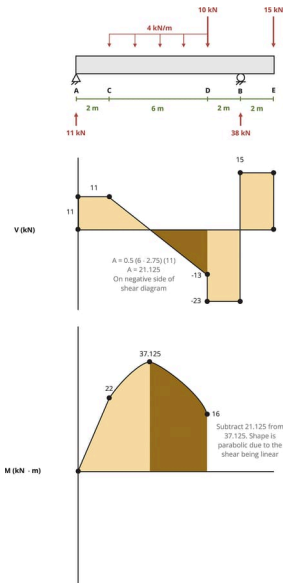


From point C to D there are two triangles, one that is on the positive side of the shear diagram and the other that is on the negative side.

To calculate the area under the curve, first calculate the distance from point C to where the shear diagram crosses the x-axis.

There are many ways to do this. We illustrate using similar triangles to find this distance. In the figure, we are comparing the larger yellow triangle and the smaller pink triangle, where unknown distance x is the base. To accomplish this we set up the following proportion:

$$\frac{24\ kN}{6\ m} = \frac{11\ kN}{x} \quad \therefore \quad x = 2.75\ \text{m}.$$



Magnitude of the Change

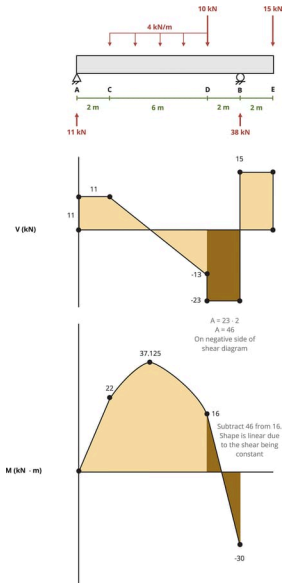
Magnitude of the change encompasses the area under the curve. The area of the triangle is 15.125 kN·m [A = ½ (2.75 m)(11 kN)].

Direction of the Change

This area is on the positive side of the shear diagram, which indicates it will go up from C to the zero point on the shear diagram. So add the area, 15.125, to the internal moment at point C (22 kN·m) to be at 37.125 kN·m at the point of zero shear.

Shape of the Segment

The shear diagram is linear, so the moment diagram will be parabolic and concave down.



We can now account for the second triangle from C to D.

Magnitude of the Change

Magnitude of the change encompasses the area under the curve. The area of the triangle is

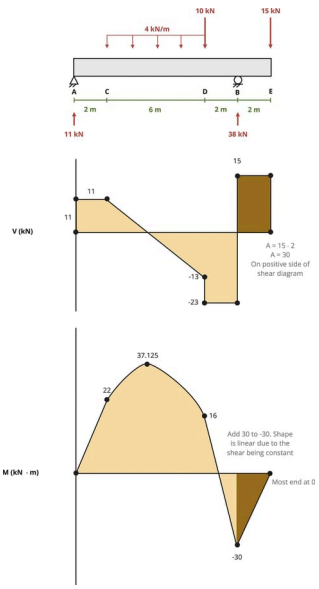
$$A = \frac{1}{2} * (6 - 2.75) * 13 = 21.125\ kN \cdot m.$$

Direction of the Change

This area is on the negative side of the shear diagram, which indicates it will go down from the point of zero shear to point D. So subtract the area 21.125 to the internal moment at the zero point (37.125 kN) to be at 16 kN·m at point D.

Shape of the Segment

The shear diagram is linear, so the moment diagram will be parabolic and concave down. The concavity of a parabolic function can be determined by examining whether the shear diagram is increasing or decreasing. In this example, the shear diagram is decreasing between points C and D, so the parabola will be concave down.



We are now able to account for the section from D to B.

Magnitude of the Change

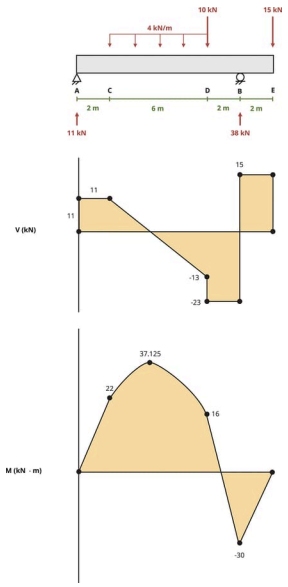
Magnitude of the change encompasses the area under the curve. The height of the rectangle is 23 kN, and the width is 2 m, so the area is 46 kN·m.

Direction of the Change

This area is on the negative side of the shear diagram, which indicates it will go down from D to B. Here subtract the area from the internal moment at D. So calculate 16 - 46 to end at negative 30.

Shape of the Segment

Because the shear diagram is constant, the moment diagram will be linear with a slope of –23.



Finally, we can build our moment diagram from B to E.

Magnitude of the Change

Magnitude of the change encompasses the area under the curve. The height of the rectangle is 15 kN and the width is 2 m, so the area is 30 kN·m.

Direction of the Change

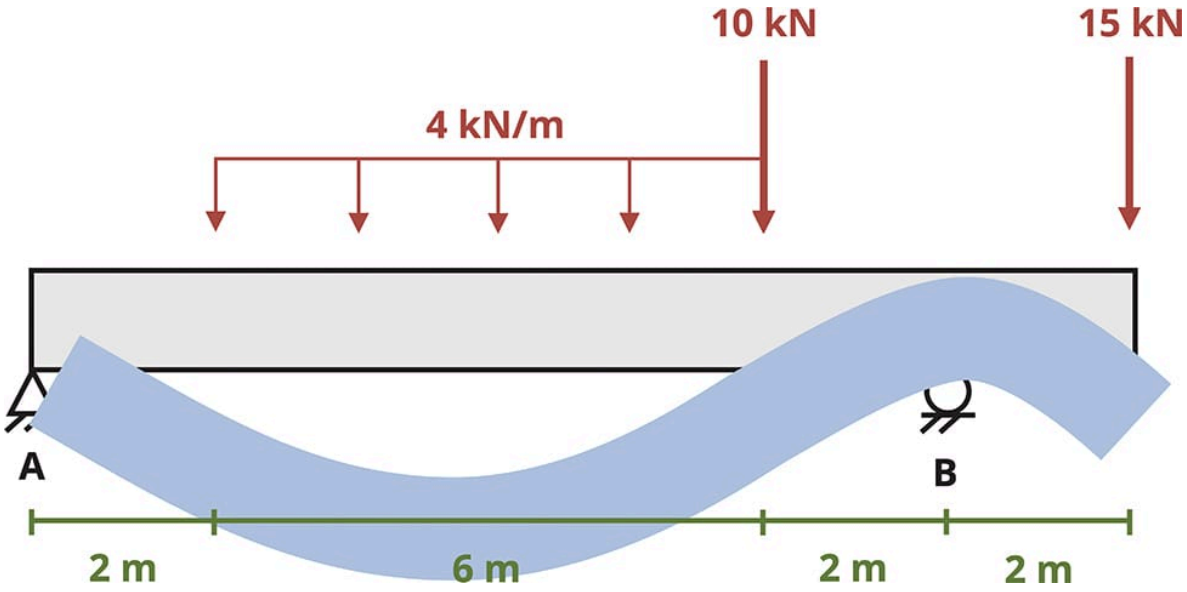
This area is on the positive side of the shear diagram, which indicates it will go up from B to E. Here add the area from the internal moment at B. So calculate (-30 + 30) to end at zero.

Shape of the Segment

The shear diagram is constant, so the moment diagram will be linear with a slope of +15.

At the end of the beam, the moment diagram “closes,” which means it ends back at zero. Moment diagrams must start and end at zero. This is a good check that you are on the right track. However, rounding reactions and areas might lead to a slightly off result at the end of the moment diagram.

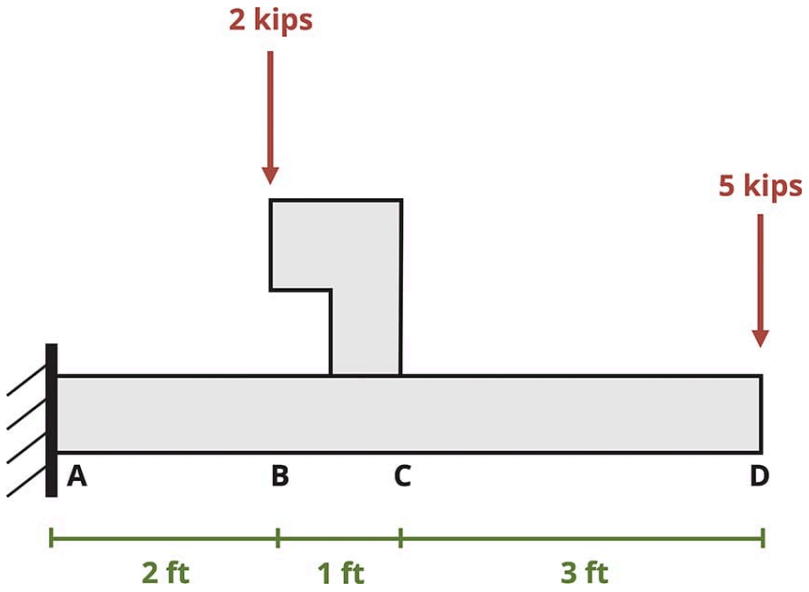
The final product is shown in the figure above. To have complete shear and moment diagrams you should label your axes, including units, and indicate the pertinent values on each diagram. As mentioned earlier, it's best to draw these diagrams right below the beam so that seeing what the internal shear and bending moment forces are in relation to a location on the beam is easy.



This example shows that this beam is subjected to both positive and negative moments. Remember that a positive moment indicates concave up bending behavior, and a negative indicates concave down. Given the beam and the external loads, it makes sense that the beam will want to bend concave up between supports in the area of positive moment. The beam will then bend concave down over support B in the area of negative moment, as shown in the figure.

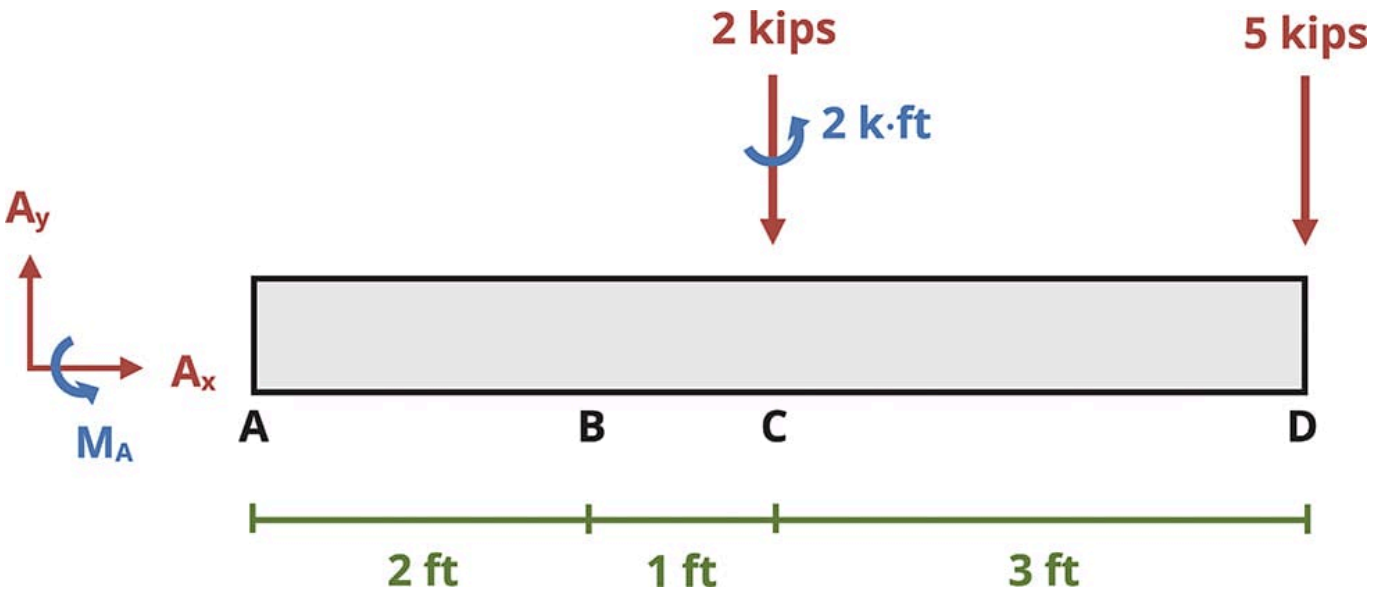
Example 7.5

Draw the shear force and bending moment diagrams for the beam shown.



Solution

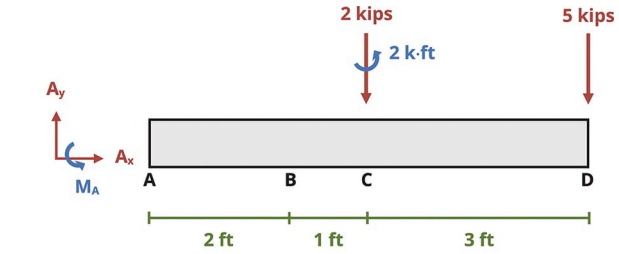
This beam differs a little from other examples in this chapter as the support is a fixed end and includes a force-couple system at point C. Nonetheless, the processes that have been presented here apply in this situation.

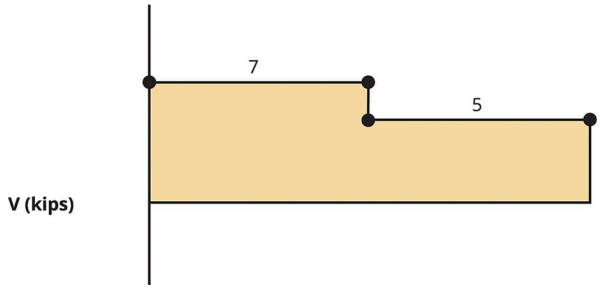


The first step is to draw an FBD of the beam, being sure to change the supports to the correct external reaction forces, as shown to the left. The 2 kips that acts on the arm at point B will cause a force on beam AD at the connection point, C. Additionally, a concentrated moment from this force also acts at point C. The magnitude of this concentrated moment is (2 kips)*(1 ft) or 2 k·ft counterclockwise.

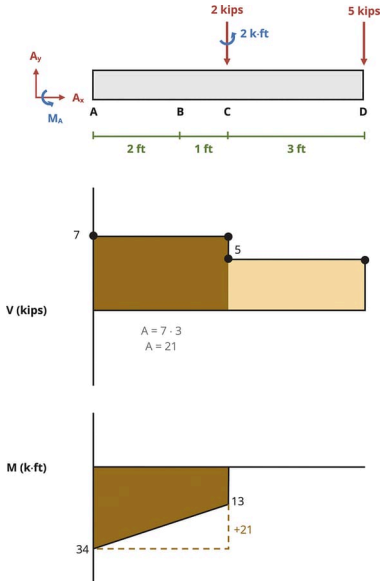
We then use static equilibrium equations to solve for the magnitude of the support reactions.

$$\sum F_x = Ax = 0$$
$$\sum F_y = A_y - 2\ kips - 5\ kips = 0 \ \rightarrow \ A_y = 7\ kips$$
$$\sum M_A = M_A + 2\ kip \cdot ft - 2\ kips\,(3\ ft) - 5\ kips\,(6\ ft) = 0 \ \rightarrow \ M_A = 34\ kip \cdot ft$$





We can now build the shear diagram. Keep in mind that the concentrated moments (at points A and C) do not contribute to the shear diagram.



We start with the vertical reaction at point A. This is going up 7 kips. Then because no loads are on the beam between A and C, the shear diagram remains constant.

The concentrated force at point C is 2 kips down. So we subtract 2 kips from 7 kips. With no loads between points C and D, the shear diagram remains constant at 5 kips.

At point D there is a concentrated load of 5 kips down. This brings the shear diagram to a close at zero.

We next build the moment diagram with the combination of the area of the shear diagram and the concentrated moments.

We start from zero and go down 34 kip·ft since the reaction at A, M_A, is counterclockwise.

Between A and B

Magnitude of the Change

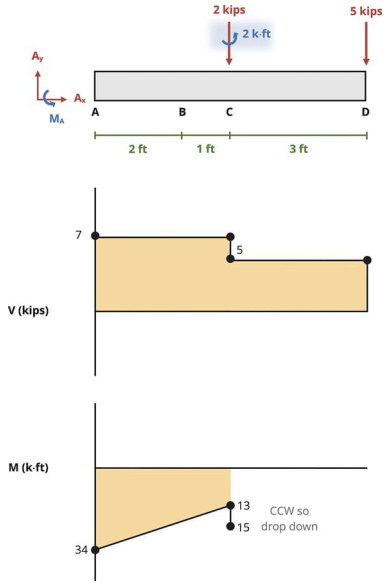
Magnitude of the change encompasses the area under the curve. The height of the rectangle is 7 kips and the width is 3 ft, so the area is 21 kip·ft.

Direction of the Change

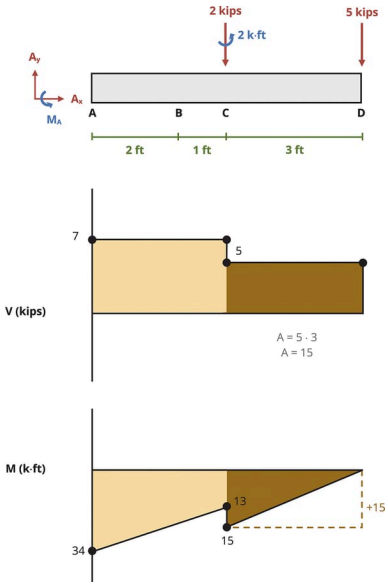
This area is on the positive side of the shear diagram, which indicates it will go up from A to C. We add the area from the internal moment at A (−34 + 21 = −13 kip·ft).

Shape of the Segment

The shear diagram is constant, so the moment diagram is linear with a slope of +7 kips/ft.



An applied concentrated moment of 2 kip·ft counterclockwise at point C creates a discontinuity in the moment diagram by dropping down 2 (–13 – 2 = –15 kip·ft).



Between C and D

Magnitude of the Change

Magnitude of the change encompasses the area under the curve. The height of the rectangle is 5 kips and the width is 3 ft, so the area is 15 kip·ft.

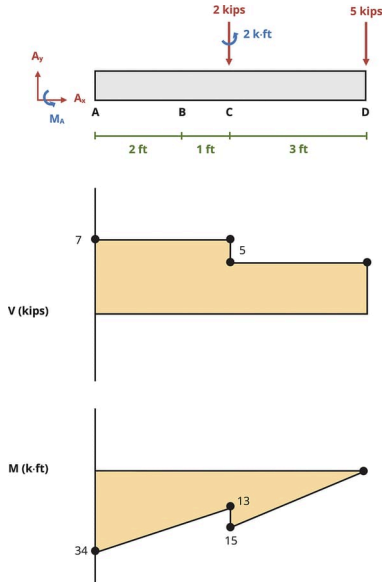
Direction of the Change

This area is on the positive side of the shear diagram, which indicates it will go up from C to D. Here we add the area from the internal moment at A (–15 + 15 = 0).

Shape of the Segment

The shear diagram is constant, so the moment diagram is linear with a slope of +5 kip/ft.

The final shear and moment diagram is in the figure to the left. Note that this cantilever structure with downward loads will bend concave down. This indicates that the entire beam will be in negative moment—which is also what our moment diagram indicates.



Step-by-Step: Shear Force and Bending Moment Diagrams (Graphical Method)

1. Use equilibrium to determine the reaction loads at any supports.
2. Use the loading diagram to draw the shear force diagram. Begin at the left end of the beam and move to the right, obeying these rules.
    - A concentrated force causes a discontinuity on the shear diagram. An upwards force causes a jump upward.
    - A distributed load causes a slope on the shear diagram equal to the size of the distributed load. An upward load causes a slope upward.
        - The total change in shear force due to a distributed load is equal to the area underneath the distributed load.
    - Anywhere there are no applied forces, the shear force does not change.
    - The shear force diagram should start and end at zero. If it does not, go back and check for errors.
3. Once the shear force diagram is complete, use it to draw the bending moment diagram. Begin at the left end of the beam and move to the right, obeying these rules.
    - An applied couple causes a discontinuity on the moment diagram. A clockwise couple causes a jump upward, and a counterclockwise couple causes a jump downward.
    - The slope of the moment diagram at any point is equal to the shear force at that point. As the shear force changes, the slope of the moment diagram will change too.
        - If the shear force is zero, the bending moment will be constant.
        - If the shear force is constant, the slope of the moment diagram will be linear.
        - If the shear force is linear, the slope of the moment diagram will be parabolic.

- If the shear force is positive, the moment diagram will slope upward.
- If the shear force is negative, the moment diagram will slope downward.
- As the magnitude of the shear force increases, the slope of the bending moment diagram becomes steeper.

◦ The bending moment diagram should start and end at zero. If it does not, go back and check for errors.

Summary

Key Takeaways

Shear and moment diagrams allow us to calculate and visualize the internal forces of beams. These internal forces are used to calculate stresses and deformations (covered in upcoming chapters). The general procedure for building the shear and moment diagrams is as follows:

1. Sketch the beam, replacing support conditions with equivalent force(s).
2. Find the support reactions using equilibrium.
3. Use the method of equations or geometry or a combination to build the shear diagram directly below your beam sketch.
4. Use the method of equations or geometry or a combination to build the moment diagram directly below your shear diagram.
5. Ensure that your diagrams are labeled, including units, and that all pertinent values are indicated.

Key Equations

Relationship between load and shear:

$$\Delta V = \Delta F$$

$$\underbrace{\Delta v}_{\substack{change\ in \\ shear}} = \int \underbrace{w dx}_{\substack{area\ under \\ loading\ curve}}$$

Relationship between shear and bending:

$$dM = V dx$$

$$\underbrace{\Delta M}_{Change\ in\ moment} = \int \underbrace{V dx}_{Area\ under\ shear\ diagram}$$

References

Figures

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- Figure 7.1: Examples of structural beams. A: Washington State Dept of Transportation. 2023. CC BY-NC-ND. https://flic.kr/p/2ojECo3. B: Government of Prince Edward Island. 2017. CC BY-NC-ND. https://flic.kr/p/R3NFRw. C: Jack E Boucher. 1985. Public domain. https://commons.wikimedia.org/wiki/File:FallingwaterCantilever570320cv.jpg.



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Chapter 8: Geometric Properties

Learning Objectives

- Define the centroids of areas
- Locate the centroid using composite parts
- Explain the principles behind determining the second moment of area
- Apply the parallel axis theorem to analyze the transformation of the second moment of area

As previous chapters have shown, geometric cross-section properties are critical to calculating stresses and deformations for members subjected to axial and torsional loads. The same is true for finding stresses and deflections in beams subjected to shear and bending loads. This chapter describes methods for finding section properties necessary to calculate the stress and deflection in beams—specifically, the centroid and the second moment of area (also known as moment of inertia and area moment of inertia).

You likely were presented with these topics in statics, so this chapter is designed to be a review. Section 8.1 reviews centroids, and Section 8.2 reviews second moment of area.

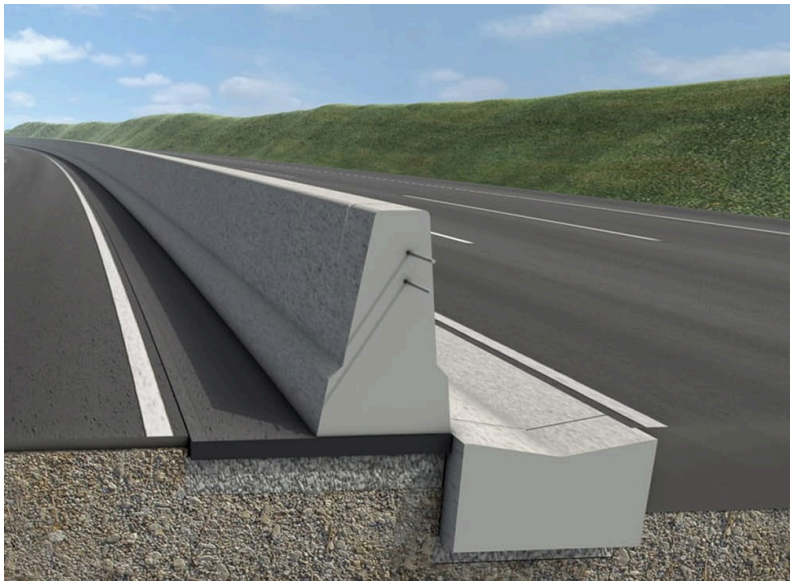


Figure 8.1: Isometric view of concrete barriers used in roadway construction.

8.1 Centroid

An object possesses weight as a result of the gravitational force acting on its constituent particles. The collective effect of these forces yields the total weight of the object, which is concentrated at a single point known as the center of gravity. If the object is composed of a homogenous (constant density) material, then the center of gravity is the same as the geometric center or centroid.

The discussion in this section focuses on the centroid of an area for common structural shapes. For a more comprehensive discussion on the center of gravity, the center of mass, calculating centroids using integration, and distributed loads, see the book *Engineering Statics* by Baker and Haynes.

The location of the centroid of an area is the point where the first moment of area equals zero. The first moment of area about

a point is calculated by multiplying the area by the perpendicular distance to the point. The centroid of an area can be found by splitting the area into a number (i) of discrete parts, summing the moment of area for these discrete parts, and then dividing by the sum of the area. These weighted averages enable us to find the location $\left(\overline{X}, \overline{Y}\right)$ of the centroid of the area.

$$\overline{X} = \frac{\sum [redacted] A_i}{\sum}$$

$$[redacted]$$

$$[redacted]$$

$$[redacted] = \frac{\sum [redacted] A_i}{\sum A_i}$$

(8.1)

where

- $\overline{X}, \overline{Y}$ = *Centroid coordinates of the overall area [m, in.]*
- $\overline{x}_i, \overline{y}_i$ = *Centroid coordinates of each discrete area [m, in.]*
- A_i = *Area of each discrete area [m^2, $in.^2$]*

We can find centroids of simple shapes easily by utilizing symmetry. When a shape possesses an axis of symmetry, each point on one side of the axis corresponds to another point symmetrically located on the opposite side. The distances between these mirrored points and the line of symmetry sum to zero because they are equal in magnitude but opposite in sign. This is true for every point in the shape, so the numerator of Equation 8.1 (the first moment of area) will be zero. Therefore, if a shape features a line of symmetry, its centroid must coincide with this line.

In cases where a shape exhibits multiple lines of symmetry, the centroid is located at their intersection, as shown in Figure 8.2.

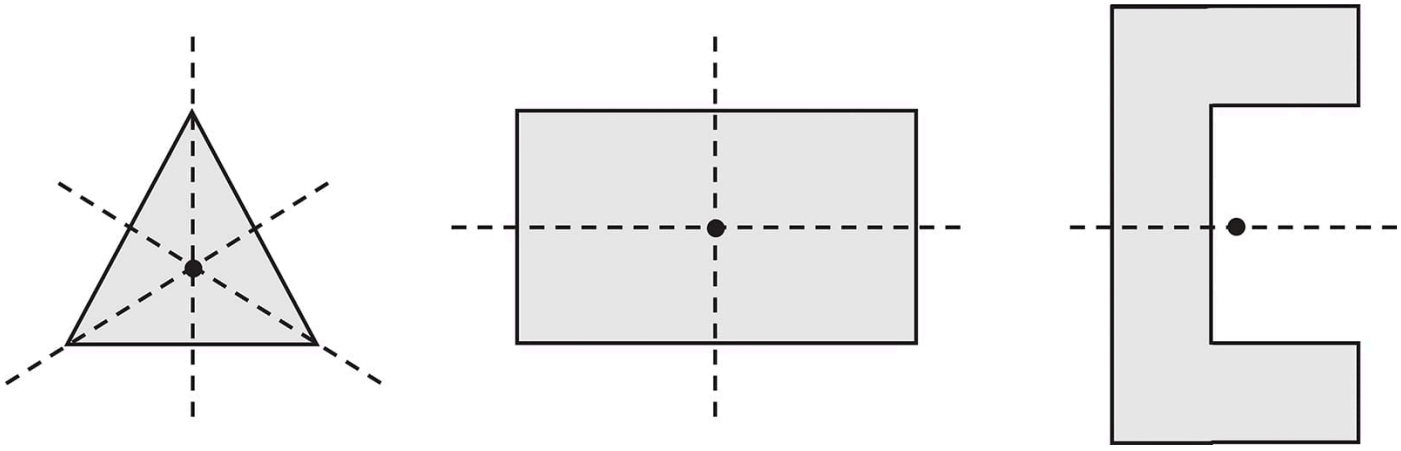


Figure 8.2: Centroids lie on axes of symmetry.

This text uses common shapes or a composite of common shapes for cross-sections. The location of the centroid of some common shapes can be found in Appendix E, part of which is reproduced in Figure 8.3. Each row of the table includes a common shape with the centroid marked and provides simple formulas for determining the centroid's x and y coordinates and calculating the area of the shape.

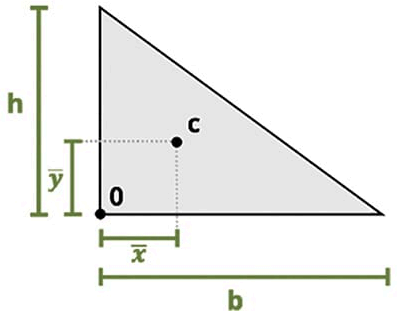
Shape	$\bar{x}$	$\bar{y}$	Area
	$\frac{b}{3}$	$\frac{h}{3}$	$\frac{1}{2}bh$

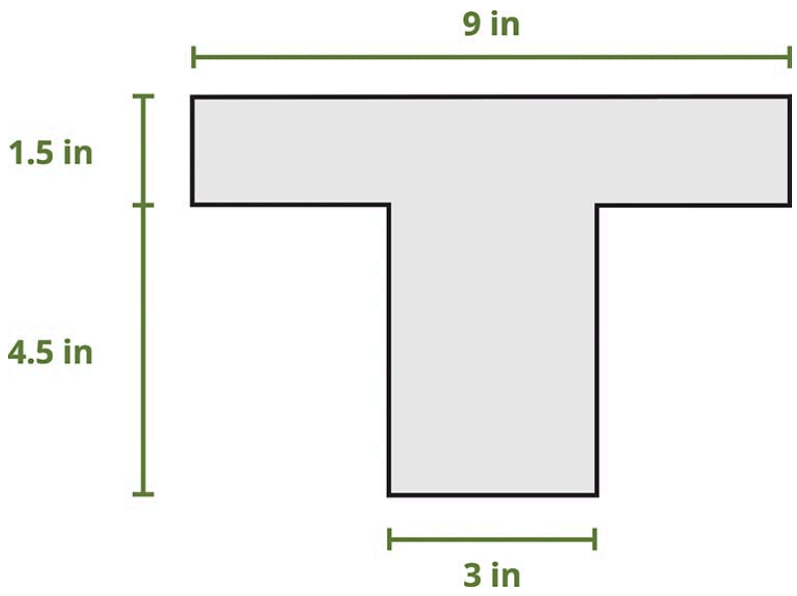
Figure 8.3: Reproduction of a shape in Appendix E. Appendix E includes a sketch of the area with the centroid marked, formulas for determining the centroid's x and y coordinates, and a formula for calculating the area of the shape.

Often structural sections are combinations of these standard shapes. These types of sections are called composite sections. As long as we know the location of the centroid of each shape in the section, we do not need to use integration to find the location of the section's overall centroid. We can use Equation 8.1 in conjunction with the table in Appendix E to find the location of the centroid for composite shapes.

Example 8.1, Example 8.2, and Example 8.3 demonstrate the process for determining the centroid coordinates of composite areas.

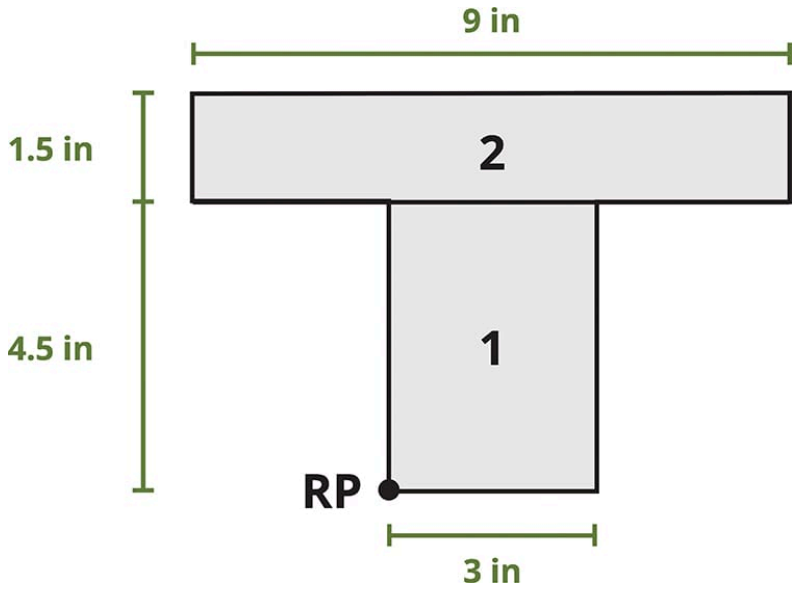
Example 8.1

Find the centroid of the shape below.



Solution

Notice that this T section is symmetric about the y-axis. This means that calculating the centroid in the x direction is unnecessary because we know it will be on that y-axis. We need only find the distance in the y direction, $\overline{Y}$.



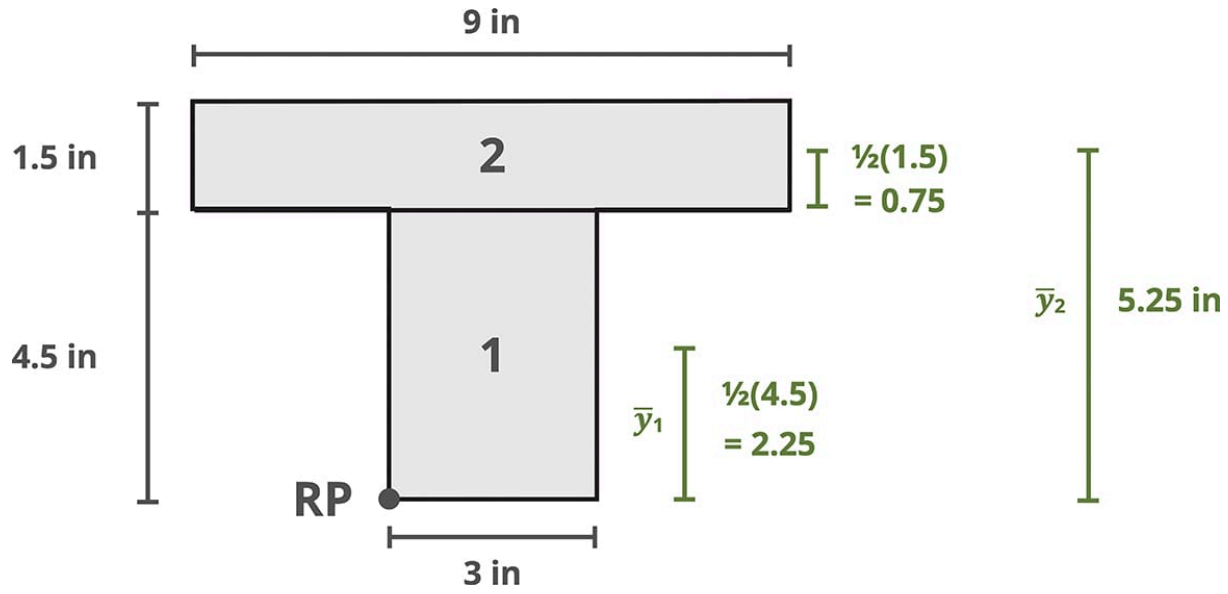
The first thing to do is establish a coordinate system where the origin is the reference point, RP. In this example we chose the bottom left corner of the section so that all y distances will be positive.

The second step is to divide the section into simpler shapes where we know the location of the centroid. In this case we chose two rectangles, labeled 1 and 2 in the figure.

Shape	A (in.^{2})	$\bar{y}$ (in.)	$A\bar{y}$ (in.^{3})
1	13.5	2.25	30.375
2	13.5	5.25	70.875
	$\sum A = 27$		$\sum A\bar{y} = 101.25$

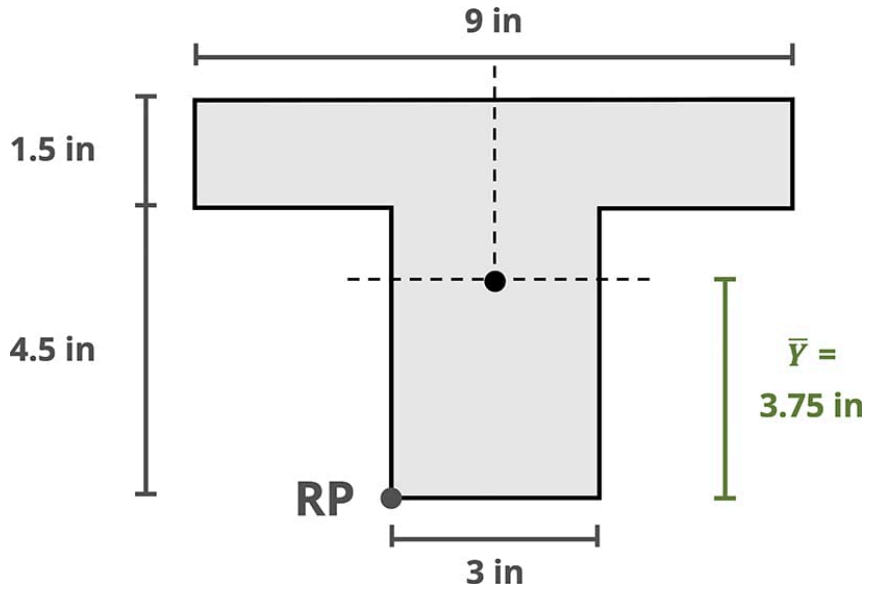
Next is to create a table with columns titled Shape, A (in.^{2}), $\bar{y}$ (in.), and $A_{\bar{y}}$ (in.^{3}). Given two shapes, there will be two rows in the table. Column A is the area of the individual shape. Column $\bar{y}$ is the perpendicular distance in the y direction from the RP to the centroid of that shape. For example, for shape 1 the $\bar{y}_1$ would be half of 4.5 in. since the bottom of shape 1 aligns with the RP. However, for shape 2 the $\bar{y}_2$ would be 4.5 in. plus one-half of 1.5 in. This formula ensures measuring from the RP as illustrated here.

$$\bar{Y} = \frac{101.25\ in.^3}{27\ in.^2} = 3.75\ in.\ (\text{from bottom})$$



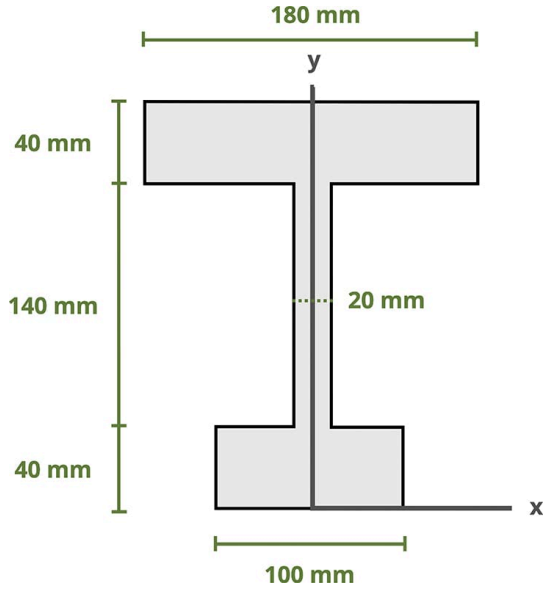
Finally, to calculate the centroid coordinates, use Equation 8.1 by summing the last column and the second column in the table. Using this approach we found the centroid to be 3.75 in. from the reference point in the y direction and on the line of symmetry in the x direction.

The following figure shows the location of the centroid for this T section.



Example 8.2

Find the centroid of the illustrated shape.

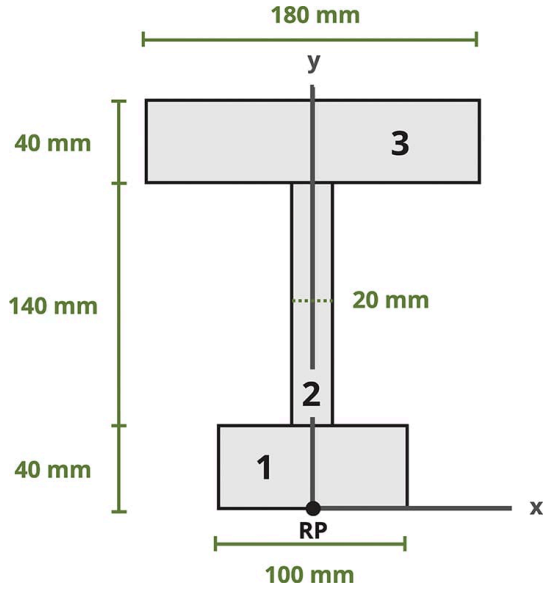


Solution

Notice that this section is symmetric about the y-axis, similar to Example 8.1. This means calculating the centroid in the x direction is unnecessary because we know it will be on that y-axis. We need only find the distance in the y direction, $\overline{Y}$.

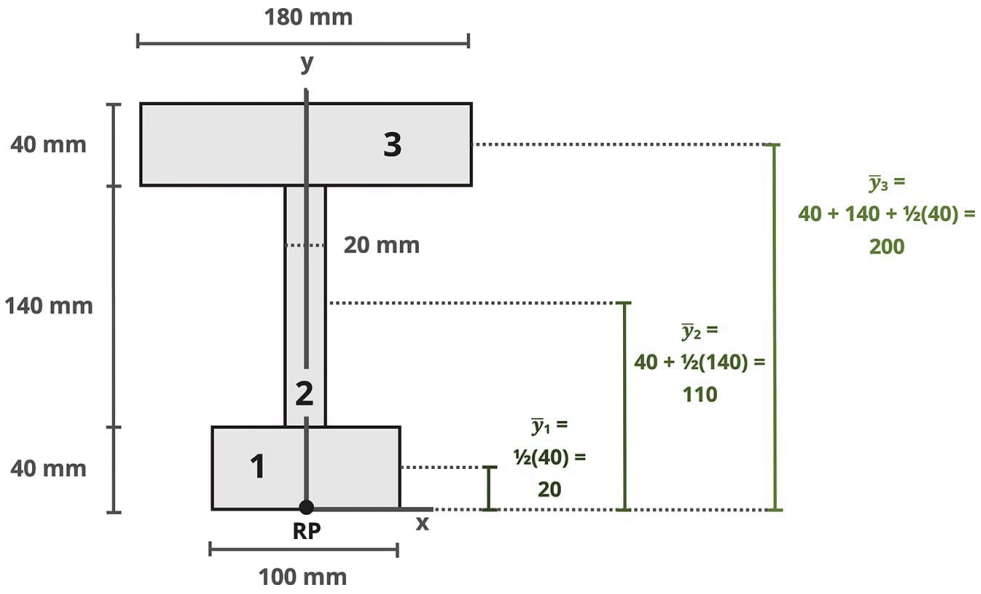
The first thing to do is establish a coordinate system whose origin is the reference point, RP. In this example, we chose the bottom of the section along the y-axis so that all y distances will be positive.

The second task is to divide the section into simpler shapes for which we know the centroid's location. In this case we chose three rectangles labeled 1, 2, and 3 in the figure.



Given three shapes, there will be three rows. The first column, A, is the area of the individual shape. The second column, $\overline{y}$, is the perpendicular distance in the y direction from the RP to the centroid of that shape.

The figure below shows how the $\overline{y}$ for each shape is calculated.



Next is to find the centroid ($\bar{x} = 0$ due to symmetry).

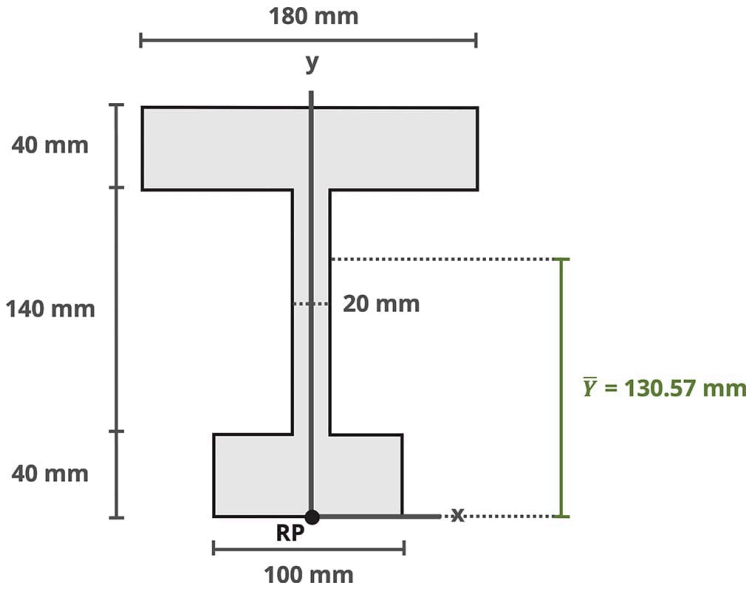
Shape	A(mm^2)	$\bar{y}$ (mm)	$A\bar{y}$ (mm^3)
1	4,000	20	80,000
2	2,800	110	308,000
3	7,200	200	1,440,000
	$\sum A = 14{,}000$		$\sum A\bar{y} = 1,828,000$

Finally, to calculate the centroid coordinates, use Equation 8.1 by summing the table's second and last columns.

$$\bar{Y} = \frac{1{,}828{,}000\ mm^3}{14{,}000\ mm^2} = 130.6\ mm \text{ from bottom}$$

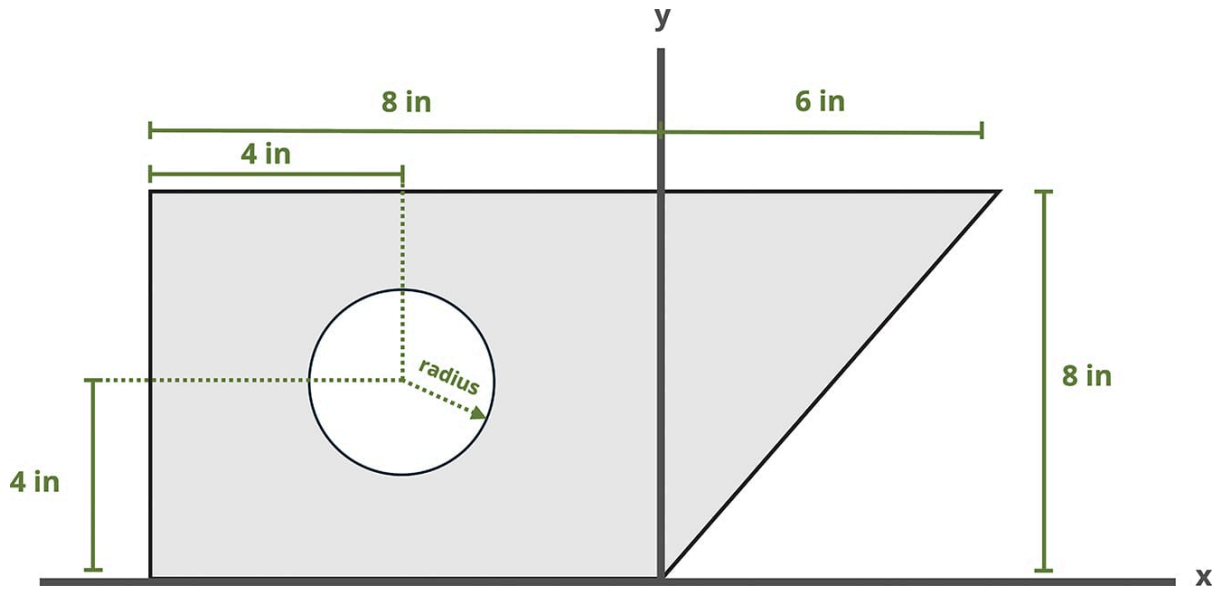
Using this approach we found the centroid to be 130.6 mm from the reference point in the y direction and on the line of symmetry in the x direction.

The figure below shows the location of the centroid for this section.



Example 8.3

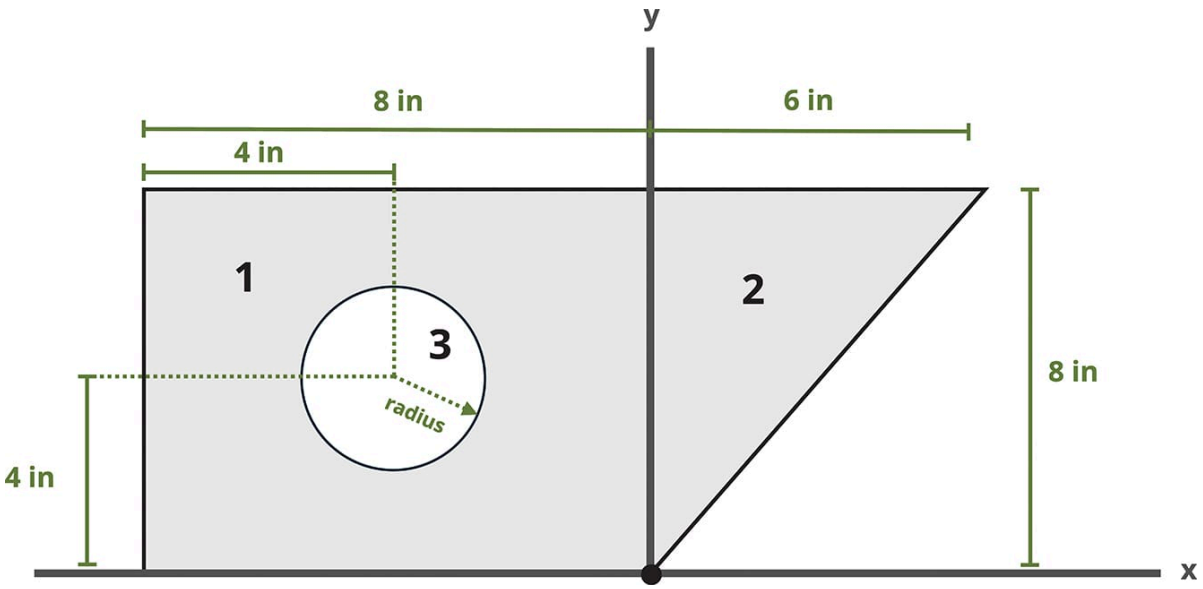
Find the centroid of the shape below.



Solution

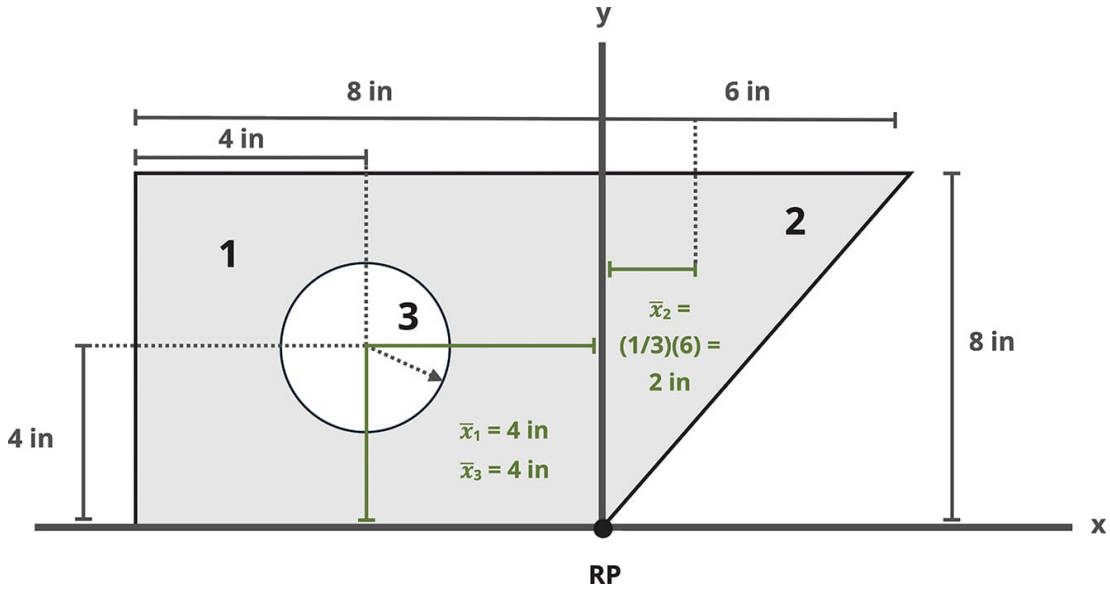
This section has no axis of symmetry, so we must find both the $\overline{x}$ and the $\overline{y}$. The first task is to establish a coordinate system whose origin is the reference point, RP. In this example the coordinate axis in the problem illustrates when $\overline{x}$ will be negative.

The second task is to divide the section into simpler shapes for which we know the centroid's location (Appendix E). In this case we will break this up into the square, triangle, and circle labeled 1, 2, and 3 in the figure. Note that the circle is a hole and the area will be represented as negative.

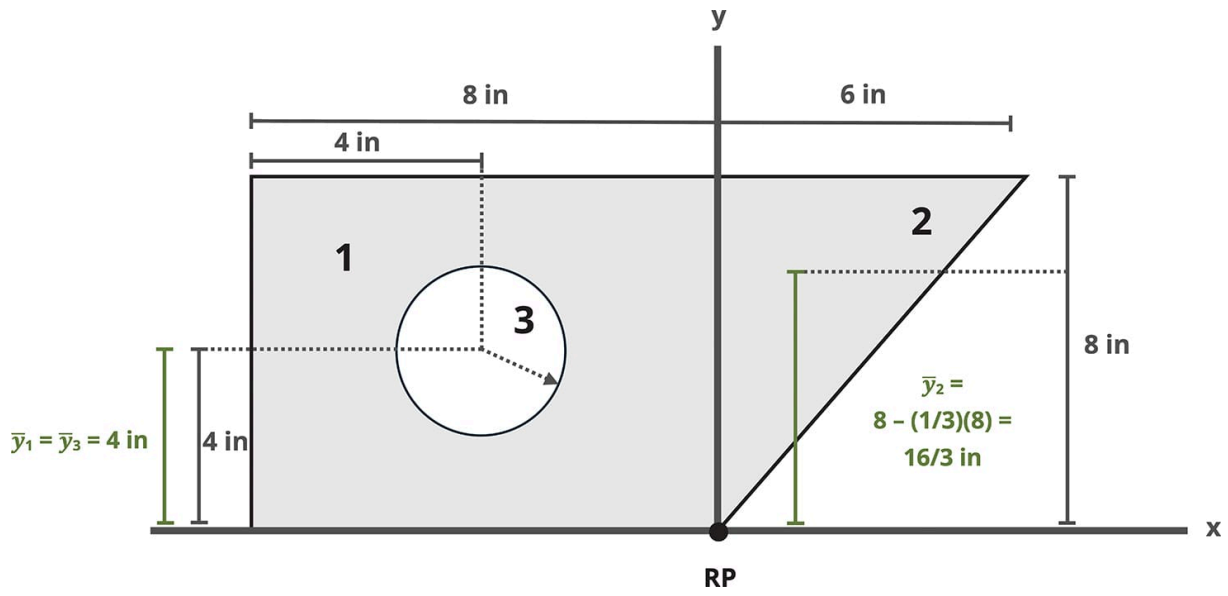


Next is to create a table with columns titled Shape, A (in^2), $\overline{x}$ (in), $\overline{y}$ (in), and $\mathrm{A}\overline{y}$ (in^3). Given three shapes, there are three rows. The first column, A, is the area of the individual shape. The second column, $\overline{x}$, is the perpendicular distance in the x direction from the RP to the centroid of that shape.

The figure below shows how the $\overline{x}$ for each shape is calculated.



The figure below shows how the $\overline{y}$ for each shape is calculated.



Finally, to calculate the centroid coordinates, use Equation 8.1 by summing the A, $A\bar{x}$, and $A\bar{y}$ columns shown in the table below.

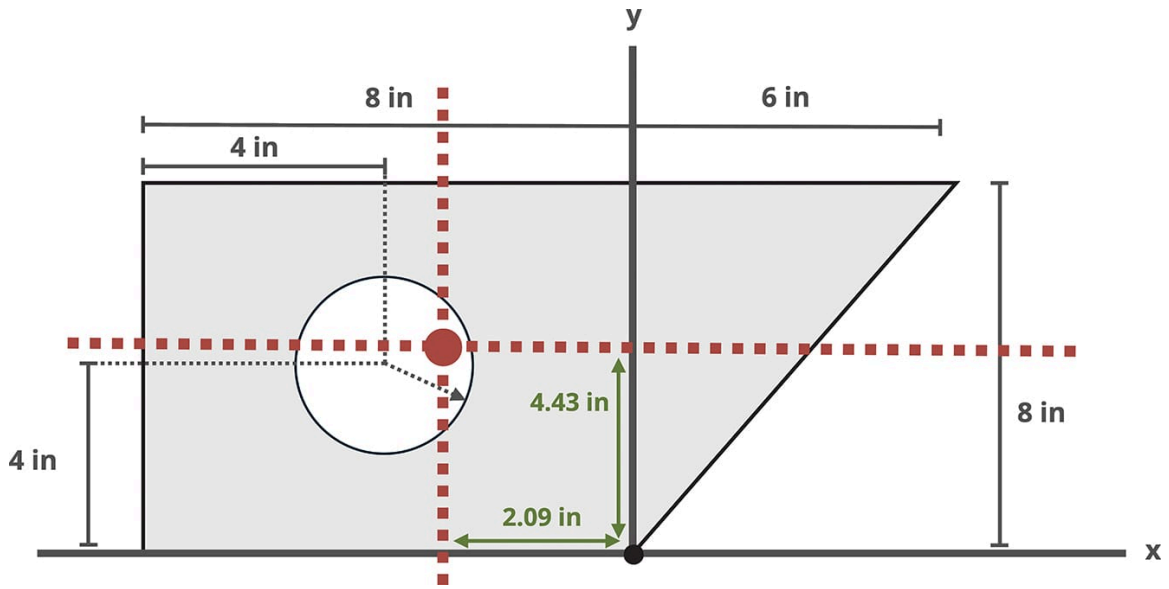
Shape	A (in.^{2})	x (in.)	$A\bar{x}(\text{in.}^3)$	$\bar{y}(\text{in.})$	$A\bar{y}(\text{in.}^3)$
1	64	-4	-256	4	256
2	24	2	48	(2/3)(8) = 16/3	128
3	–4π	-4	16π	4	–16π
	$\sum A = 75.4$		$\sum A\bar{x} = -258.3$		$\sum A\,\bar{y} = 333.7$

$$\bar{X} = \frac{\sum A\bar{x}}{\sum A} = \frac{-157.7\ in.^3}{75.4\ in.^2} \rightarrow \bar{X} = -2.09\ in.$$

$$\bar{Y} = \frac{\sum A\,\bar{y}}{\sum A} = \frac{333.7\ in.^3}{75.4\ in.^2} \rightarrow \bar{Y} = 4.43\ in.$$

Using this approach, we calculated the centroid to be –2.09 in. in the x direction from the reference point, which indicates it will be to the left of the y-axis. The y component of the centroid is 4.43 in. in the y direction from the reference point.

The figure below shows the location of the centroid for this section.



Step-by-Step: Finding the Centroid of Composite Sections

To find the location of the composite section's centroid where the location of each part's centroid is known, use the following procedure:

1. Establish a coordinate axis where the origin is the reference point (RP). This is where the centroid will be measured.

2. Break up the overall section into simpler shapes where the location of the centroid of each shape is known. It is helpful here to sketch this. If the section has a hole, include the hole as a shape with a negative area. Remember to take advantage of symmetry. If the overall section is symmetric about an axis, you already know the location of the centroid in that direction as it is on that axis.

3. Determine the area (A), the perpendicular distance from the y-axis ($\bar{x}$), and the perpendicular distance from the x-axis ($\bar{y}$) to the centroid of each shape. As sections become more complex, it might be advantageous to arrange these values in tabular form, as shown in this section.

4. Apply Equation 8.1 to find the section's overall centroid, $(\bar{X},\bar{Y})$. Remember that this location is measured from the RP established in step 1.

5. Check that your answer is reasonable. Imagine any straight line that passes through the centroid. There should be an equal amount of area on each side of this line. Mark the centroid location on the composite area and check that this appears roughly accurate.

8.2 Second Moment of Area

Another important geometric property needed to calculate stresses in beams is the second moment of area (also known as moment of inertia and area moment of inertia). You likely encountered this topic in statics, as you did the topic of centroids, so this section is designed as a review. By definition the second moment of area depicted in Figure 8.4 is

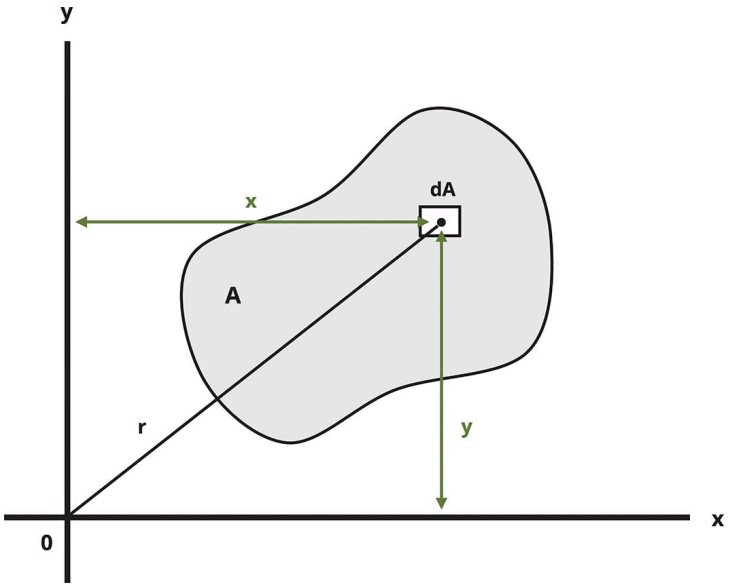


Figure 8.4: Second moment of area about dA

$$\boxed{\begin{aligned} I_x &= \int_A y^2 dA \\ I_y &= \int_A x^2 dA \\ I_z = J &= \int_A r^2 dA \end{aligned}}. \tag{8.2}$$

Keep in mind that this text uses common shapes or a composite of common shapes for cross-sections. Appendix E includes formulas to calculate the second moment of area. Part of Appendix E is reproduced in Figure 8.5. Each row of the table includes a common shape with the centroid marked and with formulas for determining the second moment of area about the centroidal axes x, y, and z. Note that these equations are valid only for the axes passing through the area's centroid and are written as $\overline{I_{x'}}$, $\overline{I_{y'}}$, and J respectively. The bar notation is used to signify that the second moment of area is being calculated around the centroidal axes.

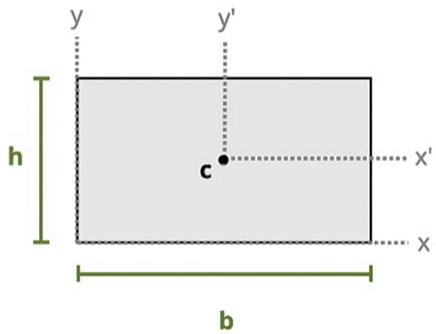
Shape	Ix	Iy
	$I_x = \frac{bh^3}{3}$ $\bar{I}_{x'} = \frac{bh^3}{12}$	$I_y = \frac{hb^3}{3}$ $\bar{I}_{y'} = \frac{hb^3}{12}$

Figure 8.5: Reproduction of a shape in Appendix E. Appendix E includes a sketch of the area with the centroid marked and formulas for determining the second moment of area about the centroidal axes x, y, and z.

As noted previously, structural sections, called composite sections, are often combinations of these standard shapes. As with calculating centroids, we break these composite sections into common shapes and find the second moment of area of these individual shapes. When calculating each individual shape's second moment of area, we must ensure that they are all about the same axis.

The parallel axis theorem provides a method for calculating the second moment of area about an axis parallel to an axis passing through the centroid, given that the second moment of area about the latter axis is known.

To establish the parallel axis theorem, we'll next examine the second moment of area about A relative to an axis x, as illustrated in Figure 8.6.

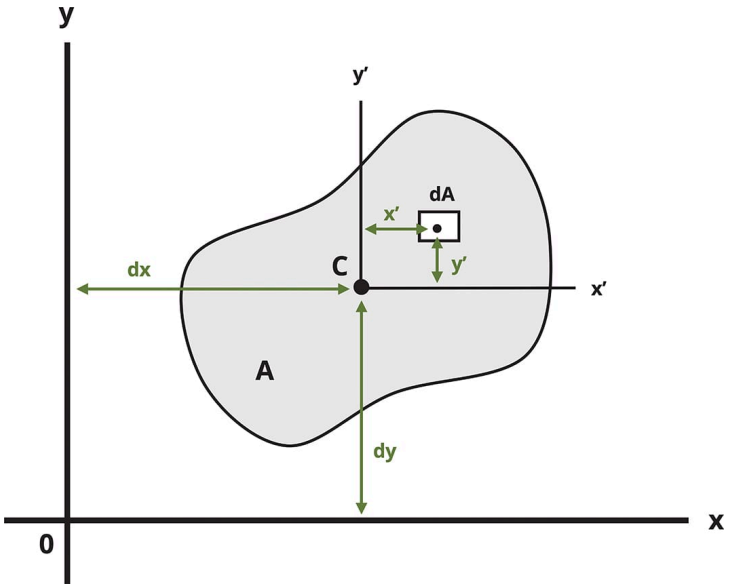


Figure 8.6: The second moment of area about A about an axis x can be derived from its second moment of area about the centroidal axis x' through a computation involving the distance dy between the axes.

The centroid of the area is at C, where the x'-y' axis is drawn; note that x' is parallel to x and y' is parallel to y. The x-y' axes will be termed the centroidal axes and are at a distance of d_x and d_y from the x-y axes respectively. The distance between the element dA and the x'-axis is denoted as y'. The result is $y = y' + d_y$. Substituting into Equation 8.2 yields

$$\begin{aligned} I_x &= \int (y' + d_y)^2 dA \\ I_x &= \int (y')^2 dA + 2d_y \int y' dA + d_y^2 \int dA. \end{aligned}$$

The first integral represents the second moment of area about the centroidal axis x'. The second integral is zero since the centroid is located on the x'-axis (i.e., $\int y' dA$ represents the first moment of area about x' centroidal axis, which is zero, as discussed in Section 8.1). The last integral represents the total area, A. So the resulting equation is

$$\boxed{I_x = \overline{I}_{x'} + Ad_y^2}.$$

(8.3)

A similar process is used to find an expression for I_y.

$$\boxed{I_y = \overline{I}_{y'} + Ad_x^2},$$

(8.4)

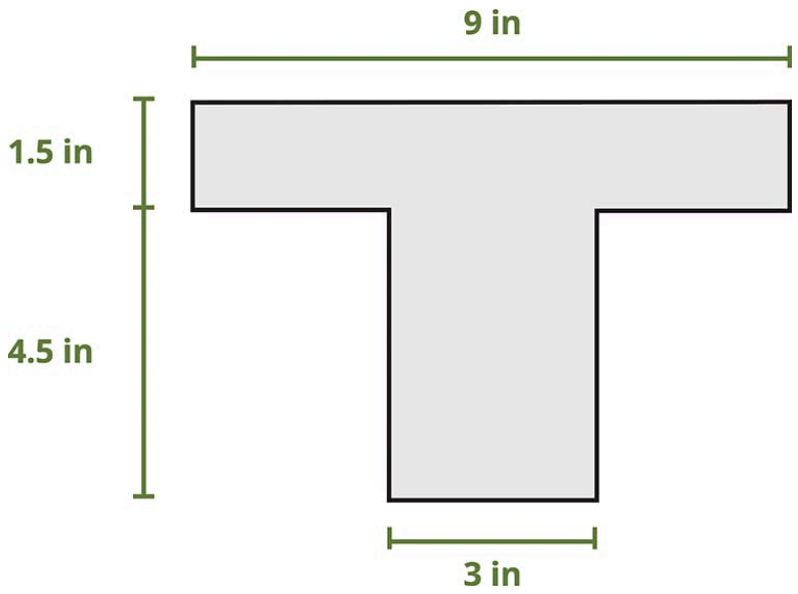
where

- I_x, I_y = *Second moment of area with respect to a given axis [m^4, $in.^4$]*
- $\overline{I_{x'}}, \overline{I_{y'}}$ = *Second moment of area with respect to a parallel axis passing through the centroid of the area [m^4, $in.^4$]*
- A = *Area [m^2, $in.^2$]*
- d_x, d_y = *Perpendicular distance between the given axis and the parallel centroidal axis [m, in.]*

While the parallel axis theorem can be used to calculate the second moment of area around any pair of axes, here the focus is primarily on calculating the second moment of area around the centroidal axes of a composite region (this is the term used in Chapter 9 and Chapter 10 when calculating bending stresses and shear stresses). Example 8.4, Example 8.5, and Example 8.6 demonstrate using the parallel axis theorem to find the second moment of area for the composite areas from the earlier examples about their centroidal axes.

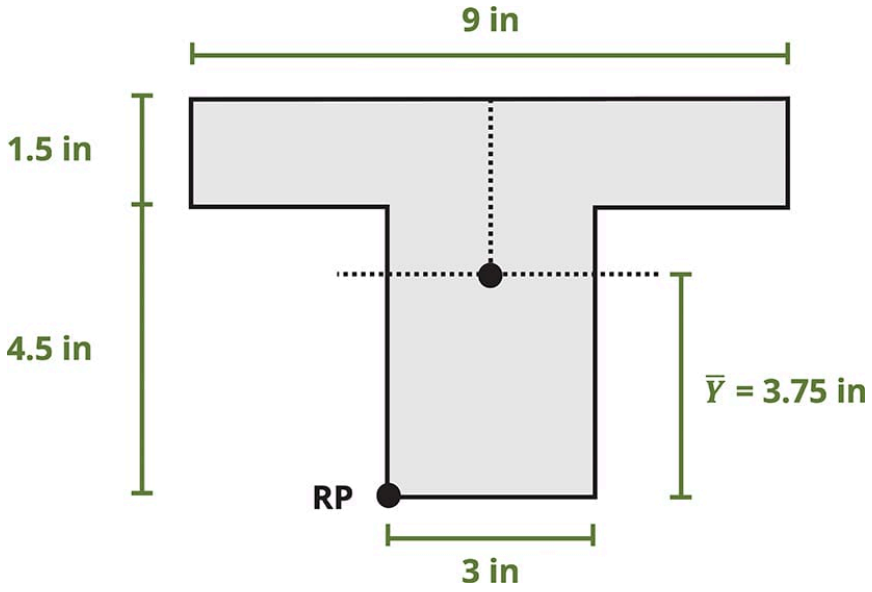
Example 8.4

Determine the I_x of the shaded area with respect to the section's centroid.



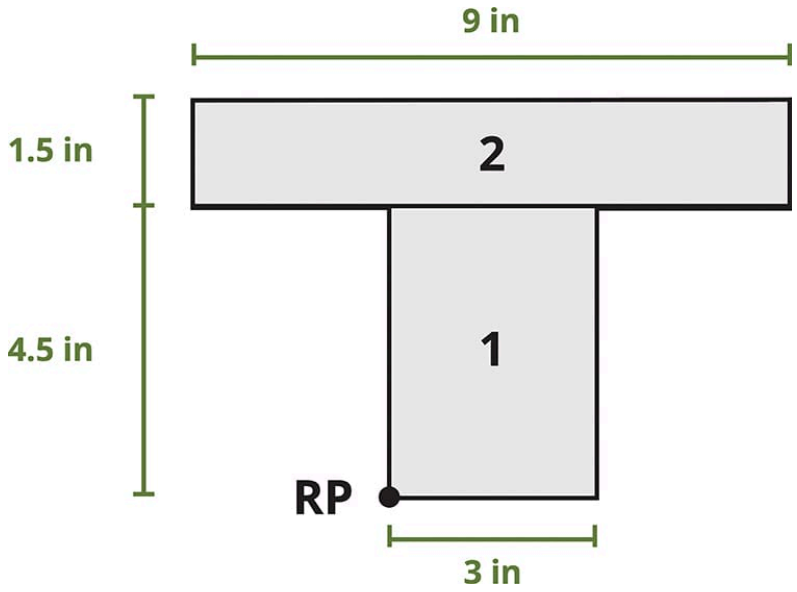
Solution

To find the second moment of area in the x direction with respect to the section's centroid, first calculate the centroid. We did this in Example 8.1, with the results as shown below.



We found that the centroid is located 3.75 in. from the bottom of the section.

The next step is to break the section into common shapes.



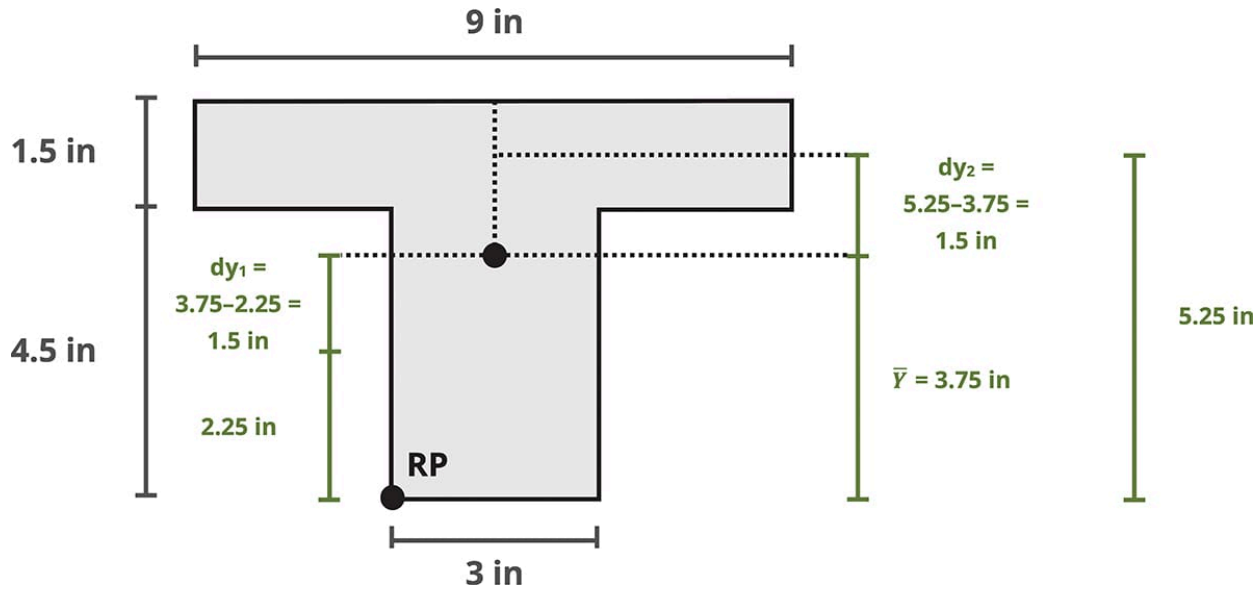
The formula for the second moment of area of a shape are with respect to the shape's centroid, not to the section's overall centroid. Make a necessary adjustment using the parallel axis theorem to transform the rotational axis from the shape's centroid to the section's centroid. Do this for each shape in the section.

Present this method in terms of a table as shown.

Shape	A (in.^{2})	I_x(in.^{4})	d_y(in.)	$I_x + Ad_y^2$ (in.^{4})
1	(3)(4.5) = 13.5	(1/12)(3)(4.5)^3 = 22.78125	1.5	22.78125 + (13.5)(1.5)^2 = 53.15625

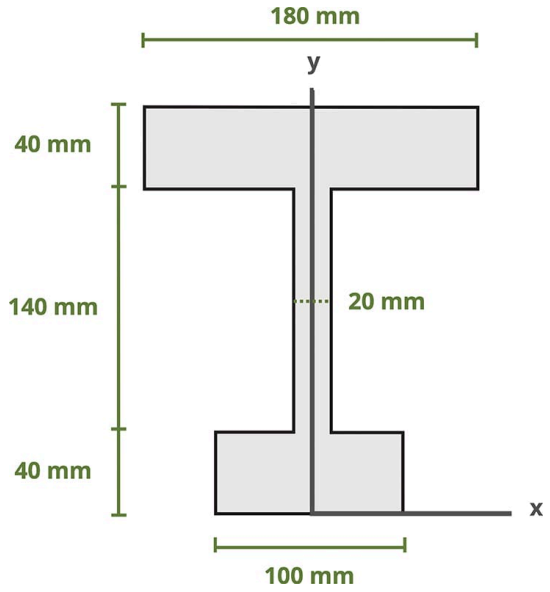
Shape	A (in.^{2})	I_x(in.^{4})	d_y(in.)	$I_x + Ad_y^2$ (in.^{4})
2	(9)(1.5) = 13.5	(1/12)(9)(1.5)^3 = 2.53125	1.5	2.53125 + (13.5)(1.5)^2 = 32.90625
				I_x = 86.0625 in.^{4}

The figure below shows the calculation and visualizations for each d_y term.



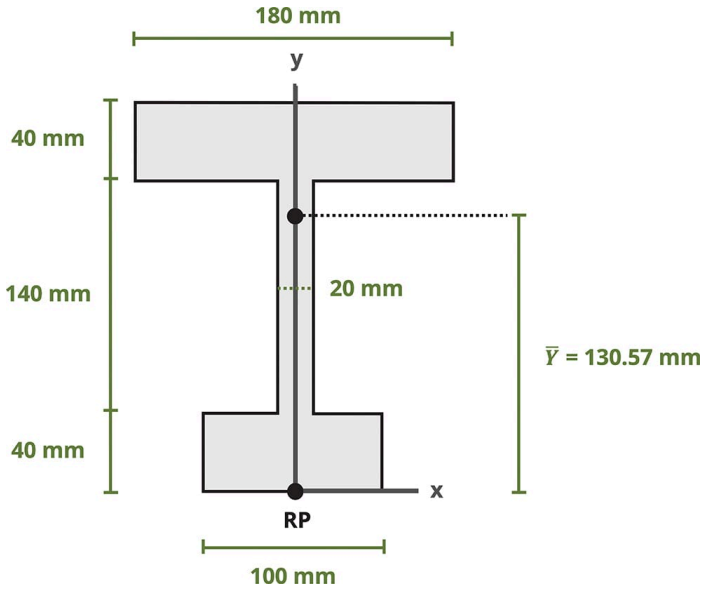
Example 8.5

Determine the I_x and I_y of the shaded area with respect to the shape's centroid.



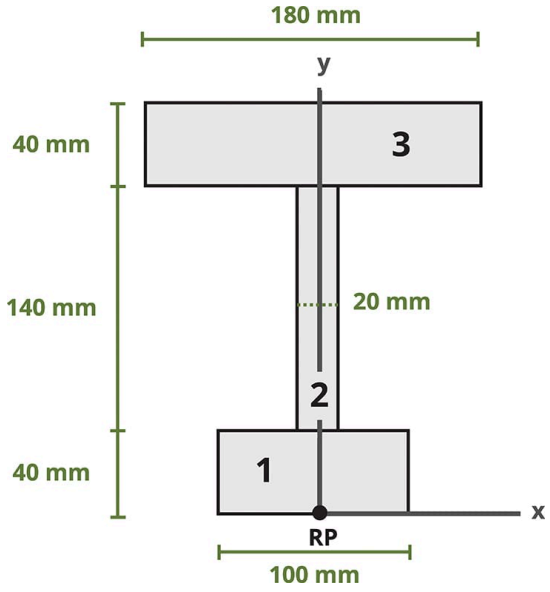
Solution

To find the second moment of area in the x and y directions with respect to the section's overall centroid, first calculate the centroid. We did this in Example 8.2, with the results shown below.



We found that the centroid is located 130.6 mm from the bottom of the section and on the y-axis.

The next step is to break the section into common shapes.

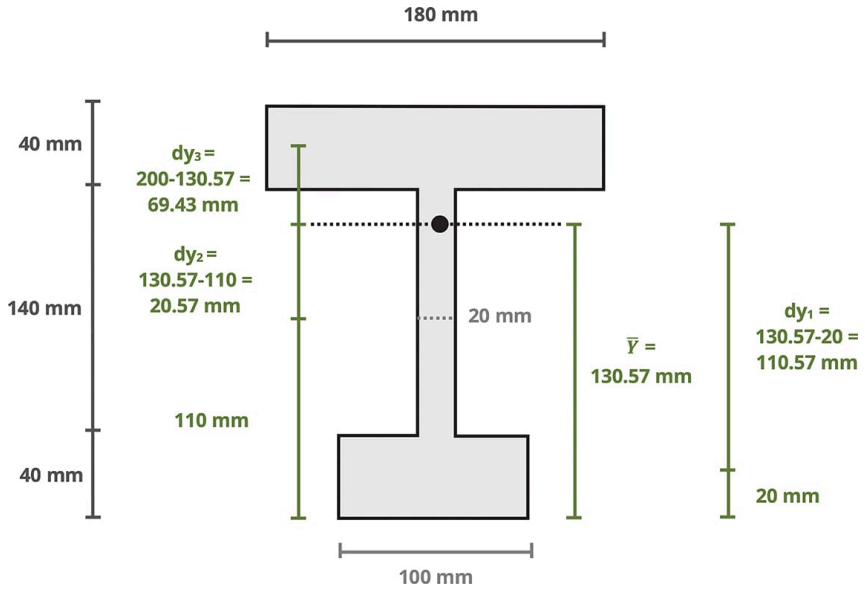


The formula for the second moment of area of a shape are with respect to the shape's centroid, not to the section's overall centroid. Make a necessary adjustment using the parallel axis theorem to transform the rotational axis from the shape's centroid to the section's centroid. Do this for each shape in the section.

Present this method in terms of a table, first for I_x.

Shape	A (in.^{2})	I_x (in.^{3})	d_y (mm)	$I_x + Ad_y^2$ (mm^4)
1	(100)(40) = 4,000	(1/12)(100)(40)^3 = 533,333	110.57	533,333 + (4,000)(110.57)^2 = 49,436,233
2	(20)(140) = 2,800	(1/12)(20)(140)^3 = 4,573,333	20.57	4573,333 + (2,800)(20.57)^2 = 5,758,083
3	(180)(40) = 7,200	(1/12)(180)(40)^3 = 960,000	69.43	960,000 + (7,200)(69.43)^2 = 35,667,779
				I_x = 90,862,095 mm^4

The calculation and visualizations for each d_y term are as shown in the figure.



Use the same process to calculate I_y.

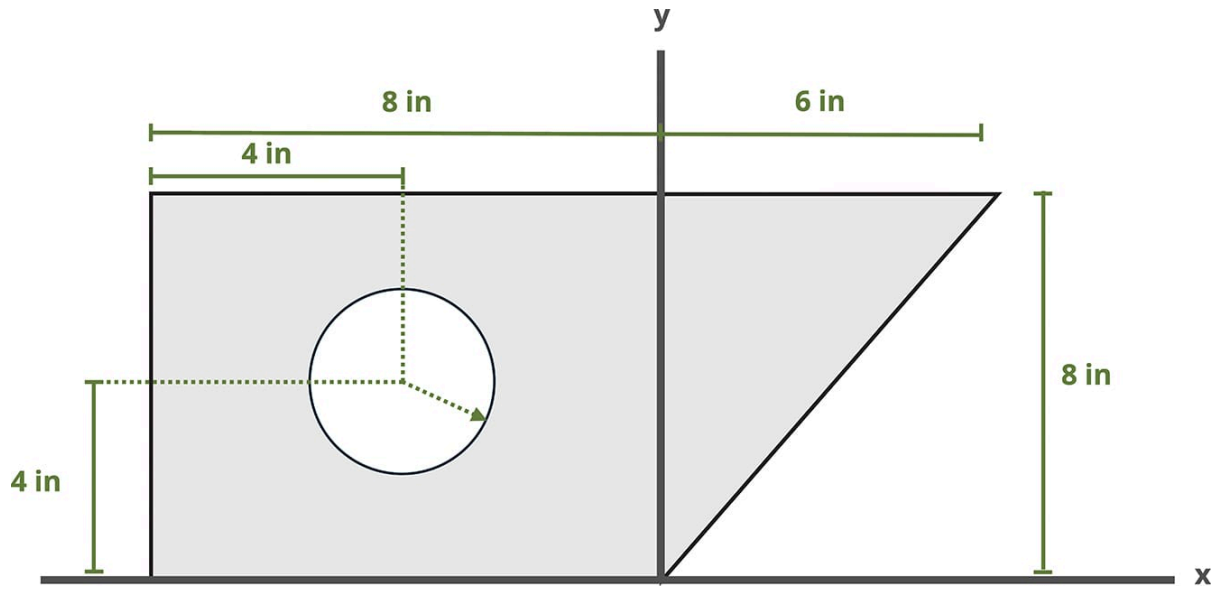
Shape	A(mm^2)	I_y(mm^4)	d_x(mm)	$I_y + Ad_x^2$ (mm^4)
1	(100)(40) = 4,000	(1/12)(40)(100)^3 = 3,333,333	0	3,333,333 + (4,000)(0)^2 = 3,333,333
2	(20)(140) = 2,800	(1/12)(140)(20)^3 = 93,333	0	93,333 + (2,800)(0)^2 = 93,333
3	(180)(40) = 7,200	(1/12)(40)(180)^3 = 19,440,000	0	19,440,000 + (7,200)(0^2 = 19,440,000

Shape	A(mm^2)	I_y(mm^4)	d_x(mm)	$I_y + Ad_x^2\ (mm^4)$
				I_y = 22,866,667 mm^4

Notice that all the values for d_x are zero when calculating I_y. This is because the centroid of each shape coincides with the centroid of the entire section, so the distance between them is zero.

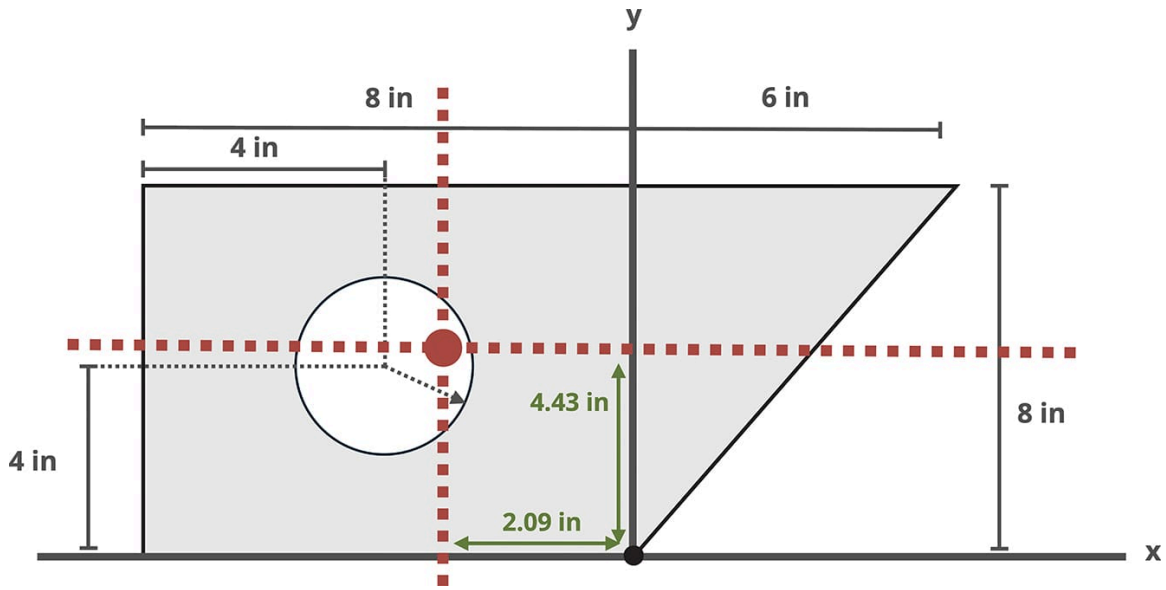
Example 8.6

Determine the I_x and I_y of the shaded area with respect to the shape's centroid.



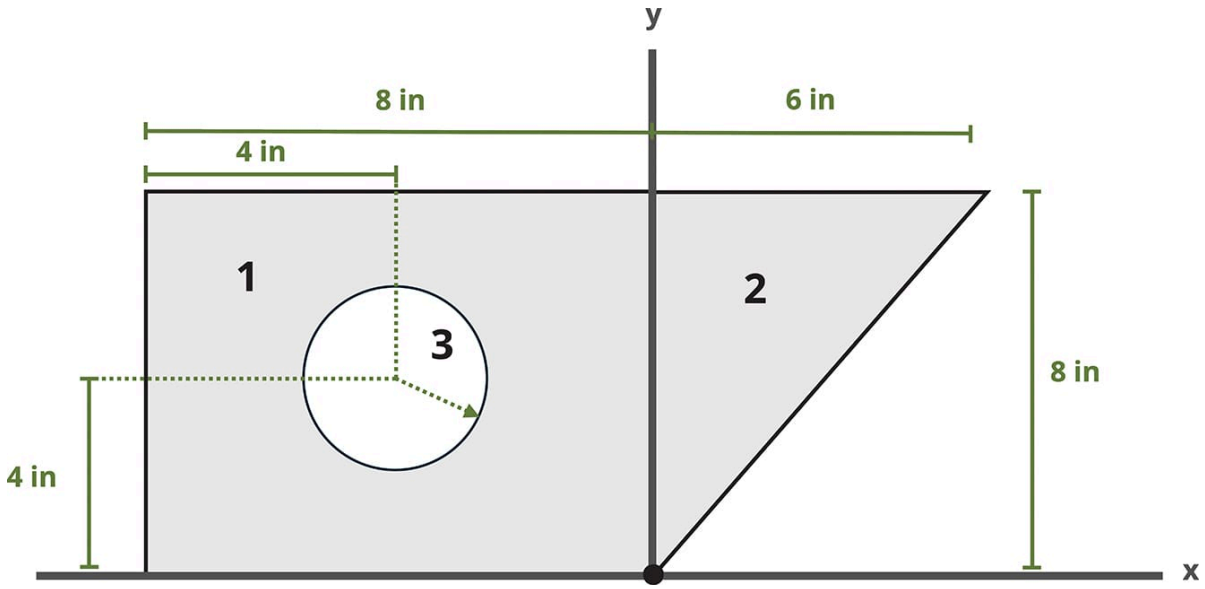
Solution

To find the second moment of area in the x and y directions with respect to the section's overall centroid, first calculate the centroid. We did this in Example 8.3, with the results as shown.



We found that the centroid is located 2.09 in. to the left of the y-axis and 4.43 in. up from the x-axis.

The next step is to break up the section into common shapes (see Appendix E) as shown in the following figure. Note that shape 3 is a hole, so we need to subtract the area and second moment of area.



The formula for the second moment of area of a shape are with respect to the shape's centroid, not to the section's overall centroid. Make a necessary adjustment using the parallel axis theorem to transform the rotational axis from the shape's centroid to the section's centroid. Do this for each shape.

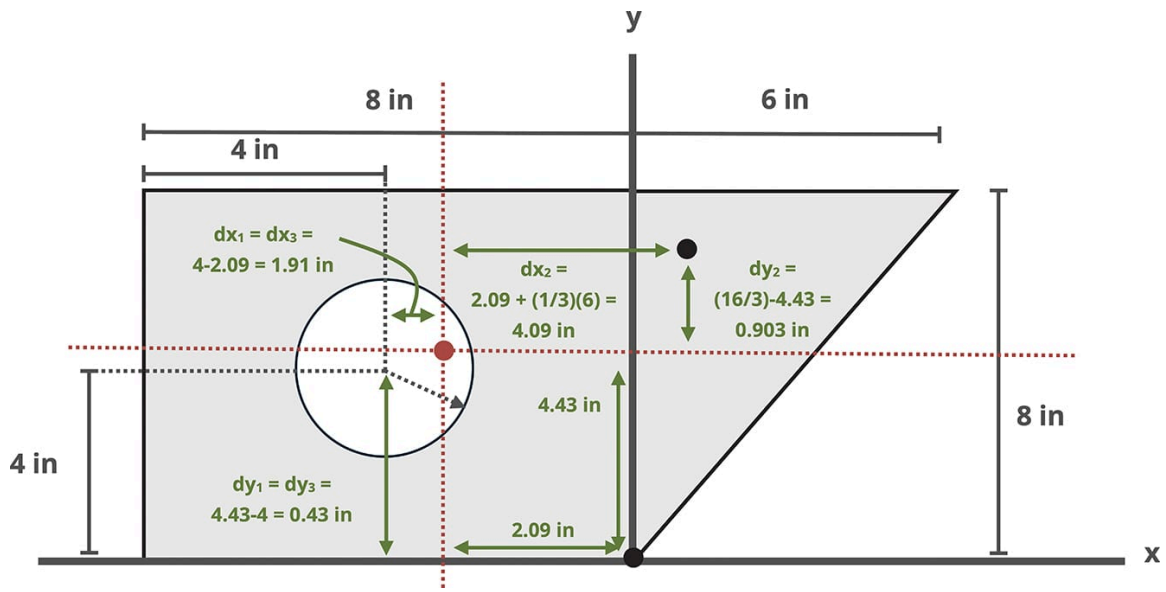
Present this method in terms of a table, first for I_x then I_y.

Shape	A (in.^{2})	I_x(in.^{4})	d_y(in.)	$I_x + Ad_y^2$ (in.^{4})
1	64	$(1/12)(8)(8)^3 = 341.3$	0.43	$341.3 + (64)(0.43)^2 = 353.1$
2	24	$(1/36)(6)(8)^3 = 85.3$	0.90	$85.3 + (24)(0.9)^2 = 104.7$
3	-4π	$(-\pi/4)(2)^4 = -12.6$	0.43	$(-12.6 + -4\pi)(0.43)^2 = -14.9$
				$I_x = 442.9$ in.^{4}

Shape	A (in.^{2})	I_y(in.^{4})	d_x(in.)	$I_y + Ad_x^2$ (in.^{4})
1	64	$(1/12)(8)(8)^3 = 341.3$	1.91	$341.3 + (64)(1.91)^2 = 574.8$

Shape	A (in.^{2})	I_y(in.^{4})	d_x(in.)	$I_y + Ad_x^2$ (in.^{4})
2	24	$(1/36)(8)(6)^3 = 48$	4.09	$48 + (24)(4.09)^2 = 449.5$
3	-4π	$(-\pi/4)(2)^4 = -12.6$	1.91	$-12.6 + (-4\pi)(1.91)^2 = -58.4$
				$I_y = 966$ in.^{4}

The calculation and visualizations for each d_x and d_y term are as shown in the figure.



Step-by-Step: Finding the Second Moment of Area

Use the following steps to find the second moment of area of a composite section:

1. Determine whether you are taking the second moment of area about a reference axis or the centroidal axis. If it is about the centroidal axis, calculate the location of the centroid.
2. Break up the overall section into simpler shapes where the location of the centroid of each shape is known. It is helpful here to sketch this. If the section has a hole, include the hole as a shape with a negative inertia.
3. Implement the parallel axis theorem for each individual shape.
4. Indicate the perpendicular distance from each part's centroid to the axis about which you are calculating the second moment of area.
5. The second moment of area of the section about an axis is the sum of the composite parts about the same axis.

Summary

Key Takeaways

- The centroid represents the geometric center of a shape.
- The center of gravity is the same as the centroid only if the material is uniform (homogenous).
- The centroid is located on any axis of symmetry for the section.
- In some cases the centroid of a shape is not located within the shape itself (as shown in the rightmost section in Figure 8.2).
- The second moment of area represents a geometric property of an area and is occasionally termed the moment of inertia or the area moment of inertia about an axis because it involves squaring the distance from the axis to each element of the area.
- The parallel axis theorem is used to compute the second moment of area of composite sections.

Key Equations

Centroid formulas:

$$\overline{X} = \frac{\sum \overline{x}_i A_i}{\sum A_i}$$
$$\overline{Y} = \frac{\sum \overline{y}_i A_i}{\sum A_i}$$

Second moment of area (moment of inertia):

$$I_x = \int_A y^2 dA$$
$$I_y = \int_A x^2 dA$$
$$I_z = J = \int_A r^2 dA$$

Parallel axis theorem:

$$I_x = \overline{I_{x'}} + A d_y^2$$
$$I_y = \overline{I_{y'}} + A d_x^2$$

References

Figures

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- Figure 8.1: Isometric view of concrete barriers used in roadway construction. Pushcreativity. 2008. CC BY-SA 3.0. https://commons.wikimedia.org/wiki/File:Concrete_step_barrier_3D_cross_section.jpg.

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Chapter 9: Bending Loads

Learning Objectives

- Describe the bending behavior of beams
- Calculate the bending stresses in beams
- Select an adequate beam, using the section modulus
- Analyze beams subjected to unsymmetric bending

Beams are structural members that support loads along their length. Typically these loads are perpendicular to the axis of the beam and cause only shear forces and bending moments.

As discussed in previous chapters, internal forces and moments and section properties are crucial to calculating stresses and deflections. The same is true for finding stresses and deflections in beams. This chapter describes calculating bending stresses in beams, such as those in Figure 9.1, with considerations for unsymmetric bending and beam design.



Figure 9.1: Looking up at bridge beams used to support a landscaped bicycle and pedestrian bridge in Seattle

Section 9.1 derives an equation for calculating bending stress. Section 9.2 discusses how to design beams to ensure they don't fail due to bending stress. Section 9.3 extends the analysis to include cases where the bending moment not only acts about the horizontal axis of the beam but has components about both the horizontal and the vertical axes.

9.1 Bending Stress

This section discusses bending behavior of straight, symmetric, homogeneous beams. Additionally, this section is limited to beams with a cross-section that is symmetric with an axis and whose bending moment is around an axis that is perpendicular to the axis of symmetry. Examine the initially unaltered beam depicted in Figure 9.2, which is characterized by a rectangular cross-section and annotated with both horizontal and vertical grid lines.



Figure 9.2: An unloaded beam with a rectangular cross-section

When a bending moment is applied, horizontal lines curve, whereas vertical lines remain straight but rotate, as illustrated in Figure 9.3. The material stretches at the bottom and compresses at the top, creating a neutral surface in between where fibers experience no length change. On a cross-section this surface appears as the neutral axis.

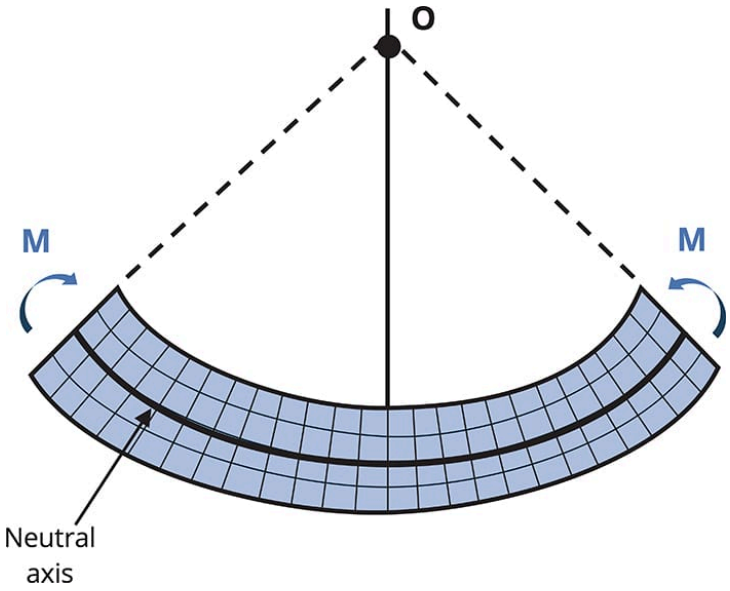


Figure 9.3: The deformation of the beam in Figure 9.2 when subjected to a symmetric bending moment

We can make a similar observation when a kid's toy undergoes bending as shown in Figure 9.4. This toy is subjected to a bending moment in the opposite direction of the beam depicted in Figure 9.3, and this direction switches the behavior of the top and the bottom surfaces. Notice in Figure 9.4 that the fibers at the top are pulled apart whereas the bottom surface fibers are pushed together.



Figure 9.4: The distortion of a kid's toy when undergoing bending

This knowledge about deformations due to bending can be applied to the failure of the column in Figure 9.5. Notice that the column was fixed at the base and subjected to a transverse force that caused failure. The fibers on the right side of the column failure point were pulled apart, whereas the fibers on the left side of the column failure point were compressed.



Figure 9.5: Column failure due to bending

We make the following assumptions about deformations due to bending:

- There must be a neutral surface parallel to both the upper and lower surfaces, where the length remains constant.
- Throughout the deformation all cross-sections of the beam remain plane and perpendicular to the longitudinal axis.
- The cross-section will keep its shape (we'll ignore the Poisson effects discussed in Section 4.4 of this text).

To understand bending stress in a beam subjected to arbitrary loads, examine a small element extracted from the beam as shown

in Figure 9.6. The derivation of the bending strain equation remains unaffected by the beam type or specific loads. Remember the fundamental definition of normal strain is

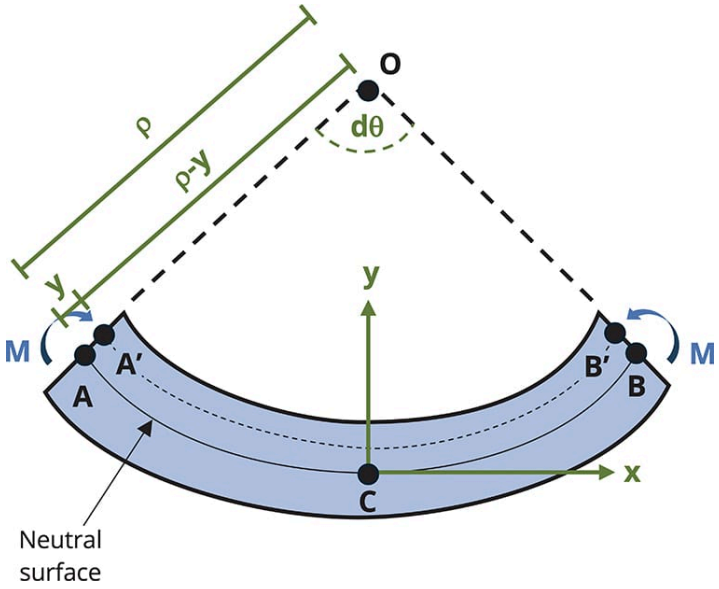


Figure 9.6: Element in beam bending

$$\varepsilon = \frac{\Delta L}{L}.$$

This equation can be used to calculate the normal strain along AB in the beam.

$$\varepsilon = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}}$$

Distance y is measured relative to the neutral surface and is positive above the neutral surface and negative below. Before bending, line AB is the same length at all y values. During bending, though, line A'B' shortens above the neutral surface and lengthens below it. By definition the length of the neutral surface doesn't change and remains the same as length AB. The lengths AB and A'B' can be described using the radius of curvature (ρ) and the differential angle (dθ) shown in Figure 9.6.

$$\overline{AB} = \rho d\theta$$

$$\overline{A'B'} = (\rho - y)\, d\theta$$

Now we substitute these lengths into the strain equation.

$$\varepsilon = \frac{(\rho - y)\, d\theta - \rho d\theta}{\rho d\theta}$$

Simplifying the equation yields

$$\varepsilon = \frac{\rho d\theta - y d\theta - \rho d\theta}{\rho d\theta}$$

$$\varepsilon = \frac{-y d\theta}{\rho d\theta}$$

$$\varepsilon = -\frac{y}{\rho}.$$

This relationship shows us that the longitudinal strain, ε, varies linearly with the distance, y, from the neutral surface. The maximum stress occurs at the outermost fibers, the extreme top and bottom of the section. This maximum distance from the neutral surface is designated c. A relationship for the maximum absolute value of the strain, ε_{max} can now be calculated.

$$\frac{\varepsilon}{\varepsilon_{max}} = -\frac{y/\rho}{c/\rho}$$
$$\varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{max}$$

Assuming that the material behaves in a linearly elastic manner, we use Hooke's law, $\sigma = \mathrm{E}\varepsilon$, to rewrite the strain relationship into a relationship of stresses.

$$\sigma = -\left(\frac{y}{c}\right)\sigma_{max}$$

Therefore, similar to the variation in normal strain, normal stress, s, will fluctuate from zero at the neutral surface to a maximum value, σ_{max}, at a distance c from the neutral surface as shown in Figure 9.7.

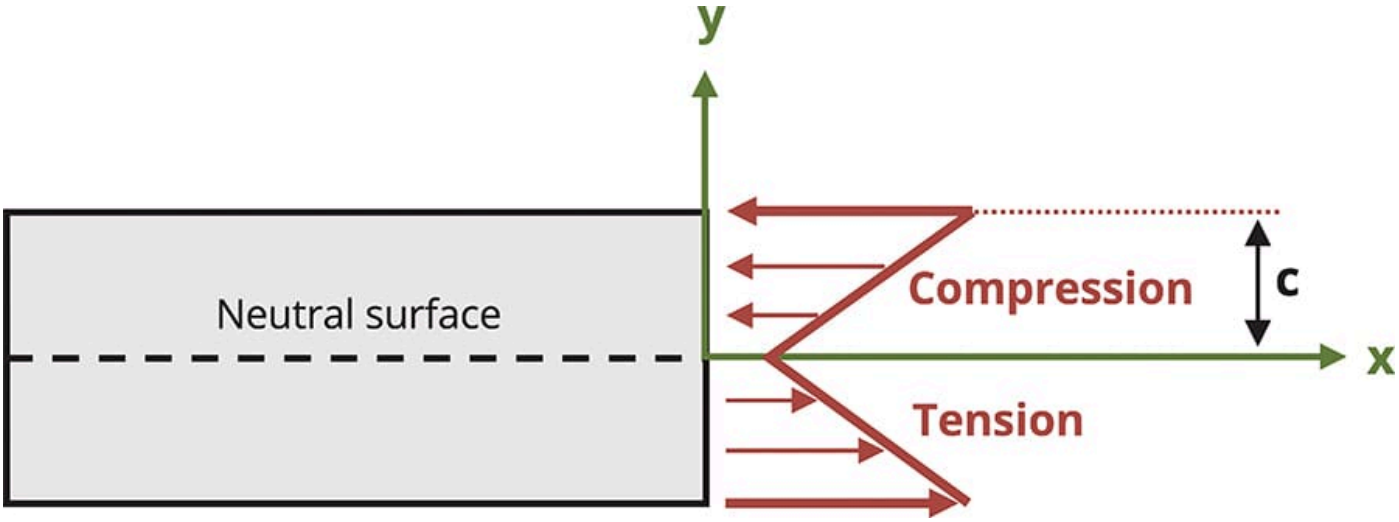


Figure 9.7: Bending stress variation

For us to determine the position of the neutral surface, the resultant force generated by the stress distribution across the cross-sectional area must be equal to zero.

$$0 = \int_A F_{resultant} = \int_A \sigma dA$$

We can now substitute the previous relationship between stress and distance from neutral axis.

$$0 = \int_A -\left(\frac{y}{c}\right)\sigma_{max} dA$$
$$0 = -\frac{\sigma_{max}}{c}\int_A y dA$$

Since $-\frac{\sigma_{\max}}{c}$ does not equal zero, what remains is

$$0 = \int_A y dA$$

This integral represents the first moment of area, as discussed in Section 8.1. This equation indicates that the first moment of the cross-section about its neutral axis must be zero. Recall that the location of the centroid was determined by $\bar{y} = \frac{\int_{\mathrm{A}} \mathrm{ydA}}{\mathrm{A}}$. Consequently, for a member experiencing pure bending and as long as the stresses remain within the elastic range, the neutral axis traverses through the centroid of the section because $\bar{y}$ (the distance from the neutral axis to the centroid) is zero.

The stress in the beam can be ascertained by setting the moment M equal to the moment produced by the stress distribution around the neutral axis.

$$dM = ydF$$

Since

$$dF = \sigma dA,$$

we can write

$$M = \int_A y dF = \int_A y\left(\sigma dA\right) = \int_A y\left(-\tfrac{y}{c}\sigma_{max}\right) dA$$
$$M = \tfrac{\sigma_{max}}{c}\int_A y^2 dA.$$

Note that the integral, $\int_A y^2 dA$, represents the second moment of area, or the moment of inertia, of the cross-sectional area about the neutral axis. Here second moment of area is denoted with I, as in Section 8.2.

Rearranging the previous equation to obtain the flexure formula yields

$$\boxed{\sigma_{max} = \frac{Mc}{I}},$$

(9.1)

where

- σ_{max} = *Maximum bending stress in the beam; note that a complete description includes magnitude, units, and tension or compression [Pa, psi]*
- M = *Internal bending moment calculated about the neutral axis of the cross-section, determined from method of sections or shear and moment diagrams [N·m, lb·in.]*
- c = *Perpendicular distance from the neutral axis to the point farthest from the neutral axis [m, in.]*
- I = *Second moment of area about the neutral axis [m^4, $in.^4$]*

Previous derivation has shown that

$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}.$$

This can be rearranged to provide the following relationship:

$$\frac{\sigma_{\max}}{c} = -\frac{\sigma}{y}.$$

We can determine a similar flexure formula to calculate the bending stress at any point along the cross-section.

$$\boxed{\sigma = -\frac{My}{I}},$$

(9.2)

where

- σ = *Bending stress [Pa, psi]*
- M = *Internal moment calculated about the neutral axis of the cross-section [N·m, lb·in.]*
- y = *Perpendicular distance from the neutral axis to the point of interest [m, in.]*
- I = *Second moment of area about the neutral axis [m^4, $in.^4$]*

As emphasized in Chapter 2, when reporting normal stress engineers must indicate whether that stress is in tension or

compression. The derivation for Equation 9.2 used a beam bending concave upward. In Chapter 7 we indicated that this type of moment is considered positive. This leads to compression above the neutral axis and tension below. The inverse is true when the beam is subjected to negative bending moment as shown in Figure 9.8.

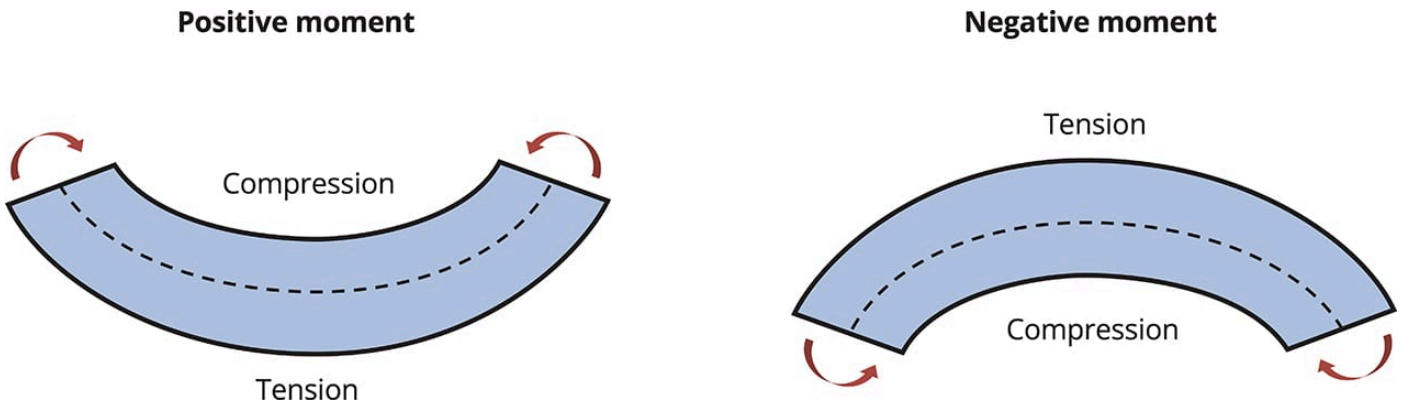
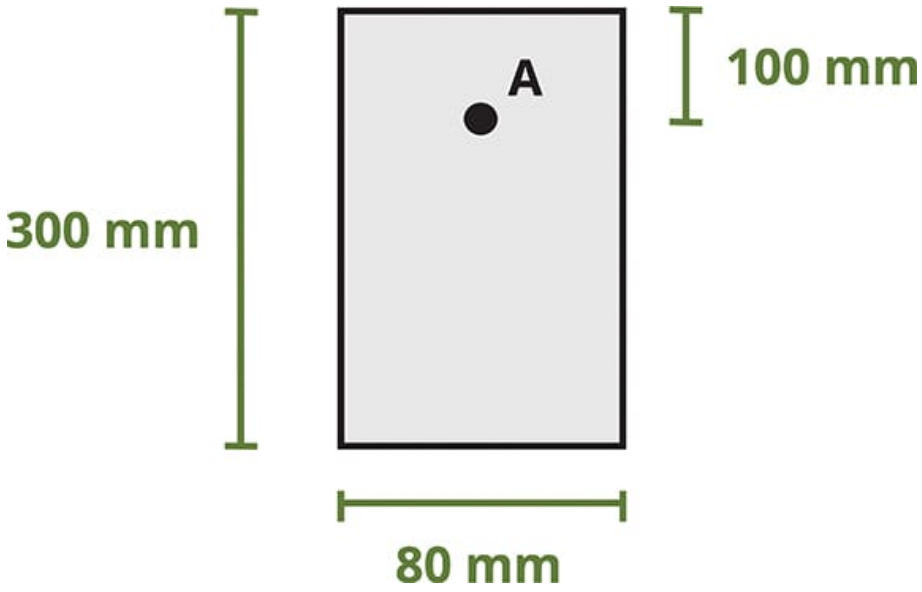


Figure 9.8: Bending stress sign convention

Example 9.2 and Example 9.3 demonstrate the calculation of bending stresses at various points on a cross-section.

Example 9.1

For the timber beam shown, calculate the normal stress at the top, at the bottom, and at point A when the beam experiences a +30 kN·m moment.

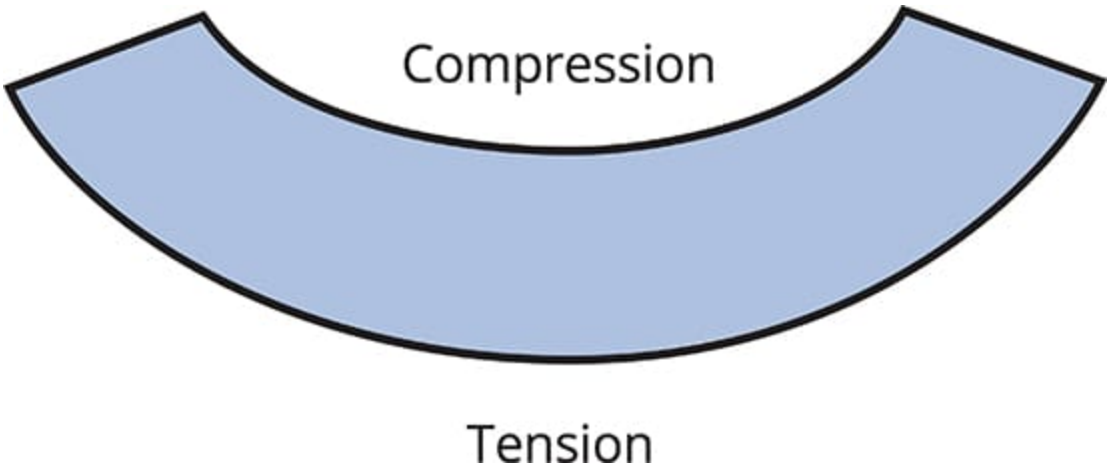


Solution

Since we are asked to find the bending stress, let's look at the bending stress formula.

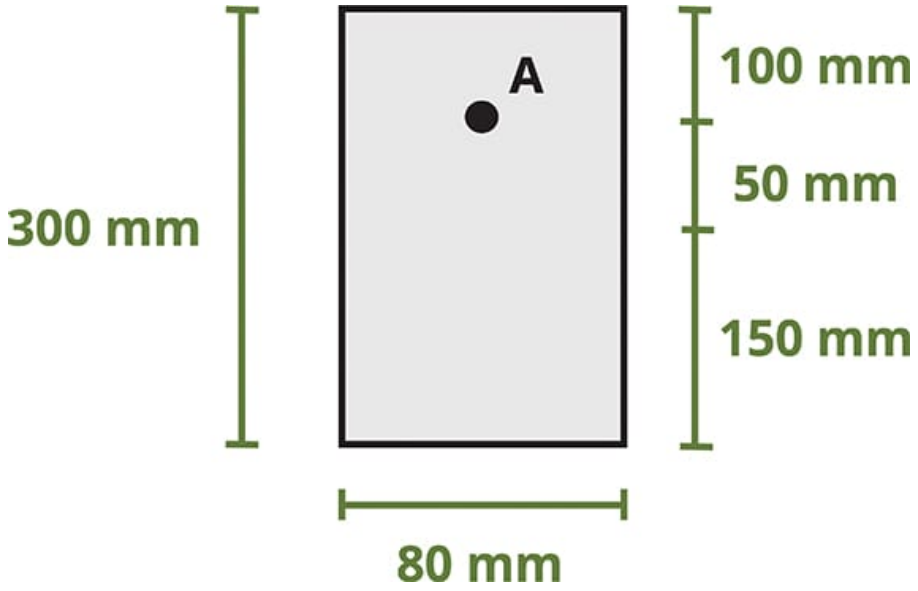
$$\sigma = -\frac{My}{I}$$

Start with the internal moment, which is a given in the problem statement to be +30 kN·m. Positive moment indicates that the beam is bending concave up, with compression above the neutral axis and tension below.



The value of y is the perpendicular distance from the neutral axis to the point of interest. This problem has three points of interest so it has three different y values—one each for the top, the bottom, and point A.

Next determine the beam properties, which are the centroid and the second moment of area. Since the cross-section is a rectangle, the centroid is located at the midpoint of the beam's height. You can then use the formula for second moment of area for a rectangle from Appendix E.



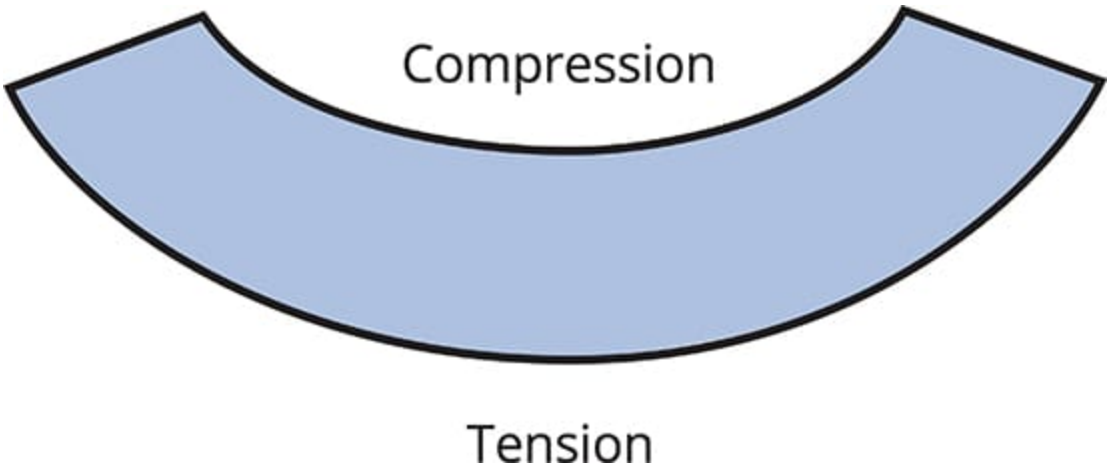
$$I_x = \frac{1}{12}(0.080m)\,(0.3m)^3$$
$$I_x = 1.8\,*10^{-4}\,*m^4$$

Now calculate stress at each of the three points requested. Start with the top.

$$\sigma_{top} = \frac{(30*10^3 Nm)\,(0.15m)}{1.8*10^{-4}*m^4} = 25\ MPa$$

A complete answer must include magnitude, units, and tension or compression. The magnitude is 25, the units will be MPa. To determine the tension or compression look at the sign of the internal moment and where along the section you are calculating the bending stress. In this example the moment is positive and we calculated the stress at the top of the section.

The calculation shows the bending stress to be in compression.



$$\sigma_{top} = 25MPa\,(C)$$

Note that as long as care is taken to use the correct signs for M and y (both positive in this case), the answer will come out with the correct sign (negative in this case, which indicates compression).

Next calculate the normal stress at the bottom of the beam. In this case all the values (M, y, and I) are the same, so the magnitude of the stress is the same (25 MPa). What changes is the type of stress (tension or compression). Since in this example the moment is positive and we calculated the stress at the bottom of the beam, the stress will be in tension.

$$\sigma_{btm} = -\frac{(30{*}10^3 Nm)(-0.15m)}{1.8{*}10^{-4}{*}m^4} = 25\ MPa$$
$$\sigma_{btm} = 25\ MPa(T)$$

Finally, calculate the stress at point A, which is located 100 mm from the top of the section. The y value measures from the neutral axis (centroid), so use y = 50 mm.

$$\sigma_A = \frac{(30{*}10^3 Nm)\,(0.05m)}{1.8{*}10^{-4}{*}m^4} = 8.33\ MPa$$

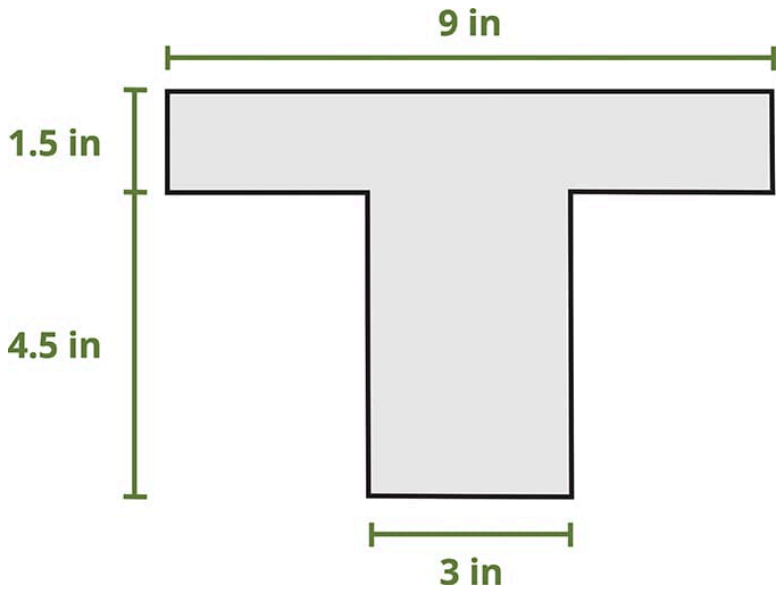
As before, evaluate whether the stress is in tension or compression. Since point A is above the neutral axis and the moment is positive, the stress at point A will be in compression.

$$\sigma_A = 8.33\ MPa\,(C)$$

Example 9.2

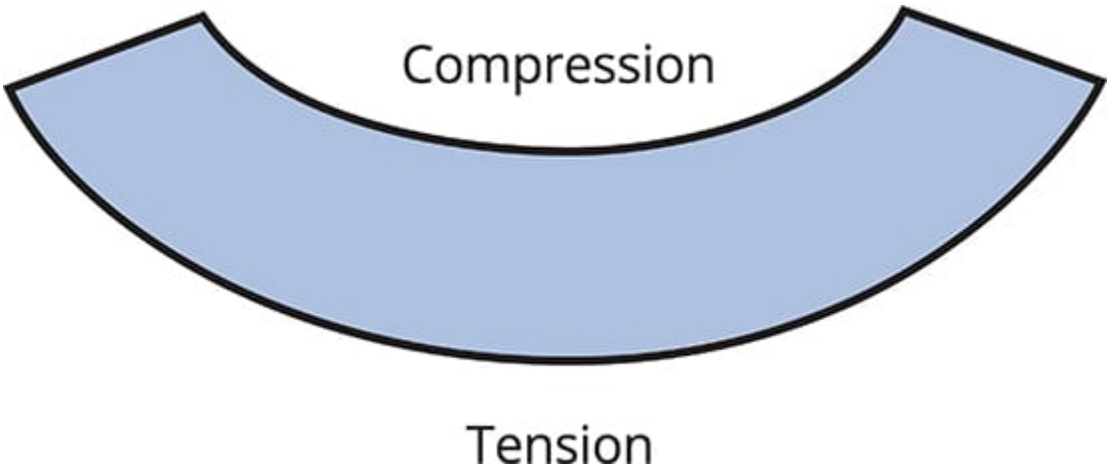
A beam with the shown cross-section is subjected to a positive moment of 50 kip·in.

Calculate the normal stress at the top, bottom, and interface between flange and web.

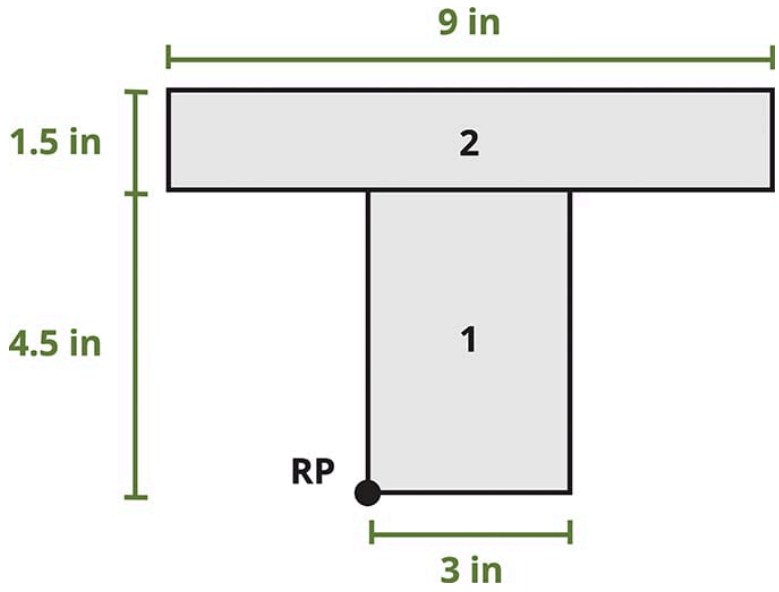


Solution

Start with the internal moment, which is given in the problem statement as +50 kip·in. The positive moment indicates that the beam is bending concave up, with compression above the neutral axis and tension below.



Next determine the beam properties, which are the centroid and the second moment of area. Since this is not a standard shape and the shape is not symmetric in the y direction, you will need to calculate this by hand. Example 8.1 and Example 8.4 cover this shape in detail.



Shape	A (in.^{2})	$\bar{y}$(in.)	A$\bar{y}$(in.^{3})	I_x(in.^{4})	d_y(in.)	$I_x + Ad_y^2$ (in.^{4})
1	13.5	2.25	30.375	22.78125	1.5	53.15625
2	13.5	5.25	70.875	2.53125	1.5	32.90625
	$\sum A = 27$		$\sum A\,\bar{y} = 101.25$			$\sum I_x = 86.0625$

$$\bar{Y} = \frac{101.25\ in.^3}{27\ in.^2} = 3.75\ in. \text{from bottom}$$
$$= 2.25\ in. \text{from top}$$

This problem has three parts that change the y value. Begin by calculating the stress at the top of the section, using the flexure formula.

$$\sigma_{top} = -\frac{My}{I}$$

Before plugging in the values, ensure that the units are consistent.

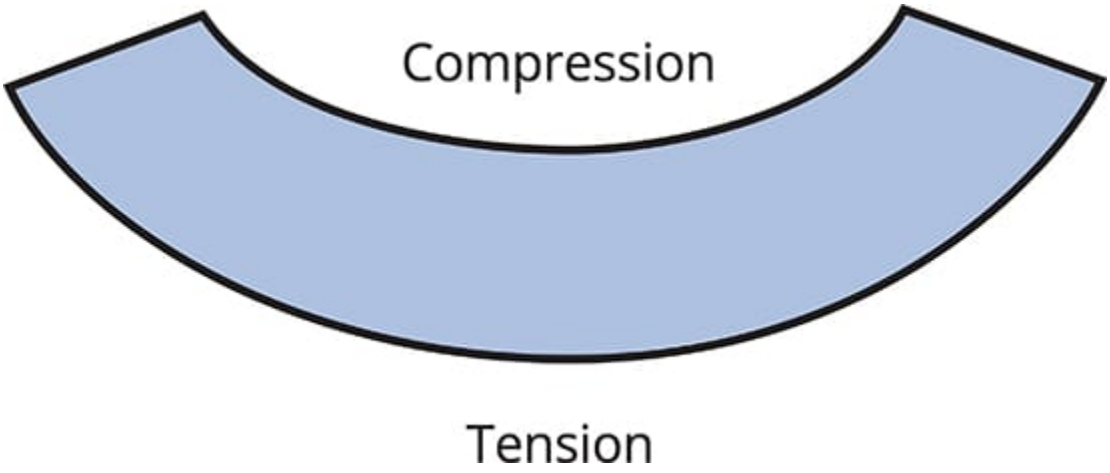
- M=50 kip $\cdot$ in.
- y=2.25 in.
- I=86.0625 in.^{4}

Since all units are in kips and inches, plug them into the flexure formula.

$$\sigma_{top} = -\frac{(50\ kip \cdot in.)\,(2.25\ in.)}{86.0625\ in.^4} = -1.31\ ksi$$

For a complete answer include magnitude, units, and tension or compression. The magnitude is 1.31, the units will be kips/in^2 or ksi. To determine the tension or compression look at the sign of the internal moment and where along the section you are

calculating the bending stress. In this example the moment is positive, so calculate the stress at the top of the section. The answer will show the bending stress to be in compression.



$$\sigma_{top} = 1.31\ ksi\ (C)$$

Note that as long as care is taken to use the correct signs for M and y (both positive in this case), the answer will come out with the correct sign (negative in this case, which indicates compression).

Now calculate the normal stress at the bottom of the beam. The only quantity that changes is the y value, so now use the distance from the neutral axis to the bottom of the beam (3.75 in.).

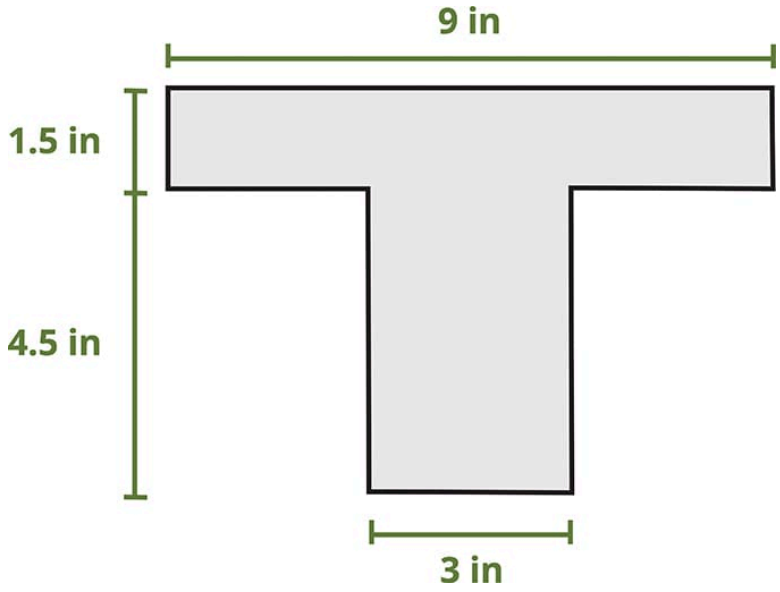
$$\sigma_{btm} = -\frac{(50\ kip \cdot in.)\,(-3.75\ in.)}{86.0625\ in.^4} = 2.18\ ksi$$

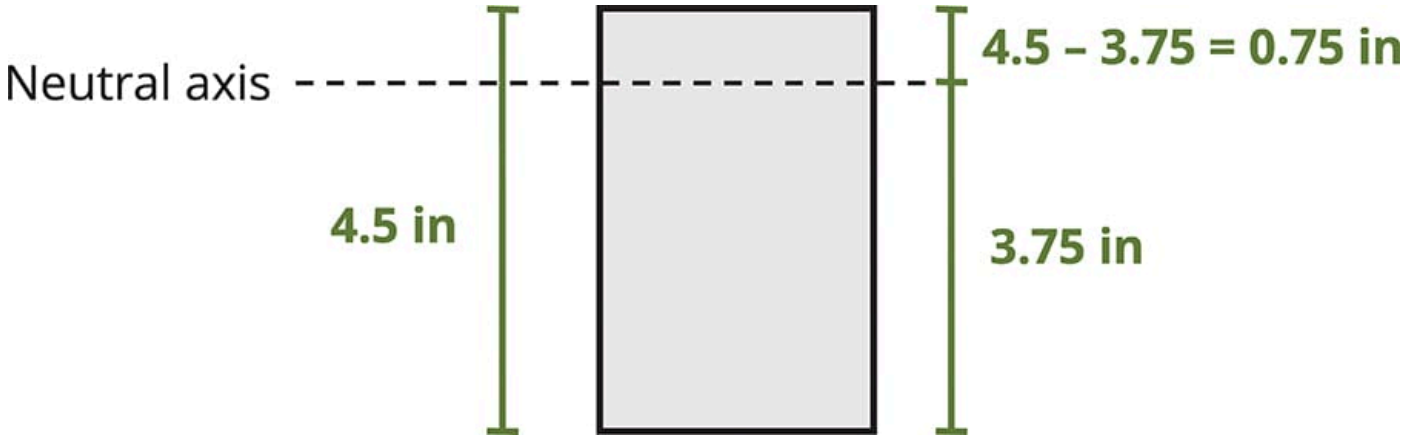
Because the moment is positive, the beam bends concave up. Anything below the neutral axis is in tension.

$$\sigma_{btm} = 2.18\ ksi\ (T)$$

This again works mathematically as long as you use the correct positive sign for the M and negative sign for y (since the point of interest is below the neutral axis). The answer shows this stress to be positive, indicating tension and matching expectation.

Finally, calculate the normal stress at the junction of the flange and the web. To do this find the distance from the neutral axis to the flange web junction.





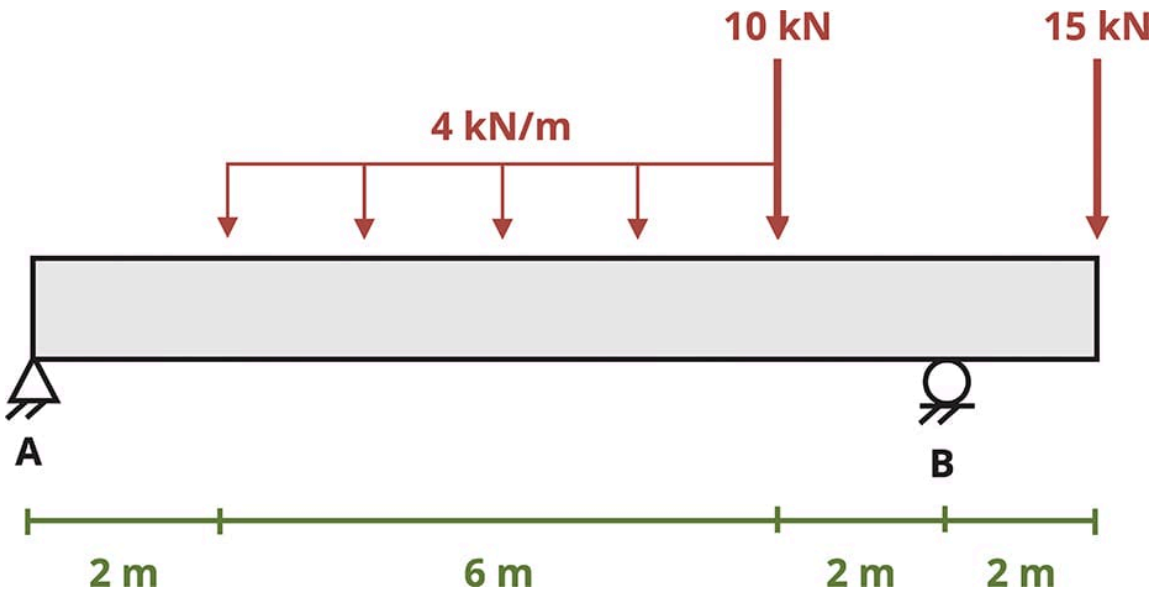
$$\sigma_{F/w} = -\frac{(50\ kip \cdot in.)\,(0.75\ in.)}{86.0625\ in.^4} = -0.436\ ksi$$

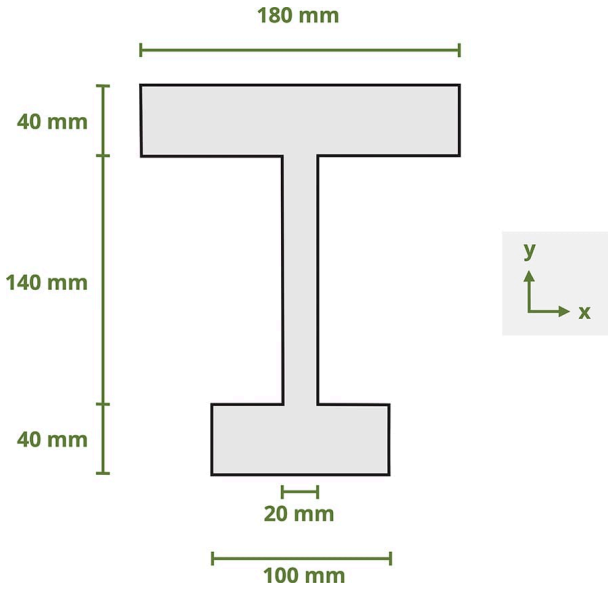
Since the flange web junction is above the neutral axis and the beam is subjected to positive moment, the bending stress will be in compression.

$$\sigma_{F/w} = 0.436\ ksi\,(C)$$

Example 9.3

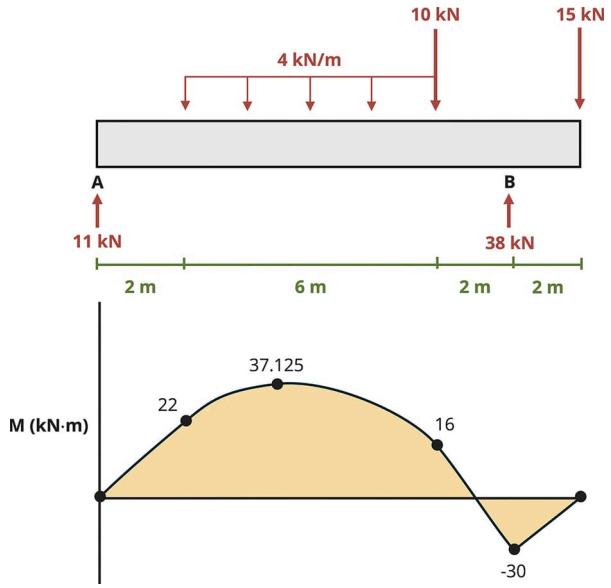
Calculate the maximum tensile and maximum compressive bending stress in the beam shown.





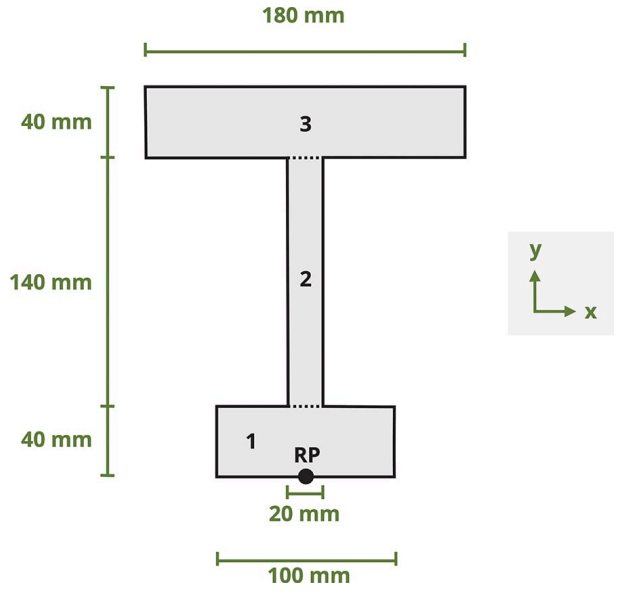
Solution

The first step is to find the beam's internal forces so that we can find the maximum positive and negative moments. The best way to do this is to draw the moment diagram, as detailed in Example 7.3.



Note that the maximum positive moment is 37.125 kN·m and the maximum negative moment is 30 kN·m.

Next determine the beam's geometric properties, which are the centroid and the second moment of area. Since this is not a standard shape and the shape is not symmetric in the y direction, calculate this by hand. Example 8.2 and Example 8.5 covered this shape in detail.



Find the centroid ($\overline{x} = 0$ due to symmetry).

Shape	A(mm^2)	$\overline{y}(mm)$	$A\overline{y}(mm^3)$
1	4,000	20	80,000
2	2,800	110	308,000
3	7,200	200	1,440,000
	$\sum A = 14{,}000$		$\sum A\,\overline{y} = 1{,}828{,}000$

$$\overline{Y} = 130.6\,mm \text{ from bottom}$$

Shape	I_x(mm^4)	d_y(mm)	$I_x + Ad_y^2\,(mm^4)$
1	533,333	110.57	49,436,233
2	4,573,333	20.57	5,758,083
3	960,000	69.43	35,667,779

Shape	$I_x (mm^4)$	$d_y (mm)$	$I_x + Ad_y^2\ (mm^4)$
			$I_x = 90{,}862{,}095\ mm^4$

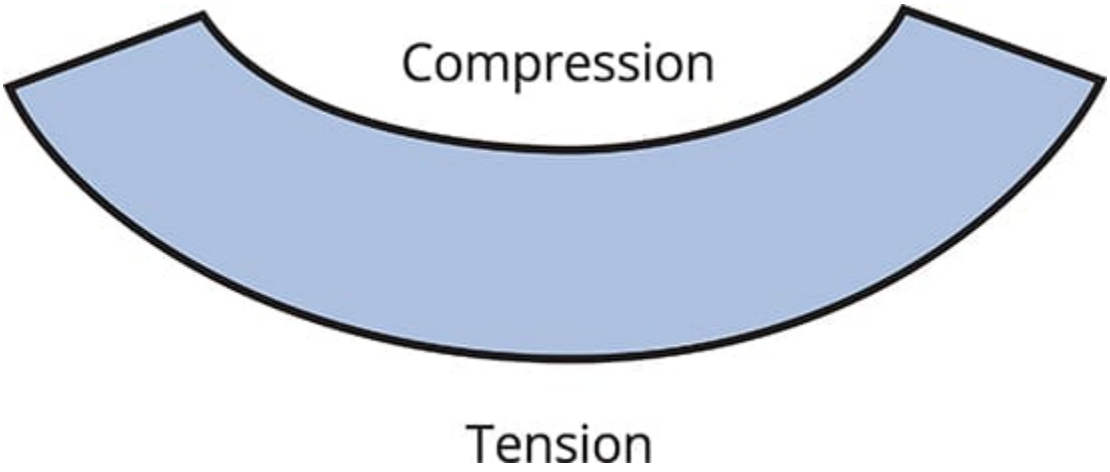
The positive bending moment will cause compression at the top of the cross-section and tension at the bottom. The negative bending moment will cause tension at the top and compression at the bottom. It's not obvious by inspection which of these two tensile stresses will be larger, nor which of the two compressive stresses will be larger. We should calculate all four to compare.

Before plugging in the values ensure that the units are consistent, so change everything to Newtons and meters.

- $M_{pos} = 37.125\ kN{\cdot}m = 37.125 \times 10^3\ N{\cdot}m$
- $M_{neg} = 30\ kN{\cdot}m = 30 \times 10^3\ N{\cdot}m$
- $y_{top} = 89.43\ mm = 0.08943\ m$
- $y_{btm} = 130.57\ mm = 0.13057\ m$
- $I_x = 90{,}862{,}095\ mm^4 = 90.862095 \times 10^{-6}\ m^4$

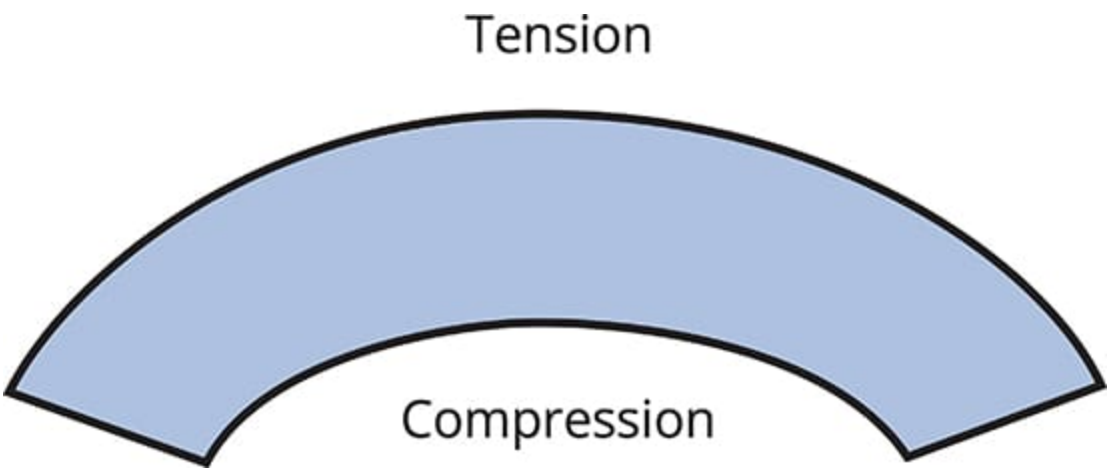
Start by calculating the section's top and bottom stress from the maximum positive moment, using the flexure formula.

$$\sigma = -\frac{My}{I}$$



$$\sigma_{top/pos} = -\frac{(37.125\times10^3\ N{\cdot}m)(0.0893\ m)}{90.86\times10^{-6}\ m^4} = -36.49\ MPa(C)$$
$$\sigma_{btm/pos} = -\frac{(37.125\times10^3\ N{\cdot}m)(-0.13057\ m)}{90.86\times10^{-6}\ m^4} = 53.35\ MPa(T)$$

Now do something similar for the maximum negative bending stress.



$$\sigma_{top/neg} = -\frac{(-30\times10^3\ N\cdot m)(0.0893\ m)}{90.86\times10^{-6}\ m^4} = 29.48\ MPa(T)$$
$$\sigma_{btm/neg} = -\frac{(-30\times10^3\ N\cdot m)(-0.13057\ m)}{90.86\times10^{-6}\ m^4} = -43.11\ MPa(C)$$

The overall maximum tensile stress for this beam occurs at the bottom of the section due to positive moment, 53.35 MPa (T). The overall maximum compressive stress (43.11 MPa (C)) also occurs at the bottom of the section in a different part of the beam but is due to the negative moment.

For many situations the overall largest stress would be sufficient. However, some materials behave differently depending on whether they are subjected to tensile or compressive stress. Concrete is a good example of this, as it's relatively strong against compressive stresses but very weak against tensile stresses. Steel rebar is placed inside concrete members to support the tensile stresses as shown in Figure 9.9. At this point in your engineering career it is good practice to report both the maximum tensile and compressive stresses at each critical point on the beam.



Figure 9.9: The end of the prestressed concrete beam showing the prestressing steel and rebar

Step-by-Step: Calculating Bending Stress

1. Determine the internal moment at the point where you want to calculate the bending stress. If you know the point along the length of the beam that you are investigating, you can cut a section and apply equilibrium equations to determine this moment. If you need to find the maximum bending stress, then draw the moment diagram to find the maximum internal moment (Chapter 7 reviews this).

2. Determine the section properties of the beam. This will require knowing the location of the centroid, neutral axis, and second moment of area. (See Chapter 8 for a review or Appendix E for a table listing I values for selected common shapes.)

3. Identify the specific location on the section for which you are computing the normal stress, y (or c, if you are calculating the maximum normal stress.)

4. Ensure that the units of M, I, and y (or c) are consistent, then use one of the flexure formulas to calculate bending stress. Ensure that the result indicates the magnitude, units, and tension or compression.

9.2 Beam Design for Bending

Frequently beam design is governed by the bending moment. This section leverages your understanding of bending stress to design two common categories of beams capable of withstanding their applied internal moments. This section starts with rolled steel beams of varying cross-sections. Appendix A provides a sampling of these standard shapes. Then follows a discussion of rectangular cross-sections, which are common for timber beams.

A safe design requires that the allowable stress of the material used is greater than the stress due to the loading. The flexure equation for maximum bending stress, $\sigma_m = \frac{M_{max}c}{I}$, has four variables: σ_m, M_{max}, c, and I. We'll start with the allowable stress of the material, σ_{all}, and use the $|M_{max}|$ derived from the loading condition. What is left are section properties, I, and c. For standard shapes these are combined into one variable, section modulus (S).

$$\boxed{S = \frac{I}{c}}$$

(9.3)

We can substitute this value into the flexure formula and rearrange for S_{min}, which is the minimum allowable value of the section modulus for the beam.

$$\boxed{S_{min} = \frac{|M_{max}|}{\sigma_{allow}}},$$

(9.4)

where

- S_{min} = *Minimum required section modulus for a beam to resist the bending stress [m^3, $in.^3$]*
- $|M_{max}|$ = *Magnitude of the maximum internal bending moment [N·m, lb·in.]*
- σ_{allow} = *Maximum allowable stress for the beam [Pa, psi]*

9.2.1 Design of Standard Steel Sections

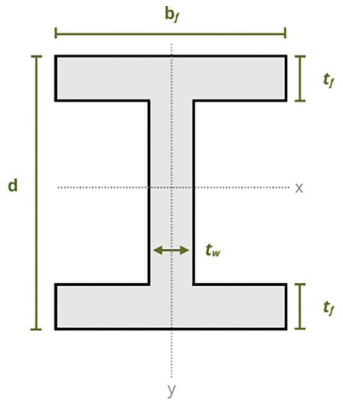
As long as the beam is designed such that its section modulus is larger than S_{min}, the beam is guaranteed not to fail due to bending stress. Rolled steel beams tend to have relatively complex cross-sections with multiple dimensions that can be varied to alter a beam's section modulus and therefore its resistance to bending stress. Designing each dimension individually would be time consuming and it would be costly to manufacture a unique beam for every loading condition.

Instead, beams are mass produced in standard sizes and engineers simply select the most appropriate beam for their specific need. Thus many safe standard steel shapes are available to satisfy this equation. Engineering is determining which of the shapes that work you will use. Often you will need to consider many criteria.

In this section we use cost to determine the shape. Steel beams are priced by their weight, and the higher the weight, the more expensive. To choose the most economical shape, we have selected the shape with the smallest weight (i.e., smallest cross-sectional area).

Basic structural steel shapes have standardized cross-sectional dimensions. W shapes (standing for wide flange) are often used

for beam design because they have an efficient cross-section. Figure 9.10 is a sample of the beam table in Appendix A.



Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
W40x431	127	41.3	16.2	2.36	1.34	34800	1690	16.6	1690	208	3.65
W40x297	87.3	39.8	15.8	1.65	0.930	23200	1170	16.3	1090	138	3.54
W40x264	77.4	40.0	11.9	1.73	0.960	19400	971	15.8	493	82.6	2.52

Figure 9.10: Portion of the W section table from Appendix A

Each beam has a designation that quickly gives us information about the section (see Figure 9.11).

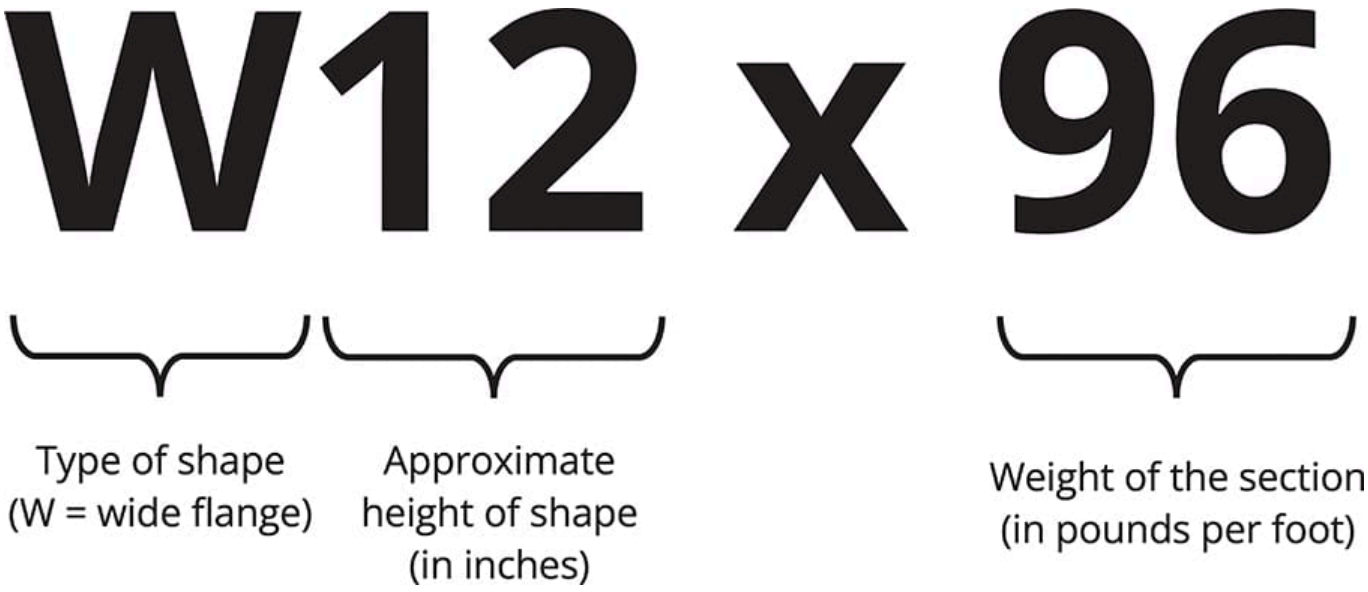


Figure 9.11: Description of the shape designation in US customary units

Notice in the sample beam section table in Figure 9.10 that all are W shapes and grouped by approximate height (first number). Within each height grouping the shapes are arranged from lightest to heaviest (second number) when going from the bottom to top of the height grouping. Given for each standard shape in this table are the cross-sectional area, depth (height), flange and web dimensions, second moment of area in the x and y directions, section modulus in the x and y direction, and radius of gyration in the x and y directions (last is a geometric property related to buckling, used in Chapter 15).

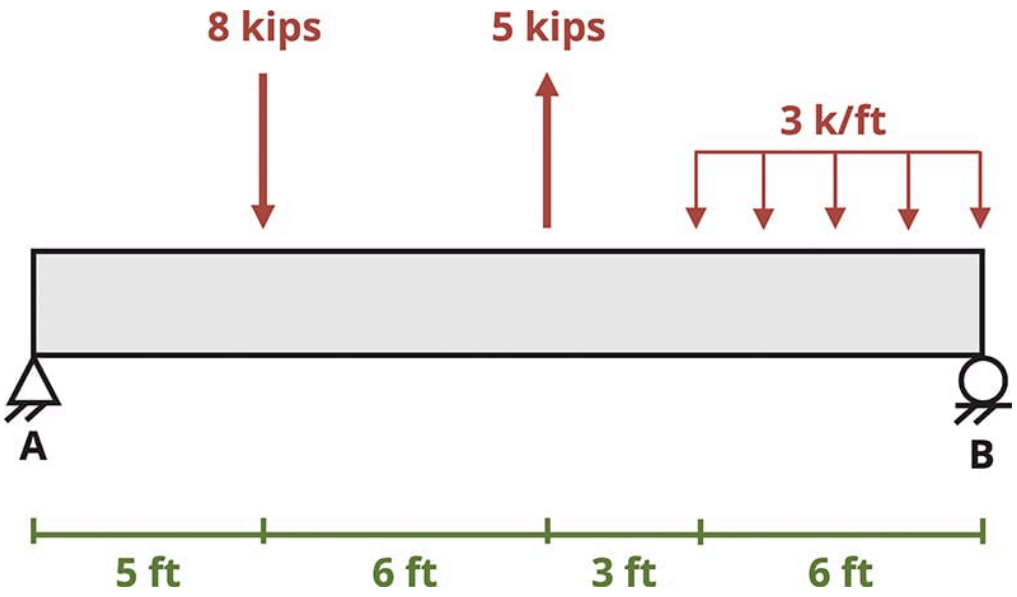
In certain instances the selection of a section might be constrained by such factors as the permissible depth of the cross-section or the allowable deflection of the beam. Here the discussion is limited to materials that behave the same in tension and compression. If this is not the case (e.g., when using concrete), then you may need to check multiple points along the length of

the beam. Additionally, if the section is not symmetric about the neutral axis, the maximum tensile and compressive stresses may not occur at the chosen place on the beam. Note that this procedure considers only bending stresses. Although bending stresses do control the design of most beams, there are instances where shear or deflection will be in control. Chapter 11 discusses this topic further.

Example 9.3 designs a wide-flange beam to resist bending stress.

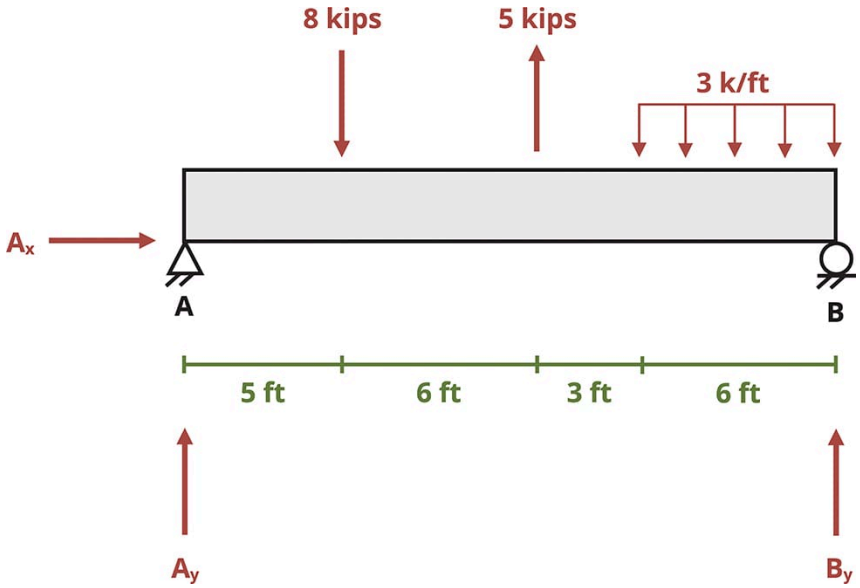
Example 9.4

Knowing that the allowable normal stress for the steel used is 24 ksi, select the most economical wide-flange beam to support the loading shown.



Solution

The first step is to find the internal forces in the beam. We'll start by calculating the reactions.



$$\sum F_x = A_x = 0$$

$$\sum M_A = -8\ kips(5\ ft) + 5\ kips(11\ ft) - \left(3\frac{kips}{ft}*6\ ft\right)(17\ ft) + B_y(20\ ft) = 0$$

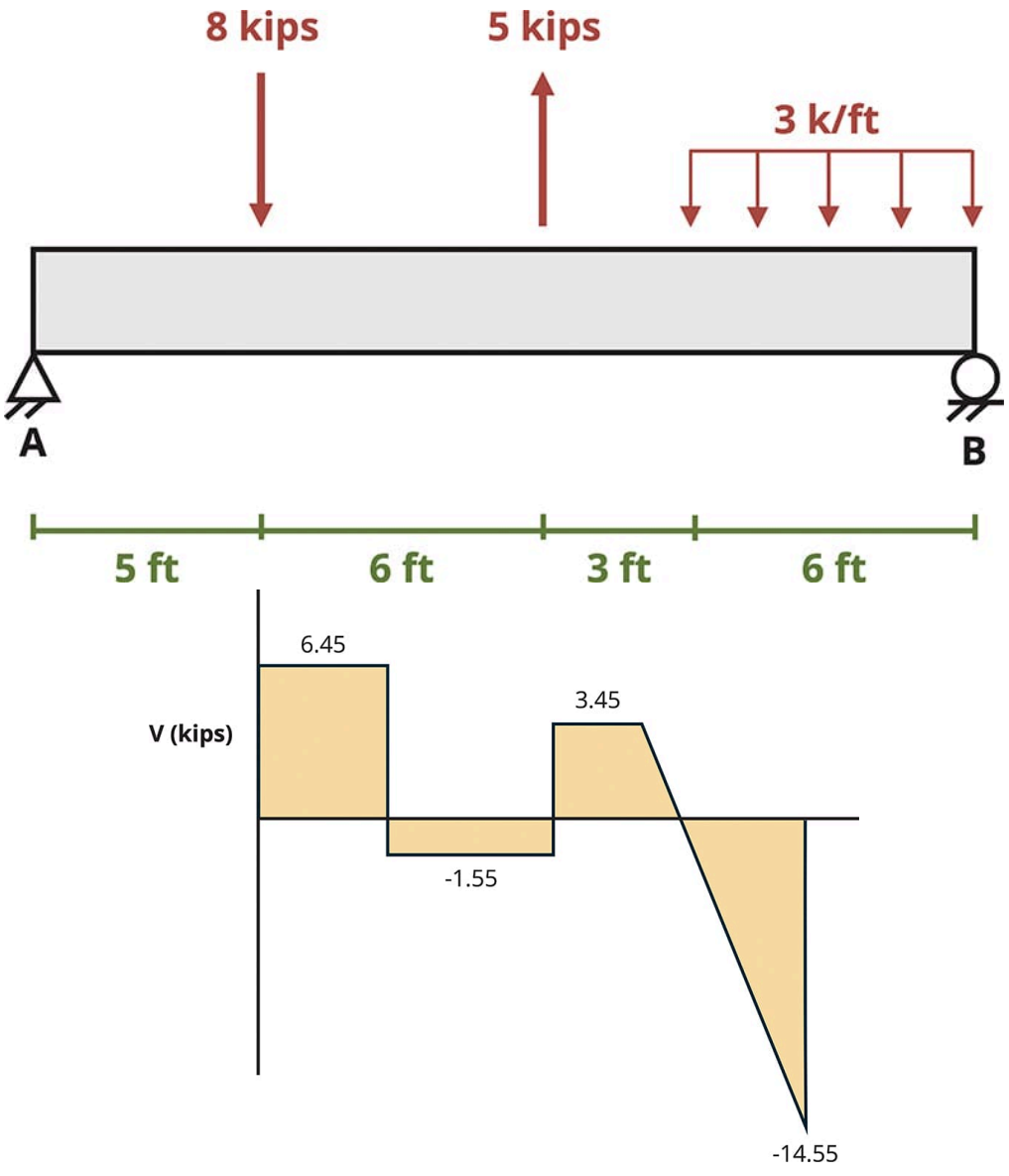
$$B_y = 14.55\ kips$$

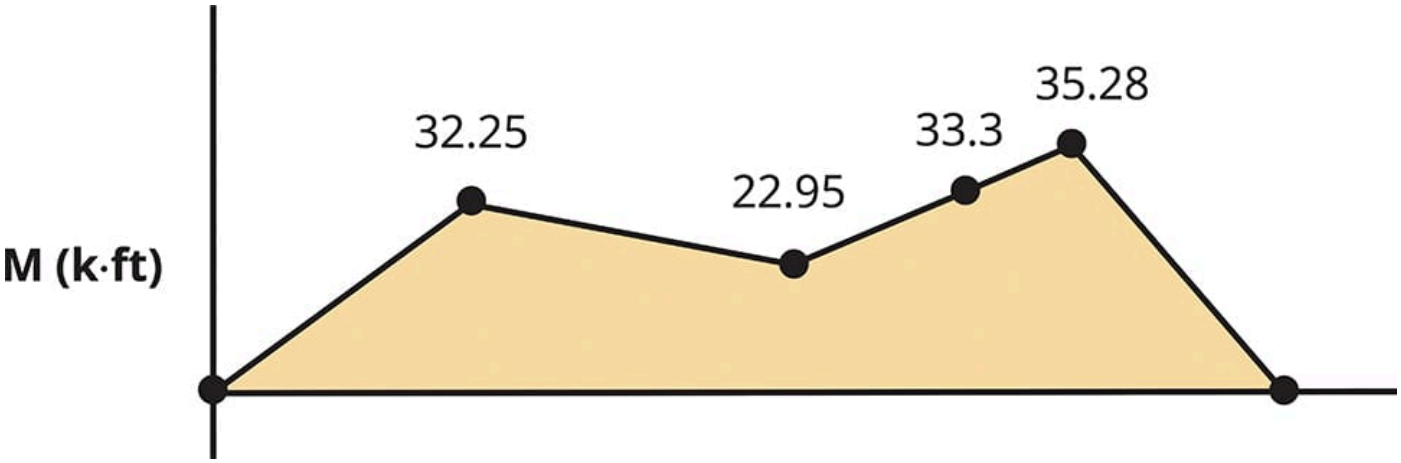
$$\sum F_y = A_y - 8\ kips + 5\ kips - \left(3\frac{kips}{ft}\right)*6\ ft) + 14.55\ kips = 0$$

$$A_y = 6.45\ kips$$

Now that we have the reactions, we can build the shear diagram.

Starting on the left side, go up the reaction at support A, 6.45 kips. Continue along the beam following the loads to finish the shear diagram.





To build the moment diagram calculate the area under the curve from the shear diagram using geometry. Calculate the area of three rectangles and two triangles.

$$A_1 = (6.45)\,(5) = 32.25$$
$$A_2 = (1.55)\,(6) = 9.3$$
$$A_3 = (3.45)\,(3) = 10.35$$
$$\tfrac{18}{6} = \tfrac{3.45}{x} \;\rightarrow\; x = 1.15\ ft, y = 6 - 1.15 = 4.85\ ft$$

$$A_4 = \tfrac{1}{2}(1.15)\,(3.45) = 1.98$$
$$A_5 = \tfrac{1}{2}(4.85)\,(14.55) = 35.28$$

From the moment diagram see that $|M_{max}|$ = 35.28 kip·ft. We will use this value and σ_{all} = 24 ksi (given in the problem statement) to calculate the minimum section modulus needed, S_{min}.

$$S_{min} = \frac{|M_{max}|}{\sigma_{allow}}$$
$$S_{min} = \frac{(35.28\ kip{\cdot}ft)\left(12\frac{in.}{ft}\right)}{24\ ksi}$$
$$S_{min} = 17.64\ in.^3$$

Now that you know the minimum section modulus needed for this beam, go to the standard beam table in Appendix A. Focusing on the S_x column, start at the bottom of the table to select one shape from each height section that is applicable.

For this beam the following sections will be adequate for bending:

- W8 × 21
- W10 × 19
- W12 × 22
- W14 × 22

Why not choose any from the W4, W5, and W6 groupings? Because none of these beams have a section modulus greater than 17.64 $in.^3$ Nor do we continue with groupings greater than W14 because those sections are far larger than what we need, as noted by the weight (second number in the designation).

Now choose the most economical beam by comparing the last number in the designation, which represents the weight per foot of the beam. The most economical section for this material and loading is the W10 × 19. This beam weighs 19 lb/ft and is the lightest of the beams selected.

Step-by-Step: Most Economical Beam Design (Standard Shapes)

1. Establish the allowable stress, σ_{all}, value of sigma for the chosen material by referring to a table of material properties

or consulting design specifications. Proceed with the following steps, assuming that σ_{all} is the same for tension and compression.

2. Determine the maximum absolute value of the bending moment $|M_{max}|$ in the beam by drawing the shear force and bending moment diagrams. (Chapter 7 reviews this).

3. Calculate the minimum allowable value of the section modulus using this equation: $S_{min} = \frac{|M_{max}|}{\sigma_{all}}$.

4. For a rolled steel beam refer to the relevant table in Appendix A. Among the accessible beam sections focus solely on those with a section modulus surpassing the minimum value calculated in step 3, $S \geq S_{min}$. Start at the bottom of the table and select one beam in each height section (if applicable). Then from this short list choose the section with the smallest weight per unit length, as it represents the most economical option for which $S \geq S_{min}$. It's essential to note that this choice may not necessarily correspond to the section with the smallest S value.

9.2.2 Design of Rectangular Sections

Rectangular cross-sections are common for timber beams. Only two dimensions need to be specified, base and height. While there are standard rectangular cross-sections too (e.g., 2 in. × 4 in.), we'll approach the design of rectangular cross-sections by assuming that one dimension scales with the other, and so we need to specify only a single dimension. Rather than select from a list of standard cross-sections, we'll specify this required dimension exactly.

Calculate the minimum section modulus as before.

$$S_{min} = \frac{|M_{max}|}{\sigma_{allow}}$$

Now set this value:

$$S_{min} = \frac{I}{c}.$$

For a rectangle, calculate

$$I = \frac{bh^3}{12}$$
$$c = \frac{h}{2}.$$

So the resulting equation is

$$S_{min} = \frac{I}{c} = \frac{2bh^3}{12h} = \frac{bh^2}{6}.$$

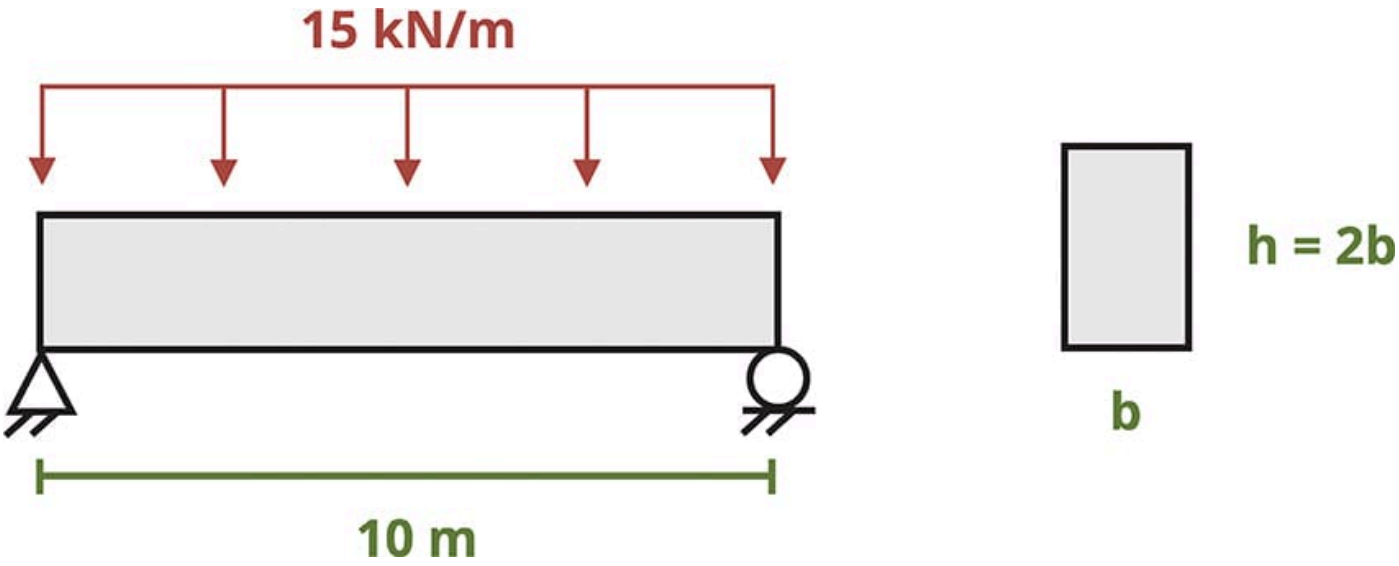
Problems will be set up such that there is a relationship between b and h, which leaves only one unknown that can be solved for directly. This will be the minimum acceptable size for this dimension—anything larger than the calculated value will not fail due to bending stress.

Example 9.4 designs a wooden beam with a rectangular cross-section to resist bending stress.

Example 9.5

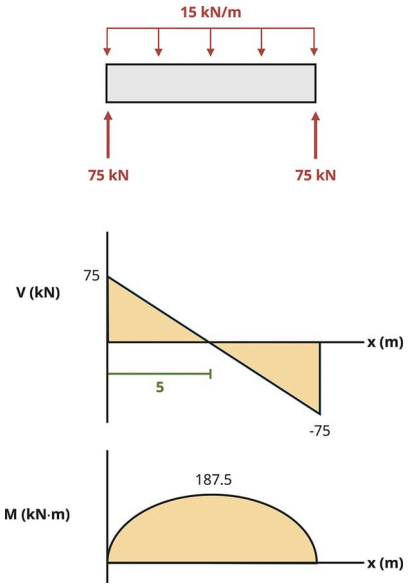
A simply supported timber beam spans a gap of 10 m. It is subjected to a uniform distributed load w = 15 kN/m as shown. The beam has a rectangular cross-section and will be manufactured such that the height is two times the base.

Determine the minimum acceptable dimensions for the cross-section of this beam if the maximum allowable bending stress of the timber is 12 MPa.



Solution

The allowable stress is given as σ_{all} = 12 MPa. Determine the magnitude of the maximum internal bending moment by drawing the shear force and bending moment diagrams. Since the loading is symmetric, the total applied load of 15 × 10 = 150 kN will be split evenly between the two supports. The reaction force at each support will be 75 kN.



The magnitude of the maximum internal bending moment $|M_{max}|$ = 187.5 kN·m.

Now determine the required section modulus. Be careful with the order of magnitude of the bending moment and allowable stress.

$$S_{min} = \frac{187.5 \times 10^3\ N \cdot m}{12 \times 10^6 \frac{N}{m^2}} = 0.015625\ m^3$$

Use this value to determine the required dimensions of the cross-section. Recall that h = 2b for this beam.

$$S_{min} = 0.015625\, m^3 = \frac{bh^2}{6} = \frac{b(2b)^2}{6} = \frac{4b^3}{6} = \frac{2}{3}b^3$$
$$b^3 = \frac{3}{2} * 0.015625\, m^3 = 0.0234375\, m^3$$
$$b = \sqrt[3]{0.0234375} = 0.286\, m$$
$$b = 286\, mm$$
$$h = 2b = 2 * 286 = 572\, mm$$

The minimum required dimensions to ensure this beam doesn't fail due to bending stress are b = 286 mm and h = 572 mm.

For the purposes of this text it is appropriate to give exact answers to three significant figures. In practice we wouldn't generally specify the required dimension down to the mm since this is unnecessarily precise to manufacture. It is more practical to design to perhaps the nearest 10 mm, remembering to always round up. For example, we calculated the required dimension b = 153 mm. We must not round this down to 150 mm because this is below the required value, but rounding up to 160 mm would be acceptable. While not included in this problem, a factor of safety (Section 4.8) is included in all designs in practice.

Step-by-Step: Most Economical Beam Design (Rectangular Sections)

1. Establish the allowable stress, σ_{all}, value of sigma for the chosen material by referring to a table of material properties or consulting design specifications. Proceed with the following steps, assuming that σ_{all} is the same for tension and compression.
2. Determine the maximum absolute value of the bending moment $|M_{max}|$ in the beam by drawing the shear force and bending moment diagrams (Chapter 7 reviews this).
3. Calculate the minimum allowable value of the section modulus using this equation: $S_{min} = \frac{|M_{max}|}{\sigma_{allow}}$.
4. Set this $S_{min} = \frac{bh^2}{6}$, substitute in the given relationship between b and h and solve for the unknown dimension.

9.3 Unsymmetric Bending—Moment Arbitrarily Applied

Occasionally a member may experience loading where the bending moment, M, doesn't align with one of the cross-section's principal axes. In such cases it is advisable to initially resolve the moment into components along the principal axes. The flexure formula can subsequently be applied to ascertain the normal stress induced by each moment component. Finally, employing the principle of superposition helps us determine the resultant normal stress at the specific point.

To systematize this process, envision a beam with a rectangular cross-section subjected to a moment, M, as depicted in Figure 9.12 (A). Here M forms an angle, q, with the maximum principal z-axis, which is the axis of maximum second moment of area for the cross-section. It is assumed that q is positive when directed from the positive z-axis toward the positive y-axis. We can resolve M into y and z components.

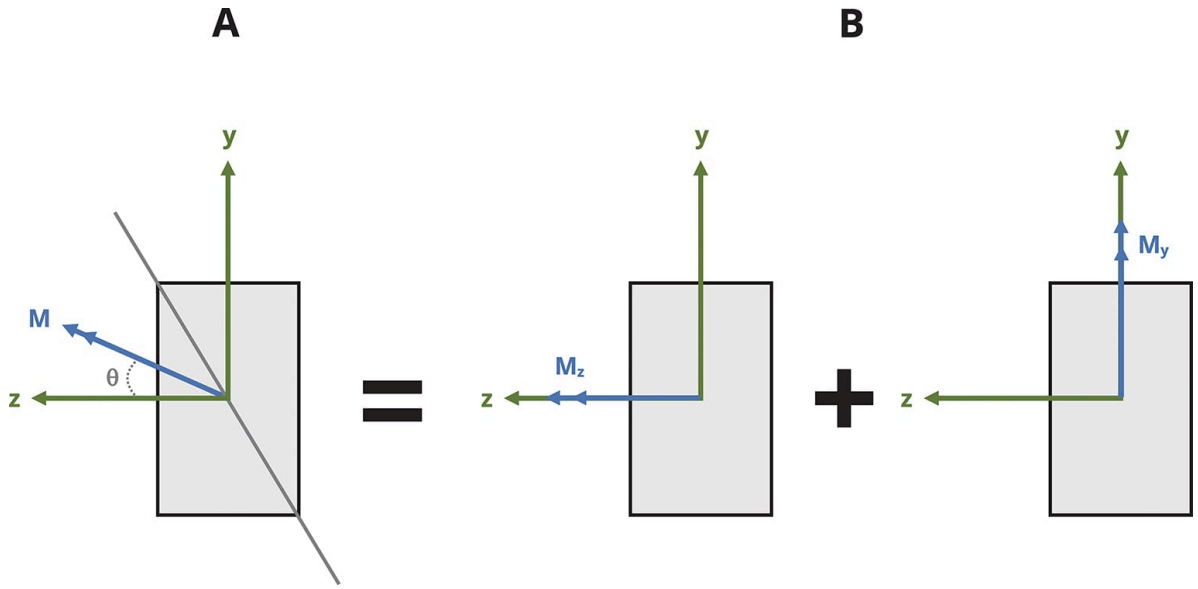


Figure 9.12: Moment arbitrarily applied to a rectangular cross-section

$$M_z = M\cos\theta$$
$$M_y = M\sin\theta$$

Figure 9.12 (B) shows the components of M acting on the rectangular cross-section. It's important to keep track of which side of the beam is in compression and which in tension when subjected to both M_z and M_y. Here we'll use tensile stress as positive and compressive stress as negative. Looking at a $+M_z$ shows us that this moment causes the top of the section to be in compression and the bottom of the section to be in tension. Inspecting the section when subjected to $+M_y$ reveals that the left side will be in tension whereas the right side will be in compression. Combining what we know about the sign convention with the flexure formula, we can calculate the resultant normal stress at any point on the cross-section by using the following equation:

$$\boxed{\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}},$$

(9.5)

where

- *σ = Bending stress at point of interest [Pa, psi]*
- *M_z, M_y = Internal bending moment components about the z-axis and the y-axis [N·m, lb·in.] y, z = Distance in the y and z directions from the centroid to the point of interest [m, in.]*
- *I_z, I_y = Second moment of area about the centroidal z-axis and y-axis [m^4, $in.^4$]*

Note that for the signs to work correctly we must assign the appropriate sign to the y and z values and to the components of the bending moment. These values are measured from the coordinate system that originates at the centroid where the x-axis is coming out of the page. It's also possible to ignore all the signs and use your spatial awareness to determine if each term is in tension or compression due to the applied moment.

Previously we identified the neutral axis—a line marking points of zero stress. When the bending moment acted around the horizontal axis, the neutral axis was also horizontal and passed through the centroid of the cross-section. When the bending moment acts at an angle (q) from the horizontal axis, the neutral axis will pass through the centroid but will not be horizontal. It too will be at an angle (a).

Let's now determine the orientation of the neutral axis, as this will commonly not coincide with the axis of bending moment.

Derive the orientation of the neutral axis by setting the previous normal stress equation equal to zero.

$$0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Then rearrange this to solve for y.

$$y = \frac{M_y I_z}{M_z I_y} z$$

Now put this equation in terms of M by using M_z=Mcos θ and M_y=Msin θ.

$$y = \left(\frac{I_z}{I_y} tan\theta\right)_z$$

Since the slope of the line is $\tan\alpha = \frac{y}{z}$, where the angle *α* represents the angle the neutral axis forms with the z-axis, this can be written as

$$\boxed{\tan(\alpha) = \left(\frac{I_z}{I_y}\tan\theta\right)},$$

(9.6)

where

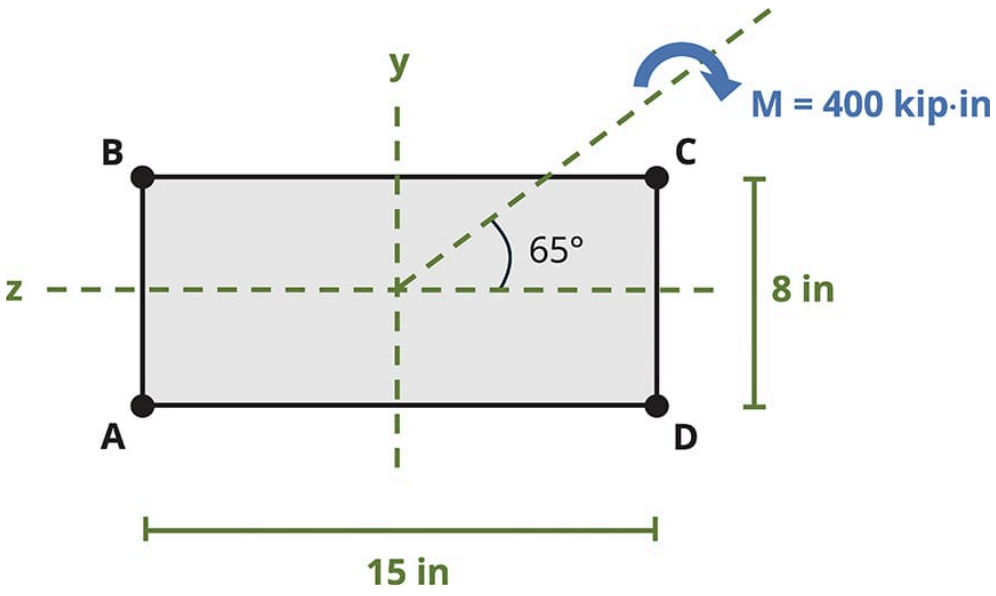
- *α = Orientation of the neutral axis, measured from the z-axis [°, rad]*
- *θ = Orientation of the internal bending moment, measured from the z-axis [°, rad]*
- *I_z, I_y = Second moment of area about the centroidal z-axis and y-axis [m^4, $in.^4$]*

Example 9.5 and Example 9.6 demonstrate how to calculate bending stress at various points on a cross-section when the moment is arbitrarily applied.

Example 9.6

A bending moment M=400 kip · in. is applied around an axis 65° above the negative z-axis as shown.

Determine the bending stress at corners A, B, C, and D. Also determine the orientation of the neutral axis with respect to the z-axis.



Solution

We first determine the components of the bending moment around the y-axis and the z-axis.

$$M_y = 400 \sin(65^\circ) = 362.5\ kip \cdot in.$$
$$M_z = -400 \cos(65^\circ) = -169.0\ kip \cdot in.$$

We'll also need the second moment of area around both axes.

$$I_y = \frac{8*15^3}{12} = 2{,}250\ in.^4$$
$$I_z = \frac{15*8^3}{12} = 640\ in.^4$$

Now we can calculate the stress at each corner by using

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Pay close attention to the signs used for each term.

Corner A:

$$\sigma_A = -\frac{-169.0\ kip{\cdot}in. \times -4\ in.}{640\ in.^4} + \frac{362.5\ kip{\cdot}in. \times 7.5\ in.}{2{,}250\ in.^4}$$
$$\sigma_A = -1.06\ ksi + 1.21\ ksi = 0.15\ ksi = 0.15\ ksi(T)$$

Corner B:

$$\sigma_B = \frac{-169.0\ kip{\cdot}in. \times 4\ in.}{640\ in.^4} + \frac{362.5\ kip{\cdot}in. \times 7.5\ in.}{2{,}250\ in.^4}$$
$$\sigma_B = 1.05\ ksi + 1.21\ ksi = 2.26\ ksi = 2.26\ ksi(T)$$

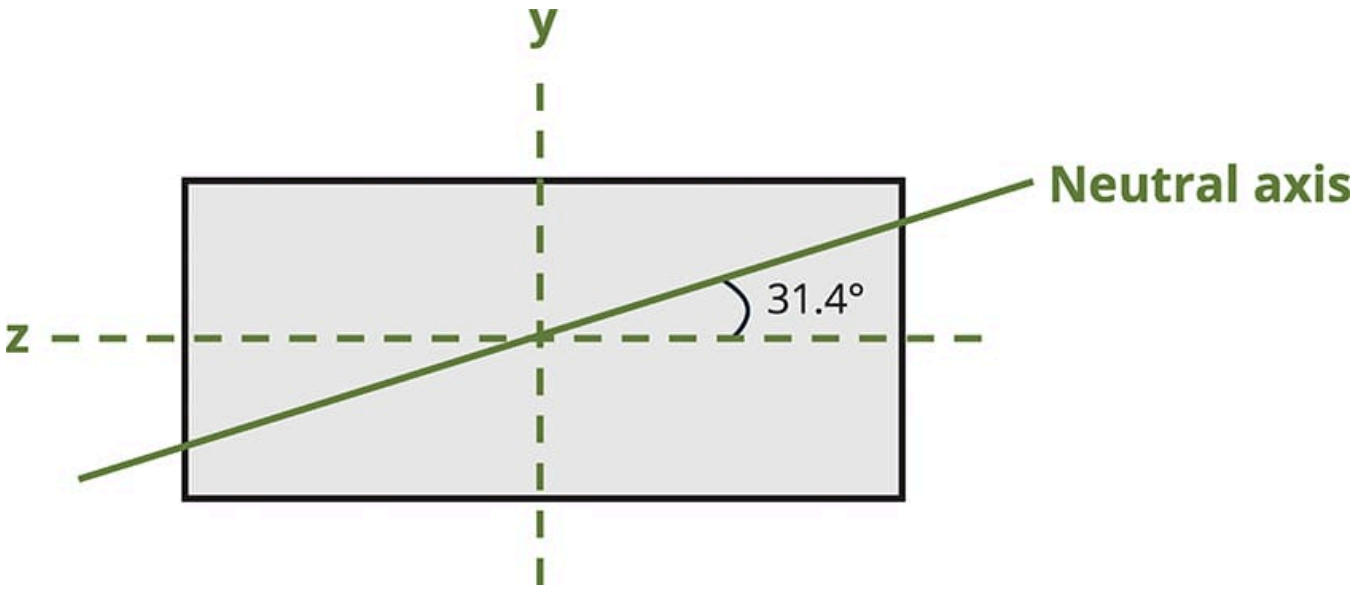
Corner C:

$$\sigma_C = -\frac{-169.0\ kip{\cdot}in. \times 4\ in.}{640\ in.^4} + \frac{362.5\ kip{\cdot}in. \times -7.5\ in.}{2{,}250\ in.^4}$$
$$\sigma_C = 1.06\ ksi - 1.21\ ksi = -0.15\ ksi = 0.15\ ksi(C)$$

Corner D:

$$\sigma_D = -\frac{-169.0\ kip{\cdot}in. \times -4\ in.}{640\ in.^4} + \frac{362.5\ kip{\cdot}in. \times -7.5\ in.}{2{,}250\ in.^4}$$
$$\sigma_D = -1.05\ ksi - 1.21\ ksi = -2.26\ ksi = 2.26\ ksi(C)$$

Note that the symmetry of the cross-section causes the stresses at diagonally opposite corners to have the same magnitude but opposite signs. This situation can shorten the problem, enabling us to calculate the stresses at only two corners (say A and B) and identify the stresses at the opposite corners.



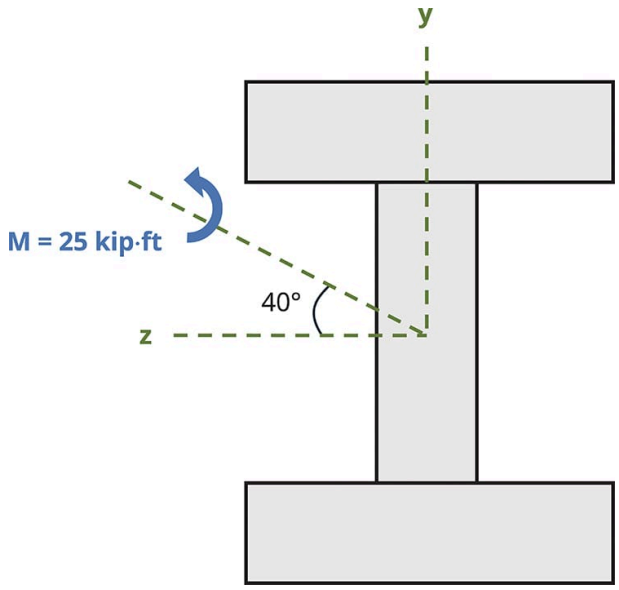
The orientation of the neutral axis is found from

$$\tan(\alpha) = \frac{I_z}{I_y}\tan(\theta)$$
$$\alpha = \tan^{-1}\left(\frac{640\ in.^4}{2{,}250\ in.^4}\tan(65°)\right) = 31.4°.$$

Example 9.7

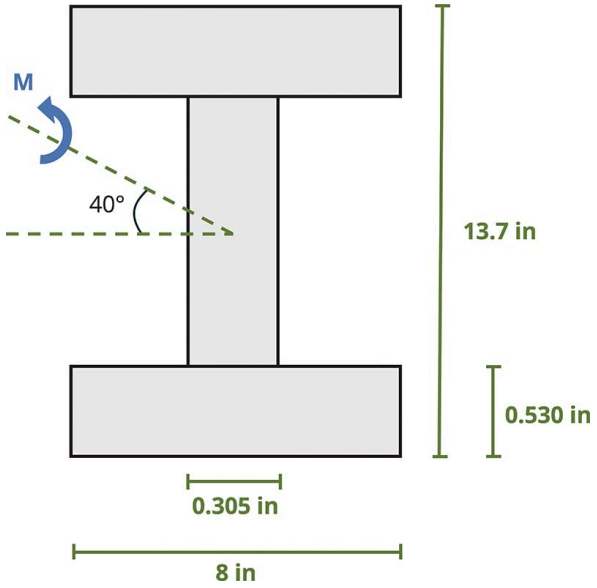
A bending moment M=25 kip · ft is applied to a W14 × 43 beam around an axis 40° above the positive z-axis as shown.

Determine the maximum tensile bending stress in the section and the orientation of the neutral axis with respect to the z-axis.



Solution

The dimensions of the W14 × 43 beam can be found in Appendix A. You can also look up the second moment of area about both axes.



$$I_y = 45.2\ in^4$$
$$I_z = 428\ in^4$$

Next determine the components of the bending moment around the y-axis and the z-axis. Since the bending moment is given in kip · ft and all the dimensions are given in inches, convert the bending moment to kip · in at this stage.

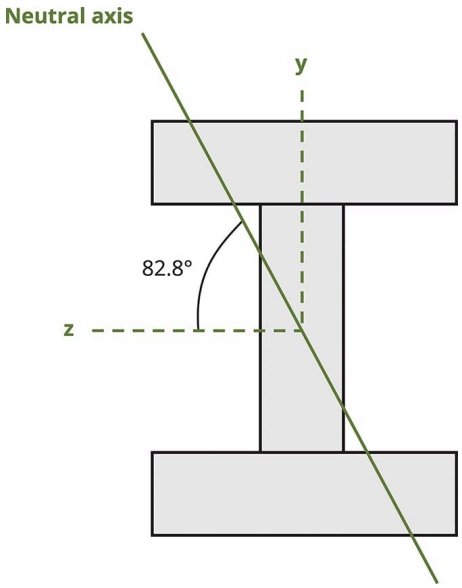
$$M_y = 25\sin(40^\circ)\ *12 = 193\ kip \cdot in.$$
$$M_z = 25\cos(40^\circ)\ *12 = 230\ kip \cdot in.$$

The bending stress at any point can be calculated using

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

The maximum tensile bending stress will occur at the corner where both of these terms are positive. Since both M_z and M_y are positive, it should be apparent that the y distance should be negative and the z distance should be positive. Thus the maximum tensile bending stress will occur at the bottom left corner. At this point $y = -\frac{13.7\ \text{in.}}{2} = -6.85\ \text{in.}$ and $z = \frac{8\ \text{in.}}{2} = 4\ \text{in.}$

$$\sigma = -\frac{230\ kip \cdot in.* -6.85\ in.}{428\ in.^4} + \frac{193\ kip \cdot in.*4\ in.}{45.2\ in.^4} = 3.68\ ksi + 17.1\ ksi = 20.8\ ksi\,(T)\,.$$



The orientation of the neutral axis is found from

$$\tan(\alpha) = \frac{I_z}{I_y}\tan(\theta)$$
$$\alpha = \tan^{-1}\left(\frac{428\ in.^4}{45.2\ in.^4}\tan(40^\circ)\right) = 82.8^\circ.$$

Step-by-Step: Unsymmetric Bending

1. Determine the geometric properties of the cross-section, including centroid location and second moment of area (I_y and I_z).
2. Determine the point of interest and the distances in the y and z directions from the centroid to this point (y and z).
3. Determine the y and z components of the internal bending moment M_y and M_z.
4. Calculate the bending stress at the point of interest from $\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$.

Summary

Key Takeaways

Beams under load experience normal stresses that vary linearly along the height of the cross-section. Assuming that the material is homogenous and linear elastic, the flexure formula can be used to calculate the normal stresses. Care must be taken to ensure that bending stresses are reported with magnitude units and tension or compression.

Engineers designing beams must consider how beams respond to bending stress, among other criteria. The design process in this chapter starts with the allowable stress and maximum moment to determine the minimum section modulus. This minimum section modulus is then used to choose an adequate, economical beam shape. (Future chapters present additional criteria for designing a beam.)

When a member experiences loading where the bending moment, M, does not align with one of the cross-section's principal axes, a method of superposition is used to determine the resultant normal stress at a specific point. It's also possible to calculate the orientation of the neutral axis due to this unsymmetric loading.

Key Equations

Flexure formulas:

$$\sigma_{max} = \frac{Mc}{I}$$
$$\sigma = \frac{My}{I}$$

Beam design for bending:

$$S = \frac{I}{c} = \frac{|M_{max}|}{\sigma_{allow}}$$

Unsymmetric bending—moment arbitrarily applied:

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

Orientation of the neutral axis:

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$

References

Figures

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- Figure 9.4: The distortion of a kid's toy when undergoing bending. Steve Berry. 2006. CC BY-NC-SA. https://flic.kr/p/4EzyRy.
- Figure 9.5: Column failure due to bending. Joshua Miller. 2005. CC BY-NC-ND. https://flic.kr/p/5RLNx.
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Chapter 10: Shear Loading in Beams

Learning Objectives

- Determine the first moment of area at various points on a cross-section
- Explain the distribution of shear stress on a cross-section
- Calculate the transverse shear stress at various points on a cross-section
- Calculate the shear flow on a built-up cross-section

Chapter 9 showed us how to calculate bending stresses in beams and how to design beams to resist bending stress. Transverse loading applied to beams will also create internal shear loads and therefore shear stresses that must be calculated and accounted for (Figure 10.1).

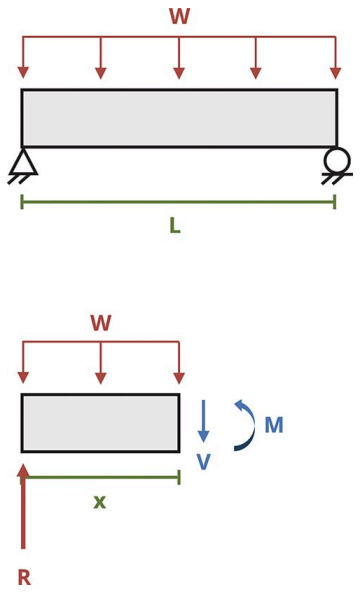


Figure 10.1: A beam subjected to transverse loading will experience both an internal bending moment and an internal shear force.

This chapter first reviews an important geometric property known as the first moment of area and demonstrates how to calculate it for common beam cross-sections in Section 10.1. Section 10.2 discusses transverse shear stresses in beams and how to calculate them. Section 10.3 covers shear flow, which is useful for beams with cross-sections formed by joining two or more parts with fasteners such as nails.

10.1 First Moment of Area

Shear stress in an object depends on the internal shear force and the shape of the cross-section. One important geometric property used in this chapter is the first moment of area. This property was introduced in Section 8.1 and was used when determining the centroid of an area. Recall we used Equation 8.1 to break down simple cross-sections into a number of composite parts. When finding the centroid of these composite areas we modified the equation slightly to

$$\bar{y} = \frac{\sum y_i A_i}{\sum A_i} = \frac{y_1 A_1 + y_2 A_2 + \ldots}{A_1 + A_2 + \ldots}.$$

We can rewrite this equation in its integral form.

$$\bar{y} = \frac{\int y dA}{\int dA}$$

The numerator in this equation is the first moment of area. It will be given the symbol Q.

$$Q = \int y dA$$

Similarly, for the purposes of this text we can write

$$\boxed{Q = \sum y_i A_i = y_1 A_1 + y_2 A_2 + ...},$$

(10.1)

where

- *Q = First moment of area [m^3, $in.^3$]*
- *y = Vertical distance from neutral axis to centroid of area A [m, in.]*
- *A = Area above (or below) point of interest [m^2, $in.^2$]*

In Section 8.1 we performed this calculation relative to some convenient reference axis to determine the location of the centroid of an area (such as in Example 8.3). A_i represents the area of each composite part, and y_i represents the distance from the reference axis to the centroid of each composite part.

The first moment of area can actually be calculated around any axis. The maximum value will always occur at the centroid of the cross-section (i.e., the neutral axis for pure bending), so many problems will involve finding the first moment of area at the neutral axis, but there are also cases where finding the first moment of area around some other axis is required. Note that the axis should be perpendicular to the shear force, so a vertical load requires us to use a horizontal axis.

To calculate the first moment of area around an axis, first draw a line through this axis. We may choose to use either the area above this axis or the area below it, whichever is simpler. The answer will be the same either way. Break down this area into composite parts (if necessary) and apply $Q = \sum y_i A_i$, where A_i is the area of each composite part and y_i is the perpendicular distance between the overall centroid of the cross-section and the centroid of each composite part. Note that to measure this distance we must first find the location of the overall centroid of the cross-section. Figure 10.2 illustrates this process for finding Q at the neutral axis and at another axis for two different cross-sections.

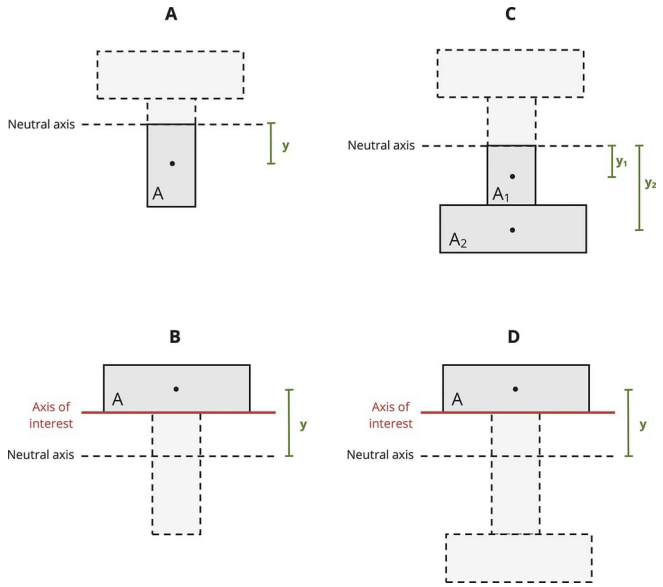


Figure 10.2: Process for finding the first moment of area, Q = yA, at the neutral axis and at a different axis of interest for two common cross-sections

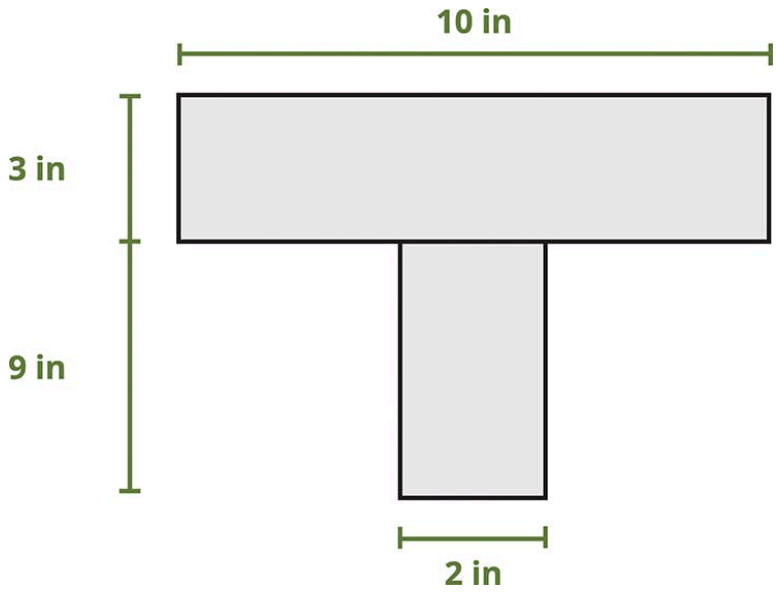
Example 10.1 demonstrates this process to find the first moment of area around the neutral axis of a T cross-section. This example is solved twice, once using the area below the neutral axis and once using the area above.

Example 10.2 uses the same cross-section and demonstrates calculating the first moment of area around a different axis on the cross-section.

Example 10.1

A beam has the T-shaped cross-section shown.

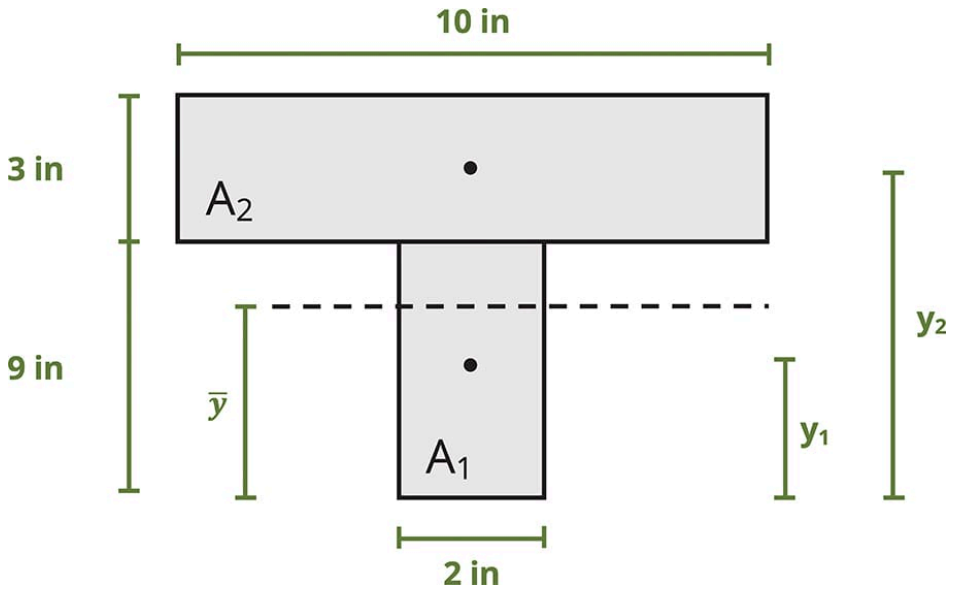
Determine the first moment of area around the neutral axis of the cross-section.



Solution

First determine the y coordinate of the centroid of the cross-section. Consider the area as two rectangles and use the method

applied in Section 8.1. We will choose a reference point at the bottom of the section to calculate the centroid.



$y_1 = 4.5\ in., A_1 = 18\ in.^2$

$y_2 = 10.5\ in., A_2 = 30\ in.^2$

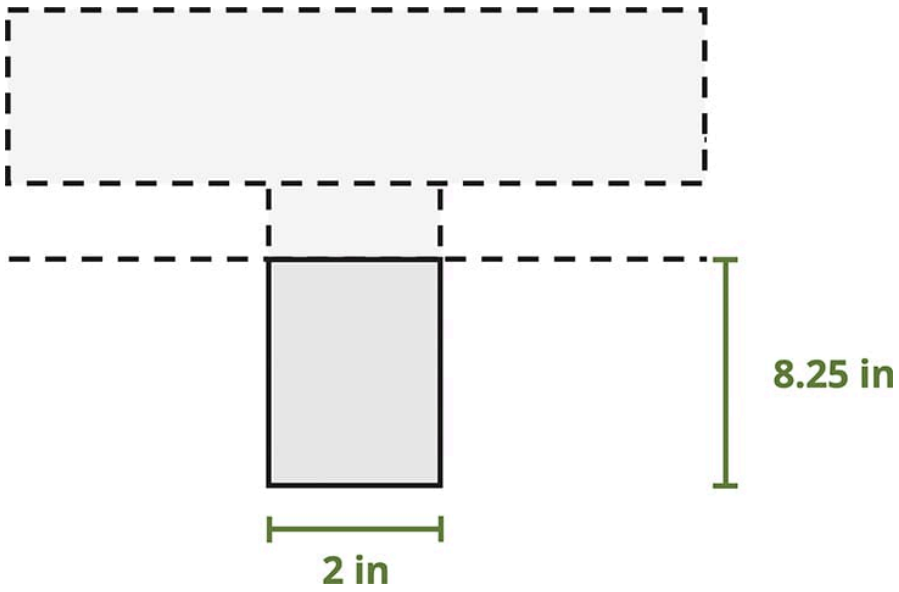
$\overline{y} = \frac{(4.5\ in.*18\ in.^2)+(10.5\ in.*30\ in.^2)}{18\ in.^2+30\ in.^2}$

$\overline{y} = 8.25\ in.$ from the bottom of the section (our reference point)

Note that since this section is simple, only two shapes, we need not create a table as we did in Section 8.1. This is a simplified version of the table, but the calculation is exactly the same.

Now draw a horizontal line through the point of interest. In this case the point of interest is the neutral axis. Use either the area above this line or the area below this line to determine the first moment of area around the neutral axis. We'll solve it both ways to demonstrate that we obtain the same answer either way.

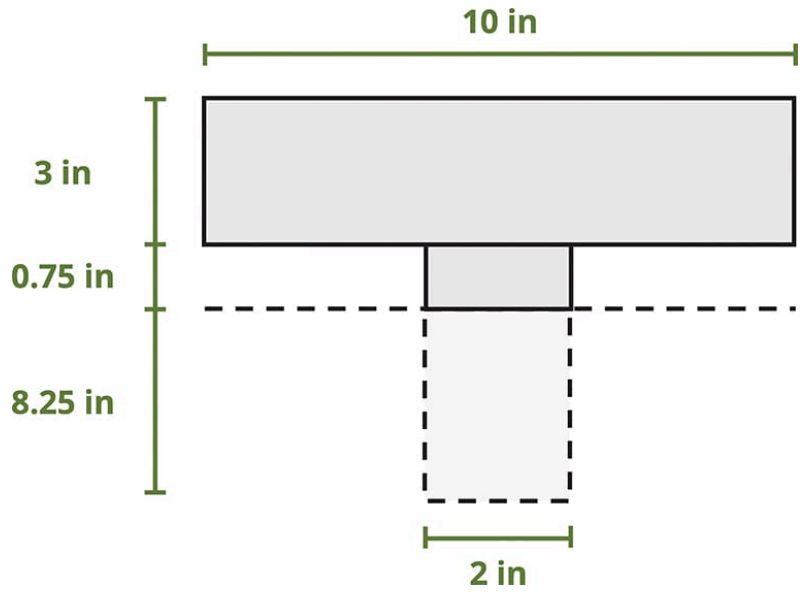
Area below the neutral axis:



This area is a simple rectangle. Recall that Q = yA, where A is the area of the rectangle and y is the vertical distance between the neutral axis and the centroid of the rectangle.

$$y = \frac{8.25\ in.}{2} = 4.125\ in.$$
$$A = 8.25\ in.\text{*}2\ in. = 16.5\ in.^2$$
$$Q = yA = 4.125\ in.\text{*}16.5\ in.^2 = 68.1\ in.^3$$

Area above the neutral axis:



This area can be split into two rectangles. Here we'll use $\mathrm{Q} = \sum \mathrm{y_i A_i} = \mathrm{y_1 A_1} + \mathrm{y_2 A_2}$.

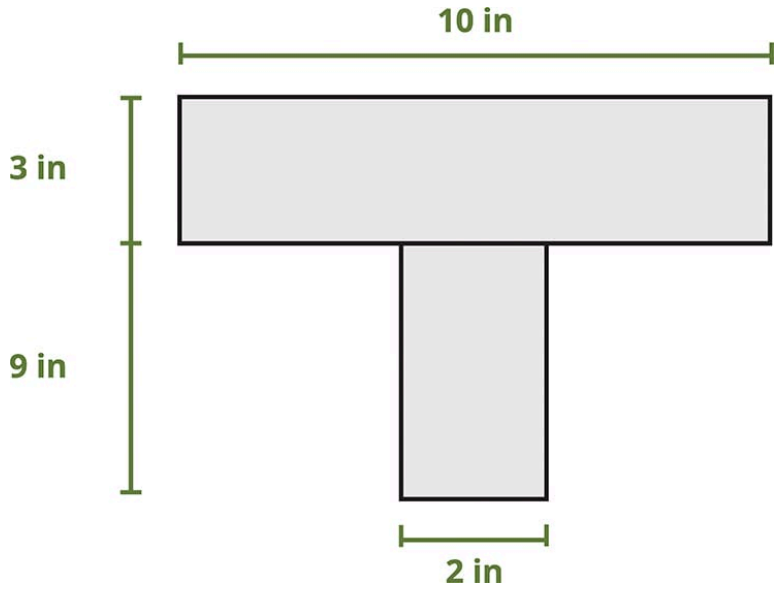
$$y_1 = \frac{0.75\ in.}{2} = 0.375\ in.$$
$$A_1 = 0.75\ in.\text{*}2\ in. = 1.5\ in.^2$$
$$y_2 = 0.75\ in. + 1.5\ in. = 2.25\ in.$$
$$A_2 = 10\ in.\text{*}3\ in. = 30\ in.^2$$
$$Q = (0.375\ in.\text{*}1.5\ in.^2) + (2.25\ in.\text{*}30\ in.^2) = 68.1\ in.^3$$

Whether we use the area above the neutral axis or the area below the neutral axis, the answer Q = 68.1 $in.^3$ is the same. We may choose to use whichever area seems simpler to work with.

Example 10.2

The T-shaped cross-section from Example 10.1 is repeated here.

Determine the first moment of area around an axis passing through the interface between the flange and the web.



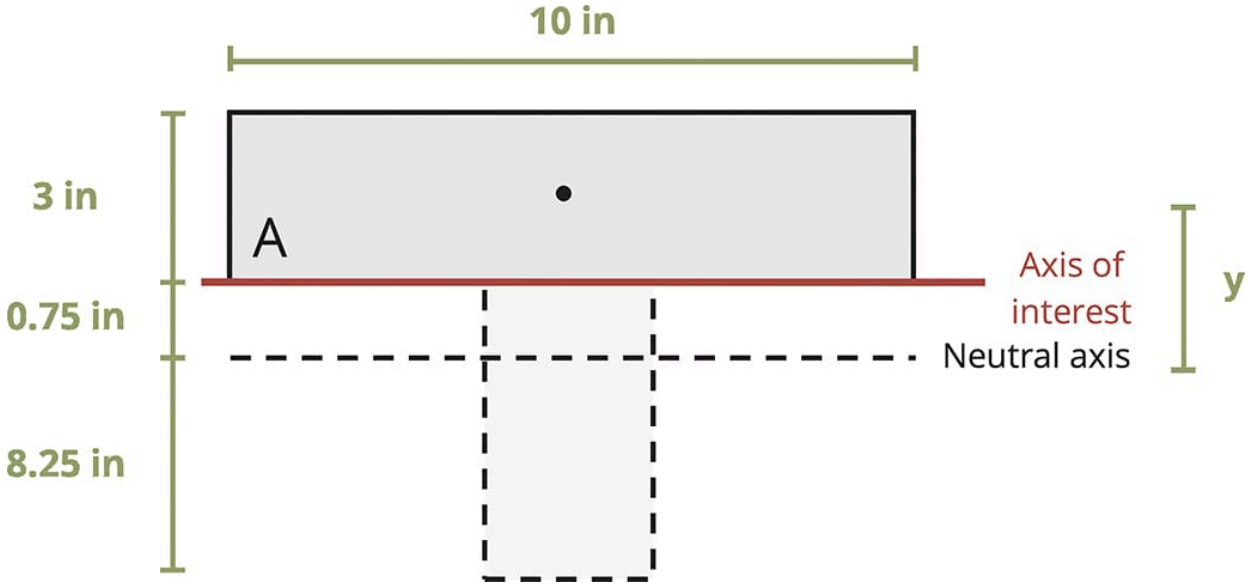
Solution

We need to first determine the y coordinate of the centroid of the cross-section, which we did in Example 10.1. The results are shown below.

$$\overline{y} = 8.25\ in. \text{from the bottom of the cross} - \text{section.}$$

Now draw a horizontal line through the point of interest. In this case the point of interest is the seam between the two boards. As demonstrated in Example 10.1, we may use either the area below this line or the area above this line to determine the first moment of area.

Area above the line:



This area is a simple rectangle. Recall that Q = yA, where *A* is the area of the rectangle and y is the vertical distance between the neutral axis and the centroid of the rectangle.

Note that the perpendicular distance is *always* measured to the neutral axis, not to the seam between the two boards.

$$y = 0.75\ in. + 1.5\ in. = 2.25\ in.$$
$$A = 10\ in.*3\ in. = 30\ in.^2$$
$$Q = yA = 2.25\ in.*30\ in.^2 = 67.5\ in.^3$$

As an additional exercise, try using the area below the seam between the two boards to calculate Q = 67.5 $in.^3$.

Simple cross-sections such as rectangles and circles are commonly used for beams. For these simple cross-sections we can derive standard solutions for the maximum first moment of area (Q_{max}). Recall that this always occurs at the neutral axis of the cross-section.

For a rectangle of base *b* and height *h* (Figure 10.3), the neutral axis will be at $\mathrm{y} = \frac{\mathrm{h}}{2}$. Drawing a line along the neutral axis and taking the area above this line results in $\mathrm{A} = \mathrm{b}\frac{\mathrm{h}}{2}$. The distance between the centroid of this area and the neutral axis of the cross-section will be $\mathrm{y} = \frac{\mathrm{h}}{4}$. Thus for a rectangular cross-section the equation is

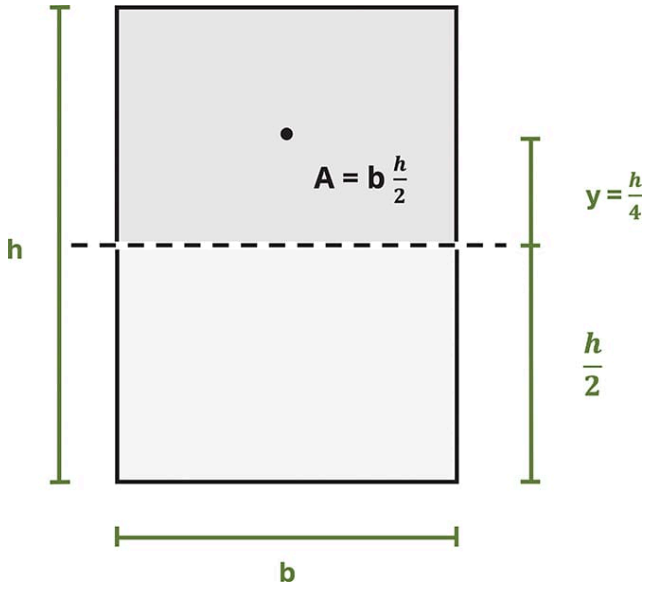


Figure 10.3: Determining the first moment of area around the neutral axis of a rectangular cross-section

$$Q_{max} = \frac{h}{4}\frac{bh}{2} = \frac{bh^2}{8}.$$

For a circular cross-section of radius r (Figure 10.4), the area above the neutral axis is semicircle, so $\mathrm{A} = \frac{1}{2}\pi\mathrm{r}^2$. The distance from the centroid of the semicircle to the neutral axis is $\mathrm{y} = \frac{4\mathrm{r}}{3\pi}$. Thus for a circular cross-section the equation is

$$Q_{max} = \frac{4r}{3\pi}\frac{1}{2}\pi r^2 = \frac{2}{3}r^3.$$

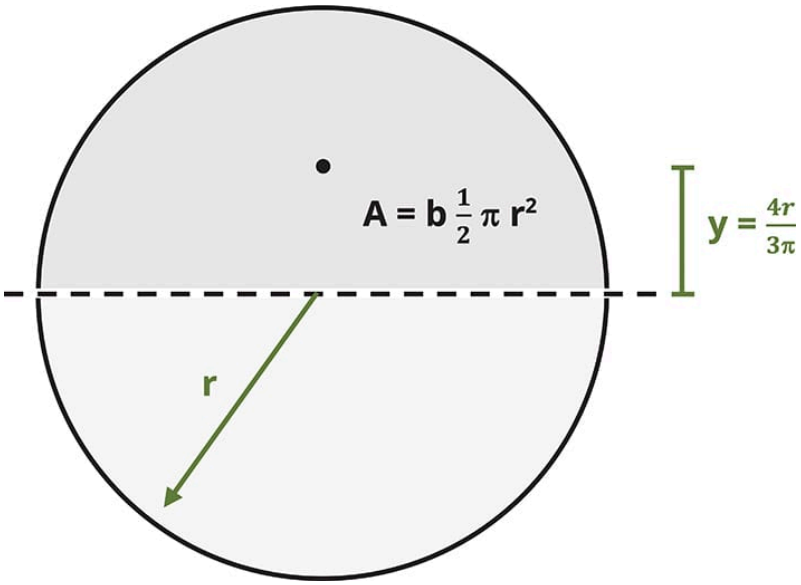


Figure 10.4: Determining the first moment of area around the neutral axis of a circular cross-section

Note that both of these approaches are used only for finding the first moment of area at the neutral axis (Q_{max}). If the first moment of area around any other axis is required, these shortcuts will not work and the general process will need to be used.

Step-by-Step: First Moment of Area

1. Locate the centroid of the cross-section.
2. Draw a line through the cross-section perpendicular to the internal shear force. Decide whether to use the area above this line or the area below this line.
3. Determine the area, A.
4. Determine the vertical distance between the centroid of area A and the neutral (centroidal) axis of the cross-section, y.
5. Calculate the first moment of area, Q = yA.
6. For composite areas, repeat steps 2 through 4 and calculate $\mathrm{Q} = \sum \mathrm{y_i A_i}$.

10.2 Shear Stress

Section 2.2 taught us about average shear stress.

$$\tau_{avg} = \frac{V}{A}$$

This is the average shear stress on a cross-section, but shear stress actually varies through the height of the section (Figure 10.5). For a vertical internal shear force, the shear stress is zero at the top and bottom of a cross-section and maximum at the neutral axis. Shear stress varies in a parabolic fashion. Because average shear stress does not provide a full picture, it is often necessary to calculate the shear stress at a specific point on the cross-section instead.



Figure 10.5: Parabolic variation in shear stress over a cross-section. For a vertical shear force on a uniform cross-section, the shear stress is zero at the top and bottom of the section and maximum at the neutral axis.

To derive the shear stress formula begin by examining a small 2D element on a body subjected to an internal shear force (Figure 10.6). If this element is in equilibrium, a shear force applied to one face of this element must also produce shear forces (and therefore shear stresses) on the other faces of the element.

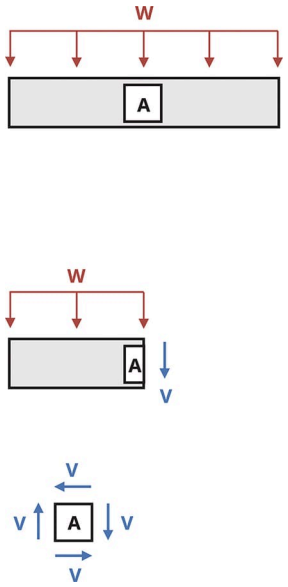


Figure 10.6: Element A on the beam is subjected to shear force V on the right side. For the element to be in equilibrium, there must also be an equal and opposite force V on the other side. However, this element will rotate, so there must be two additional forces to create an equal moment in the opposite direction.

Thus an element experiencing a vertical shear stress also experiences a horizontal shear stress. This is known as the complementary property of shear. This effect can be seen by forming a “beam” out of a stack of paper and applying a vertical force. The pieces of paper slide horizontally relative to each other (Figure 10.7).

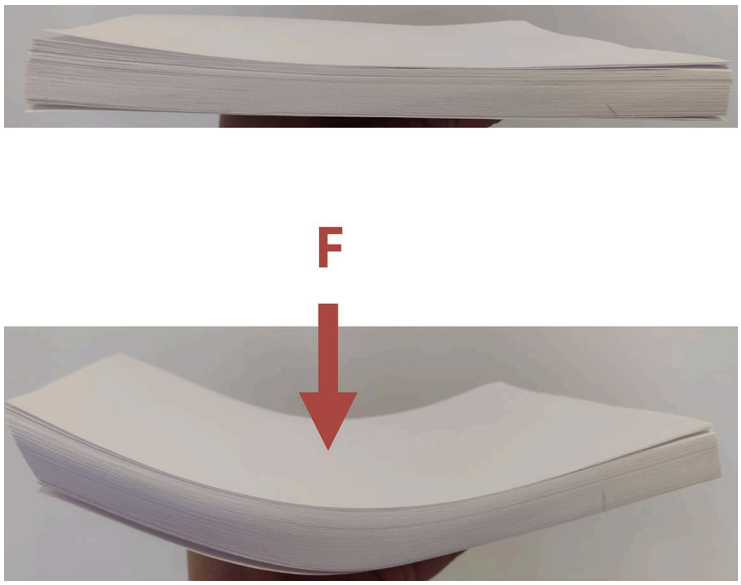


Figure 10.7: A stack of papers representing a beam. The ends of the beam are initially vertical. Load F is applied, creating an internal vertical shear force. The papers slide relative to each other horizontally, indicating that there must also be a horizontal shear force created.

Another example is a wooden beam that fails due to transverse shear. In such beams cracks tend to appear horizontally even though the applied load is vertical. This is due to horizontal shear stress between the layers of the wood (Figure 10.8).



Figure 10.8: Horizontal shear failure in a wooden beam subjected to a vertical load.

Imagine a beam subjected to an arbitrary vertical load. Consider a small cross-section of width Δx (Figure 10.9). This cross-section is subjected to an external load, an internal shear force on each side of the cross-section, and an internal bending moment on each side of the cross-section. The two shear forces and two bending moments are not necessarily equal.

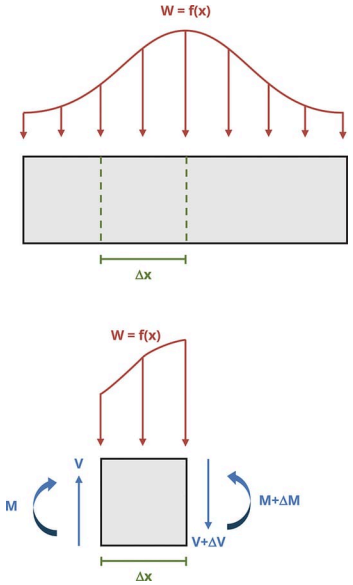


Figure 10.9: Internal loads on a small cross-section of width Δx on a beam subjected to an arbitrary vertical load, w = f(x). The internal loads on the right side of the section are slightly larger than those on the left.

By considering horizontal equilibrium, we do not need to account for the vertical loads (V, V + ΔV, and w). From the flexure formula (Equation 9.2) the bending moments (M and M + ΔM) will create horizontal stresses (Figure 10.10) equal to

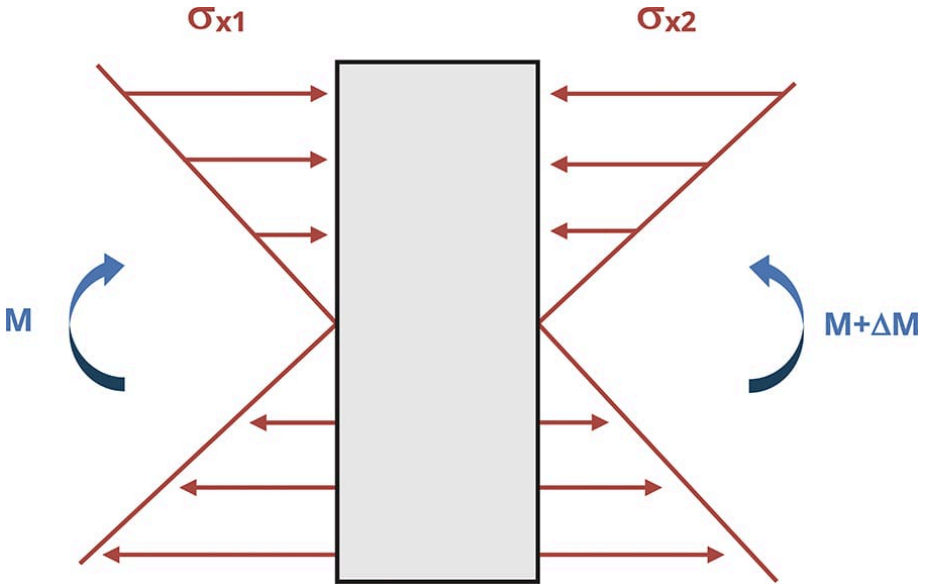


Figure 10.10: Horizontal stresses created by moments M and (M + ΔM)

$$\sigma_{x1} = \frac{My}{I}$$

and

$$\sigma_{x2} = \frac{(M + \Delta M)\, y}{I}.$$

Since $\mathrm{F} = \sigma \mathrm{A}$, we can determine the total force applied by each bending moment.

$$F_1 = \int \frac{My}{I} dA$$

and

$$F_2 = \int \frac{(M + \Delta M)\, y}{I} dA.$$

Each force is tensile in part of the beam and compressive in part of the beam. When calculated across the entire cross-section, these forces will sum to zero. However, if we consider only part of the cross-section, the forces do not sum to zero (Figure 10.11). For the beam to be in equilibrium, there must be a horizontal shear force. For now, we'll call this force F' and derive an equation for the shear stress it creates.

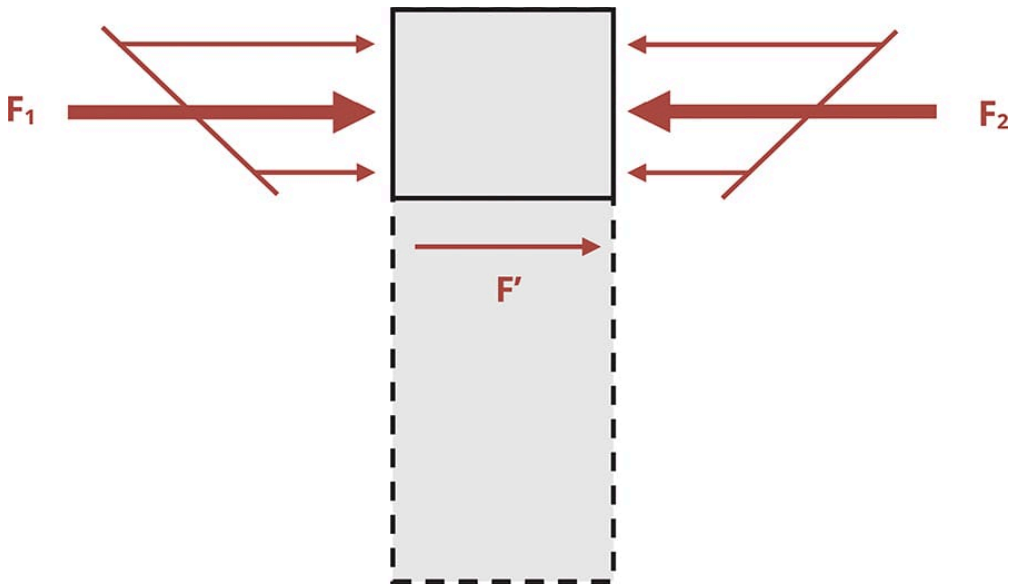


Figure 10.11: Forces F_1 and F_2 do not balance when only part of the element is considered. There must be an additional internal shear force F'.

By equilibrium obtain

$$\sum F_x = F' + \int \frac{My}{I} dA - \int \frac{(M+\Delta M)y}{I} dA = 0$$
$$F' = -\int \frac{My}{I} dA + \int \frac{My}{I} dA + \int \frac{\Delta My}{I} dA$$
$$F' = \int \frac{\Delta My}{I} dA = \frac{\Delta M}{I} \int y dA.$$

Once the constants are taken out of the integral, the remaining integral is just the first moment of area, Q.

$$F' = \frac{\Delta M}{I} Q$$

We can calculate the transverse shear stress from $\tau = \frac{V}{A}$, where V = F' and A is the area of the cross-section that force F' acts on. This area will be the width, Δx, multiplied by the thickness of the beam, t.

$$\tau = \frac{V}{A} = \frac{F'}{\Delta x t} = \frac{\Delta M Q}{\Delta x I t}$$

From Section 7.2 we know that the change in moment with respect to x is equal to the shear force, V.

$$\frac{\Delta M}{\Delta x} = V$$
$$\boxed{\tau = \frac{VQ}{It}},$$

(10.2)

where

- τ = *Transverse shear stress at a point on the cross-section [Pa, psi]*
- *V = Internal shear force at the cross-section [N, lb]*
- *Q =First moment of area at the point of interest on the cross-section [m^3, $in.^3$]*
- *I = Second moment of area of the cross-section [m^4, $in.^4$]*
- *t = Thickness of the cross-section [m, in.]*

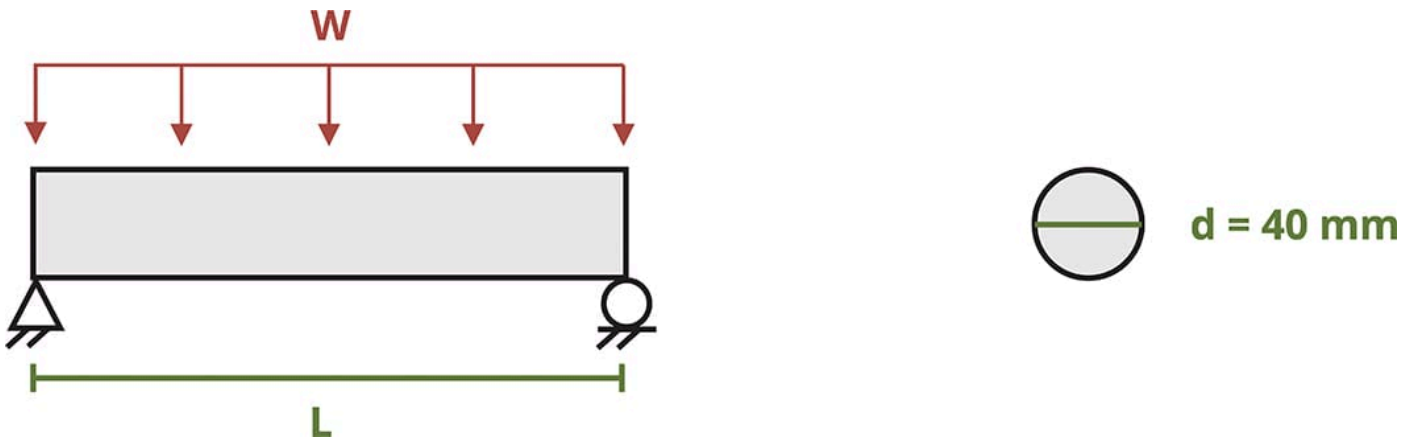
This is the transverse shear stress equation, allowing us to calculate the transverse shear stress at any point on a cross-section. This is often more useful than simply calculating the average shear stress across the entire cross-section.

Example 10.3 demonstrates calculation of the maximum transverse shear stress in a simply supported beam with a circular cross-section.

Example 10.3

A simply supported beam with the circular cross-section shown is subjected to a uniformly distributed load w = 30 kN/m across its entire length L = 8 m.

Determine the maximum shear stress that occurs in the beam.

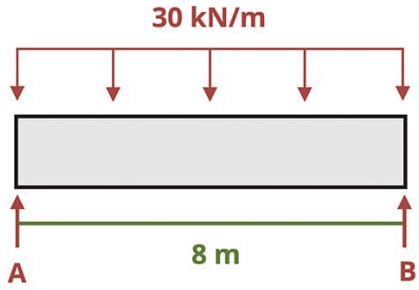


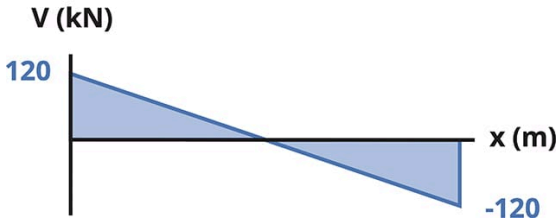
Solution

The shear stress can be calculated from $\tau = \frac{VQ}{It}$. The maximum shear stress will occur when V and Q are at their maximum values.

The maximum shear force, V_{max}, can be found by drawing a shear force diagram. Start with a free body diagram (FBD) of the beam and use this to sketch the shear force diagram, employing the graphical method of Section 7.4. The loading and supports are symmetric, so each support will resist half the applied load.

$$A = B = \frac{wL}{2} = \frac{30\frac{kN}{m} * 8\ m}{2} = 120\ kN$$





It is apparent that the maximum shear force in the beam is V_{max} = 120 kN. This occurs at the supports.

The second moment of area for the circular cross-section is given by

$$I = \frac{\pi}{4}r^4 = \frac{\pi}{4}(0.02\ m)^4 = 1.257 \times 10^{-7}\ m^4.$$

The maximum first moment of area, Q_{max}, occurs at the neutral axis of the cross-section. For a circular cross-section, as demonstrated in the text, that is

$$Q_{max} = \frac{2}{3}r^3 = \frac{2}{3}(0.02\ m)^3 = 5.33 \times 10^{-6}\ m^3.$$

The thickness of the cross-section at the neutral axis equals the diameter, t = 0.04 m.

With all the terms known, we can now calculate the maximum shear stress in the beam.

$$\tau = \frac{VQ}{It} = \frac{120{,}000\ N{*}5.33 \times 10^{-6}\ m^3}{1.257 \times 10^{-7}\ m^4{*}0.04\ m} = 127 \times 10^6 \frac{N}{m^2} = 127\ MPa$$

Note that the first moment of area will always be at its maximum value at the cross-section's neutral axis. Since shear stress $\tau = \frac{VQ}{It}$, the maximum shear stress often occurs at the neutral axis too. This will be true if the thickness, t, of the cross-section is uniform (e.g., a rectangular cross-section) or at its thinnest at the neutral axis (e.g., a wide-flange beam).

However, if thickness t is larger at the neutral axis than at another point in the beam, the maximum shear stress will possibly occur at the point where the thickness decreases even though Q will be smaller at this location than at the neutral axis. In these cases it's best to calculate the shear stress at both the neutral axis and the location where the thickness decreases.

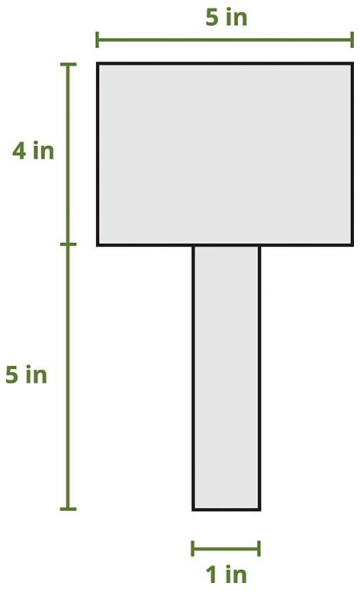
Example 10.4 demonstrates how to calculate the maximum transverse stress at a cross-section of varying thickness.

Example 10.4

A beam with the cross-section shown is subjected to an internal shear force of 20 kips.

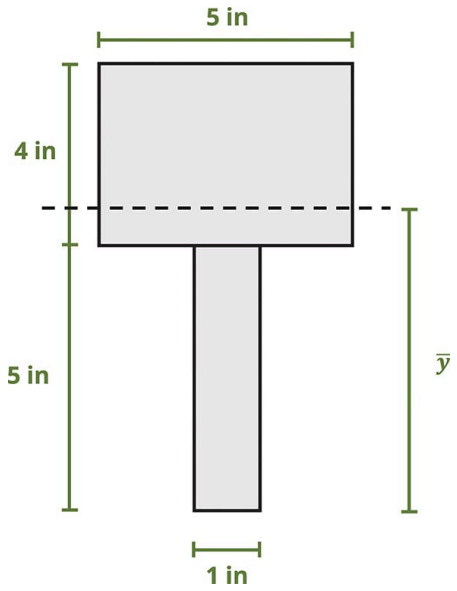
Determine the following:

1. The shear stress at the neutral axis of the cross-section
2. The shear stress at the point where the thickness of the cross-section changes



Solution

Determine the y coordinate of the centroid using the method of Section 8.1. Choose the reference point to be at the bottom of the section.



$y_1 = 2.5\ in., A_1 = 5\ in.^2$

$y_2 = 7\ in., A_2 = 20\ in.^2$

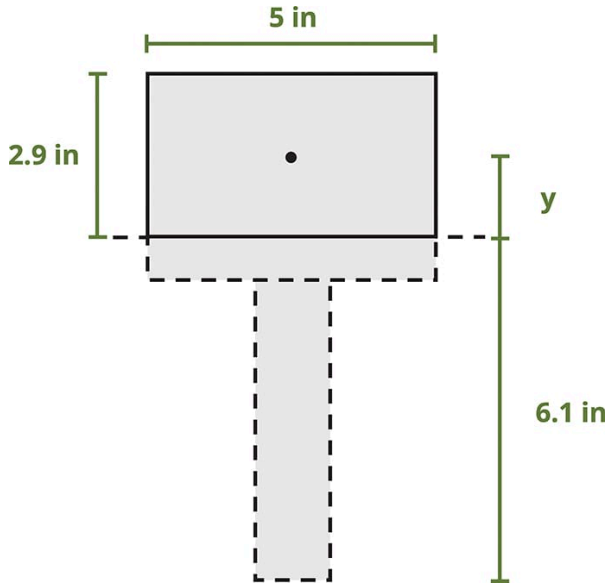
$\overline{y} = \frac{(2.5\ in.*5\ in.^2)+(7\ in.*20\ in.^2)}{5\ in.^2+20\ in.^2} = 6.1\ in.$ from the bottom of the section.

Determine the second moment of area of the entire cross-section using the method of Section 8.2.

$$I = \left(\frac{1*5^3}{12} + 5*(2.5-6.1)^2\right) + \left(\frac{5*4^3}{12} + 20*(7-6.1)^2\right) = 118.08\ in.^4$$

The shear stress at the neutral axis of the cross-section:

Draw a line through the point of interest and use the area above this line to calculate the first moment of area.



$$Q = yA = \frac{2.9\ in.}{2} * (2.9\ in.*5\ in.) = 21.025\ in.^3$$

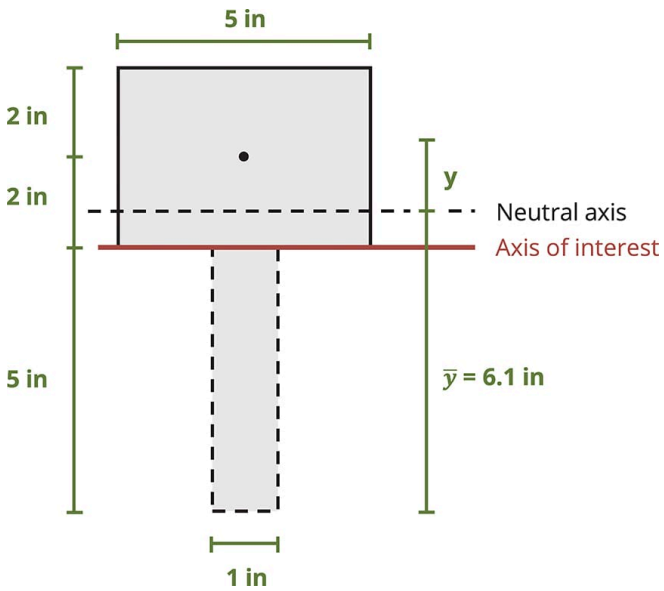
The thickness of the cross-section at this point is t = 5 in.

The shear stress at the neutral axis is then given by

$$\tau = \frac{VQ}{It} = \frac{20{,}000\ lb*21.025\ in.^3}{118.08\ in.^4*5\ in.} = 712\ psi.$$

The shear stress at the point where the thickness of the cross-section changes:

Draw a line through the point of interest and use the area above this line to calculate the first moment of area.



$$Q = yA = 0.9\ in.* (4\ in.*5\ in.) = 18\ in.^3$$

The thickness of the cross-section at this point is t = 1 in.

The shear stress at this line is then given by

$$\tau = \frac{VQ}{It} = \frac{20{,}000\ lb*18\ in.^3}{118.08\ in.^4*1\ in.} = 3{,}049\ psi.$$

Note that even though the first moment of area is smaller at this axis, the stress is larger because the thickness of the cross-section decreases significantly at this point.

Step-by-Step: Shear Stress

1. Cut a cross-section through the point of interest and determine the internal shear force acting on this cross-section. Do this either through equilibrium or by drawing a shear force diagram (Chapter 7).
2. Locate the centroid of the cross-section as shown in Section 8.1.
3. Calculate the second moment of area, I, for the cross-section. Use the entire cross-section for this calculation as shown in Section 8.2.
4. Draw a horizontal line through the point of interest and calculate the first moment of area, Q, as shown in Section 10.1.
5. Determine the thickness of the cross-section, t, at the point of interest.
6. Calculate the transverse shear stress, $\tau = \frac{\mathrm{VQ}}{\mathrm{It}}$.

10.3 Shear Flow

Some cross-sections are formed by connecting multiple pieces together with fasteners like nails or bolts (Figure 10.12). In such members we must analyze not only the stresses in the cross-section itself but also the stresses in the fasteners.

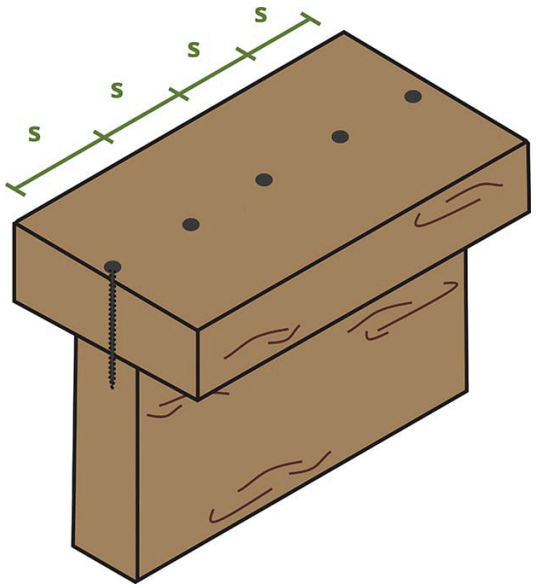


Figure 10.12: T cross-section formed by connecting two boards together with nails. The spacing s between nails along the length of the beam is consistent.

A useful quantity in this regard is shear flow, q. Shear flow measures how much shear force is applied per the beam's unit length and is calculated in units of force/length [N/m, lb/in.]. For example, if a nail is capable of resisting a shear force up to 10 kN and there is a shear flow of 30 kN/m, it is trivial to determine that we need at least three nails for every meter of beam.

Recall from Section 10.2 that the internal horizontal shear force, F', may be written as

$$F' = \frac{\Delta M}{I} Q.$$

Dividing both sides by Δx results in

$$\frac{F'}{\Delta x} = \frac{\Delta MQ}{\Delta x I}.$$

Similar to the derivation in the previous section, we may replace $\frac{\Delta \mathrm{M}}{\Delta \mathrm{x}} = \mathrm{V}$. We may also replace $\frac{F'}{\Delta x} = \mathrm{q}$. The result is

$$q = \frac{VQ}{I}.$$

This equation is similar to the shear stress equation, but the thickness of the cross-section doesn't appear here because shear flow is measured per unit length. We will generally want to determine the shear flow at the seam where the individual pieces of the cross-section are joined together as this is where the fasteners must resist the horizontal shear force.

Often it's necessary to determine how far apart to space the nails along the length of the beam. Since shear flow is the shear force per unit length, if we know how much shear force each fastener can resist, we can say that the shear flow is equal to the force resisted by each fastener (F_{nail}) divided by the spacing between the fasteners (s). Sometimes there will be multiple rows of nails side by side (Figure 10.13). In this case multiply the force resisted by each fastener by the number of rows (r).

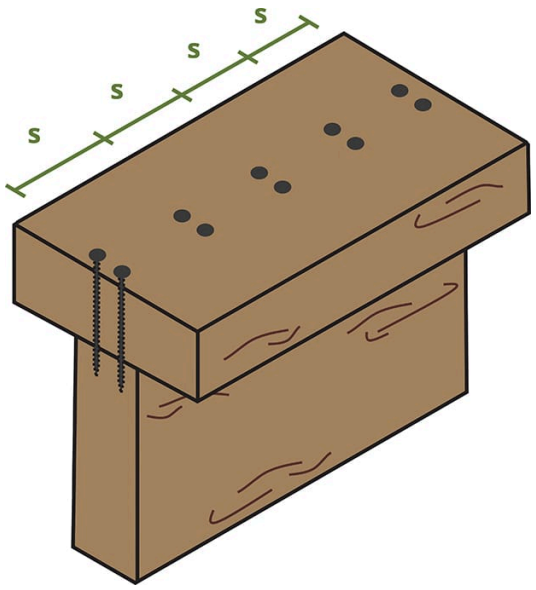


Figure 10.13: T cross-section held together by two rows of nails. The rows are spaced evenly along the length of the beam by distance s.

$$\boxed{q = \frac{VQ}{I} = \frac{F_{nail} r}{s}},$$

(10.3)

where

- *q = Shear flow at a line on the cross-section [N/m, lb/in.]*
- *V = Internal shear force at the cross-section [N, lb]*
- *Q = First moment of area at the point of interest on the cross-section [m^3, $in.^3$]*
- *I = Second moment of area of the cross-section [m^4, $in.^4$]*
- *F_{nail} = Horizontal shear force resisted by each nail [N, lb]*

- *r = Number of rows of nails resisting the shear force*
- *s = Spacing between nails [m, in.]*

Sometimes fasteners may be placed horizontally (Figure 10.14). In these cases still draw a line along the seam perpendicular to the fastener (so vertically here). Then use the area either to the left or right of this line to determine the first moment of area. Note that we still use the vertical distance between the centroid of the chosen area and the neutral axis of the cross-section. For cross-sections with an axis of symmetry, choose to draw either one vertical line and use the area to the left (or right) of this line or two vertical lines, one at each seam, and use the area between (or outside) these lines.

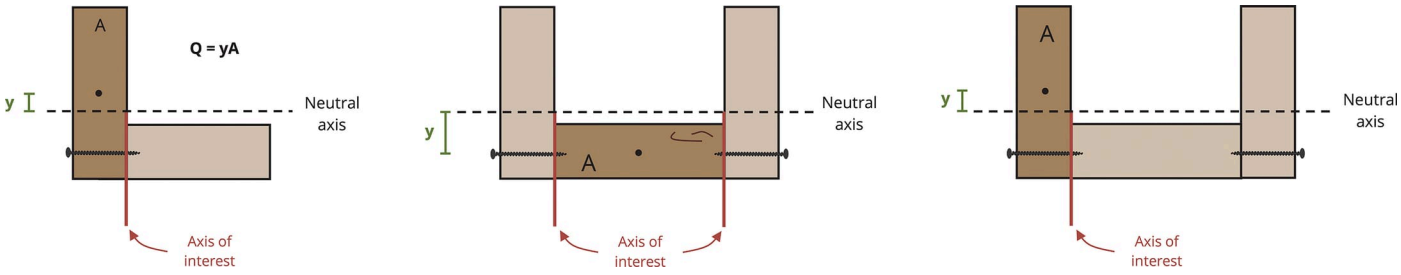


Figure 10.14: A horizontal fastener requires a vertical line along the seam and the selection of the area either to the left or right of this line. When the first moment of area is calculated by Q = yA, the distance y remains in the vertical direction.

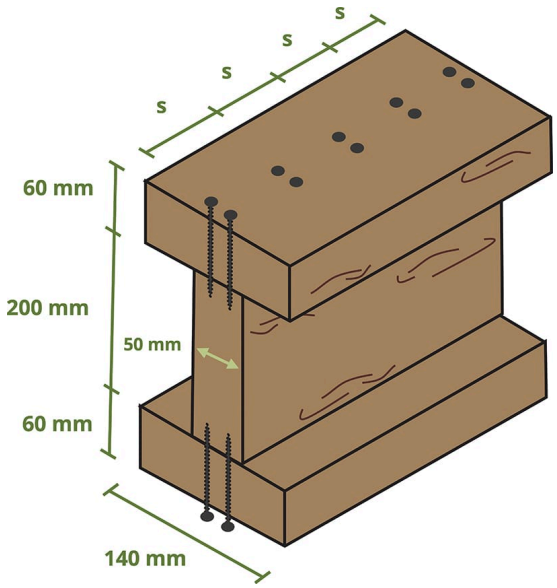
Example 10.5 works through a problem with two rows of nails at each seam.

Example 10.6 involves an L-beam with a horizontal nail.

Example 10.5

A beam is made by nailing together three boards to form the cross-section shown. At each connection, two rows of nails are used to secure the boards. Each nail can resist a horizontal shearing force of 1,500 N.

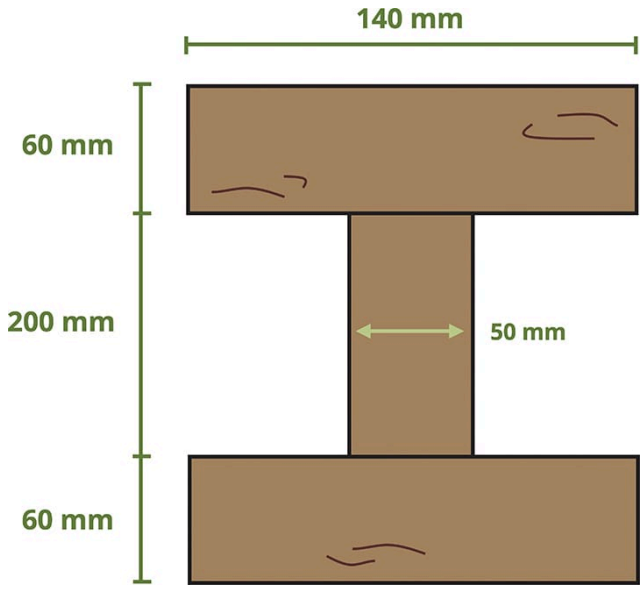
If the beam is subjected to an internal vertical shear force of 4,500 N, determine the minimum required spacing, s, between the rows of nails.



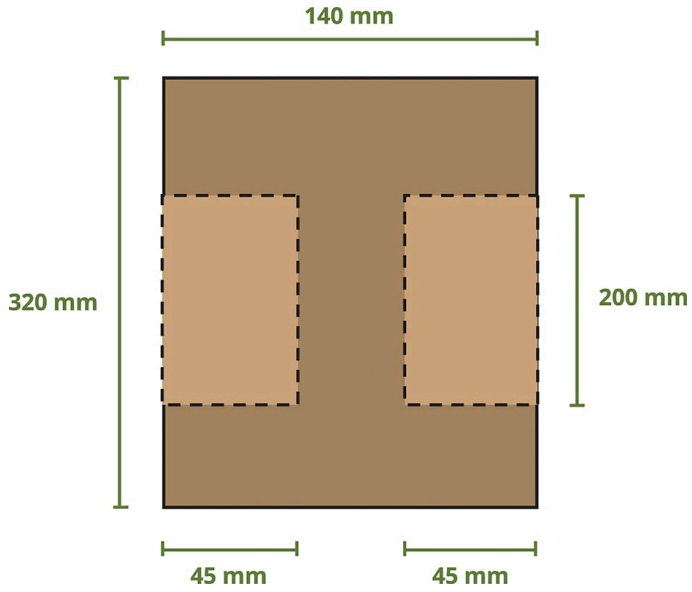
Solution

We need to determine the shear flow at one of the points where the beams are connected. Use $q = \frac{VQ}{I}$ where V = 4,500 N. Since the section is symmetric, determine the location of the centroid by inspection, $\overline{y} = 160 \text{ mm}$ from the top and bottom. Now

determine the second moment of area, I, for the cross-section.

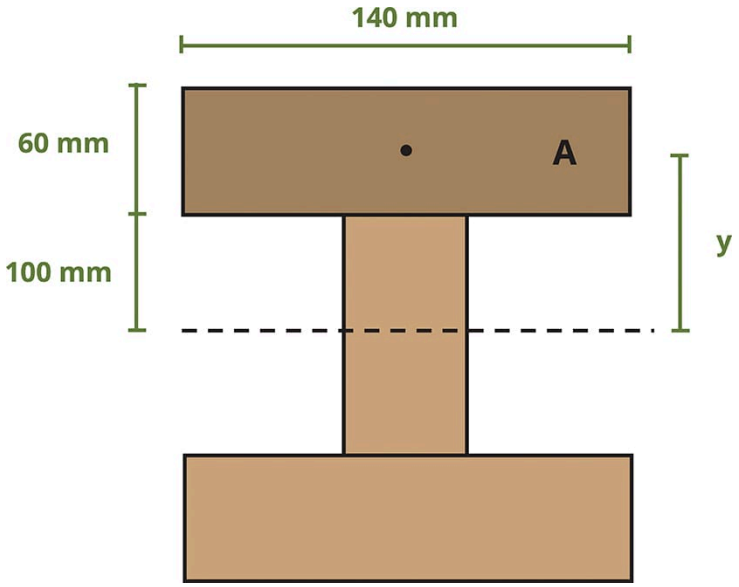


Consider the area as a large rectangle with two smaller rectangles cut out.



$$I = \frac{140*320^3}{12} - 2\left(\frac{45*200^3}{12}\right) = 322.3 \times 10^6\ mm^4 = 322.3 \times 10^{-6}\ m^4$$

Next draw a line through the point of interest (along the seam where the nails are holding the boards together) and determine the first moment of area, Q, at this line.



$$Q = yA = 130\ mm^* (60\ mm^*140\ mm) = 1{,}092 \times 10^3\ mm^3 = 1{,}092 \times 10^{-6}\ m^3$$

Shear flow is then

$$q = \frac{VQ}{I} = \frac{4{,}500\ N^*1{,}092 \times 10^{-6}\ m^3}{322.3 \times 10^{-6}\ m^4} = 15{,}247\ \frac{N}{m}.$$

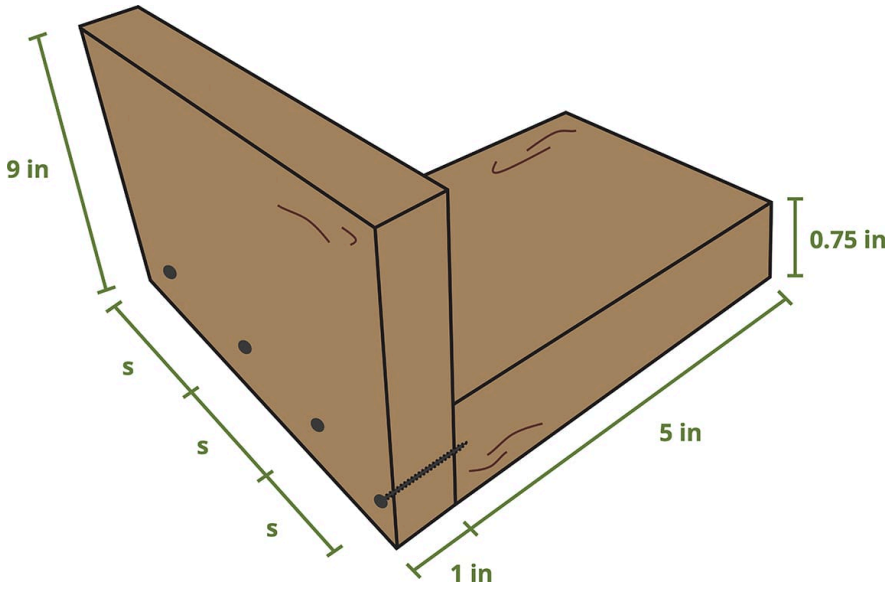
Finally, set the shear flow, $\mathrm{q} = \frac{\mathrm{F_{nail}r}}{\mathrm{s}}$, and solve for the spacing, s.

$$s = \frac{F_{nail}r}{q} = \frac{1{,}500\ N^*2}{15{,}247\ \frac{N}{m}} = 0.197\ m = 197\ mm$$

Example 10.6

A beam is formed by nailing two boards together to form the L-shaped cross-section shown. The nails are spaced a distance s = 1.5 in. apart along the length of the beam, and each nail can resist a force of 700 lb in shear.

Determine the maximum allowable internal shear force in the beam.



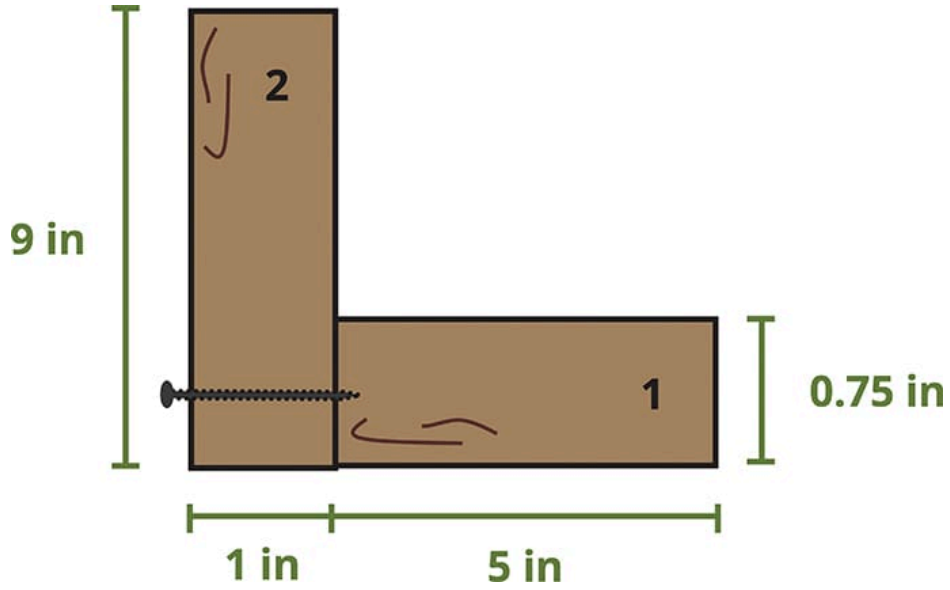
Solution

Determine the allowable shear flow in the beam given the nails.

$$q = \frac{F_{nail} r}{s} = \frac{700\ lb*1}{1.5\ in.} = 467\ \frac{lb}{in.}$$

This shear flow can be set equal to $\mathrm{q} = \frac{\mathrm{VQ}}{\mathrm{I}} = 467\ \frac{\mathrm{lb}}{\mathrm{in.}}$.

Find the y coordinate of the centroid of the cross-section, setting the reference point at the bottom of the section.



$y_1 = 0.375\ in., A_1 = 5\ in.*0.75\ in. = 3.75\ in.^2$

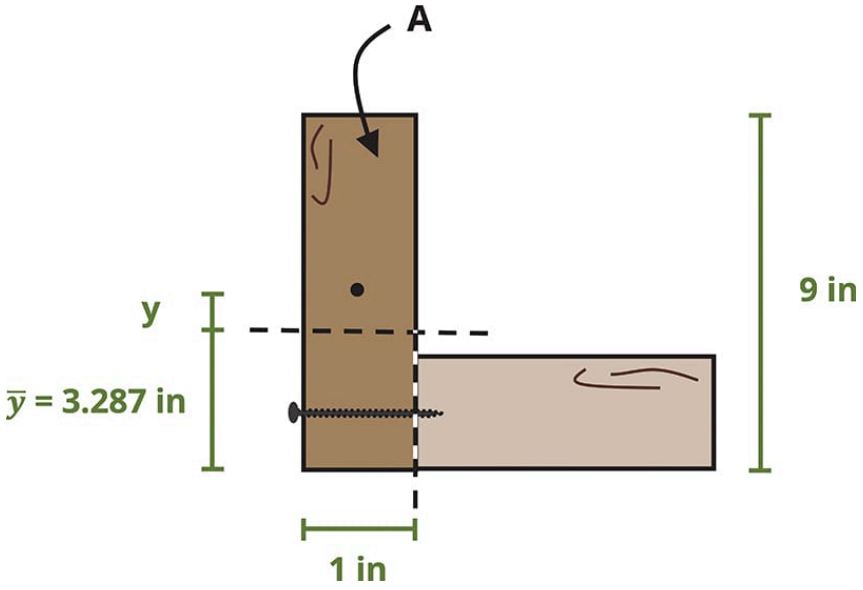
$y_2 = 4.5\ in., A_2 = 1\ in.*9\ in. = 9\ in.^2$

$\overline{y} = \frac{(0.375\ in.*3.75\ in.^2)+(4.5\ in.*9\ in.^2)}{3.75\ in.^2+9\ in.^2} = 3.287\ in.$ from the bottom of the section.

Find the second moment of area for the entire cross-section.

$$I = \left(\frac{5*0.75^3}{12} + 3.75*(3.287-0.375)^2\right) + \left(\frac{1*9^3}{12} + 9*(4.5-3.287)^2\right) = 106\ in.^4$$

Draw a line along the seam where the nails are holding the two boards together. Note that this is a vertical line. To determine the first moment of area, Q, we use the area either to the left or to the right of this line. Note that when calculating Q = yA, the distance y is still measured in the vertical direction between the centroid of area A and the neutral axis of the cross-section.



$$y = 4.5\ in. - 3.287\ in. = 1.213\ in.$$
$$A = (1\ in.*9\ in.) = 9\ in.^2$$
$$Q = 1.213\ in.*9\ in.^2 = 10.9\ in.^3$$

Finally, solve for the maximum allowable internal shear force in the beam by rearranging the shear flow equation.

$$V = \frac{qI}{Q} = \frac{467\ \frac{lb}{in.}*106\ in.^4}{10.9\ in.^3} = 4{,}541\ lb$$

Step-by-Step: Shear Flow

1. Cut a cross-section through the point of interest and determine the internal shear force acting on this cross-section. This can be accomplished either through equilibrium or by drawing a shear force diagram (Chapter 7).
2. Locate the centroid of the section as shown in Section 8.1.
3. Calculate the second moment of area, I, for the cross-section. Use the entire cross-section for this calculation, as shown in Section 8.2.
4. Draw a horizontal line through the point of interest and calculate the first moment of area, Q, as shown in Section 10.1.
5. Calculate the shear flow using $\mathrm{q} = \frac{\mathrm{VQ}}{\mathrm{I}}$.
6. If required, calculate the maximum allowable spacing between the fasteners using $\mathrm{q} = \frac{\mathrm{F_{nail}r}}{\mathrm{s}}$. Alternatively, we may be given the spacing and asked to find the force resisted by each fastener, F_{nail}.

Summary

Key Takeaways

Objects subjected to vertical shear stresses also experience horizontal shear stresses.

The first moment of area (Q) is an important quantity when considering shear. It can be calculated along any axis. Generally the internal shear force is vertical, and so the first moment of area is calculated around a horizontal axis. The maximum first moment of area will occur around the neutral axis of the cross-section.

Transverse shear stress varies parabolically across the cross-section. For a vertical shear force it is zero at the top and bottom of the section. For a uniform cross-section the maximum transverse shear stress occurs at the neutral axis because this is where the largest first moment of area is located. If the cross-section varies in thickness, the maximum transverse shear stress may not

be at the neutral axis since the transverse shear stress increases as the thickness of the cross-section decreases.

Shear flow is a measure of the shear force over a unit length. It is useful when creating a cross-section by fastening two or more individual pieces together. The shear flow can be used to determine the maximum allowable spacing between the fasteners to ensure the fasteners do not break. In practice, this should be determined alongside a calculation of the maximum shear stress in the cross-section.

Key Equations

First moment of area:

$$Q = \int ydA = \sum y_i A_i$$

For a rectangle:

$$Q_{max} = \frac{bh^2}{8}$$

For a circle:

$$Q_{max} = \frac{2}{3}r^3$$

Shear stress:

$$\tau_{avg} = \frac{V}{A}$$
$$\tau = \frac{VQ}{It}$$

Shear flow:

$$q = \frac{VQ}{I} = \frac{F_{nail}r}{s}$$

References

Figures

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- Figure 10.7: A stack of papers representing a beam. James Lord. 2024. CC BY.
- Figure 10.8: Horizontal shear failure in a wooden beam subjected to a vertical load. Figure 8 in Adam Derkowski, Marcin Kuliński, Adrian Trociński, Jakub Kawalerczyk, and Radosław Mirski. 2022. Mechanical Characterization of Glued Laminated Beams Containing Selected Wood Species in the Tension Zone. Materials 15(18), 6380. https://doi.org/10.3390/ma15186380. CC BY 4.0.

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Chapter 11: Beam Deflection

Learning Objectives

- Derive an equation for the elastic curve of a loaded beam
- Calculate the slope and deflection at any point of a loaded beam
- Use standard solutions and the method of superposition to determine deflection in more complex problems
- Use knowledge of deflection to solve statically indeterminate problems
- Design beam cross-sections that meet specifications for bending stress, shear stress, and deflection

We now know how to calculate both the bending stress (Chapter 9) and shear stress (Chapter 10) in a beam. This chapter considers the deformation of a beam subjected to bending and shear stresses. Under load, beams deflect from their original position (Figure 11.1). Some amount of deflection is unavoidable, but it is usually desirable to limit deflection as much as possible. Although deflection often pose no safety risk (unless the allowable stress of the beam is also exceeded), too much deflection can render a beam unfit for purpose.



Figure 11.1: A model bridge showing severe deflection along its span (left). Although the bridge hasn't broken, there is too much deflection for the application. The deflection in a hockey stick during a shot (right). In this application the deflection is helpful as it stores and releases energy during the shot.

The deflected shape of the beam is known as the elastic curve (Figure 11.2). Exaggerated sketches of the elastic curve provide a quick visual reference of how the beam will deflect and can be used to check that our numerical answers are realistic.

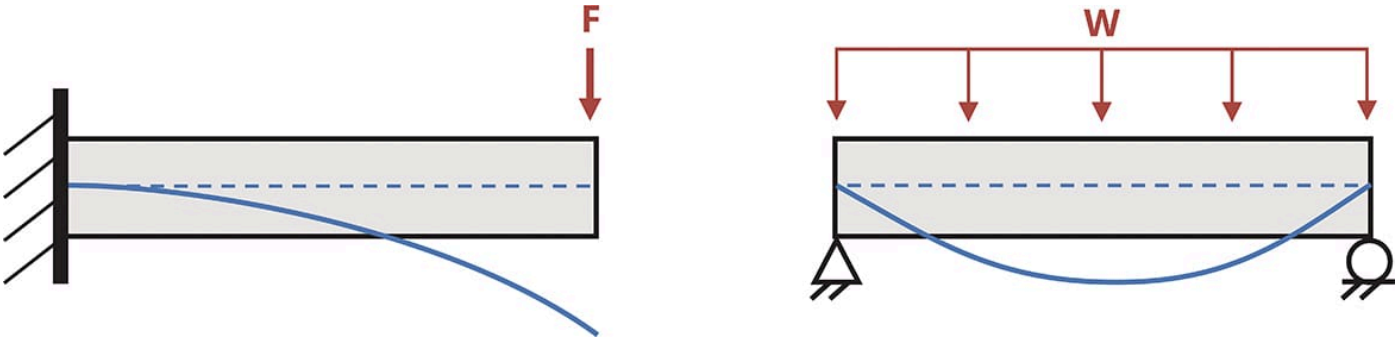


Figure 11.2: Two example beams showing the undeflected position of the centerline (dotted line) and the deflected position (elastic curve) of the beam (solid line).

Section 5.3 showed us how to calculate deformation due to axial load $\left(\Delta \mathrm{L}=\frac{\mathrm{FL}}{\mathrm{AE}}\right)$. Section 6.2 showed us how to calculate deformation due to torsional load $\left(\phi=\frac{\mathrm{TL}}{\mathrm{JG}}\right)$. Recall that both types of deformation depend on these elements:

- Applied/internal load (F, T)
- Length of the object (L)
- Cross-section geometry (A, J)
- Material properties (E, G)

It is probably unsurprising, then, that beam deflection also depends on these four elements.

This chapter introduces two common techniques for calculating deflection: the method of integration (Section 11.1 and Section 11.2) and the method of superposition (Section 11.3).

Being able to calculate deflection is vitally important because it allows us to predict how much a beam will deflect before we build a structure. Additionally, being able to calculate deflection will help us solve some more statically indeterminate problems (Section 11.4).

Finally, Section 11.5 enables us to build on our beam design work of Section 9.2 by expanding the specifications to include limitations on shear stress and deflection as well as bending stress.

11.1 Integration of the Moment Equation

We'll begin by deriving an equation for calculating the deflection at any point in a loaded beam. In Section 9.1 we derived equations relating to bending strain and bending stress.

$$\varepsilon=-\frac{y}{\rho}$$
$$\sigma=-\frac{My}{I}$$

Applying Hooke's law to the strain equation produces

$$\begin{aligned}\varepsilon=-\frac{y}{\rho} &= \frac{\sigma}{E}\\ \sigma &= -\frac{Ey}{\rho}.\end{aligned}$$

Setting the two stress equations equal produces

$$\frac{Ey}{\rho}=\frac{My}{I}.$$

Then rearranging the result yields

$$\frac{1}{\rho} = \frac{M(x)}{EI}.$$

The radius of curvature (ρ) can be related to the deflection (y) by the equation

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{\frac{3}{2}}}.$$

This equation describes the exact deflection (y) at any distance (x) along the beam for elastic deformations. In most engineering applications the deflection (y) is very small. We can simplify this equation significantly by assuming that the deflection (y) will be small and thus that $\frac{\mathrm{dy}}{\mathrm{dx}}$ is very small and $\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2$ is negligible. This simplifies the equation to

$$\frac{1}{\rho} = \frac{\frac{\partial^2 y}{\partial x^2}}{[1+0]^{\frac{3}{2}}} = \frac{\partial^2 y}{\partial x^2}.$$

Finally, the equation becomes

$$\frac{1}{\rho} = \frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI}.$$

To solve for deflection first find an equation for the internal bending moment as a function of (x), using equilibrium just as in Section 7.3 for drawing the bending moment diagram. Then set this equation equal to $\mathrm{EI}\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}$ and integrate twice.

$$\boxed{\begin{gathered} M(x) = EI\frac{\partial^2 y}{\partial x^2} \\ \int M(x)dx + C_1 = EI\frac{\partial y}{\partial x} \\ \iint M(x)dx + C_1 x + C_2 = EIy \end{gathered}}$$

(11.1)

In these equations y represents the deflection of the beam and $\frac{\partial \mathrm{y}}{\partial \mathrm{x}}$ represents the slope of the deflected beam. Since the equations are functions of *x*, we can find the slope and deflection at any point along the beam.

Note that each successive integral introduces a constant of integration that we must solve for. We do this through the use of boundary conditions—that is, points on the beam where we already know the slope and/or deflection. There are two common boundary conditions for our applications:

- At any support the deflection (y) is zero.
- At a fixed support the slope $\left(\frac{\partial \mathrm{y}}{\partial \mathrm{x}}\right)$ is zero.

Figure 11.3 illustrates these boundary conditions. Once the constants of integration are known, we can define equations for the slope and deflection of the beam in terms of distance (x) along the beam and calculate the slope and deflection at any value of x. Example 11.1 and Example 11.2 show this process applied to a simply supported beam and a cantilever beam respectively.

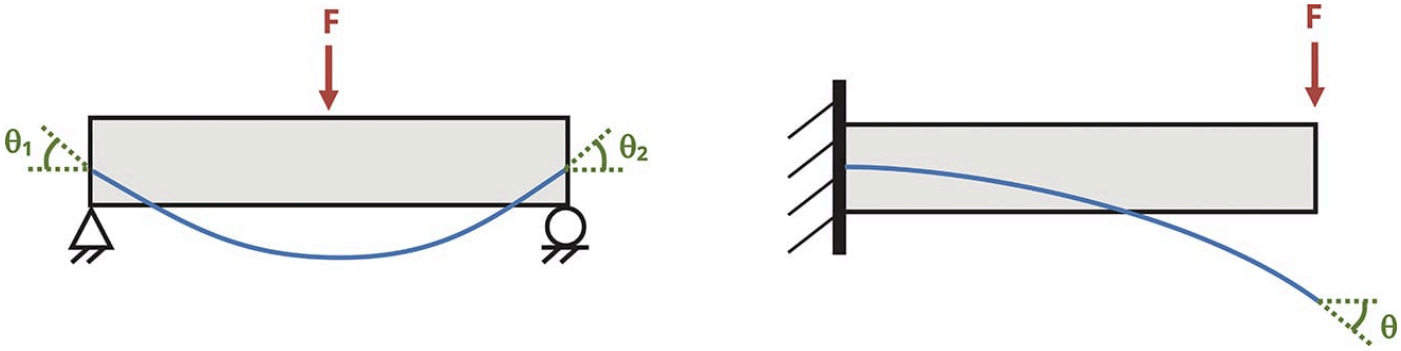
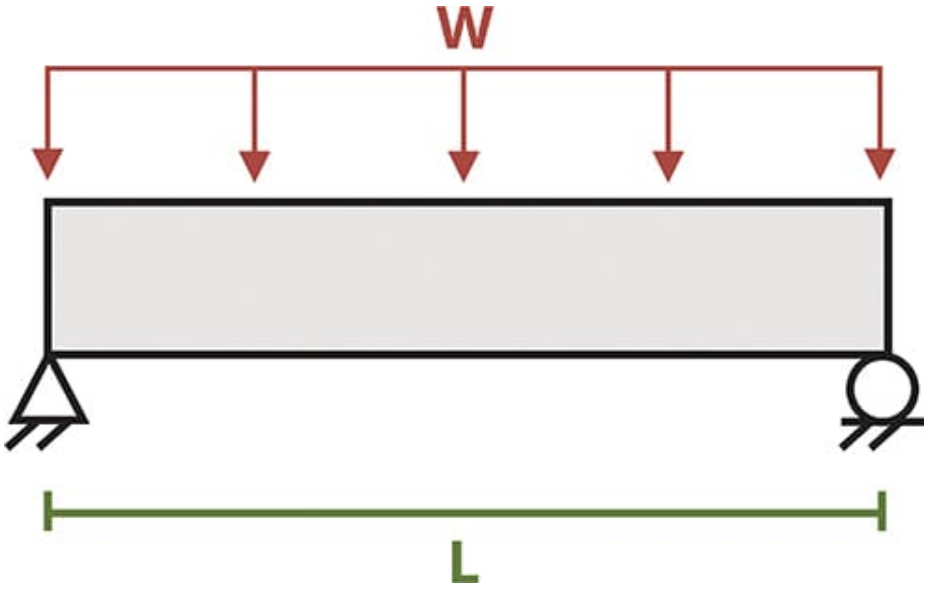


Figure 11.3: At pin and roller supports the deflection is zero but the slope is not. At a fixed support both the slope and deflection are zero. A statically determinate beam in equilibrium always has two boundary conditions.

Example 11.1

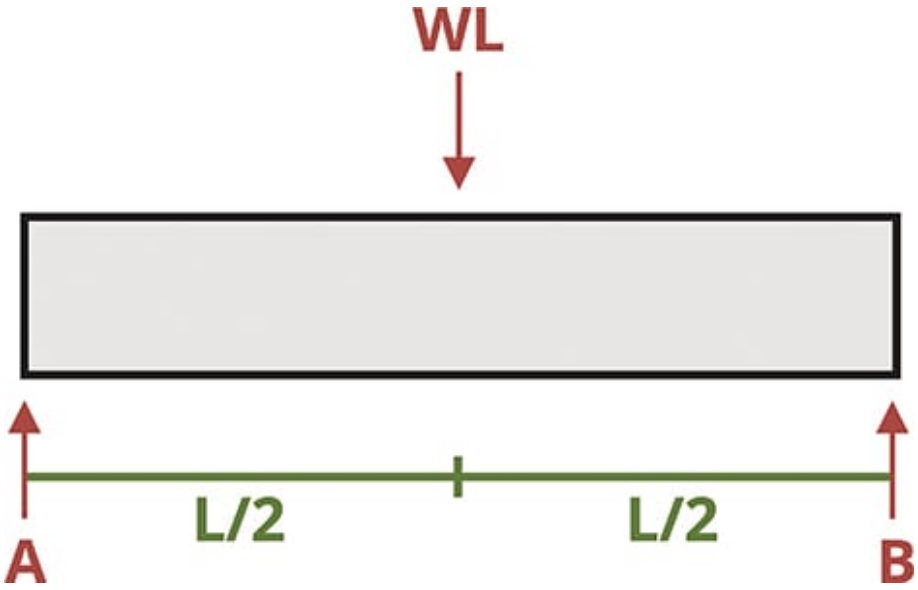
A simply supported beam of length L = 10 m is subjected to a uniform distributed load of w = 20 kN/m.

Determine the equation of the elastic curve and use this to find the deflection of the beam at the midpoint, x = 5 m. Assume E = 200 GPa and I = 350 × 10^6 mm^4.



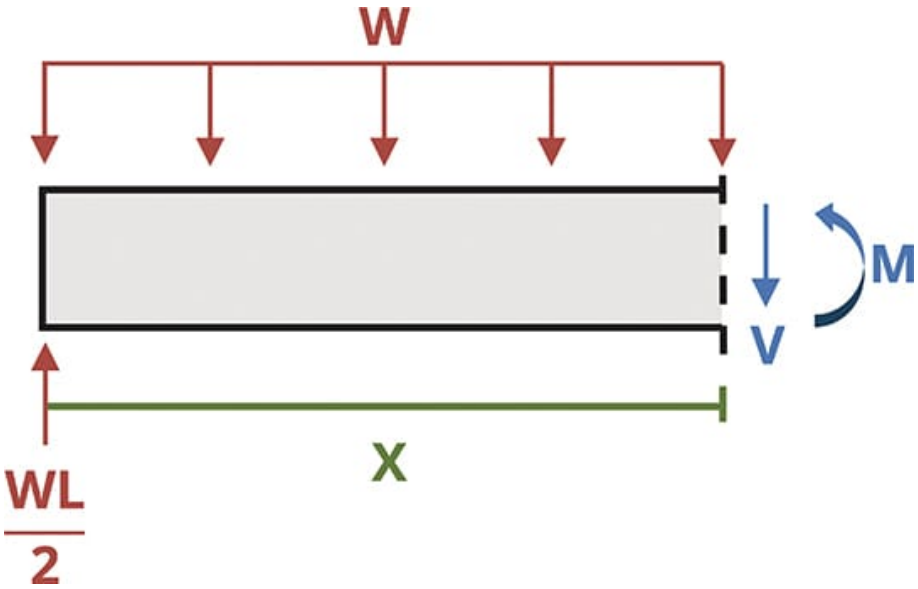
Solution

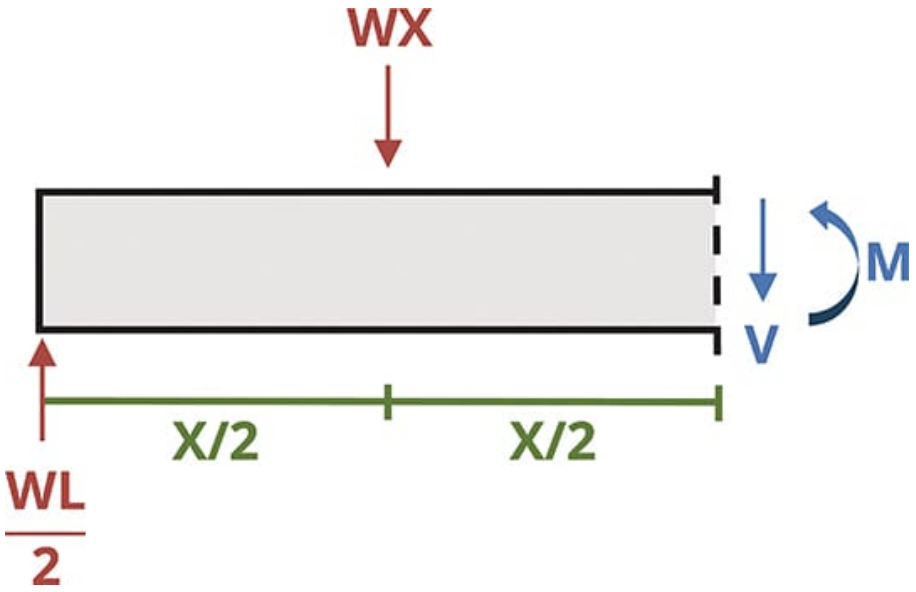
Sketch a free body diagram (FBD) of the beam and use equilibrium equations to solve for the reaction forces at the supports.



$$\sum M_A = -wL\left(\frac{L}{2}\right) + BL = 0 \;\rightarrow\; B = \frac{wL}{2}$$
$$\sum Fy = A - wL + \frac{wL}{2} = 0 \;\rightarrow\; A = \frac{wL}{2}$$

Cut a cross-section through the beam at distance x and draw an FBD of everything to the left of the cut, including the internal loads. Use equilibrium to determine an equation for the internal bending moment, M, as a function of x.





$$\sum M_x = M + wx\left(\tfrac{x}{2}\right) - \tfrac{wL}{2}(x) = 0$$
$$M = \tfrac{wLx}{2} - \tfrac{wx^2}{2} = EI\tfrac{\partial^2 y}{\partial x^2}$$

Integrate this equation twice.

$$\tfrac{wLx^2}{4} - \tfrac{wx^3}{6} + c_1 = EI\tfrac{\partial y}{\partial x}$$
$$\tfrac{wLx^3}{12} - \tfrac{wx^4}{24} + c_1 x + c_2 = EIy$$

Apply boundary conditions at the supports.

$$\text{At } x = 0, y = 0 \;\rightarrow\; C_2 = 0$$
$$\text{At } x = L, y = 0 \;\rightarrow\; \tfrac{wL(L)^3}{12} - \tfrac{w(L)^4}{24} + C_1 L = 0$$
$$C_1 = \tfrac{wL^3}{24} - \tfrac{wL^3}{12} = -\tfrac{wL^3}{24}$$

Substitute these constants into the deflection equation and rearrange.

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3}{24}x\right]$$

Substitute in the given values for and solve this equation at the midpoint (x = 5m).

$$y = \frac{1}{200 \times 10^9 \frac{N}{m^2} \times 350 \times 10^{-6} m^4}\left[\frac{20,000\,\frac{N}{m} \times 10m \times (5m)^3}{12} - \frac{20,000\,\frac{N}{m} \times (5m)^4}{24} - \frac{20,000\,\frac{N}{m} \times (10m)^3 \times 5m}{24}\right]$$

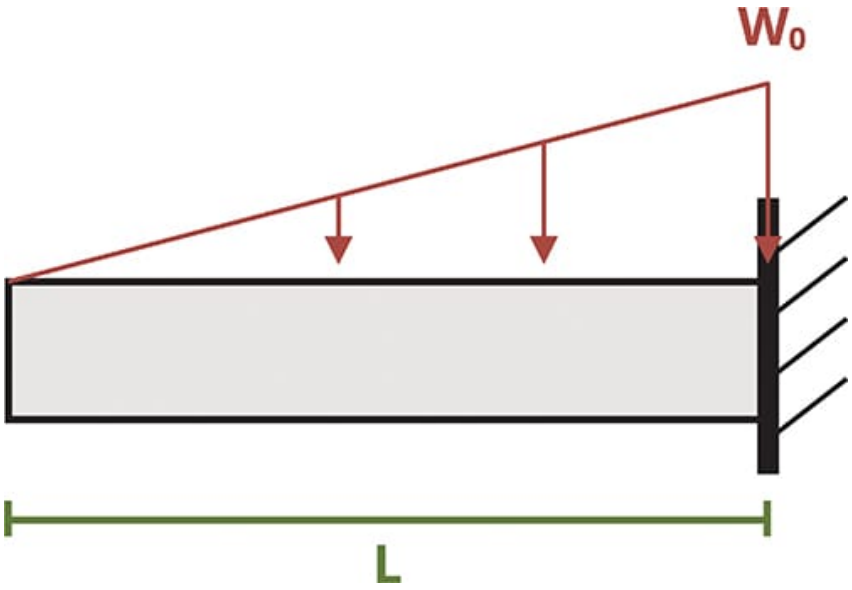
$$y = -0.0372\,m$$

$$y = -37.2\,mm$$

Example 11.2

A cantilever beam of length L = 8 ft is subjected to a linear distributed load where w_0 = 30 kip/ft.

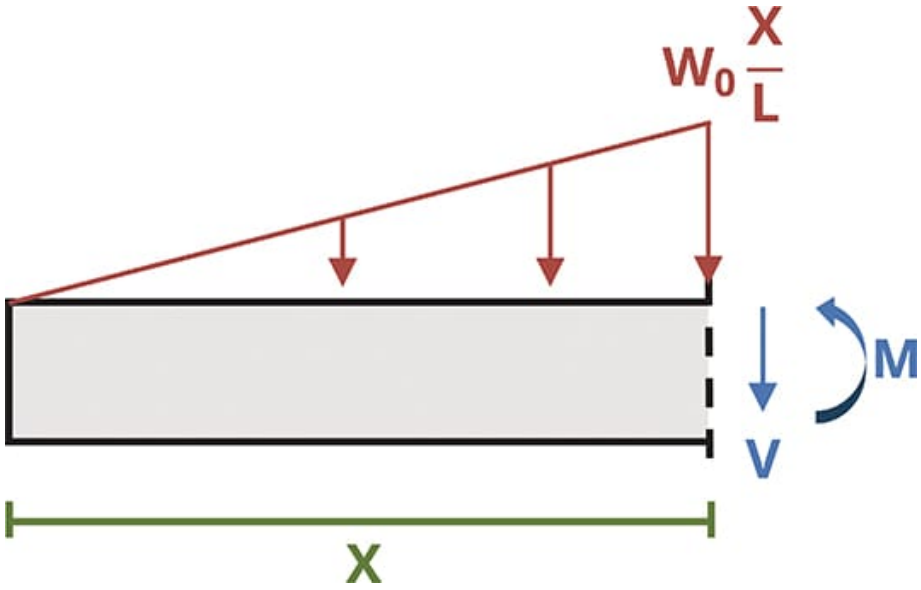
Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end. Assume E = 29 × 10^6 psi and I = 375 $in.^4$

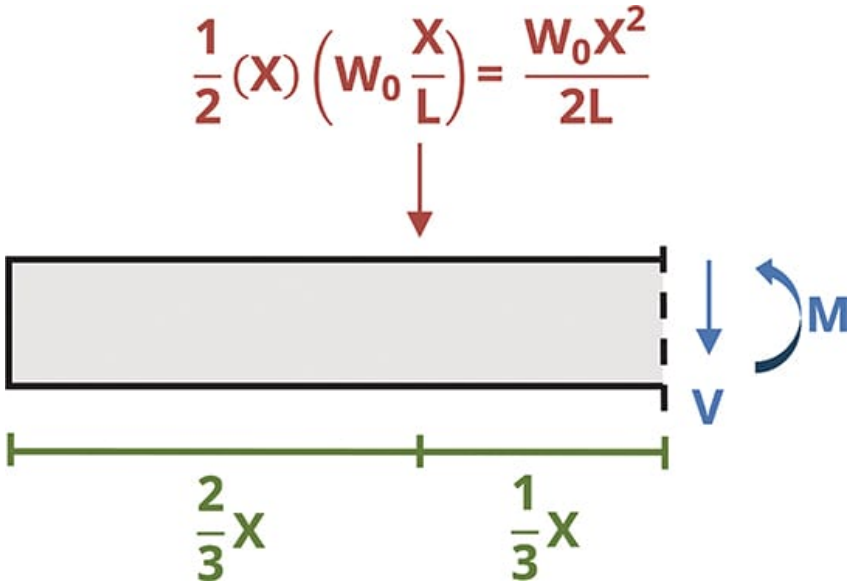


Solution

Cut a cross-section through the beam at distance x and draw an FBD of everything to the left of the cut, including the internal loads. This diagram omits the reactions at the fixed support, so there's no need to solve for these first.

Use equilibrium to determine an equation for the internal bending moment, M, as a function of x. Then integrate this equation twice.





$$\begin{aligned}
\sum M_x = \quad & M + \tfrac{w_0 x^2}{2L}\left(\tfrac{x}{3}\right) = 0 \\
M \quad & = -\tfrac{w_0 x^3}{6L} = EI\tfrac{\partial^2 y}{\partial x^2} \\
\int M\,dx \quad & = -\tfrac{w_0 x^4}{24L} + C_1 = EI\tfrac{\partial y}{\partial x} \\
\int\int M\,dx \quad & = -\tfrac{w_0 x^5}{120L} + C_1 x + C_2 = EIy
\end{aligned}$$

Apply boundary conditions at the fixed support.

$$\begin{aligned}
\text{At } x = L, \tfrac{\partial y}{\partial x} = 0 \quad \rightarrow \quad - \quad & \tfrac{w_0 L^4}{24L} + C_1 = 0 \\
C_1 \quad & = \tfrac{w_0 L^3}{24} \\
\text{At } x = L, y = 0 \quad \rightarrow \quad - \quad & \tfrac{w_0 L^5}{120L} + \tfrac{w_0 L^3}{24}(L) + C_2 = 0 \\
C_2 \quad & = \tfrac{w_0 L^4}{120} - \tfrac{w_0 L^4}{24} \\
& = -\tfrac{4 w_0 L^4}{120} \\
& = -\tfrac{w_0 L^4}{30}
\end{aligned}$$

Substitute these constants into the deflection equation and rearrange.

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length.

$$y = \frac{1}{29\times10^6\ \frac{lb}{in.^2} * 375\ in.^4}\left[-\frac{\frac{30{,}000\ \frac{lb}{ft}}{12\ \frac{in.}{ft}} * \left(8\ ft * 12\ \frac{in.}{ft}\right)^4}{30}\right]$$

$$y = -0.651\ in.$$

These examples have single continuous loads. If the loading is discontinuous, however, we must modify our approach because no single moment equation can describe the entire beam. Section 7.3 showed that when drawing shear force and bending moment diagrams we can find a different equation for the internal bending moment in each loading region (i.e., a piecewise function over the length of the beam). This requires finding separate moment equations for each loading region and integrating each separately. Do this each time the loading changes, considering

- Both sides of a concentrated load

- A distributed load that begins or ends

This text covers deflection of beams made of a single material with uniform cross-sections (i.e., where elastic modulus, E, and second moment of area, I, remain constant). In practice, changes in E and I necessitate additional separate equations for each change.

Figure 11.4 shows a beam with two loading regions. Fully describing the deflection in this beam requires us to determine two internal bending moment equations—one in segment AB and one in segment BC. Each equation must be integrated twice, thereby introducing four constants of integration. Although two of these can be determined by using boundary conditions, we must determine the remaining two by using continuity conditions.

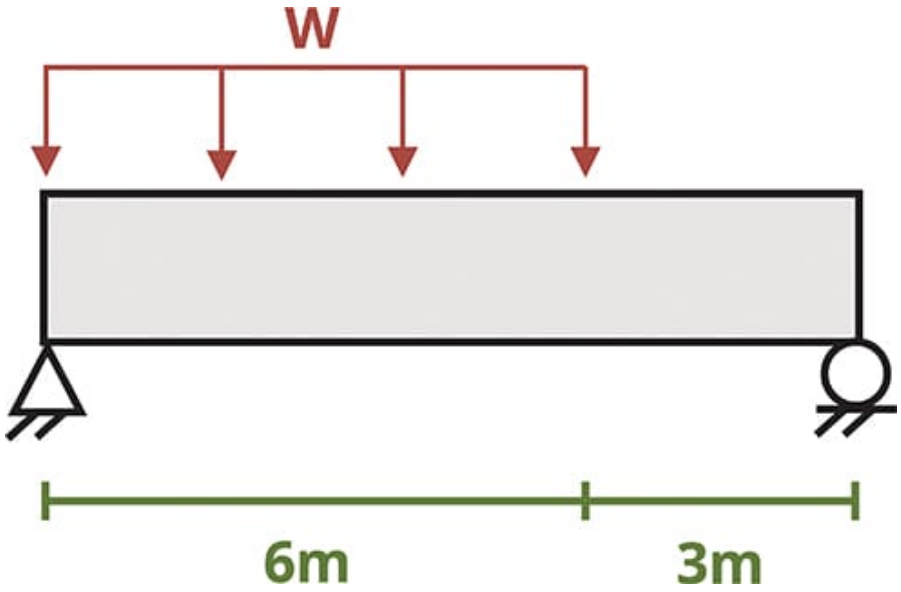


Figure 11.4: The loading on the simply supported beam is different in segment AB than in segment BC. Each segment has its own bending moment equation that must be found and integrated.

Equation set 1 describes the internal moment, slope, and deflection of the beam from point A to point B, and equation set 2 describes the internal moment, slope, and deflection from point B to point C. Note that both sets of equations describe point B, where the loading changes. They must therefore return the same results at point B. So we can say that

$$\left(\frac{\partial y}{\partial x}\right)_1 = \left(\frac{\partial y}{\partial x}\right)_2$$
$$y_1 = y_2.$$

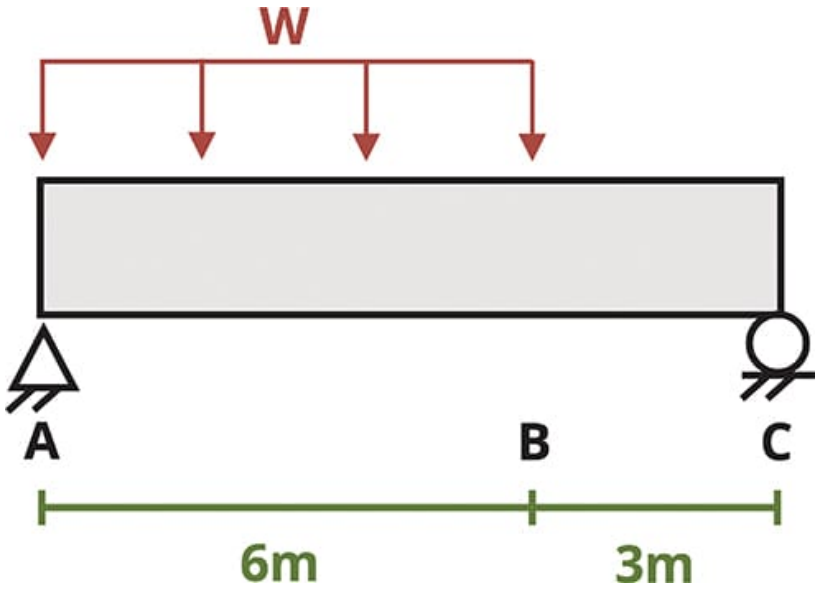
See Example 11.3 for the full solution for this beam.

Example 11.3

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown.

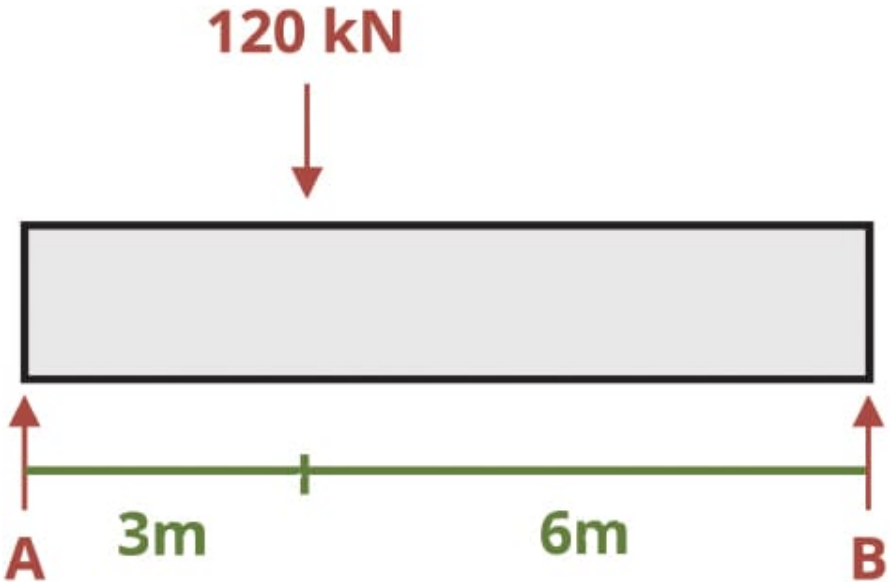
Determine the equations of the elastic curve between 0 ≤ x ≤ 6 m and between 6 m ≤ x ≤ 9 m.

Then determine the deflection at x = 5 m. Assume E = 200 GPa and I = 394 × 10^6 mm^4.



Solution

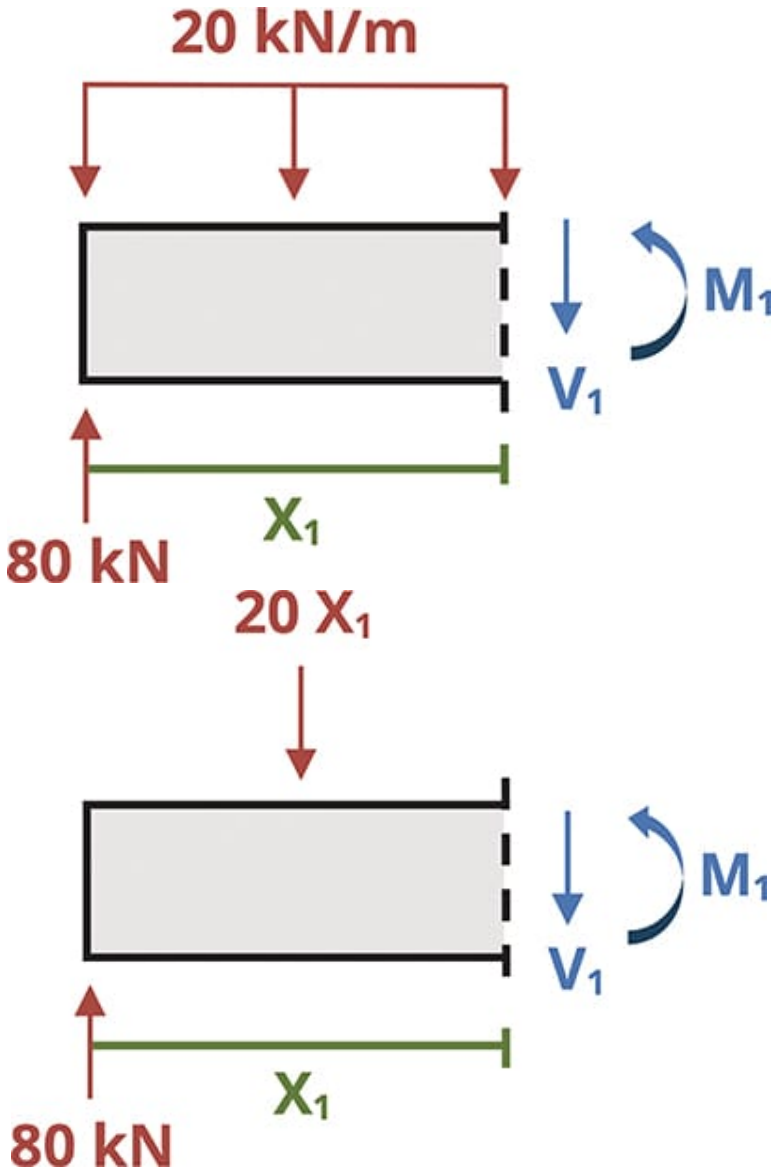
Sketch an FBD of the beam and use equilibrium equations to solve for the reaction forces at the supports.



$$\sum M_A = (120\ kN * 3\ m) + (B * 9\ m) = 0\ \rightarrow\ B = 40\ kN$$
$$\sum Fy = -120\ kN + 40\ kN = 0\ \rightarrow\ A = 80\ kN$$

Cut a cross-section in the first loading region at distance x_1. Draw an FBD of everything to the left of this cut, including the internal loads.

Use equilibrium to determine an equation for the internal bending moment, M_1, as a function of x_1. Then integrate this equation twice.



$$\sum M_x = M_1 + 20x_1\left(\frac{x_1}{2}\right) - 80x_1 = 0$$

$$M_1 = 80x_1 - 10x_1^2 = EI\frac{\partial^2 y_1}{\partial x_1^2}$$

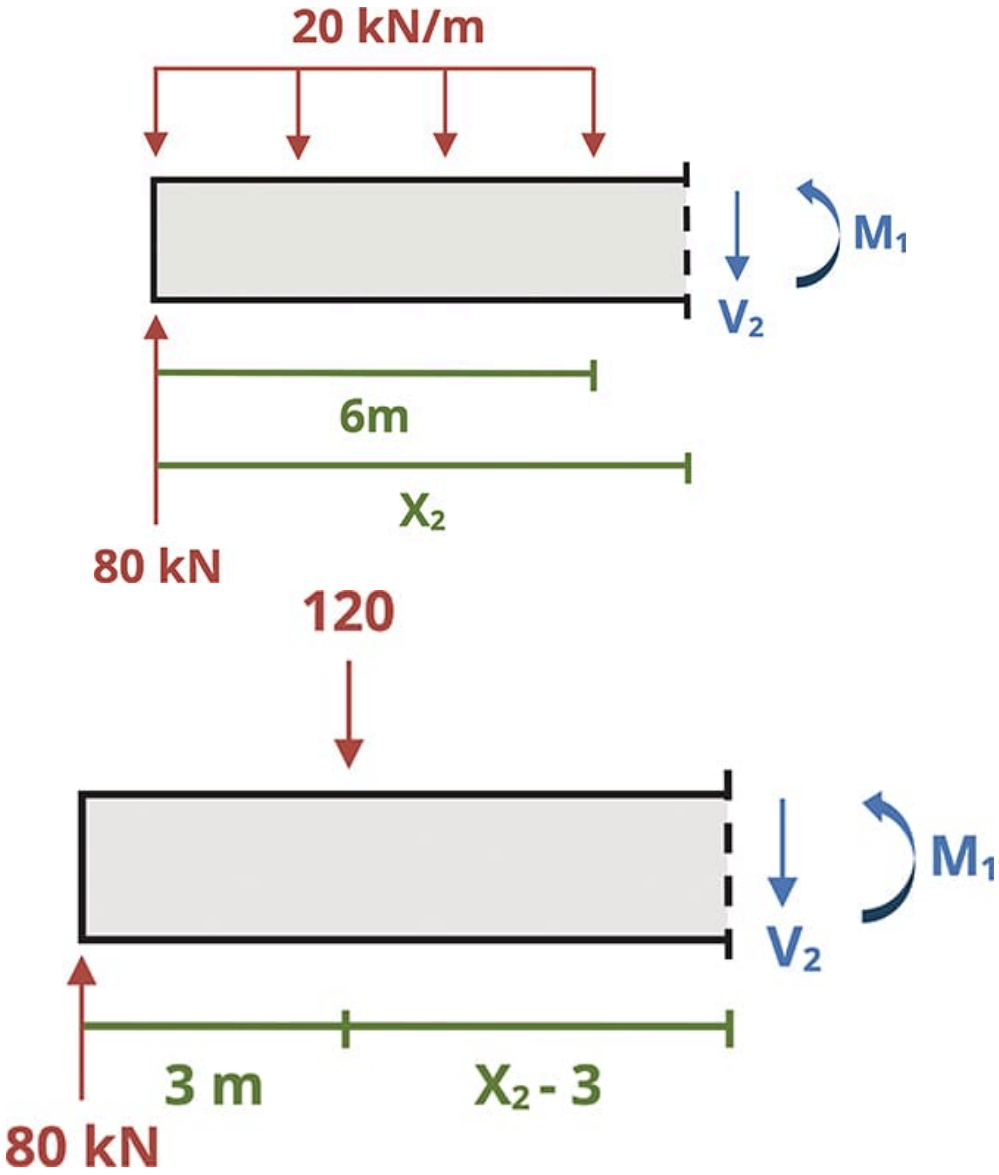
$$\int M_1\,dx = \frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$

$$\int\int M_1\,dx = \frac{80x_1^3}{6} - \frac{10x_1^4}{12} + c_1x_1 + c_2 = EIy_1$$

These equations are valid from 0 ≤ x_1 ≤ 6 m.

Set these equations aside and cut a cross-section in the second loading region at distance x_2. Draw an FBD of everything to the left of this cut, including the internal loads.

Use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice.



$$\begin{aligned}
\sum M_x = \quad & M_2 + 120(x_2 - 3) - 80x_2 = 0 \\
M_2 \quad & = 80x_2 - 120x_2 + 360 \\
M_2 \quad & = 360 - 40x_2 = EI\tfrac{\partial^2 y_2}{\partial x_2^2} \\
\int M_2\, dx \quad & = 360x_2 - \tfrac{40x_2^2}{2} + c_3 = EI\tfrac{\partial y_2}{\partial x_2} \\
\int\int M_2\, dx \quad & = \tfrac{360x_2^2}{2} - \tfrac{40x_2^3}{6} + c_3x_3 + c_4 = EIy_2
\end{aligned}$$

These equations are valid from 6 m ≤ x_2 ≤ 9 m.

Apply bounding conditions at the supports. One support is in loading region 1, and the other is in region 2. Be careful to use the appropriate deflection equation when applying the boundary conditions.

$$\begin{aligned}
\text{At } x_1 = 0, y_1 = 0 \quad & \rightarrow \quad c_2 = 0 \\
\text{At } x_2 = 9, y_2 = 0 \quad & \rightarrow \quad \tfrac{360(9)^2}{2} - \tfrac{40(4)^3}{6} + c_3(9) + C_4 = 0 \\
& \qquad\quad c_4 = -9c_3 - 9,720
\end{aligned}$$

Apply continuity conditions at the point where the loading changes (x_1 = x_2 = 6 m).

$$\text{At } x_1 = x_2 = 6 \text{ m}, \frac{\partial y_1}{\partial x_1} = \frac{\partial y_2}{\partial x_2}$$
$$\frac{80(6)^2}{2} - \frac{10(6)^3}{3} + c_1 = 360(6) - \frac{40(6)^2}{2} + c_3$$
$$1,440 - 720 + c_1 = 2,160 - 720 + c_3$$
$$C_1 = 720 + C_3$$

$$\text{At } x_1 = x_2 = 6 \text{ m}, y_1 = y_2$$
$$\frac{80(6)^3}{6} - \frac{10(6)^4}{12} + c_1(6) = \frac{360(6)^2}{2} - \frac{40(6)^3}{6} + c_3(6) + C_4$$
$$2,880 - 1,080 + 6C_1 = 6,480 - 1,440 + 6C_3 + C_4$$
$$1,800 + 6C_1 = 5,040 + 6C_3 + C_4$$

Substitute in $C_4 = -9C_3 - 9{,}720$ and $C_1 = 720 + C_3$.

$$1,800 + 6(720 + c_3) = 5,040 + 6c_3 + c_4$$
$$1,800 + 4,320 + 6c_3 = 5,040 + 6c_3 + c_4$$
$$6,120 - 5,040 = c_4$$
$$c_4 = 1,080$$

$$c_4 = -9c_3 - 4,720$$
$$c_3 = \frac{-1,080 - 9,720}{9}$$
$$c_3 = -1,200$$

$$c_1 = 720 + c_3 = 720 - 1,200$$
$$c_1 = -480$$

Substitute constants into deflection equations.

$$y_1 = \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right]$$
$$y_2 = \frac{1}{EI}\left[\frac{360x_2^2}{2} - \frac{40x_2^3}{6} + 1,200x_3 + 1,080\right]$$

Deflection at midpoint occurs when $x_1 = 5$ m.

$$y_1 = \frac{1}{EI}\left[\frac{80\,kN \times (5m)^3}{6} - \frac{10\frac{kN}{m} \times (5m)^4}{12} - 480\,kN \cdot m^2 \times (5m)\right]$$
$$= -\frac{1,254 \times 10^3 N \cdot m^3}{200 \times 10^9 \frac{N}{m^2} \times 394 \times 10^{-6} m^4}$$
$$y_1 = -0.0159\,m$$
$$y_1 = -15.9\,mm$$

Note that we are not required to make both cuts from the left to find the internal bending moment. In some cases making one cut from the left and the other from the right may be the easier approach. Consider the beam in Figure 11.5. Given a cut in the region not under the distributed load, an FBD of the left side of the cut shows many more forces than an FBD of the right side.

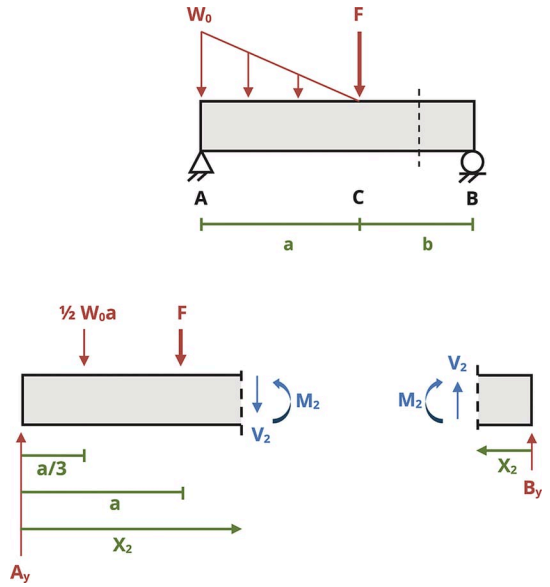


Figure 11.5: Simple beam showing two options for finding the internal bending moment. Note that the moment equation will be much simpler if we draw the right-hand side of the cut.

The moment equation is less complex if we draw the right-hand side of the cut rather than the left-hand side. This in turn simplifies the integrations and the math involved in finding the constants of integration.

If, however, we do measure distance x_2 from the right, we must make one change to the continuity conditions. Recall that when both cuts are from the left, at point C the slope and deflection calculated from equation 1 must be the same as those calculated from equation 2.

$$\left(\frac{\partial y}{\partial x}\right)_1 = \left(\frac{\partial y}{\partial x}\right)_2$$
$$y_1 = y_2$$

When distance x_2 is measured from the right, $y_1 = y_2$ at point C because the deflection is downward whether we measure the horizontal distance from the left or from the right. The magnitude of the slope will also be the same at point C regardless of which equation we use to calculate it. However, the direction of rotation of the slope will change from counterclockwise to clockwise because we have changed the coordinate system (Figure 11.6). Thus our second boundary condition must become

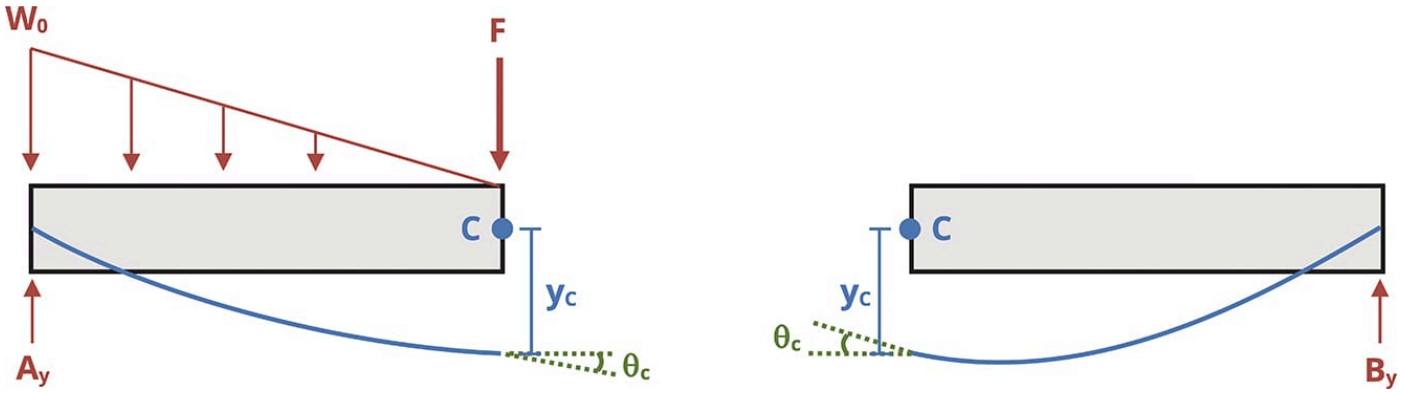


Figure 11.6: Depending on which side of the cut we draw, the deflection at point C will have the same magnitude and direction. The slope at point C will have the same magnitude but opposite directions if one cut is from the left and the other from the right.

$$\left(\frac{\partial y}{\partial x}\right)_1 = -\left(\frac{\partial y}{\partial x}\right)_2.$$

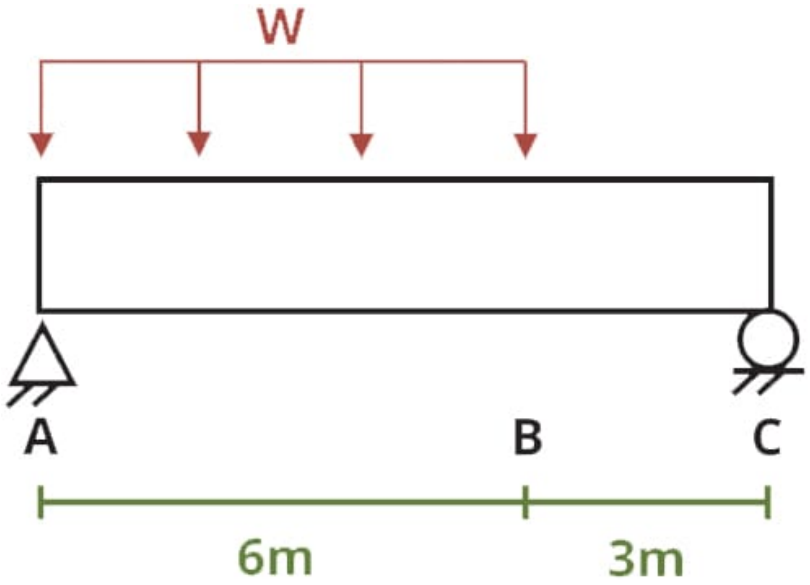
This negative sign is vitally important, and we will not obtain the correct answer if we forget to include it. Example 11.3 has been reworked, this time with one cut made from the right-hand side. Compare this with the previous solution to see the differences.

Example 11.3 (Reworked)

A simply supported beam is subjected to a distributed load of w = 20 kN/m as shown.

Determine the equation of the elastic curve between 0 ≤ x ≤ 6 m and between 6 m ≤ x ≤ 9 m.

Then determine the deflection at x = 5 m. Assume E = 200 GPa and I = 394 × 10^6 mm^4.

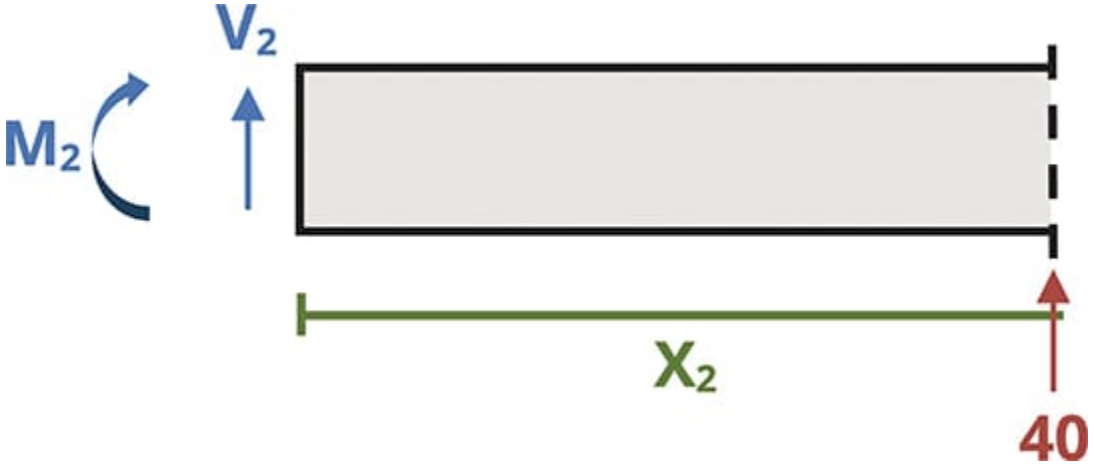


Solution

Proceed through the first cut as before and find

$$\frac{80x_1^2}{2} - \frac{10x_1^3}{3} + c_1 = EI\frac{\partial y_1}{\partial x_1}$$
$$\frac{80x_1^3}{6} - \frac{10x_1^4}{12} + c_1x_1 + c_2 = -EIy_1.$$

Make a second cut from the right hand side. Draw an FBD and use equilibrium to determine an equation for the internal bending moment, M_2, as a function of x_2. Then integrate this equation twice.



$$\begin{aligned}\sum M_x = &\quad 40x_2 - M_2 = 0\\ M_2 &= 40x_2 = EI\frac{\partial^2 y_2}{\partial x_2^2}\\ \int M_2\,dx &= \frac{40x_2^2}{2} + c_3 = EI\frac{\partial y_2}{\partial x_2}\\ \int\int M_2\,dx &= \frac{40x_2^3}{6} + c_3x_2 + c_4 = EIy_2\end{aligned}$$

Apply boundary conditions as before, although this time the roller support is at $x_2 = 0$ since we're measuring x_2 from the right side.

$$\begin{aligned}&\text{At } x_1 = 0, y_1 = 0 \;\rightarrow\; c_2 = 0 \text{ (as before)}\\ &\text{At } x_2 = 0, y_2 = 0 \;\rightarrow\; c_4 = 0\end{aligned}$$

Apply continuity conditions at the point where the loading changes (x_1 = 6 m, x_2 = 3 m).

$$\begin{aligned}&\text{At } x_1 = 6\,m, x_2 = 3\,m, y_1 = y_2\\ &\frac{80(6)^3}{6} - \frac{10(6)^4}{12} + c_1(6) = \frac{40(3)^3}{6} + c_3(3)\\ &2,880 - 1,080 + 6c_1 = 180 + 3c_3\\ &540 + 2c_1 = c_3\\ &\text{At } x_1 = 6\,m, x_2 = 3\,m, \frac{dy_1}{dx_1} = -\frac{dy_2}{dx_2}\\ &\frac{80(6)^2}{2} - \frac{10(6)^3}{3} + c_1 = -\frac{40(3)^2}{2} - c_3\\ &1,440 - 720 + c_1 = -180 - c_3\\ &900 + c_1 = -c_3\end{aligned}$$

Substitute in $540 + 2c_1 = c_3$.

$$\begin{aligned}540 + 2c_1 &= -900 - c_1\\ 3c_1 &= -1440\\ c_1 &= -480 \text{ (as before)}\\ 900 - 480 &= -c_3\\ c_3 &= -420\end{aligned}$$

Substitute constants into deflection equations.

$$\begin{aligned}y_1 &= \frac{1}{EI}\left[\frac{80x_1^3}{6} - \frac{10x_1^4}{12} - 480x_1\right] \quad \text{(as before)}\\ y_2 &= \frac{1}{EI}\left[\frac{40x_2^3}{6} - 420x_2\right]\end{aligned}$$

Deflection at midpoint occurs when x_1 = 5 m.

$$\begin{aligned}&y_1 = \frac{1}{EI}\left[\frac{80\,kN\times(5m)^3}{6} - \frac{10\frac{kN}{m}\times(5m)^4}{12} - 480\,kN\cdot m^2\times(5m)\right]\\ &= -\frac{1{,}254\times10^3 N\cdot m^3}{200\times10^9\frac{N}{m^2}\times394\times10^{-6}m^4}\\ &y_1 = -0.0159\,m\\ &y_1 = -15.9\,mm\end{aligned}$$

Step-by-Step: Deflection by Integration of Moment Equation

1. Solve for reaction loads using equilibrium.
2. Cut a cross-section through the beam and find the internal bending moment as a function of x. If the external loading changes, repeat this in each segment of the beam.
3. Set the internal bending moment $\mathrm{M(x)} = \mathrm{EI}\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ and integrate twice.
4. Use boundary conditions to determine two constants of integration.

1. At a fixed support slope $\left(\frac{\partial y}{\partial x}\right) = 0$.
2. At any support deflection (y) = 0.

5. In cases with more than one region of load, apply continuity conditions to determine additional constants of integration. At the point where the loading changes find
   1. $\left(\frac{\partial y_1}{\partial x_1}\right) = \left(\frac{\partial y_2}{\partial x_2}\right)$
   2. $y_1 = y_2$
6. Determine the slope and deflection at any value of x as required.

11.2 Integration of the Load Equation

In the previous section we used equilibrium to determine the equation of the internal bending moment and integrated this equation twice. This approach works well for relatively simple loading such as concentrated loads and uniformly or linearly distributed loads. For more complex loads, such as that shown in Figure 11.7, determining the internal bending moment equation in this way can be difficult. However, it's possible to avoid the need to do so.

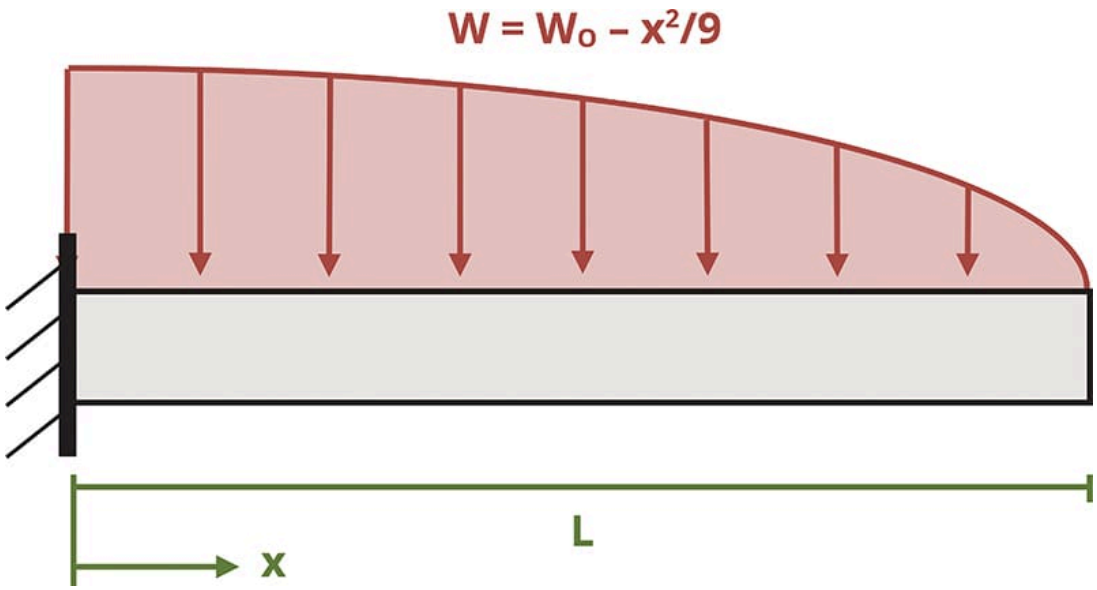


Figure 11.7: Complex external loads make it difficult to use equilibrium to determine the internal bending moment equation.

In Section 11.1 we used equilibrium equations to determine the internal bending moment in a beam as a function of x. We related this bending moment to the deflection by

$$M(x) = EI\frac{\partial^2 y}{\partial x^2}.$$

In Section 7.2 we learned that there is a relationship between the internal bending moment, the internal shear force, and the external load. For an external distributed load, w, we found

$$V = \int_0^L w(x)dx$$

$$M = \int_0^L V(x)dx.$$

We may therefore relate the deflection of a beam not only the internal bending moment but also to the internal shear force and external distributed load through successive integrations. For some loading configurations it can be simpler to integrate the distributed load equation four times instead of finding the equation of the internal bending moment through equilibrium and integrating twice. When taking the simpler approach, remember that loads acting downward (as most loads on beams do) are negative.

$$\begin{aligned} w(x) &= EI\frac{\partial^4 y}{\partial x^4} \\ V(x) &= EI\frac{\partial^3 y}{\partial x^3} = \int w(x)dx + C_1 \\ M(x) &= EI\frac{\partial^2 y}{\partial x^2} = \int V(x)dx + C_2 \\ EI\theta(x) &= EI\frac{\partial y}{\partial x} = \int M(x)dx + C_3 \\ EIy(x) &= \int \theta(x)dx + C_4 \end{aligned}$$

(11.2)

Note that each successive integral will introduce a new constant of integration. To solve for these constants we must again apply boundary conditions. Now with four constants four boundary conditions are required. Two of these conditions are the same as those in Section 11.1:

- At any support the deflection is 0.
- At a fixed support both slope and deflection are zero.

The other two boundary conditions come from knowing the value of the shear force and bending moment at a point in the beam. Our work in Chapter 7 enables us to find them. Although we can use known values at any point in the beam, it is generally easiest to use the shear force and bending moment at x = 0. Note the two conditions at this point:

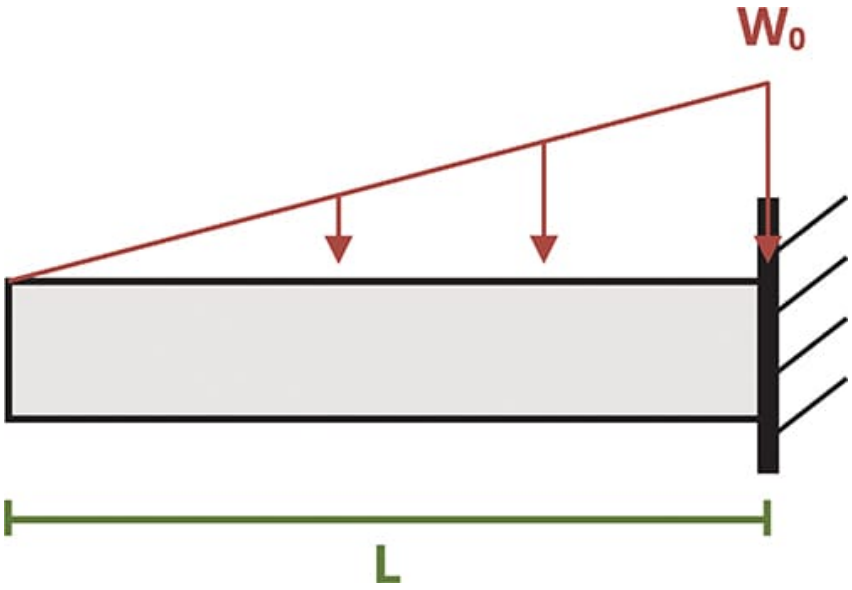
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

Example 11.4 demonstrates how to find the deflection of a beam by integrating the load equation. Although this method can be used for any beam, it is best applied to beams with a single continuous distributed load acting over the length of the beam. Beams with discontinuous loading require integration of multiple load equations.

Example 11.4

The cantilever beam from Example 11.2 is repeated here. Length L = 8 ft is subjected to a linear distributed load where w_0 = 30 kip/ft.

Determine the equation of the elastic curve and use this to find the deflection of the beam at the free end.



Solution

First define the equation of the load, which increases linearly from zero to w_0.

$$\omega = -\omega_0 \frac{x}{L} = EI\frac{\partial^4 y}{\partial x^4}$$

Integrate this equation four times.

$$\begin{aligned}
-\tfrac{w_0 x^2}{2L} + c_1 &= EI\tfrac{\partial^3 y}{\partial x^3} \\
-\tfrac{w_0 x^3}{6L} + c_1 x + c_2 &= EI\tfrac{\partial^3 y}{\partial x^2} \\
-\tfrac{w_0 x^4}{24L} + \tfrac{c_1 x^2}{2} + c_2 x + c_3 &= EI\tfrac{\partial y}{\partial x} \\
-\tfrac{w_0 x^5}{120L} + \tfrac{c_1 x^3}{6} + \tfrac{c_2 x^2}{2} + c_3 x + c_4 &= EIy
\end{aligned}$$

Apply boundary conditions at the fixed support. At the free end the internal shear force and bending moment will be zero. At the fixed support (at x = L) the slope and deflection will be zero.

$$\begin{aligned}
&\text{At } x = 0, v = 0 \quad \rightarrow \quad c_1 = 0 \\
&\text{At } x = 0, M = 0 \quad \rightarrow \quad c_2 = 0 \\
&\text{At } x = L, \tfrac{dy}{dx} = 0 \quad \rightarrow \quad c_3 = \tfrac{w_0 L^3}{24} \\
&\text{At } x = L, y = 0 \quad \rightarrow \quad c_4 = -\tfrac{w_0 L^4}{30}
\end{aligned}$$

This is the same result found in Example 11.2, and the problem proceeds in the same way to find the deflection at the free end.

Substitute these constants into the deflection equation and rearrange.

$$y = \frac{1}{EI}\left[-\frac{w_0 x^5}{120L} + \frac{w_0 L^3}{24}x - \frac{w_0 L^4}{30}\right]$$

Substitute in the given values and solve this equation at the free end (x = 0). Remember to convert feet to inches in both the distributed load and the length.

$$y = \frac{1}{29\times10^6 \frac{lb}{in.^2} * 375\ in.^4}\left[-\frac{\frac{30{,}000\ \frac{lb}{ft}}{12\ \frac{in.}{ft}} * \left(8\ ft \times 12\ \frac{in.}{ft}\right)^4}{30}\right]$$

$$y = -0.651\ in.$$

Step-by-Step: Deflection by Integration of the Load Equation

1. Define an equation that describes the external distributed load as a function of x. Remember that loads acting downward are negative.
2. Integrate this equation four times to determine equations for internal shear force, internal bending moment, slope, and deflection respectively.
3. Apply boundary conditions to determine the four constants of integration:
    1. At x = 0 the shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
    2. At x = 0 the bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.
    3. At a fixed support slope $\left(\frac{\partial y}{\partial x}\right) = 0.$
    4. At any support deflection(y) = 0.
4. Determine the slope and deflection at any value of x as required.

11.3 Superposition

It should be apparent from the previous sections that finding deflection using the method of integration can become quite complicated and time-consuming if there are multiple loads. In such cases, the method of superposition is a good alternative.

Beams are typically loaded in a relatively small number of standard configurations, and the deflection behavior of a beam subjected to one of these standard loads is very well understood. The slope and deflection equations for many common loading conditions have been derived through the methods described in Section 11.1 and Section 11.2 and recorded. A subset of these beam deflections has been included in Appendix B. Figure 11.8 shows an example of the information provided there for one such beam.

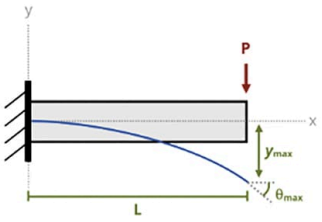
Beam	Slope	Max. Deflection	Elastic Curve
	$\theta_{max} = -\frac{PL^2}{2EI}$	$y_{max} = -\frac{PL^3}{3EI}$	$y = -\frac{Px^2}{6EI}(3L - x)$

Figure 11.8: Appendix B contains some standard loading configurations along with a sketch of the elastic curve, equations for the maximum deflection and maximum slope, and the general equation of the elastic curve.

This information can be useful even for more complex loading. When multiple loads act on a beam, the effect of each load on the deflection of the beam may be considered independently. Thus even for loading more complex than the loads in Appendix B, it may be possible to simplify the problem by considering each load separately and calculating the deflection caused at a point by each load (Figure 11.9). These deflections can then be summed to find the overall deflection at that point. We must, of course, calculate the deflection at the same point for each load in order to add them together. See Example 11.5 for a demonstration.

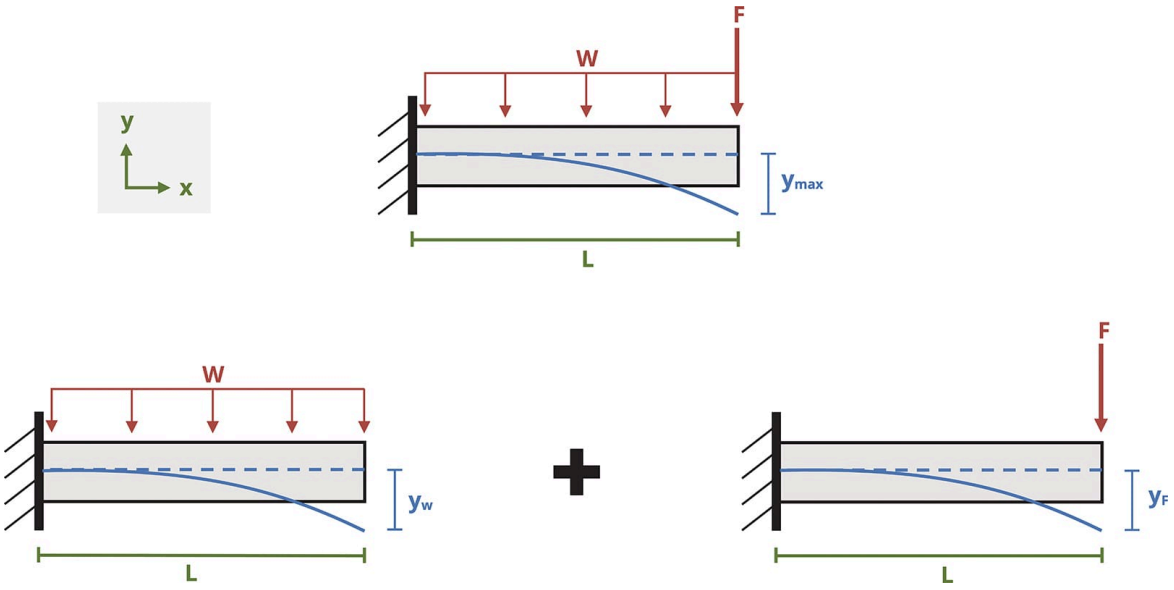


Figure 11.9: Although Appendix B doesn't contain a cantilever beam with both a distributed load and a concentrated load, it does contain cantilever beams with each load separately. We may find the combined deflection by finding the deflection of each load separately and adding them together.

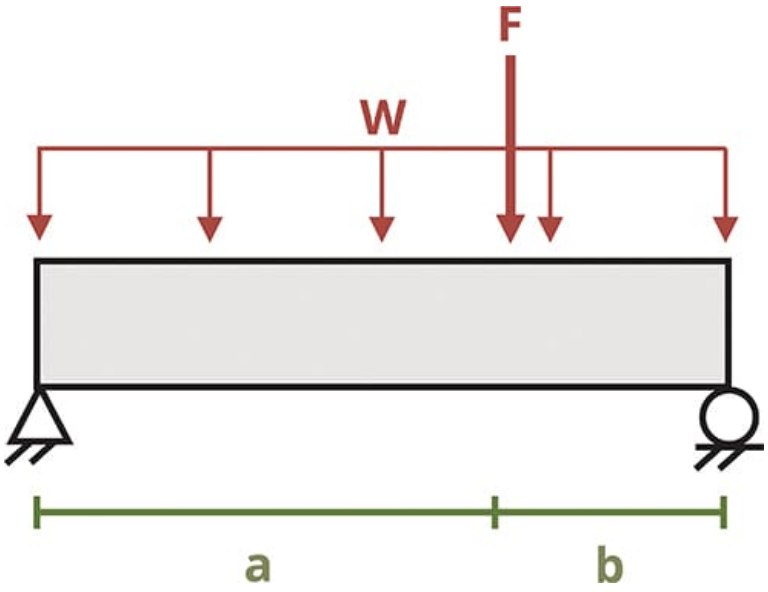
$$y_w = -\frac{wL^4}{8EI} \qquad y_F = -\frac{FL^3}{3EI}$$

$$y_{max} = -\frac{wL^4}{8EI} - \frac{FL^3}{3EI}$$

Example 11.5

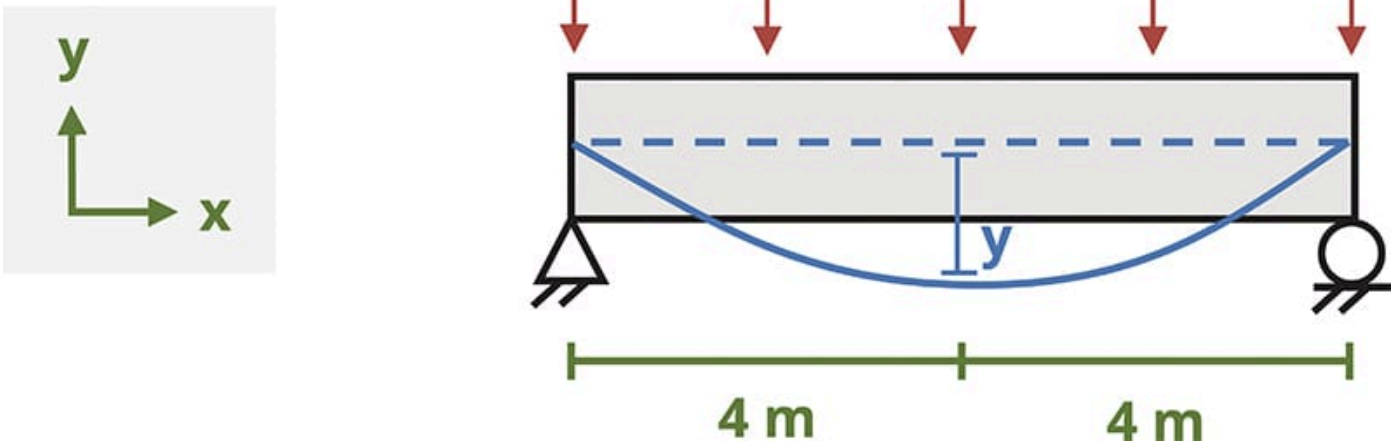
A simply supported beam is subjected to a distributed load w = 20 kN/m and a concentrated load F = 50 kN.

If distance a = 6 m and b = 2 m, determine the deflection at the midpoint of the beam (x = 4 m). Assume E = 210 GPa and I = 275 × 10^6 mm^4.



Solution

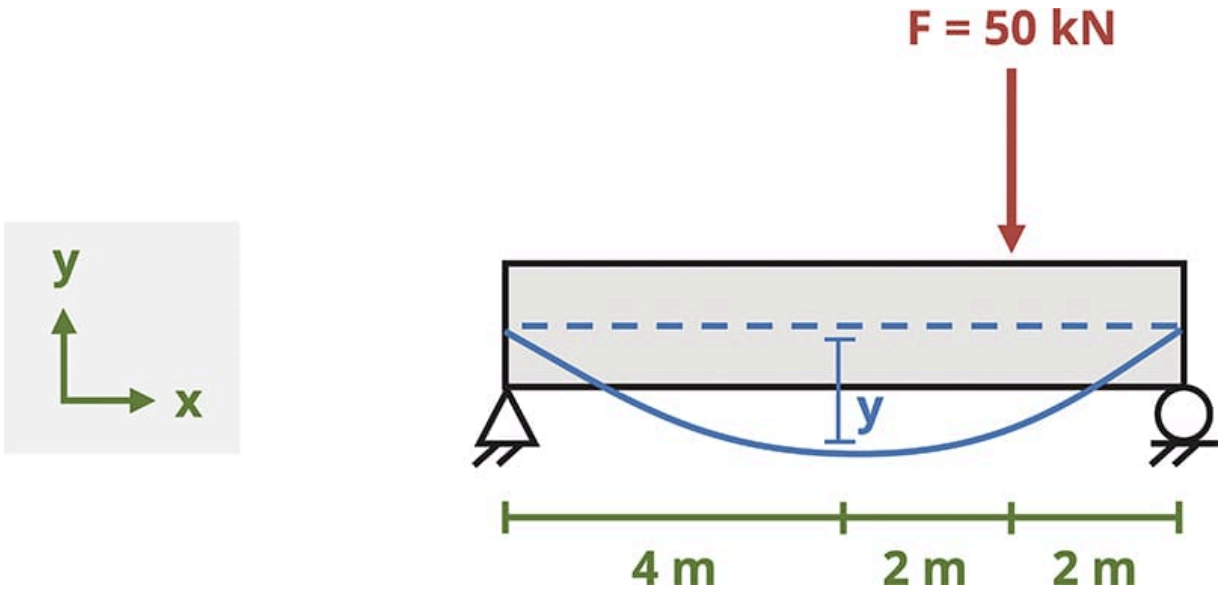
Consider each load separately. We'll start with the distributed load (although this step can be completed in any order).



Appendix B shows that the maximum deflection for this load occurs at the center of the beam, which is the point of interest.

$$\begin{aligned} y &= -\frac{5wL^4}{384EI} \\ &= -\frac{5*20{,}000\ \frac{N}{m}*(8\ m)^4}{384*210\times10^9\ \frac{N}{m^2}\times275\times10^{-6}\ m} \\ &= -0.01847\ m \\ &= -18.47\ mm \end{aligned}$$

Now consider the concentrated load.



Here the maximum deflection does not occur at the center of the beam. Use the elastic curve equation for x < a to find the deflection at x = 4 m when the load is applied at a = 6 m.

$$\begin{aligned} y &= -\frac{Fbx}{6EIL}\left[L^2-b^2-x^2\right] \\ &= -\frac{50{,}000\ N*2\ m*4\ m}{6*210\times10^9\ \frac{N}{m^2}*275\times10^{-6}\ m^4*8\ m}\left[(8\ m)^2-(2\ m)^2-(4\ m)^2\right] \\ &= -0.00635\ m \\ &= -6.35\ mm \end{aligned}$$

The total deflection at the beam's midpoint is

$$y = -18.47\ mm - 6.35\ mm$$
$$y = -24.8\ mm.$$

Step-by-Step: Deflection by Superposition

1. Sketch the beam with just one load acting on it.
2. Use Appendix B to find a beam with the same supports and loading.
3. Use the relevant equation in Appendix B to find the deflection at the point of interest.
4. Repeat this process until all loads have been considered. Add the individual deflections at the point of interest to find the overall deflection at this point.

11.4 Statically Indeterminate Deflection

A problem is statically indeterminate if there are more unknown support reactions than there are equilibrium equations to solve for them. These problems were introduced in Section 5.5 and Section 6.3. In both sections, to solve statically indeterminate problems we took advantage of our knowledge of deformation to solve for one of the reaction forces before using equilibrium equations to find the others. We'll do something similar here. We may use either an integration method like that used in Section 11.1 or a superposition method like that used in Section 11.3.

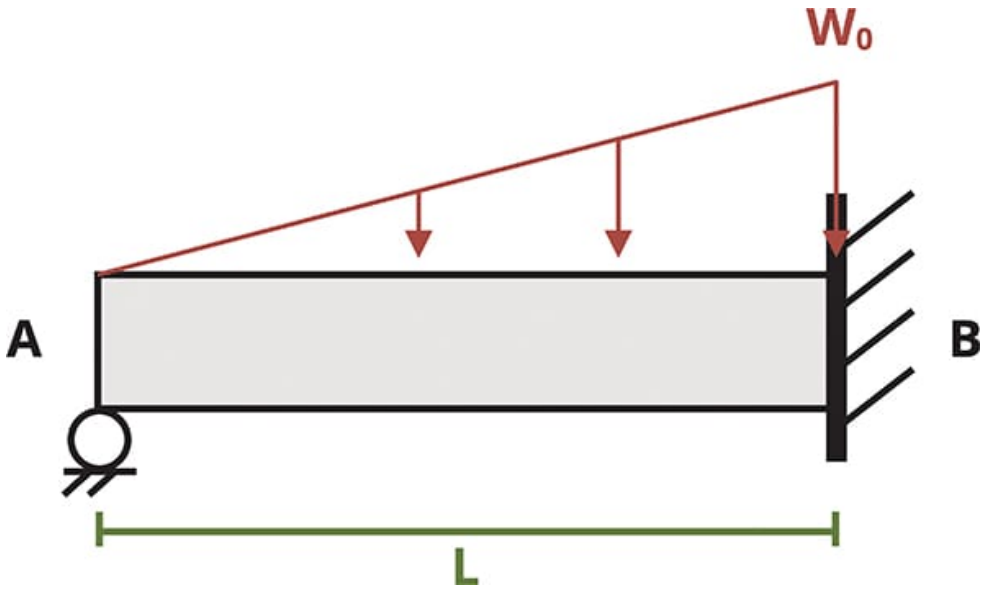
In practice engineers design structures with redundancies so that if one part fails it can be replaced without the entire structure collapsing. As such, static indeterminacy is the norm.

In Section 11.1 we cut a cross-section and found the internal bending moment as a function of x, M = f(x). This was integrated twice to determine the equation of the elastic curve. For statically indeterminate problems, we'll write the moment equation as a function of both x and A, where A is the unknown force at the redundant support. Integrating this equation twice results in three unknowns, but also three boundary conditions. These unknowns and boundary conditions enable us to solve for the redundant reaction force, and then we can use equilibrium equations to solve for the other reactions. See Example 11.6.

Example 11.6

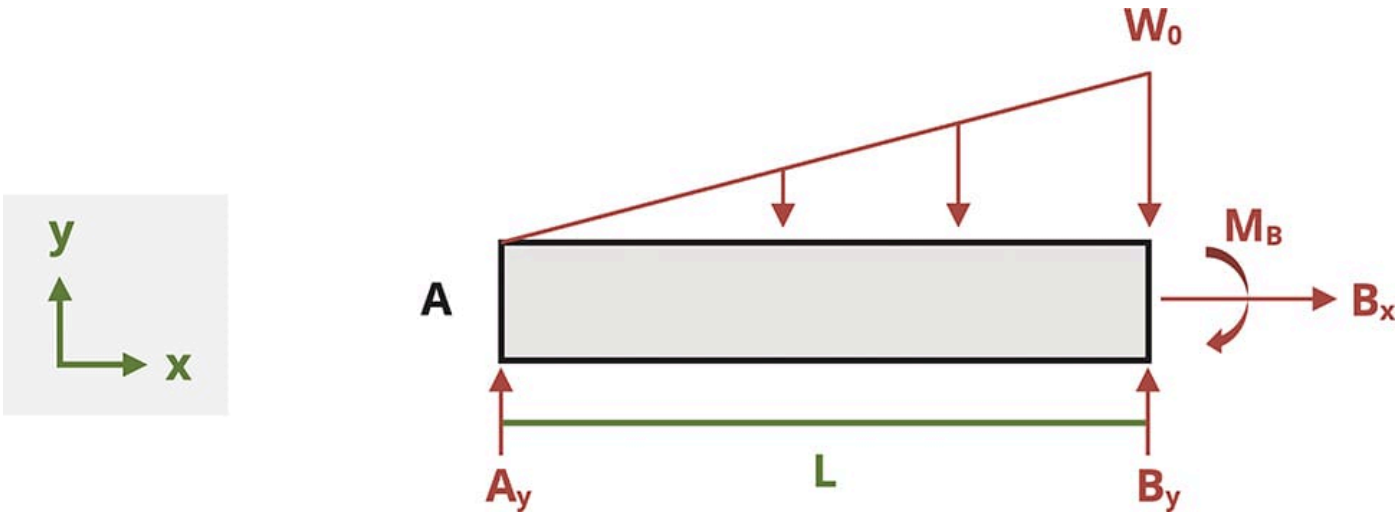
A propped cantilever beam of length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft.

Determine the reactions at supports A and B.

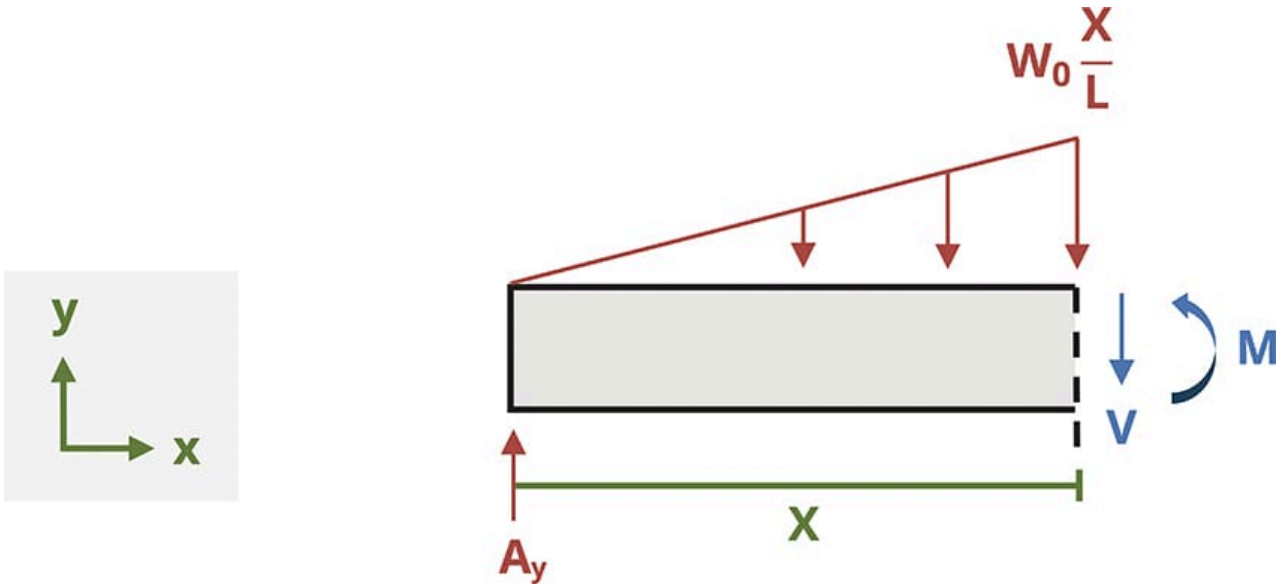


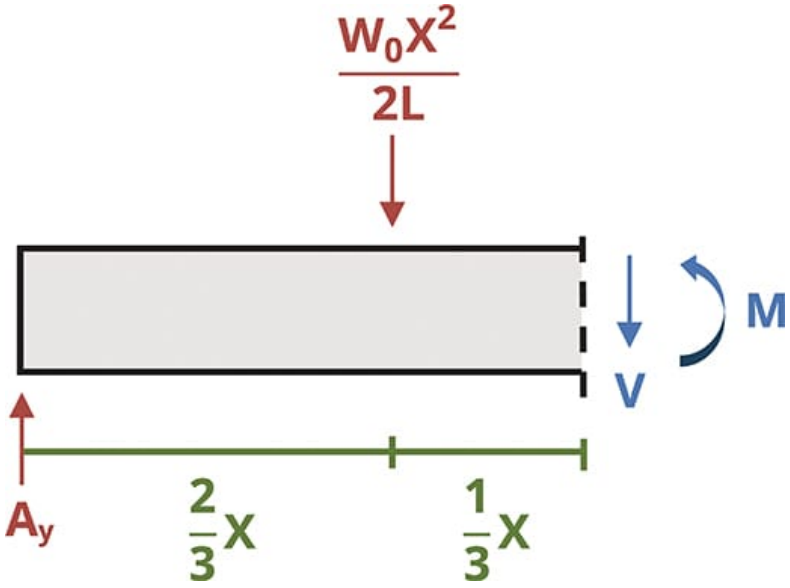
Solution

An FBD of the beam shows four unknowns and only three equilibrium equations. We can solve $\sum F_x = 0$ to find $B_x = 0$, but that still leaves us with three unknowns and two equations.



Cut a cross-section and determine the internal bending moment in terms of distance x and redundant force A_y. Set this equation equal to $EI\frac{\partial^2 y}{\partial x^2}$ and integrate twice.





$$\begin{aligned}
\sum M_x = \quad & M + \tfrac{w_0 x^2}{2L}\left(\tfrac{x}{3}\right) - A_y x = 0 \\
M \quad & = A_y x - \tfrac{w_0 x^3}{6L} = EI\tfrac{\partial^2 y}{\partial x^2} \\
\int M\,dx \quad & = \tfrac{A_y x^2}{2} - \tfrac{w_0 x^4}{24L} + c_1 = EI\tfrac{\partial y}{\partial x} \\
\int\int M\,dx \quad & = \tfrac{A_y x^3}{6} - \tfrac{w_0 x^5}{120L} + c_1 x + c_2 = EIy
\end{aligned}$$

Now we have three unknowns (A_y, c_1, c_2) and three boundary conditions (at the roller, deflection equals zero; at the fixed support, both slope and deflection equal zero).

Apply the boundary conditions.

$$\begin{aligned}
&\text{At } x = 0, y = 0 \quad \rightarrow \quad c_2 = 0 \\
&\text{At } x = L, y = 0 \quad \rightarrow \quad \tfrac{A_y L^3}{6} - \tfrac{w_0 L^4}{120} + c_1 L = 0 \\
&\text{At } x = L, \tfrac{\partial y}{\partial x} = 0 \quad \rightarrow \quad \tfrac{AyL^2}{2} - \tfrac{w_0 L^3}{24} + c_1 = 0
\end{aligned}$$

Rearrange and substitute.

$$\begin{gathered}
c_1 = \tfrac{w_0 L^3}{24} - \tfrac{A_y L^2}{2} \\
\tfrac{A_y L^3}{6} - \tfrac{w_0 L^4}{120} + \tfrac{w_0 L^4}{24} - \tfrac{A_y L^3}{2} = 0 \\
\tfrac{-2A_y L^3}{6} + \tfrac{4w_0 L^4}{120} = 0 \\
A_y = \tfrac{w_0 L}{10}
\end{gathered}$$

We can now reach a numerical answer and find the other reactions by equilibrium.



$$A_y = \frac{20\ \frac{kips}{ft} * 6\ ft}{10} = 12\ kips$$
$$\sum F_y = 12\ kips - 60\ kips + B_y = 0 \quad \rightarrow \quad B_y = 48\ kips$$
$$\sum M_A = -(60\ kips * 4\ ft) + (48\ kips * 6\ ft) - M_B = 0 \quad \rightarrow \quad M_B = 48 \quad kip \cdot ft$$

This method works well for beams with a single continuous load. While it can be used for more complex loads, it quickly becomes complicated and time-consuming. For these cases superposition is recommended.

The superposition method is like the methods used to solve other statically indeterminate problems in Section 5.5 and Section 6.3. To solve a statically indeterminate deflection problem using superposition, first identify the redundant support. We have three equilibrium equations and so can solve for three unknown support reactions. Any support reactions beyond these three are redundant.

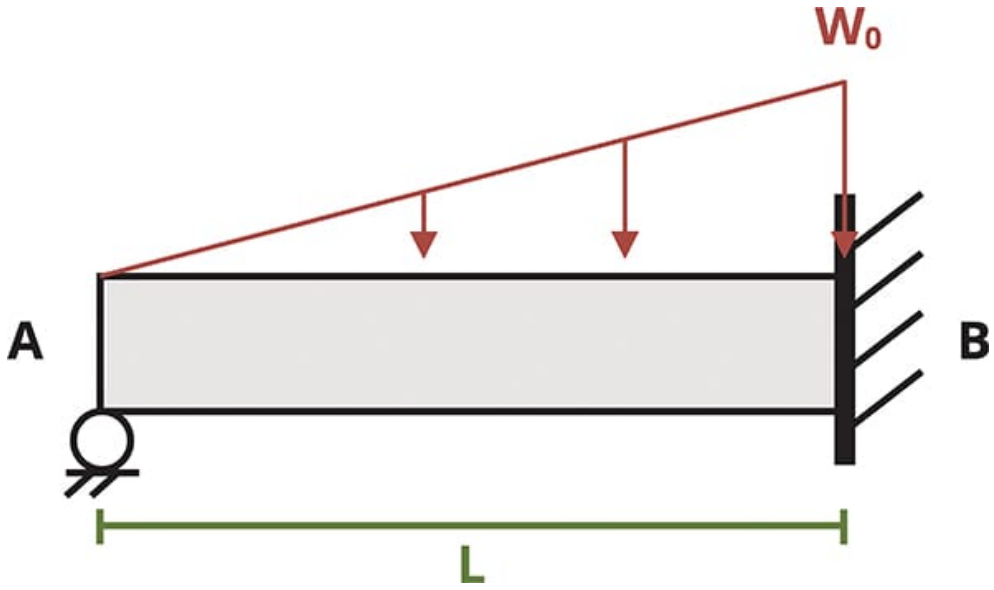
Remove each redundant reaction and determine the deflection resulting from the applied loads at the point where the reaction was removed. Without fail do this for each redundant reaction. It is best to remove a support such that the remaining beam is either simply supported or cantilever, given that these are the only support configuration in Appendix B.

Then replace each redundant reaction and determine the resulting deflection where the reaction is applied. Since there is actually a support here, the total deflection must be zero. Sum both deflections and set them equal to zero. This enables you to calculate the value of the redundant support. Once this is known, the other reaction loads can be determined using equilibrium. See Example 11.7 for a demonstration.

Example 11.7

The propped cantilever beam of Example 11.6 is shown again here. Length L = 6 ft and is subjected to a linear distributed load where w_0 = 20 kips/ft.

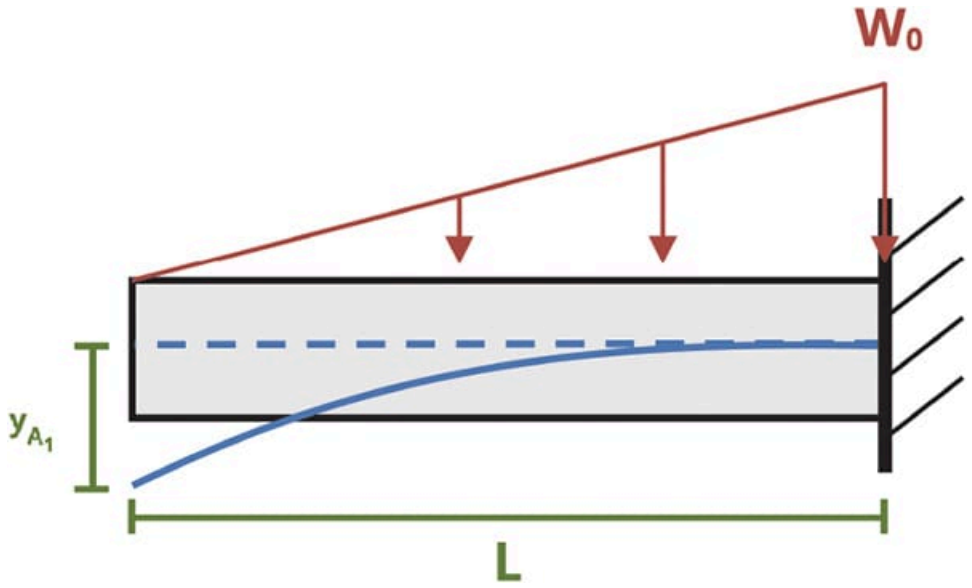
Determine the reactions at supports A and B.



Solution

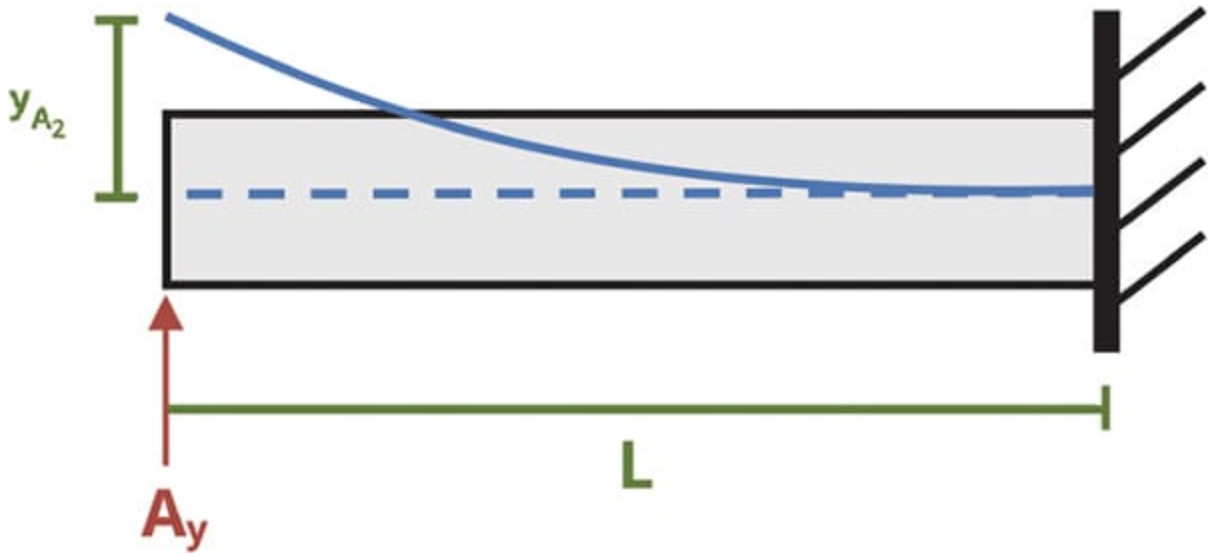
As demonstrated in Example 11.6, this problem is statically indeterminate.

This time remove redundant support A_y and use Appendix B to find the deflection at point A.



$$y_{A_1} = -\frac{w_0 L^4}{30EI}$$

Now replace A_y and use Appendix B to find the deflection at point A caused by A_y.



$$y_{A_2} = \frac{A_y L^3}{3EI}$$

Since there is actually a support at A, the deflection at A must be zero.

$$-\frac{w_0 L^4}{30EI} + \frac{A_y L^3}{3EI} = 0 \quad \rightarrow \quad A_y = \frac{W_0 L}{10}$$

This is the same answer found in Example 11.6, and the rest of the problem proceeds in the same way.



$$A_y = \frac{20\ \frac{kips}{ft} * 6\ ft}{10} = 12\ kips$$
$$\sum F_y = 12\ kips - 60\ kips + B_y = 0 \quad \rightarrow \quad By = 48\ kips$$
$$\sum M_A = (60\ kips * 4\ ft) + (48\ kips * 6\ ft) - M_B = 0 \quad \rightarrow \quad M_B = 48\ kip \cdot ft$$

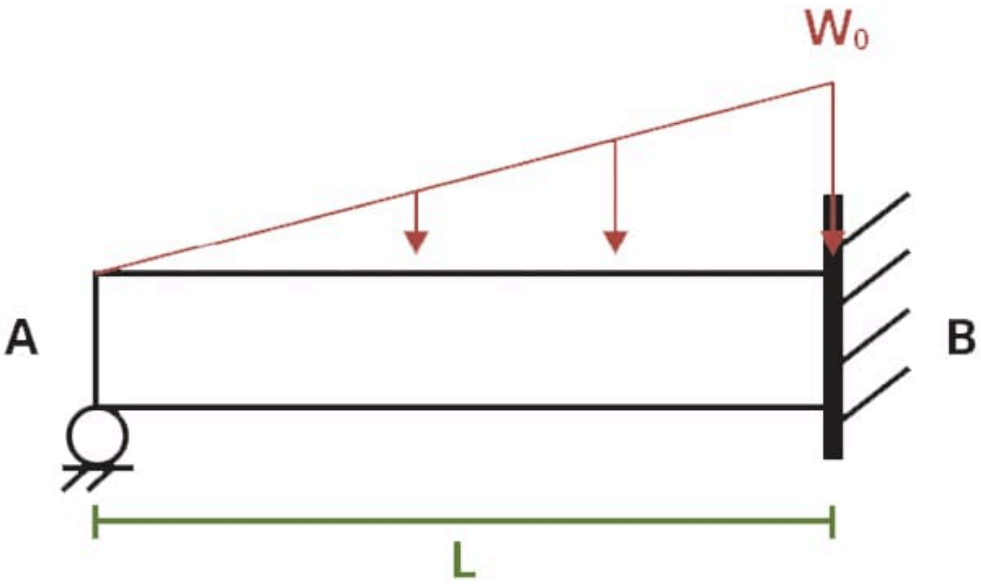
It's possible to remove a moment reaction so that what remains is a simply supported beam. In this case, determining the deflection at the point where the load was removed isn't helpful because removing the moment reaction from a fixed support effectively leaves a pin support and the deflection with the moment removed will still be zero. In this situation look at the slope at the point where the reaction was removed.

Example 11.7 is repeated with the moment at B removed instead of the force at A.

Example 11.7 (REWORKED)

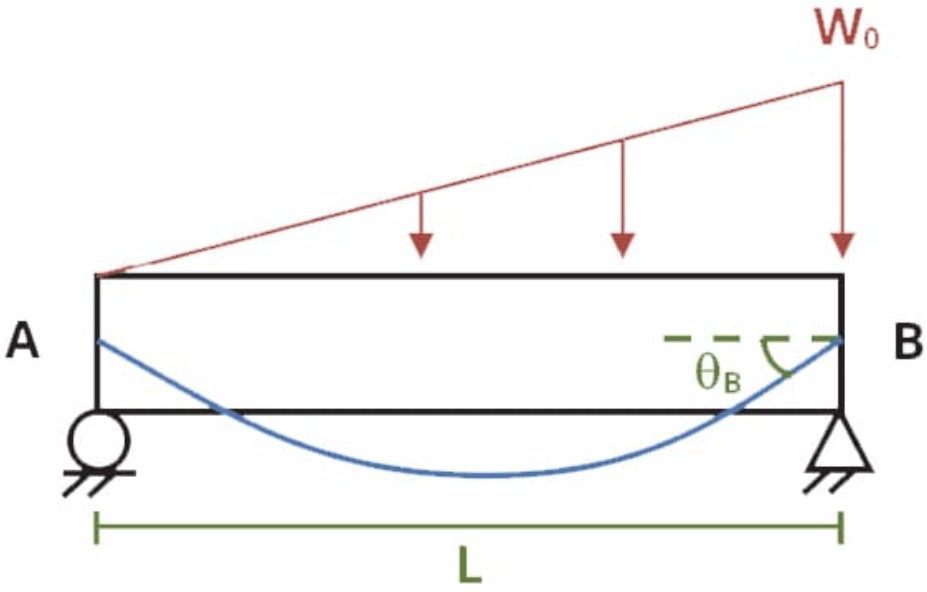
The propped cantilever beam of Example 11.6 is shown again here. Length L = 6 ft is subjected to a linear distributed load where w_0 = 20 kips/ft.

Determine the reactions at supports A and B.



Solution

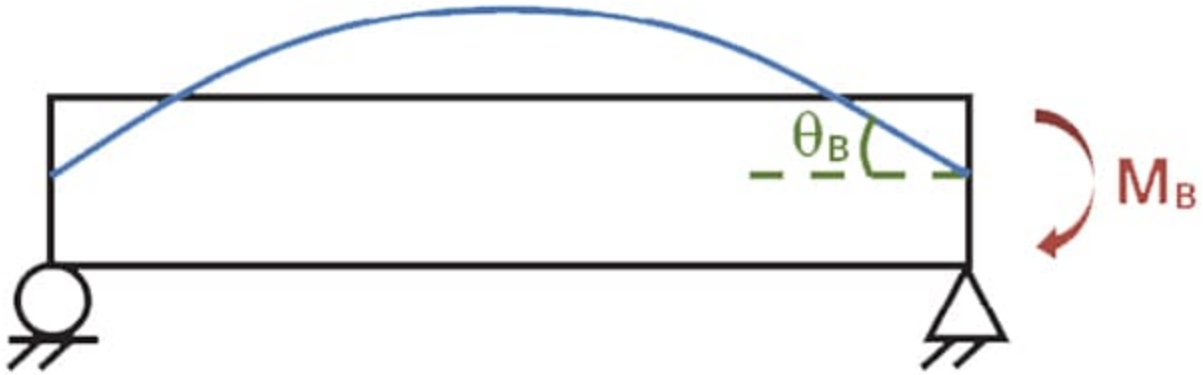
This time remove the moment reaction at B to leave a simply supported beam.



Finding the deflection at B would not help as it would simply be zero. Instead, use Appendix B to find the slope at B.

$$\theta_B = \frac{W_0 L^3}{45EI}$$

Now replace the moment reaction at B and use Appendix B to find the slope at B due to this moment.



$$\theta_B = -\frac{M_B L}{3EI}$$

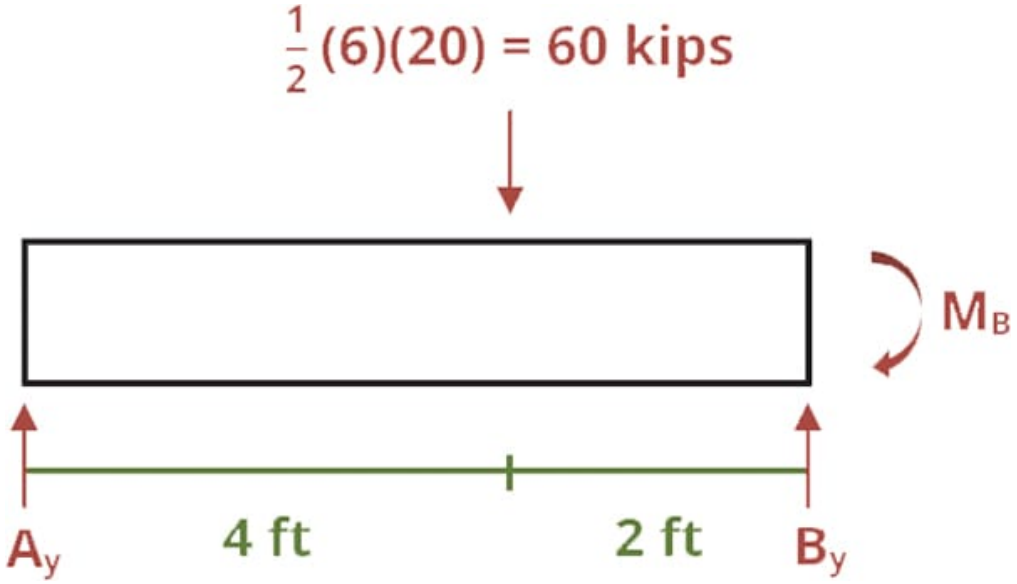
Since there is actually a fixed support at B, the slope here must be zero.

$$\frac{W_0 L^3}{45EI} - \frac{M_{BL}}{3EI} = 0$$
$$M_B = \frac{W_0 L^2}{15}$$

Adding numbers yields

$$M_B = \frac{20\ \frac{kips}{ft} * (6\ ft)^2}{15} = 48\ kip \cdot ft.$$

This is the same answer found in Example 11.6. Reactions A_y and B_y can be found by equilibrium and will also result in the same answers as in Example 11.6.



$$\sum M_A = (60\ kips * 4\ ft) + 48\ kip \cdot ft - (B_y * 6\ ft) = 0 \quad \rightarrow \quad B_y = 48\ kips$$
$$\sum F_y = A_y + -60\ kips + 48\ kips = 0 \quad \rightarrow \quad A_y = 12\ kips$$

Note that some problems may involve multiple redundant reactions. In such cases remove all redundant reactions and determine the deflection from the external loads at every point where a reaction was removed. Then replace the redundant reactions one at a time and determine the deflection from each load at every point where a reaction was removed. As before, the total deflection at each of these points must be zero. Sum the deflection at each point and set it equal to zero. There will be one equation for every redundant reaction, and so all redundant reactions can be calculated.

Step-by-Step: Statically Indeterminate Deflection

1. Identify and remove redundant reaction(s).
2. Determine the deflection at the point where the reaction was removed. Use superposition if possible, or integration otherwise.
    - If the removed reaction was a moment, determine the slope at this point instead.
3. Replace the redundant reaction and determine the deflection caused by this load at the point where the load acts.
    - If the removed reaction was a moment, determine the slope at this point instead.
4. These deflections (or slopes) must add up to the total deflection at this point. This is typically zero due to the support unless indicated otherwise. Use this to determine the magnitude of the redundant reaction.
5. Use equilibrium to solve for the other reactions. Then solve for anything else required as normal.

11.5 Beam Design for Bending, Shear, and Deflection

Section 9.2 showed us how to use the section modulus to design a beam to meet bending stress specifications. In practice, beams must also meet specifications for shear stress and deflection. Limitations are placed on all three of these criteria, and we must design our beams to meet all three simultaneously. We'll begin with rectangular cross-sections and then move on to standard beams using Appendix A.

As in Section 9.2, problems involving rectangular cross-sections typically relate the base and height of the cross-section such that only one dimension must be determined. Previously we designed the cross-section to resist bending stress using

$$S_{min} = \frac{M_{max}}{\sigma_{allow}} = \frac{I}{c}.$$

Since $\mathrm{I} = \frac{\mathrm{bh}^3}{12}$ and $\mathrm{c} = \frac{\mathrm{h}}{2}$, we were able to determine the required dimensions to meet the minimum required section modulus. Note that the equation $\mathrm{S} = \frac{\mathrm{I}}{\mathrm{c}}$ simplifies to $\mathrm{S} = \frac{\frac{\mathrm{bh}^3}{12}}{\frac{\mathrm{h}}{2}} = \frac{2\mathrm{bh}^3}{12\mathrm{h}} = \frac{\mathrm{bh}^2}{6}$ for a rectangle.

To meet the minimum required shear stress, use $\tau = \frac{\mathrm{VQ}}{\mathrm{It}}$. Applying the method of Section 10.1 to a rectangle (Figure 11.10) shows that $\mathrm{Q} = \mathrm{yA} = \frac{\mathrm{h}}{4}\frac{\mathrm{bh}}{2} = \frac{\mathrm{bh}^2}{8}$.

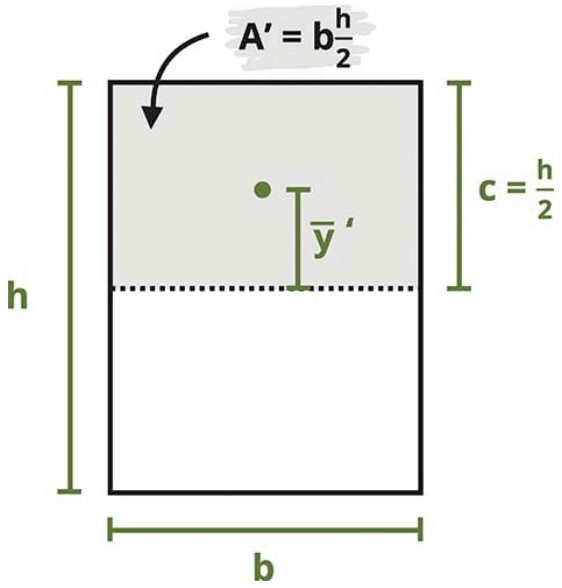


Figure 11.10: Rectangular cross-section showing the dimensions used for calculating the section modulus (S) and first moment of area (Q)

With $I = \frac{bh^3}{12}$ and $t = b$, rewrite the shear stress equation to find the maximum shear stress in a rectangular cross-section as $\tau_{max} = \frac{V\frac{bh^2}{8}}{\frac{bh^3}{12}b} = \frac{12Vbh^2}{8b^2h^3} = \frac{12V}{8bh} = \frac{3}{2}\frac{V}{A}$.

Thus set $\tau_{max} = \frac{3}{2}\frac{V}{A}$ and solve for the required dimensions to not exceed the maximum allowable shear stress.

The maximum deflection of the beam under a given loading configuration can be found using the methods shown in this chapter. Wherever possible use superposition and Appendix B. If it's not possible to do so, then use the method of integration instead. In either case the equation for the beam's maximum deflection will include the second moment of area, I. Set the maximum allowable deflection equal to this equation and replace $I = \frac{bh^3}{12}$.

Then solve for the cross-section's required dimensions to not exceed the maximum allowable deflection of the beam.

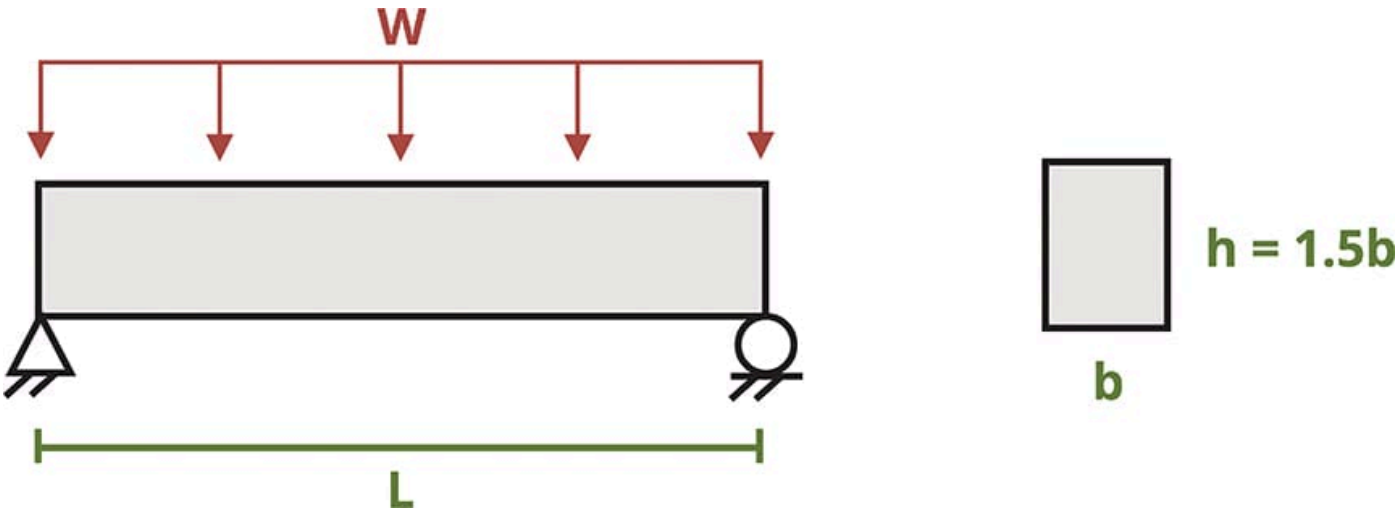
You now have three different dimensions: one required to not exceed the maximum bending stress, one required to not exceed the maximum shear stress, and one required to not exceed the maximum deflection. Since none of these limitations may be exceeded, select the largest of the three potential answers. See Example 11.8 for a demonstration.

Example 11.8

The simply supported wooden beam (E = 1,700 ksi) of length L = 12 ft is subjected to a uniform distributed load of w = 1.5 kips/ft.

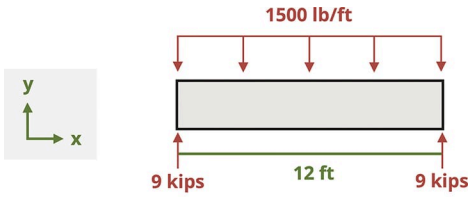
Assume the beam has an allowable bending stress of 900 psi, an allowable shear stress of 180 psi, and a deflection limited to beam span/240. The beam has a rectangular cross-section where h = 1.5b.

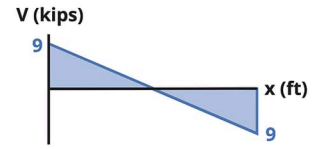
Determine the minimum required dimensions of the beam's cross-section to meet these specifications. Give your answer rounded up to the nearest inch to allow for realistic manufacturing tolerances.

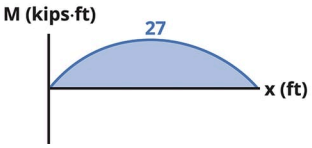


Solution

Start by drawing shear force and bending moment diagrams to determine the maximum internal loads.







- V_{max} = 9 kips
- M_{max} = 27 kip·ft

Design for bending by using the minimum required section modulus, S_{min}.

$$S_{min} = \frac{M_{max}}{\sigma_{allow}} = \frac{27{,}000\ lb{\cdot}ft{*}12\ \frac{in.}{ft}}{900\ \frac{lb}{in.^2}} = 360\ in.^3$$

$$360\ in.^3 = \frac{bh^2}{6} = \frac{b(1.5b)^2}{6} = \frac{2.25\ b^3}{6}$$

$$b = 9.865\ in.$$

Design for shear.

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

$$180\ \frac{lb}{in.^2} = \frac{3}{2} * \frac{9{,}000\ lb}{A}$$

$$A = \frac{3}{2} * \frac{9{,}000\ lb}{180\ \frac{lb}{in.^2}} = 75\ in.^2$$

$$75\ in.^2 = bh = b(1.5b) = 1.5b^2$$

$$b = 7.071\ in.$$

Design for deflection.

From Appendix B note that

$$y_{max} = -\frac{5wL^4}{384EI}.$$

$$-\frac{Span}{240} = -\frac{\left(12\ ft{*}12\ \frac{in.}{ft}\right)}{240} = -\frac{5{*}\frac{1{,}500\ \frac{lb}{ft}}{12\ \frac{in.}{ft}}{*}\left(12\ ft{*}12\frac{in.}{ft}\right)^4}{384{*}1{,}700{,}000\ \frac{lb}{in.^2}{*}I}$$

$$I = \frac{bh^3}{12} = \frac{b(1.5b)^3}{12} = \frac{3.375b^4}{12} = \frac{5{*}\frac{1{,}500\ \frac{lb}{ft}}{12\frac{in.}{ft}}+\left(12\ ft{*}12\frac{in.}{ft}\right)^4}{384{*}1{,}700{,}000\ \frac{lb}{in.^2}} * \frac{240}{\left(12\ ft{*}12\frac{in.}{ft}\right)}$$

$$b = 7.028\ in.$$

Select the largest dimension required to meet all these criteria.

$$b_{req} = 9.865\ in.$$

$$h_{req} = 1.5b_{req} = 14.8\ in.$$

Calculate to the next nearest inch.

$$b_{req} = 10\,in.$$
$$h_{req} = 15\,in.$$

For standard beams again use the method of Section 9.2 to select a beam from Appendix A that meets the specifications for bending stress. Use $\mathrm{S_{min}} = \frac{\mathrm{M_{max}}}{\sigma_{\mathrm{allow}}}$ and select a beam with $\mathrm{S} > \mathrm{S_{min}}$.

Next determine the equation for the beam's maximum deflection under the given loading conditions. As before, use superposition and Appendix B whenever possible. Set the maximum allowable deflection equal to this equation and solve for the minimum required second moment of area, $\mathrm{I_{min}}$. Now select the lightest beam from Appendix A that meets both $\mathrm{S_{min}}$ and $\mathrm{I_{min}}$.

Steel W-beams rarely fail from shear stress, but shear must nonetheless be checked. An exact value for the maximum shear stress in the beam can be found from $\tau_{\mathrm{max}} = \frac{\mathrm{VQ}}{\mathrm{It}}$. However, the beam's web resists most of the shear stress. A reasonable estimate of the maximum shear stress can be found using $\tau_{\mathrm{estimate}} = \frac{\mathrm{V}}{\mathrm{A_{web}}}$. This calculation is relatively simple, but it's important to note that it is only an estimate. In most cases the maximum shear stress will be significantly less than the allowable shear stress, and this estimate will suffice to demonstrate that. In the rare case that the maximum shear stress is within 20 percent of the allowable shear stress, an exact value must be determined using $\tau_{\mathrm{max}} = \frac{\mathrm{VQ}}{\mathrm{It}}$.

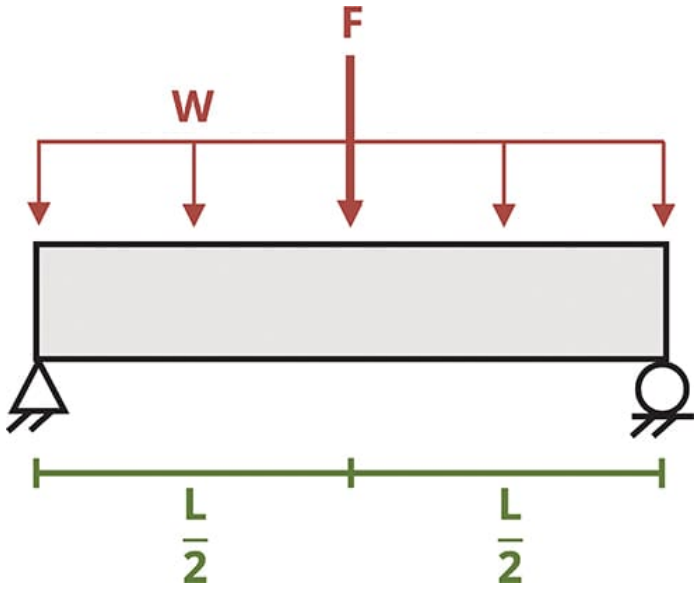
The beam was selected to meet the bending stress and deflection requirements. As long as the maximum shear stress is less than the allowable shear stress, the beam can be used. See Example 11.9 for a demonstration.

Example 11.9

A steel (E = 210 GPa) W-beam of length L = 10 m needs to support the load shown, where w = 5 kN/m and F = 100 kN.

Assume the beam has an allowable bending stress of 150 MPa, an allowable shear stress of 100 MPa, and a maximum deflection limited to 40 mm.

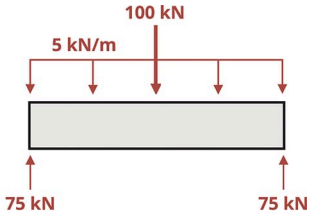
Select the lightest weight W-beam from Appendix A that meets these specifications.

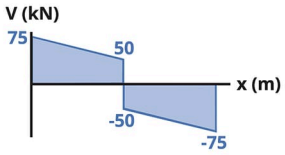


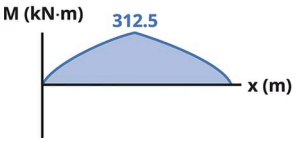
Solution

Load is symmetric, so each support will hold half the load. Draw shear force and bending moment diagrams to find the maximum internal loads.

- V_{max} = 75 kN
- M_{max} = 312.5 kN·m







Design for bending by using the minimum required section modulus S_{min}.

$$S_{min} = \frac{M_{max}}{\sigma_{Allow}} = \frac{312.5\times10^3\ N\cdot m}{150\times10^6\ \frac{N}{m^2}} = 0.002083\ m^3$$
$$= 2,083 \times 10^3\ mm^3$$

Design for deflection.

We know the maximum deflection is limited to 40 mm (0.04 m) and the beam will deflect downward. From Appendix B we can see the maximum deflection for both the concentrated load and the distributed load occur in the same place. We can therefore add the two maximum deflection equations together and solve for the second moment of area.

$$y_{max} = -0.04\ m = -\frac{5wL^4}{384EI} - \frac{FL^3}{48EI}$$
$$I_{req} = \frac{1}{0.04\ m}\left[\frac{5*5{,}000\ \frac{N}{m}*(10\ m)^4}{384*(210\times10^9)\frac{N}{m^2}} + \frac{100{,}000\ N*(10\ m)^3}{48*(210\times10^9)\frac{N}{m^2}}\right]$$
$$I_{req} = 0.0003255\ m^4$$
$$= 325.5 \times 10^6\ mm^4$$

Select the lightest beam from Appendix A with

$$S > 2,083 \times 10^3\ mm^3$$
$$I > 325.5 \times 10^6\ mm^4.$$

Options include

$$W310 \times 158 (S = 2,380 \times 10^3\ mm^3, I = 388 \times 10^6\ mm^4)$$
$$W360 \times 134 (S = 2,340 \times 10^3\ mm^3, I = 416 \times 10^6\ mm^4)$$
$$W460 \times 113 (S = 2,390 \times 10^3\ mm^3, I = 554 \times 10^6\ mm^4)$$
$$W530 \times 101 (S = 2,290 \times 10^3\ mm^3, I = 616 \times 10^6\ mm^4)$$
$$W610 \times 125 (S = 3,210 \times 10^3\ mm^3, I = 986 \times 10^6\ mm^4)$$
$$W760 \times 147 (S = 4,410 \times 10^3\ mm^3, I = 1,660 \times 10^6\ mm^4).$$

Select the lightest of these beams.

Use W530 × 101.

Check the estimated shear stress to ensure there's no risk of exceeding the allowable shear stress.

$$\begin{aligned} \text{For } W530 \times 101, A_{web} &= t_w d \\ &= 0.01098\ m * 0.536\ m \\ &= 0.0058853\ m^2 \\ \tau = \frac{v_{max}}{A_{web}} = \frac{75{,}000\ N}{0.0058853\ m^2} &= 12.7\ MPa \end{aligned}$$

Since $\tau_{Allow} = 100\ MPa$, this will be fine. The estimated shear stress is not close to the allowable shear stress.

Step-by-Step: Intermediate Beam Design

Rectangular cross-sections:

1. Determine minimum required section modulus, $\mathrm{S_{min}} = \frac{\mathrm{M_{max}}}{\sigma_{\mathrm{allow}}} = \frac{\mathrm{bh^2}}{6}$ and determine the minimum dimensions to meet the required $\mathrm{S_{min}}$.
2. Set the maximum allowable shear stress, $\tau_{\mathrm{max}} = \frac{3}{2}\frac{\mathrm{V}}{\mathrm{A}}$, and determine the minimum dimensions to not exceed τ_{max}.
3. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. This equation will contain I, which depends on dimensions b and h. Determine the minimum dimensions to not exceed the maximum allowable deflection.
4. Use the largest of the three calculated dimensions, given that these will be the smallest dimensions that meet all three requirements.

Standard beams:

1. Determine minimum required section modulus, $\mathrm{S_{min}} = \frac{\mathrm{M_{max}}}{\sigma_{\mathrm{allow}}}$.
2. Determine the maximum deflection for the applied load(s) and set this equal to the maximum allowable deflection. Determine the minimum required second moment of area, $\mathrm{I_{min}}$, to not exceed the maximum allowable deflection.
3. Select the lightest beam from Appendix A that meets both the $\mathrm{S_{min}}$ and the $\mathrm{I_{min}}$ requirements.
4. Estimate the maximum shear stress in the selected beam using $\tau = \frac{\mathrm{V}}{\mathrm{A_{web}}}$. If this is within approximately 20 percent of the maximum allowable shear stress, calculate the maximum shear stress using $\tau_{\mathrm{max}} = \frac{\mathrm{VQ}}{\mathrm{It}}$.
5. If the maximum shear stress is less than the maximum allowable value, use this beam. If it's greater than the maximum allowable value, return to step 3 and select the next lightest beam that meets both the $\mathrm{S_{min}}$ and $\mathrm{I_{min}}$ requirements.

Summary

Key Takeaways

- Beams under load will experience a type of deformation known as deflection. It is typically required to limit deflection to some small amount.
- Deflection can be determined using either the method of integration or the method of superposition.
- Both methods can be used for both statically determinate and statically indeterminate beams, though superposition is recommended whenever possible as it is generally quicker than integration.
- Engineers designing beams must consider the response of the beam to bending stress, shear stress, and deflection simultaneously.

Key Equations

Deflection by integration:

$$w(x) = EI\frac{d^4y}{dx^4}$$
$$V(x) = EI\frac{d^3y}{dx^3} = \int w(x)dx + C1$$
$$M(x) = EI\frac{d^2y}{dx^2} = \int V(x)dx + C2$$
$$\theta(x) = EI\frac{dy}{dx} = \int M(x)dx + C3$$
$$y(x) = \int \theta(x)dx + C4$$

Boundary conditions:

- At any support the deflection is 0.
- At a fixed support both the slope and deflection are zero.
- The shear force will be equal to the applied force (including a force applied by a support) if there is one, or zero otherwise.
- The bending moment will be equal to the applied moment (including a moment applied by a fixed support) if there is one, or zero otherwise.

Beam design for rectangular cross-sections:

$$S = \frac{bh^2}{6}$$
$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

Beam design for W-beams:

$$S_{min} = \frac{M_{max}}{\sigma_{allow}}$$
$$\tau_{max} = \frac{VQ}{It}$$
$$\tau_{estimate} = \frac{V}{A_{web}}$$

References

Figures

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- Figure 11.1: A model bridge showing severe deflection along its span (left). Left: Denise Krebs. 2012. CC BY. https://flic.kr/p/bnc8qp. Right: clyde. 2012. CC BY-NC. https://flic.kr/p/bpkcyR.

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Chapter 12: Stress Transformation

Learning Objectives

- Use shear and normal stress states calculated in a given plane to find the shear and normal stress state in a different (rotated) plane using equations and using Mohr's circle
- Use shear and normal stress states calculated in a given plane to find the planes of maximum and minimum stresses, as well as the corresponding stress magnitudes, by using both equations and Mohr's circle
- Calculate the absolute maximum shear stress using both equations and Mohr's circle

The preceding chapters discussed how to calculate normal and shear stresses that arise from various loading situations. Also discussed was how those stresses can vary from point to point on a body or structure. The combination of normal stress and shear stress that exists at a point is referred to as the general stress state. When we restrict analysis to 2D states of stress, we can represent the stress state on a 2D stress element as discussed in Section 4.7 and illustrated in Figure 4.9 for the x-y plane.

The plane of the stress state depends on the coordinate system chosen for calculating the stresses. The coordinate system most often chosen is one considered to be the most straightforward system for the given loading directions and geometrical orientation of the body. However, the plane for which the stresses are most easily calculated is not always the only plane for which we would want to know the stress state.

For example, Figure 12.1 shows an air duct that is part of a heat pump system and is subject to internal air pressure. The welded seams are wound in a helical pattern that forms an angle with the horizontal (x) axis. Chapter 13 develops basic equations to calculate stress in the x-y plane due to pressure. However, in this case it's additionally important to know the stresses in the directions parallel and perpendicular to the weld seams (the x'-y' plane) to ensure that the weld could tolerate the loading, so being able to transform the stresses we can most easily calculate in x and y directions to the x' and y' directions would be useful.

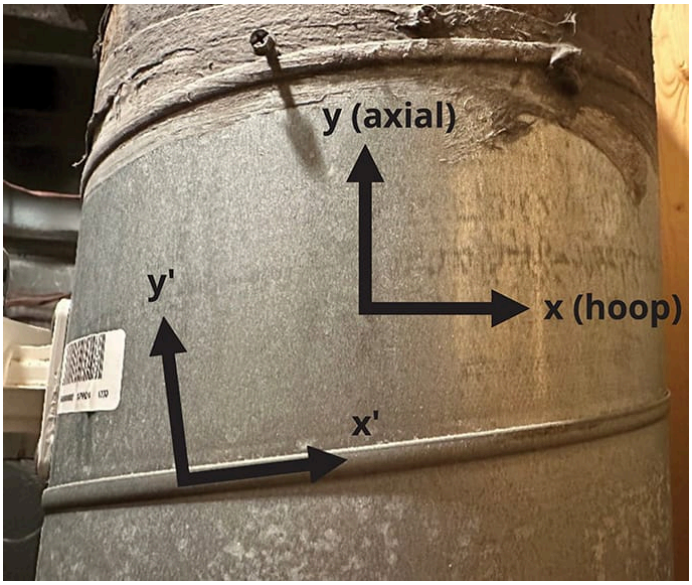


Figure 12.1: A pressurized duct is fabricated with welding in a helical pattern.

This chapter begins by detailing how to represent a 2D stress state on a stress element (Section 12.1).

Then it addresses how to derive stress transformation equations that allow us to use a given stress state in one plane to find the corresponding stress state in different planes (Section 12.2). This process is known as stress transformation.

Also addressed is how to derive equations that allow us to determine the planes on which the maximum and minimum normal

and shear stresses occur and the corresponding stress states (Section 12.3). The maximum and minimum normal stresses are considered principal stresses.

Section 12.4 discusses an alternative graphical method for performing stress transformation, known as Mohr's circle.

Section 12.5 prompts us to use Mohr's circle to expand our analysis and look not only at the shear stress in the original 2D plane but also out-of-plane shear stress.

12.1 Stress Elements for 2D Stress States

A square stress element shows the normal stress and shear stress at a point for a 2D stress state (see Figure 12.4). Here we discuss the important aspects of the stress element:

1. Plane of the element

   The coordinate axes on the stress element represent the plane on the body where the point is located, with the positive directions indicated by arrows and axes labels. To determine the plane a point is on, we observe the axes on a square element on the side of the body when viewing it straight on. Figure 12.2 illustrates three points on a body, each on a different plane (x-y, y-z, and x-z).

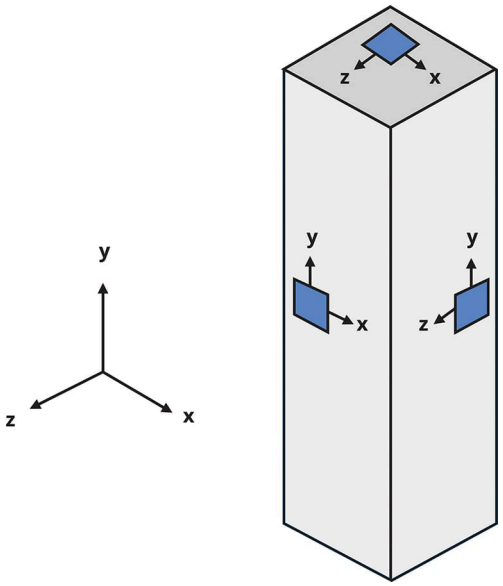


Figure 12.2: Points on three planes of a body

2. Positive and negative faces

   We refer to the edges of the element as positive faces and negative faces depending on which way the coordinate axes are oriented. For example, if the vertical coordinate axis is positive upward, then the top edge of element is the positive face for that coordinate axis and the bottom edge is the negative face. This is illustrated for two different planes in Figure 12.3.

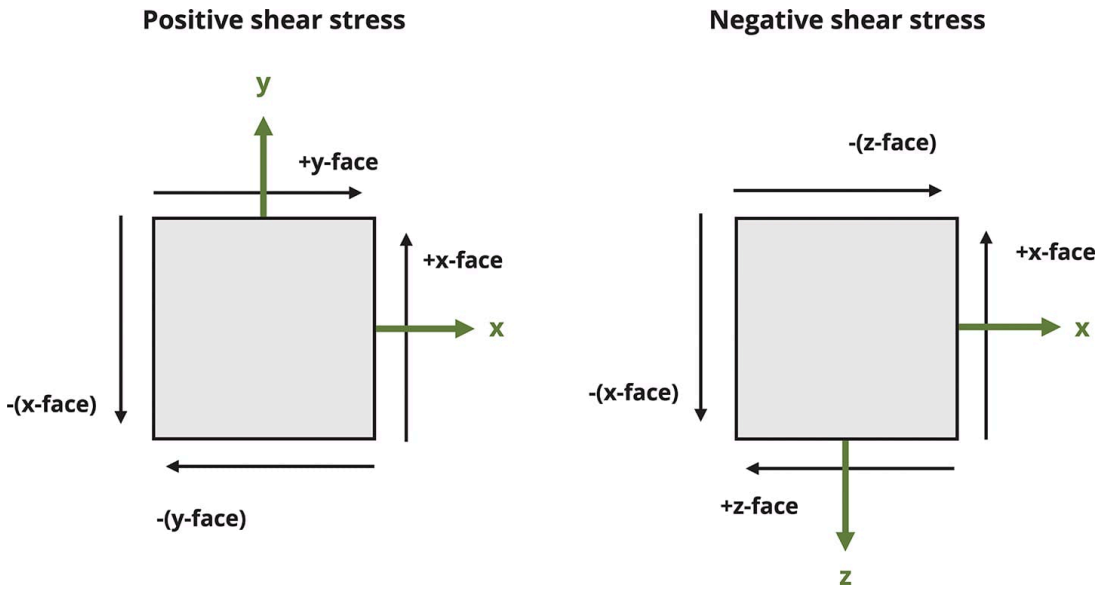


Figure 12.3: Positive and negative faces and shear stress signs

3. Normal stresses

   Normal stresses that are positive (tensile) are illustrated by arrows that point away from the element. Negative normal stresses (compressive) point toward the element. Normal stresses directed in the horizontal direction are shown on the left and right faces of the element, and normal stresses directed in the vertical direction are shown on the top and bottom faces.

4. Shear stress

   As explained in Section 4.7, the shear stress will be the same on all sides of the stress element. The manner in which the shear stress arrows are oriented around the body indicates the sign. A positive shear stress is one for which the shear stress arrow points in a positive direction on a positive face or in a negative direction on a negative face. The arrows are drawn such that no matter which face you choose to examine, you should reach the same conclusion. The magnitude of the shear stress is usually specified in one corner of the element.

For example, Figure 12.3 shows that for the positive shear stress case, the shear stress arrow on the +x-face points in the +y direction and the shear stress arrow on the –x-face points in the –y direction. Either way a positive shear stress is indicated and would yield the same result as for the y-faces.

For the negative shear stress case, the shear stress arrow on the +x-face points in the –z direction and the shear stress arrow on the –x-face points in the +z direction. Either way a negative shear stress is indicated and would yield the same result as for the z-faces.

An example of a fully drawn stress element can be found in Figure 12.4. In this case the shear stress is τ_{xz} = +16 MPa (note the positive z–directed shear stress arrow on the positive x-face), and the normal stresses are σ_x = –150 MPa and σ_z = +90 MPa.

12.2 Stress Transformation Equations

The Section 12.2.1 stress transformation equations allow us to use a set of stresses in one given plane (e.g., σ_x, σ_y, and τ_{xy}) as inputs to determine stresses in any other plane of interest. Understanding the derivations will help lend clarity to the inputs and outputs of the equations. If you wish to skip the derivations, go directly to Equation 12.1 and/or Equation 12.2. Section 12.2.2 covers the details of how to use the equations.

12.2.1 Derivation of Equations

Transformation of a uniaxial stress state was examined in Section 2.4. In that case a known force acting in the axial direction enabled us to determine the stress on an inclined plane. The current derivation for a more general stress state is based on the same idea of cutting a section and applying equilibrium.

Figure 12.5 shows a stress element representing a general stress state in the x-y plane and the full body diagram (FBD)—but with stresses instead of forces—of the element cut along the plane of interest. The n and t directions are normal and parallel to the incline respectively, and θ and β are complementary angles.

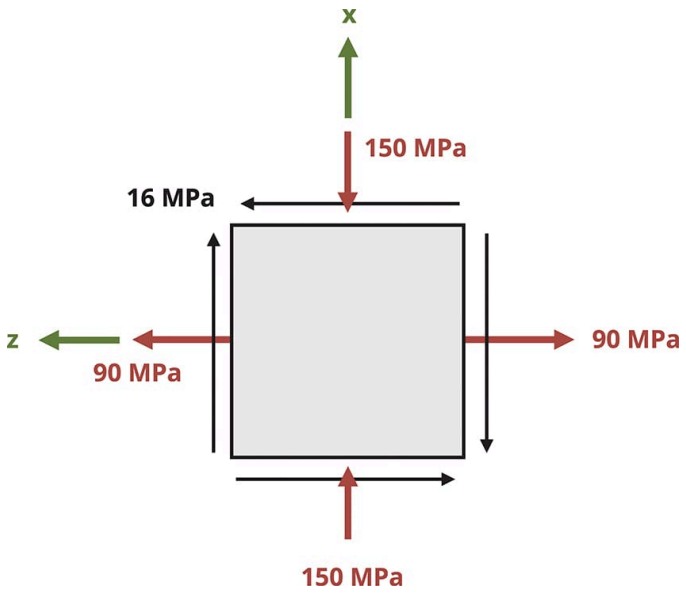


Figure 12.4: Example of a 2D stress element in the x-z plane

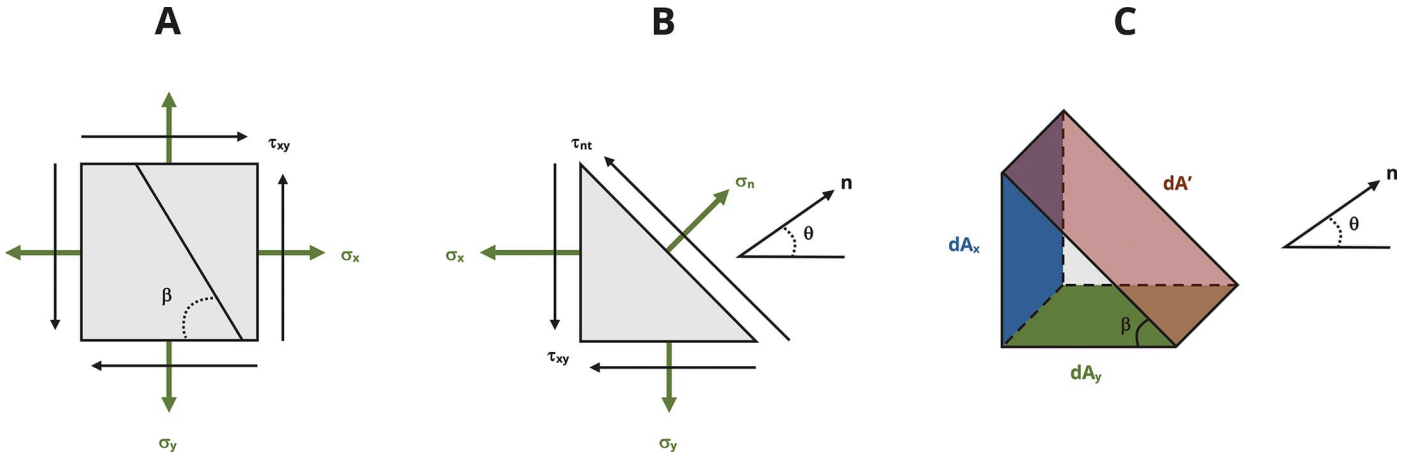


Figure 12.5: Stress element in the x-y plane (A), FBD of the cut section of the stress element (B), and surface areas the stresses of the cut stress element act on (C).

We can proceed as follows:

1. Write expressions for the forces that correspond to the stresses acting on the cut element. Do this by multiplying the stresses with the area of the side of the element they act on. These areas are illustrated in Figure 12.5 (C). Note that s_y and the t_{xy} on the bottom face of the element act on area dA_y, while s_x and the t_{xy} on the left face of the element act on area dA_x. The internal forces normal and parallel to the incline act on area dA'.

On the left face we have

$$F_x = -\sigma_x dA_x$$
$$F_y = -\tau_{xy} dA_x.$$

On the bottom face we have

$$F_y = -\sigma_y dA_y$$
$$F_x = -\tau_{xy} dA_y.$$

Both faces exhibit negative signs because the stresses act to the left (negative x) and down (negative y).

On the inclined face we have

$$F_x = \sigma_n \, dA' \cos(\theta) - \tau_{nt} \, dA' \sin(\theta)$$
$$F_y = \sigma_n \, dA' \sin(\theta) + \tau_{nt} \, dA' \cos(\theta).$$

2. Express dA_x and dA_y in terms of dA'.

   Using trigonometry results in

$$dA_x = dA' \sin(\beta)$$
$$= dA' \cos(\theta)$$
$$dA = dA' \cos(\beta)$$
$$= dA' \sin(\theta).$$

3. Substitute these area expressions into the force equations and write the force equilibrium equations.

$$\sum F_x = -\sigma_x \, dA' \cos(\theta) - \tau_{xy} \, dA' \sin(\theta) + \sigma_n \, dA' \cos(\theta) - \tau_{nt} \, dA' \sin(\theta) = 0$$
$$\sum F_y = -\tau_{xy} \, dA' \cos(\theta) - \sigma_y \, dA' \sin(\theta) + \sigma_n \, dA' \sin(\theta) + \tau_{nt} \, dA' \cos(\theta) = 0$$

4. Solve the equilibrium equations for σ_n and τ_{nt} and simplify.

$$\boxed{\begin{aligned} \sigma_n &= \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{xy} \sin(\theta) \cos(\theta) \\ \tau_{nt} &= -(\sigma_x - \sigma_y) \cos(\theta) \sin(\theta) + \tau_{xy} (\cos^2(\theta) - \sin^2(\theta)) \end{aligned}}$$

(12.1)

These equations are the stress transformation equations.

Alternatively, implementing the trigonometric identities results in

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$
$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

The transformation equations can also be expressed as

$$\boxed{\begin{aligned} \sigma_n &= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \\ \tau_{nt} &= -\frac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta) \end{aligned}},$$

(12.2)

where

- σ_n = *Average normal stress perpendicular to the inclined plane [Pa, psi]*

- τ_{nt} = *Average shear stress parallel to the inclined plane [Pa, psi]*
- σ_x = *Average normal stress in the x direction before any transformation [Pa, psi]*
- σ_y = *Average normal stress in the y direction before any transformation [Pa, psi]*
- τ_{xy} = *Average shear stress in the x-y plane before any transformation [Pa, psi]*
- θ = *Angle between x-axis and n-axis [°, rad]*

Either Equation 12.1 or Equation 12.2 will work to perform stress transformations. Which set to use is determined by personal preference.

12.2.2 Using the Stress Transformation Equations

Whereas the n and t directions have referred to the directions normal and parallel to an incline, a planar element will have two normal directions, as represented by x, y and x', y' in Figure 12.6. In transforming stresses from the x-y plane to the x'-y' plane, we can distinguish between the two normal stresses by the angle used in the stress transformation equation for σ_n. Given how θ was defined in the derivation of the transformation equations, the following is true:

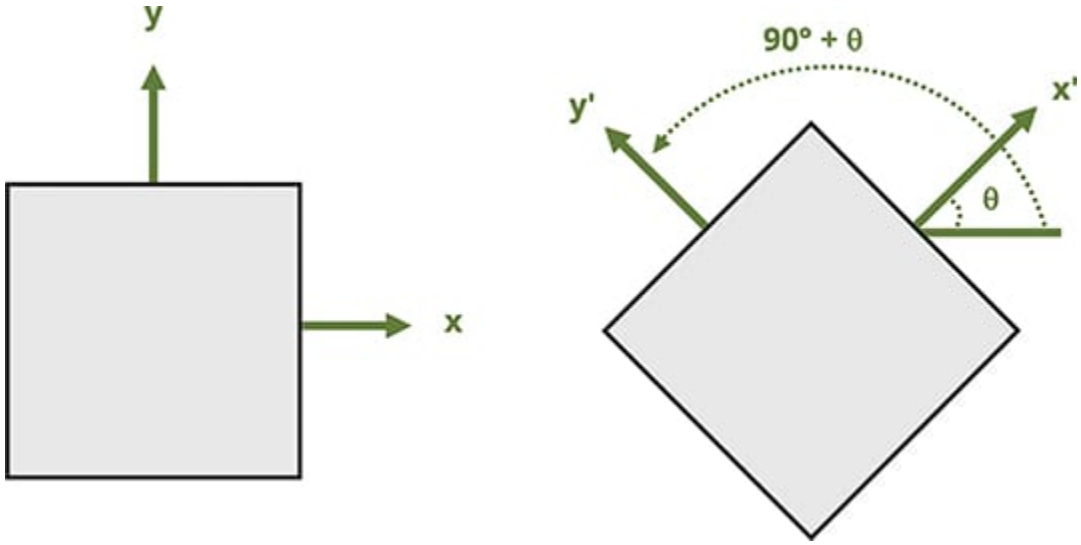


Figure 12.6: The angle θ is defined to be the angle between the original x-axis and the angle normal to the plane of interest. Subbing θ into the stress transformation equations gives $\sigma_{x'}$ and $\tau_{x'y'}$ while subbing in (90° + q) gives $s_{y'}$.

$$\begin{aligned}\sigma_{x'} &= \sigma_n(\theta)\\ \sigma_{y'} &= \sigma_n(\theta + 90°)\\ \tau_{x'y'} &= \tau_{nt}(\theta)\,.\end{aligned}$$

Another way to quickly determine $s_{y'}$ once $s_{x'}$ is known is to apply the concept of stress invariance, where the sum of the normal stresses remains constant across different planes.

$$(\sigma_x + \sigma_y) = constant$$

or

$$\boxed{(\sigma_{x'} + \sigma_{y'}) = (\sigma_x + \sigma_y)}\,.$$

(12.3)

It is imperative to understand which angle to substitute into the given transformation equations, including whether the angle is

positive or negative. To ensure the correct angle and sign are used, refer to Figure 12.7 and take the following steps:

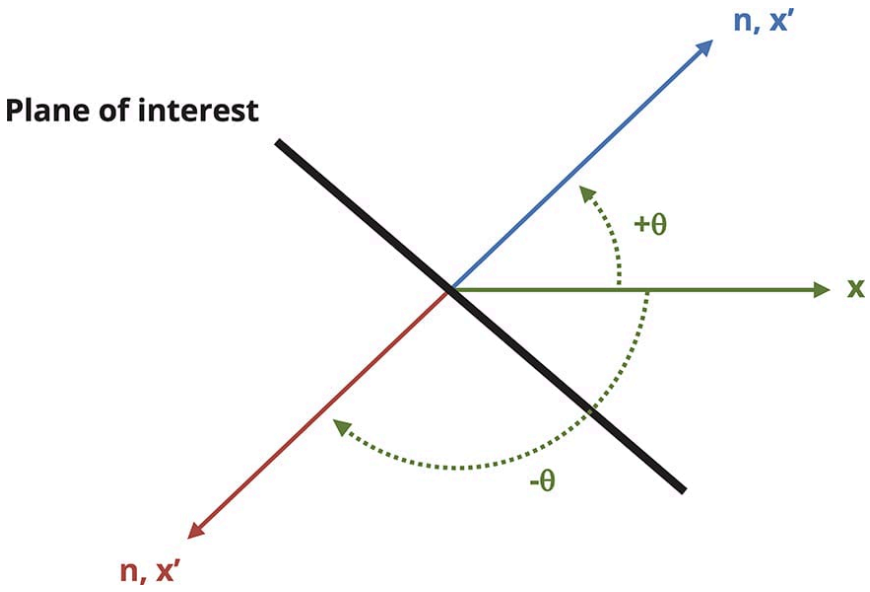


Figure 12.7: Draw a sketch of the x-axis and the normal axis to ensure you use the correct magnitude and sign for θ in the stress transformation equations.

1. Draw a line that represents one side of the plane of interest.
2. Draw the x-axis (positive to the right).
3. Draw the axis normal to the plane of interest. Whether you draw the normal axis oriented counterclockwise or clockwise from the x-axis doesn't matter as long as you use the correct magnitude and sign for the corresponding θ (see step 4).
4. Determine the angle that leads from the x-axis to the normal axis. This angle is q. If the direction from the x-axis to the normal axis is counterclockwise, then θ is positive. If the direction from the x-axis to the normal axis is clockwise, then θ is negative.

12.2.3 Equation Inputs for Planes Other than x-y

In some cases the reference stress state may not be in the x-y plane, or the x-axis and the y-axis may be oriented differently from the orientation in Figure 12.5. In these cases adjustments are required to implement the transformation equations to ensure consistency of inputs with how the equations were derived.

As an example, consider the reference stress state in Figure 12.8. As given, you might question which stress to substitute for σ_x and which to substitute for σ_y. Keeping in mind that the transformation equations were derived from a coordinate system in which the horizontal axis is positive to the right and the vertical axis is positive upward, we can ensure consistency with the inputs by reorienting the reference stress element in the same way, as is also shown in Figure 12.8. We can then attribute the coordinate designated to the horizontal direction to correspond to the x terms in the transformation equations, and likewise the coordinate designated to the vertical direction to correspond to the y terms in the transformation equations. In the case of the stress state in Figure 12.8, we would substitute -150 MPa for σ_x, 90 MPa for σ_y, and 16 MPa for τ_{xy}.

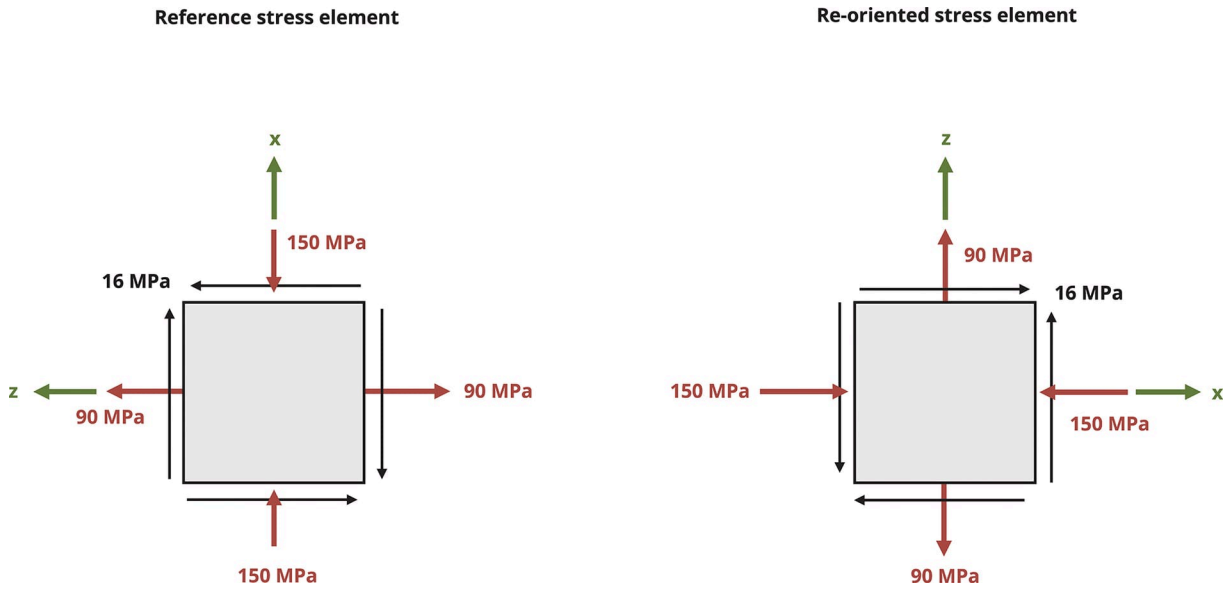


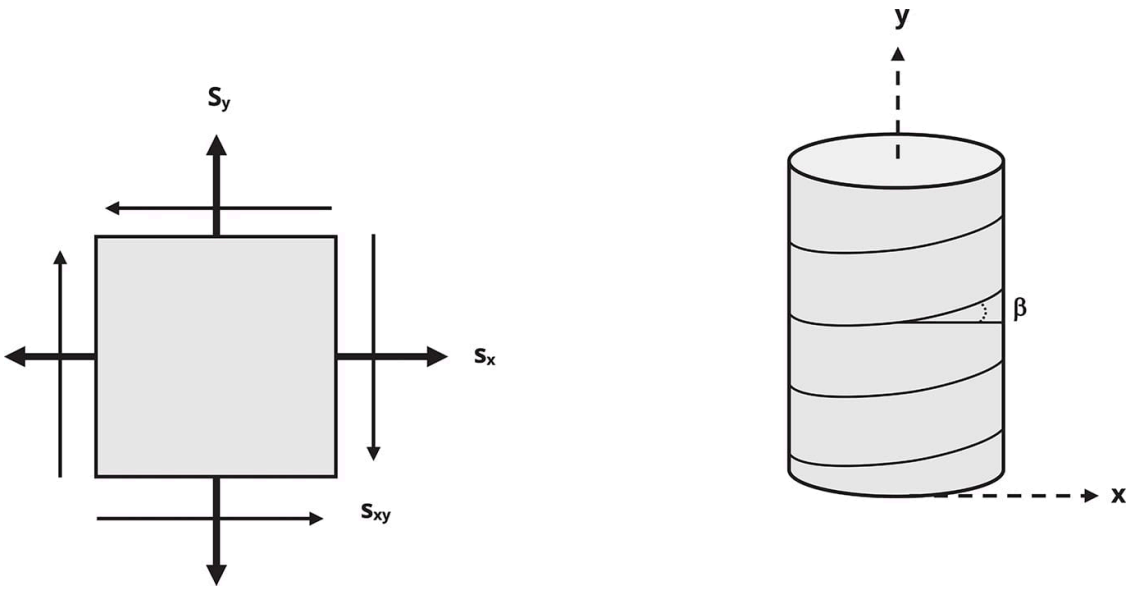
Figure 12.8: Stress element drawn in the x-z plane based on given coordinate axes and then reoriented

Example 12.1 and Example 12.2 illustrate the use of the stress transformation equations.

Example 12.1

The general stress state at a point on a hollow pipe formed with spiral welds inclined to an angle β with respect to the horizontal is shown.

For β = 42° and stress magnitudes S_x = 128 MPa, S_y = 80 MPa, and S_{xy} = 75.45 MPa, determine the stress parallel and perpendicular to the weld.



Solution

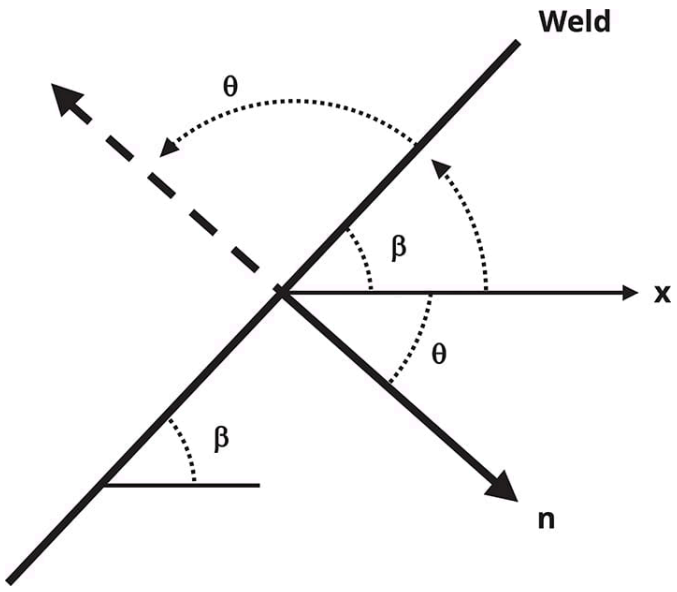
The stress parallel and perpendicular to the weld can be found by applying the stress transformation equations with σ_x = 128 MPa, σ_y = 80 MPa, and τ_{xy} = −75.45 MPa.

$$\sigma_n = \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{xy}\sin(\theta)\cos(\theta)$$
$$\tau_{nt} = -(\sigma_x - \sigma_y)\cos(\theta)\sin(\theta) + \tau_{xy}[redacted](\theta) - \sin^2(\theta)$$

Note that the normal stresses are positive since they are both shown to be tensile on the given stress element but that the shear stress is negative because the shear stress arrow points in the negative y direction on the positive x-face. (The same conclusion can be drawn by observing any of the other faces.)

To find the angle θ to substitute into the equations, follow the process described in Section 12.2.2:

1. Draw the line that represents the plane of interest. In this case the plane of interest is the plane of the weld that is parallel to the spiral.
2. Draw the x-axis leading positive to the right.
3. Draw the axis normal (n) to the plane of interest.
4. Determine the angle that leads from the x-axis to the normal axis. This angle is q.
    - If we use the angle from the x-axis to the solid normal axis, the angle will be negative since that is a clockwise rotation: θ = –(90 – β) = –48°.
    - We could also base θ on the dashed normal axis, in which case the magnitude of θ would be θ = 180° – 48° = 132°. This angle is positive since the rotation from the x-axis to the dashed normal axis is counterclockwise.



Calculate.

$$\sigma_n = 128\ MPa \cos^2(-48^\circ) + 80\ MPa \sin^2(-48^\circ) + 2 * -75.45\ MPa \sin(-48^\circ)\cos(-48^\circ)$$
$$\sigma_n = 176.5\ MPa$$
$$\tau_{nt} = -(128\ MPa - 80\ MPa)\cos(-48^\circ)\sin(-48^\circ) - 75.45\ MPa\,(\cos^2(-48^\circ) - \sin^2(-48^\circ))$$
$$\tau_{n_t} = 31.76\ MPa$$

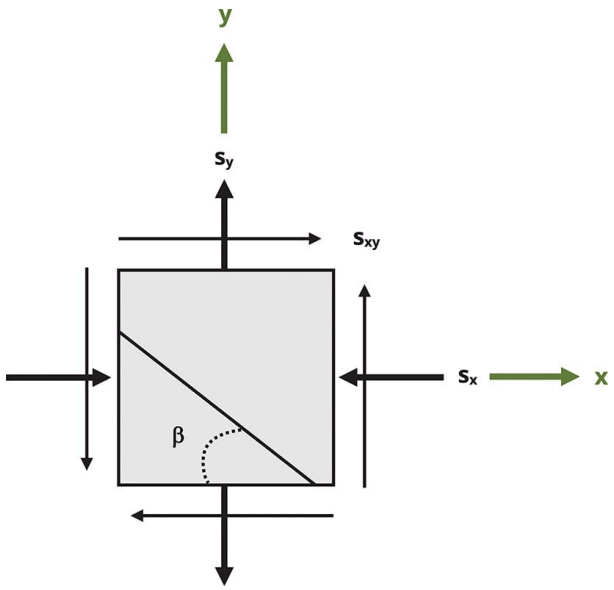
Answer:

With σ_x = 128 MPa, σ_y = 80 MPa, τ_{xy} = –75.45 MPa, and θ = –48° (or +132°), the stress normal to the weld is σ_n = 176.5 MPa. The stress parallel to the weld is τ_{nt} = 31.76 MPa.

Example 12.2

Stresses at a point on a body are calculated and represented on the stress element given below. Normal and shear stress magnitudes acting on horizontal and vertical planes at the point are S_x = 1,700 psi, S_y = 800 psi, and S_{xy} = 790 psi.

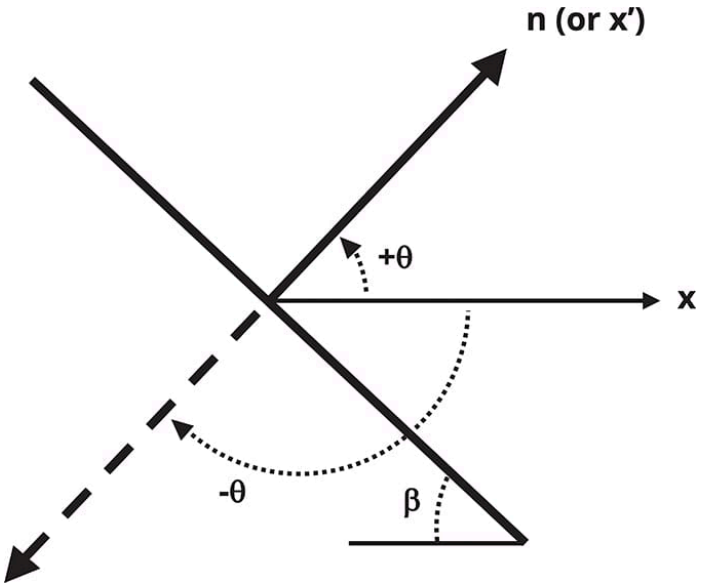
Determine the normal and shear stresses at this point for the plane rotated β = 36° from the horizontal axis as shown. Represent the stress state for the rotated plane on a stress element.



Solution

First determine the angle θ needed for the stress transformation equations. Recall the process described in Section 12.2.2:

1. Draw a line that represents the plane of interest.
2. Draw the x-axis positive going toward the right.
3. Draw the axis normal (n) to the plane of interest.
4. Determine the angle leading from the x-axis to the normal axis. This angle is θ.
    - Either angle labeled θ in the illustration will work and will produce the same results.
    - The angle rotated counterclockwise from the x-axis is θ = 90 – β = 54°.
    - The angel rotated clockwise from the x-axis is θ = –(90 + β) = –126°.



With θ determined, the only thing left to do is make the substitutions into the stress transformation equations and calculate. Note that the σ_n equation will give us the normal stress on the rotated x-face ($\sigma_{x'}$). We will also need to calculate σ_n with the angle (θ + 90°) to get $\sigma_{y'}$ or use the stress invariance concept: ($\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$).

Substituting σ_x = –1,700 psi (negative since it is shown to be compressive), σ_y = 800 psi, τ_{xy} = 790 psi, and θ = 54° into the stress transformation equations

$$\begin{aligned}\sigma_n &= \tfrac{\sigma_x+\sigma_y}{2} + \tfrac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)\\ &= -\tfrac{-1{,}700\,psi+800\,psi}{2} + \tfrac{-1{,}700\,psi-800\,psi}{2}\cos(2*54^\circ) + 790\,psi*\sin(2*54^\circ)\\ &= 687.6\,psi\\ \tau_{nt} &= -\tfrac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)\\ &= -\tfrac{-1{,}700\,psi-800\,psi}{2}\sin(2*54^\circ) + 790\,psi*\cos(2*54^\circ)\\ &= 944.7psi\end{aligned}$$

yields $\sigma_{x'}$ = 687.6 psi and $\tau_{x'y'}$ = 945 psi.

Applying stress invariance –1,700 psi + 800 psi = 687.6 psi + $\sigma_{y'}$ yields $s_{y'}$ = –1,588 psi.

Answer:

σx' = 687.6 psi

τx'y' = 945 psi

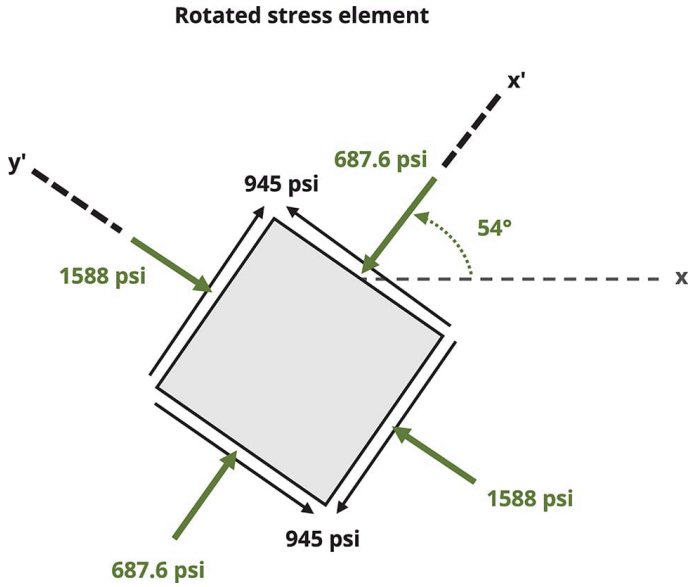
σy' = –1,588 psi

Stress element for rotated plane:

The 687.5 psi stress goes on the face rotated 54° from the horizontal because this was the value obtained by substituting that angle into the stress transformation equation. It was determined to be tensile (positive) so it points away from the x' face.

The 1,588 psi stress is compressive (negative), so it points toward the y' face.

The shear stress transformation equation yielded a positive result, indicating a positive shear stress so the arrows are arranged on the stress element accordingly.



Step-by-Step: Stress Transformation

Given a general stress state (σ_x, σ_y, and τ_{xy}) at a point in the x-y plane* where x is the horizontal axis positive to the right and y is the vertical axis positive upward:

1. Establish the angle θ that corresponds to the angle of rotation between the positive x-axis and the axis normal (x') to the plane of interest. Remember to make the angle negative if the rotation is clockwise from the x-axis to the normal axis and positive if the rotation is counterclockwise.

2. Substitute the stresses and the angle θ with the appropriate signs into the stress transformation equations to find σ_n and τ_{nt}. Recall that normal stresses are negative if compressive and positive if tensile and that shear stresses are positive if the shear stress arrow on the positive x-face points in the positive y direction and negative otherwise. This will result in $\sigma_{x'}$ and $\tau_{x'y'}$.

3. Use (θ + 90°) in the σ_n transformation equation or apply the concept of stress invariance ($\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$) to find $\sigma_{y'}$.

*If the general stress state is given in some other plane, reorient (rotate) the stress element so that one of the axes is positive to the right (this axis will correspond to the x-axis in the equations) and the other axis is positive upwards (this axis will correspond to the y-axis in the equations).

12.3 Principal Stresses and Max/Min Shear Stress

Often the plane we are the most concerned with is the one where the normal stress or the shear stress is at a maximum because it is where failure is most likely to occur. The maximum and minimum normal stresses are referred to as the principal stresses (σ_1 and σ_2), and the planes on which they act are the principal planes. The principal planes are described by angles of rotation, θ_{p1} and θ_{p2}, from the reference plane for which the general stress state as obtained. The maximum and minimum shear stresses are simply referred to as the maximum and minimum shear stress respectively, and the planes on which they occur are referred to simply as the planes of maximum and minimum shear stress (denoted as θ_{s1} and θ_{s2}).

Important to note here is that only rotations within a two-dimensional plane are considered in this section. As discussed in Section 12.5, the principal planes and principal stresses are the same even when we consider out-of-plane rotation of the stress element (see Figure 12.9), but the maximum shear stress could be different. Therefore the maximum and minimum sheer stress as determined by the derived equations below will be referred to as the max/min in-plane shear stress.

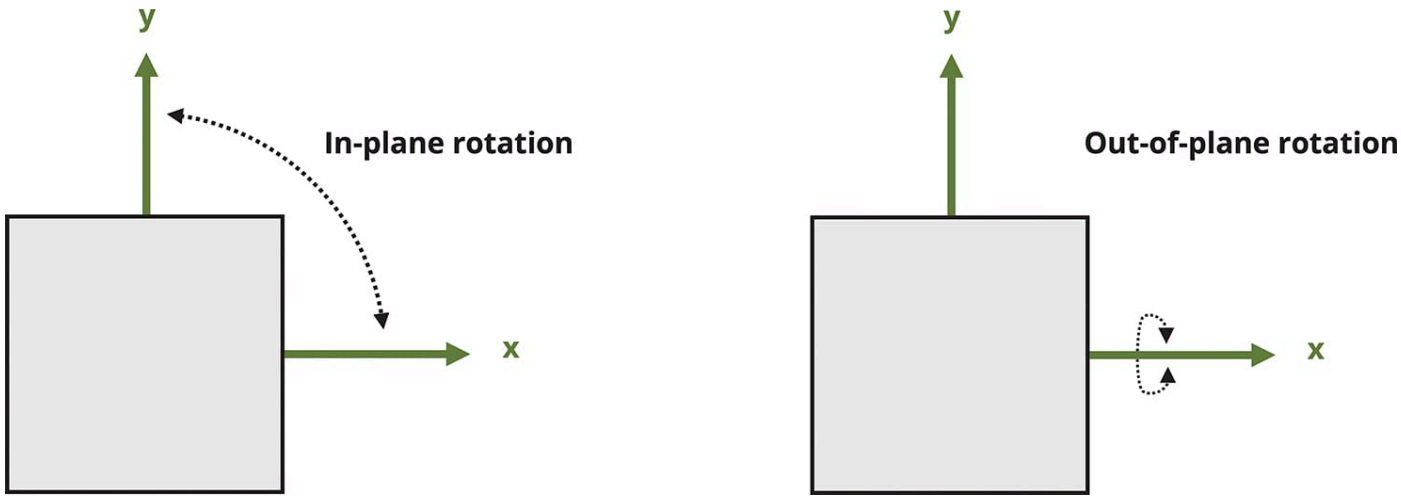


Figure 12.9: The maximum shear stress obtained from in-plane rotations of the stress element may not be the same as the maximum shear stress when considering out-of-plane rotation as well. This section describes how to find the max/min in-plane shear stress.

As we rotate the stress element, the normal stress changes in a manner predictable with the stress transformation equations. Figure 12.10 shows the variation of normal stress $\sigma_{x'}$ with angle q. A similar graph could be drawn for $\tau_{x'y'}$. Based on the equations and the plots, we know that the max and min stresses will occur at angles 90° apart from one another.

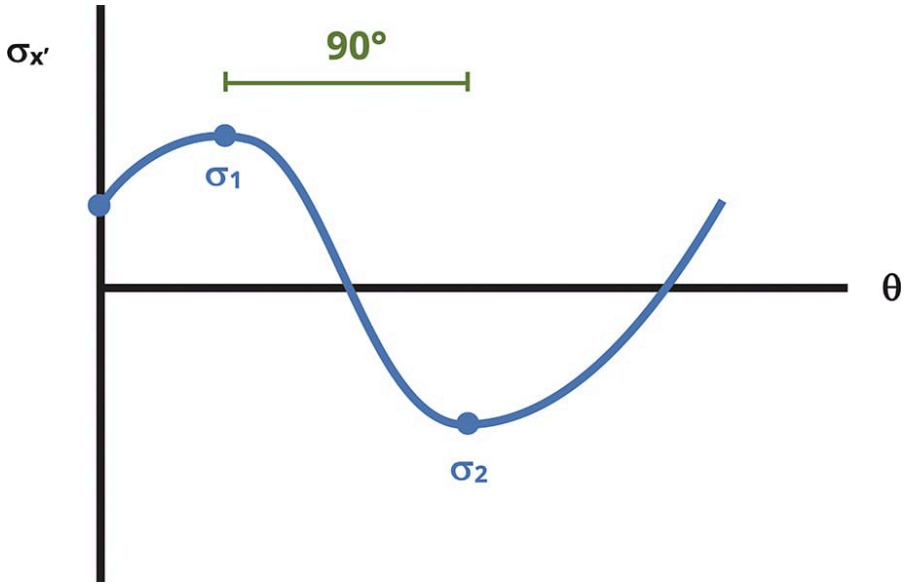


Figure 12.10: Plot showing how normal stress $\sigma_{x'}$ varies with angle θ

From calculus we know that the maximums and minimums of a function that result from varying a single variable can be found by setting the derivative of the function with respect to that variable equal to 0 and then solving the resulting equation for the variable in question. Plugging the variable back into the original equation gives the corresponding maximum or minimum value of the function. This is also equivalent to finding where the slope of the function is equal to 0. Thus, to find the angles for which the normal stress and in-plane shear stresses are at extremes, we can simply set the derivatives of the transformation equations (Equation 12.1 or Equation 12.2) with respect to θ equal to 0, and solve for the θ values.

Using Equation 12.2 (though Equation 12.1 would yield the same results)

$$\frac{d\sigma_n}{d\theta} = -(\sigma_x - \sigma_y)\sin(2\theta_p) + 2\tau_{xy}\cos(2\theta_p) = 0$$
$$\frac{d\tau_{nt}}{d\theta} = -(\sigma_x - \sigma_y)\cos(2\theta_s) - 2\tau_{xy}\sin(2\theta_s) = 0.$$

Note that θ is subscripted with a p in the first equation because this angle represents a principal plane as defined. Similarly, θ is subscripted with an S in the second equation because this angle represents a plane of maximum (max) or minimum (min) in-plane shear stress.

These equations can be rearranged to isolate tan(2θ) on one side and subsequently used to find θ_p and θ_s values.

$$\boxed{\begin{aligned}\tan(2\theta_p) &= \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \\ \tan(2\theta_s) &= -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}\end{aligned}},$$

(12.4)

where

- θ_p = *Orientation of a principal plane with respect to the x-axis [°, rad]*
- θ_s = *Orientation of the maximum or minimum in-plane shear plane with respect to the x-axis [°, rad]*
- τ_{xy} = *Average shear stress in the x-y plane before any transformation [Pa, psi]*
- σ_x = *Average normal stress in the x direction before any transformation [Pa, psi]*
- σ_y = *Average normal stress in the y direction before any transformation [Pa, psi]*

Once the angles associated with the planes of extreme stresses are determined, they can be substituted into Equation 12.1 or Equation 12.2 along with the stresses in the given plane to find the principal stresses and max/min in-plane shear stress.

Here are a few calculation details that are helpful to remember:

1. The inverse tangent function has two possible solutions. One solution is the angle that will be reported by a calculator, and the other possible answer is that angle plus or minus 180°. So if we take the inverse tangent of the expressions of Equation 12.4 and then divide that result by 2, we get one of the solution angles. Then adding or subtracting 90° to that angle gives the second. This applies for the θ_p and the θ_s values.

2. By convention the most positive of the two principal stresses is denoted as σ_1. The other principal stress is then σ_2. The angle that produces that σ_1 stress is θ_{p1}, and the angle that produces σ_2 is θ_{p2}.

3. The max/min in-plane shear stress will always turn out to be equal in magnitude but opposite in sign—in other words, the minimum shear stress is always the negative of the maximum shear stress. As with the principal planes and stresses, substitute one of the θ_s angles into the shear stress transformation equation to determine which angle corresponds to the maximum in-plane shear stress and which corresponds to the minimum.

4. The angles of max/min shear stress are 90° apart from each other and 45° apart from the principal directions. For example, if one principal direction is found to be 40°, then the other principal direction can be determined as either 40° + 90° = 130° or 40° − 90° = −50°. The angles of max/min in-plane shear stress can then be determined as either 40° + 45° = 95° or 40° − 45° = −5°. It is also acceptable to add or subtract 45° from 130° or from −50°.

5. Given the variety of possible correct angles, here we use the convention that principal directions and angles of max/min shear stress are reported as the ones in the range −90° ≤ θ ≤ +90°.

If desired, the principal stresses and max/min shear stress can be calculated directly without first finding the principal directions. The following equations arise from some equation manipulation which will not be detailed here, but can be used generally. Adding the term under the square root will give either the max or min stress and subtracting will give the other.

$$\boxed{\begin{array}{c}\sigma_{1,2}=\frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+{\tau_{xy}}^2}\\ \tau_{x'y'}=\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+{\tau_{xy}}^2}\end{array}},$$

(12.5)

where

- $\sigma_{1,2}$ = *Principal stresses [Pa, psi]*
- σ_x = *Average normal stress in the x direction before any transformation [Pa, psi]*
- σ_y = *Average normal stress in the y direction before any transformation [Pa, psi]*
- τ_{xy} = *Average shear stress in the x-y plane before any transformation [Pa, psi]*

Some useful conclusions that can be drawn simply by studying the principal plane and plane of max/min shear stress equations are as follows:

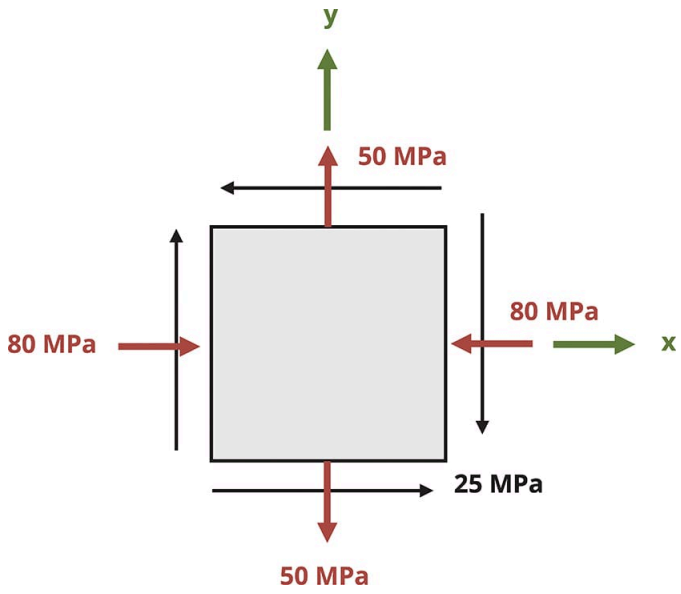
1. Setting the shear stress transformation equation given in Equation 12.1 or Equation 12.2 equal to 0 results in the same equation for tan(2θ) as the tan(2θ_p) equation in Equation 12.4. We can then conclude that *shear stress is 0 on the principal planes*. It can also be said that if the shear stress is found to be 0 on any given plane, then that plane must be a principal plane.
2. The equation for tan(2θ_s) is the negative inverse of the equation for tan(2θ_p). This means that 2θ_s = 2θ_p ± 90° or θ_s = θ_p ± 45°. Recognizing this can enable us to calculate one set of angles more efficiently once the other is found.
3. When either of the θ_s angles is substituted into the normal stress transformation equation, the normal stress will always work out to be the average stress $\left(\frac{\sigma_x+\sigma_y}{2}\right)$. Once again this recognition can help make determining the total stress state on the plane of in-plane max/min shear stress more efficient.

Example 12.3 demonstrates the calculation of the principal stresses, the maximum in-plane shear stress, and the planes where these stresses occur.

Example 12.3

The general stress state at a point in the x-y plane is represented by the stress element shown.

Determine the principal stresses and the maximum in-plane shear stress. Then draw the stress element rotated to one of the principal planes and the plane of max or min in-plane shear stress.



Solution

1. Find the principal planes by substituting σ_x = −80 MPa, σ_y = 50 MPa, and τ_{xy} = −25 MPa into Equation 12.4.

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$
$$= \frac{2*-25MPa}{(-80MPa-50MPa)}$$
$$= 0.3846$$
$$2\theta_p = \tan^{-1}(0.3846) = 21.038°$$
$$\theta_p = 10.52°$$

So one of the principal angles is θ_p = 10.52°.

The other angle is then θ_p = 10.52° + 90° = 100.52°.

Thus the principal plane angles are 10.52° and 100.52°.

2. Substituting θ_p = 10.52° into Equation 12.2 yields

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$
$$\sigma_{n_1} = \frac{-80\ MPa+50\ MPa}{2} + \frac{-80\ MPa-50\ MPa}{2}\cos(2*10.52°) - 25\ MPa*\sin(2*10.52°)$$
$$\sigma_{n_1} = -84.65\ MPa.$$

Substituting θ_p = 100.52° into the same equation yields

$$\sigma_{n_2} = \frac{-80\ MPa+50\ MPa}{2} + \frac{-80\ MPa-50\ MPa}{2}\cos(2*100.52°) - 25\ MPa*\sin(2*100.52°)$$
$$\sigma_{n_2} = 54.65*MPa.$$

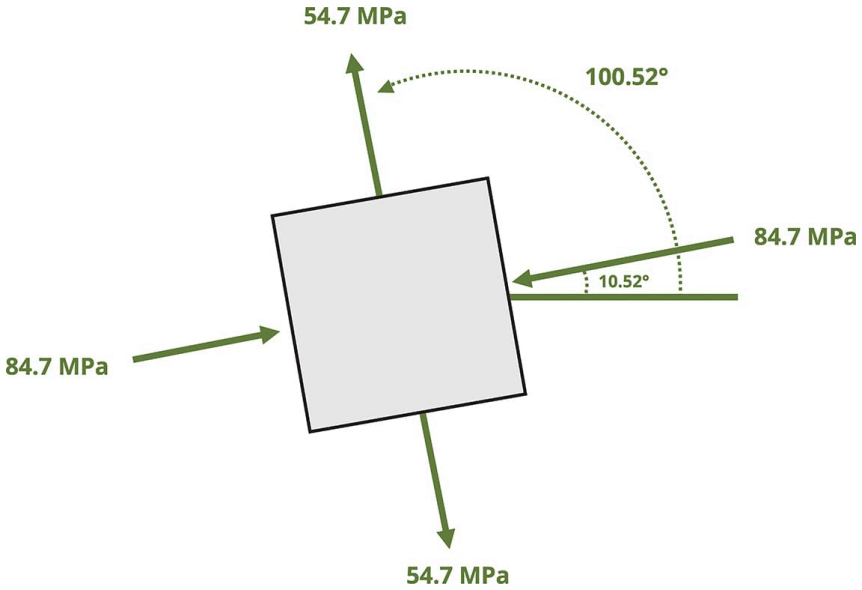
Since 54.7 MPa is the more positive of the two principal stresses, σ_1 = 54.7 MPa and σ_2 = −84.7 MPa.

Given these results, we know that rotating the stress element counterclockwise by 10.52° results in σ_2 = −84.7 MPa on the x'-face and σ_1 = 54.7 MPa on the y'-face.

Conversely, if the stress element is rotated counterclockwise by 100.52°, the stress on the x'-face would be 54.7 MPa and the stress on the y' face would be −84.7 MPa.

The element rotated to the principal planes is shown below.

Note that the stress element looks the same no matter which angle is used for the rotation. Recall the discussion at the end of Section 12.3 that the shear stress on the principal plane is 0.



3. Next find the planes of max/min in-plane shear planes as well as the corresponding shear stresses. Note that we could use

$$\tan(2\theta_s) = -\frac{(\sigma_x - \sigma_y)}{2\tau_{xy}},$$

but since we already have the θ_p angles, we can also more easily use

$$\theta_S = \theta_p \pm 45^\circ.$$

Choose either θ_p angle to add and subtract the 45°. The final orientation of the element would be the same in either case. Since visualizing angle rotations smaller in magnitude than 90° is generally easier than working with larger magnitudes, use the smaller θ_p value.

So θ_s = 10.52° ± 45° = 55.5° and –34.5°.

4. Substituting θ_s = 55.5° into

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\theta)\Big) + \tau_{xy}\cos(2\theta)\Big)$$

yields

$$\tau_{nt} = -\frac{-80\ MPa - 50\ MPa}{2}\sin(2 * 55.5^\circ) - 25\ MPa\cos(2 * 55.5^\circ)$$
$$\tau_{nt} = 69.6\ MPa.$$

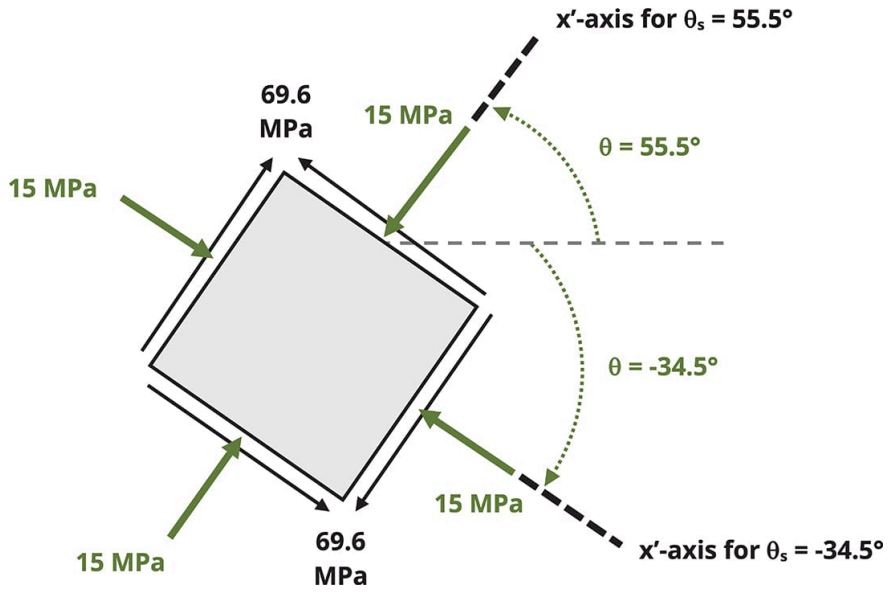
Since this is a positive value, this is the maximum in-plane shear stress and so θ_s = –34.5° should yield τ_{nt} = –69.6 MPa = τ_{min} (in-plane).

Rotating the stress element counterclockwise by 55.5° will result in a positive shear stress, and rotating clockwise 34.5° will result in a negative shear stress.

5. The normal stress on the plane of max/min in-plane shear stress is

$$\sigma = \sigma_{avg} = \frac{(\sigma_x + \sigma_y)}{2}$$
$$\sigma_{avg} = \frac{-80\ MPa + 50\ MPa}{2}$$
$$\sigma_{avg} = -15\ MPa.$$

Draw the stress rotated to the planes of max/min shear stress.



Once again notice that the stress element looks fundamentally the same whichever θ_s angle is used.

Answer:

- θ_{p1} = 100.52° and θ_{p2} = 10.52°
- σ_1 = 54.7 MPa and σ_2 = −84.7 MPa
- θ_s = 55.5° and −34.5°
- τ_{max} (in-plane) = 69.6 MPa
- τ_{min} (in-plane) = −69.6 MPa

Elements are drawn above.

Step-by-Step: Principal Stresses and Max/Min Shear Stress

Given a general stress state (σ_x, σ_y, and τ_{xy}) at a point in the x-y plane where x is the horizontal axis positive to the right and y is the vertical axis positive upward, do the following:

1. Use the tan($2\theta_p$) equation to find the planes of principal stresses. One of the principal angles is found using the inverse tan function on the calculator. The other angle will be 90° from that one.
2. Substitute the two θ_p angles into the σ_n transformation equation to determine the principal stresses. The most positive of the two is deemed σ_1 and the other is σ_2. Alternatively, substitute only one angle into the stress transformation to find one of the principal stresses. The other principal stress can be calculated using stress invariance.
3. Use the tan($2\theta_s$) equation to find the planes of max/min in-plane shear stress. Alternatively, add or subtract 45° from one of the principal plane angles.
4. Substitute one of the θ_s angles into the τ_{nt} equation to determine whether it leads to the maximum in-plane shear stress or the minimum in-plane shear stress. The other angle will lead to a shear stress of equal magnitude but opposite sign.
5. Calculate the average stress. This stress is the normal stress on the x'-face and the y'-face when the element is rotated to the planes of max/min in-plane shear stress.

12.4 Mohr's Circle

As an alternative to using equations to determine principal planes and stresses as well as max/min shear planes and stresses, use a graphical method known as Mohr's circle. This method acknowledges that the stress transformation equations are actually parametric equations of a circle. The perimeter of the circle consists of coordinate points that correspond to normal stress on the horizontal axis and shear stress on the vertical axis. The advantage of using Mohr's circle is that it doesn't require recalling equations and with some practice can be quick and efficient to use. It is also used more extensively in more advanced topics on material behavior.

To illustrate the construction of Mohr's circle, we'll use the reference stress state element in Figure 12.11.

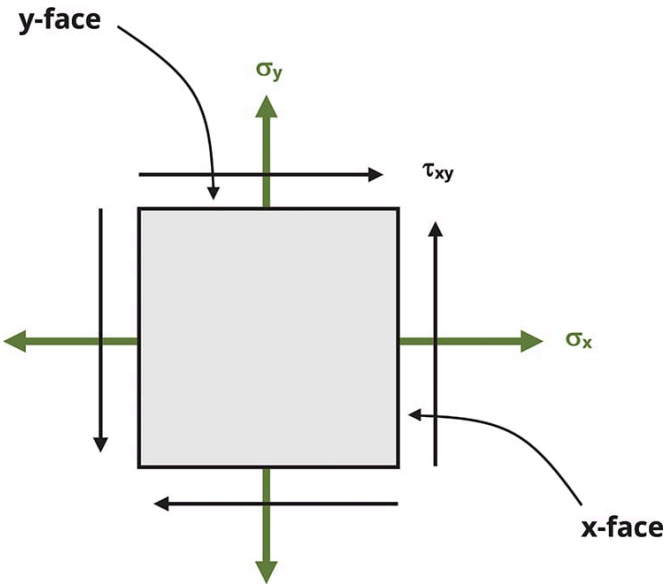


Figure 12.11: Reference stress state element

Now we can take the following steps (reference the fully constructed circle in Figure 12.16 for each step):

1. To start, draw a set of coordinate axes where normal stress will be on the horizontal axis and shear stress will be on the vertical axis. In the figures containing the circle below, shear stress is labeled with a (CW) for clockwise on the positive side and a (CCW) for counterclockwise on the negative side. The reason is explained in step 2.

2. Plot the x point.

   The x point represents the stresses applied to the x-face of the stress element, as shown in Figure 12.12, with coordinates (σ_x, τ_{xy}).

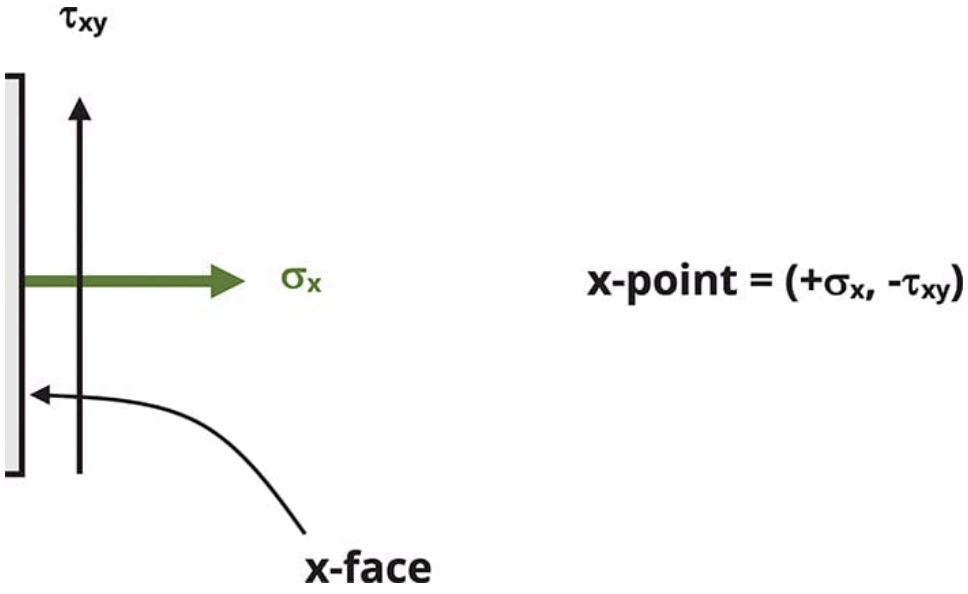


Figure 12.12: Stresses applied to the positive x-face from the x point

- The normal stress coordinate is positive if tensile and negative if compressive.
- The sign of the shear stress coordinate depends on the direction of rotation caused by the shear stress on the x-face. If the shear stress causes the element to rotate clockwise, it is a positive coordinate, and if counterclockwise, it is negative.

  In Figure 12.11 the normal stress on the x-face is tensile and the upward shear stress tends to cause the stress element to rotate counterclockwise, so the x point is $(+\sigma_x, -\tau_{xy})$. This point is shown plotted in Figure 12.14.

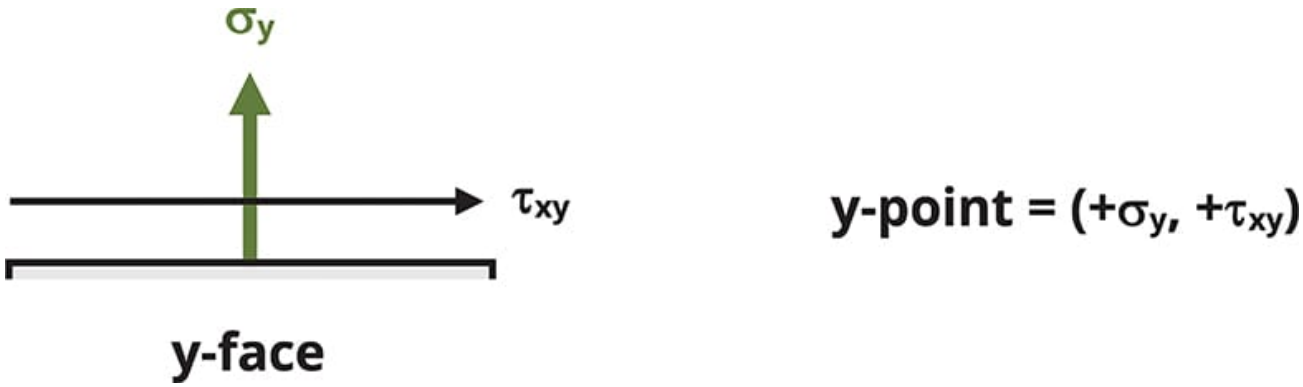


Figure 12.13: Stresses applied to positive y-face from the x-y point

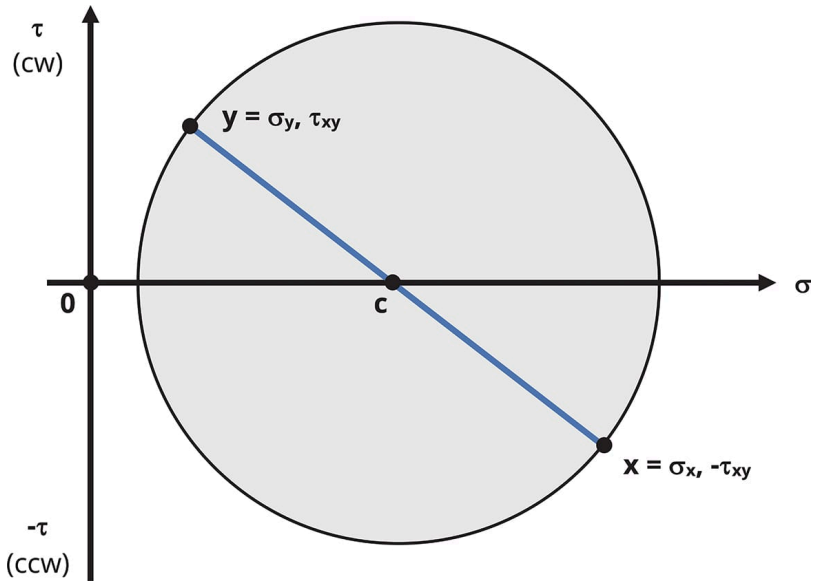


Figure 12.14: Mohr's circle with the x point, y point, and center point plotted

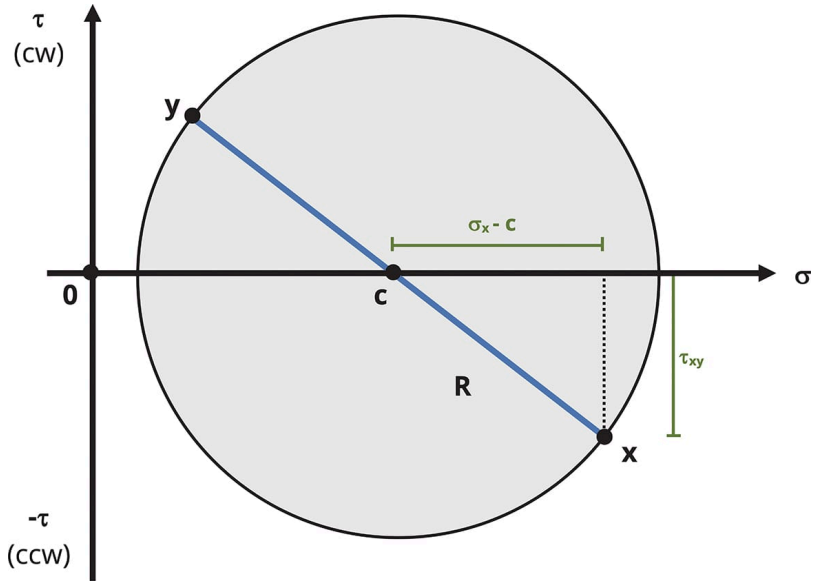


Figure 12.15: The radius of the circles can be found using the Pythagorean theorem

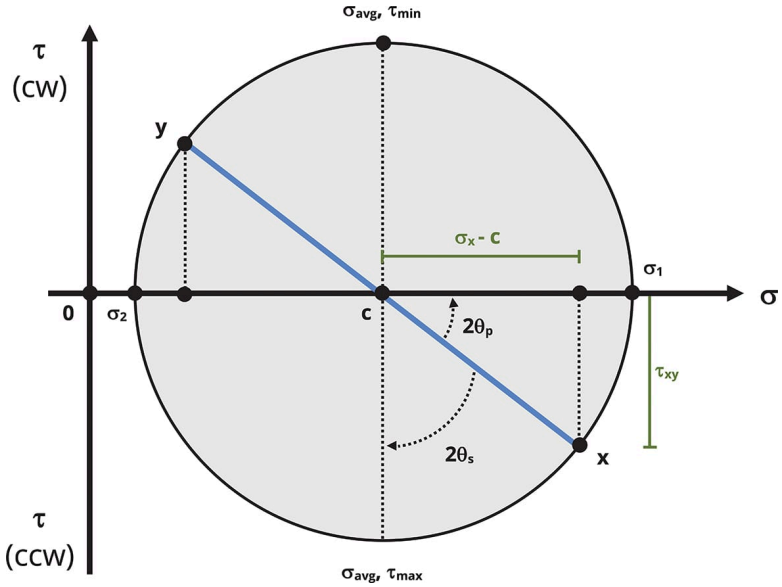


Figure 12.16: Fully constructed Mohr's circle

3. Plot the y point.

   Like the x point, the y point consists of coordinates (σ_y, τ_{xy}). The sign convention is the same as that described for the x point. As illustrated in Figure 12.13 for the stress element given in Figure 12.11, the normal stress on the y-face is tensile and the rightward shear stress tends to cause the element to rotate clockwise, so the y point is ($+\sigma_y$, $+\tau_{xy}$). This point is shown plotted in Figure 12.14.

4. Draw the circle around the diameter formed by connecting the x and y points. The center of the circle is the average normal stress and is denoted with the letter C.

$$C = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right)$$

   Note that the radial line that leads from C to the x point represents the reference x-axis and the radius from C to the y point represents the reference y-axis.

5. The radius of the circle, R, can be determined using the Pythagorean theorem on the triangle formed between C, σ_x, and the x point.

$$R = \sqrt{(\sigma_x - C)^2 + \tau_{xy}^2}$$

6. Recalling that the shear stress is zero on the principal plane, note that the principal plane is simply the normal stress axis. Thus we can identify σ_1 to be the point on the right extreme of the circle and σ_2 to be the point on the left extreme of the circle.

$$\sigma_1 = C + R$$
$$\sigma_2 = C - R$$

   These points can be found on Figure 12.16.

7. The max/min in-plane shear stress is at the bottom and top of the circle respectively.

$$\tau_{max} = +R$$
$$\tau_{min} = -R$$

   The maximum shear stress is at the bottom in accordance with the shear stress plotting convention discussed in steps 2 and

3.

The vertical diameter that connects τ_{max} and τ_{min} is the plane of max/min in-plane shear stress.

Since that plane passes through the center of the circle, we can confirm that the normal stress on the plane of max/min shear stress is $C = \sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$.

These points can be found on Figure 12.16.

8. Rotations around the circle represent twice the actual amount of rotation between planes. Recall that rotations counterclockwise from the x point are positive angles and rotations clockwise from the x point are negative angles.

   So the angle between the principal plane and the x point on the circle is $2\theta_p$. Determine this angle by applying right-triangle trigonometry principles to the right triangle used to find the radius.

$$\tan(2\theta_p) = \frac{\tau_{xy}}{\sigma_x - C}$$

   Figure 12.16 shows that in this case the $2\theta_p$ angle has a counterclockwise rotation from the reference x-axis to the principal axis. Therefore if we were to draw the stress element rotated counterclockwise to θ_p, σ_1 would be drawn on the x'-face and σ_2 would be drawn on the y'-face.

9. The angle to the plane of max/min in-plane shear stress, θ_s, can be found in a similar manner. Visual inspection shows that $2\theta_s$ will be the complement of $2\theta_p$. Recalling the convention used to apply a sign to the shear stress coordinate, note that the angle from the reference x-axis to the negative side of the shear stress axis will correspond to the rotation that results in the maximum (positive) in-plane shear stress. The angle from the reference x-axis to the positive side of the shear stress axis will correspond to the angle that results in the minimum (negative) in-plane shear stress. The maximum and minimum in-plane shear stresses will have the same magnitude but opposite sign.

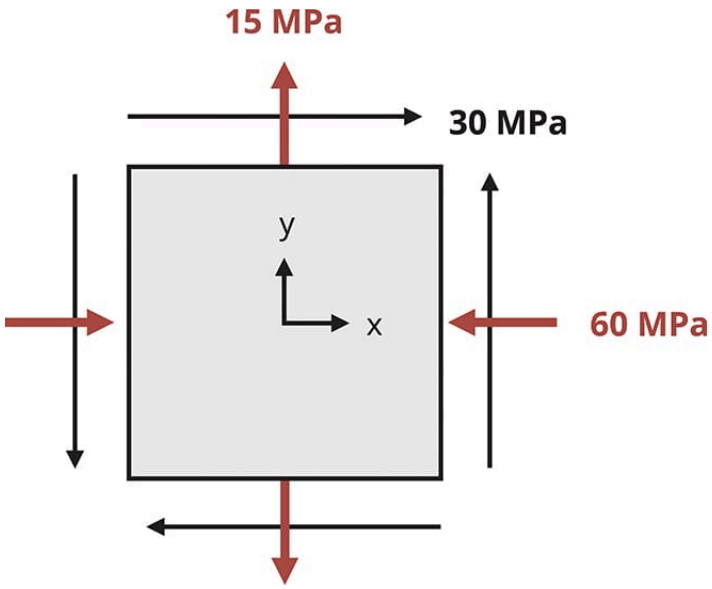
   Once these quantities are known, the use of the circle to perform stress transformations at arbitrary planes requires making similar trigonometric calculations.

Example 12.4 demonstrates the use of Mohr's circle to find maximum and minimum stresses and to perform stress transformations.

Example 12.4

For the general stress state in the x-y plane represented on the stress element shown, use Mohr's circle to find the following:

1. The principal planes and principal stresses. Sketch the stress element rotated to these planes.
2. The maximum in-plane shear stress and the plane where it acts. Sketch the stress element rotated to this plane.
3. The stress state on the plane rotated clockwise 40°. Sketch the stress element rotated to this plane.



Solution

For parts a and b, follow the steps given in Section 12.4 to construct the circle.

Illustrations of a partial Mohr's circle are provided at various intermediate steps. The complete circle with values can be found at the end of step 9.

1. Draw the s and t axes.
2. Identify and plot the x point:

   Looking at the positive x-face, note that the normal stress is compressive, so the horizontal coordinate will be –60 MPa. The shear stress arrow points upward, which direction tends to rotate the element counterclockwise, so the vertical coordinate will be negative, or –30 MPa. The x point is then (–60, –30) MPa.



3. Identify and plot the y point:

   Looking at the positive y-face, note that the normal stress is tensile, so the horizontal coordinate will be +15 MPa. The shear stress arrow points to the right, which direction tends to rotate the element clockwise, so the vertical coordinate will be positive, or +30 MPa. The y point is then (15, 30) MPa.



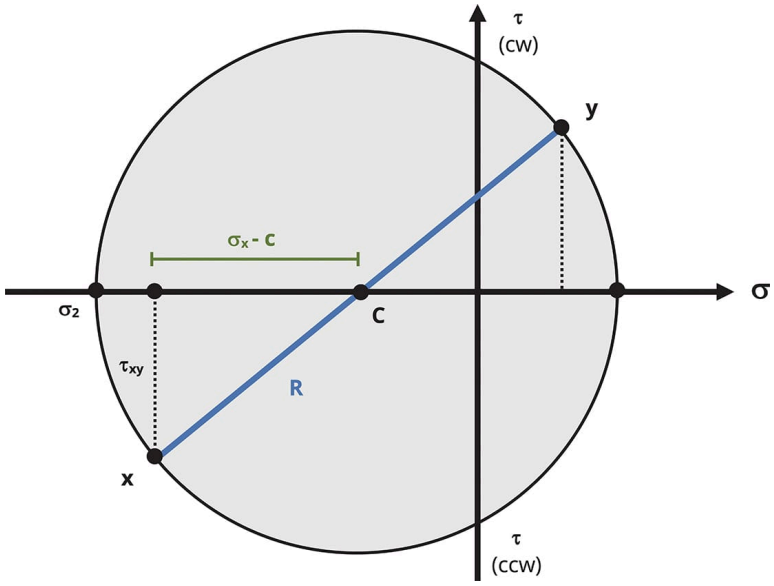
4. Connect the two points on the circle and identify the center.

$$C = \frac{\sigma_x + \sigma_y}{2}$$
$$C = \frac{-60\ MPa + 15\ MPa}{2}$$
$$C = -22.5\ MPa$$

5. Find the radius of the circle, R.

   Note that because the given loading involves negative normal stresses, we will use absolute values when calculating distances and lengths.

$$R = \sqrt{(|\sigma_x| - |C|)^2 + \tau_{xy}^2}$$
$$R = \sqrt{(60\ MPa - 22.5\ \text{MPa})^2 + (30\ MPa)^2}$$
$$R = 48.02\ MPa$$



6. Find σ_1 and σ_2.

   σ_1 is the distance from the origin to the rightmost point of the circle on the σ-axis. Since C is negative, use the absolute value.

$$\sigma_1 = R - |C|$$
$$\sigma_1 = 48.02\ MPa - 22.5\ MPa$$
$$\sigma_1 = 25.52\ MPa$$

   σ_2 is the distance from the origin to the leftmost point of the circle on the σ-axis. The distance is calculated by absolute value and the negative sign is applied to the total based on noting by inspection of the circle that σ_2 must be negative.

$$\sigma_2 = -(R + |C|)$$
$$\sigma_2 = -(48.02\ \text{MPa} + 22.5\ MPa)$$
$$\sigma_2 = -70.52\ MPa$$

7. Find τ_{max} (in-plane) and the corresponding normal stress.

   The maximum in-plane shear stress is the bottommost point of the circle, which is simply R = 48.02 MPa, or τ_{max} (in-plane) = 48.0 MPa.

   The normal stress on this plane is C = –22.5 MPa.

8. Find the principal planes.

   One of the principal planes can be found by using the triangle drawn in the third quadrant of the circle (see figure below). The angle in this triangle is twice the angle between the reference x-axis and the principal plane that corresponds to σ_2.

   Absolute values are used to calculate the triangle lengths and signs are assigned to the resulting angles based on visual inspection of clockwise versus counterclockwise rotation.

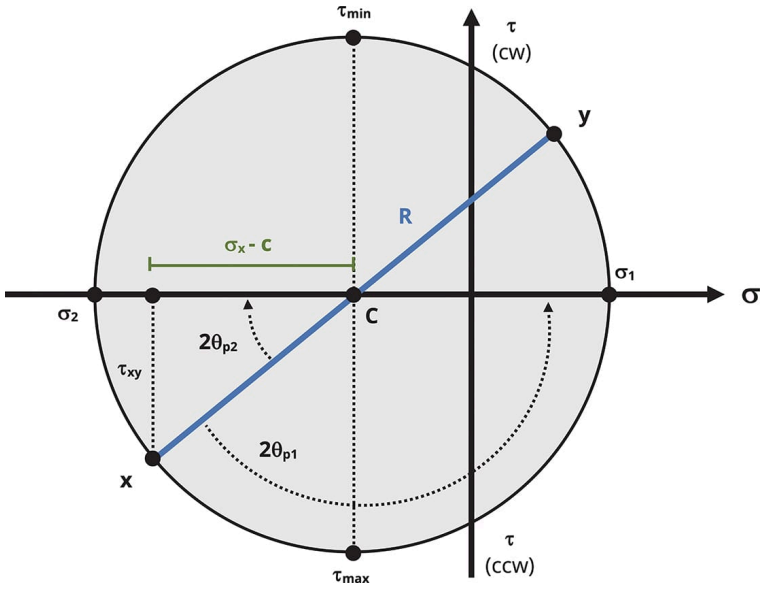
$$|\tan(2\theta_{p2})| = \frac{|\tau_{xy}|}{|\sigma_x| - |C|}$$
$$|\tan(2\theta_{p2})| = \frac{30\ MPa}{60\ MPa - (-22.5\ MPa)}$$
$$|\tan(2\theta_{p2})| = 0.8$$

$$|\theta_{p2}\,| = 19.33^\circ$$

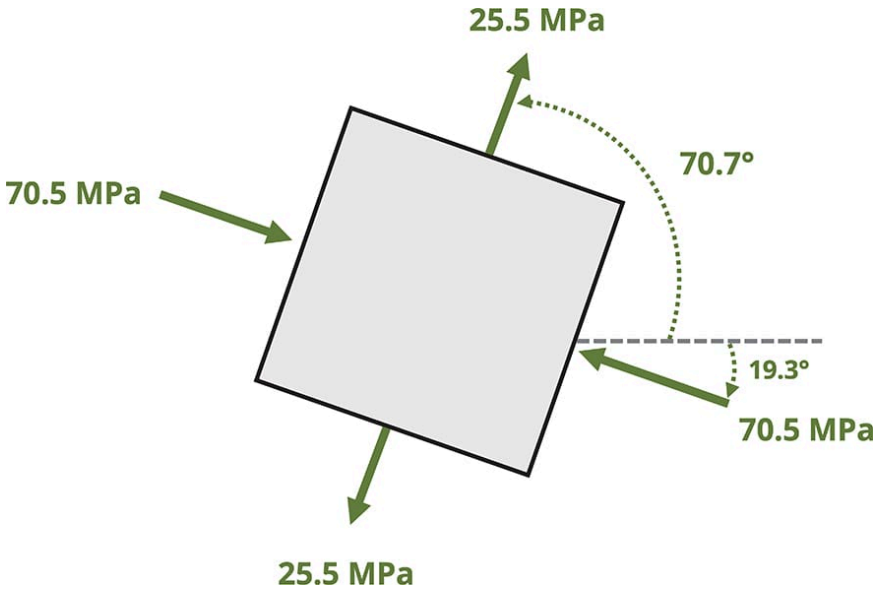
   Looking at the circle, we can see that this angle is a clockwise rotation from the reference state, so θ_{p2} = –19.33°. If we rotate the stress element clockwise 19.33°, σ_2 would be shown on the x'-face. The other principal plane (θ_{p1}) is given by half the angle that goes from the reference x point to σ_1 on the circle.

$$|2\theta_{p1}| = 180 - |2\theta_{p2}|$$
$$|\theta_{p1}| = 70.67$$

   This is the angle of rotation for which σ_1 appears on the rotated x'-face. It is a counterclockwise rotation so θ_{p1} = +70.67°.



   The stress state with the element rotated to both the principal planes is shown.

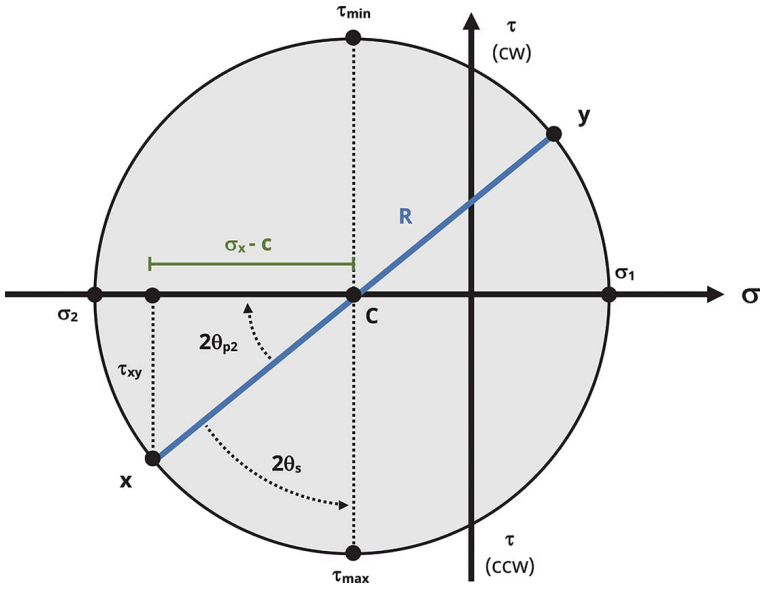


9. Find θ_s.

   With the θ_p angles known, we can see that

$$|2\theta_s| = 90 - |2\theta_{p2}|$$
$$|2\theta_s| = 90 - |2 * -19.33^\circ|$$
$$|2\theta_s| = 51.34^\circ$$
$$|\theta_s| = 25.67^\circ.$$

   Since θ_{p2} was used to calculate θ_s, we get the angle between the x point and the lower side of the shear stress axis, which is a counterclockwise rotation. The lower side of the shear stress axis corresponds to a positive overall shear stress based on the plotting convention used. We can conclude that τ_{max} occurs on the plane θ_s = +25.67°.

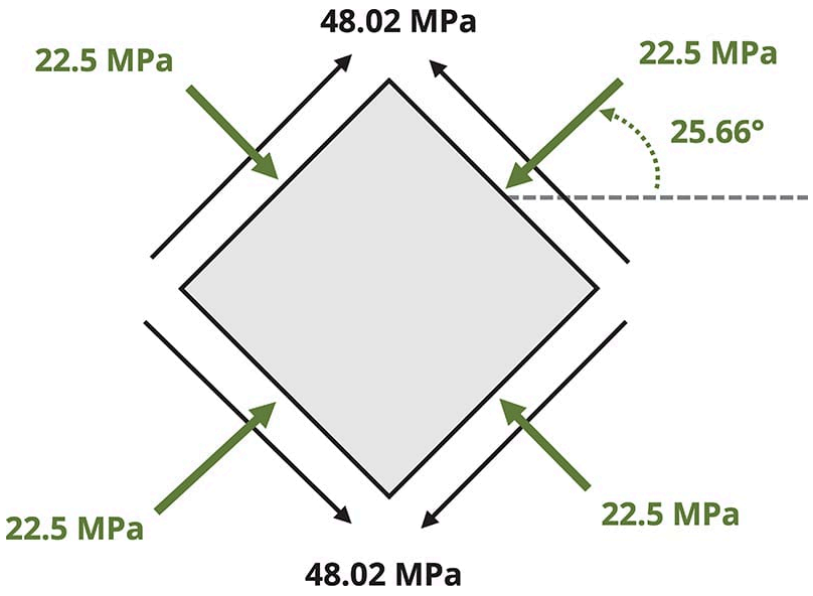


Conversely, the angle between the x point and the upper part of the shear stress axis is

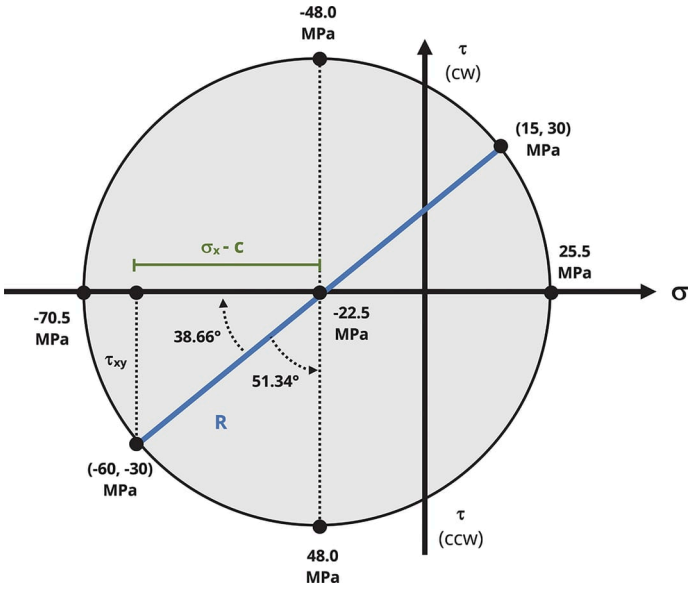
$$|2\theta_s| = 90 + |2\theta_{p2}|$$
$$|2\theta_s| = 90 + |2 * -19.33^\circ|$$
$$|2\theta_s| = 128.66^\circ.$$

This is a clockwise rotation, so τ_{min} would occur at θ_s = (–128.66°)/2= –64.33°.

The stress element rotated to the plane of max shear stress is shown.

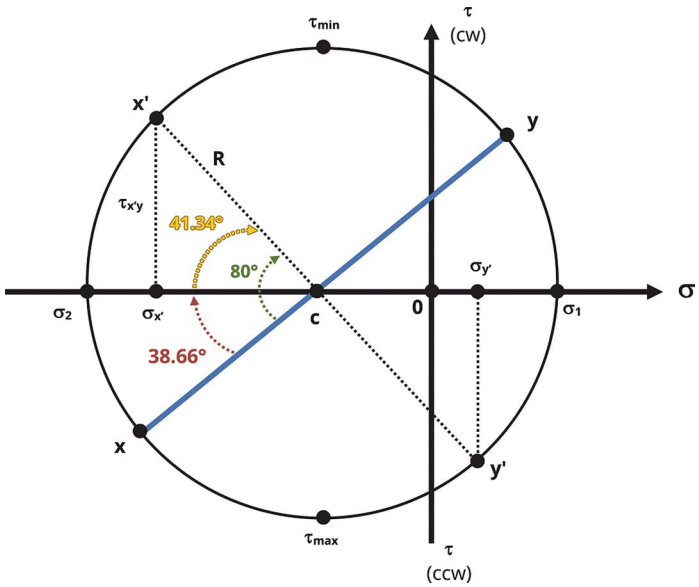


The completed Mohr's circle is as shown.



For part c, note that stress transformation is 40° clockwise.

Recalling that rotations around the circle are twice the actual rotation of the plane, draw in a point (denoted as x') that is rotated 80° clockwise from the x point. This point has coordinates ($\sigma_{x'}$, $\tau_{xy'}$).



Since we know from step 7 that the x point is 38.66° below σ_2 on the circle, we find that x' is 80° – 38.66° = 41.34° above σ_2.

The y' point is drawn diagonally opposed to the x' point, or at an 80° clockwise rotation from the reference y point.

Since the x' point has a negative s coordinate, the normal stress on the x'-face is compressive.

Using trigonometry based on the circle, normal stress on the x'-face is

$$\begin{aligned}\sigma_{x'} &= -\left(|C| + R\cos(28.66^\circ)\right)\\ \sigma_{x'} &= \left(22.5\ MPa + 48.02\ MPa\left(\cos(41.34^\circ)\right)\right)\\ \sigma_{x'} &= -58.57\ MPa.\end{aligned}$$

The normal stress on the y'-face can be similarly found.

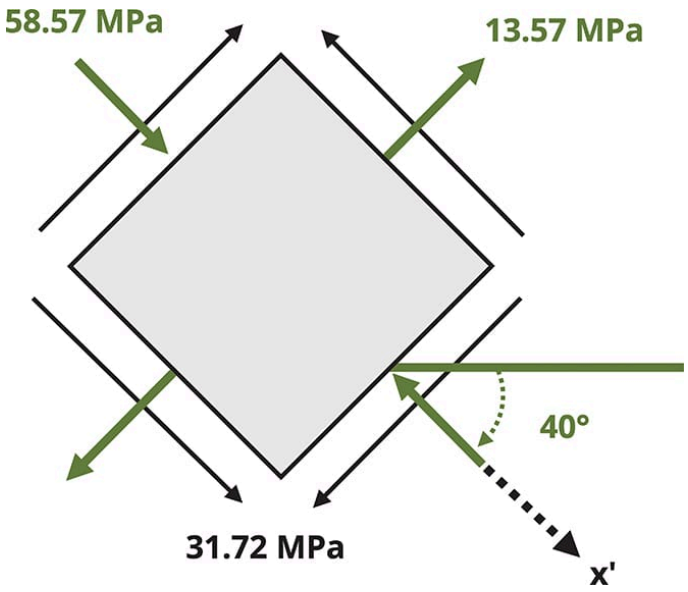
$$\begin{aligned}\sigma_{y'} &= \mathrm{R}\cos\left(41.34^\circ\right) - |\mathrm{C}|\\ \sigma_{y'} &= 48.02\ MPa\ ^*\cos\left(41.36^\circ\right) - 22.5\ MPa\\ \sigma_{y'} &= 13.57\ MPa\end{aligned}$$

Since x' has a positive τ coordinate, the shear stress arrow on the x'-face would tend to rotate the element clockwise, which corresponds to a negative shear stress.

Using trigonometry based on the circle

$$\begin{aligned}\tau_{x'y'} &= -\left(R\sin(41.34^\circ)\right)\\ \tau_{x'y'} &= \left(48.02\ MPa^*\sin(41.34^\circ)\right)\\ \tau_{x'y'} &= -31.72\ MPa.\end{aligned}$$

The rotated stress element is as shown.



Answer:

- σ_1 = 25.5 MPa on the principal plane θ_p = 70.67°
- σ_2 = -70.5 MPa on the principal plane θ_p = –19.33°
- τ_{max} (in-plane) = 48.0 MPa on the plane θ_s = 25.66°

On the plane rotated 40° clockwise from the given plane:

- $\sigma_{x'}$ = -58.6 MPa
- $\sigma_{y'}$ = 13.57 MPa
- $\tau_{x'y'}$ = -31.71 MPa

12.5 Mohr's Circle in 3D and Absolute Maximum Shear Stress

Many materials yield or fail in shear rather than normal stress, so knowing the absolute maximum shear stress the material is subjected to is important. The methods used in Section 12.3 and Section 12.4 provide the maximum in-plane shear stress that arises from rotations in the 2D plane (rotation around the z-axis in Figure 12.17). However, as discussed in Section 12.3, the actual maximum shear stress may result from an out-of-plane rotation (i.e., rotation around the x-axis or y-axis as in Figure 12.17). To account for this potential circumstance, we need to evaluate a 3D stress element despite still restricting analysis to 2D stress states.

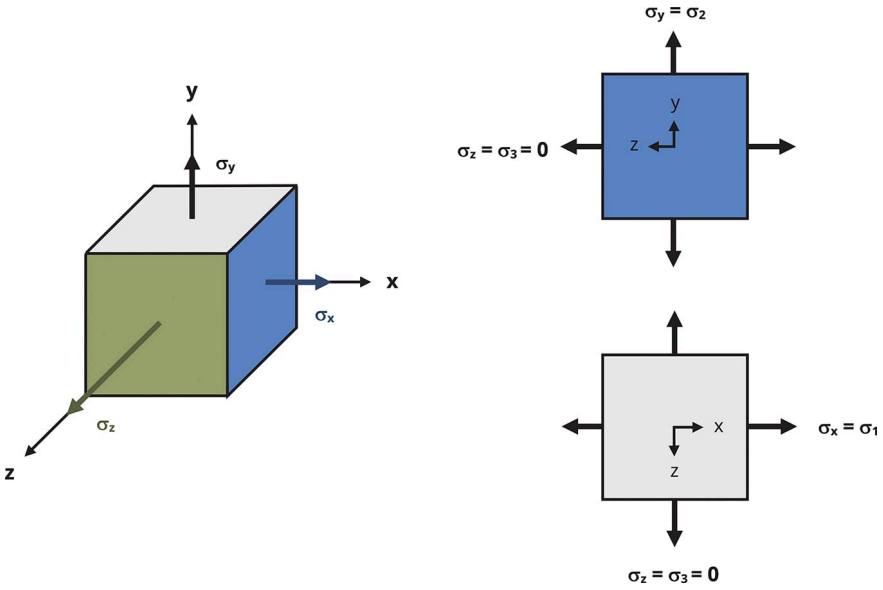


Figure 12.17: 3D stress element with stress acting in only two of the three principal directions

For the sake of this discussion we'll consider a reference stress state with stresses only in the x-y plane, though it could generally be any other 2D plane. To further simplify the discussion we'll say that the stress element is already rotated around the z-axis to the principal stress state such that $\sigma_x = \sigma_1$ and $\sigma_y = \sigma_2$. The resulting 3D stress cube and representations of the x-z and y-z planes are shown in Figure 12.17. Given the plane stress restriction, $\tau_{xz} = \tau_{yz} = 0$, which means that the x-z and y-z planes are also at principal stress states. Designating the principal stress in the z direction to be σ_3 means that the principal stresses in the x-z plane are σ_2 and σ_3 and that the principal stresses in the y-z plane are σ_1 and σ_3. For the reference stress state (plane stress), $\sigma_3 = 0$ on all planes.

As the stress cube rotates around the x-axis, σ_z varies between 0 at $\theta = 0°$ and σ_2 at $\theta = 90°$. Meanwhile σ_y varies in the opposite way and σ_x remains unchanged. Similarly, as the stress cube rotates around the y-axis, σ_z varies between 0 at $\theta = 0°$ and σ_1 at $\theta = 90°$, whereas σ_x varies in the opposite way and σ_y remains unchanged.

We can draw Mohr's circle for each of the three planes in Figure 12.17 on one plot. Just as the circle represents rotation of the stress element around the z-axis for stresses in the x-y plane, it represents rotation around the y-axis for the x-z plane and rotation around the x-axis for the y-z plane. We'll consider three cases:

1. σ_1 and σ_2 are both positive.
2. σ_1 and σ_2 are both negative.
3. σ_1 and σ_2 have opposite signs.

Case 1: σ_1 and σ_2 are both positive.

Referring to Figure 12.18, the larger inner circle is Mohr's circle for the x-y plane, as evidenced by the maximum and minimum normal stresses on that circle, σ_1 and σ_2 respectively. The smaller inner circle is Mohr's circle for the y-z plane, which has 0 and σ_2 at the extremes. The largest circle is Mohr's circle for the x-z plane, which has 0 and σ_1 at the extremes. In this case the largest shear stress is given by the radius of the largest circle, which is $\sigma_1/2$.

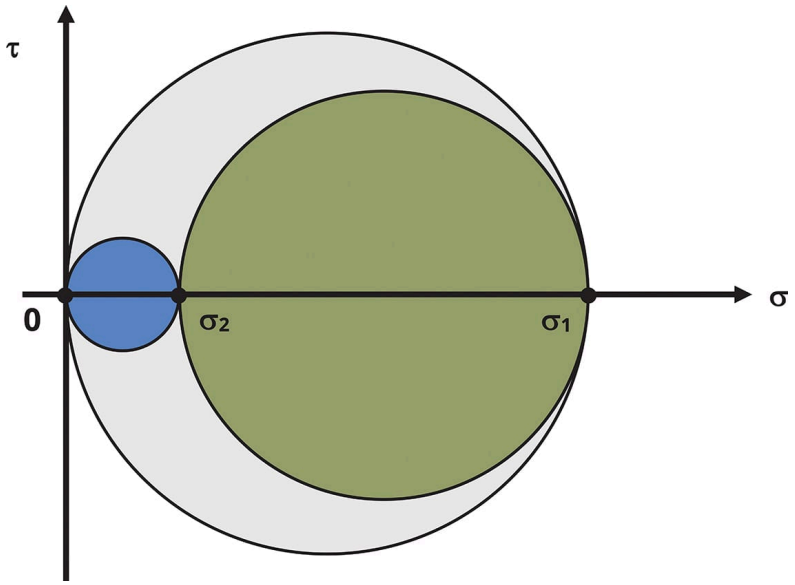


Figure 12.18: The three planes represented on Mohr's circle for the case where both in-place principal stresses in the x-y plane are positive

So in the case that σ_1 and σ_2 are both positive, the absolute maximum shear stress is

$$\tau_{(\max)absolute} = \left|\frac{\sigma_1}{2}\right|$$

The absolute value is there since we know the maximum shear stress must be positive, given that the maximum and minimum shear stresses are negatives of each other.

Case 2: σ_1 and σ_2 are both negative.

Referring to Figure 12.19, note that all the statements made in case 1 hold true here except that now the largest circle is that of the y-z plane for which the radius is $\sigma_2/2$.

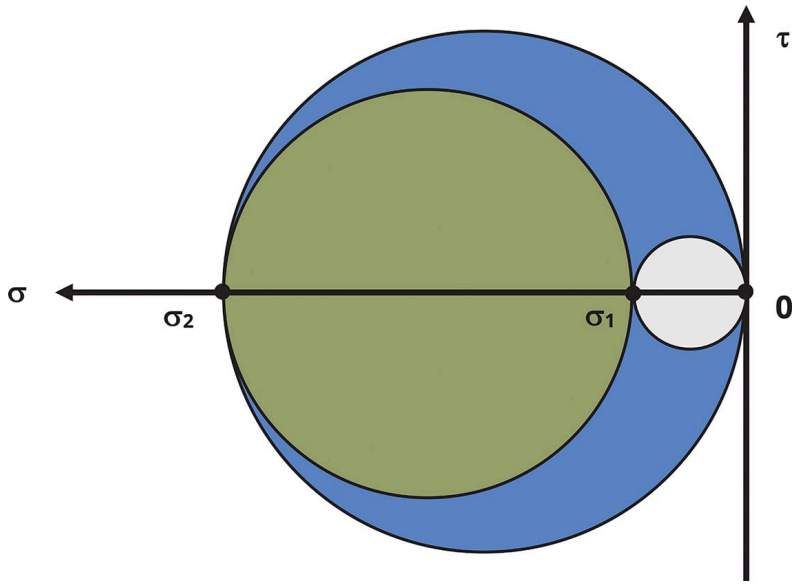


Figure 12.19: The three planes represented on Mohr's circle for the case where both principal stresses in the x-y plane are negative

So in the case that σ_1 and σ_2 are both negative, the absolute max shear stress is

$$\tau_{(\max)absolute} = \left|\frac{\sigma_2}{2}\right|$$

Case 3: σ_1 and σ_2 have opposite signs.

Referring to Figure 12.20, note that the Mohr's circle for the x-y plane is the largest circle. This means that the absolute maximum shear stress and the maximum in-plane shear stress are the same.

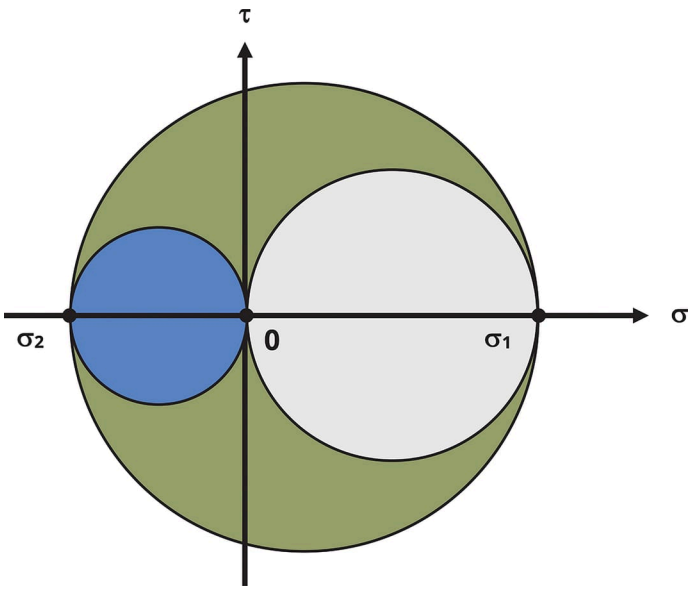


Figure 12.20: The three planes represented on Mohr's circle for the case where the principal stresses in the x-y plane are opposite signs

So in the case that σ_1 and σ_2 are opposite in sign, the absolute maximum shear stress is

$$\tau_{(\max)absolute} = \left| \frac{\sigma_1 - \sigma_2}{2} \right|.$$

More generally, given principal stresses σ_1, σ_2, and $\sigma_3 = 0$, the absolute maximum shear stress can be determined from

$$\boxed{\tau_{(\max)absolute} = \left| \frac{\sigma_{max} - \sigma_{min}}{2} \right|},$$

(12.6)

where

- $\tau_{(max)absolute}$ = *Absolute maximum shear stress [Pa, psi]*
- σ_{max} = *Maximum principal stress [Pa, psi]*
- σ_{min} = *Minimum principal stress [Pa, psi]*

This works for any of the three cases:

- Case 1: $\sigma_{max} = \sigma_1$, $\sigma_{min} = 0$.
- Case 2: $\sigma_{max} = 0$, $\sigma_{min} = \sigma_2$.
- Case 3: $\sigma_{max} = \sigma_1$, $\sigma_{min} = \sigma_2$.

Step-by-Step: Determine Absolute Maximum Shear Stress

To determine the absolute maximum shear stress, it's not necessary to draw all three circles as drawn above for illustrative purposes. Instead simply take these steps:

1. Determine the principal stresses in the x-y plane (or whichever 2D plane is being used for the general stress state).
2. Determine the absolute maximum shear stress to be $\tau_{(\max)\text{absolute}} = \left| \frac{\sigma_{\max} - \sigma_{\min}}{2} \right|$.

Summary

Key Takeaways

- Stress transformation equations can be used to take stresses calculated in a specific plane and determine the stresses on a different plane of interest.
- Mohr's circle provides a graphical method that can also be used and requires minimal need to recall equations.
- You can also use specific equations or Mohr's circle to determine the planes of max/min normal stress and shear stress and the corresponding stress values.
    - For planar (2D) loading conditions, principal stresses (max/min normal stresses) will not be affected by out-of-plane rotations, so those values can be calculated in the given plane.
    - However, to determine the absolute max/min shear stress, out-of-plane rotations must be considered. The determination of the absolute maximum or minimum shear stress can ultimately be made using the signs and values of the principal stresses and including the out-of-plane principal stress to be 0.

Key Equations

Stress transformation:

$$\sigma_n = \sigma_x \cos^2(\theta) + \sigma_y \sin^2(\theta) + 2\tau_{xy} \sin(\theta)\cos(\theta)$$
$$\tau_{nt} = -(\sigma_x - \sigma_y)\cos(\theta)\sin(\theta) + \tau_{xy}(\cos^2(\theta) - \sin^2(\theta))$$
$$\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$$

or

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$
$$\tau_{nt} = -\frac{\sigma_x-\sigma_y}{2}\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

Principal plane and stress:

$$\tan(2\theta_p) = \frac{2\tau_{xy}}{(\sigma_x-\sigma_y)}$$
$$\sigma_1, \sigma_2 = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

In-plane max/min shear planes and stresses:

$$\tan(2\theta_s) = -\frac{(\sigma_x-\sigma_y)}{2\tau_{xy}}$$
$$\theta_s = \theta_p \pm 45^\circ$$
$$\tau_{(max)in-plane} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left|\frac{\sigma_1-\sigma_2}{2}\right|$$
$$\sigma_{average} = \left(\frac{\sigma_x+\sigma_y}{2}\right)$$

Absolute maximum shear stress:

$$\tau_{(max)absolute} = \left|\frac{\sigma_{max} - \sigma_{min}}{2}\right|$$

Mohr's circle:

$$\mathrm{C} = \left(\frac{\sigma_x+\sigma_y}{2}\right)$$
$$\mathrm{R} = \sqrt{(C+\sigma_x)^2 + \tau_{xy}^2}$$
$$\tan(2\theta p) = \frac{\tau_{xy}}{\sigma_x - C}$$
$$\tan(2\theta s) = \frac{\sigma_x - C}{\tau_{xy}}$$
$$\sigma_{1,2} = \mathrm{R} \pm \mathrm{C}$$

References

Figures

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- Figure 12.1: A pressurized duct is fabricated with welding in a helical pattern. Sneha Davison. 2024. CC BY.



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Chapter 13: Thin-Walled Pressure Vessels

Learning Objectives

- Define thin-walled pressure vessels
- Describe the stresses that exist in thin-walled pressure vessels
- Calculate the stresses generated in the walls of thin-walled pressure vessels
- Perform stress transformation on these stresses

Pressure vessels are storage tanks that are used for storing fluids at high pressure and must therefore be designed to resist this pressure. They are commonly found in factories, power plants, vehicles, and even your own home. Figure 13.1 shows some common applications.

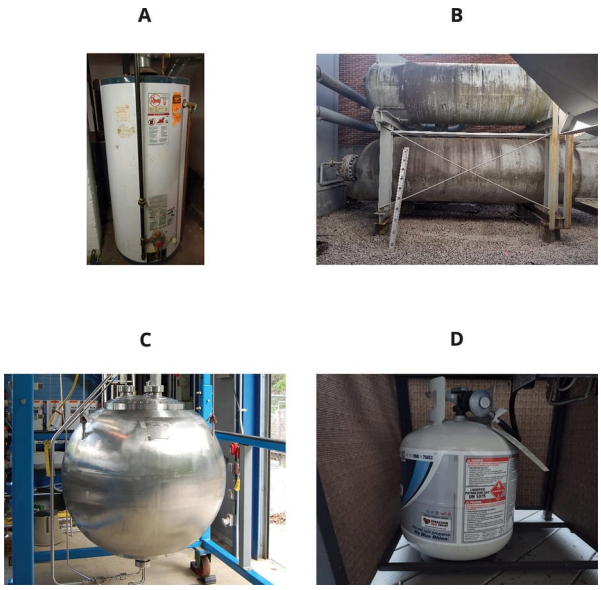


Figure 13.1: Some common examples of pressure vessels. (A) standard home water heater; (B) industrial compressed gas storage; (C) propellant tank for aerospace applications; (D) butane tank for home use.

Pressure vessels are necessarily hollow, and a thin-walled pressure vessel is one where the ratio

$$\frac{Inner\ radius}{Wall\ thickness} > 10.$$

This implies that the wall thickness is relatively small compared to the vessel radius. This simplifies our analysis in two ways:

1. We can assume that stresses in the vessel walls are uniform across the thickness of the wall.
2. We can ignore stress in the radial direction (perpendicular to the vessel wall).

The internal pressure P is typically reported as the gauge pressure, which is defined as the internal pressure minus the atmospheric pressure. This allows us to calculate the stresses directly without needing to account for atmospheric pressure. As we'll see in this chapter, stresses are generated in the walls of thin-walled pressure vessels in two directions known as the axial stress and hoop stress (Figure 13.2). There is a third stress in the radial direction, but this is small relative to the other stresses and is assumed to be zero in this text.

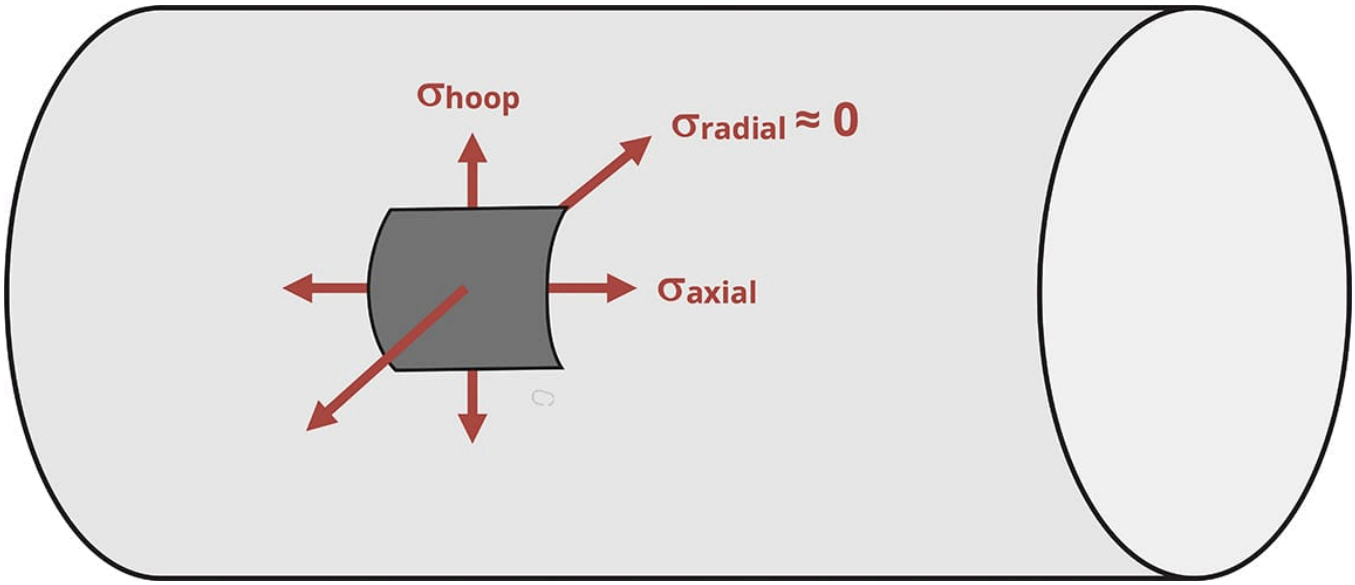


Figure 13.2: A stress element on the wall of a thin-walled pressure vessel. Stresses exist in the axial and hoop directions. The radial stress is very small in comparison and is assumed to be zero.

Figure 13.1 shows that thin-walled pressure vessels are commonly either cylindrical or spherical. Section 13.1 discusses the stresses in cylindrical vessels. Section 13.2 discusses the stresses in spherical vessels.

13.1 Cylindrical Pressure Vessels

The equations for stresses in the wall of a cylindrical pressure vessel can be derived with the help of a free body diagram (FBD). A first step is to cut a cross-section through the vessel of interest to expose the fluid inside (Figure 13.3).

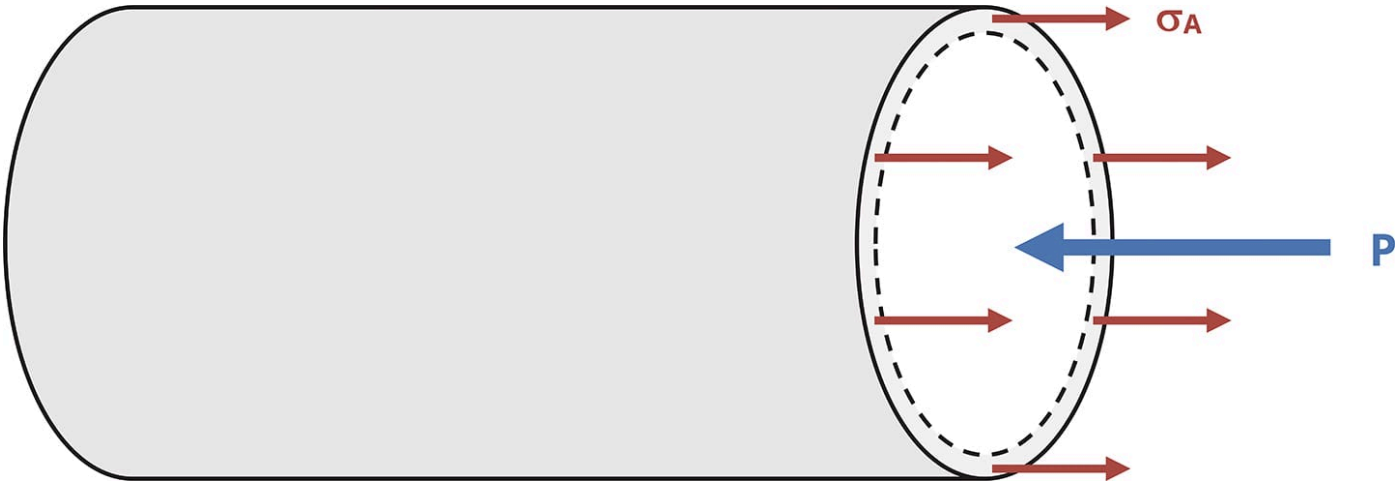


Figure 13.3: Cross-section through the cylindrical vessel showing the axial stress in the walls (s_A) and the fluid pressure P

There will be an axial stress from the internal fluid pressure (P) and an axial stress in the walls of the vessel (σ_A). Since $\text{stress} = \frac{\text{force}}{\text{area}}$, we can determine the two forces. The fluid pressure acts over a circular area $\mathrm{A} = \pi \mathrm{r}_\mathrm{i}^2$. The stress in the walls acts over an area equal to the circumference of the vessel multiplied by the wall thickness, $\mathrm{A} = 2\pi \mathrm{r}_\mathrm{i} \mathrm{t}$.

By equilibrium we calculate the following:

$$\sum F_x = (\sigma_A)(2\pi r_i t) - (P)(\pi {r_i}^2) = 0$$
$$\boxed{\sigma_A} \boxed{= \frac{P r_i}{2t}},$$

(13.1)

where

- σ_A = *Axial stress in the wall of the vessel [Pa, psi]*
- *P = Fluid pressure inside the vessel [Pa, psi]*
- r_i = *Inner radius of the vessel [m, in.]*
- *t = Wall thickness of the vessel [m, in.]*

Similarly, there will be a stress in the lateral direction, referred to as the hoop stress σ_H. A vertical cross-section (Figure 13.4) is cut to derive an equation for this stress.

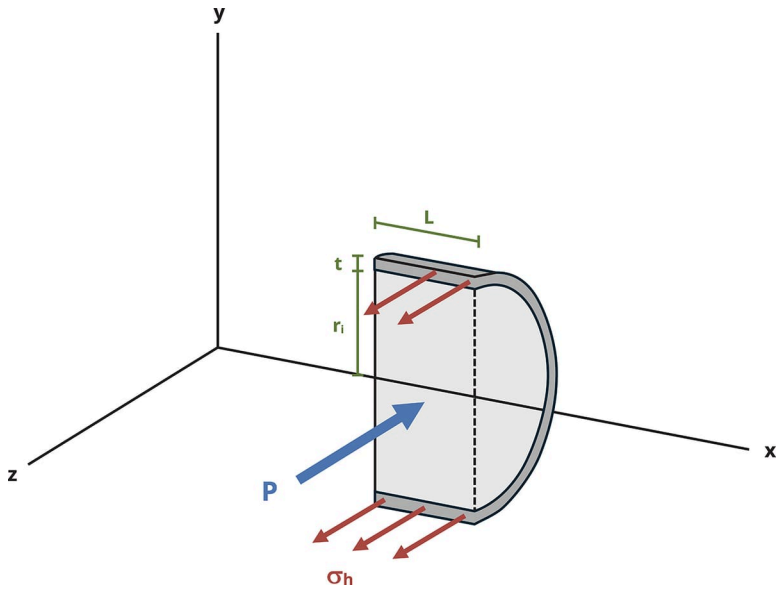


Figure 13.4: Cross-section through the cylindrical vessel showing the hoop stress in the walls (s_H) and the fluid pressure P

There will again be a stress from the internal fluid pressure (P) and a stress in the walls of the vessel (σ_H). The fluid pressure acts over an area of $\mathrm{A} = 2\mathrm{r_i L}$. The stress in the walls acts over an area of $\mathrm{A} = 2\mathrm{tL}$.

By equilibrium we calculate the following:

$$\sum F_x = (\sigma_H)(2tL) - (P)(2r_i L) = 0$$
$$\boxed{\sigma_H} \boxed{= \frac{P r_i}{t}},$$

(13.2)

where

- σ_H = *Hoop stress in the wall of the vessel [Pa, psi]*

- *P = Fluid pressure inside the vessel [Pa, psi]*
- *r_i = Inner radius of the vessel [m, in.]*
- *t = Wall thickness of the vessel [m, in.]*

Since the radial stress is assumed to be zero, the axial and hoop stresses represent a state of plane stress such as discussed in Chapter 12. Further, since there is no shear stress, the axial and hoop stresses represent the principal stresses. Because the hoop stress is the larger of the stresses, it is the first principal stress (σ_1). The axial stress is the second principal stress (σ_2). The third principal stress acts in the radial direction and has already been assumed to be zero. Figure 13.5 shows a plane stress element representing the stresses in the wall of a cylindrical thin-walled pressure vessel.

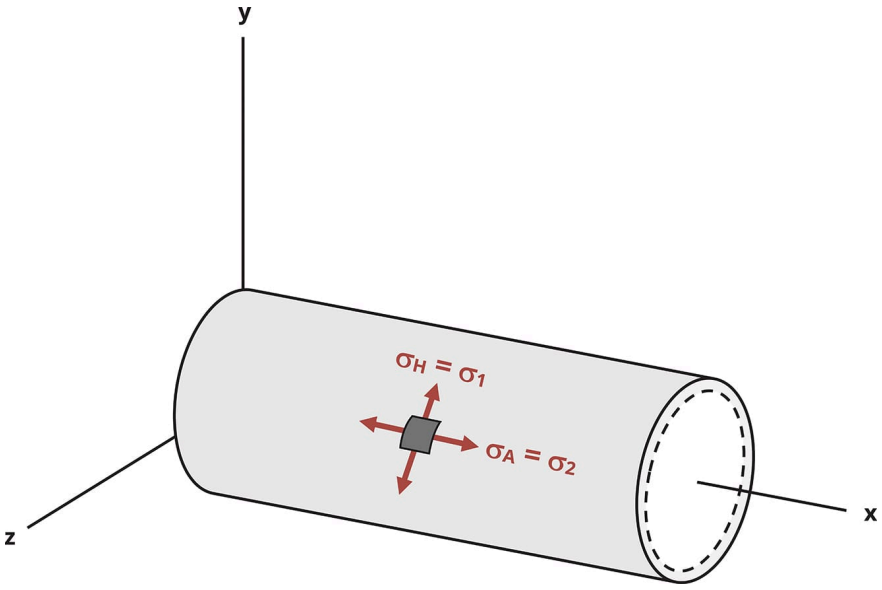


Figure 13.5: The hoop stress represents the first principal stress σ_1. The axial stress represents the second principal stress σ_2. There is no shear stress.

Using the methods presented in Chapter 12, we may perform stress transformation on these stresses. Figure 13.6 shows Mohr's circle for the in-plane and out-of-plane stresses.

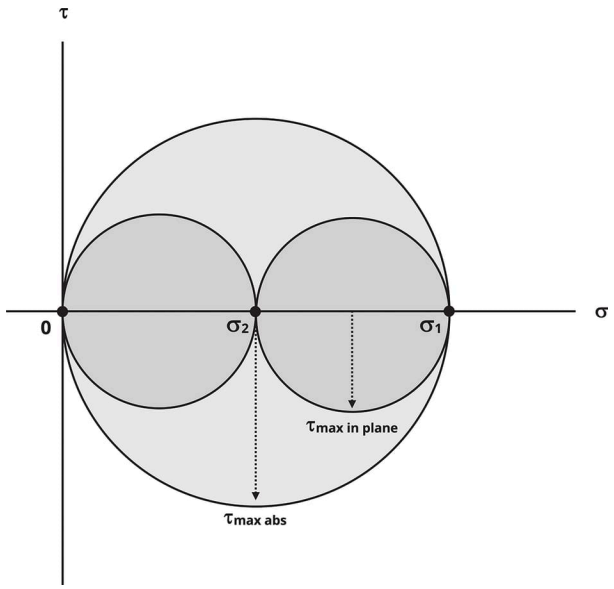


Figure 13.6: Mohr's circle for the stresses in the walls of a cylindrical thin-walled pressure vessel, showing principal stresses σ_1 and σ_2. The third principal stress σ_3 = 0.

Since the two principal stresses are positive and the third principal stress is zero, it should be apparent that the maximum in-plane shear stress can be determined as

$$\tau_{max\ in\ plane} = \frac{\sigma_1 - \sigma_2}{2}.$$

It should also be apparent that the maximum out-of-plane shear stress will be larger and can be calculated from Equation 12.4.

$$\tau_{(max)absolute} = \left| \frac{\sigma_{max} - \sigma_{min}}{2} \right| = \frac{\sigma_1}{2}$$

Example 13.1 demonstrates calculation of the principal stresses and maximum in-plane shear stress for a cylindrical thin-walled pressure vessel. Example 13.2 combines the content from this section with stress transformation from Chapter 12 to find the stresses at specific orientations.

Example 13.1

A cylindrical pressure vessel has an outer diameter of 8 ft and a wall thickness of 2 in. The vessel contains propane at a gauge pressure of 250 psi.

Determine the principal stresses and maximum in-plane shear stress in the walls of the vessel.

Solution

First determine the inner radius of the vessel. It will be easiest to convert this to inches since the other length units in the problem are given in inches.

$$\text{Outer radius, } \mathrm{r_o} = \frac{8\ \text{ft}}{2} = 4\ \text{ft} = 48\ \text{in.}$$

$$\text{Inner radius, } \mathrm{r_i} = 48\ \text{in.} - 2\ \text{in.} = 46\ \text{in.}$$

Now calculate the axial and hoop stresses.

$$\sigma_A = \frac{Pr_i}{2t} = \frac{250\ psi * 46\ in.}{2 * 2\ in.} = 2{,}875\ psi$$
$$\sigma_H = \frac{Pr_i}{t} = \frac{250\ psi * 46\ in.}{2\ in.} = 5{,}750\ psi$$

Since there is no shear stress, there are two principal stresses.

$$\sigma_1 = 5{,}750\ psi$$
$$\sigma_2 = 2{,}875\ psi$$

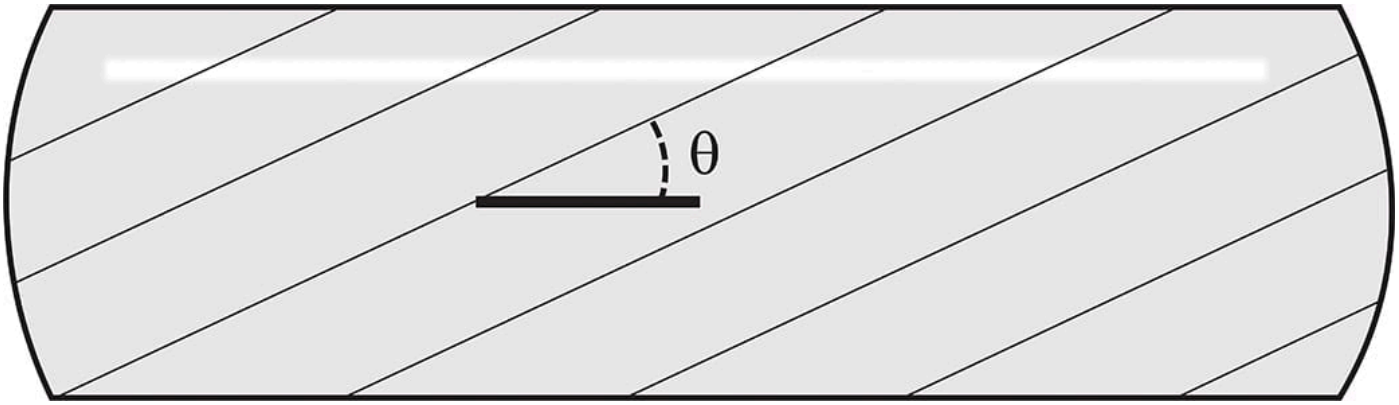
Finally, the maximum in-plane shear stress can be found from

$$\tau_{max\ in\ plane} = \frac{\sigma_1 - \sigma_2}{2} = \frac{5{,}750\ psi - 2{,}875\ psi}{2} = 1{,}438\ psi$$

Example 13.2

A cylindrical pressure vessel is made by welding a steel plate at an angle of θ = 25° as shown. The vessel has an outer diameter of 1.5 m and a wall thickness of 40 mm. The vessel contains fluid at a gauge pressure of 5 MPa.

Determine the normal stress perpendicular to the weld and the shear stress parallel to the weld.



Solution

First determine the inner radius of the vessel. Wall thickness t = 40 mm = 0.04 m.

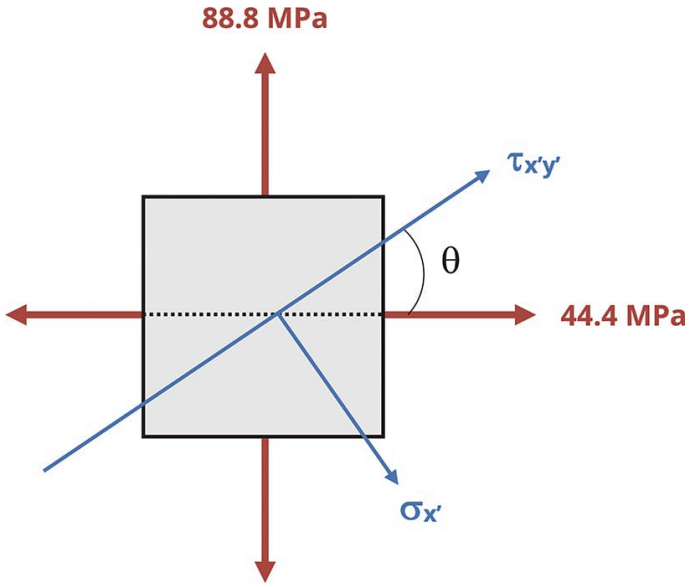
$$\text{Outer radius, } r_o = \frac{1.5\ \text{m}}{2} = 0.75\ \text{m}.$$

$$\text{Inner radius, } r_i = 0.75\ \text{m} - 0.04\ \text{m} = 0.71\ \text{m}.$$

Now calculate the axial and hoop stresses.

$$\sigma_A = \frac{Pr_i}{2t} = \frac{5\ MPa \times 0.71\ m}{2 \times 0.04\ m} = 44.4\ MPa$$
$$\sigma_H = \frac{Pr_i}{t} = \frac{5\ MPa \times 0.71\ m}{0.04\ m} = 88.8\ MPa$$

Using the methods shown in Section 12.1, calculate the normal stress perpendicular to the weld. Draw a stress element of a point on the pressure vessel. The axial stress will act horizontally and the hoop stress vertically for this element.



The weld is at an angle of 25° measured counterclockwise from horizontal. The normal stress perpendicular to this weld is therefore at an angle of 65° clockwise from horizontal. Determine the normal stress at this orientation from

$$\sigma_x^{'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta)$$
$$\sigma_x^{'} = \frac{44.4\ MPa+88.8\ MPa}{2} + \frac{44.4\ MPa-88.8\ MPa}{2}\cos(2*-65^\circ) + 0$$
$$\sigma_x^{'} = 66.6\ MPa - 22.2\ MPa*\cos(-130^\circ)$$
$$\sigma_x^{'} = 80.9\ MPa.$$

The shear stress occurs parallel to the weld at an angle of 25° counterclockwise from the horizontal. Determine the shear stress at this orientation from

$$\tau_{x'y'} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin(2\theta) + \tau_{xy}\cos(2\theta)$$
$$\tau_{x'y'} = -\left(\frac{44.4\ MPa-88.8\ MPa}{2}\right)\sin(2*25^\circ) + 0$$
$$\tau_{x'y'} = 22.2\ MPa*\sin(50^\circ)$$
$$\tau_{x'y'} = 17.0\ MPa.$$

Step-by-Step: Cylindrical Thin-Walled Pressure Vessels

1. Determine the inner radius, wall thickness, and internal gauge pressure for the vessel.
2. Calculate the axial $\left(\sigma_{\mathrm{A}} = \frac{\mathrm{Pr_i}}{2\mathrm{t}}\right)$ and hoop $\left(\sigma_{\mathrm{H}} = \frac{\mathrm{Pr_i}}{\mathrm{t}}\right)$ stresses—these are the principal stresses.
3. Perform any stress transformation calculations as required.

13.2 Spherical Pressure Vessels

A spherical vessel will yield the same cross-section regardless of the axis it is cut along (Figure 13.7). The stress in the walls of a spherical vessel is the same as the axial stress in a cylindrical vessel.

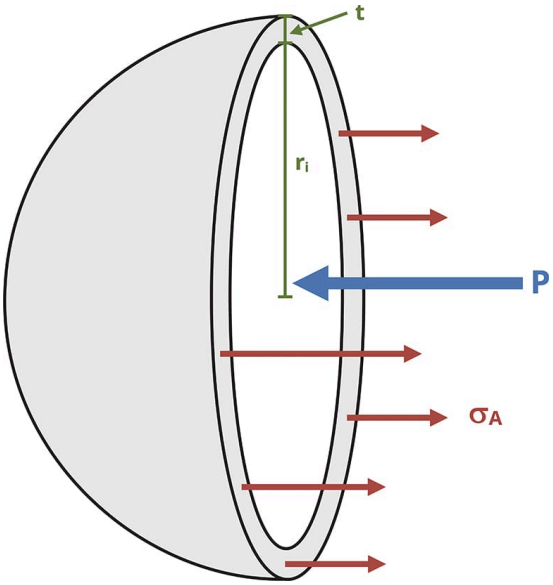


Figure 13.7: Cross-section through the spherical vessel showing the axial stress in the walls (σ_A) and the fluid pressure P

By equilibrium calculate the following:

$$\sum F_x = (\sigma_A)(2\pi r_i t) - (P)(\pi r_i^2) = 0$$

$$\boxed{\sigma_A = \frac{P r_i}{2t}},$$

(13.3)

where

- σ_A = *Axial stress in the wall of the vessel [Pa, psi]*
- *P = Fluid pressure inside the vessel [Pa, psi]*
- r_i = *Inner radius of the vessel [m, in.]*
- *t = Wall thickness of the vessel [m, in.]*

In this case the same stress exists in both principal directions. Again assume that the radial stress is zero, meaning that these stresses again form a state of plane stress (Figure 13.8). There is again no shear stress, so these stresses are the principal stresses. Thus the two principal stresses here are the same.

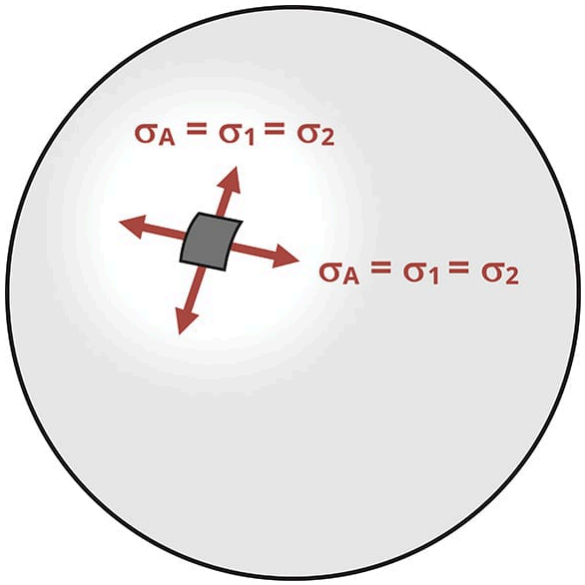


Figure 13.8: Spherical vessels experience the same axial stress in both principal directions. There is no shear stress.

Note that these principal stresses are equal to σ_2 in the cylindrical vessel. As such, spherical vessels are stronger than cylindrical vessels because the axial stress (σ_2) is half the hoop stress (σ_1) and there is no hoop stress in a spherical vessel. However, because spherical vessels are more expensive to manufacture, cylindrical vessels remain very common.

We may again use the methods employed in Chapter 12 to perform stress transformation. Figure 13.9 shows Mohr's circle for the in-plane and out-of-plane stresses.

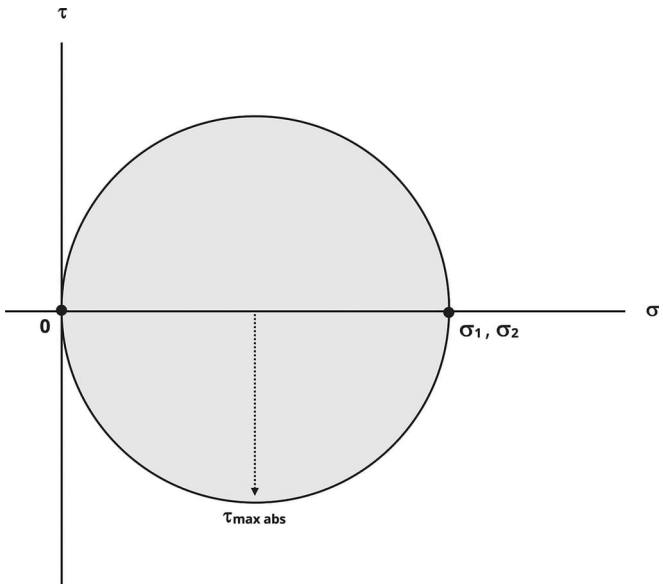


Figure 13.9: Mohr's circle for the stresses in the walls of a spherical thin-walled pressure vessel. Principal stresses $\sigma_1 = \sigma_2$, and the in-plane circle reduces to a single point. The third principal stress is $\sigma_3 = 0$.

This time $\sigma_1 = \sigma_2$ and so the in-plane Mohr's circle reduces to a single point. The maximum in-plane shear stress is therefore zero. The maximum out-of-plane shear stress can be calculated from

$$\tau_{(max)absolute} = \frac{\sigma_1}{2}.$$

Example 13.3 demonstrates the design of a spherical vessel, based on the allowable stress in the vessel walls.

Example 13.3

A spherical pressure vessel is made of steel with a yield strength of 250 MPa. The vessel has an outer diameter of 4 m and a wall thickness of 25 mm.

If the factor of safety with respect to yield is 2.5, determine the maximum allowable gauge pressure of the fluid within the vessel.

Solution

Wall thickness, $\mathrm{t} = 25\ \mathrm{mm} = 0.025\ \mathrm{m}$

Outer radius, $\mathrm{r_o} = 2\,m$

Inner radius, $\mathrm{r_i} = 2\ \mathrm{m} - 0.025\ \mathrm{m} = 1.975\ \mathrm{m}$

Start by using the factor of safety to find the maximum allowable stress in the walls of the vessel.

$$\sigma_{allow} = \frac{250\ MPa}{2.5} = 100\ MPa$$

Spherical vessels experience the same stress in both directions.

$$\sigma_A = \frac{Pr_i}{2t}$$

Rearrange the equation to solve for P.

$$P = \frac{\sigma_A * 2t}{r_i}$$

$$P = \frac{100\ MPa * 2 * 0.025\ m}{1.975\ m} = 2.53\ MPa$$

Step-by-Step: Spherical Thin-Walled Pressure Vessels

1. Determine the inner radius, wall thickness, and internal gauge pressure for the vessel.
2. Calculate the axial stress $\left(\sigma_\mathrm{A} = \frac{\mathrm{Pr_i}}{\mathrm{2t}}\right)$. This stress will be the same in the horizontal and vertical directions.
3. Perform any stress transformation calculations as required.

Summary

Key Takeaways

Pressure vessels contain fluids at high pressure. A thin-walled pressure vessel is one where $\frac{\text{Inner radius}}{\text{Wall thickness}} > 10$.

For thin-walled pressure vessels, the stresses in the vessel walls are assumed to be constant across the thickness of the wall and the stress in the radial direction is assumed to be zero. The stresses fluid pressure creates in the axial and hoop directions are the principal stresses.

Pressure vessels are commonly cylindrical or spherical. Cylindrical vessels experience both axial and hoop stresses. The axial stress is half the hoop stress. Spherical vessels experience only an axial stress. As such, spherical vessels experience less stress than cylindrical vessels.

No shear stress is generated in the axial and hoop directions. However, stress transformation can help determine both in-plane and out-of-plane shear stresses.

Key Equations

Cylindrical thin-walled pressure vessels

Hoop stress:

$$\sigma_H = \frac{Pr_i}{t} = \sigma_1$$

Axial stress:

$$\sigma_A = \frac{Pr_i}{2t} = \sigma_2$$

Maximum in-plane shear stress:

$$\tau_{max\ in\ plane} = \frac{\sigma_1 - \sigma_2}{2}$$

Absolute maximum shear stress:

$$\tau_{(max)absolute} = \frac{\sigma_1}{2}$$

Spherical thin-walled pressure vessels

Axial stress:

$$\sigma_A = \frac{Pr_i}{2t} = \sigma_1 = \sigma_2$$

Absolute maximum shear stress:

$$\tau_{(max)absolute} = \frac{\sigma_1}{2}$$

References

Figures

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Chapter 14: Combined Loads

Learning Objectives

- Determine total normal stress resulting from combinations of normal forces, bending moments, and pressure
- Determine total shear stress when transverse loads in one or both transverse directions are combined with torsional loads
- Determine the general stress state when any combination of the above is applied and use those results to determine principal stresses and principal stress planes

This chapter uses all the previously discussed concepts to determine the total normal stress and the total shear stress for a body subjected to various types of loading simultaneously. Axial forces (Chapter 5), bending moments (Chapter 9), and pressure (Chapter 13) may be applied in various combinations to give total normal stress. Transverse loads (Chapter 10) in multiple directions and torsion (Chapter 6) can combine to give total shear stress.

Understanding which loads lead to which types of stresses and in which directions requires having a good grasp of coordinate axes. In particular being able to distinguish between the longitudinal axis and the cross-sectional axes is important. This can change from body to body as different coordinate axes are used, but it can also change within an individual body according to changes in geometry. Section 14.1 reviews how to distinguish between the coordinate axes as well as defines generic terms such that we can express equations in a more generalized manner.

Section 14.2 reviews and expands on the various sources of normal stress. These normal stresses will be reframed in terms of the general axes discussed in Section 14.1 so as to clarify which other stresses they may combine with. Section 14.2.1 revisits bending stress to express the unsymmetric bending stress equation in terms of general cross-sectional axes. Section 14.2.3 considers eccentric loading, which is a specific instance of combined normal stress in which normal forces result in bending.

Section 14.3 focuses on combining the various sources of shear stress to determine the total shear stress. To expand on the concepts discussed and used in Chapter 6 and Chapter 10, here we perform calculations based on both possible transverse loading directions as well as torsion. As in Section 14.2, this is accomplished using a general set of coordinate axes.

Section 14.4 further expands the above cases to situations in which all combinations of loading are possible and the general stress state is determined.

Note that determining the internal reactions for the general problems presented in this chapter requires the use of 3D equilibrium equations, particularly for determining bending moments and torsion. You may thus find it helpful to review Section 1.4 to understand these calculations.

14.1 Coordinate Axes and Directions

To generalize the discussion for any set of coordinate axes, we will use generic terms for the various directions, as shown in Figure 14.1. The vertical axis of the cross-section will be referred to as the V-axis, the horizontal axis of the cross-section will be referred to as the H-axis, and the longitudinal (axial) direction will be referred to as the L-axis.

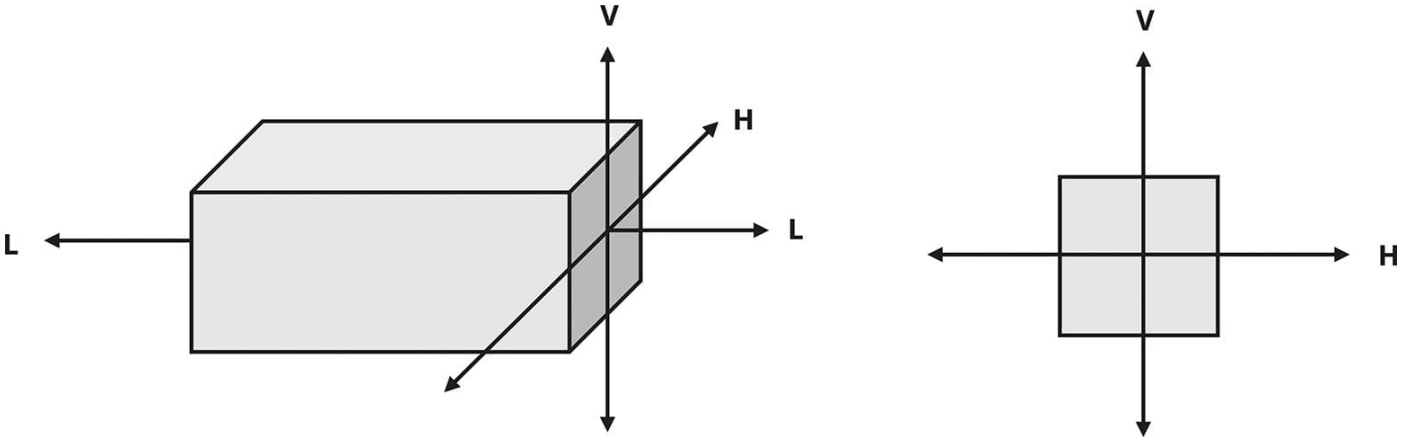


Figure 14.1: To maintain generality, we use V to denote the vertical axis of the cross-section, H to denote the horizontal axis of the cross-section, and L to denote the longitudinal axis of the body based on the cross-section of interest.

The longitudinal direction, synonymous with axial direction, is the direction perpendicular to a cross-section that is cut at a point of interest on a body. The cross-sectional axes are the axes drawn on the cross-section based on viewing the cross-section down the longitudinal axis.

These directions could change from point to point depending on the location of the point of interest and the geometry of the body. For example, consider the structure in Figure 14.2 with the x-, y-, z-axes oriented at the origin as shown. To examine point P on the structure, we make a vertical cut at point P. The direction normal to the cut surface is the x direction, so the x-axis is the longitudinal axis. When viewing the cut surface looking down the x-axis, you see the y-axis and the z-axis on the cross-section, so those are the cross-sectional axes. In this view, the y-axis appears to be the vertical (V) axis of the cross-section and the z-axis is horizontal (H).

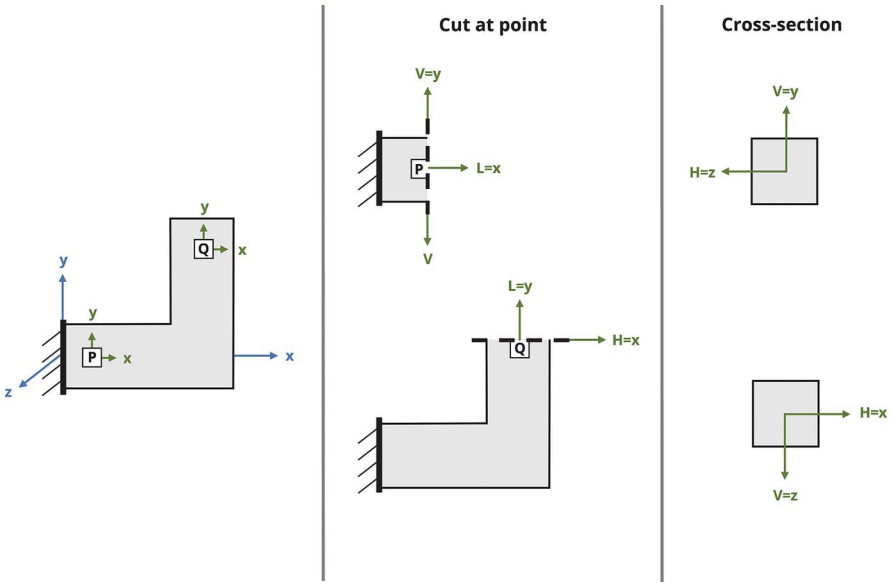


Figure 14.2: The longitudinal (L), horizontal (H), and vertical (V) axes depend on how the axes are arranged globally for the structure as well as on the specific location of the point in question.

In contrast, for point Q we must make a horizontal cut to examine that point. The normal direction to the cut is then the vertical, or y-axis, and the cross-sectional axes as viewed looking down the y-axis are the x-axis and z-axis. In this view, the x-axis would appear to be the horizontal axis of the cross-section and the z-axis would appear to the vertical axis of the cross-section.

Depending on your perspective, you may switch the cross-sectional axes you consider to be horizontal versus vertical. Ultimately

this will not matter as long as you correctly distinguish between the longitudinal and cross-sectional directions.

In the following sections we will express equations in terms of these generic directions so that they will apply to any orientation of coordinate axes.

14.2 Combined Normal Stress

Previous chapters considered individual sources of longitudinal normal stress: axial loads (Chapter 5), bending loads (Chapter 9), and pressure loads (Chapter 13). With all three of these forms of loading present we can express the generalized equation for total longitudinal stress as

$$\sigma_L = \frac{\sum F_L}{A} + \sigma_{bending} + \frac{pr}{2t},$$

where

- σ_L = *Total normal stress in the longitudinal direction [Pa, psi]*
- F_L = *Force in the longitudinal direction [N, lb]*
- A = *Cross-sectional area [m^2, $in.^2$]*
- $\sigma_{bending}$ = *Bending stress (discussed in more detail below) [Pa, psi]*
- p = *Internal pressure [Pa, psi]*
- r = *Inner radius of pipe or pressure vessel [m, in.]*
- t = *Wall thickness of pipe or pressure vessel [m, in.]*

If there is pressure loading in addition to the contribution to the longitudinal normal stress indicated above, there will also be normal stress in the cross-sectional directions equal to the hoop stress.

$$\sigma_V = \sigma_H = \sigma_{hoop} = \frac{pr}{t}$$

Of course, for problems without pressure, the pressure terms would be excluded.

Of the three types of longitudinal normal stress, bending stress tends to be the most complex to understand in terms of calculation inputs and signs, especially with varying coordinate axes directions. Section 14.2.1 revisits the concept of bending stress to discuss how to generalize the Chapter 9 equations to any orientation of body with any arrangement of coordinate axes.

14.2.1 Generalized Bending Stress Equation

Section 9.3 derived an equation for bending stress in the x direction based on bending around the cross-sectional axes y and z. That equation is

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}.$$

The minus sign in front of the M_z term and the plus sign in front of the M_y term are part of the derived equation. When the correct signs for M_z, M_y, y, and z are inserted, the correct sign of the stress is obtained (positive for tensile and negative for compressive). Note that Chapter 9 considered no other cross-sectional axes so that focus on the general concept of bending stress could be more easily maintained.

When the cross-sectional axes are not y and z, we apply the same general form of the above bending stress equation. However, for a different set of axes, we not only need to change the bending moment and second moment of area axes in the equation but also need to reconsider the signs in front of each term. In any given bending load situation the bending moments are the moments around the centroidal axes of the cross-section (V and H) and the resulting normal stress is in the longitudinal direction

(L).

Given this information, we can then reframe the bending stress equation in a generalized form.

$$\boxed{\sigma_{(bending)} = \pm \left| \frac{M_V \, d_H}{I_V} \right| \pm \left| \frac{M_H \, d_V}{I_H} \right|}, \tag{14.1}$$

where

- $\sigma_{(bending)}$ = *Bending stress in the longitudinal direction [Pa, psi]*
- M_V = *Bending moment around the vertical axis [N×m, lb×in.]*
- M_H = *Bending moment around the horizontal axis [N×m, lb×in.]*
- d_H = *Horizontal distance between the point of interest on the cross-section and the V-axis [m, in.]*
- d_V = *Vertical distance between the point of interest on the cross-section and the H-axis [m, in.]*
- I_V = *Second moment of area about the vertical axis [m^4, $in.^4$]*
- I_H = *Second moment of area about the horizontal axis [m^4, $in.^4$]*

Figure 14.3 illustrates the distances d_H and d_V for the case where the stress at the example point P on the cross-section is being calculated.

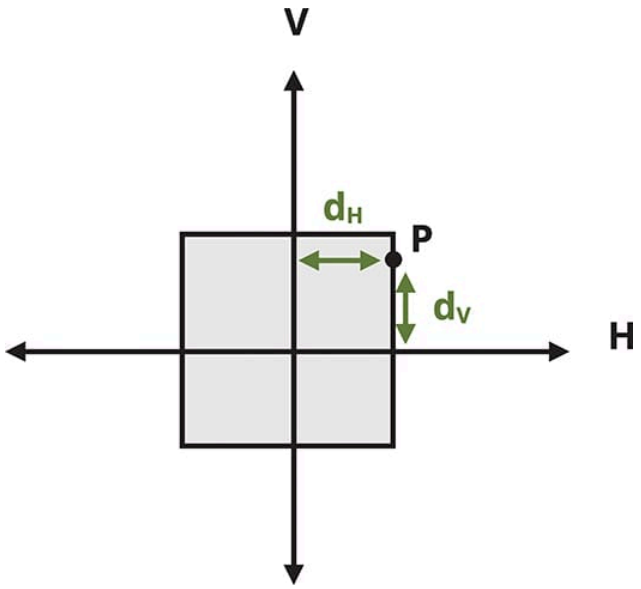


Figure 14.3: The distances d_H and d_V that would go in Equation 14.1 when calculating the bending stress for point P on the cross-section

The absolute values used for the bending stress terms with the ± in front indicate that we will calculate the magnitude of each bending stress term and then assess the signs of the terms independently. We assess the signs by visualizing the bending direction or by using visualization tools, described below. Recall that compressive stresses are considered negative and tensile stresses are positive.

Also helpful to note is that M_H does not affect the normal stress of points directly on the H-axis and M_V does not affect the normal stress of points directly on the V-axis. This is evident from the fact that d_H = 0 for a point on the V-axis and d_V = 0 for a point on

the H-axis.

14.2.2 Assessing Bending Stress Signs for Generalized Bending Stress Equation

The sign on each of the two bending stress terms in Equation 14.1 is determined by inspection of the bending direction. Section 9.1 showed that a bending moment around the horizontal axis causes the normal stress to vary along the vertical axis of the cross-section. Section 9.3 showed that a bending moment around the vertical axis causes normal stress to vary across the cross-section's horizontal axis. Which side is in tension versus compression in each case depends on the direction (clockwise versus counterclockwise) of the bending moment as well as the location of the point on the body.

If we can visualize the expected bending direction due to loading, we can assess whether a point will be on the “inside” of the bend (compression) or on the “outside” of the bend (tension). For example, in Figure 14.4 the rightward horizontal force results in bending around the V-axis of the cross-section. The direction of the bending is such that all points to the right of the V-axis on the cross-section are on the inside of the bend (compression) and all points to the left of the V-axis on the cross-section are on the outside of the bend (tension).

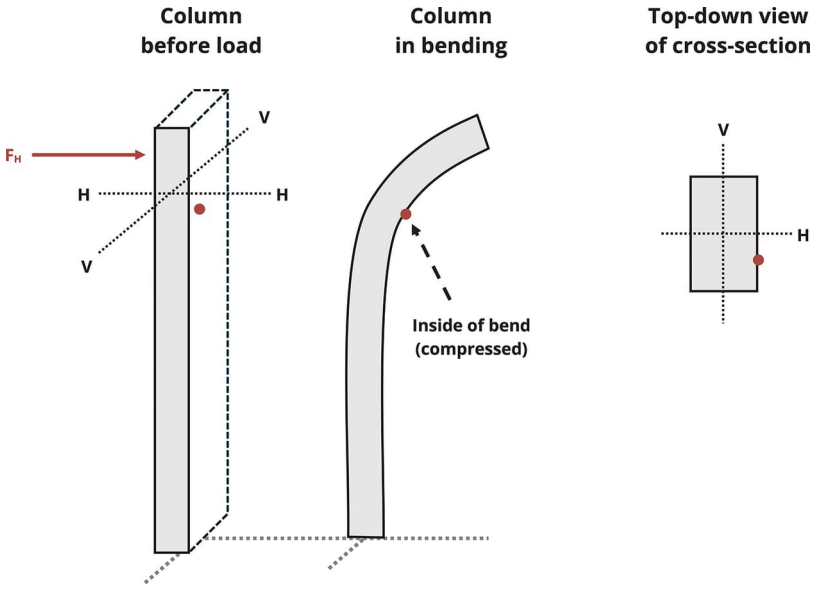


Figure 14.4: Bending around the V-axis

In Figure 14.5, the force in the V direction results in a bending moment around the H-axis. The direction of the bending is such that all points above the H-axis on the cross-section are on the inside of the bend (compression) and all points below the H-axis on the cross-section are on the outside of the bend (tension).

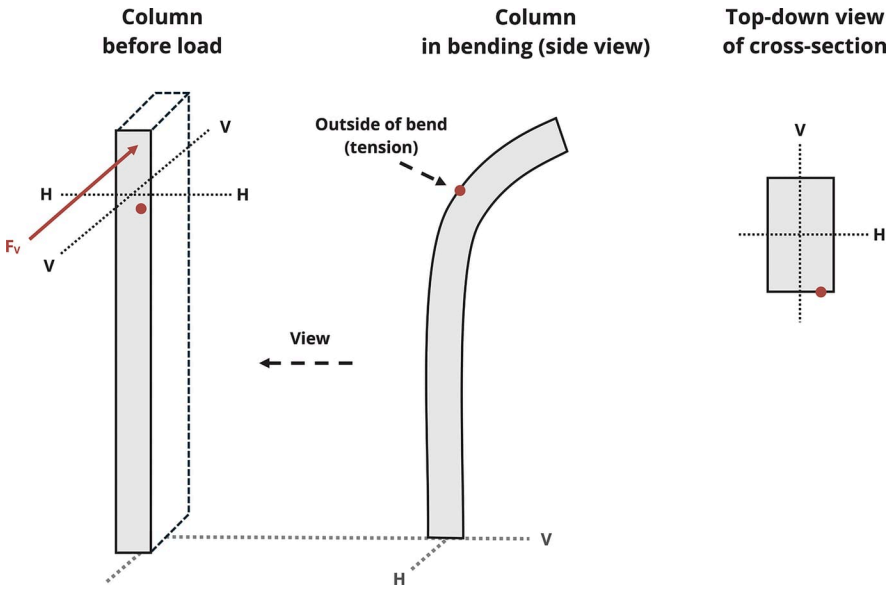


Figure 14.5: Bending around the H-axis

If visualization is difficult, use the following tool (refer to Figure 14.6):

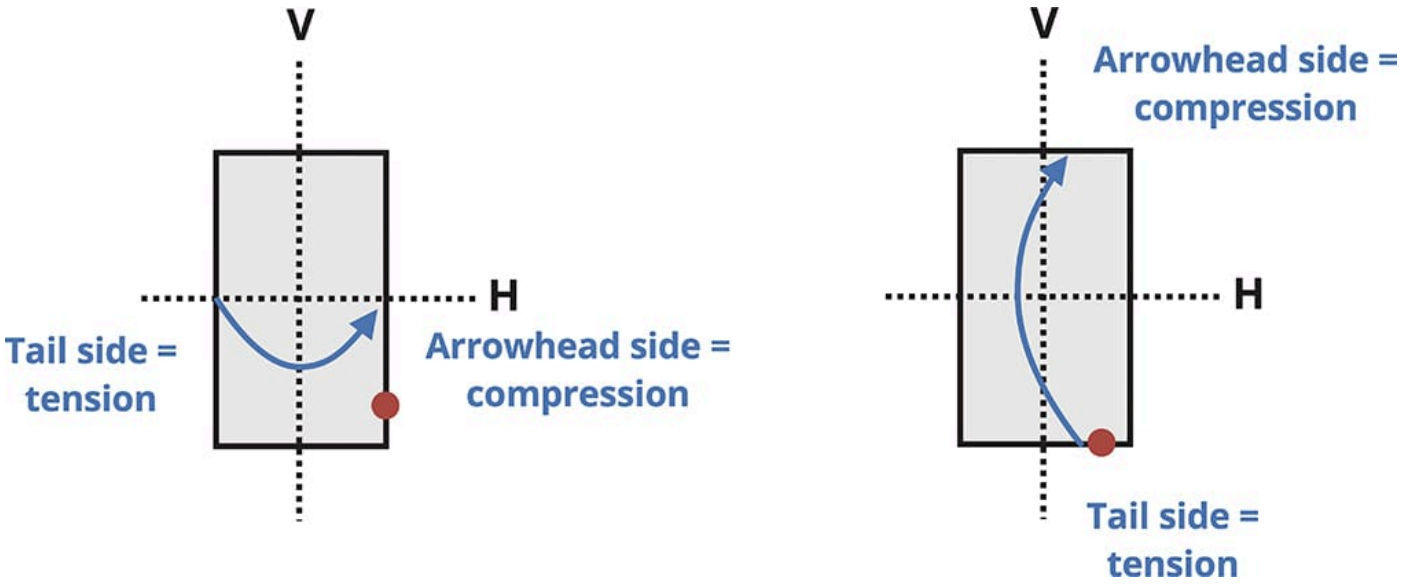


Figure 14.6: Bending moments around vertical and horizontal axes shown with exaggerated moment arrows to assess tension versus compression.

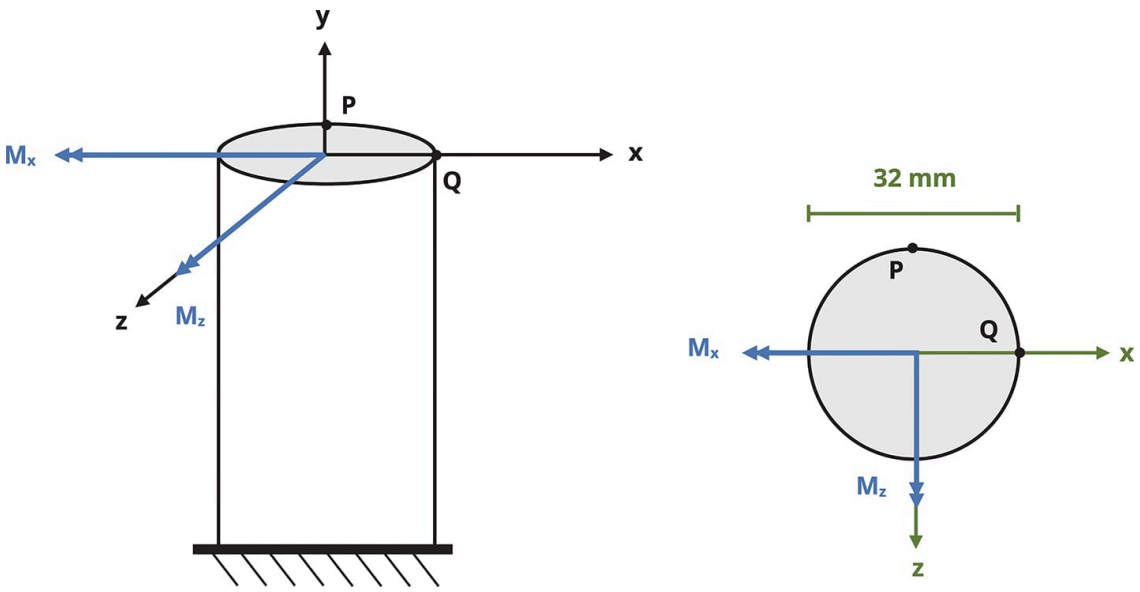
1. Draw the cross-section as viewed from the free end or cut end toward the origin. For example, in the case of the columns in Figure 14.4 and Figure 14.5, we would draw the top-down view of the cross-section.
2. Draw the bending moment arrows around the V-axis and the H-axis on the sketch of the cross-section. Exaggerate the arrows so they are drawn all the way across.
    - Make sure to draw the bending moment arrows such that they cross over (or in front of) the bending axis (as opposed to under or behind).
3. The side the arrowhead points to is the compressive side, and the side the tail of the arrow is on is the tensile side.

This approach is demonstrated for the bending stress calculation in Example 14.1.

Example 14.1

The internal bending moments on a 32 mm diameter solid steel rod are found to be M_x = 376 kN·m (clockwise) and M_z = 204 kN·mm (counterclockwise) as shown.

Determine the bending stress at points P and Q.



Solution

1. Normally, the first step is to draw the free body diagram (FBD) of the cut section to find the internal bending moments. In this case, for simplicity, the internal moments are already given.

2. Write the general bending stress equation and eliminate unnecessary terms.

$$\sigma_L = \pm \left| \frac{M_H \, d_V}{I_H} \right| \pm \left| \frac{M_V \, d_H}{I_V} \right|$$

Looking at the cross-section, we can see that the cross-sectional vertical axis is the z-axis and the cross-sectional horizontal axis is the x-axis. The longitudinal axis is the y-axis.

$$\sigma_y = \pm \left| \frac{M_z \, d_x}{I_z} \right| \pm \left| \frac{M_x \, d_z}{I_x} \right|$$

Evaluating point P:

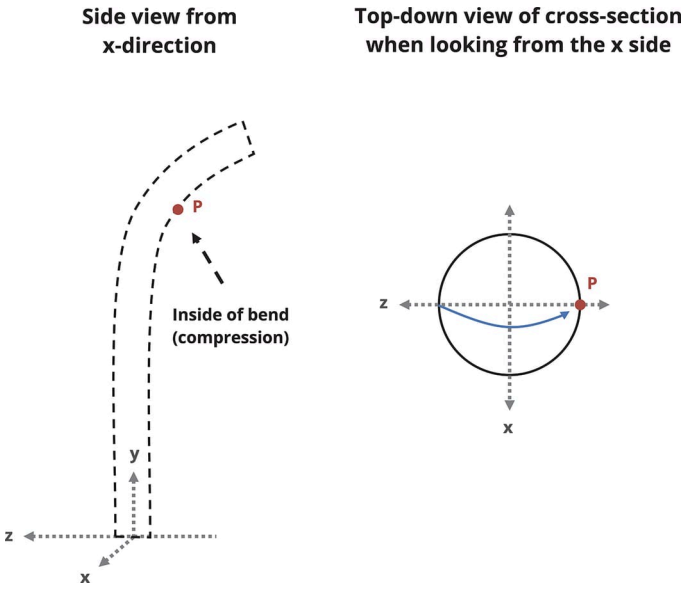
Because P is on the centroidal z-axis, d_x = 0, and the general bending stress equation reduces to

$$\sigma_y = \pm \left| \frac{M_x \, d_z}{I_x} \right|$$

3. Assess the sign.

With the clockwise bending moment around the x-axis, the beam can be visualized as bending backward such that point P will be on the inside of the bend, or in compression. An approximation of the resulting shape is sketched here as it would appear from the side view (looking down the x-axis).

Alternatively, draw the cross-section with the exaggerated moment arrow clockwise around the x-axis. Since the arrowhead would be pointed toward P, the point is in compression.



4. Calculate the values.

$$|M_x| = \text{given} = (376kN \cdot mm)(1,000N/kN) = 376,000N \cdot mm$$
$$d_z = \text{radius} = 16\,mm$$
$$\mathrm{I_x} = \tfrac{\pi}{64} d^4 = \tfrac{\pi}{64}(32\,mm)^4 = 51.47 \times 10^3 mm^4$$
$$\sigma_{y_P} = -\frac{376{,}000\ N{\cdot}mm(16mm)}{51.47{\times}10^3\ mm^4}$$
$$= -117\ N/mm^2$$
$$= -117\ MPa$$

Evaluating Point Q:

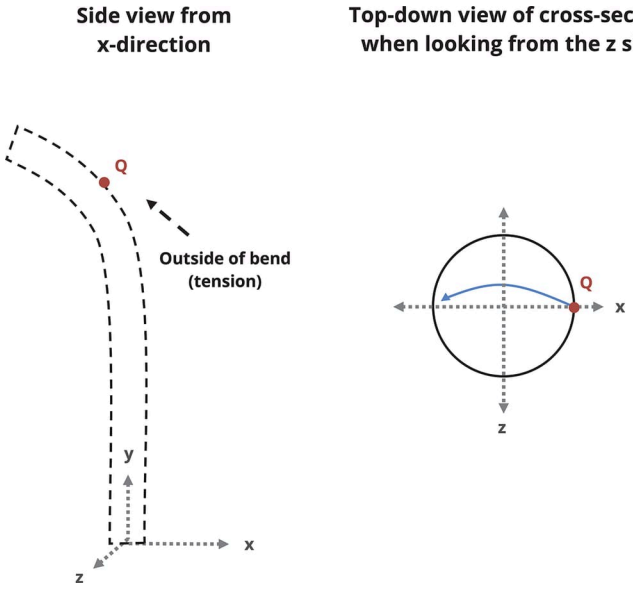
Because point Q is on the centroidal x-axis, d_z = 0, so the general bending stress equation reduces to

$$\sigma_{y_Q} = \pm \left| \frac{M_z\, d_x}{I_z} \right|$$

5. Assess the sign.

With the counterclockwise bending moment around the z-axis, the beam can be visualized as bending into somewhat of a backward “C” shape as viewed from the front (down the z-axis). Point Q would be on the outside of the bend, or in tension.

Alternatively, draw the cross-section with the exaggerated moment arrow counterclockwise around the z-axis. Since Q is at the tail side, the point is in tension.



6. Calculate the values.

$$|M_z| = \text{given} = (204kN \cdot mm)\,(1,000N/kN) = 204,000\ N \cdot mm$$

$$d_x = \text{radius} = 16\ mm$$

For the circular cross-section, $I_z = I_x = 51.47 \times 10^3\ mm^4$.

$$\sigma_{y_Q} = \frac{204{,}000\ N{\cdot}mm^*(16\ mm)}{51.47{\times}10^3\ mm^4}$$
$$= 63.4\ N/mm^2$$
$$= 63.4\ MPa$$

Answer:

- σ_{yP} = 116.9 MPa (Compression)
- σ_{yQ} = 63.4 MPa (Tension)

14.2.3 Eccentric Loading

When the line of action of an axial load doesn't pass through the centroid of a cross-section, it results in both a normal reaction force and a bending moment. For example, consider the traffic light and corresponding FBD shown in Figure 14.7.

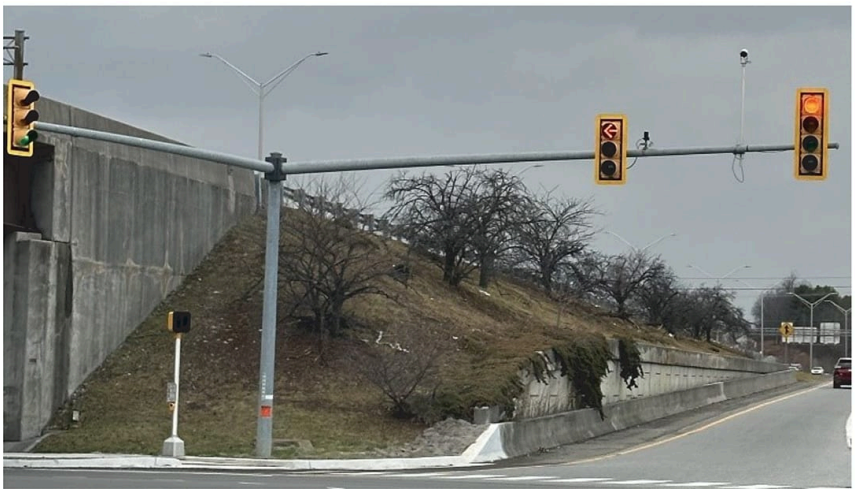


Figure 14.7: The vertical pole is subject to direct axial stress and bending stress.

The vertical support pole is subjected to normal forces from the weights of the traffic lights (the weight of the horizontal poles, cameras, etc. are assumed to be negligible for this example). Also, bending moments around both cross-sectional axes (x and z) result from the weights not being directly centered on the support pole.

The total normal stress in the longitudinal direction, s_L, at any given point on the pole would be given by

$$\sigma_L = \frac{\sum F_L}{A} + \sigma_{bending}.$$

In terms of the general axes discussed in Section 14.1, and substituting in Equation 14.1 for the bending stress, we have

$$\boxed{\sigma_L = \pm \left| \frac{F_L}{A} \right| \pm \left| \frac{M_V d_H}{I_V} \right| \pm \left| \frac{M_H d_V}{I_H} \right|},$$

(14.2)

where

- σ_L = *Bending stress in the longitudinal direction [Pa, psi]*
- F_L = *Internal normal force in the longitudinal direction [N, lb]*
- A = *Cross-sectional area [m^2, $in.^2$]*
- M_V = *Bending moment around the vertical axis of the cross-section [N×m, lb×in.]*
- M_H = *Bending moment around the horizontal axis of the cross-section [N×m, lb×in.]*
- d_H = *Horizontal distance between the point of interest on the cross-section and the V-axis [m, in.]*
- d_V = *Vertical distance between the point of interest on the cross-section and the H-axis [m, in.]*
- I_V = *Second moment of area about the vertical axis of the cross-section [m^4, $in.^4$]*
- I_H = *Second moment of area about the horizontal axis of the cross-section [m^4, $in.^4$]*

The sign for the first term is positive if the F_L is tensile (directed away from the cross-section) and negative if F_L is compressive

(pointed toward the cross-section).

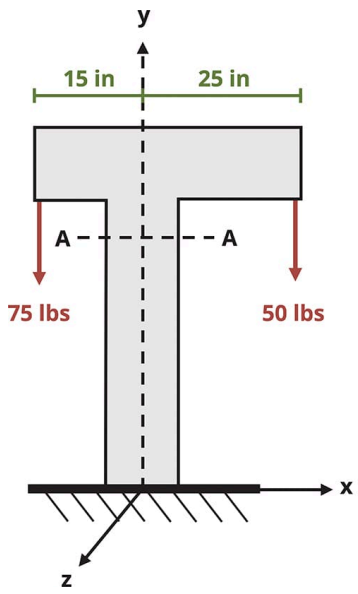
Example 14.2 and Example 14.3 demonstrate the calculation of normal stress in eccentric loading problems.

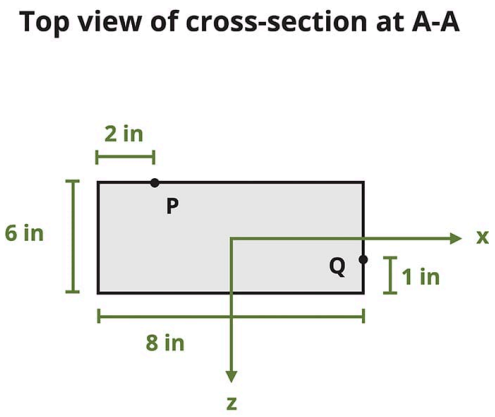
Example 14.4 demonstrates a loading that includes pressure as part of the loading combination and drawing the resulting stress element.

Example 14.2

The structure is fixed to the ground and subjected to the 75 lb and 50 lb vertical forces shown.

Determine the total normal stress at points P and Q on the cross-section located at A-A. Assume the 40 in. horizontal section at the top remains rigid.





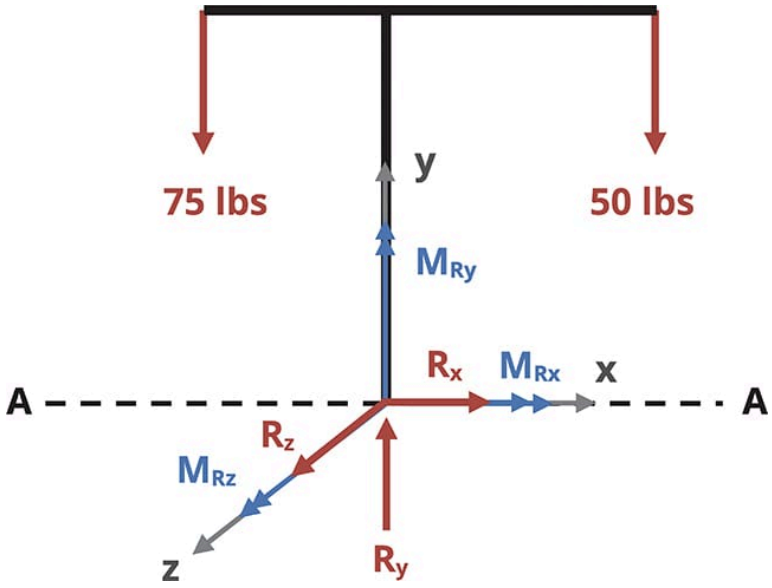
Solution

1. Draw the FBD of the structure cut at A-A and write the equilibrium equations to find the internal reactions.

Below is the FBD for the section of the structure above the cut. Using the section above the cut allows us to avoid needing to calculate the reactions at the fixed support.

In this case reaction forces R_X and R_Z are 0 by inspection (note that no forces are applied in those directions).

Similarly, reaction moments M_{Rx} and M_{Ry} are 0 by inspection: All the forces act only in the y direction, so $M_{Ry} = 0$. All the forces pass through the x-axis, so $M_{Rx} = 0$.



Calculate R_y and M_{Rz}.

$$\sum F_y = R_y - 75\ lb - 50\ lb = 0 \quad \rightarrow \quad R_y = 125\ lb$$
$$\sum M_z = \sum \pm (Fy{*}x) + \sum \pm (Fx{*}y) + M_{Rz} = 0$$
$$= (75\ lb{*}15\ in.) - (50\ lb{*}25\ in.) + M_{Rz} = 0 \quad \rightarrow \quad M_{Rz} = 125\ lb \cdot in.$$

Remember when calculating reaction moments to calculate moments about the centroidal axes of the cross-section no matter where on the cross-section the point of interest lies.

2. Write the generalized normal stress equation and eliminate unnecessary terms.

$$\sigma_L = \pm \left| \frac{F_L}{A} \right| | \pm \left| \frac{M_V d_H}{I_V} \right| \pm \left| \frac{M_H d_V}{I_H} \right|$$

Here the longitudinal axis is the y-axis. The vertical axis of the cross-section is the z-axis, and the horizontal axis is the x-axis. With $M_{Rx} = 0$, we have

$$\sigma_y = \pm \left| \frac{R_y}{A} \right| \pm \left| \frac{M_{Rz}\, d_x}{I_z} \right|$$

3. Assess signs.

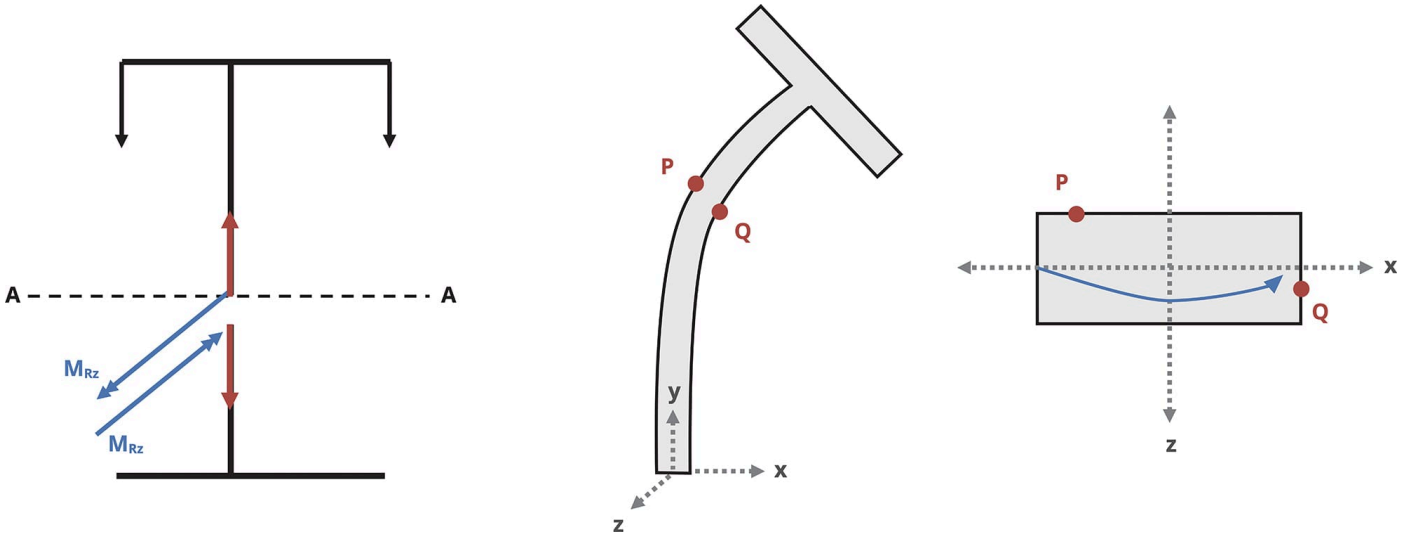
According to the equilibrium equations applied in step 2, the following is true:

- R_y points toward the cut section, so it is compressive (negative). In addition, the force affects all points on the cross-section in the same way, so the R_y/A term is negative for both points P and Q.
- M_{Rz} is counterclockwise for the section of the body above the cut at A-A.

However, correctly visualizing bending direction tends to be easier if we consider the direction of the bending moment for the section of the body attached to the fixed support. In this case that is the section below the cut. Because the reactions on either side of the cut act in opposing directions (equal/opposite reactions), the section attached to the fixed support is subjected to a clockwise bending moment.

The opposing reactions at the cut are shown here, as well as a sketch of the body bending clockwise around the z-axis and a top down view of the cross-section with the moment arrow going clockwise around the z-axis.

Given the above, note that point P will be in tension (outside of the bend) from the bending stress and point Q will be compression (inside of bend).



4. Calculate the stress.

In addition to R_y and M_{Rz}, the following are required:

- A = (6in.)(8in.) = 48 $in.^2$
- d_x for point P = x distance between P and the z-axis = 2 in.
- d_x for point Q = x distance between Q and the z-axis = 4 in.

$$I_z = \frac{1}{12}bh^3 = \frac{1}{12}(6\ in.)\ (8\ in.)^3 = 256\ in.^4$$

Evaluate point P.

$$\sigma_{yP} = -\left|\frac{125\ lb}{48\ in.^2}\right| + \left|\frac{(125\ lb{\cdot}in.)(2\ in.)}{256\ in.^4}\right|$$
$$\sigma_{yP} = -1.63\ psi$$

Evaluate point Q.

$$\sigma_{yQ} = -\left|\frac{125\ lb}{48\ in.^2}\right| - \left|\frac{(125\ lb{\cdot}in.)(4\ in.)}{256\ in.^4}\right|$$
$$\sigma_{yQ} = -4.56\ psi$$

The negative answers indicate that the stress at both points is overall compressive.

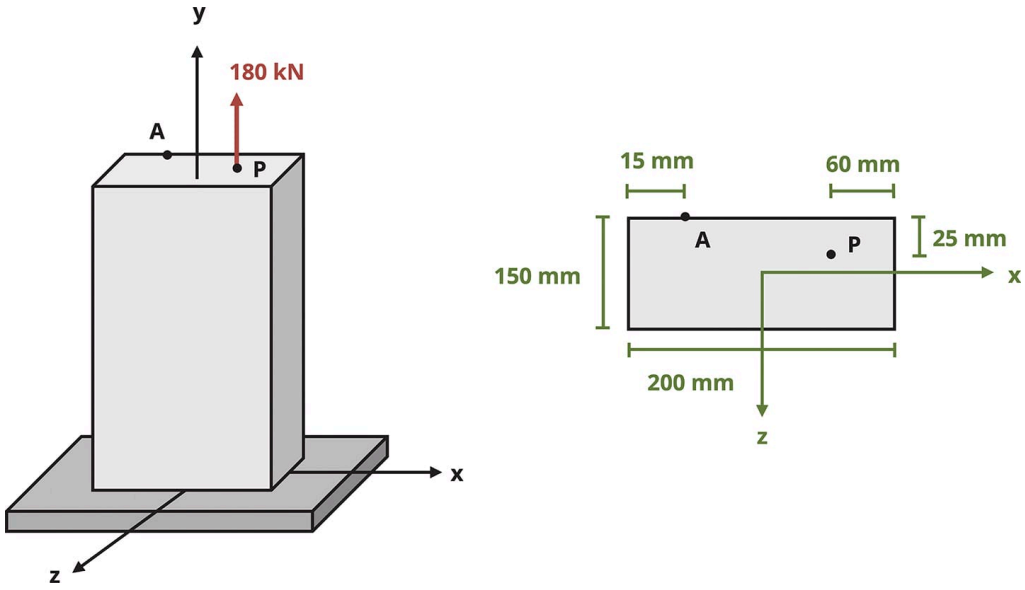
Answer:

- σ_{yP} = –1.63 psi
- σ_{yQ} = –4.56 psi

Example 14.3

A post is fixed to the ground and subjected to a 180 kN force pulling up at point P, shown on the cross-section.

Determine the normal stress at point A.

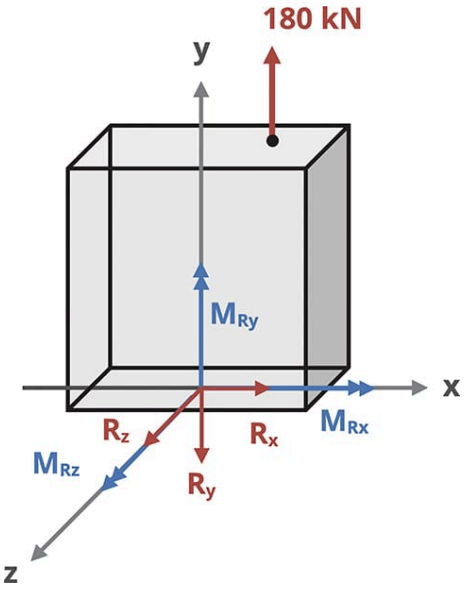


Solution

1. Draw the FBD of the structure cut at an arbitrary y location and write the equilibrium equations to find the internal reactions.

For the given loading the internal reactions will be the same for any y location, so we can make a cut anywhere to determine the bending moments and normal force reaction.

Draw an FBD above the cut as shown.



The reaction forces R_z and R_x are 0 by inspection, and the reaction moment M_{Ry} is also zero by inspection since the only applied force acts in the y direction.

$$\begin{aligned} & \sum F_y = 180kN - R_y = 0 \quad \rightarrow \quad R_y = 180kN \\ \sum M_x &= \sum \pm(Fy^*z) + \sum \pm(Fz^*y) + M_{Rx} \\ &= (180\ kN^*50\ mm) + M_{Rx} = 0 \quad \rightarrow \quad M_{Rx} = -9000\ kN \cdot mm \\ \sum M_z &= \sum \pm(F_x^*y) + \sum \pm(F_y^*x) + M_{Rz} = 0 \\ &= (180kN^*40mm) + M_{Rz} = 0 \quad \rightarrow \quad M_{Rz} = -7200\ kN \cdot mm \end{aligned}$$

2. Write the generalized normal stress equation.

Apply the generalized normal stress equation (with the pressure term omitted).

$$\sigma_L = \pm\left|\frac{F_L}{A}\right| \pm \left|\frac{M_V d_H}{I_V}\right| \pm \left|\frac{M_H d_V}{I_H}\right|$$

In this case the longitudinal axis of the post is the y-axis, the vertical axis of the cross-section is the z-axis, and the horizontal axis of the cross-section is the x-axis. Thus

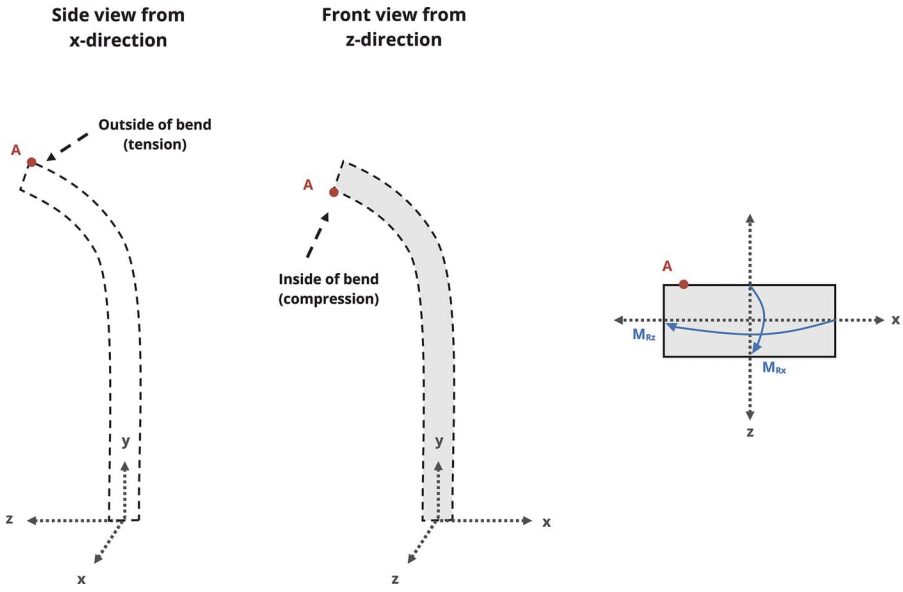
$$\sigma_{yA} = \pm\left|\frac{R_y}{A}\right| \pm \left|\frac{M_z d_x}{I_z}\right| \pm \left|\frac{M_x d_z}{I_x}\right|.$$

3. Assess the signs.

According to the equilibrium analysis, R_y points away from the cross-section, so that force results in tensile stress (positive).

The bending moments were both found to be clockwise for the section of the body above the cut. As discussed in Example 14.2, to more easily visualize the bending use the direction of the bending moments on the section of the body below the, for which the reaction moments will be opposite in direction to what we found in step 1. Thus, we visualize the column to bend counterclockwise around the x-axis and the z-axis.

The bending shapes that correspond to the two bending moments are shown here along with a sketch of the cross-section with the bending moment arrows represented. We can see that the bending moment around the x-axis results in tension for point A but the bending moment around the z-axis results in compression.



4. Calculate.

$$\sigma_{yA} = \left|\frac{R_y}{A}\right| - \left|\frac{M_z d_x}{I_z}\right| + \left|\frac{M_x d_z}{I_x}\right|$$

$$\sigma_{yA} = \left|\frac{180{,}000\ N}{(0.2\ m)(0.15\ m)}\right| - \left|\frac{(7{,}200\ N{\cdot}m)(0.085\ m)}{\frac{1}{12}(0.15\ m)(0.2\ m)^3}\right| + \left|\frac{(9{,}000\ N{\cdot}m)(0.075\ m)}{\frac{1}{12}(0.2\ m)(0.15\ m)^3}\right|$$

$$\sigma_{yA} = 11.9\ MPa$$

The stress works out to be positive and therefore tensile.

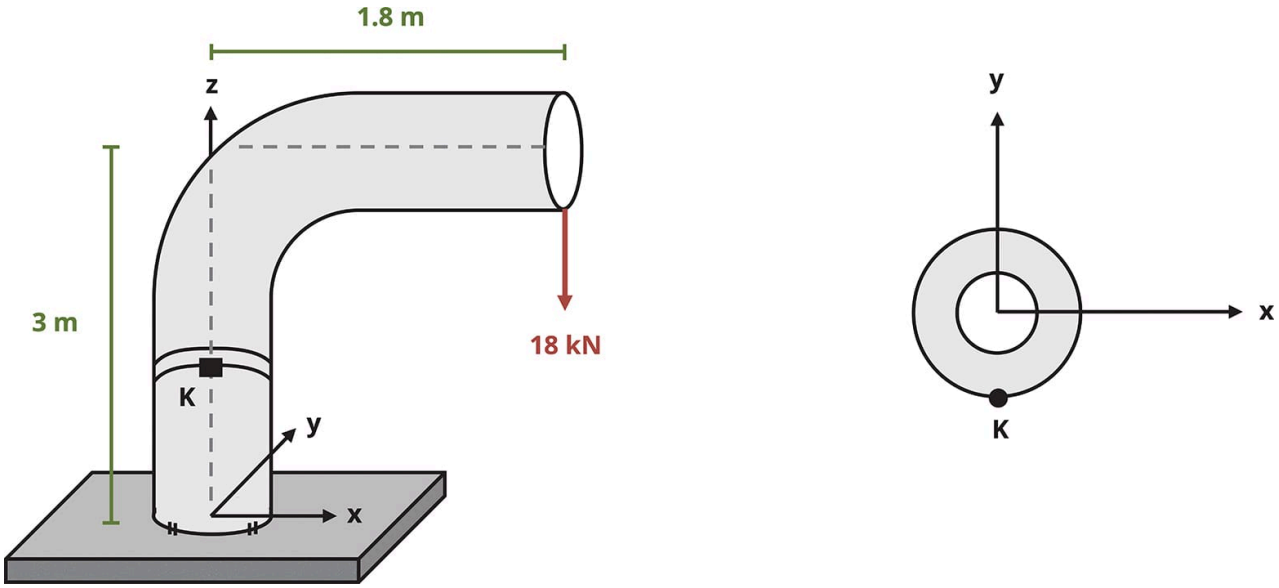
Answer:

- σ_{yA} = 11.9 MPa

Example 14.4

For the pressurized pipe loaded as shown, determine the total normal stress at point K. Noting that there would be no shear stress at point K for this loading, represent the general stress state of point K on a stress element.

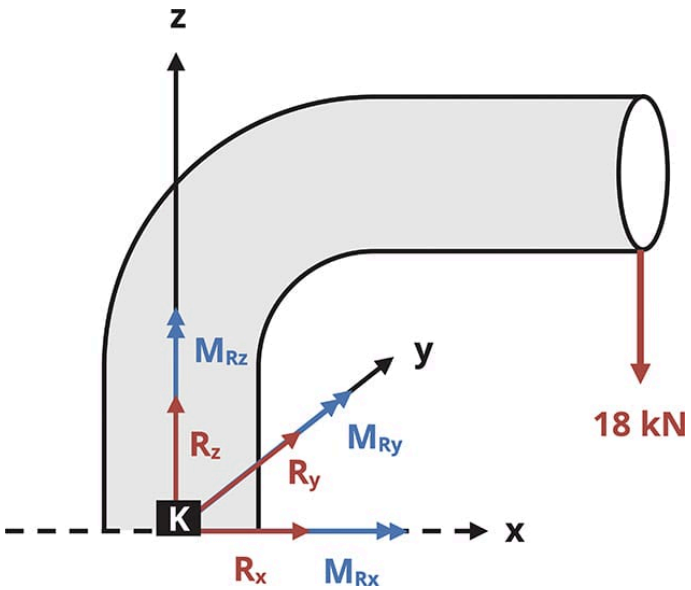
The pipe has an outside diameter of 200 mm and a wall thickness of 10 mm. The load P_z = 18 kN, and the internal pressure in the pipe is 1,500 kPa.



Solution

1. Draw the FBD of the structure cut at point K and write the equilibrium equations to find the internal reactions.

The FBD above the cut is as shown.



There are no forces in the x or y direction, so $R_x = R_y = 0$.

The 18 kN force is parallel to the z-axis and traverses the x-axis, so $M_{Rz} = M_{Rx} = 0$.

M_{Ry} is not zero; however, point K is on the y-axis, so this point will be unaffected by M_{Ry} (the x distance from the y-axis to point K

on the cross-section is 0).

Thus, we don't need to calculate moment reactions.

Calculate the reaction force in the y direction.

$$\sum F_y = R_y - 18\ kN = 0 \quad \rightarrow \quad R_y = 18\ kN$$

2. Write the general normal stress equation and eliminate unnecessary terms. Note that the pressure loading will contribute to the longitudinal normal stress and will also result in normal stress in the cross-sectional directions (hoop stress).

$$\sigma_L = \pm \left| \frac{F_L}{A} \right| \pm \left| \frac{M_V d_H}{I_V} \right| \pm \left| \frac{M_H d_V}{I_H} \right| + \frac{pr}{2t}$$
$$\sigma_V = \sigma_H = \sigma_{hoop} = \frac{pr}{t}$$

In this problem the longitudinal direction is the z direction and the cross-sectional directions are x and y. Eliminate both bending stress terms, as discussed in step 1.

$$\sigma_{zK} = \pm \left| \frac{R_z}{A} \right| + \frac{pr}{2t}$$
$$\sigma_x = \sigma_y = \frac{pr}{t}$$

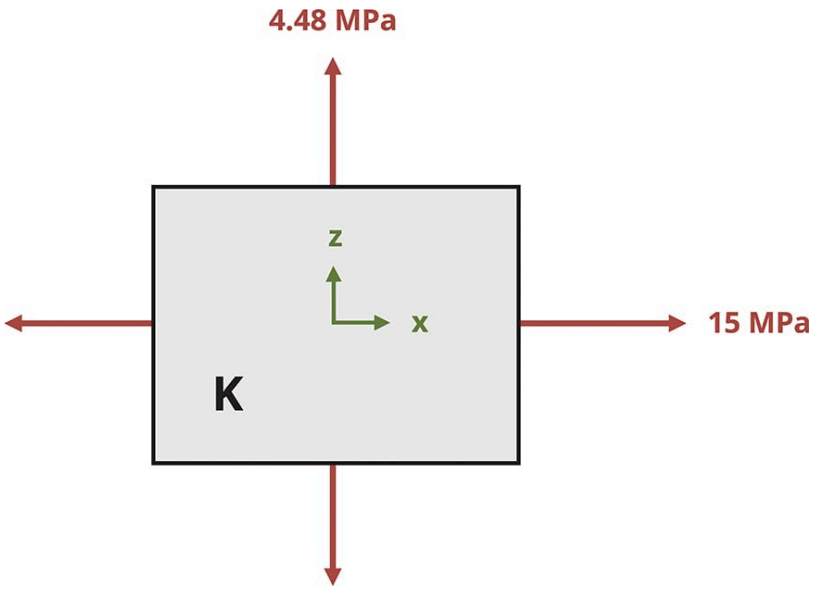
3. Assess signs.

According to the equilibrium analysis, R_z points toward the cut section, so the first term in the stress equation will be negative. Internal pressure stress will always be positive.

4. Calculate stress.

$$\sigma_{zK} = -\left| \frac{R_z}{A} \right| + \frac{pr}{2t}$$
$$= -\frac{18{,}000\ N}{\pi(0.1^2 - 0.09^2)m^2} + \frac{(1{,}500{,}000\ Pa)(0.1\ m)}{2(0.01\ m)}$$
$$= 4.48\ MPa$$
$$\sigma_x = \frac{pr}{t} = \frac{(1{,}500{,}000\ Pa)(0.1\ m)}{0.01\ m} = 15\ MPa$$

5. Sketch the stress element, noting that point K is in the x-z plane.



Step-by-Step: Combined Normal Stress

1. Cut the structure at the point of interest and determine the internal reactions.

- The sum of the forces in the longitudinal direction will give the total normal force.
- The sum of the moments about each of the cross-sectional centroidal axes will give the bending moments: M_V about the vertical cross-sectional axis and M_H about the horizontal cross-sectional axis.

2. Write the general normal stress equation for the longitudinal and cross-sectional directions and eliminate unnecessary terms (i.e., terms from loading that is not applied or that doesn't affect the point in question).

$$\sigma_L = \pm\left|\frac{F_L}{A}\right| \pm \left|\frac{M_V d_H}{I_V}\right| \pm \left|\frac{M_H d_V}{I_H}\right| + \frac{pr}{2t}$$
$$\sigma_V = \sigma_H = \sigma_{hoop} = \frac{pr}{t}$$

3. Assess signs, assigning a positive sign to tensile terms and a negative sign to compressive terms.
   - For longitudinal forces this is determined by whether the internal normal reaction force points to the cut section (compression) or away from it (tensile).
   - For bending moments assess by visualizing whether the point will be on the inside part of the bend (compression) or outside part of the bend (tension). Drawing the bending moment arrow on the cross-section can be helpful in making this determination.
   - Internal pressure is always positive.
4. Substitute values and signs from steps 1 and 3 into the equations from step 2.
5. If needed, represent the stress state on a stress element.

14.3 General Shear Stress

Shear stress due to torsion was discussed in Chapter 6 while shear stress due to transverse loading was discussed in Chapter 10. This section discusses the additional considerations that go into determining the total shear stress when both forms of loading are present.

14.3.1 Shear Stress due to Torsion

As discussed in Section 6.1, shear stress due to torsion in a circular shaft can be calculated as Equation 6.1.

$$\tau_{torsional} = \frac{T\rho}{J},$$

where

- $T = Torque = \sum M_L$ [N·m, lb·in.]
- *ρ = Radial distance from center to the point of interest on the cross-section [m, in.]*
- *J = Polar second moment of area for circular cross-section [m^4, $in.^4$]*

Once again, as was done for the normal stress equations, the ± and absolute value is used to signify that we will calculate the magnitude of the stress and assess the sign independently.

General problems may require us to sum moments to calculate T, so it's important to remember that torque is the moment about a body's longitudinal axis, or that moment component causes the body to twist. Since this form of shear stress is combined with transverse shear stress to obtain the total stress, we must understand the sign of the shear stress.

Section 12.1 showed that the sign of shear stress depends on the orientation of the shear stress arrows on the stress element that represents the stress state of the point of interest. To determine in which direction to draw a shear stress arrow on a stress element for a particular point, cut the body at one edge of the stress element and visualize the direction in which the shear load would act across the cut.

For example, Figure 14.8 shows the stress element to be in the L-H plane. When the cylinder is cut at the top of the stress element, examine the bottom section of the cylinder and note that the reaction torque at the cut surface should be opposite in direction to that applied at the bottom surface. This results in a counterclockwise (from the top view perspective) reaction torque at the cut, which means the top section tends to want to twist (and exert a force) from left to right across the stress element's top edge.

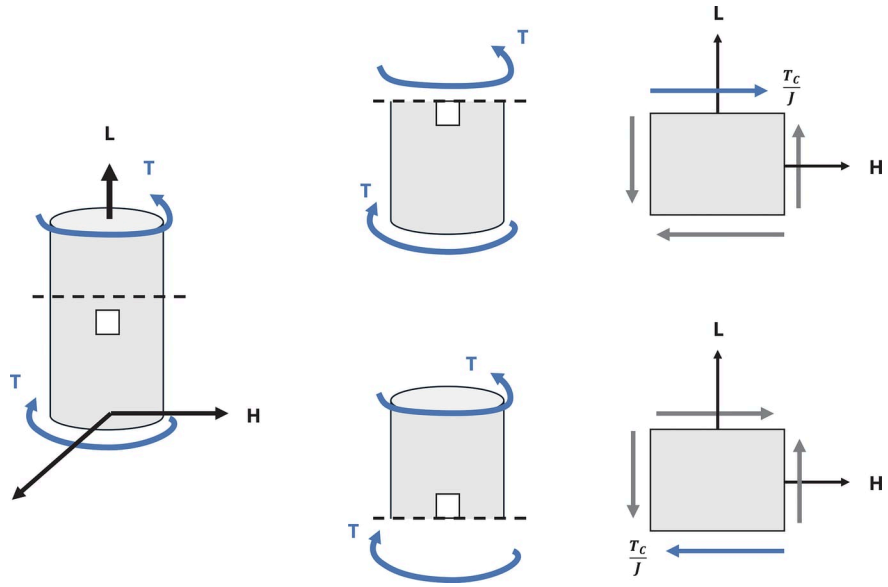


Figure 14.8: Observation of the direction of the shear load acting on the stress element indicates the sign of the shear stress.

If the longitudinal axis is positive upward, the top face of the element is a positive L face. In addition, if the horizontal axis is positive to the right, the rightward shear force arrow acts in in the positive H direction. A positive H force on the positive L face constitutes a positive shear stress.

Similarly, when the cut is made at the bottom edge of the element, we can examine the cylinder's top section, for which equilibrium dictates there is a clockwise (again, based on top view) reaction moment at the cut edge. This means that the bottom section tends to want to twist from right to left across the bottom element's edge. The right to left arrow on the bottom face of the element represents a negative H force on a negative L face. The double negative indicates a positive shear stress, confirming the result obtained above from studying the bottom section of the cylinder.

If the H-axis were to be oriented to be positive to the left instead, then this same shear stress would be negative. The same can be said if the H-axis is positive to the right but the L-axis is positive downward.

14.3.2 Shear Stress due to Transverse Shear Force

In Chapter 10 the equation $\tau = \frac{VQ}{It}$ (Equation 10.2) was evaluated to find the shear stress due to transverse shear force V that arises from a nonconstant bending moment. In that chapter only one direction of shear force was considered for each problem and the sign of the shear stress was not considered.

For general loading we may have two directions of transverse shear force corresponding to the two cross-sectional axes directions, as shown on the illustration of the bar in Figure 14.9. In this case, we will need to apply the transverse shear stress equation for both directions and also understand the signs so that we can combine them with each other and with the torsional shear stress. The direction of the shear force will affect the direction of the cut made across the cross-section to find Q, t, and the axis the second moment of area is based on.

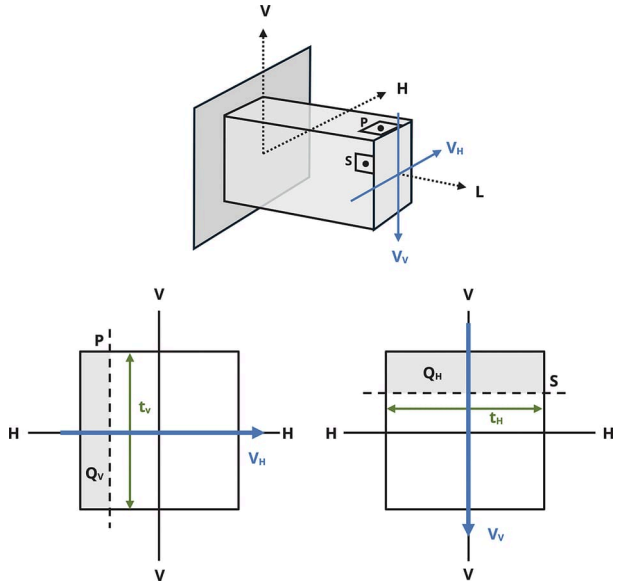


Figure 14.9: Both directions of transverse shear force applied to a bar and the corresponding cuts on the cross-section used to find Q

When applying the transverse shear stress equations for specific directions of shear force, keep the following in mind:

1. The cut on the cross-section to determine the first moment of area Q will be perpendicular to the direction of shear force direction (illustrated on the cross-sections in Figure 14.9).
2. The second moment of area (I) will be based on the axis perpendicular to the shear force direction.
3. The thickness (t), as before, refers to the thickness of the solid part of the cross-section that is cut through to calculate Q (illustrated on the cross-sections in Figure 14.9).

Using the same reference axes as in the general normal stress equation—H for horizontal cross-sectional axis and V for vertical cross-sectional axis—the transverse shear stress equation can be written as follows:

$$\tau_{transverse} = \pm \left| \frac{V_H Q_V}{I_V t_V} \right| \pm \left| \frac{V_V Q_H}{I_H t_H} \right|,$$

(14.3)

where

- $\tau_{(transverse)}$ = *Transverse shear stress at point of interest [Pa, psi]*
- V_H = *Internal horizontal shear force [N, lb]*
- V_V = *Internal vertical shear force [N, lb]*
- Q_V = *First moment of area about the vertical axis passing through the point of interest [m^3, $in.^3$]*
- Q_H = *First moment of area about the horizontal axis passing through the point of interest [m^3, $in.^3$]*
- I_V = *Second moment of area about the vertical centroidal axis of the cross-section [m^4, $in.^4$]*
- I_H = *Second moment of area about the horizontal centroidal axis of the cross-section [m^4, $in.^4$]*
- t_V = *Vertical thickness of the cross-section cut through [m, in.]*

- t_H = *Horizontal thickness of the cross-section cut through [m, in.]*

When assessing the contributions to shear stress present for any given point, consider these simplifying observations:

1. If a point is on the top or bottom edge of a noncircular cross-section or is the top or bottom point of a circular cross-section, it will experience no transverse shear stress from a vertical shear force.

2. If a point is on the left or right edge of a noncircular cross-section or is the leftmost or rightmost point of a circular cross-section, it will feel no transverse shear stress from a horizontal shear force.

For example, in Figure 14.9, point P will experience no shear stress from V_V and point S will feel no shear stress from V_H. Figure 14.10 further illustrates the preceding statements.

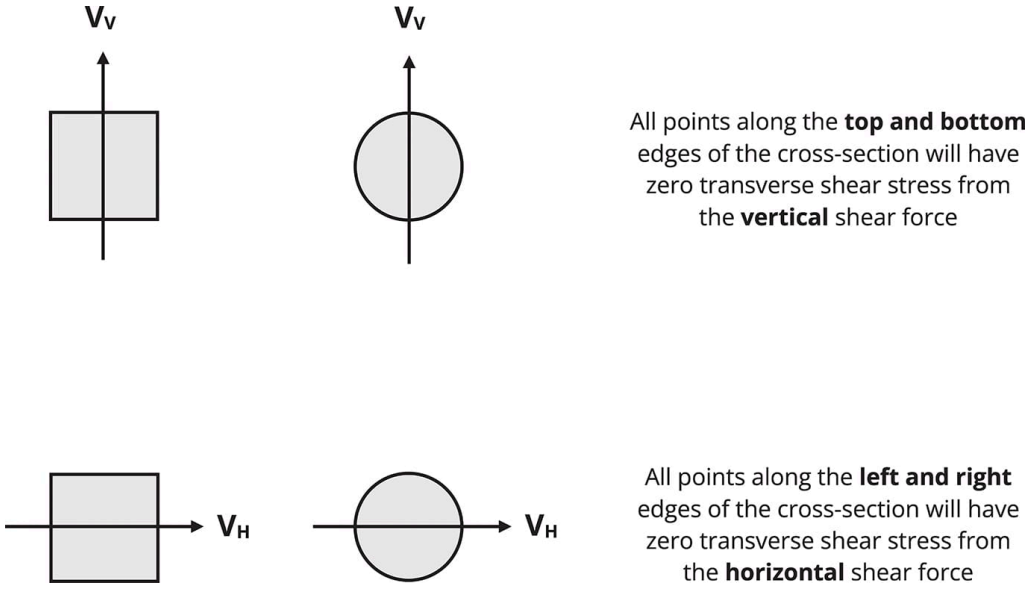


Figure 14.10: Locations of zero transverse shear stress based on direction of shear force

The reason is that in the case of a vertical shear force bending is around the H-axis so the top and bottom edges of the cross-section are free edges. Similarly, in the case of a horizontal shear force bending is around the V-axis so the left and right edges of the cross-section are free surfaces. For a circular cross-section the idea is similar but applies only to the points at the top/bottom and left/right positions along the centroidal axes on the perimeter. In other words, $Q = 0$ for the top and bottom edges in the case of a vertical shear force and for the left and right edges for a horizontal shear force.

The signs on the shear stress are determined as described in Section 14.3.1 for torsion. The direction of the internal shear force is determined by cutting the body at the point of interest, drawing the FBD of the cut section, and applying equilibrium. The resulting shear stress is considered to act in that same direction on the cut surface of the element. For example, in Figure 14.9, point S is on the V-L plane. Figure 14.11 shows the element drawn from the side view, looking down the H-axis. The shear force V_V acts downward on the stress element's right side. If the axes are oriented such that the longitudinal axis is positive to the right, the right side of the stress element is the +L-face. If the V-axis of the element is positive upward, then V_V is acting in the negative V direction on the +L-face, indicating a negative shear stress. Similarly, point P is on the H-L plane. In Figure 14.11 the stress element is drawn from the top view (viewing down the V-axis). The shear force V_H acts upward on the stress element's right side. If the right side is the +L-face and the H-axis is positive upward, then V_H acts in the +H direction on the +L-face, indicating a positive shear stress.

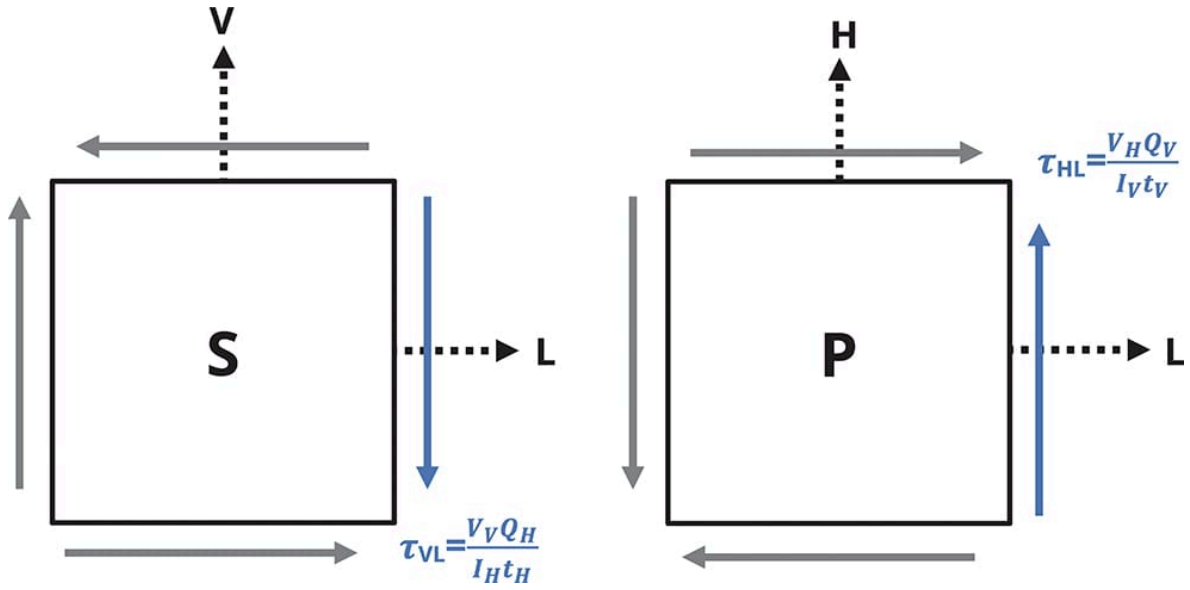


Figure 14.11: Stress elements for points S and P in Figure 14.9. The blue arrows correspond to the direction of the shear stress inducing shear force for the respective point on the cut face.

14.3.3 Combined Shear Stress

Now we can consider the case of bodies simultaneously subjected to both torsion and transverse shear stresses. Given a general loading situation, we may need to calculate the torque and also identify which of the forces constitute transverse shear forces. It will be helpful to remember:

1. The transverse shear forces are the internal forces in the directions of the cross-sectional axes.
2. Torque is the internal moment about the longitudinal axis.
    - Also recall that torsion is only considered for circular cross-sections.

The total shear stress acts in the plane of the point of interest. It can be calculated as

$$\boxed{\tau = \pm \left| \frac{T\rho}{J} \right| \pm \left| \frac{V_H Q_V}{I_V t_V} \right| \pm \left| \frac{V_V Q_H}{I_H t_H} \right|},$$

(14.4)

where

- τ = *Shear stress at point of interest [Pa, psi]*
- T = *Internal torque [N·m, lb·in.]*
- ρ = *Radial distance from centroid to point of interest [m, in.]*
- J = *Polar second moment of area [m^4, $in.^4$]*
- V_H = *Internal horizontal shear force [N, lb]*
- V_V = *Internal vertical shear force [N, lb]*
- Q_V = *First moment of area about the vertical axis passing through the point of interest [m^3, $in.^3$]*

- Q_H = *First moment of area about the horizontal axis passing through the point of interest [m^3, $in.^3$]*
- I_V = *Second moment of area about the vertical centroidal axis of the cross-section [m^4, $in.^4$]*
- I_H = *Second moment of area about the horizontal centroidal axis of the cross-section [m^4, $in.^4$]*
- t_V = *Vertical thickness of the cross-section at the cut made for Q_V [m, in.]*
- t_H = *Horizontal thickness of the cross-section at the cut made for Q_H [m, in.]*

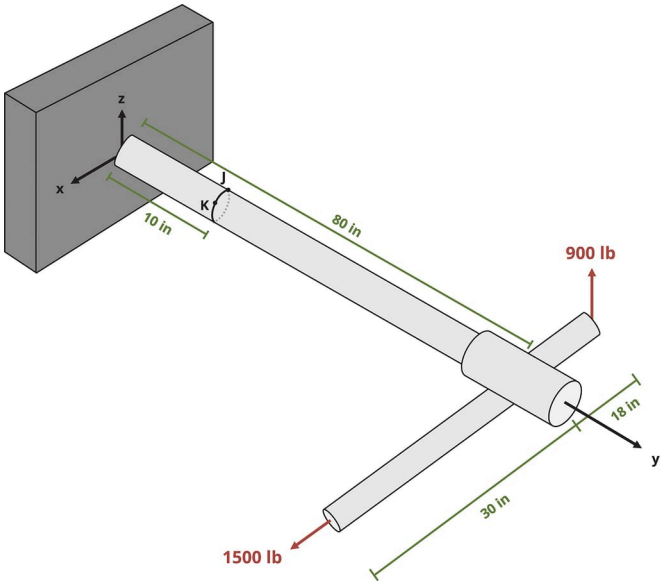
Example 14.5 demonstrates the calculation of total shear stress when both torsion and transverse shear are acting.

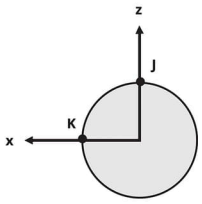
Example 14.6 demonstrates the calculation of transverse shear stress with multiple directions of shear force.

Example 14.5

A solid steel bar with an outside diameter of 6.7 in. is fixed to the wall. A 900 lb force acts vertically, and a 1,500 lb force acts in the positive x direction as shown.

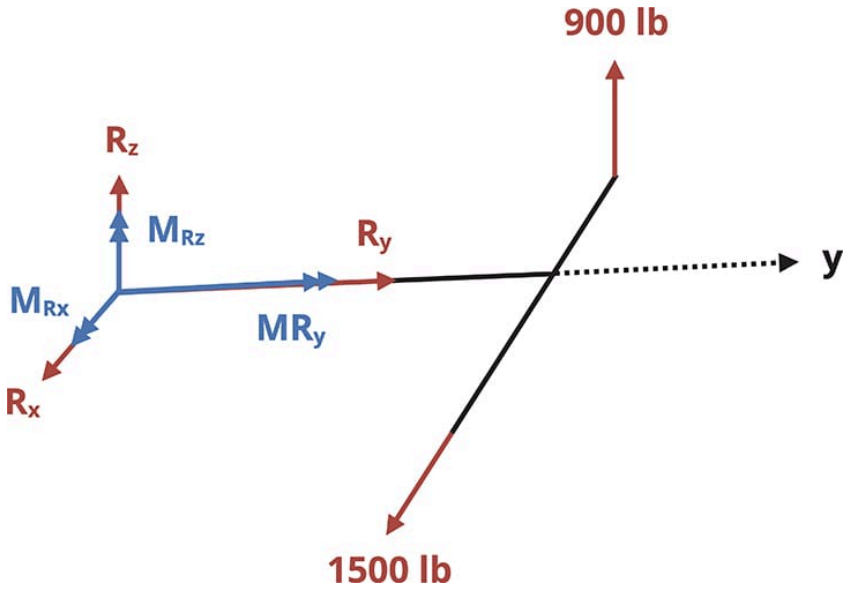
Determine the maximum shear stress at points J and K on the outside of the pipe.





Solution

1. Cut the body at the cross-section where K and J are located, then apply equilibrium to the FBD of the cut section to find the internal reactions. Examine the section to the right of the cut to avoid needing to find the reactions at the wall.



The cross-sectional axes are the x-axis and the z-axis, so these are the bending moment and transverse shear directions. The y-axis is perpendicular to the cross-section, so it is the longitudinal axis.

Before proceeding with equilibrium calculations, note that we are asked to find only shear stress. Therefore, we don't need to know the bending moments M_{Rx} and M_{Rz} nor the normal force R_y.

The shear forces are the forces in the cross-sectional directions (R_x and R_z). Find them by applying the force equilibrium equations.

$$\begin{aligned} \sum F_x = R_x + 1{,}500\ lb = 0 \quad &\rightarrow \quad R_x = V_x = -1{,}500\ lb \\ \sum F_z = R_z + 900\ lb = 0 \quad &\rightarrow \quad R_z = V_z = -900\ lb \end{aligned}$$

The torque is the moment about the longitudinal axis (M_{Ry}). Find it by applying the moment equilibrium equation for the y-axis. Note that on the FBD the direction of M_{Ry} is assumed to act counterclockwise around the y-axis when viewed from the positive end to the negative (right to left).

$$\sum M_y = M_{Ry} + (900\ lb)\,(18\ in.) = 0 \ \rightarrow \ M_{Ry} = T = -16{,}200\ lb \cdot in.$$

The negative result means the torque at the cut side of the FBD acts clockwise around the y-axis when viewed from the positive side toward the negative (right to left).

2. Write the general shear stress equation and eliminate unnecessary terms.

$$\tau = \pm \left| \frac{T\rho}{J} \right| \pm \left| \frac{V_H Q_V}{I_V t_V} \right| \pm \left| \frac{V_V Q_H}{I_H t_H} \right|$$

The horizontal axis of the cross-section is the x-axis and the vertical axis is the z-axis.

Point K is on the y-z plane. Since point K is on the x-axis, it will not experience shear stress from V_x.

Point J is on the x-y plane. Since point J is on z-axis, it will not experience shear stress from V_z.

$$\begin{aligned} \tau_{Kyz} &= \pm \left| \tfrac{T\rho}{J} \right| \pm \left| \tfrac{V_z Q_x}{I_x t_x} \right| \\ \tau_{Jxy} &= \pm \left| \tfrac{T\rho}{J} \right| \pm \left| \tfrac{V_x Q_z}{I_z t_z} \right| \end{aligned}$$

3. Assess the signs.

For the torsional stress term the following is true:

If we consider the stress element for each point to be just inside the cut section used to find the internal reactions, as shown in the figure below, we can see that the left side of each element will be the side that the calculated internal shear loads act on.

Given the evident direction of M_{RY} found in step 1, illustrated in the diagram below, it acts in an upwards direction across the left face of both stress elements. This is indicated with the blue arrow and T label on the stress elements drawn. For point K this results in a negative contribution to the shear stress (positive z force on negative y face), but for point J it will be a positive shear stress (negative x force on negative y-face).

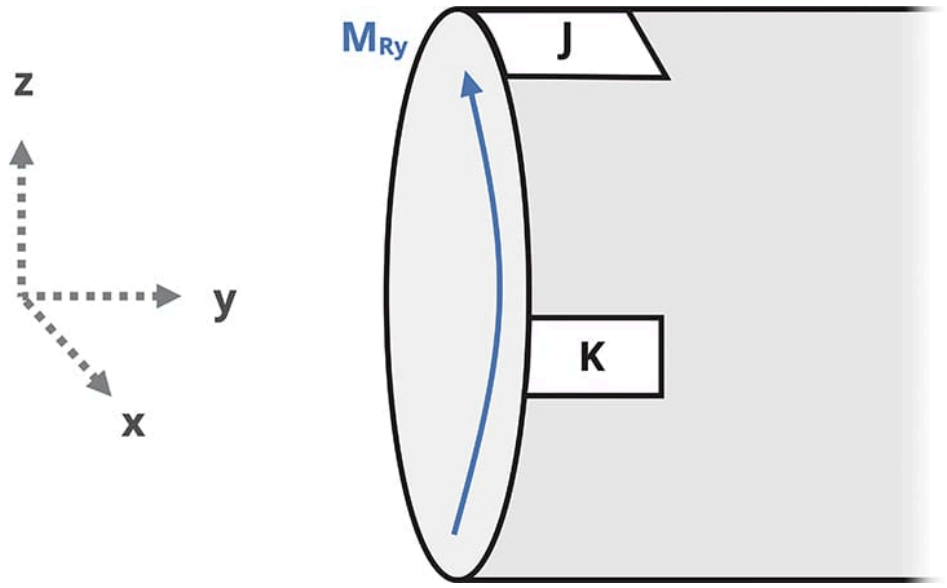
For the shear forces the following is true:

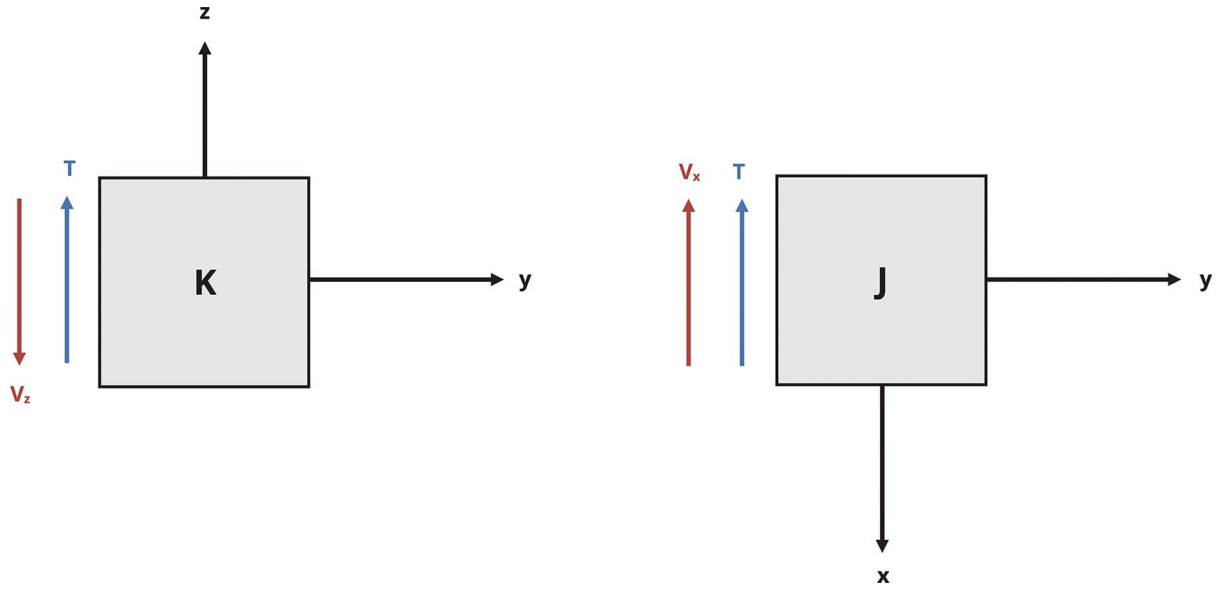
R_Z was found to act in the negative z direction, so it will act downward on the left face of the stress element for point K. The resulting shear stress will be positive.

R_X was found to act in the negative x direction, so it will act upward on the left face of the stress element for point J. This shear stress will also be positive.

The result is

$$\tau_{Kyz} = -\frac{T\rho}{J} + \frac{V_z Q_x}{I_x t_x}$$
$$\tau_{Jxy} = \frac{T\rho}{J} + \frac{V_x Q_z}{I_z t_z}.$$





4. Calculate the equations.

For a circular cross-section:

$$Q_x = Q_z = \frac{1}{12}d^3 = \frac{1}{12}(6.70\ in.)^3 = 25.06\ in.^3$$
$$I_x = I_z = \frac{\pi}{64}d^4 = \frac{\pi}{64}(6.7\ in.)^4 = 98.92\ in.^4$$
$$J = \frac{\pi}{32}d^4 = \frac{\pi}{32}(6.7\ in.)^4 = 197.83\ in.^4$$

For point K:

$$\begin{aligned}\tau_{Kyz} &= -\frac{T\rho}{J} + \frac{V_z Q_x}{I_x t_x} \\ &= \frac{(-16{,}200\ lb\ in.)(\frac{6.70\ in.}{2})}{197.83\ in.^4} + \frac{(900\ lb)(25.06\ in.^3)}{(98.92\ in.^4)(6.70\ in.)} \\ &= -274.3\ psi + 34.04\ psi \\ &= -240.3\ psi.\end{aligned}$$

For point J, the torsional shear stress will have the same magnitude.

$$\begin{aligned}\tau_{Jxy} &= 274.3\ psi + \frac{(1500\ lb)(25.06\ in.^3)}{(98.92\ in.^4)(6.7\ in.)}. \\ &= 331.0\ psi\end{aligned}$$

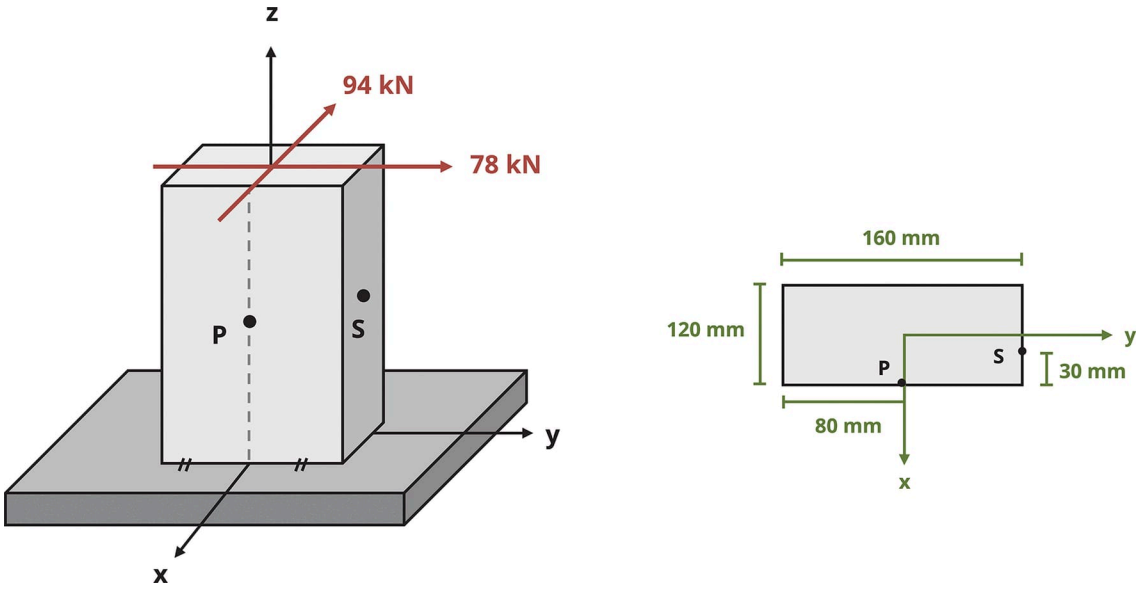
Answer:

- τ_{Kyz} = –240.27 psi
- τ_{Jxy} = 331.02 psi

Example 14.6

Two forces are applied to a rectangular post fixed to the ground support.

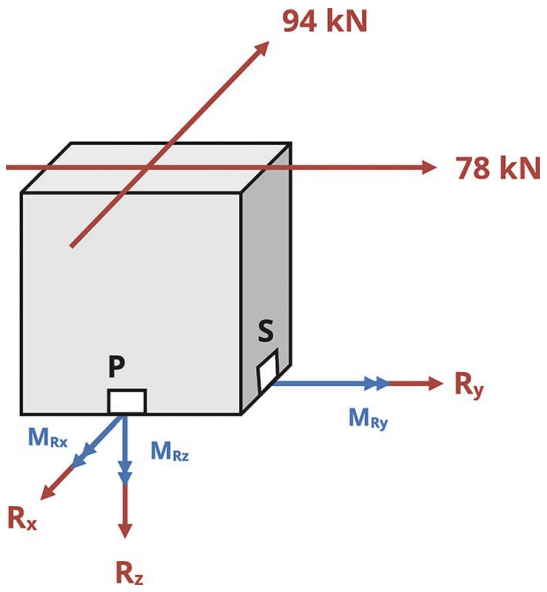
Determine the shear stress at points P and S.



Solution

1. Draw the FBD of the structure cut at the vertical location of P and S, then apply equilibrium to find the internal reactions.

The FBD of the section above the cut is as shown.



The cross-sectional axes are the x-axis and the y-axis, so these are the bending moment and transverse shear directions. The z-axis is perpendicular to the cross-section, so it is the longitudinal axis.

While we can see $R_z = 0$, it is also not necessary to find since it only contributes to normal stress.

Similarly, the bending moments M_{Rx} and M_{Ry} are not necessary to find for this problem since they would also only contribute to the normal stress. However, this problem is revisited in Example 14.7, where those moments are found.

Since both of the applied forces pass through the centroidal z-axis, $M_{Rz} = T = 0$. Find the internal shear forces using force equilibrium equations.

$$\sum F_x = R_x - 94\,kN = 0 \quad \rightarrow \quad R_x = V_x = 94\,kN$$
$$\sum F_y = R_y + 78\,kN = 0 \quad \rightarrow \quad R_y = V_y = -78\,kN$$

2. Write the general shear stress equation and eliminate unnecessary terms. Ignoring the torsion term since T = 0 (and we have a noncircular cross-section), the general shear stress equation can be written

$$\tau = \pm \left| \frac{V_H Q_V}{I_V t_V} \right| \pm \left| \frac{V_V Q_H}{I_H t_H} \right|,$$

where the horizontal axis of the cross-section is the y–axis and the vertical axis of the cross-section is the x-axis.

Recall that Section 14.3.2 noted that since point P is on the bottom edge of the cross-section (from the top view), it will not experience shear stress from V_x. Similarly, point S is on the right edge of the cross-section (also from the top view), so it will not experience shear stress from V_y.

This leaves us with

$$\tau_{Pyz} = \pm \left| \frac{V_y Qx}{I_x t_x} \right|$$
$$\tau_{Sxz} = \pm \left| \frac{V_x Qy}{I_y t_y} \right|.$$

3. Assess the signs.

Since section above the cut was used to calculate internal reactions, consider the stress elements to be at the bottom of the section, just inside the cut surface, as shown on the FBD in step 1. The shear forces then act on the bottom face of the stress elements.

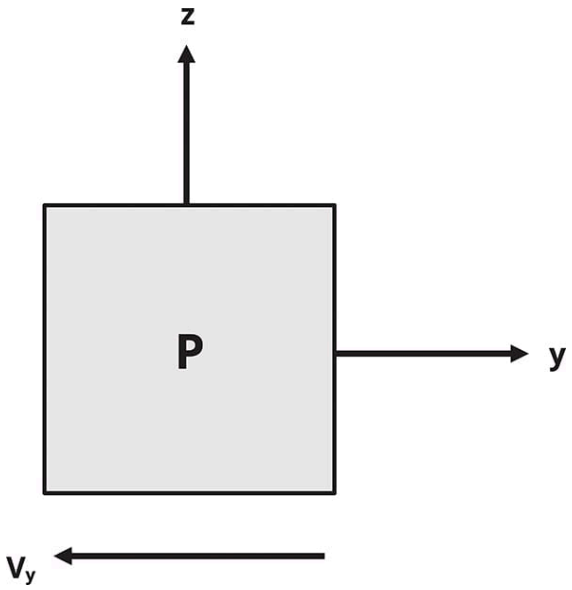
Point P is on the x-y plane and point S is on the x-z plane. Both stress elements are shown below with axes positive in the indicated directions.

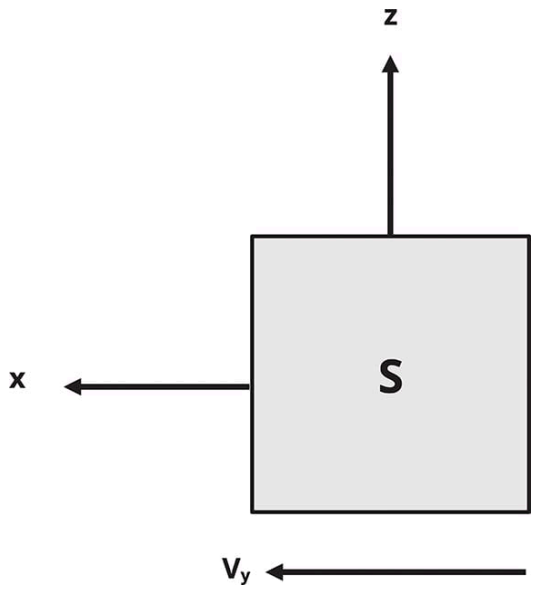
V_y was found to act in the negative y direction, so it is drawn directed to the left at the bottom of the P stress element. This results in a positive shear stress for point P.

$$\tau_{Pyz} = \frac{V_y Qx}{I_x t_x}$$

V_x was found to act in the positive x direction, so it is drawn directed to the left at the bottom of the S stress element. This results in a negative shear stress for point S.

$$\tau_{Sxz} = -\frac{V_x Qy}{I_y t_y}$$





4. Calculate stress.

$$V_H = V_y = 78\ kN$$

$$V_V = V_x = 94\ kN$$

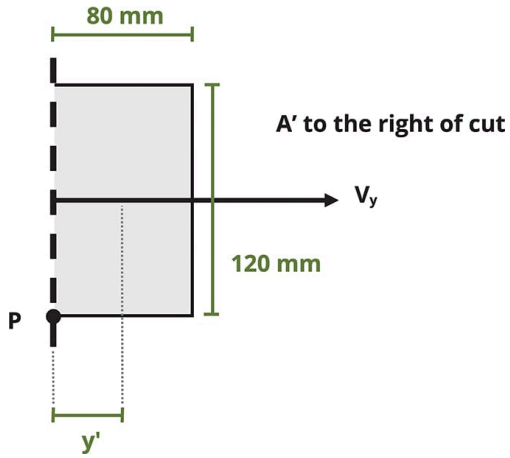
$$I_H = I_y = \frac{1}{12}(160\ mm)\,(120\ mm)^3 = 23.04 \times 10^6\ mm^4 = 23.04 \times 10^{-6}\ m^4$$

$$I_V = I_x = \frac{1}{12}(160\ mm)^3\,(120\ mm) = 40.96 \times 10^6\ mm^4 = 40.96 \times 10^{-6}\ m^4$$

For point P we need Q_x and t_x. Cutting vertically through point P on the cross-section produces

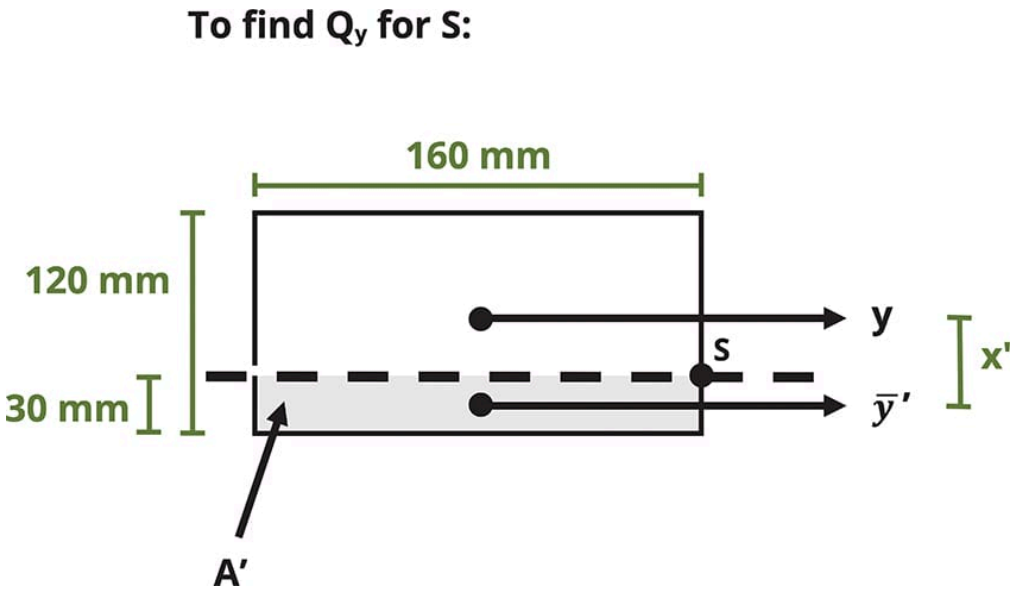
$$Q_x = A'y' = \tfrac{1}{2}(160\ mm)\,(120\ mm) * \left(\tfrac{1}{4}\right)(160\ mm) = 384{,}000\ mm^3 = 384 \times 10^{-6}\ m^3$$
$$t_x = 120\ mm = 0.12\ m.$$

To find Q_x for point P:



For point S we need Q_y and t_y. Cutting horizontally through point S on the cross-section produces

$$Q_y = A'x' = (160\,mm)\,(30\,mm)\,*\,(60\,mm - 15\,mm) = 216{,}000\,mm^3 = 216 \times 10^{-6}\,m^3$$
$$t_y = 160\,mm = 0.16\,m.$$



So the resulting equations are

$$\tau_{Pyz} = \frac{V_y Q_x}{I_x t_x} = \frac{(78{,}000\,N)(384\times10^{-6}\,m^3)}{(40.96\times10^{-6}\,m^4)(0.12\,m)} = 6.09\,MPa$$
$$\tau_{Sxz} = -\frac{V_x Q_y}{I_y t_y} = -\frac{(94{,}000\,N)(216\times10^{-6}\,m^3)}{(23.04\times10^{-6}\,m^4)(0.16\,m)} = -5.51\,MPa.$$

Answer:

- τ_{Pyz} = 6.09 MPa
- τ_{Sxz} = –5.51 MPa

Step-by-Step: Combined Shear Stress

1. Cut the body at the location of the point of interest and determine the internal reactions at that point.
2. Write the generalized shear stress equation and eliminate unnecessary terms (e.g., terms from loading that are not applied or that don't affect the point in question).

$$\tau = \pm\left|\frac{T\rho}{J}\right| \pm \left|\frac{V_H Q_V}{I_V t_V}\right| \pm \left|\frac{V_V Q_H}{I_H t_H}\right|$$

Remember that if the point in question is on an edge through which the transverse shear force passes, the shear stress at that point from that direction of shear force will be 0.

Also remember that for this text, torsion will only be applicable for circular cross-sections.

3. Assess the signs by drawing the direction in which the shear stress acts on the cut side of the stress element.
4. Calculate the total combined sheer stress.

Remember when finding Q and t to cut through the point in the direction perpendicular to the shear force.

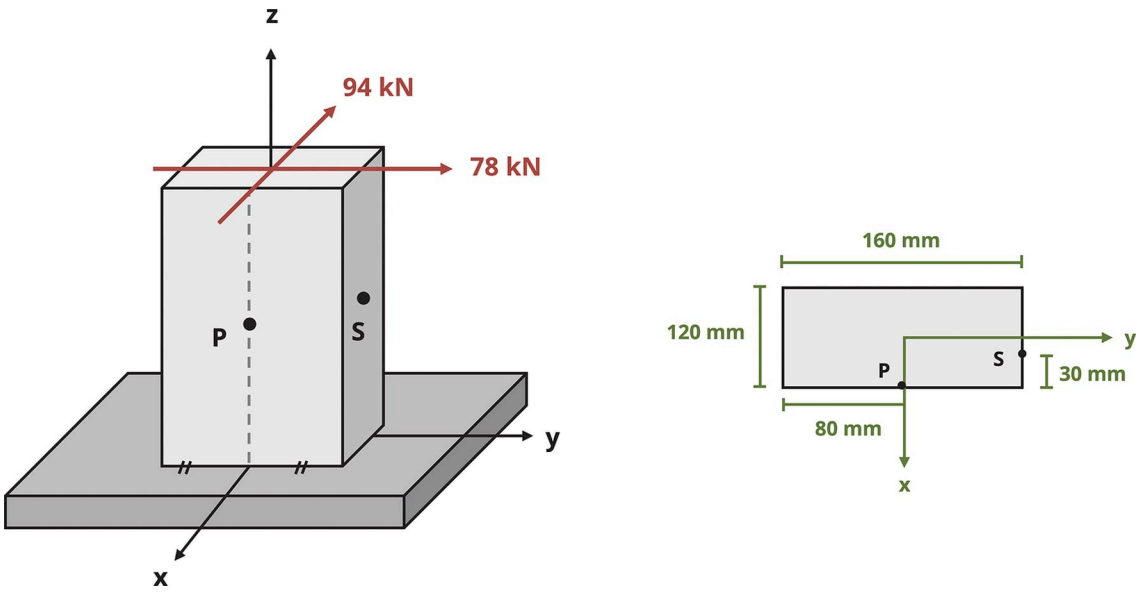
14.4 General Combined Stress

Once the total normal stress and total shear stress at a specified point are calculated, the general stress state can be represented on a stress element and used to determine principal stresses, principal planes, and max/min shear stresses.

Example 14.7 extends the problem from Example 14.6 to find the principal stresses and max/min shear stresses.

Example 14.7

Reexamine Example 14.6 to find the principal stresses and max/min shear stresses for point P on the post.



Solution

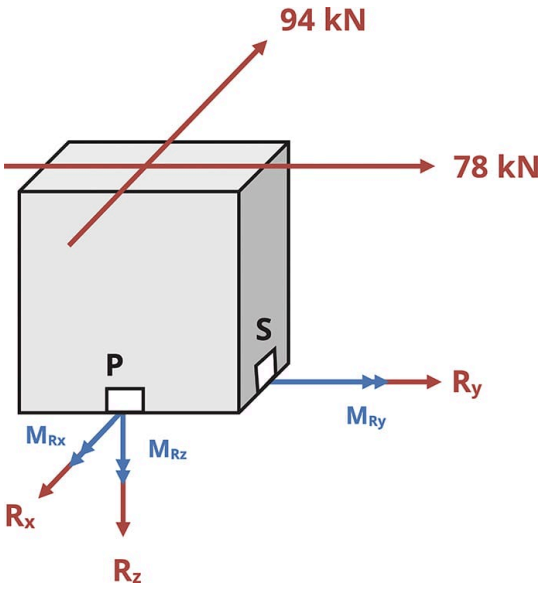
In Example 14.6 the total shear stress for point P was found to be positive 6.09 MPa.

To find the principal stresses we must take two steps:

1. Finish finding the general stress state by also finding the total normal stress.
2. Use the principal stress equations or Mohr's circle.

To complete the first task:

1. Again use the FBD of the section of the post above the cut made at the vertical location of P and S to find the relevant internal reactions.



We know from Example 14.6 that R_x = 94 kN and R_y = –78 kN.

Inspection shows us that R_z = 0.

Point P is on the x-axis, so it will not experience any bending stress from M_{Rx}.

As noted in the previous example, M_{Ry} = 0 by inspection, but it would not contribute to the normal stress in any case since that would be a torsional moment.

Thus the only bending moment reaction to calculate is M_{Ry}. Do this using the moment equilibrium equation for the y-axis.

$$\begin{aligned}\sum M_y &= (-94\,kN)(150\,mm) + M_{Ry} = 0 \\ &= -14,100\,kN\,mm + M_{Ry} = 0 \quad \rightarrow \quad M_{Ry} = 14,100\,kN\,mm\end{aligned}$$

2. Write the generalized normal stress equation (noting that there is no pressure loading).

$$\sigma_L = \pm\left|\frac{F_L}{A}\right| \pm \left|\frac{M_V d_H}{I_V}\right| \pm \left|\frac{M_H d_V}{I_H}\right|$$

From the top down view of the cross-section shown, we see that the vertical axis of the cross–section is the x-axis and the horizontal axis is the y-axis. The z-axis is the axis perpendicular to the cut surface, so it is the longitudinal axis. The generalized normal stress equation can be rewritten with the given axes.

$$\sigma_z = \pm\left|\frac{R_z}{A}\right| \pm \left|\frac{M_{Rx} d_y}{I_x}\right| \pm \left|\frac{M_{Ry} d_x}{I_y}\right|$$

With R_z = 0 and point P located on the x-axis (see discussion in step 1), the equation can be reduced to

$$\sigma_{Pz} = \pm\left|\frac{(M_{Ry})(x)}{I_y}\right|.$$

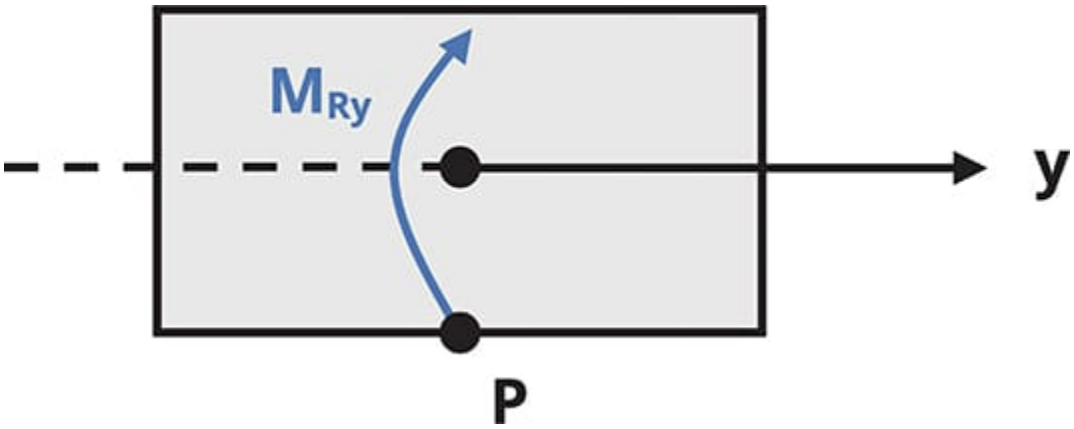
3. Assess the sign.

To assess the sign recall that the reaction moment at the cut surface of the upper section of post was determined to be positive, or counterclockwise. That means the reaction moment at the cut surface for the lower section of post (beneath the cut at P and S) will be clockwise. As mentioned in Example 14.2 and Example 14.3, use the bending moment on the section attached to the fixed support (the lower section in this case) to visualize the bending.

With a clockwise bending moment around the y-axis, looking from the side view down the y-axis, the post would appear to curve to

the right, placing point P on the outside of the bend (tension). Alternatively, drawing the cross-section from the top view with the clockwise moment arrow drawn around the y-axis shows that point P is on the tail side of the cross-section, confirming tension at this point.

So $\sigma_z = \frac{M_{Ry}x}{I_y}$



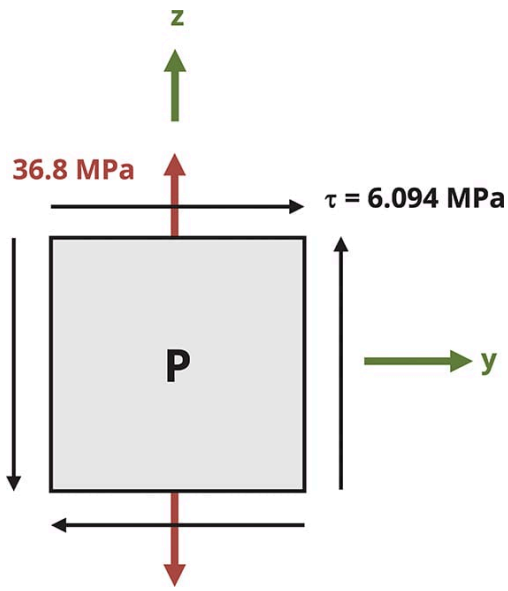
4. Calculate.

Recalling $I_y = 23.04 \times 10^{-6}\ m^4$ from Example 14.6, note that the normal stress at P is then

$$\sigma_{Pz} = \frac{M_{Ry}d_x}{I_y}$$
$$\sigma_{Pz} = \frac{14{,}100\ N{\cdot}m(0.06\ m)}{23.04{\times}10^{-6}\ m^4}$$
$$\sigma_{Pz} = 36.77\ MPa.$$

To complete the second task:

Now find the principal stresses and max/min shear stress. Start by representing the general stress state on a stress element.



We can use the simplified version of the principal stress and the max/min shear equation (Equation 12.5); however, these equations were generally written for stresses in the x-y plane.

$$\sigma_{1,2} = \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$
$$\tau_{max,min} = \pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + {\tau_{xy}}^2}$$

In this problem point P is on the y-z plane. Based on the discussion at the end of Section 12.2.2, we can replace the x subscripts with y and the y subscripts with z. Thus the equations become

$$\sigma_{1,2} = \frac{0+36.77\ MPa}{2} \pm \sqrt{\left(\frac{0-36.77\ MPa}{2}\right)^2 + (6.094\ MPa)^2}$$
$$\sigma_{1,2} = 37.75\ MPa, -0.984\ MPa$$
$$\tau_{\max,\min} = \sqrt{\left(\frac{0-36.77\ MPa}{2}\right)^2 + (6.094\ MPa)^2}$$
$$\tau_{max,min} = 19.37\ MPa, -19.37\ MPa.$$

Answer:

- σ_1 = 37.75 MPa
- σ_2 = -0.984 MPa
- τ_{min} = -19.37 MPa
- τ_{max} = 19.37 MPa

Summary

Key Takeaways

The overall general stress state for any given point on a loaded body is found by calculating and combining all forms of normal stress and all forms of shear stress.

The combined normal stress at a point comes from normal forces, bending stress, and pressure.

- Normal forces are those perpendicular to the cross-section where the point in question is located. These are also the forces in the longitudinal direction.
    - Normal forces that pull on the body are tensile (positive).
    - Normal forces that push on the body are compressive (negative).
- The bending moment contribution to normal stress comes from bending moments around the centroidal cross-sectional axes.
    - In bending, visualizing the bending shape of a body can determine if a point will be in tension or compression. A point that is located on the outside part of the bend will be in tension and a point on the inside part of the bend will be in compression.
    - Alternatively, use the visual aid described in Section 14.1 to help determine if the bending results in tension or compression.
- Both normal forces and bending moments will result in stress in the longitudinal direction with respect to the location of the cross-section.

Pressure will result in normal stress in two directions (hoop and longitudinal/axial).

- The longitudinal stress will combine with the normal stress from normal forces and bending moments to result in an overall longitudinal stress.

- The hoop stress will be the only normal stress in the cross-sectional directions for the situations considered in this text.
- Stress from internal pressure will always be tensile.

The combined shear stress at a point comes from torsion and transverse shear forces.

- The torsional moment is the moment about the longitudinal axis.
- The transverse shear forces lead in the directions of the cross-sectional axes.
- The signs for shear stress are determined by drawing the stress element and considering the direction of the internal torque or shear force on the element's cut edge relative to the element's positive and negative faces.

Key Equations

$$\sigma_L = \pm\left|\frac{F_L}{A}\right| \pm \left|\frac{M_V d_H}{I_V}\right| \pm \left|\frac{M_H d_V}{I_H}\right| + \frac{pr}{2t}$$
$$\sigma_{hoop} = \sigma_V = \sigma_H = \frac{pr}{t}$$
$$\tau = \pm\left|\frac{Tc}{J}\right| \pm \left|\frac{V_H Q_V}{I_V t_V}\right| \pm \left|\frac{V_V Q_H}{I_H t_H}\right|,$$

where

- *L = Longitudinal axis*
- *V = Vertical centroidal axis of the cross-section*
- *H = Horizontal centroidal axis of the cross-section*
- *σ = Normal stress [Pa, psi]*
- *A = Cross-sectional area [m^2, $in.^2$]*
- *F_L = Internal normal force in the longitudinal direction [N, lb]*
- *A = Cross-sectional area [m^2, $in.^2$]*
- *M_V = Bending moment around the vertical axis of the cross-section [N×m, lb×in.]*
- *M_H = Bending moment around the horizontal axis of the cross-section [N×m, lb×in.]*
- *d_H = Horizontal distance between the point on the cross-section for which the stress is being calculated and the V-axis [m, in.]*
- *d_V = Vertical distance between the point on the cross-section for which the stress is being calculated and the H-axis [m, in.]*
- *I_V = The second moment of area about the vertical axis of the cross-section [m^4, $in.^4$]*
- *I_H = The second moment of are about the horizontal axis of the cross-section [m^4, $in.^4$]*
- *p = Internal pressure [Pa, psi]*
- *r = Inner radius of pressure vessel [m, in.]*
- *t (for the pressure term) = Wall thickness of pressure vessel [m, in.]*
- *τ = Shear stress at point of interest [Pa, psi]*
- *T = Internal torque [N·m, lb·in.]*

- ρ = *Radial distance from centroid to point of interest [m, in.]*
- J = *Polar moment of inertia [m^4, $in.^4$]*
- V_H = *Internal horizontal shear force [N, lb]*
- V_V = *Internal vertical shear force [N, lb]*
- Q_V = *First moment of area about the vertical axis passing through the point of interest [m^3, $in.^3$]*
- Q_H = *First moment of area about the horizontal axis passing through the point of interest [m^3, $in.^3$]*
- t_V = *Vertical thickness of the cross-section [m, in.]*
- t_H = *Horizontal thickness of the cross-section [m, in.]*

References

Figures

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Chapter 15: Columns

Learning Objectives

- Describe column buckling behavior
- Construct the equation known as Euler's formula for columns
- Differentiate end conditions to calculate an appropriate effective length

Until now our focus has been primarily on meeting strength and deflection criteria. While these aspects are crucial, it's equally important to ensure a structure's stability. This chapter therefore delves into the stability analysis of long and slender members subjected to compressive loading, commonly referred to as columns. Some examples are shown in Figure 15.1. When the compressive load reaches a critical point, columns may undergo sudden sideways deflection, a phenomenon known as buckling. After buckling, a column is unable to support any further load. This chapter focuses on long, slender, homogeneous, axial compression members, otherwise known as columns.



Figure 15.1: Examples of columns

Section 15.1 derives the buckling formula for columns that are supported by pin connections at both ends. Section 15.2 extends this to consider other common end conditions.

15.1 Euler's Formula for Buckling

Even in ideal conditions, buckling often occurs for long, slender members at loads significantly lower than the material's crushing or compression limit. When a long, slender object is subjected to axial loading, it may buckle once the load reaches a critical threshold known as the critical load, P_{cr}. At this point the object experiences significant lateral deflection perpendicular to the load direction, so columns should be designed such that the applied load does not exceed the critical load.

Buckling of a ruler (long, slender member) is depicted in Figure 15.2. The ruler in the left and center photos is subjected to axial load, P, that is less than critical load, P_{cr}. This is known as stable equilibrium. In the photo on the right the axial load P exceeded the critical load P_{cr}, causing a significant lateral deflection. This is known as unstable equilibrium.

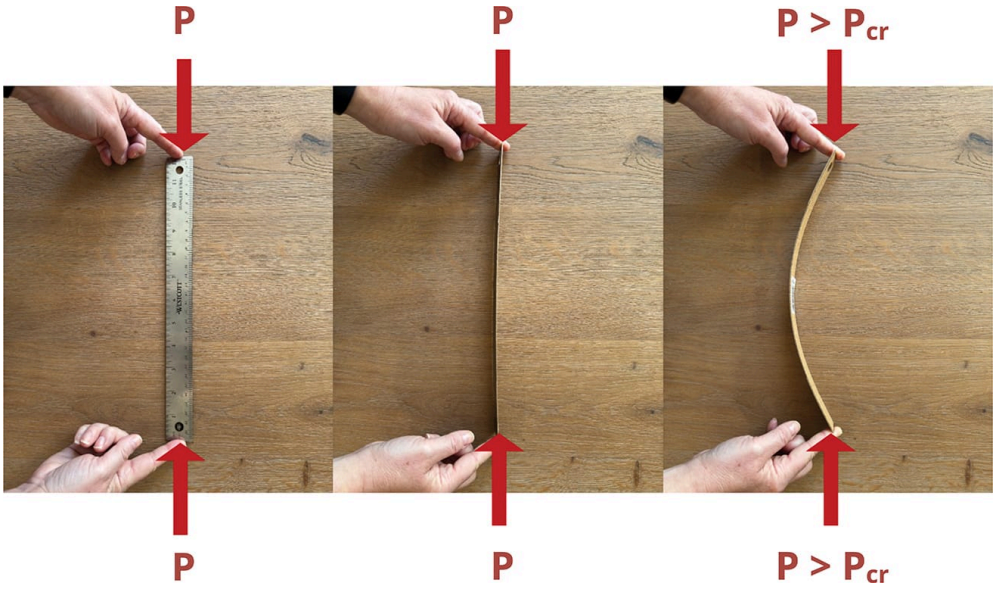


Figure 15.2: Buckling of a ruler

15.1.1 Stability

We can consider stability by using a simplified model of a pinned-pinned column. This column consists of two ridged bars connected with a pin and a spring with a stiffness k, as the free body diagram (FBD) shows in Figure 15.3 (A). When the load, P, is small, the system remains vertical and the spring is unstretched. If point C is displaced to the right a small amount, Δ, the spring will produce a force F = kΔ, as shown in Figure 15.3 (B). This force is used to resist the horizontal forces, P_x = Ptanθ, as shown in Figure 15.3 (C).

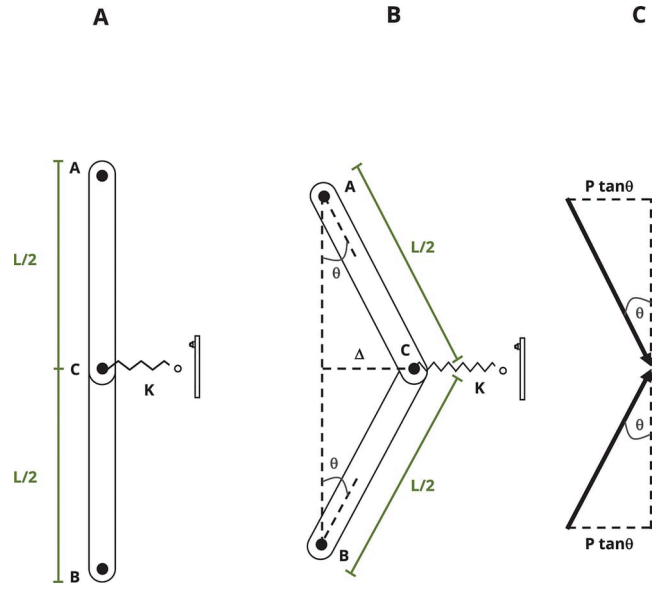


Figure 15.3: Free body diagram of a simplified column. (A) Load P is small, the column remains vertical, and the spring is unstretched. (B) The column is displaced from its vertical centerline and the spring is compressed. (C) The compressed spring produces a force that resists the horizontal components of the forces acting along the line of the column.

Since θ is small

$$\sin\theta \sim \theta \text{and } \tan\theta \sim \theta.$$

Consequently

$$\Delta = \theta\frac{L}{2} \text{and} 2P_x = 2P\theta.$$

Using the spring restoring force equation F = kΔ, then substituting in for Δ, results in

$$F = k\theta\frac{L}{2}.$$

If the spring restoring force is greater than the axial force, then equilibrium is stable.

$$F > P$$
$$k\theta\tfrac{L}{2} > 2P\theta$$

The θ cancels out, and we can solve for P.

$$P < \frac{kL}{4}$$

Similarly, we can solve for the expression for unstable equilibrium F < P.

$$P > \frac{kL}{4}$$

The point at which F = P is crucial because it is the line between stable and unstable equilibrium. This is known as the critical load P_{cr}.

$$P_{cr} = \frac{kL}{4}$$

Thus if the load P exceeds P_{cr}, the system will be unstable. If P is less than P_{cr}, the system will be stable. P_{cr} is used in the next section to determine the buckling load of columns.

15.1.2 Euler's Formula

As mentioned in earlier sections, members can fail due to material yielding or fracturing. However, instability represents another critical failure mechanism that requires consideration. Here Euler's formula can be used to calculate the theoretical buckling load. To do this we'll start with the simplest of columns, pinned-pinned end connections. This means that the column cannot translate but can rotate as shown in Figure 15.4.

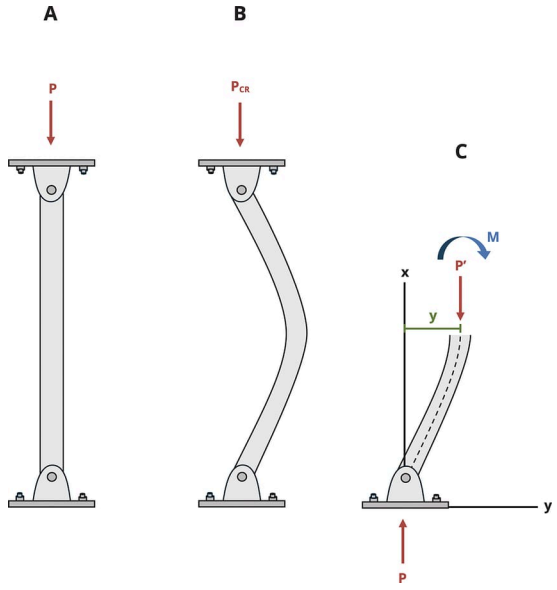


Figure 15.4: Pinned-pinned axially loaded column

Columns are similar to beams rotated 90°, so we can use what we already know about the elastic curve.

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

We can use the FBD in Figure 15.4 (C) to sum moments about any point on the lower column section.

$$\sum M = 0 = M + Py$$
$$M = -Py$$

We can now substitute this expression into the elastic curve equation.

$$\frac{d^2y}{dx^2} = \frac{-Py}{EI}$$
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right) y = 0$$

This is a linear, homogeneous, second-order differential equation with constant coefficients, which has a general solution of

$$y = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right)$$

The two constants, C_1 and C_2, can be determined by imploring boundary conditions. Since y = 0 at x = 0 and y = 0 when x = L, the first condition results in

$$0 = C_1 (0) + C_2 (1)$$
$$C_2 = 0.$$

The second boundary condition, x = L when y = 0, results in

$$0 = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + 0.$$

The equation is satisfied if C_1 = 0. However, this is a trivial solution that implies y = 0 and there is no lateral deflection. For the nontrivial solution, the sine function must be equal to zero, which requires

$$\sqrt{\frac{P}{EI}}L = n\pi$$

wheren=1,2,3,4,...

The smallest value of P occurs when n = 1; substituting this and rearranging gives us Euler's formula.

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}},$$

(15.1)

where

- P_{cr} = *Critical axial load on the column just before it buckles [N, lb]. (This load must be less than the load that would cause the stress in the column to exceed the yield stress; otherwise due to yield stress the column will fail before it buckles.)*
- *E = Elastic modulus of the material [Pa, psi]*
- *I = Second moment of area for the column [m^4, $in.^4$]. (If I_x and I_y differ, use the smaller value.)*
- *L = Unsupported length of the column (pinned-pinned connection) [m, in.]*

This formula is named after the Swiss mathematician, Leonhard Euler, who developed it in 1744. Other buckling modes can exist (n = 2, n = 3, n = 4, etc.) but are less commonly used because the load required for the higher modes would be large. The first three buckling loads can be seen in Figure 15.5.

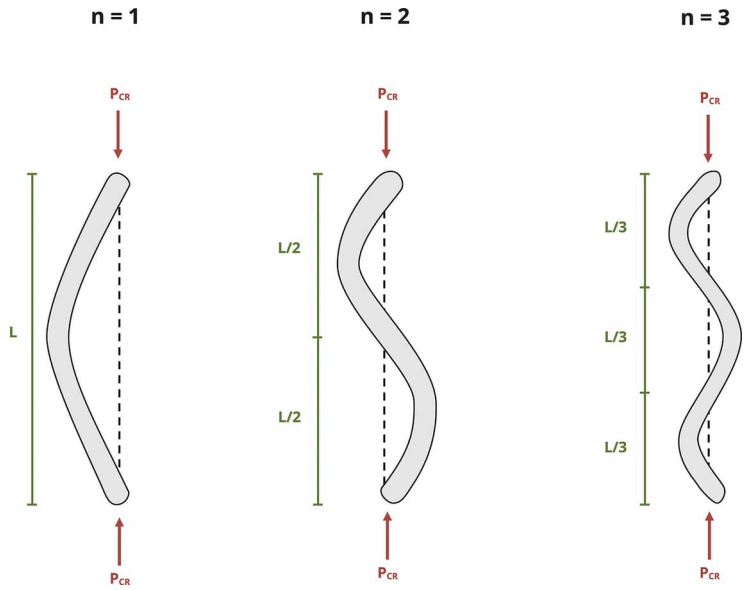


Figure 15.5: First three buckling modes

15.1.3 Buckling Direction

Columns pinned at both ends with circular or square cross-sections (equal second moments of area) can buckle in any direction. For columns with asymmetric cross-sections, buckling occurs in the plane perpendicular to the axis with the smallest second moment of area. For example, a ruler in Figure 15.2 with a rectangular cross-section buckles about its weaker axis under compression. For nonsymmetric sections, both axes should be analyzed to find the smallest second moment of area, which is then used to calculate the critical buckling load. Figure 15.6 shows buckling in wood supports.



Figure 15.6: Buckling of wooden studs

15.1.4 Critical Stress

The Euler buckling equation can be used to calculate the critical stress for a column by setting $I = Ar^2$, where A is the cross-sectional area and r is the radius of gyration. For a more comprehensive discussion on the radius of gyration, see the *Engineering Statics* book by Baker and Haynes.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E\left(Ar^2\right)}{AL^2}$$

Rearranging produces

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2}},$$

(15.2)

where

- σ_{cr} = *Critical stress on the column just before it buckles [Pa, psi]. (This stress must be less than that of the yield stress; otherwise due to yield stress the column will fail before it buckles.)*
- *E = Elastic modulus of the material [Pa, psi]*
- *L = Unsupported length of the column (pinned-pinned connection) [m, in.]*
- *r = Smallest radius of gyration of the column [m, in.]*

The quantity in the denominator, L/r, is called the slenderness ratio of the column. Many times this ratio is used to classify columns as short, intermediate, or long. The smallest value of the radius of gyration is used to find the critical stress.

A plot of the critical stress versus the slenderness ratio is shown in Figure 15.7. For larger slenderness ratios the curve is hyperbolic. For smaller slenderness ratios the critical stress is equal to the material's yield stress. This graph illustrates that if the critical stress calculated using Euler's equation is greater than the yield stress, the stress is of no interest because the column will yield before it has the chance to buckle.

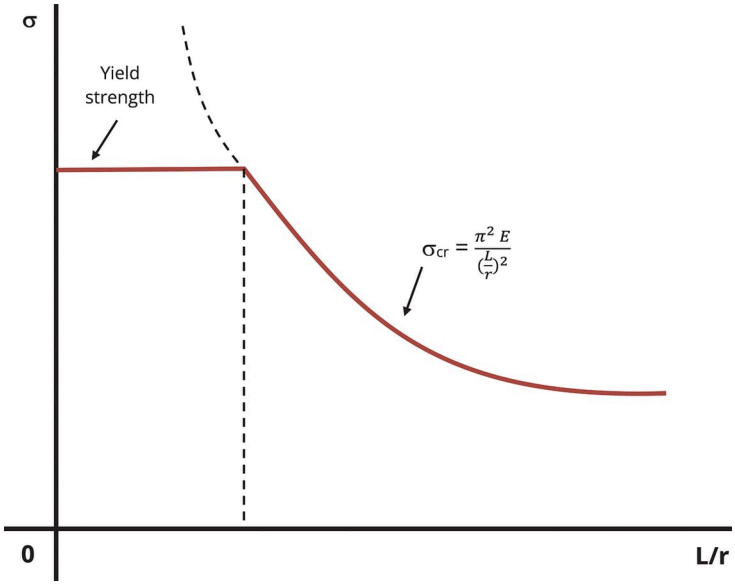


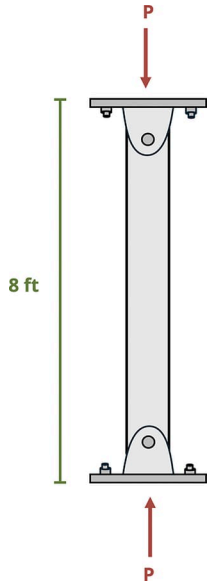
Figure 15.7: Plot of critical stress

Example 15.1 demonstrates calculation of the critical buckling load and critical buckling stress.

Example 15.1

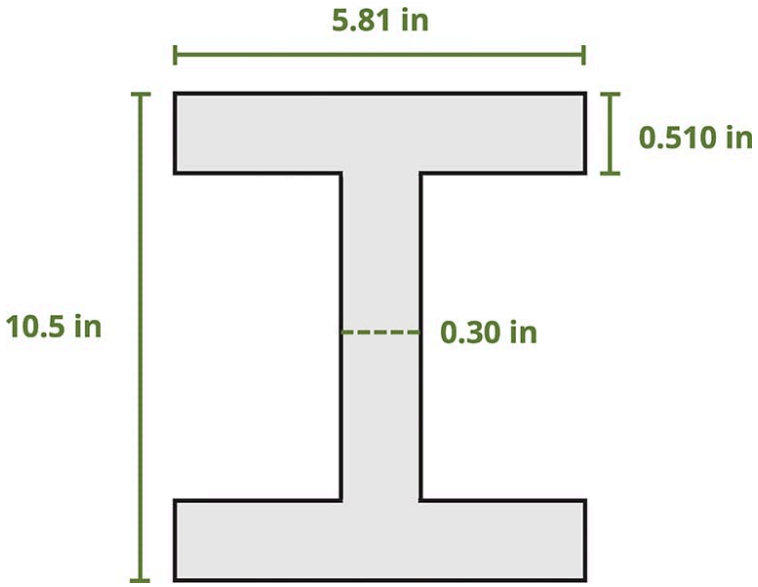
A pinned-pinned W10 × 30 column is 8 ft long and made of A492 steel (σ_y = 50 ksi and E = 29,000 ksi).

What is the largest axial load this column can support?



Solution

First consult the table in Appendix A to get the section properties of a W10 × 30 section.



$$A = 8.84\ in.^2$$
$$I_x = 170\ in.^4$$
$$r_y = 4.38\ in.$$
$$I_y = 16.7\ in.^4$$
$$r_y = 1.37\ in.$$

Then start by calculating the critical load using Euler's buckling formula. Note that we can confidently say that this column will buckle about the y-axis since the second moment of area in the y direction is the smallest.

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(29{,}000\ ksi)(16.7\ in.^4)}{\left(8\ ft * 12\frac{in.}{ft}\right)^2}$$

P_{cr}=518.6*kips*

This represents the load at which any increase in applied load will cause the column to buckle.

Next calculate the stress associated with this critical load.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{518.6\ kips}{8.84\ in.^2} = 58.7\ ksi$$

Since this stress is larger than the yield stress of the steel (σ_y = 50 ksi), the column will not buckle first. It will fail due to yielding.

Calculate the force using the stress of an axial member formula discussed in Chapter 2.

$$\sigma_{cr} = \frac{P}{A}$$
$$50\ ksi = \frac{P}{8.84\ in.^2}$$

p=442kips

Step-by-Step: Buckling of Pin-Supported Columns

1. Determine the geometric properties of the cross-section, including A, I_x, and I_y.
2. Apply Euler's formula to calculate the critical buckling load around the weak axis.
3. Calculate the critical buckling stress and check that this is less than the yield stress.
4. If the critical stress is larger than the yield stress, calculate the applied load that will cause yielding instead.

15.2 Effect of Supports

We derived Euler's equation for buckling with pinned-pinned supports (the simplest case). However, in many cases columns are supported by other end conditions. Just as Euler's formula for buckling was derived by solving a differential equation for the pinned-pinned conditions, the same can be done for other types of support cases. We will derive one more here.

Let's consider the case where a column is fixed at one end and free at the other end as shown in Figure 15.8 (A). Summing moments at the cut end of the FBD in Figure 15.8 (B) yields M = P(δ-y).

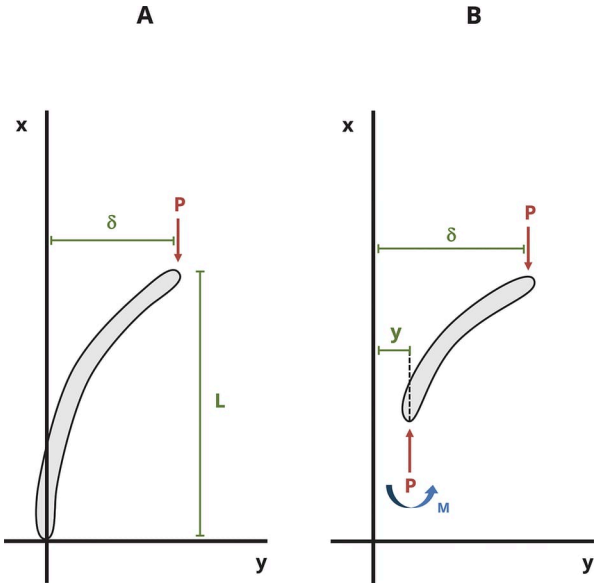


Figure 15.8: Column with a fixed base and a free end

We can now write the differential equation using the elastic curve equation.

$$EI\frac{d^2y}{dx^2} = P(\delta - y)$$
$$\frac{d^2y}{dx^2} + \left(\frac{P}{EI}\right)y = \left(\frac{P}{EI}\right)\delta$$

Since the right side is not equal to zero, this equation is nonhomogeneous (note this differs from the earlier derivation for the pinned-pinned connection). There will be a particular and complementary solution.

$$y = C_1 \sin\left(\sqrt{\frac{P}{EI}}x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}x\right) + \delta$$

We can now employ boundary conditions to solve for the constants. At x = 0, y = 0, so C_2 = δ. Also at x = 0, dy/dx = 0, so C_1 = 0. This leads us to the following deflection curve:

$$y = \delta\left(1 - \cos\left(\sqrt{\frac{P}{EI}}x\right)\right).$$

At the free end of the column, x = L, we know that y = δ, and therefore

$$0 = \delta\cos\left(\sqrt{\frac{P}{EI}}L\right).$$

The trivial solution δ = 0 shows that no matter what the P value, no buckling will occur. Thus

$$0 = \cos\left(\sqrt{\frac{P}{EI}}L\right)$$
$$\sqrt{\frac{P}{EI}}L = \frac{n\pi}{2}$$
$$\text{where n} = 1, 3, 5, \ldots$$

The smallest critical load happens at n = 1 and results in

$$P_{cr} = \frac{\pi^2 EI}{(2L)^2}.$$

Other support conditions can be derived in a similar manner.

15.2.1 Effective Length

The critical load equation for a fixed-free column differs from a pinned-pinned column only by multiplying the length L by 2. The effective length, L_e, is the distance between points where the moment is zero. For a pinned-pinned column, L = L_e, as illustrated in Figure 15.9, whereas for a fixed-free column, L_e = 2L.

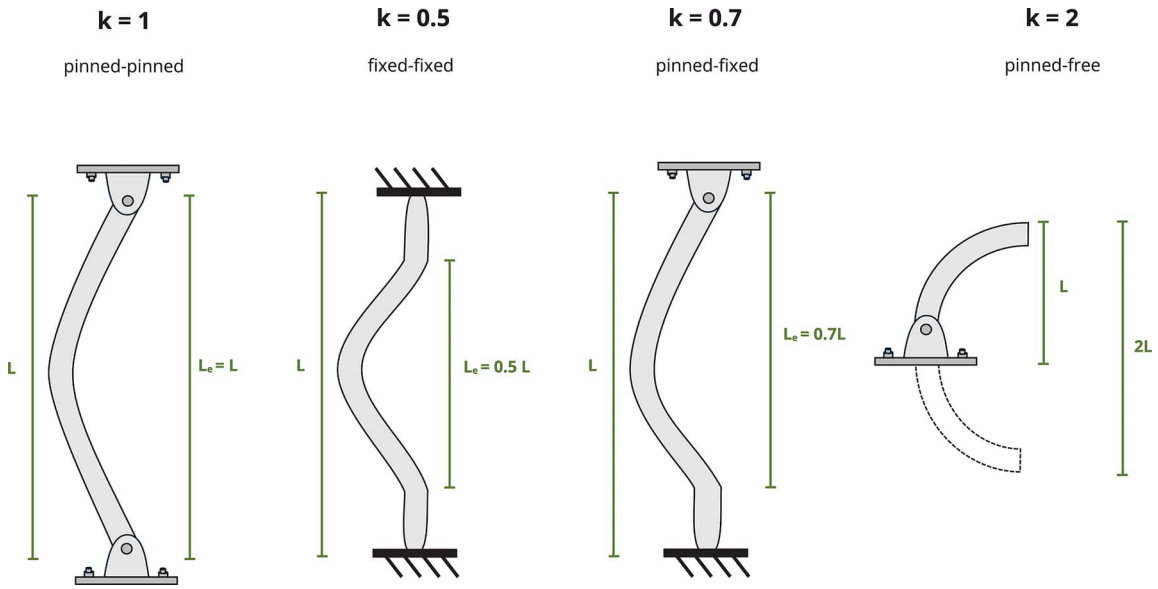


Figure 15.9: Effective length of columns for various end conditions

Other common end conditions also relate L to L_e. A general equation applies to any support condition by replacing the effective length with KL, where K is the effective length factor.

$$\boxed{P_{cr} = \frac{\pi^2 EI}{(KL)^2}}$$

(15.3)

and

$$\boxed{\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2}}.$$

(15.4)

Find the effective length factor, K, below each of the end conditions in Figure 15.9.

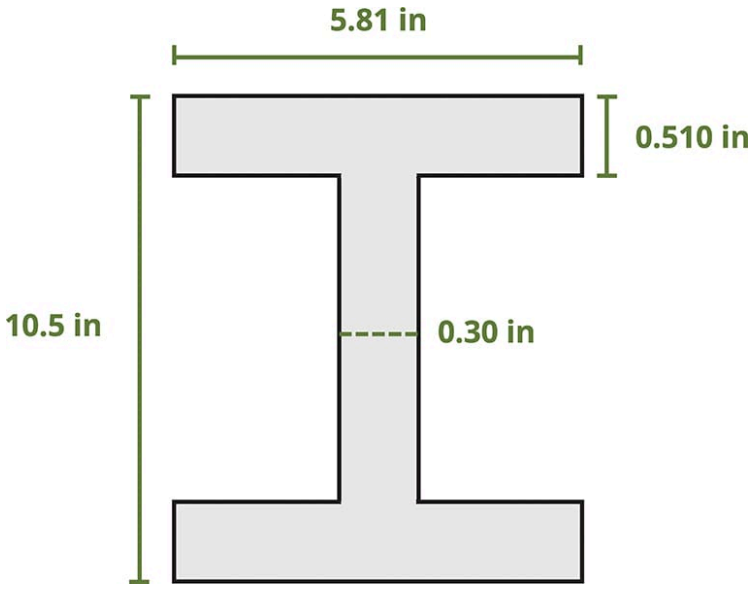
Example 15.2 and Example 15.3 demonstrate how to calculate the critical buckling load in columns with various different supports.

Example 15.2

Redo Example 15.1 with a fixed-fixed column.

Solution

Consult the table in Appendix A to find the section properties of a W10 × 30 section.



$$A = 8.84\ in.^2$$
$$I_x = 170\ in.^4$$
$$r_y = 4.38\ in.$$
$$I_y = 16.7\ in.^4$$
$$r_y = 1.37\ in.$$

Start by calculating the critical load using Euler's buckling formula. Note that we can confidently say that this column will buckle about the y-axis since the second moment of area in the y-direction is the smallest.

$$P_{cr} = \frac{\pi^2 EI}{(kL)^2} = \frac{\pi^2(29{,}000\ ksi)(16.7\ in.^4)}{\left(2{*}8\ ft{*}12\ \frac{in.}{ft}\right)^2}$$

P_{cr}=29.7*kips*

This represents the load at which any increase in applied load will cause the column to buckle.

Calculate the stress associated with this critical load.

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{129.7\ kips}{8.84\ in.^2} = 14.7\ ksi$$

Since this stress is smaller than the yield stress of the steel (σ_y = 50 ksi), the column will buckle first at a critical load of 129.7 kips.

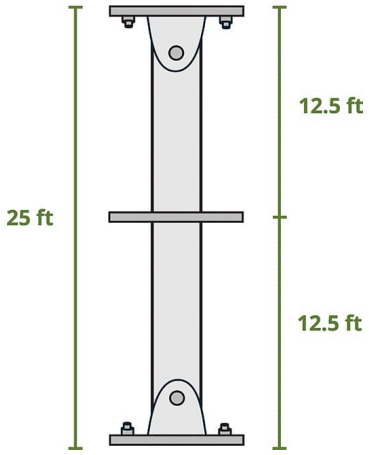
Example 15.3

A W12 × 50 column carries a load of 650 kips.

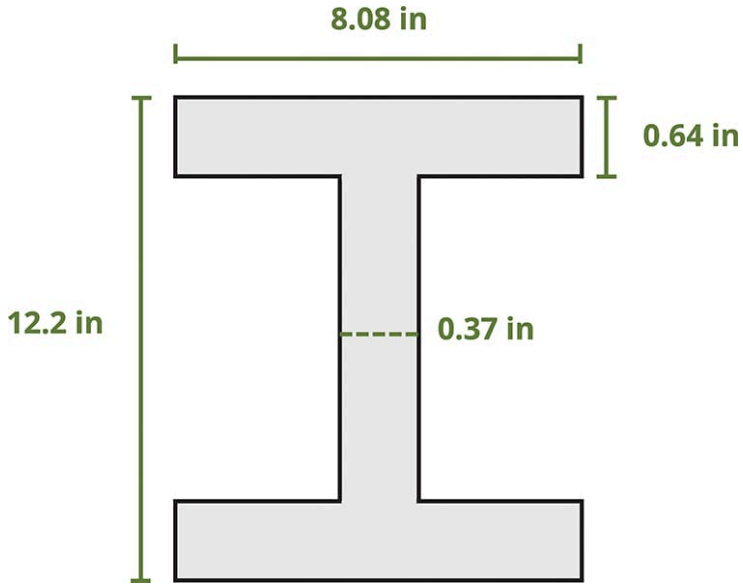
If the column is 25 ft long, has fixed-pinned end conditions, is braced at the midpoint in the weak direction, and is made from A492 steel (σ_y = 50 ksi and E = 29,000 ksi), is it adequate?

Solution

k = 0.7



$$A = 14.6\ in.^2$$
$$I_x = 391\ in.^4$$
$$r_y = 5.18\ in.$$
$$I_y = 56.3\ in.^4$$
$$r_y = 1.96\ in.$$



Consult the table in Appendix A to find the section properties of a W12 × 50 section.

Start by calculating the critical load using Euler's buckling formula. The brace at the midpoint of the column changes the length of the column in the weak direction. This doesn't make obvious which axis the column will buckle about, so you need to check both the strong and the weak directions.

Strong axis

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(29{,}000\ ksi)(391\ in.^4)}{\left(0.7{*}25\ ft{*}\ 12\ \frac{in.}{ft}\right)^2}$$

P_{cr}=2,538*kips*

Use the general form of Euler's buckling equation. Since the end supports are pinned-fixed, use K = 0.7. The critical load in the strong direction is calculated to be 2,538 kips.

Compare this to buckling about the weak axis. The values of I, K, and L will change. The L value changes because the column is braced at the midpoint, so the column has a shorter distance to buckle about. The K value can be tricky, though. There were no details about the brace, so you don't know if the brace will provide an end condition like a fixed or pinned end. So calculate the worst case, which would be a pinned-pinned end condition (this makes the denominator the largest, so the result will be the smallest).

Weak axis

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(29{,}000\ ksi)(56.3\ in.^4)}{\left(1{*}12.5\ ft{*}12\frac{in.}{ft}\right)^2}$$

P_{cr}=716*kips*

The critical load in the weak direction is calculated to be 716 kips. This means that if the column were to buckle, it would be about the weak axis since this load was smaller. Because the critical buckling load is greater than the 650 kip applied load, the column is adequate. However, you also need to calculate the load that would cause yielding of the steel.

Yield

$$\sigma_{yield} = \frac{P}{4} \quad \rightarrow \quad P = (50\ ksi)(14.6\ in.^2)$$
$$P_{yield} = 730\ kips$$

Given a yielding load of 730 kips, the column is adequate for a 650 kip load. Failure would first occur by buckling in the weak direction if the load were increased to 716 kips. Note that we did not apply any safety factors. In civil engineering, design codes for steel columns designate how to apply safety factors to the loading and the capacity.

What would happen to the capacity of the columns if the brace were not there? To find out, let's recalculate the critical load for buckling about the weak direction.

No brace

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(29{,}000\ ksi)(56.3 in.^4)}{\left(0.7{*}25\ ft{*}12\ \frac{in.}{ft}\right)^2}$$

P_{cr}=365*kips*

Notice how much lower this is than the braced condition. If the column were not braced, it would be inadequate for the applied load.

Step-by-Step: Buckling of Columns with Different Supports

1. Determine the geometric properties of the cross-section, including A, I_x, and I_y.
2. Determine the effective length of the column, given the supports.
3. Apply Euler's formula to calculate the critical buckling load. If the supports differ for each axis, calculate the critical load for both the strong and weak axes.
4. Calculate the critical buckling stress and check that this is less than the yield stress.
5. If the critical stress is larger than the yield stress, calculate the applied load that will cause yielding instead.

Summary

Key Takeaways

- Structural members that are long, slender, and subjected to compressive axial loads are categorized as columns.
- In addition to checking stress and deflection, engineers need to check buckling as a form of member failure.
- The critical load refers to the maximum axial load a column can withstand just before buckling occurs.
- The effective length is the distance on the column between the points where the moment is zero. This accounts for various end conditions.

Key Equations

Euler's buckling formula (general case):

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

Critical stress formula (general case):

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{KL}{r}\right)^2}$$

References

Figures

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- Figure 15.1: Examples of columns. A: Unknown author. 2020. Pixabay license. https://pixabay.com/photos/bridge-tunnel-columns-structure-5441571/. B: ds_30. 2020. Pixabay license. https://pixabay.com/photos/buildings-concrete-construction-4839276/. C: G41rn8. 2009. CC BY-SA. https://commons.wikimedia.org/wiki/File:Hangzhou_2009_1794.jpg.
- Figure 15.2: Buckling of a ruler. Amy Richardson. 2024. CC BY.
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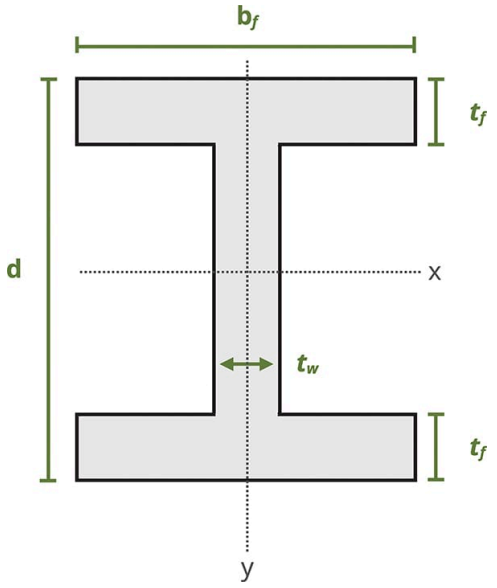
APPENDICES

Appendix A: Geometric Properties of Standard Beams
Appendix B: Slope and Deflection Tables
Appendix C: Material Properties
Appendix D: Units and Order of Magnitude
Appendix E: Centroids and Second Moment of Area for Common Shapes

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Appendix A: Geometric Properties of Standard Beams

A.1 Wide-Flange (W) Shapes. US Customary Units



Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
W40×431	127.00	41.30	16.20	2.36	1.34	34,800.0	1,690.0	16.60	1,690.0	208.0	3.65
W40×297	87.30	39.80	15.80	1.65	0.93	23,200.0	1,170.0	16.30	1,090.0	138.0	3.54
W40×264	77.40	40.00	11.90	1.73	0.96	19,400.0	9,71.0	15.80	493.0	82.6	2.52
W36×487	143.00	39.30	17.10	2.68	1.50	36,000.0	1,830.0	15.80	2,250.0	263.0	3.96
W36×247	72.50	36.70	16.50	1.35	0.80	16,700.0	913.0	15.20	1,010.0	123.0	3.74
W36×210	61.90	36.70	12.20	1.36	0.83	13,200.0	719.0	14.60	411.0	67.5	2.58
W30×326	95.90	32.40	15.40	2.05	1.14	16,800.0	1,040.0	13.20	1,240.0	162.0	3.60

Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
W30×173	50.90	30.40	15.00	1.07	0.66	8,230.0	541.0	12.70	598.0	79.8	3.42
W30×99	29.00	29.70	10.50	0.67	0.52	3,990.0	269.0	11.70	128.0	24.5	2.10
W24×279	81.90	26.70	13.30	2.09	1.16	9,600.0	718.0	10.80	823.0	124.0	3.17
W24×131	38.60	24.50	12.90	0.96	0.61	4,020.0	329.0	10.20	340.0	53.0	2.97
W24×84	24.70	24.10	9.02	0.77	0.47	2,370.0	196.0	9.79	94.4	20.9	1.95
W21×201	59.30	23.00	12.60	1.63	0.91	5,310.0	461.0	9.47	542.0	86.1	3.02
W21×122	35.90	21.70	12.40	0.96	0.60	2,960.0	273.0	9.09	305.0	49.2	2.92
W21×68	20.00	21.10	8.27	0.69	0.43	1,480.0	140.0	8.60	64.7	15.7	1.80
W18×130	38.30	19.30	11.20	1.20	0.67	2,460.0	256.0	8.03	278.0	49.9	2.70
W18×76	22.30	18.20	11.00	0.68	0.43	1,330.0	146.0	7.73	152.0	27.6	2.61
W18×40	11.80	17.90	6.02	0.53	0.32	612.0	68.4	7.21	19.1	6.35	1.27
W16×67	19.60	16.30	10.20	0.67	0.40	954.0	117.0	6.96	119.0	23.2	2.46
W16×40	11.80	16.00	7.00	0.51	0.31	518.0	64.7	6.63	28.9	8.25	1.57
W16×31	9.13	15.90	5.53	0.44	0.28	375.0	47.2	6.41	12.4	4.49	1.17
W14×90	26.50	14.00	14.50	0.71	0.44	999.0	143.0	6.14	362.0	49.9	3.70

Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
W14×53	15.60	13.90	8.06	0.66	0.37	541.0	77.8	5.89	57.7	14.3	1.92
W14×34	10.00	14.00	6.75	0.46	0.29	340.0	48.6	5.83	23.3	6.91	1.53
W14×22	6.49	13.70	5.00	0.34	0.23	199.0	29.0	5.54	7.0	2.80	1.04
W12×106	31.20	12.90	12.20	0.99	0.61	933.0	145.0	5.47	301.0	49.3	3.11
W12×53	15.60	12.10	10.00	0.58	0.35	425.0	70.6	5.23	95.8	19.2	2.48
W12×50	14.60	12.20	8.08	0.64	0.37	391.0	64.2	5.18	56.3	13.9	1.51
W12×26	7.65	12.20	6.49	0.38	0.23	204.0	33.4	5.17	17.3	5.34	1.51
W12×22	6.48	12.30	4.03	0.43	0.26	156.0	25.4	4.91	7.0	2.80	1.04
W10×68	20.00	10.40	10.10	0.77	0.47	394.0	75.7	4.44	134.0	26.4	2.59
W10×39	11.50	9.92	7.99	0.53	0.32	209.0	42.1	4.27	45.0	11.3	1.98
W10×30	8.84	10.50	5.81	0.51	0.30	170.0	32.4	4.38	16.7	5.75	1.37
W10×19	5.62	10.20	4.02	0.40	0.25	96.3	18.8	4.14	4.29	2.14	0.87
W8×40	11.70	8.25	8.07	0.56	0.36	146.0	35.5	3.53	49.1	12.2	2.04
W8×31	9.12	8.00	8.00	0.44	0.29	110.0	27.5	3.47	37.1	9.27	2.02
W8×21	6.16	8.28	5.27	0.40	0.25	75.3	18.2	3.49	9.77	3.71	1.26

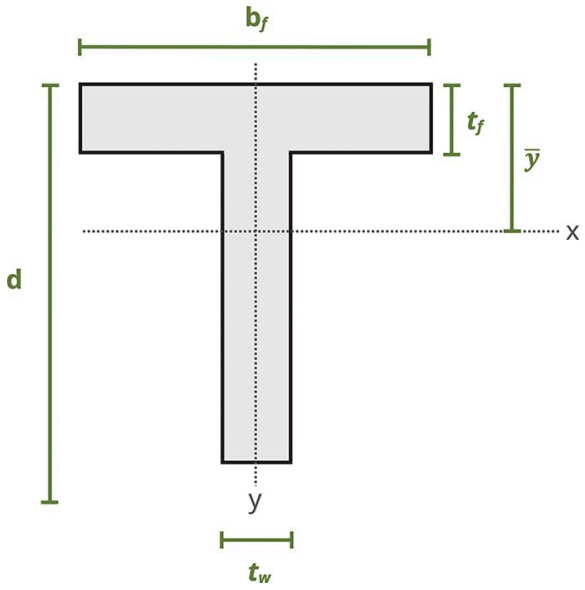
Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
W8×13	3.84	7.99	4.00	0.26	0.23	39.6	9.91	3.21	2.73	1.37	0.84

A.2 Wide-Flange (W) Shapes. SI Units

Label	Area, A (mm^2)	Depth, d (mm)	Flange Width, b_f (mm)	Flange Thickness, t_f (mm)	Web Thickness, t_w (mm)	I_x ($\times 10^6$ mm^4)	S_x ($\times 10^3$ mm^3)	r_x (mm)	I_y ($\times 10^6$ mm^4)	S_y ($\times 10^3$ mm^3)	r_y (mm)
W1000×642	81900	1050	411	59.9	34.0	14,500	27,700	422	703	3,410	92.7
W1000×443	56300	1010	401	41.9	23.6	9,660	19,200	414	454	2,260	89.9
W1000×393	49900	1020	302	43.9	24.4	8,070	15,900	401	205	1,350	64.0
W920×725	92300	998	434	68.1	38.1	15,000	30,000	401	937	4,310	101.0
W920×368	46800	932	419	34.3	20.3	6,950	15,000	386	420	2,020	95.0
W920×313	39900	932	310	34.5	21.1	5,490	11,800	371	171	1,110	65.5
W760×484	61900	823	391	29.0	52.1	6,990	17,000	335	516	2,650	91.4
W760×257	32800	772	381	27.2	16.6	3,430	8,870	323	249	1,310	86.9
W760×147	18700	754	267	17.0	13.2	1,660	4,410	297	53.3	401	53.3
W610×415	52800	678	338	53.1	29.5	4,000	11,800	274	343	2,030	80.5
W610×195	24900	622	328	24.4	15.4	1,670	5,390	259	142	869	75.4
W610×125	15900	612	229	19.6	11.9	986	3,210	249	39.3	342	49.5

Label	Area, A (mm^2)	Depth, d (mm)	Flange Width, b_f (mm)	Flange Thickness, t_f (mm)	Web Thickness, t_w (mm)	I_x ($´10^6$ mm^4)	S_x ($´10^3$ mm^3)	r_x (mm)	I_y ($´10^6$ mm^4)	S_y ($´10^3$ mm^3)	r_y (mm)
W530×300	38300	584	320	41.4	23.1	2,210	7,550	241	226	1,410	76.7
W530×182	23200	551	315	24.4	15.2	1,230	4,470	231	127	806	74.2
W530×101	12900	536	210	17.4	11.0	616	2,290	218	26.9	257	45.7
W460×193	24700	490	284	30.5	17.0	1,020	4,200	204	116	818	68.6
W460×113	14400	462	279	17.3	10.8	554	2,390	196	63.3	452	66.3
W460×60	7610	455	153	8.0	13.3	255	1,120	183	7.95	104	32.3
W410×100	12600	414	259	16.9	10.0	397	1,920	177	49.5	380	62.5
W410×60	7610	406	178	12.8	7.8	216	1,060	168	12	135	39.9
W410×46.1	5890	404	140	11.2	7.0	156	773	163	5.16	73.6	29.7
W360×134	17100	356	368	18.0	11.2	416	2,340	156	151	818	94.0
W360×79	10100	353	205	16.8	9.4	225	1,270	150	24	234	48.8
W360×51	6450	356	171	11.6	7.2	142	796	148	9.70	113	38.9
W310×158	20100	328	310	25.1	15.5	388	2,380	139	125	808	79.0
W310×79	10100	307	254	14.6	8.8	177	1,160	133	39.9	315	63.0
W310×38.7	4940	310	165	9.7	5.8	64.9	547	131	7.20	87.5	38.4

A.3 WT Shapes. US Customary Units



Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	y_ (in. below top)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
WT22×184	54.00	22.20	16.00	2.17	1.22	5.57	2,380	143.0	6.64	675	84.4	3.54
WT22×145	42.60	21.80	15.80	1.58	0.865	5.26	1,830	111.0	6.54	521	65.9	3.49
WT22×115	33.90	21.50	15.80	1.22	0.710	5.17	1,440	88.6	6.53	398	50.5	3.43
WT18×143	42.00	18.90	12.30	1.93	1.06	5.02	1,360	98.0	5.69	301	48.9	2.68
WT18×105	30.90	18.30	12.20	1.36	0.830	4.87	985	73.1	5.65	206	33.8	2.58
WT18×85	25.00	18.10	12.00	1.10	0.680	4.73	786	58.9	5.61	160	26.6	2.53
WT15×117.5	34.70	15.70	15.10	1.50	0.830	3.41	674	55.1	4.41	427	5.8	3.51
WT15×74	21.80	15.30	10.50	1.18	0.650	3.84	466	40.6	4.63	114	21.7	2.28
WT15×58	17.10	15.00	10.50	0.85	0.565	3.94	373	33.7	4.67	82.1	15.6	2.19

Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	y_- (in. below top)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
WT13.5×89	26.30	13.90	14.10	1.19	0.725	3.04	414	38.2	3.97	278	39.4	3.25
WT13.5×57	16.80	13.60	10.10	0.93	0.570	3.42	289	28.3	4.15	79.3	15.8	2.18
WT13.5×47	13.80	13.50	10.00	0.75	0.490	3.41	239	23.8	4.16	62.0	12.4	2.12
WT12×73	21.50	12.40	12.90	1.09	0.650	2.66	264	27.2	3.50	195	30.3	3.01
WT12×52	15.30	12.00	12.80	0.75	0.500	2.59	189	20.0	3.51	130	20.3	2.91
WT12×42	12.40	12.10	9.02	0.77	0.470	2.97	166	18.3	3.67	47.2	10.5	1.95
WT10.5×66	19.40	10.90	12.40	1.04	0.650	2.33	181	21.1	3.06	166	26.7	2.93
WT10.5×50.5	14.90	10.70	12.30	0.80	0.500	2.18	135	15.8	3.01	124	20.2	2.89
WT10.5×34	10.00	10.60	8.27	0.51	0.430	2.59	103	12.9	3.20	32.4	7.83	1.80
WT9×79	23.20	9.86	11.30	1.44	0.810	2.17	160	20.8	2.63	174	30.7	2.74
WT9×53	15.60	9.37	11.20	0.94	0.590	1.97	104	14.1	2.59	110	19.7	2.66
WT9×38	11.10	9.11	11.00	0.68	0.425	1.80	71.8	9.83	2.54	76.2	13.8	2.61
WT7×88	25.90	7.61	15.70	1.31	0.830	1.43	80.5	13.0	1.76	419	53.5	4.02
WT7×60	17.70	7.24	14.70	0.94	0.590	1.24	51.7	8.61	1.71	247	33.7	3.74
WT7×41	12.00	7.16	10.10	0.86	0.510	1.39	41.2	7.14	1.85	74.1	14.6	2.48

Label	Area, A ($in.^2$)	Depth, d (in.)	Flange Width, b_f (in.)	Flange Thickness, t_f (in.)	Web Thickness, t_w (in.)	y_- (in. below top)	I_x ($in.^4$)	S_x ($in.^3$)	r_x (in.)	I_y ($in.^4$)	S_y ($in.^3$)	r_y (in.)
WT5×44	13.00	5.42	10.30	0.99	0.605	1.06	20.8	4.77	1.27	89.9	17.4	2.63
WT5×30	8.84	5.11	10.10	0.68	0.420	0.88	12.9	3.04	1.21	58.1	11.5	2.57
WT5×15	4.42	5.24	5.81	0.51	0.300	1.10	9.28	2.24	1.45	8.35	2.87	1.37

A.4 WT Shapes. SI Units

Label	Area, A (mm^2)	Depth, d (mm)	Flange Width, b_f (mm)	Flange Thickness, t_f (mm)	Web Thickness, t_w (mm)	y_- (mm below top)	I_x ($\times 10^6$ mm^4)	S_x ($\times 10^3$ mm^3)	r_x (mm)	I_y ($\times 10^6$ mm^4)	S_y ($\times 10^3$ mm^3)	r_y (mm)
WT550×274	34800	564	406	50.0	27.9	141	991	2,340	169	281.6	1,380	89.9
WT550×216.5	27500	554	401	40.1	22.0	134	762	1,820	166	217	1,080	88.6
WT550×171.5	21900	546	401	31.0	18.0	131	599	1,450	166	166	828	87.1
WT460×212.5	27100	480	312	49.0	26.9	128	566	1,610	145	125	801	68.1
WT460×156.5	19900	465	310	34.5	21.1	124	410	1,200	144	85.7	554	65.5
WT460×126.5	16100	460	305	27.9	17.3	120	327	965	142	66.6	436	64.3
WT380×175	22400	399	384	38.1	21.1	86.6	281	903	112	178	931	89.2
WT380×110	14100	389	267	30.0	16.5	97.5	194	665	118	47.5	356	57.9
WT380×86.5	11000	381	267	21.6	14.4	100	155	552	119	34.2	256	55.6

Label	Area, A (mm^2)	Depth, d (mm)	Flange Width, b_f (mm)	Flange Thickness, t_f (mm)	Web Thickness, t_w (mm)	y_- (mm below top)	I_x ($´10^6$ mm^4)	S_x ($´10^3$ mm^3)	r_x (mm)	I_y ($´10^6$ mm^4)	S_y ($´10^3$ mm^3)	r_y (mm)
WT345×132.5	17000	353	358	30.2	18.4	77.2	172	626	101	116	646	82.6
WT345×85	10800	345	257	23.6	14.5	86.9	120	464	105	33.0	259	55.4
WT345×70	8900	343	254	18.9	12.4	86.6	99.5	390	106	25.8	203	53.8
WT305×108.5	13900	315	328	27.7	16.5	67.6	110	446	88.9	81.2	497	76.5
WT305×77.5	9870	305	325	19.1	12.7	65.8	78.7	328	89.2	54.1	333	73.9
WT305×62.5	8000	307	229	19.6	11.9	75.4	69.1	300	93.2	19.6	172	49.5
WT265×98	12500	277	315	26.4	16.5	59.2	75.3	346	77.7	69.1	438	74.4
WT265×75	9610	272	312	20.3	12.7	55.4	56.2	259	76.5	51.6	331	73.4
WT265×50.5	6450	269	210	17.4	10.9	65.8	42.9	211	81.3	13.5	128	45.7
WT230×117.5	15000	250	287	36.6	20.6	55.1	66.6	341	66.8	72.4	503	69.6
WT230×79	10100	238	284	23.9	15.0	50.0	43.3	231	65.8	45.8	323	67.6
WT230×56.5	7160	231	279	17.3	10.8	45.7	29.9	161	64.5	31.7	226	66.3
WT180×131	16700	193	399	33.3	21.1	36.3	33.5	213	44.7	174	877	102.0
WT180×89.5	11400	184	373	23.9	15.0	31.5	21.5	141	43.4	103	552	95.0
WT180×61	7740	182	257	21.7	13.0	35.3	17.1	117	47.0	30.8	239	63.0

Label	Area, A (mm^2)	Depth, d (mm)	Flange Width, b_f (mm)	Flange Thickness, t_f (mm)	Web Thickness, t_w (mm)	y_- (mm below top)	I_x ($´10^6$ mm^4)	S_x ($´10^3$ mm^3)	r_x (mm)	I_y ($´10^6$ mm^4)	S_y ($´10^3$ mm^3)	r_y (mm)
WT125×65.5	8390	138	262	25.1	15.4	26.9	8.66	78.2	32.3	37.2	285	66.8
WT125×44.5	5700	130	257	17.3	10.7	22.5	5.37	49.8	30.7	24.2	188	65.3
WT125×224	2850	133	148	13.0	7.62	27.9	3.86	36.7	36.8	3.48	47.0	34.8

References

Table data

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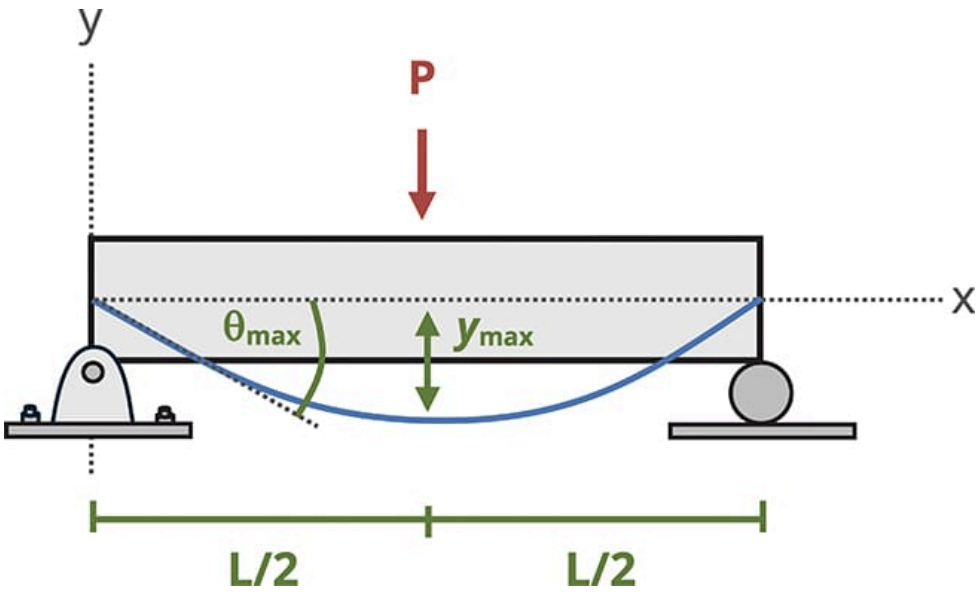
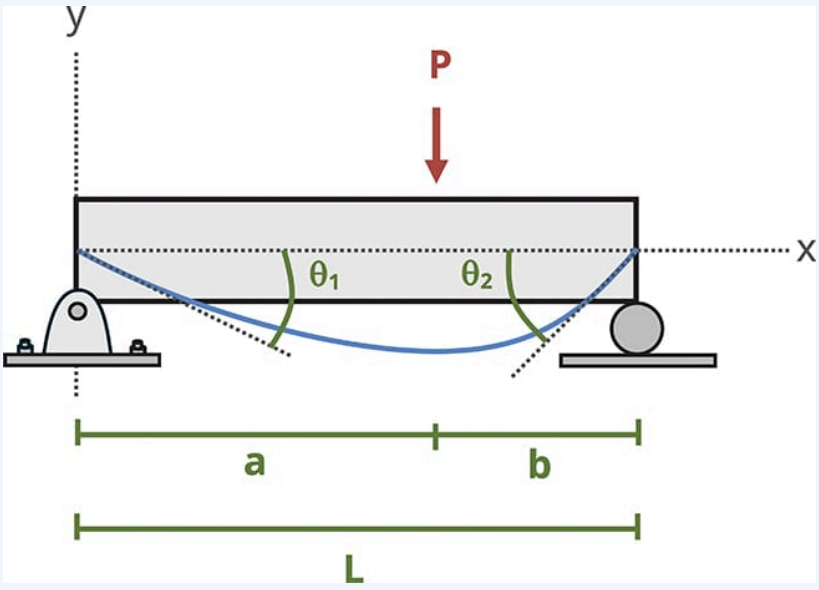
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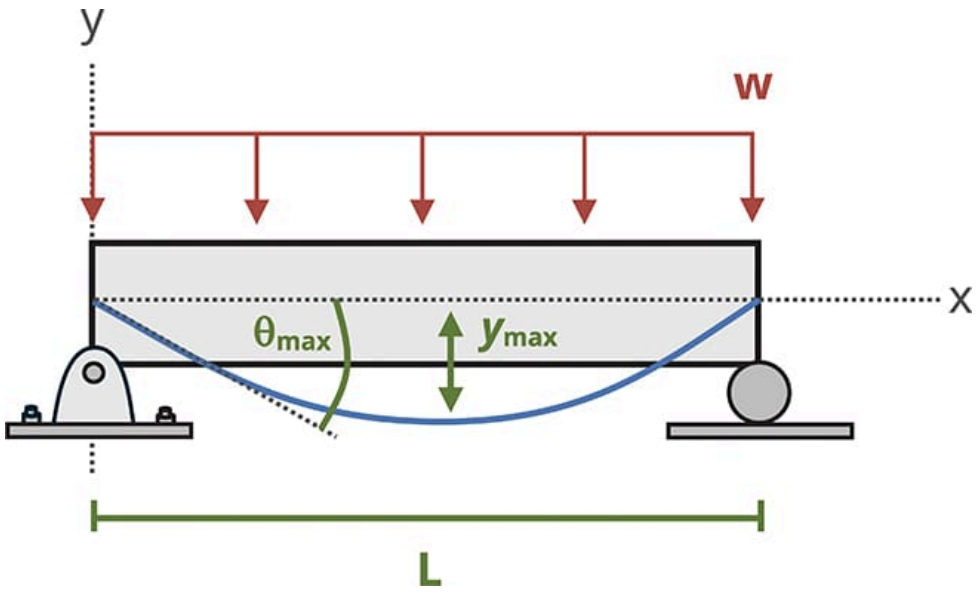
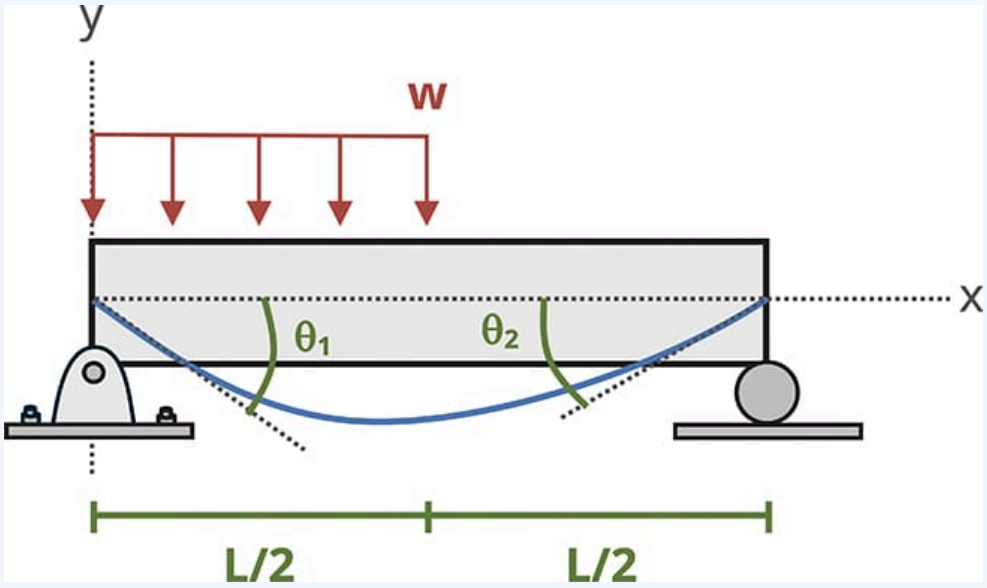
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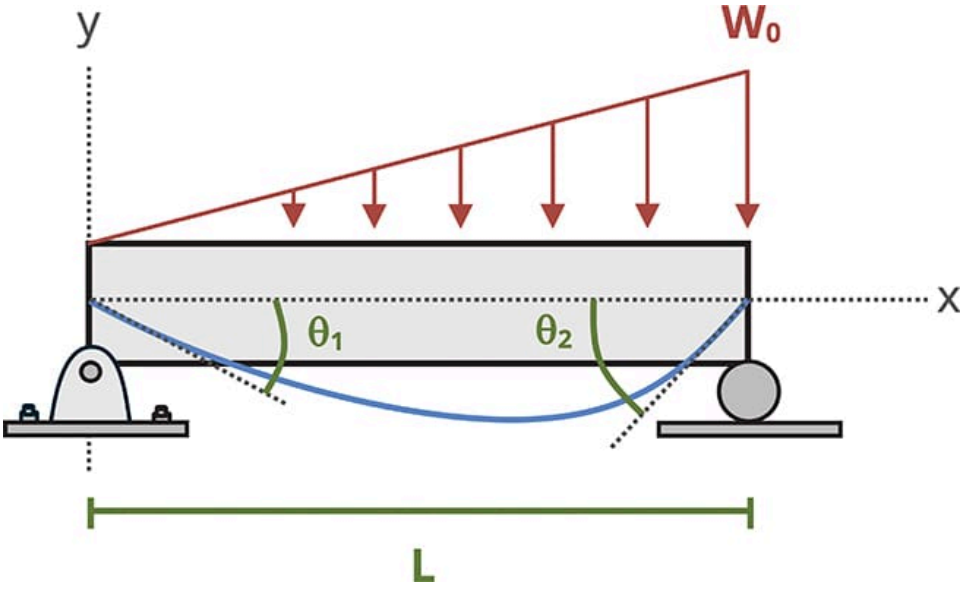
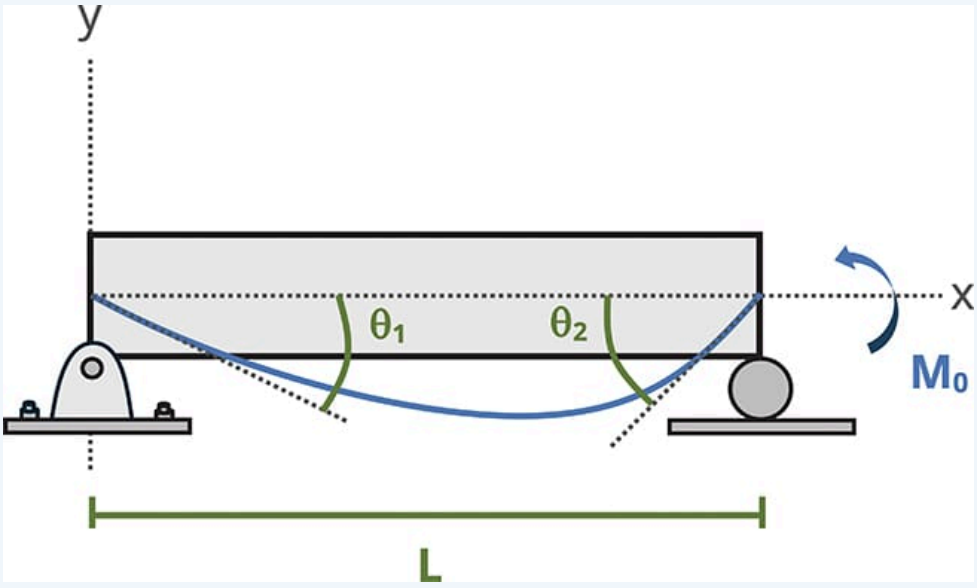
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Appendix B: Slope and Deflection Tables

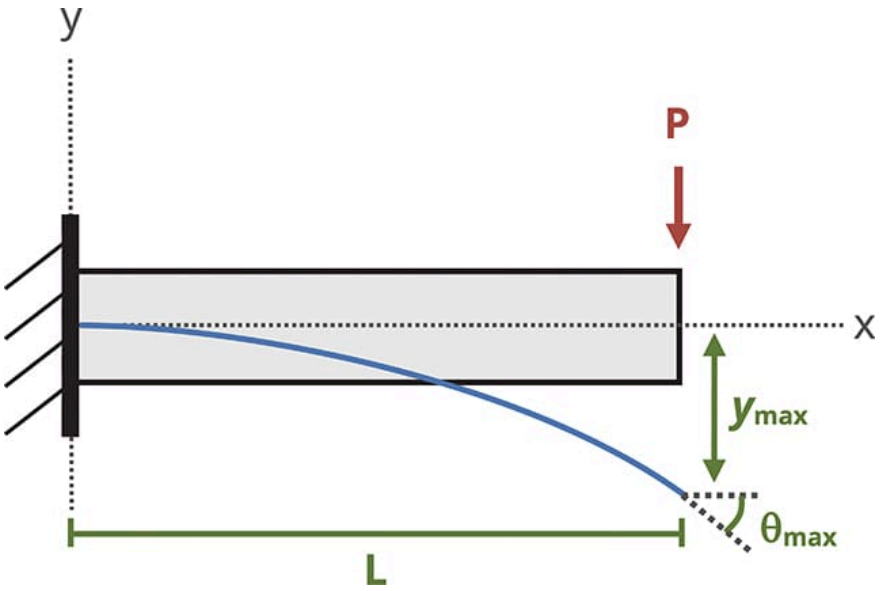
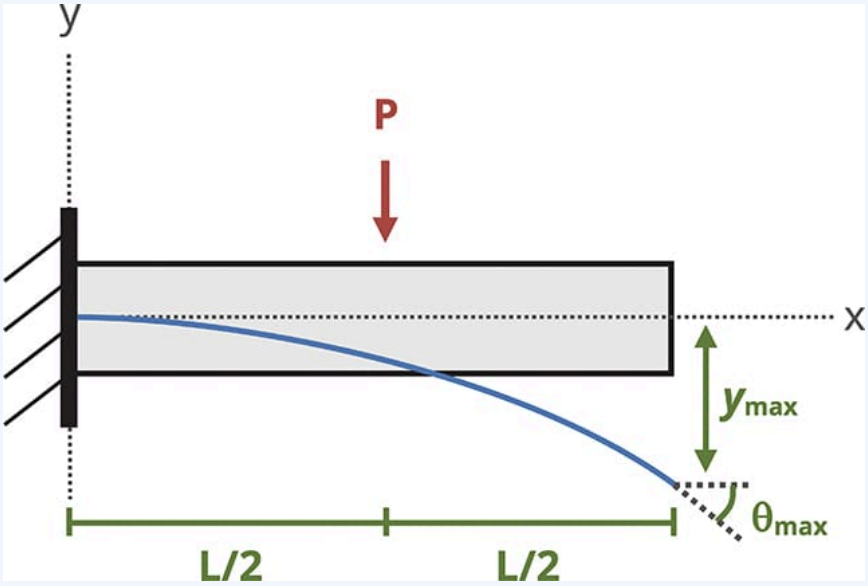
B.1 Simply Supported Beams

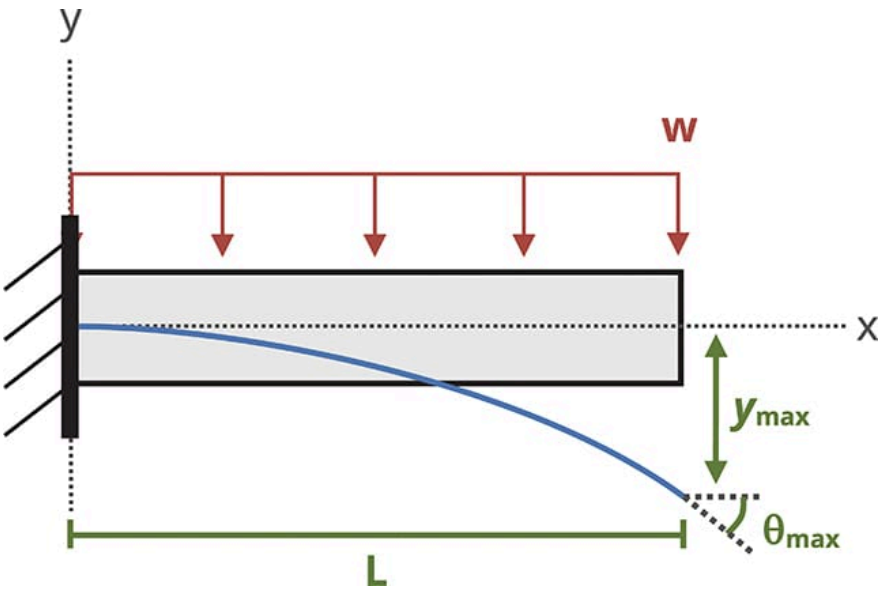
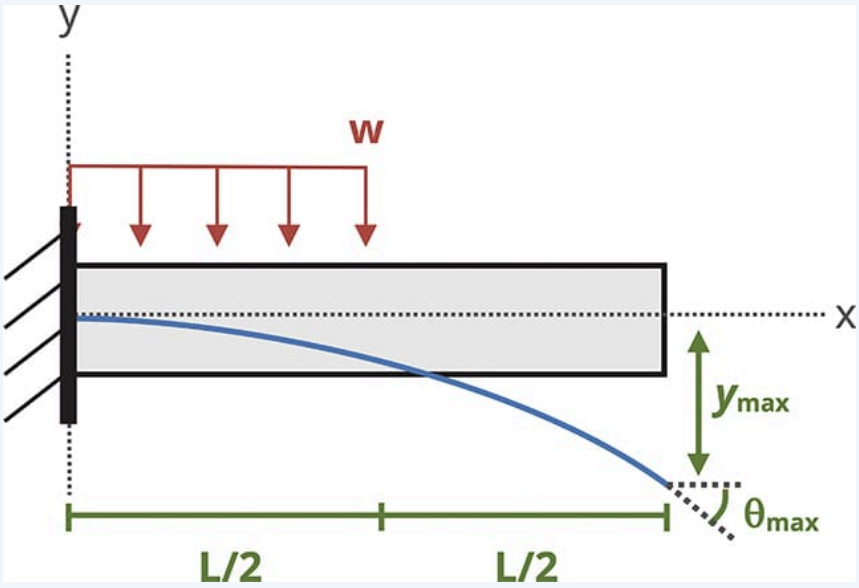
Beam	Slope	Max. Deflec
	$\theta_{max} = -\frac{PL^2}{16EI}$	$y_{max} =$
	θ_A = [redacted] θ_B = $\frac{Pab(L+a)}{6EIL}$	$y_{max} =$ [redacted] At x = $\sqrt{}$ [redacted]

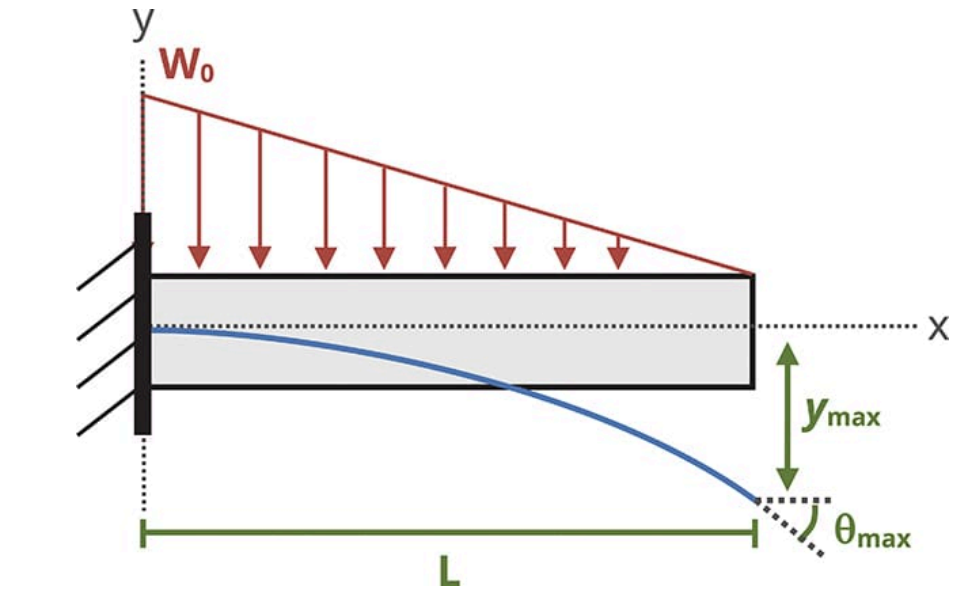
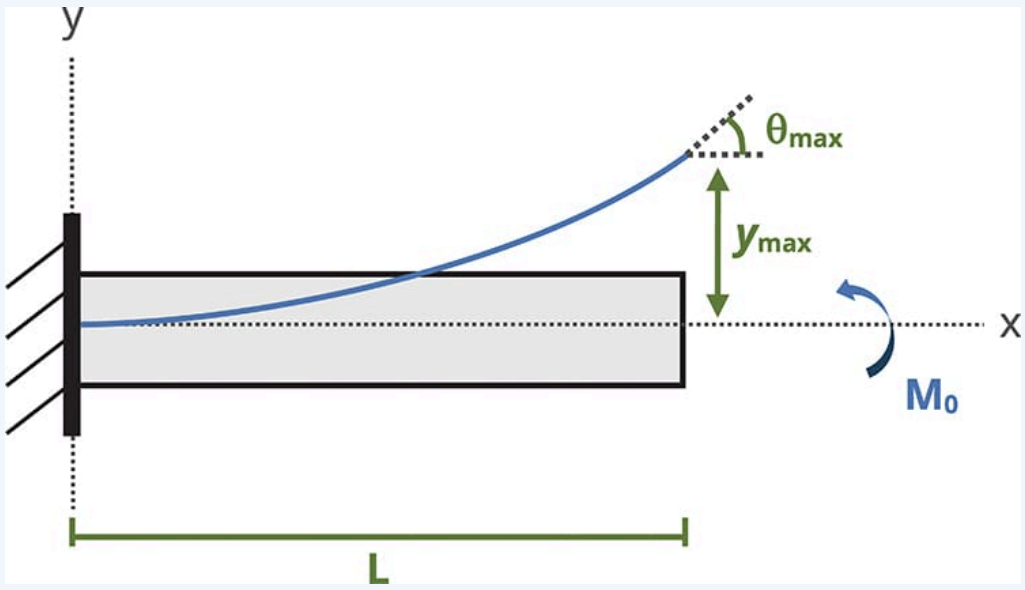
Beam	Slope	Max. Deflec
	$\theta_{max} = -\frac{wL^3}{24EL}$	$y_{max} =$
	$\theta_A = -\frac{3wL^3}{128EI}$ $\theta_B = \frac{7wL^3}{384EI}$	$y_{max} =$ At x =

Beam	Slope	Max. Deflec
	$\theta_A = -\frac{7wL^3}{360EI}$ $\theta_B = \frac{wL^3}{45EI}$	$y_{max} =$ At x =
	$\theta_A = -\frac{ML}{6EI}$ $\theta_B = \frac{ML}{3EI}$	$y_{max} =$ At x =

B.2 Cantilever Beams

Beam	Slope	Max. Deflection	Elastic (
	$\theta_{max} = -\frac{PL^2}{2EI}$	$y_{max} = -\frac{PL^3}{3EI}$	$y = -\frac{Px^2}{6EI}$
	$\theta_{max} = -\frac{PL^2}{8EI}$	$y_{max} = -\frac{5PL^3}{48EI}$	$y = -\frac{Px}{12E}$ Validfor $\frac{L}{2}$ y $= -\frac{PL^2}{48EI}$ Validfor

Beam	Slope	Max. Deflection	Elastic
	$\theta_{max} = -\frac{wL^3}{6EI}$	$y_{max} = -\frac{wL^4}{8EI}$	$y = -\frac{wx}{24E}$
	$\theta_{max} = -\frac{wL^3}{48EI}$	$y_{max} = -\frac{7wL^4}{384EI}$	$y = -\frac{wx}{24I}$ Valid for $0 \leq x \leq$ Valid for

Beam	Slope	Max. Deflection	Elastic
	$\theta_{max} = -\frac{wL^3}{24EI}$	$y_{max} = -\frac{w_0 L^4}{30EI}$	$y = -\frac{w}{120}$ $+ 5Lx^2 -$
	$\theta_{max} = \frac{ML}{EI}$	$y_{max} = \frac{ML^2}{2EI}$	$y = \frac{Mx^2}{2EI}$

References

Figures

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Appendix C: Material Properties

C.1 US Customary Units

Material	Specific Weight ($lb/in.^3$)	Elastic Modulus (10^6 psi)	Shear Modulus (10^6 psi)	Poisson's Ratio	Coefficient of Thermal Expansion (10^{-6} / °F)	Yield Strength (Tens.) (ksi)	Yield Strength (Comp.) (ksi)	Ultimate Strength (Tens.) (ksi)	Ultimate Strength (Comp.) (ksi)
Aluminum (**2014-T6**)	0.101	10.6	3.9	0.35	12.8	60	60	68	68
Aluminum (**6061-T6**)	0.098	10.0	3.7	0.35	13.1	37	37	42	42
Cast Iron (**Gray ASTM 20**)	0.260	10.0	3.9	0.28	6.70	-	-	26	96
Cast Iron (**Malleable**)	0.264	24.0	9.3	0.28	6.70	33	-	40	83
Copper (**C83400**)	0.316	14.6	5.4	0.35	9.80	11.4	11.4	35	35
Copper (**C86100**)	0.319	15.2	5.6	0.34	12.2	50	50	95	95
Steel (**A-36**)	0.284	29.0	11.2	0.32	6.50	36	36	58	58
Steel (**A992**)	0.284	29.0	11.2	0.32	6.50	50	50	65	65
Steel (**Stainless 304**)	0.284	28.0	11.0	0.27	9.60	30	30	75	75

Material	Specific Weight ($lb/in.^3$)	Elastic Modulus (10^6 psi)	Shear Modulus (10^6 psi)	Poisson's Ratio	Coefficient of Thermal Expansion (10^{-6} / °F)	Yield Strength (Tens.) (ksi)	Yield Strength (Comp.) (ksi)	Ultimate Strength (Tens.) (ksi)	Ultimate Strength (Comp.) (ksi)
Titanium Alloy	0.160	17.4	6.4	0.36	5.20	134	134	145	145
Concrete (**Low Strength**)	0.086	3.20	-	0.15	6.0	-	-	-	3.0
Concrete (**High Strength**)	0.086	4.20	-	0.15	6.0	-	-	-	6.0
Douglas Fir	0.017	1.9	-	0.29	-	-	-	15.0	7.2
White Oak	0.025	1.8	-	0.31	-	-	-	-	7.4

C.2 SI Units

Material	Density (kg/m^3)	Elastic Modulus (GPa)	Shear Modulus (GPa)	Poisson's Ratio	Coefficient of Thermal Expansion (10^{-6} / °C)	Yield Strength (Tens.) (MPa)	Yield Strength (Comp.) (MPa)	Ultimate Strength (Tens.) (MPa)	Ultimate Strength (Comp.) (MPa)
Aluminum (**2014-T6**)	2,790	73.1	27	0.35	23	414	414	469	469
Aluminum (**6061-T6**)	2,710	68.9	26	0.35	24	255	255	290	290
Cast Iron (**Gray ASTM 20**)	7,190	67.0	27	0.28	12	-	-	179	669
Cast Iron (**Malleable**)	7,300	165	65	0.28	12	230	-	345	620

Material	Density (kg/m^3)	Elastic Modulus (GPa)	Shear Modulus (GPa)	Poisson's Ratio	Coefficient of Thermal Expansion (10^{-6} / °C)	Yield Strength (Tens.) (MPa)	Yield Strength (Comp.) (MPa)	Ultimate Strength (Tens.) (MPa)	Ultimate Strength (Comp.) (MPa)
Copper (**C83400**)	8,740	101	37	0.35	18	70	70	241	241
Copper (**C86100**)	8,830	103	38	0.34	17	345	345	655	655
Steel (**A-36**)	7,850	200	75	0.32	12	250	250	400	400
Steel (**A992**)	7,850	200	75	0.32	12	345	345	450	450
Steel (**Stainless 304**)	7,860	193	75	0.27	17	207	207	517	517
Titanium Alloy	4,430	120	44	0.36	9.4	924	924	1,000	1,000
Concrete (**Low Strength**)	2,380	22.1	-	0.15	11	-	-	-	20
Concrete (**High Strength**)	2,370	29.0	-	0.15	11	-	-	-	30
Douglas Fir	470	13.1	-	0.29	-	-	-	100	48
White Oak	690	12.0	-	0.31	-	-	-	-	51

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Appendix D: Units and Order of Magnitude

D.1 Commonly Used Orders of Magnitude

Multiplication factor	Prefix	Symbol	Example
$10^{12} = 1{,}000{,}000{,}000{,}000$	tera	T	1 terawatt = 1 TW = 1×10^{12} watts
$10^{9} = 1{,}000{,}000{,}000$	giga	G	1 gigabyte = 1 GB = 1×10^{9} bytes
$10^{6} = 1{,}000{,}000$	mega	M	1 megapascal = 1 MPa = 1×10^{6} pascals
$10^{3} = 1{,}000$	kilo	k	1 kilonewton = 1 kN = 1×10^{3} Newtons
$10^{-3} = 0.001$	milli	m	1 millisecond = 1 ms = 1×10^{-3} seconds
$10^{-6} = 0.000001$	micro	μ	1 micrometer = 1 μm = 1×10^{-6} meters
$10^{-9} = 0.000000001$	nano	n	1 nanovolt = 1 nV = 1×10^{-9} volts
$10^{-12} = 0.000000000001$	pico	p	1 picoamp = 1 pA = 1×10^{-12} amps

D.2 Commonly Used Units in this Text

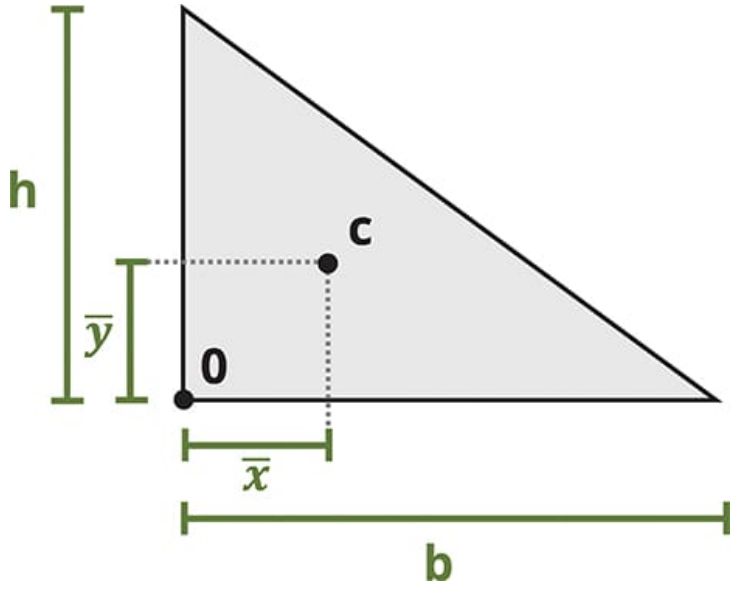
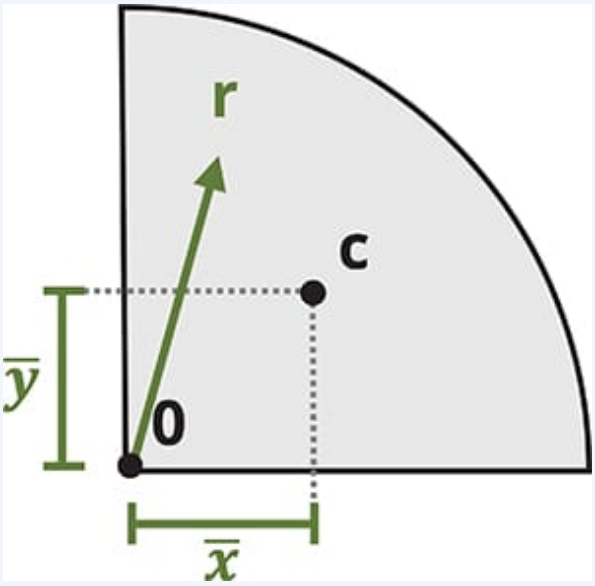
Quantity	SI Unit (symbol)	US Unit (symbol)	Conversion factor
First Moment of Area (Q)	Meters^3 (m^3)	Inches^3 ($in.^3$)	1 $in.^3$ = 1.639×10^{-5} m^3
Second Moment of Area (I)	Meters^4 (m^4)	Inches^4 ($in.^4$)	1 $in.^4$ = 4.162×10^{-7} m^4
Area (A)	Meters^2 (m^2)	Inches^2 ($in.^2$)	1 $in.^2$ = 6.452×10^{-4} m^3

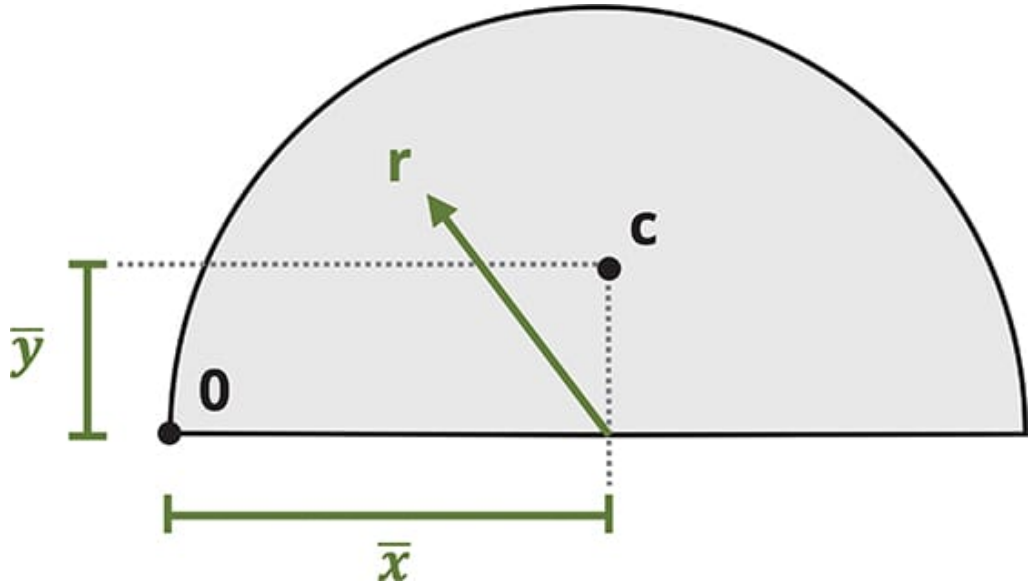
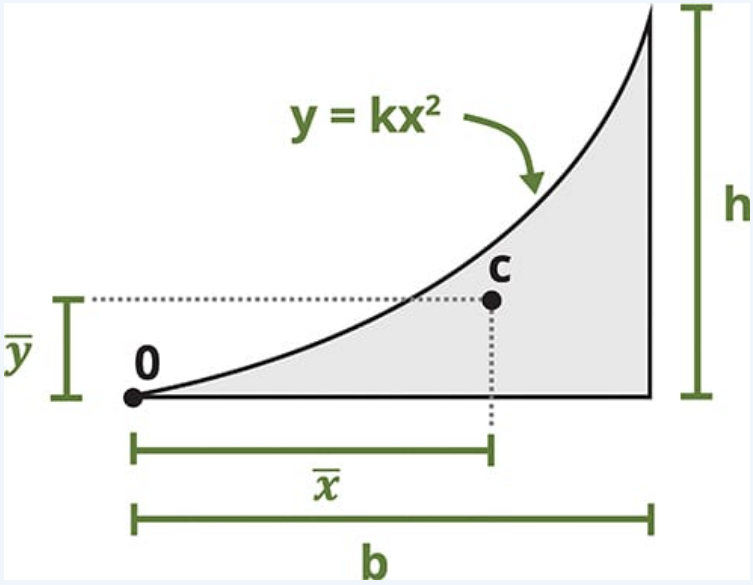
Quantity	SI Unit (symbol)	US Unit (symbol)	Conversion factor
Force (F)	Newton (N)	Pound (lb)	1 lb = 4.448 N
Length (L)	Meter (m)	Foot (ft) or inch (in.)	1 ft = 12 in. = 0.3048 m
Moment of a force (M)	Newton-meter (N·m)	Pound-inch (lb·in.)	1 lb·in. = 0.1130 N·m
Power (P)	Watt (W)	Horsepower (hp) or lb·ft/s	1 hp = 550 lb·ft/s = 1.356 W
Pressure (P)	Pascal (Pa)	Pound per square inch (psi)	1 psi = 6,895 Pa
Strain (e)	Unitless	Unitless	-
Stress (s)	Pascal (Pa) = N/m^2	Pound per square inch (psi) = $lb/in.^2$	1 psi = 6,895 Pa
Temperature (T)	°C or K	°F or °R	°F = (°C × 9/5) + 32 K = 273.15 + °C °R = °F + 459.67
Time (t)	Second (s)	Second (s)	-
Torque (T)	Newton-meter (N·m)	Pound-inch (lb·in.)	1 lb·in. = 0.1130 N·m

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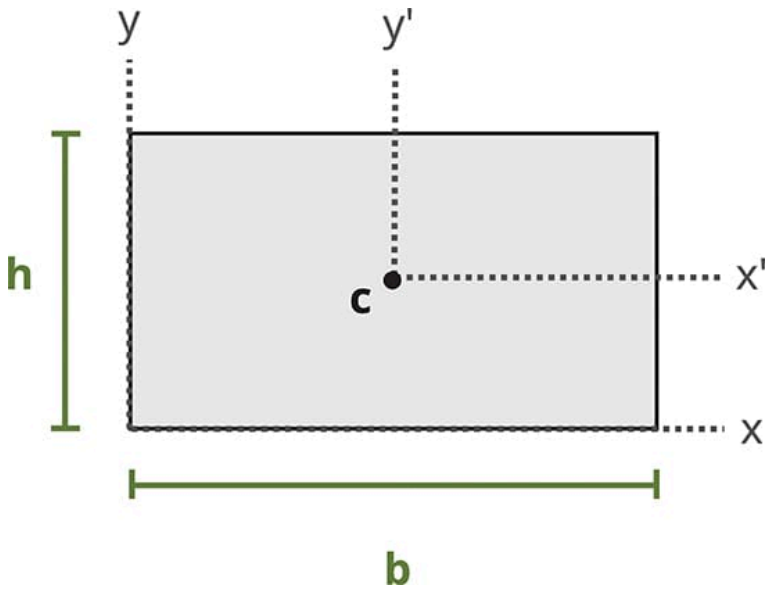
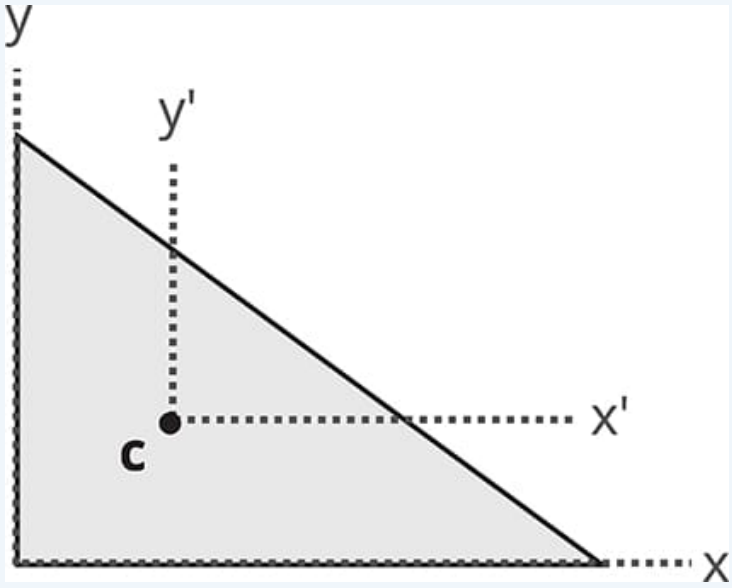
Appendix E: Centroids and Second Moment of Area for Common Shapes

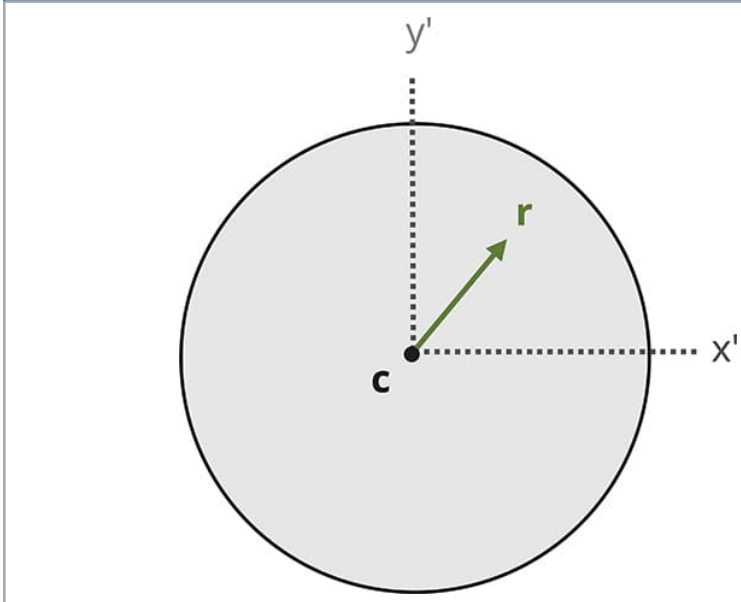
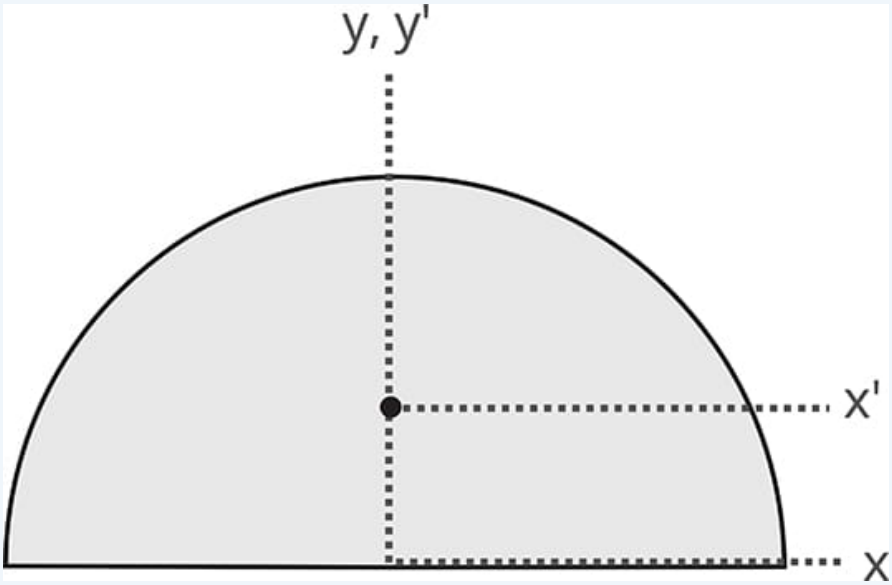
E.1 Centroid Coordinates for Common Shapes

Shape	$\bar{x}$	$\bar{y}$	Area
	$\frac{b}{3}$	$\frac{h}{3}$	$\frac{1}{2}bh$
	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{1}{4}\pi r^2$

Shape	$\bar{x}$	$\bar{y}$	Area
	r	$\frac{4r}{3\pi}$	$\frac{1}{2}\pi r^2$
	$\frac{3}{4}b$	$\frac{3}{10}h$	$\frac{1}{3}bh$

E.2 Second Moment of Area for Common Shapes

Shape	I_x	I_y	J
	$I_x = \frac{bh^3}{3}$ [redacted] $I_{x'} = \frac{bh^3}{12}$	$I_y = \frac{hb^3}{3}$ [redacted] $I_{y'} = \frac{hb^3}{12}$	$\bar{J}_C = \frac{bh}{12}(h^2)$
	$I_x = \frac{bh^3}{12}$ [redacted] $I_{x'} = \frac{bh^3}{36}$	$I_y = \frac{hb^3}{12}$ [redacted] $I_{y'} = \frac{hb^3}{36}$	$\bar{J}_C = \frac{bh}{36}(h^2)$

Shape	I_x	I_y	J
	$\bar{I}_{x'} = \frac{\pi r^4}{4}$	$\bar{I}_{y'} = \frac{\pi r^4}{4}$	$\bar{J}_C = \frac{\pi r^4}{2}$
	$I_x = \frac{\pi r^4}{8}$	$I_y = \frac{\pi r^4}{8}$	$J_{base} = \frac{\pi r^4}{4}$

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GLOSSARY

S

Sample Word 1 | Sample Definition 1

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