# LINEAR SYSTEMS



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# Linear Systems

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Detailed Licensing


Detailed Licensing

# Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

Some of the course content is developed from the courses and seminar book designed by Professor Alan V. Oppenheim of the Massachusetts Institute of Technology, who served as John Buck's advisor. In particular, we acknowledge *Signals and Systems* by Alan V. Oppenheim and A. S. Willsky (Prentice Hall, 1982) as a foundational source. The corresponding sections of this text were typically consulted, with additional material occasionally supplemented by notes from John Buck's recorded lectures. We also note that MIT makes related course content publicly available through its OpenCourseWare program, which provides further resources for readers interested in the original course materials: Introduction | Signals and Systems | Electrical Engineering and Computer Science | MIT OpenCourseWare

# CHAPTER OVERVIEW

## 1: Discrete-Time Linear Systems

### Course Description

This course introduces students to discrete-time signal analysis and linear systems. Topics include time-domain analysis of discrete-time linear time-invariant (LTI) systems; solutions of difference equations; system functions and digital filters; stability and causality; discrete-time Fourier series; discrete-time Fourier transforms and discrete Fourier transforms; z-transforms; sampling and the sampling theorem; discrete-time state equations; and communication systems. Students use analysis tools to design systems that meet functional specifications.

### Course Learning Objectives

The objectives of the course are to learn to analyze existing discrete-time linear systems and to modify them or create new ones to satisfy specific requirements. By the end of this course, students should be able to:

1. Prove whether systems are linear, time-invariant, stable, and causal. Explain the implications of these properties.
2. Determine periodicity of discrete-time signals.
3. Graph mathematical descriptions of discrete signals and write mathematical descriptions for graphed functions.
4. Develop an accurate mathematical description of a discrete linear system and represent it in a block diagram. Determine the difference equation representing a discrete-time system from its block diagram representation.
5. Explain the conditions required for discrete samples of a continuous-time signal to capture all of the information in the original signal. Sketch the resulting discrete-time frequency spectrum from the spectrum of the continuous-time signal.
6. Translate among the different representations of a system, including difference equation, impulse response, frequency response, and transfer function.
7. Convolve two discrete signals together either graphically or analytically.
8. Compute the Discrete-Time Fourier Series (DTFS) for periodic signals and sketch their spectra. Explain conceptually how the DTFS relates to time-domain properties of the signal.
9. Compute the Discrete-Time Fourier transform (DTFT) of basic signals such as the unit sample, sinusoids, and exponentials. Recognize and exploit DTFT properties to find the transform and inverse transform of more complicated signals.
10. Apply the frequency response of a Linear-Time-Invariant (LTI) system to both the DTFT and DTFS to find the system output.
11. Determine the transfer function of an LTI system using the z-transform, and solve for the system output for a given input, system description, and initial conditions.
12. Determine system stability from the z-transform.
13. Verify analytically obtained solutions using MATLAB and analyze system/signal behavior in time and frequency.


1.1: Discrete-Time Signals
1.2: Discrete-Time Systems
1.3: Convolution
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# 1.1: Discrete-Time Signals

## Learning Objectives: **Discrete-Time Signals**

1. Distinguish discrete-time (DT) signals from continuous-time (CT) signals, and generate new examples of DT signals.
2. Recognize DT signals vs sampled CT signals inherently.
3. Graph simple transformations of a discrete-time signal.
4. Recognize graphs of complex signals as superpositions of simple signals.
5. Match analytic expressions for exponential signals to visual representations of the signals.

### YouTube Videos

DT Signal Fundamentals



Common DT Signals



Basic Operations on DT Signals



DT Periodic Signals



Even and Odd Symmetry



## Summary

Definition: Discrete-Time Signals are signals that describe a wide variety of physical phenomena. The information in a signal contains a pattern of some form. Signals can be mathematically represented by functions of one or more independent variables. Time is an example of the independent variable. In fact, time is the actual independent variable in many cases, included signals (i.e., source voltage and capacitor voltage) in circuits, see Fig. 1.1.1.

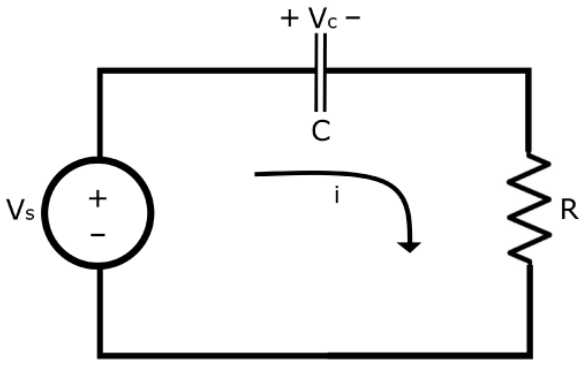



**Figure 1.1.1:** Simple RC circuit illustrating a source voltage (Vs) and capacitor voltage (Vc) as example discrete-time signals.

**Course notation:** discrete-time signals $x[n]$

Discrete-time signals are defined only at specific points in time rather than continuously, so their independent variable takes on discrete values. In other words, the discrete-time signal $x[n]$ is only defined for n = 0, ±1, ±2, ±3, .... As a result, $x[n]$ is a discrete sequence.

### Where do DT signals come from?

1. Phenomena where the independent variable is inherently discrete, such as the stock market (Fig. 1.1.2).

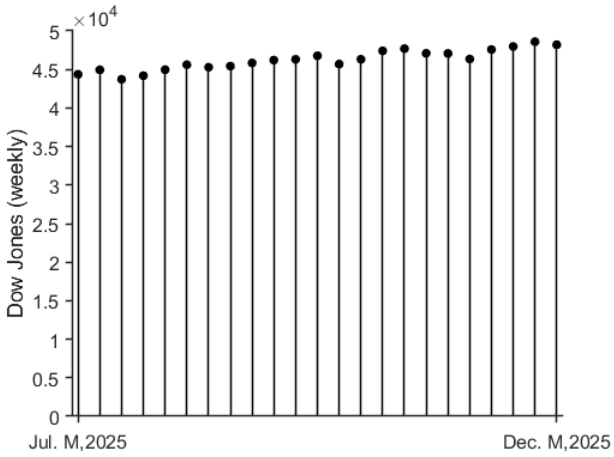



**Figure 1.1.2:** An example of an inherent discrete-time signal showing the weekly Dow-Jones stock market index from July, 2025, to December, 2025.

2. Sampling of continuous-time signals $x(t)$ where the discrete-time signal $x[n]$ represents successive samples of $x(t)$, see Fig. 1.1.3.

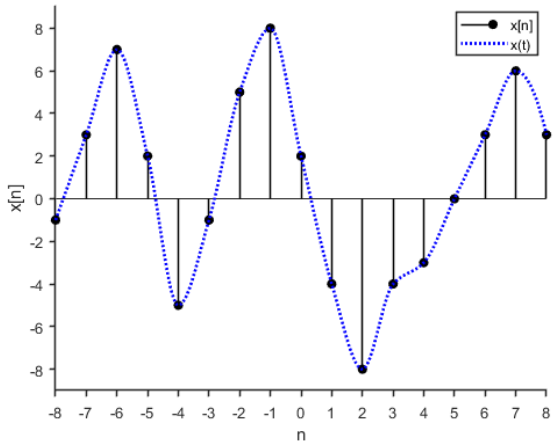



**Figure 1.1.3:** Example of how a CT signal is sampled to form a DT sequence.

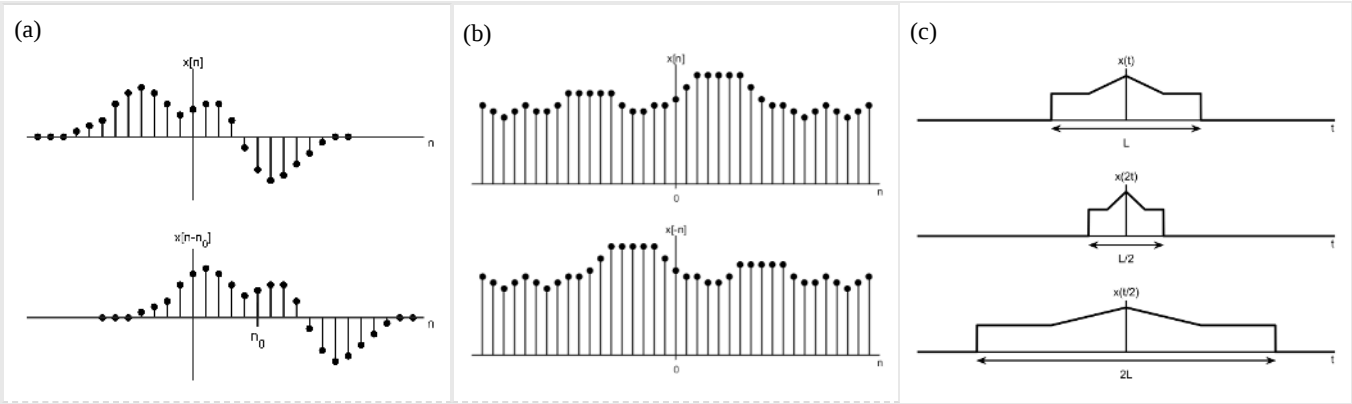
Definition of Discrete-Time average power

$$P = \lim_{N\to\infty}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}|x[n]|^2 \tag{1.1.1}$$

- **Energy signal:** finite total energy ($E<\infty$) $\Rightarrow$ average power is zero.
- **Power signal:** finite average power ($P>0$) $\Rightarrow$ total energy is infinite.

Transformations of the Independent Variable

- **Time shift:** $x[n-n_0]$ where $x[n-n_0]$ has the same shape as $x[n]$, but shifted by $n_0$ samples to the right direction.
- **Time reversal:** $x[-n]$ is obtained by reflecting $x[n]$ about the vertical axis (i.e., $n=0$).
- **Time scaling (discrete-time):** Time scaling is more restrictive in DT than CT. Common forms include compressing/expanding via indexing like $x[kn]$ where $k$ is an integer.




**Figures 1.1.4** Illustration of transformation in a discrete-time signal: (a) time shifting; (b) time reversal (middle); and (c) time scaling.

Periodic Signals:

A periodic DT signal $x[n]$ has a period $N$ such that $x[n]=x[n+N]$, for all $n$. If a signal is periodic, then there is a smallest positive integer $N$ (the fundamental period).

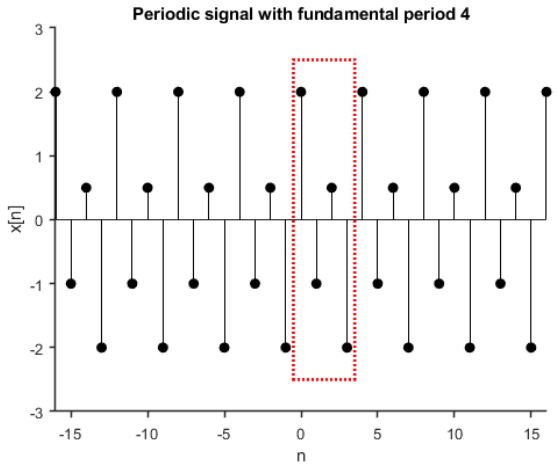



**Figure 1.1.5:** Example of a periodic signal with a fundamental period of $N = 4$.

### Even and Odd Signals

Even and Odd signals are defined based on symmetry under time reversal:

- **Even:** $x[-n] = x[n]$
- **Odd:** $x[-n] = -x[n]$ and $x[0] = 0$

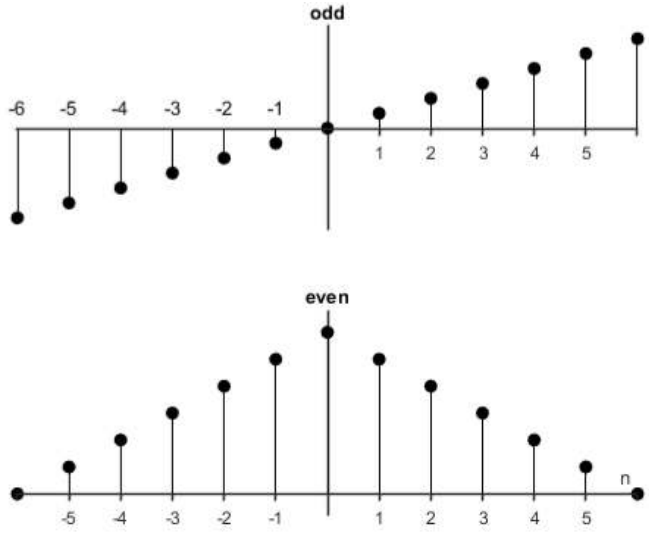



**Figure 1.1.6:** Examples of an Odd (Top) and Even (Bottom) signal.

Any signal can be decomposed into an even and an odd part:

$$x_e[n] = \frac{x[n] + x[-n]}{2} \tag{1.1.2}$$

$$x_o[n] = \frac{x[n] - x[-n]}{2} \tag{1.1.3}$$

### Complex exponential

General complex exponential: $x[n] = C\,a^n$, where $C$ and $a$ can be real or complex.

- If $0 < a < 1$, then $|x[n]|$ decays exponentially as $n$ increases.
- If $a > 1$, then $|x[n]|$ grows exponentially as $n$ increases.
- If $a < 0$, the sequence alternates sign.
- If $a = 1$, then $x[n] = C$ (constant).

- If $a = -1$, then $x[n]$ alternates between $C$ and $-C$.

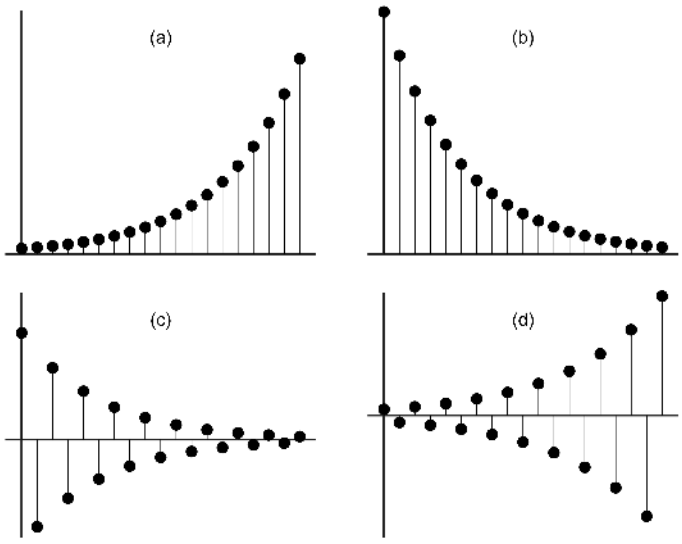


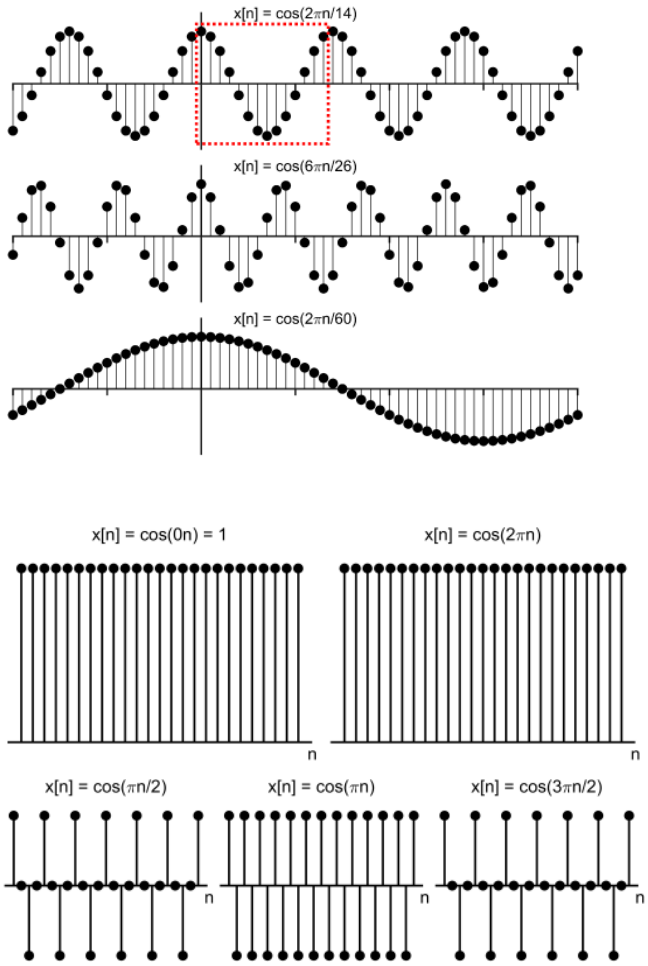

**Figure 1.1.7:** Examples of 4 exponential signals: (a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; and (d) $\alpha < -1$.

## Sinusoidal signal

Sinusoid:

\[x[n] = A\cos(\omega_0 n + \phi).$$.


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**Figures 1.1.8:** Examples of sinusoidal signals with differing frequencies.

Euler relationship (complex exponential to sinusoid)

$$e^{j\theta} = \cos(\theta) + j\sin(\theta). \tag{1.1.4}$$

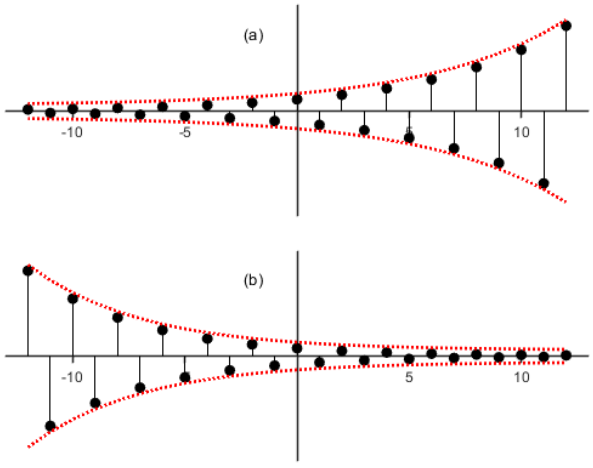



**Figure 1.1.9:** Stem plot of an exponential signal with growth (a) and decay (b).

Discrete-Time Unit Impulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.1.5}$$

Discrete-Tine Unit step

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{1.1.6}$$

Note that $\delta[n] = u[n] - u[n-1]$

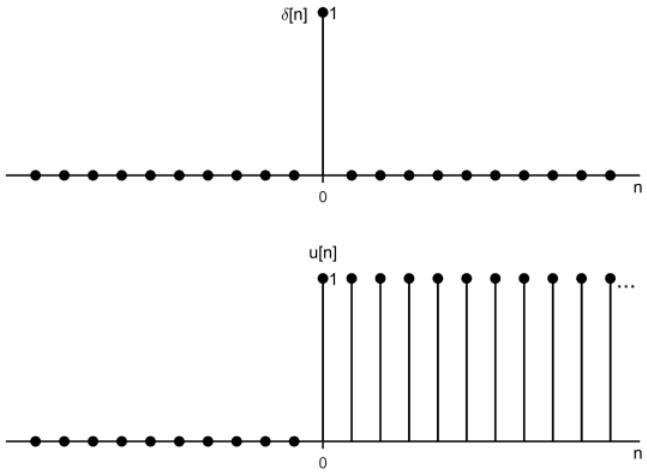



***Figure 1.1.10:*** *Stem plot of the DT unit impulse (top panel) and DT unit step (bottom panel).*

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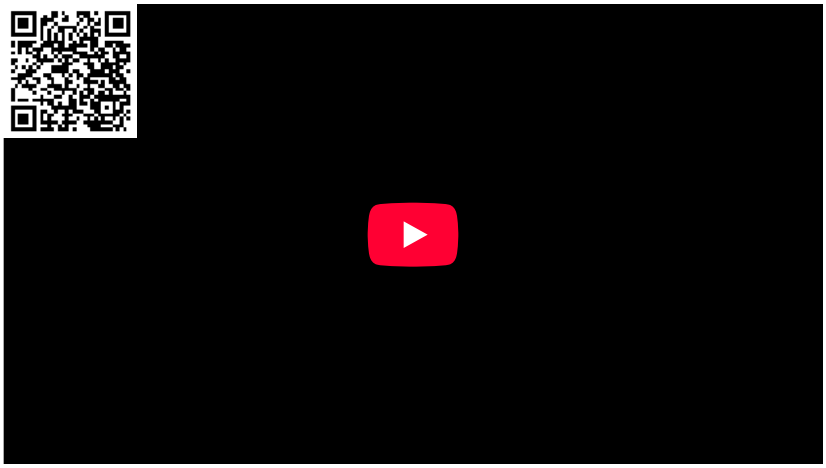
## 1.2: Discrete-Time Systems

### Learning Objectives: Discrete-Time System

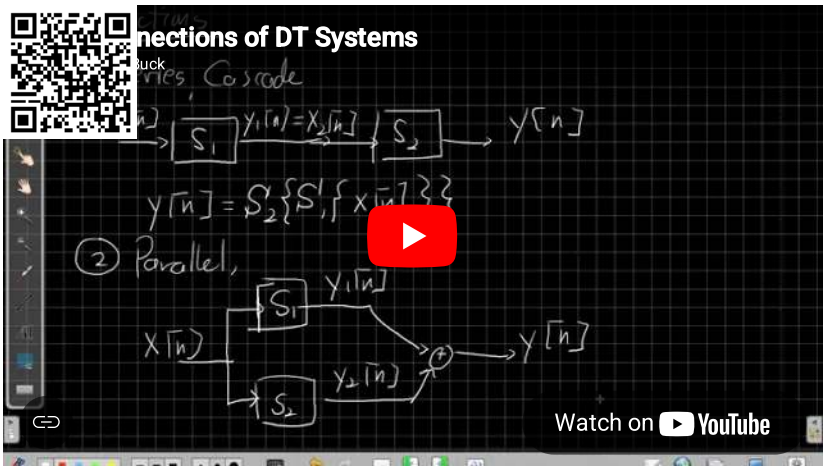
1. Translate verbal descriptions of systems into equations.
2. Demonstrate whether systems satisfy the properties of linearity, time-invariance, causality, and stability.
3. Argue whether a system can be linear and time-invariant (LTI) by examining a collection of inputs and outputs.

### YouTube Videos

Systems Fundamentals



Connection of DT Systems




Systems Properties: Linearity



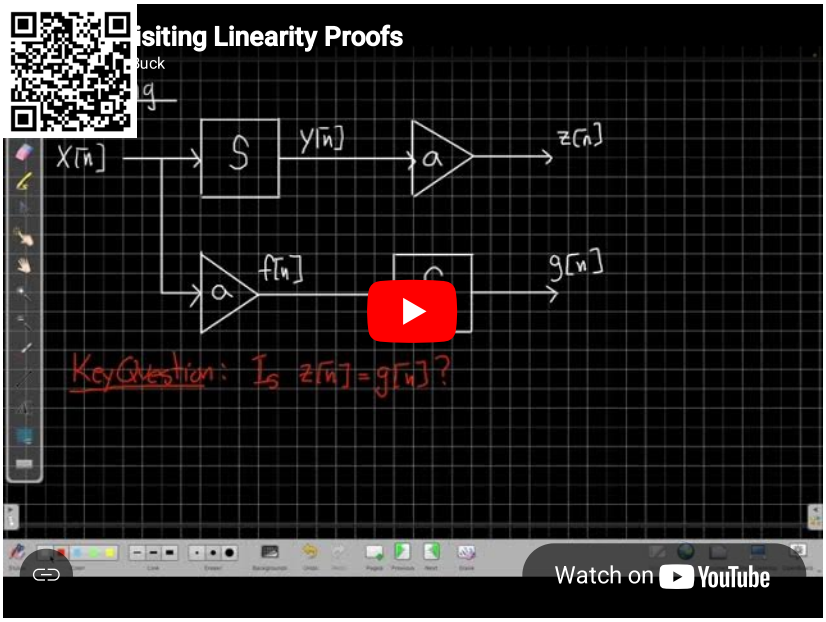
DT System Properties: Time Invariance



System Properties: Stability (Revisit)



Revisiting Linearity Proofs




Revisiting Time-Invariance Proofs



Proving Linearity



Proving Time Invariance



System Properties: Causality



## Summary

### Discrete-Time Systems




**Definition:** A discrete-time (DT) system transforms an input signal $x[n]$ into an output signal $y[n]$.

***Figure 1.2.1:*** *Basic discrete system flow diagram.*

The system **S** is an equation relating the input signal $x[n]$ to the output signal $y[n]$.

**Example**

$$y[n] = \frac{1}{7}\left(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6]\right) \tag{1.2.1}$$

The output signal is given by the average of the most recent input signal values (a *running average*). For example, if $x[n]$ represents the number of new cases of a disease on a given day, then $y[n]$ is the running average of new cases. This system is **non-recursive** because $y[n]$ does not depend on past or future values of $y[n]$.

**Example**

$$y[n] = 1.03\,y[n-1] + x[n] \tag{1.2.2}$$

This system can be interpreted as a bank balance with positive 3% interest and daily transactions. This system is **recursive** because the current value of $y[n]$ depends on a past value of the output $y[n-1]$.

### Connections of Discrete-Time Systems

There are three ways to connect systems:

**Series (Cascade)** - The output from one system is fed into another system.

$$y[n] = S_2\{S_1\{x[n]\}\} \tag{1.2.3}$$

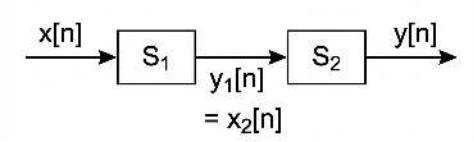



***Figure 1.2.2:*** *Basic discrete series (cascade) system flow diagram.*

**Parallel** - The same input signal is applied to both systems and then combined.

$$y[n] = y_1[n] + y_2[n] = S_1\{x[n]\} + S_2\{x[n]\} \tag{1.2.4}$$

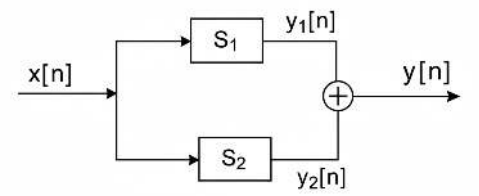



***Figure 1.2.3:*** *Basic discrete parallel system flow diagram.*

**Feedback** - One system output is fed back and combined with the input to repeat the process.

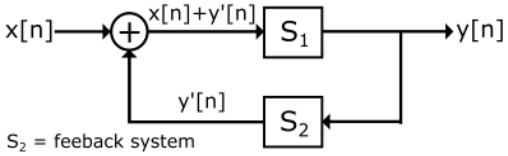



***Figure 1.2.4:*** *Basic discrete feedback system flow diagram.*

---

System Properties

**Linearity**

A linear system must satisfy both the **scaling** and **superposition** properties.

Scaling:

$$T\{a\,x[n]\} = a\,T\{x[n]\} \tag{1.2.5}$$

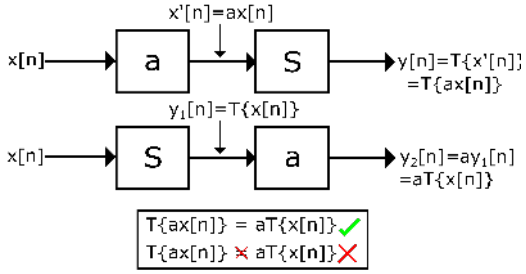



***Figure 1.2.5:*** *Discrete flow diagram showing a visual proof for the scaling property.*

Superposition:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} \tag{1.2.6}$$

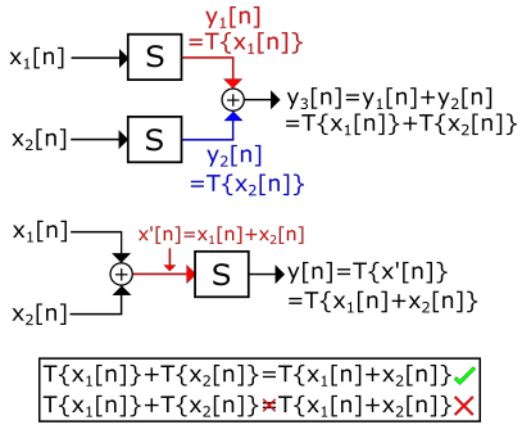



***Figure 1.2.6:*** *Discrete flow diagram showing a visual proof for the superposition property.*

**Time-Invariance**

If $y[n] = S\{x[n]\}$, then the system is time-invariant if: $S\{x[n - n_0]\} = y[n - n_0]$

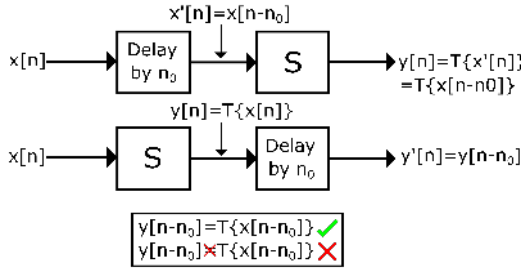



***Figure 1.2.5:*** *Discrete flow diagram showing a delay in series with a system.*

## Casuality

A system is causal if all output values depend **only** on past and present values. I,e, the index for the output must **always** be greater than the index of the system.

## Stability

For a system to be stable, given any input that is bounded, the output must **always** be bounded as well.

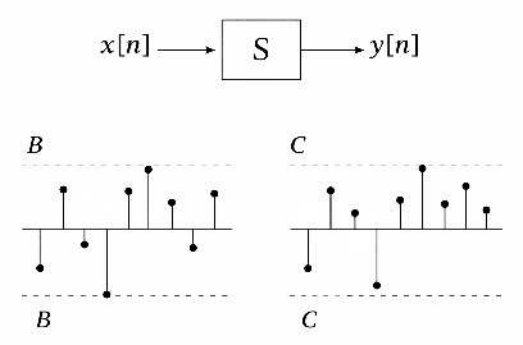



***Figure 1.2.6:*** *Graphical stability representation.*

### Inverteability

A system is invertible if distinct inputs lead to distinct outputs. A system is invertible if an inverse system exists that recovers the input from the output.

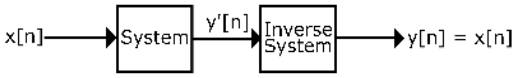



***Figure 1.2.7:*** *Flow diagram showing a system and it's inverse system in parallel producing the initial input.*

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# 1.3: Convolution

## Learning Objectives: Convolution

1. Understand how the convolution equation is a consequence of linearity and time invariance.
2. Practice computing convolutions using both the equation directly and the *flip-and-shift* (ticker tape) method.
3. Begin to develop insight into how the optimal approach to computing a convolution depends on the signals being convolved.

## YouTube Videos

LTI Systems and Convolution



Convolution Properties



Convolution Example - Flip and Shift



Convolution using Tickertape



Convolution with a Unit Step



Discrete-time Convolution Sum and Example



## Summary

### Convolution and LTI Systems

For discrete-time systems that are **linear** and **time-invariant (LTI)**, the output $y[n]$ can be expressed as the convolution of the input signal $x[n]$ with the system's impulse response $h[n]$

$$y[n] = x[n] * h[n] \tag{1.3.1}$$

where $*$ denotes convolution.

### Discrete-Time Convolution Equation

The discrete-time convolution sum is given by:

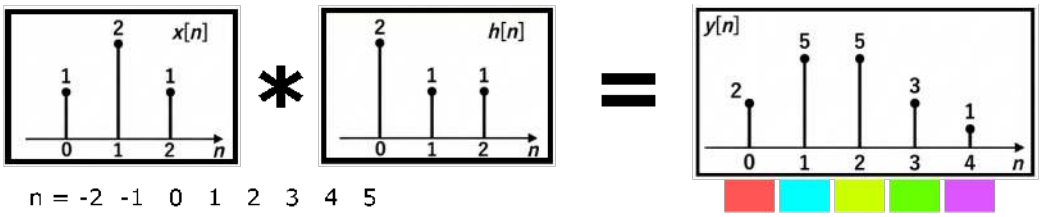
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \tag{1.3.2}$$

This equation follows directly from the properties of linearity and time invariance.

### Flip-and-Shift (Ticker Tape) Method

An alternative way to compute convolution is the **flip-and-shift** method:

1. Flip one of the signals (usually $h[n]$) to obtain $h[-k]$.
2. Shift the flipped signal by $n$, giving $h[n-k]$.
3. Multiply point-by-point with $x[k]$.
4. Sum over all $k$.




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| n = | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | |
| x[n] = | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | |
| h[-n+0] = | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | y[0] = (1x0)+(1x0)+(2x1)+(0x2)+(0x1)+(0x0)+(0x0)+(0x0) = 2 |
| h[-n+1] = | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | y[1] = (0x0)+(1x0)+(1x1)+(2x2)+(0x1)+(0x0)+(0x0)+(0x0) = 5 |
| h[-n+2] = | 0 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | y[2] = (0x0)+(0x0)+(1x1)+(1x2)+(2x1)+(0x0)+(0x0)+(0x0) = 5 |
| h[-n+3] = | 0 | 0 | 0 | 1 | 1 | 2 | 0 | 0 | y[3] = (0x0)+(0x0)+(0x1)+(1x2)+(1x1)+(2x0)+(0x0)+(0x0) = 3 |
| h[-n+4] = | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 0 | y[4] = (0x0)+(0x0)+(0x1)+(0x2)+(1x1)+(1x0)+(2x0)+(0x0) = 2 |

**Figure 1.3.1:** Example of the ticker-tape method of estimating the convolution between two discrete-time signals.

### Choosing a Convolution Method

The most efficient way to compute a convolution depends on the characteristics of the signals involved:

- Finite-length signals often benefit from the flip-and-shift graphical approach.
- Signals with simple analytic expressions may be easier to compute directly using the convolution sum.

### Properties

**Commutative:** The output (i.e., convolution) of an **LTI** system with input $x[n]$ and unit impulse response $h[n]$ is the same to the output of an **LTI** system with input $h[n]$ and unit impulse response $x[n]$

$$x[n] * h[n] = h[n] * x[n]$$

**Distributive:** The output to $x[n]$ convolved with two different unit impulse responses added together is the same to the output to $x[n]$ convolved separately with each unit impulse response and then both convolutions added together after

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

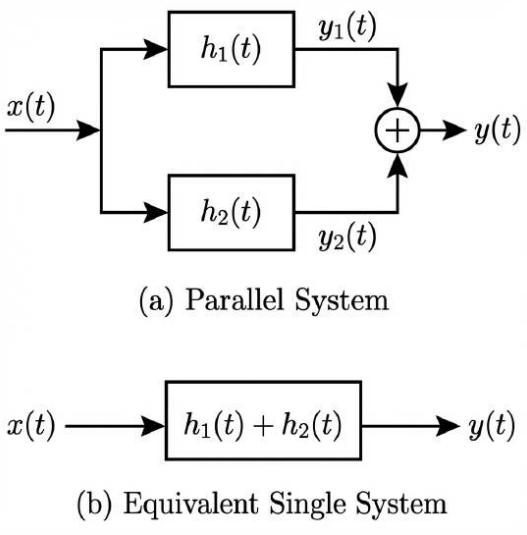



**Figure 1.3.2:** Flow diagram representing two parallel systems and their equivalent single systems

**Associative:** The output to $x[n]$ convolved with two different unit impulse responses convolved together is the same to the output to $x[n]$ convolved with either unit impulse response first, followed by that output being convolved with the other unit impulse response

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n]$$

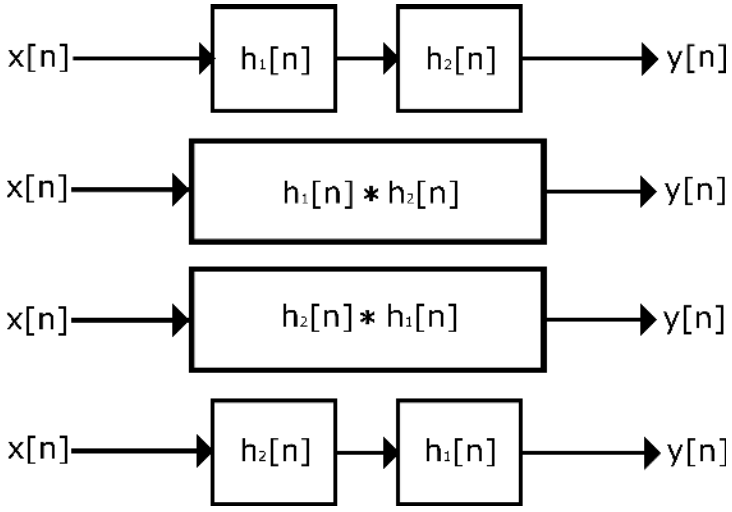


**Figure 1.3.3:** Flow diagram representing the associative property of convolution. Remember that the order of the LTI systems is irrelevant due to the commutative property.

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# 1.4: LTI System Properties

## Learning Objectives: Causality and Stability for LTI Systems

1. Identify whether an LTI system is causal from the impulse response.
2. Identify whether an LTI system is stable from the impulse response.
3. Apply convolution properties to efficiently compute the outputs of compound systems.
4. Improve our understanding of how features in the input affect features in the convolution output.

## YouTube Videos

Testing Causality of LTI Systems with the Impulse Response



Testing Stability of LTI Systems with the Impulse Response



## Summary

### LTI Systems With and Without Memory

A system is *memoryless* if its output at time $n$ depends only on the input value at that same time.

For a discrete-time (DT) LTI system, the output is $y[n] = x[n] * h[n]$, where $*$ denotes convolution.

A DT LTI system is memoryless if its impulse response is a scaled impulse:

$$h[n] = k\,\delta[n] \quad \Rightarrow \quad y[n] = x[n] * \big(k\,\delta[n]\big) = k\,x[n]. \tag{1.4.1}$$

So, in a memoryless LTI system, the output is simply a scaled replica of the input.

If a DT LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory.

$$h[n] = 0 \text{ for } n \neq 0 \tag{1.4.2}$$

### Invertibility of LTI Systems

A system is *invertible* if an inverse system exists that recovers the input from the output.

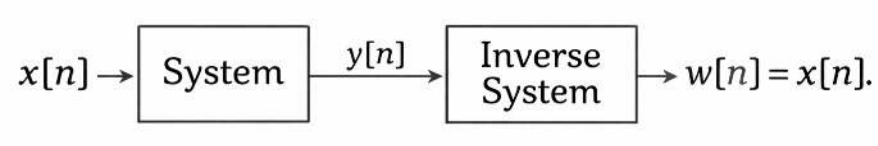



**Figure 1.4.1:** Flow diagram showing that the combination of a discrete-time system and its inverse system in cascade produces the initial input.

If the original system has impulse response $h[n]$, and the inverse has impulse response $h_i[n]$, then cascading them gives:

$$x[n] \xrightarrow{h[n]} y[n] \xrightarrow{h_i[n]} w[n] \tag{1.4.3}$$

$$w[n] = y[n] * h_i[n] = (x[n] * h[n]) * h_i[n] = x[n] * (h[n] * h_i[n]). \tag{1.4.4}$$

For perfect recovery ($w[n] = x[n]$ for all inputs), we must have:

$$h[n] * h_i[n] = \delta[n]. \tag{1.4.5}$$

#### Example: Accumulator System and Its Inverse

Let $h[n] = u[n]$ (unit step), then:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} x[k]\, u[n-k] = \sum_{k=-\infty}^{n} x[k]. \tag{1.4.6}$$

This system is a running-sum/accumulator system whose output accumulates all present and past samples/values of the input.

The inverse system (to “undo” accumulation) has impulse response:

$$h_i[n] = \delta[n] - \delta[n-1]. \tag{1.4.7}$$

Quick sanity check:

$$u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]. \tag{1.4.8}$$

### Causality for LTI Systems

A system is *causal* if the output at time n depends only on the present and past input values (not future values).

For an LTI system, causality is equivalent to the impulse response being zero for negative time:

**Causality condition:**

$$h[n] = 0 \;\text{ for } n < 0. \tag{1.4.9}$$

### Stability for LTI Systems (BIBO Stability)

A system is *stable* if every bounded input produces a bounded output (BIBO stability).

If $|x[n]| \leq B$ for all $n$, then for an LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \Rightarrow |y[n]| \leq \sum_{k=-\infty}^{\infty} |x[k]|\, |h[n-k]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|\,. \tag{1.4.10}$$

Therefore, an LTI system is stable if its impulse response is absolutely summable:

**Stability condition:**

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty. \tag{1.4.11}$$

Examples noted:

- $h[n] = \delta[n - n_0]$ is stable because $\sum |h[k]| = 1$.
- $h[n] = u[n]$ is unstable because $\sum_{k=-\infty}^{\infty} |u[k]| = \infty$.

### Unit Step Response of an LTI System

The unit step response (i.e., the step response) is the output/response of an LTI system when the input is the unit step u[n]:

$$s[n] = u[n] * h[n]. \tag{1.4.12}$$

Using convolution:

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]\, u[n-k] = \sum_{k=-\infty}^{n} h[k]. \tag{1.4.13}$$

Therefore, the step response is the running sum of the impulse response.

The impulse response is the first difference of its step response:

$$h[n] = s[n] - s[n-1]. \tag{1.4.14}$$

---

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# 1.5: Block Diagrams and Difference Equations

## Learning Objectives: Block Diagram and Difference Equations

1. Solve for the impulse response of a causal LTI system satisfying a linear constant-coefficient difference equation (LCCDE) by iterating the equation forward in time.
2. Identify the order of a LCCDE and recognize whether it is a recursive or non-recursive system implementation.
3. Draw block diagrams to represent a LCCDE.

## YouTube Videos

Linear Constant-Coefficient Difference Equations



Examples of Difference Equations



Block Diagrams for Linear Constant-Coefficient Difference Equations



## Summary

### Causal LTI Systems Described by Difference Equations

The input and output signals of a causal LTI system can be related through a **linear constant-coefficient difference equation** (LCCDE).

An **N$^{th}$-order** linear constant-coefficient difference equation has the form:

\[\displaystyle \sum_{k=0}^{N} a_k\,y[n-k] = \sum_{k=0}^{M} b_k\,x[n-k]$$

This equation describes the relationship between the input $x[n]$ and the output $y[n]$.

Solving for $y[n]$:

\[\displaystyle y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k\,x[n-k] - \sum_{k=1}^{N} a_k\,y[n-k]\right)$$

The solution $y[n]$ can be written as the sum of a particular solution and a homogeneous solution:

\[y[n] = y_p[n] + y_h[n]$$

The homogeneous solution satisfies: $\sum_{k=0}^{N} a_k\, y_h[n-k] = 0$ with the initial rest condition $y[n] = 0$ for $n < 0$, assuming $x[n] = 0$ for $n < 0$.

### Non-Recursive Systems (FIR)

When $N = 0$, the difference equation becomes:

\[\displaystyle y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0}\,x[n-k]$$

In this case, the impulse response is:

\[h[n] =

$$\begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases} \tag{1.5.1}$$

$$

The impulse response has finite duration, so the system is a finite impulse response (FIR) system.

This is a non-recursive equation because the present output does not depend on past values of $y[n]$.

**Recursive systems** involve feedback and generally produce infinite-length impulse responses (IIR).

✔ Example: Recursive Systems (IIR)

Assume

$$y[n] - \frac{1}{3}y[n-1] = x[n] \tag{1.5.2}$$

Rearrange:

$$y[n] = x[n] + \frac{1}{3}y[n-1] \tag{1.5.3}$$

We need the previous value of the output $y[n-1]$ to compute the present value $y[n]$.

Initial condition:

$$x[n] = k\,\delta[n] \tag{1.5.4}$$

This means that $x[n] = 0$ for $n \leq -1$ and $y[n] = 0$ for $n \leq -1$

Solving for $n = 0$:

$$\begin{aligned} y[0] &= x[0] + \frac{1}{3}y[-1] = k + \frac{1}{3}(0) = k \\ y[1] &= x[1] + \frac{1}{3}y[0] = 0 + \frac{1}{3}k = \left(\frac{1}{3}\right)k \\ y[2] &= x[2] + \frac{1}{3}y[1] = 0 + \frac{1}{3}\left(\frac{1}{3}\right)k = \left(\frac{1}{3}\right)^2 k \end{aligned} \tag{1.5.5}$$

Continuing this pattern, for $n \geq 0$:

$$y[n] = \left(\frac{1}{3}\right)^n k \tag{1.5.6}$$

Setting $k = 1$ gives the impulse response:

$$h[n] = \left(\frac{1}{3}\right)^n u[n] \tag{1.5.7}$$

This is an impulse response of infinite duration → an IIR system.

Block Diagram Representations of 1st-Order Systems Described by Difference Equations

A block diagram is composed of:

- Adders (summation)
- Multiplication by constants
- Unit delays

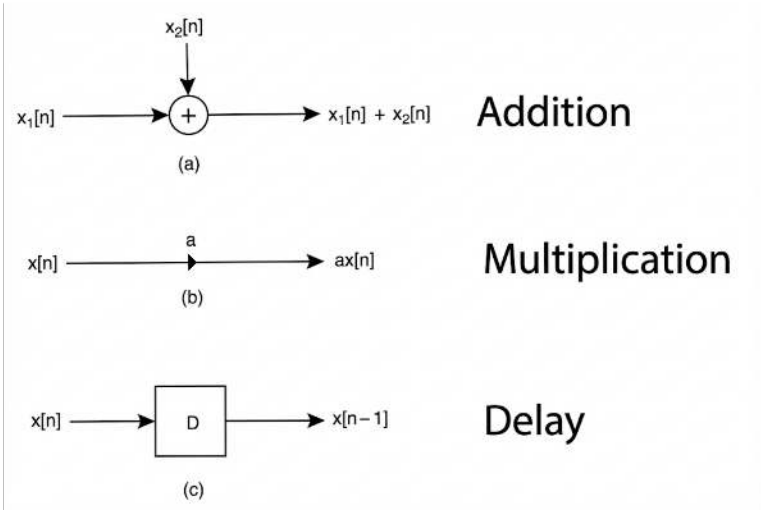


*Figure 1.5.1:* Illustrations of the basic elements for representing causal systems using a block diagram representation: (a) adder; (b) multiplier by a constant coefficient; and (c) delay.

Consider a first-order difference equation:

\[y[n] + a\,y[n-1] = b\,x[n]$$

Solving for $y[n]$:

\[y[n] = -a\,y[n-1] + b\,x[n]$$

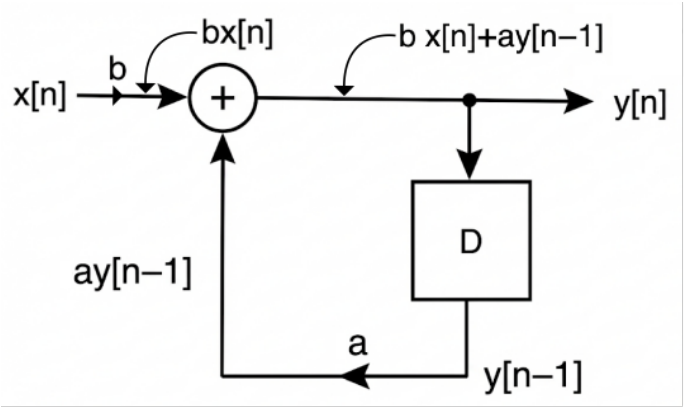



**Figure 1.5.2:** Block diagram of a first-order LCCDE

## Example: Block Diagram Construction

Given:

\[y[n] + \frac{1}{4}y[n-1] = x[n] - \frac{1}{8}x[n-1] + \frac{1}{3}x[n-2]$$

Solving for $y[n]$:

\[y[n] = x[n] - \frac{1}{8}x[n-1] + \frac{1}{3}x[n-2] - \frac{1}{4}y[n-1]$$

This equation can be implemented using delays, multipliers, and adders as shown in the corresponding block diagram.

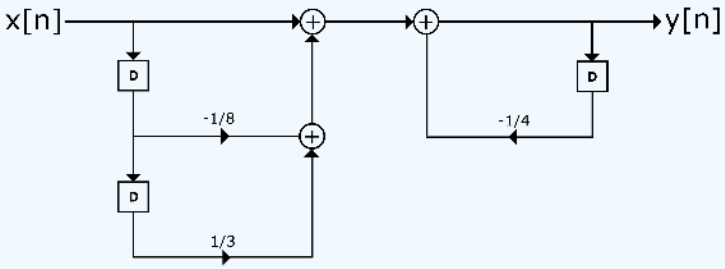



**Figure 1.5.3:** Block diagram for the given example

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# 1.6: Eigenfunctions, DT Fourier Series

## Learning Objectives: Eigenfunctions and Discrete-Time Fourier Series (DTFS)

1. Apply the eigenfunction property of LTI systems to exponential inputs to find the output.
2. Compute a periodic signal $x[n]$ from its Fourier series coefficients.
3. Compute the Fourier series coefficients from a periodic signal $x[n]$.
4. Decide whether a system could be LTI based on whether inputs/outputs satisfy the eigenfunction relationship.

### YouTube Videos

Eigenfunction property



Discrete-Time Fourier Series defined



Discrete-Time Fourier Series Synthesis Example



Discrete Time Fourier Series Example



# Summary

### Response of LTI Systems to Complex Exponentials

Consider the set of complex exponential signals $z^n$ where $z$ is a complex number. The complex exponential signals are useful because:

1. They can be used to construct a wide variety of useful discrete-time signals.
2. If an input is expressed as a linear combination of exponential signals, the output of a LTI system can also be expressed as a linear combination of the exponential signals.

> **Key idea (eigenfunction property):**
> If $x[n] = z^n$ is applied to an LTI system, the output is the *same exponential* scaled by a complex factor: $z^n \longrightarrow H(z)\, z^n$.

**Derivation (from convolution):**

\[y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]\[

$$\text{If } x[n] = z^n, \;\; y[n] = \sum_{k=-\infty}^{\infty} z^{\,n-k} h[k] = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}. \tag{1.6.1}$$

Define the scaling factor:

\[H(z) \;=\; \sum_{k=-\infty}^{\infty} h[k]\,z^{-k}$$

Therefore:

\[y[n] = H(z)\,z^n.$$

### LTI Output for Inputs That Are Sums of Exponentials

If the input is a linear combination of exponentials,

\[x[n] = \sum_k a_k\,z_k^{\,n},$$

then, the LTI output is a linear combination of the *same exponentials*:

\[y[n] = \sum_k a_k\,H(z_k)\,z_k^{\,n}.$$

The coefficients in the output signals are scaled from $a_k$ to $a_k H(z_k)$.

### Periodic Complex Exponentials and the DT Fourier Series (DTFS)

Recall: $x[n]$ is periodic with period $N$ if $x[n] = x[n+N]$.

A complex exponential $e^{j\omega_0 n}$ is periodic with period $N$ when $\omega_0 = \frac{2\pi}{N}$.

Define the DTFS basis functions:

\[\phi_k[n] = e^{j k \omega_0 n} = e^{j\frac{2\pi}{N}kn}, \qquad k=0,\pm1,\pm2,\ldots$$

Because $\phi_{k+rN}[n] = \phi_k[n]$, there are only $N$ distinct exponentials for an $N$-periodic sequence. Therefore, the DTFS uses a **finite** sum with $N$ terms.

**DTFS Synthesis (build $x[n]$ from $a_k$)**

\[x[n] = \sum_{k=0}^{N-1} a_k\,e^{j\frac{2\pi}{N}kn}$$

**DTFS Analysis (compute $a_k$ from $x[n]$)**

\[a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{-j\frac{2\pi}{N}kn}$$

---

### When is $x[n] = \sin(\omega_0 n)$ Periodic?

To have periodicity with period $N$, we need: $\sin(\omega_0 n) = \sin(\omega_0(n+N)) = \sin(\omega_0 n + \omega_0 N)$. This occurs when $\omega_0 N = 2\pi m$ for some integer $m$.

> **Periodicity condition:** $x[n] = \sin(\omega_0 n)$ is periodic if $N = \dfrac{2\pi m}{\omega_0}$ is an integer for some integer $m$.

Using Euler's identity:

$$\sin(\omega_0 n) = \frac{1}{2j}\,e^{j\omega_0 n} - \frac{1}{2j}\,e^{-j\omega_0 n}. \tag{1.6.2}$$

If $\omega_0 = \dfrac{2\pi}{N}$, then DTFS coefficients are:

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}, \qquad a_k = 0 \text{ otherwise.} \tag{1.6.3}$$

---

**Example: DTFS Coefficients for a Mixed Signal**

Discrete-Time periodic signal with period $N$:

\[x[n]=1+\sin\!\Big(\frac{2\pi}{N}n\Big)+2\cos\!\Big(\frac{2\pi}{N}n\Big)+ 2 \sin\!\Big(\frac{4\pi}{N}n\Big).$$

Converting each term to an exponential using Euler equations

$$x[n] = 1 + \frac{1}{2j}e^{j\frac{2\pi}{N}n} - \frac{1}{2j}e^{-j\frac{2\pi}{N}n} + \frac{2}{2}e^{j\frac{2\pi}{N}n} + \frac{2}{2}e^{-j\frac{2\pi}{N}n} + \frac{2}{2j}e^{j\frac{2\pi}{N}2n} - \frac{2}{2j}e^{-j\frac{2\pi}{N}2n}. \tag{1.6.4}$$

Simplifying the above equation and comparing to $x[n] = \sum_k a_k e^{j\frac{2\pi}{N}kn}$, the DTFS coefficients are

$$a_0 = 1, \tag{1.6.5}$$

$$a_1 = 1 + \frac{1}{2j}, \qquad a_{-1} = 1 - \frac{1}{2j}, \tag{1.6.6}$$

$$a_2 = \frac{1}{j} = -j, \qquad a_{-2} = -\frac{1}{j} = j. \tag{1.6.7}$$

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## 1.7: DTFS Properties

### Learning Objectives: Discrete-Time Fourier Series (DTFS) Properties

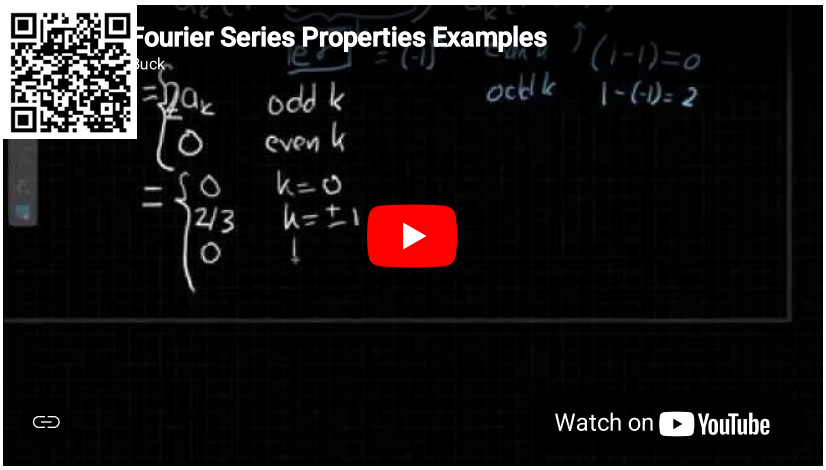
1. Solve for Fourier series coefficients for new signals using DTFS properties by relating the new signal to signals with known FS coefficients.

### YouTube Videos

Discrete-Time Fourier Series Properties



DT Fourier Series Properties Examples




Discrete Time Fourier Series Example



### Summary

Core DTFS equations

If x[n] is periodic with period N and fundamental frequency equal to $\omega_0 = \frac{2\pi}{N}$, we write:

$$x[n] \iff a_k$$

where $\iff$ represents a Fourier Series.

**Synthesis (reconstruction) Equation:**

$$x[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\omega_0 n} \tag{1.7.1}$$

**Analysis (coefficients) Equation:**

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \, e^{-jk\omega_0 n} \tag{1.7.2}$$

Key DTFS properties

- **Multiplication in time**

If $x[n] \iff a_k$ and $y[n] \iff b_k$ (same period $N$), then

$$x[n]\, y[n] \iff d_k = \sum_{\ell=\langle N \rangle} a_\ell \, b_{k-\ell} \tag{1.7.3}$$

*(This looks like a discrete convolution-style summation in the coefficient index.)*

- **First difference**

If $x[n] \iff a_k$, then

$$x[n] - x[n-1] \iff \left(1 - e^{-j\frac{2\pi}{N}k}\right) a_k \tag{1.7.4}$$

- **Parseval's relation (average power)**

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2 \tag{1.7.5}$$

Left side = average power in one period of $x[n]$. Right side = sum of powers in each harmonic (via $a_k$).

**Table 1.7.1** Properties of the Discrete-Time Fourier Series (DTFS)

| Periodic Signal | | Fourier Series Coefficients |
|---|---|---|
| $x[n]$, $y[n]$ periodic with period $N$ and fundamental frequency $\omega_0 = \frac{2\pi}{N}$ | | $a_k$, $b_k$ periodic with period $N$ |
| Parseval's Relation for Periodic Signals | | |
| $\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x[n] \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$ | | |
| Linearity | | |
| $A\,x[n] + B\,y[n]$ | | $A\,a_k + B\,b_k$ |
| Time Shifting | | |
| $x[n - n_0]$ | | $a_k \, e^{-j\frac{2\pi}{N}kn_0}$ |
| Frequency Shifting | | |
| $e^{j\frac{2\pi}{N}mn}\, x[n]$ | | $a_{k-m}$ |
| Conjugation | | |
| $x^*[n]$ | | $a^*_{-k}$ |
| Time Reversal | | |
| $x[-n]$ | | $a_{-k}$ |
| Time Scaling | | |
| $x_m[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period $mN$) | | $\frac{1}{m} a_k$ (viewed as periodic with period $mN$) |
| Periodic Convolution | | |
| $\sum_{r=\langle N \rangle} x[r]\, y[n-r]$ | | $N\, a_k\, b_k$ |
| Multiplication | | |
| $x[n]\, y[n]$ | | $\sum_{\ell=\langle N \rangle} a_\ell \, b_{k-\ell}$ |
| Conjugate Symmetry for Real Signals | | |
| $x[n]$ real | | $a_k = a^*_{-k}$<br>$\Re\{a_k\} = \Re\{a_{-k}\}$<br>$\Im\{a_k\} = -\Im\{a_{-k}\}$<br>$\lvert a_k \rvert = \lvert a_{-k} \rvert$<br>$\angle a_k = -\angle a_{-k}$ |
| Real and Even Signals | | |
| $x[n]$ real and even | | $a_k$ real and even |
| Real and Odd Signals | | |
| $x[n]$ real and odd | | $a_k$ purely imaginary and odd |

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# 1.8: Filtering and the DTFS

## Learning Objectives: Filtering and the Discrete-Time Fourier Series (DTFS)

1. Identify discrete-time frequency-selective filters such as lowpass, highpass, and bandpass filters.
2. Solve for the output of LTI filters using the Fourier series of a periodic input signal $x[n]$.

## YouTube Videos

DTFS Filtering Lowpass Example



## Summary

### Fourier Series and LTI Systems

Reminder: if the input to a discrete-time LTI system is a complex exponential,

$$x[n] = z^n, \tag{1.8.1}$$

then the output is

$$y[n] = H(z)\, z^n, \tag{1.8.2}$$

where the system function is

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}. \tag{1.8.3}$$

On the unit circle $z = e^{j\omega}$, the system function becomes the **frequency response**:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}. \tag{1.8.4}$$

If the input $x[n]$ is periodic with period $N$, its DTFS is

$$x[n] = \sum_{k=0}^{N-1} a_k\, e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}. \tag{1.8.5}$$

Because each exponential $e^{jk\omega_0 n}$ is an eigenfunction of an LTI system, the output is

$$y[n] = \sum_{k=0}^{N-1} a_k\, H(e^{jk\omega_0})\, e^{jk\omega_0 n}. \tag{1.8.6}$$

**DTFS + LTI "filtering rule":**
If $x[n] \Longleftrightarrow a_k$ and $y[n] \Longleftrightarrow b_k$, then

$$b_k = a_k H(e^{jk\omega_0}). \tag{1.8.7}$$

Note that the frequency response gives the gain applied to each harmonic.

### What “Filtering” Means (Frequency-Shaping)

**Filtering** means changing (or removing) the relative amplitudes of the frequency components in a signal. LTI systems that modify the spectrum of an input signal are called **frequency-shaping filters**.

Frequency-selective filters pass certain frequencies (ideally without attenuation) and attenuate/reject others.

The magnitude response is often plotted in dB:

$$20 \log_{10} \left| H(e^{j\omega}) \right|. \tag{1.8.8}$$

### Frequency-Selective Filters

- **Lowpass Filter:** passes low frequencies; attenuates/rejects high frequencies.

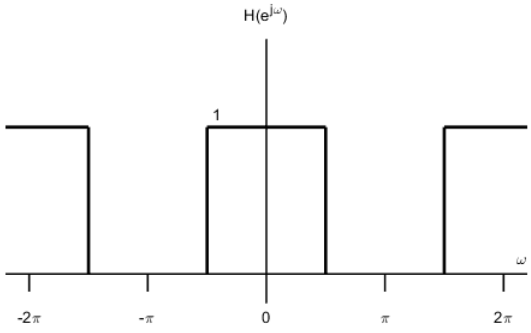



**Figure 1.8.1:** Illustration of an ideal lowpass filter (LPF).

- **Highpass Filter:** passes high frequencies; attenuates/rejects low frequencies.

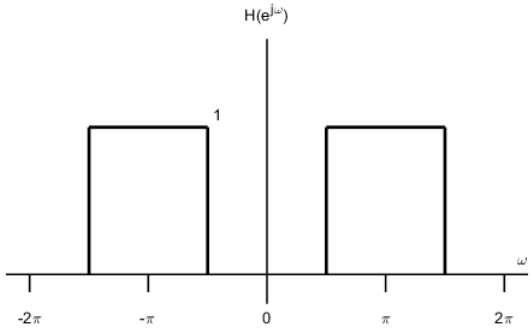



**Figure 1.8.2:** Illustration of an ideal highpass filter (HPF)

- **Bandpass Filter:** passes a band of frequencies; attenuates/rejects frequencies below and above that band.

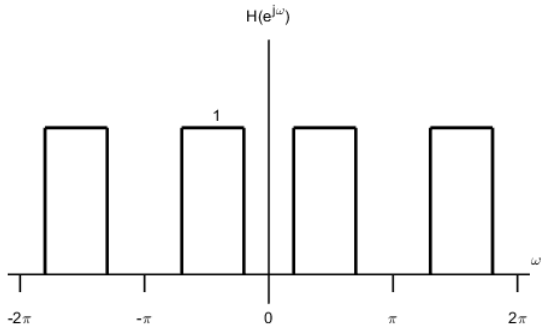



**Figure 1.8.3:** Ilustration of a bandpass filter.

**Cutoff frequencies** define the boundaries between the *passband* (passing frequencis) and *stopband* (rejected frequencies).

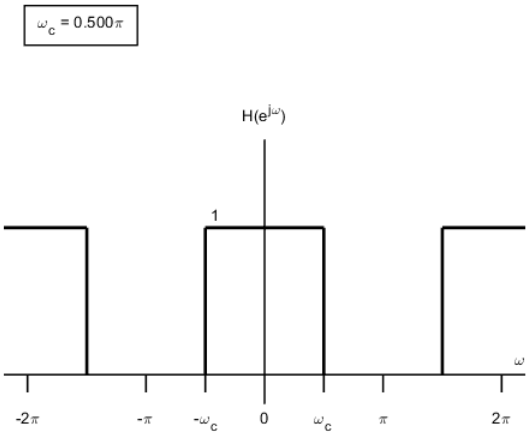



**Figure 1.8.4:** Illustration of the cutoff frequency equal to $w_c = 0.5\pi$ in an ideal lowpass filter.

## Moving-Average FIR Filters (Nonrecursive DT Filters)

A common nonrecursive (FIR) filter has the form:

$$y[n] = \sum_{k=-M}^{M} b_k\, x[n-k]. \tag{1.8.9}$$

This means $y[n]$ is a weighted combination of $(2M+1)$ input samples around time $n$.

### Example A: Two-Point Average

Consider

$$y[n] = \frac{1}{2}\big(x[n] + x[n-1]\big) \quad \Longrightarrow \quad h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]. \tag{1.8.10}$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{2}\left(1 + e^{-j\omega}\right) = e^{-j\omega/2}\cos\left(\frac{\omega}{2}\right), \tag{1.8.11}$$

so

$$|H(e^{j\omega})| = \left|\cos\left(\frac{\omega}{2}\right)\right|. \tag{1.8.12}$$

### Example B: Three-Point Moving Average

Consider

$$y[n] = \frac{1}{2}\big(x[n-1] + x[n] + x[n+1]\big) \quad \Longrightarrow \quad h[n] = \frac{1}{2}\delta[n-1] + \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n+1]. \tag{1.8.13}$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{2}\big(e^{-j\omega} + 1 + e^{j\omega}\big) = \frac{1}{2}\big(1 + 2\cos\omega\big). \tag{1.8.14}$$

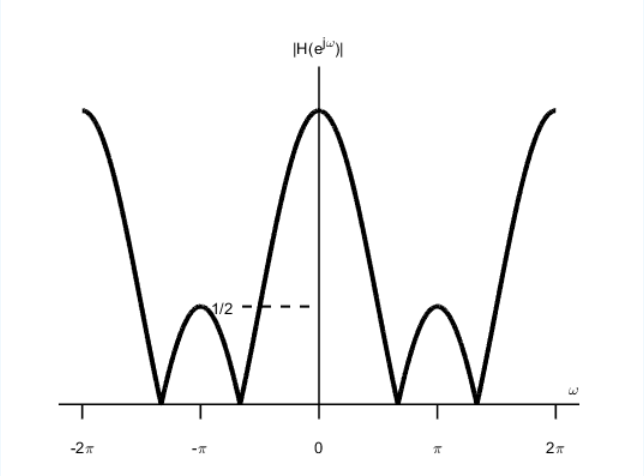



**Figure 1.8.5:** Frequency response for a three-point moving-average lowpass filter.

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# 1.9: DT Fourier Transform

## Learning Objectives: Discrete-Time Fourier Transform (DTFT)

1. Compute the DTFT of signals such as decaying exponentials, finite rectangular pulses, and simple finite-length signals.
2. Explain how time-domain properties of a signal impact the magnitude of its Fourier Transform.

## YouTube Videos

Discrete-Time Fourier Transform



More DT Fourier Transform Examples



DT Fourier Transform for Simple Finite Length Signals

Fourier Transform for Discrete-Time Symmetric Rectangular Pulse




## Summary

### From Fourier Series (DTFS) to Fourier Transform (DTFT)

Principle: start with a finite-duration (nonperiodic) signal $x[n]$, create a periodic extension $\tilde{x}[n]$ with period $N$, write a DTFS for $\tilde{x}[n]$, and then let $N \to \infty$.

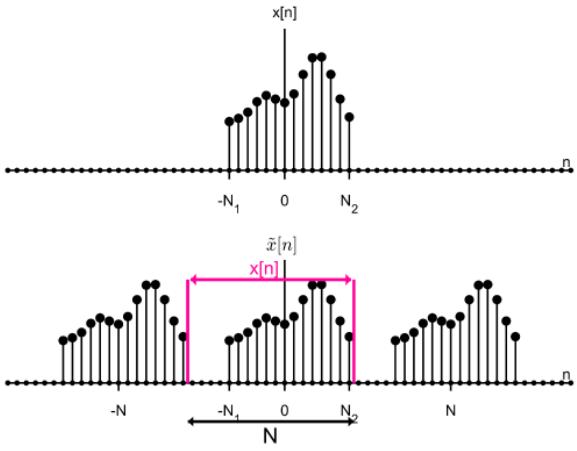



**Figure 1.9.1:** Illustration of how to generate a periodic extended signal $\tilde{x}[n]$ from a nonperiodic signal $x[n]$.

DTFS for the periodic extension:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k \, e^{jk\omega_0 n}, \qquad \omega_0 = \frac{2\pi}{N}. \tag{1.9.1}$$

The coefficients relate to samples of the DTFT:

$$a_k = \frac{1}{N} X\left(e^{jk\omega_0}\right). \tag{1.9.2}$$

Substitute into the DTFS synthesis:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}. \tag{1.9.3}$$

As $N \to \infty$, $\omega_0 \to 0$ and the sum becomes an integral.

**DTFT Synthesis equation (reconstruct $x[n]$ from $X(e^{j\omega})$):**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega. \tag{1.9.4}$$

DTFT Analysis Equation (Definition)

**DTFT pair:**

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega. \tag{1.9.5}$$

Interpretation: $X(e^{j\omega})$ is the "spectrum" of $x[n]$. In other words, the spectrum describes how $x[n]$ is composed in terms of complex exponentials at different frequencies $\omega$.

## Key DTFT Properties

### Periodicity in frequency

The DTFT is periodic in $\omega$ with period $2\pi$:

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right). \tag{1.9.6}$$

### Existence convergence condition

A sufficient condition for the DTFT to converge is absolute summability:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty. \tag{1.9.7}$$

### DTFT vs DTFS

- **DTFT** represents (possibly) nonperiodic $x[n]$ using an **integral** over a continuum of $\omega$.
- **DTFS** represents a periodic $x[n]$ using a **finite sum** of $N$ harmonics $k\omega_0$.

**Important Example DTFTs**

### Decaying exponential

If $x[n] = a^n u[n]$ with $|a| < 1$, then

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}. \tag{1.9.8}$$

A common magnitude form:

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1-2a\cos\omega+a^2}}. \tag{1.9.9}$$

### Finite rectangular pulse

If

$$x[n] = \begin{cases} 1, & |n| \le M \\ 0, & \text{otherwise} \end{cases} \tag{1.9.10}$$

then

$$X(e^{j\omega}) = \sum_{n=-M}^{M} e^{-j\omega n} = \frac{\sin\left(\omega\left(M+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}. \tag{1.9.11}$$

### Unit impulse

$$x[n] = \delta[n] \quad \Longrightarrow \quad X(e^{j\omega}) = 1. \tag{1.9.12}$$

### Pure complex exponential

$$x[n] = e^{j\omega_0 n} \quad \Longrightarrow \quad X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k). \tag{1.9.13}$$

### Relationship Between DTFS Coefficients and DTFT of a Periodic Signal

If $x[n]$ is periodic with DTFS coefficients $a_k$ (period $N$ and $\omega_0 = 2\pi/N$), then its DTFT is a line spectrum:

$$X(e^{j\omega}) = 2\pi \sum_{k=0}^{N-1} a_k\, \delta(\omega - k\omega_0). \tag{1.9.14}$$

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# 1.10: DT Fourier Transform Properties

## Learning Objectives: Discrete-Time Fourier Transform Properties

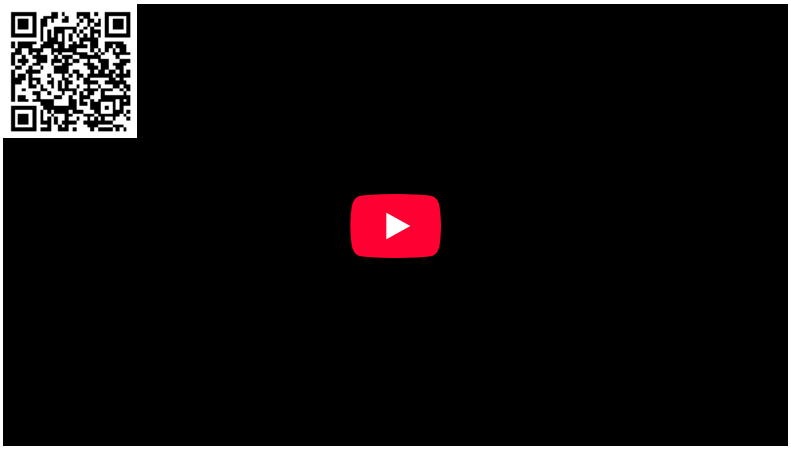
1. Exploit Fourier Transform properties to explain how manipulating a signal in time changes its Fourier transform (and vice versa).
2. Find inverse Fourier Transforms of new frequency-domain expressions by linking them to known transforms via properties.

## YouTube Videos

Properties of the Discrete-Time Fourier Transform



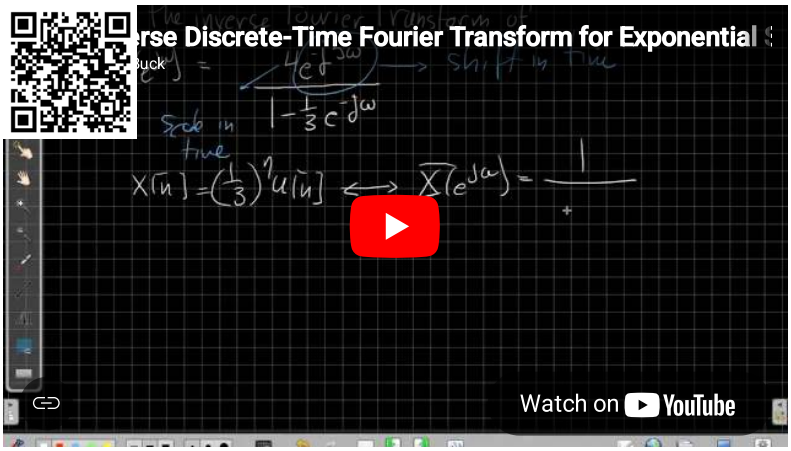
Summarizing the properties of the continuous impulse function



Discrete-Time Fourier Transform of Sinusoidal Signals



Inverse Discrete-Time Fourier Transform for Exponential Signals




Inverse Discrete-Time Fourier Transform of a Rectangular Pulse



## Summary

DTFT Pair (Reminder) $x[n] \iff X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \iff \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega. \tag{1.10.1}$$

**Periodicity:**

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}). \tag{1.10.2}$$

Key Properties of the DTFT

- **Linearity**

If $x_1[n] \iff X_1(e^{j\omega})$ and $x_2[n] \iff X_2(e^{j\omega})$, then

$$a\,x_1[n] + b\,x_2[n] \iff a\,X_1(e^{j\omega}) + b\,X_2(e^{j\omega}). \tag{1.10.3}$$

- **Time shifting**

$$x[n - n_0] \iff e^{-j\omega n_0}X(e^{j\omega}). \tag{1.10.4}$$

- **Frequency shifting (modulation)**

$$e^{j\omega_0 n}x[n] \iff X\left(e^{j(\omega-\omega_0)}\right). \tag{1.10.5}$$

- **Conjugation + symmetry for real signals**

$$x^*[n] \iff X^*(e^{-j\omega}). \tag{1.10.6}$$

If $x[n]$ is **real**, then $X(e^{j\omega})$ has **conjugate symmetry**:

$$X(e^{j\omega}) = X^*(e^{-j\omega}). \tag{1.10.7}$$

Therefore:

$$\Re\{X(e^{j\omega})\} \text{ is even in } \omega, \qquad \Im\{X(e^{j\omega})\} \text{ is odd in } \omega. \tag{1.10.8}$$

- **Time reversal**

$$x[-n] \iff X(e^{-j\omega}). \tag{1.10.9}$$

- **Differencing in time**

$$x[n] - x[n-1] \iff (1 - e^{-j\omega})X(e^{j\omega}). \tag{1.10.10}$$

- **Accumulation (running sum)**

If $y[n] = \sum_{m=-\infty}^{n} x[m]$, then (up to a possible impulse-train term at $\omega = 2\pi k$ depending on DC),

$$Y(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}\, X(e^{j\omega}). \tag{1.10.11}$$

- **Time expansion**

Define

$$x_k[n] = \begin{cases} x[n/k], & n \text{ is a multiple of } k \\ 0, & \text{otherwise} \end{cases} \quad \Longleftrightarrow \quad X_k(e^{j\omega}) = X(e^{jk\omega}). \tag{1.10.12}$$

- **Differentiation in frequency**

$$n\,x[n] \Longleftrightarrow j\,\frac{d}{d\omega} X(e^{j\omega}). \tag{1.10.13}$$

- **Parseval's relation**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega, \tag{1.10.14}$$

where $|X(e^{j\omega})|^2$ is the energy-density spectrum.

- **Convolution property**

If $y[n] = x[n] * h[n]$, then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}), \tag{1.10.15}$$

where $H(e^{j\omega})$ is the **frequency response** of the LTI system.

- **Multiplication in time**

If $y[n] = x_1[n]\,x_2[n]$, then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})\,d\theta \tag{1.10.16}$$

(periodic convolution in frequency).

**Ideal Lowpass Example**

If

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \tag{1.10.17}$$

then the impulse response is

$$h[0] = \frac{\omega_c}{\pi}, \qquad h[n] = \frac{\sin(\omega_c n)}{\pi n},\ n \neq 0. \tag{1.10.18}$$

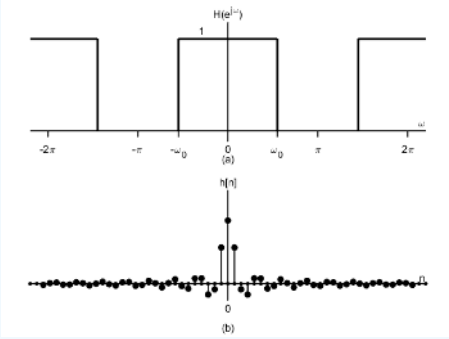



**Figure 1.10.1:** (a) Frequency response of ideal lowpass filter and (b) its impulse response.

**Table 1.10.1** Properties of the Discrete-Time Fourier Transform (DTFT)

| Discrete-Time Signal | Fourier Transform |
|---|---|
| $x[n]$, $y[n]$ | $X(e^{j\omega})$, $Y(e^{j\omega})$ periodic with period $2\pi$ |
| Parseval's Relation for Aperiodic Signals | |
| $\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{2\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$ | |
| Linearity | |
| $a\,x[n] + b\,y[n]$ | $a\,X(e^{j\omega}) + b\,Y(e^{j\omega})$ |
| Time Shifting | |
| $x[n-n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| Frequency Shifting | |
| $e^{j\omega_0 n}\,x[n]$ | $X\left(e^{j(\omega-\omega_0)}\right)$ |
| Conjugation | |
| $x^*[n]$ | $X^*(e^{-j\omega})$ |
| Time Reversal | |
| $x[-n]$ | $X(e^{-j\omega})$ |

| Discrete-Time Signal | Fourier Transform |
|---|---|
| $x[n],\ y[n]$ | $X(e^{j\omega}),\ Y(e^{j\omega})$ periodic with period $2\pi$ |
| Time Expansion | |
| $x_k[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| Convolution | |
| $x[n] * y[n]$ | $X(e^{j\omega})\,Y(e^{j\omega})$ |
| Multiplication | |
| $x[n]\,y[n]$ | $\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\,Y(e^{j(\omega-\theta)})\,d\theta$ |
| Differentiation in Frequency | |
| $n\,x[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |

**Table 1.10.2** Basic Discrete-Time Fourier Transform (DTFT) Pairs


https://eng.libretexts.org/@go/page/128231

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{N}kn}$ | $2\pi\sum_{k=-\infty}^{\infty} a_k \delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi\sum_{\ell=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi\ell)$ | (a) $\omega_0=\frac{2\pi m}{N}$<br>$a_k=\begin{cases}1, & k=m, m\pm N, m\pm 2N, \ldots\\ 0, & \text{otherwise}\end{cases}$ (1.10.19)<br>(b) $\frac{\omega_0}{2\pi}$ irrational → The signal is aperiodic |
| $\cos(\omega_0 n)$ | $\pi\sum_{\ell=-\infty}^{\infty}\left[\delta(\omega-\omega_0-2\pi\ell)+\delta(\omega+\omega_0-2\pi\ell)\right]$ | (a) $\omega_0=\frac{2\pi m}{N}$<br>$a_k=\begin{cases}\frac{1}{2}, & k=m, m\pm N, m\pm 2N, \ldots\\ \frac{1}{2}, & k=-m, -m\pm N,\\ & -m\pm 2N, \ldots\\ 0, & \text{otherwise}\end{cases}$ (1.10.20)<br>(b) $\frac{\omega_0}{2\pi}$ irrational → The signal is aperiodic |
| $\sin(\omega_0 n)$ | $\frac{\pi}{j}\sum_{\ell=-\infty}^{\infty}\left[\delta(\omega-\omega_0-2\pi\ell)-\delta(\omega+\omega_0-2\pi\ell)\right]$ | (a) $\omega_0=\frac{2\pi r}{N}$<br>$a_k=\begin{cases}-\frac{j}{2}, & k=r, r\pm N, r\pm 2N, \ldots\\ \frac{j}{2}, & k=-r, -r\pm N,\\ & -r\pm 2N, \ldots\\ 0, & \text{otherwise}\end{cases}$ (1.10.21)<br>(b) $\frac{\omega_0}{2\pi}$ irrational → The signal is aperiodic |
| $x[n]=1$ | $2\pi\sum_{\ell=-\infty}^{\infty}\delta(\omega-2\pi\ell)$ | $a_k=\begin{cases}1, & k=0, \pm N, \pm 2N, \ldots\\ 0, & \text{otherwise}\end{cases}$ (1.10.22) |
| Periodic square wave<br>$x[n]=\begin{cases}1, & \lvert n\rvert\le N_1\\ 0, & N_1<\lvert n\rvert\le N/2\end{cases}$<br>$x[n+N]=x[n]$ | $2\pi\sum_{k=-\infty}^{\infty} a_k\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k=\begin{cases}\frac{\sin\left(\frac{2\pi k}{N}\left(N_1+\frac{1}{2}\right)\right)}{N\sin\left(\frac{\pi k}{N}\right)}, & k\neq 0, \pm N, \pm 2N, \ldots\\ \frac{2N_1+1}{N}, & k=0, \pm N, \pm 2N, \ldots\end{cases}$ (1.10.23) |
| $\sum_{k=-\infty}^{\infty}\delta[n-kN]$ | $\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k=\frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \lvert a\rvert<1$ | $\frac{1}{1-ae^{-j\omega}}$ | — |
| $x[n]=\begin{cases}1, & \lvert n\rvert\le N_1\\ 0, & \lvert n\rvert>N_1\end{cases}$ | $\frac{\sin\left(\omega\left(N_1+\frac{1}{2}\right)\right)}{\sin(\omega/2)}$ | — |
| $\frac{\sin\left(\frac{\omega_m}{\pi}n\right)}{n}=\frac{\omega_m}{\pi}\operatorname{sinc}\left(\frac{\omega_m}{\pi}n\right)$<br>$0<\omega_m<\pi$ | $X(\omega)=\begin{cases}1, & 0\le\lvert\omega\rvert\le\omega_m\\ 0, & \omega_m<\lvert\omega\rvert\le\pi\end{cases}$<br>$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$ | — |
| $\delta[n-n_0]$ | $e^{-j\omega n_0}$ | — |

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# 1.11: DTFS and Difference Equations

## Learning Objectives: Difference Equations and Frequency Responses

1. Solve for the frequency response of an LTI system specified by a difference equation, and vice versa.
2. Exploit the Fourier transform properties to solve for the output of an LTI system for an exponential input.
3. Find the impulse response of a *causal* LTI system specified by a difference equation via the frequency domain.

## YouTube Videos

Inverse Discrete-Time Fourier Transform using Partial Fractions



Difference Equation to Impulse Response



Solving for the Frequency Response of an LTI system from a Difference Equation



Finding the Difference Equation from a Rational Frequency Response



Using Fourier Transforms to find the output of an LTI system



Zooming in on partial fraction



## Summary

### General LCCDE (Linear Constant-Coefficient Difference Equation)

A linear constant-coefficient difference equation for an LTI system can be written as:

$$\sum_{k=0}^{N} a_k\, y[n-k] \;=\; \sum_{k=0}^{M} b_k\, x[n-k]. \tag{1.11.1}$$

Two common ways to get the frequency response $H(e^{j\omega})$:

1. Eigenfunction property: complex exponentials are eigenfunctions of LTI systems:

$$e^{j\omega n} \xrightarrow{\text{LTI}} H(e^{j\omega})\, e^{j\omega n}. \tag{1.11.2}$$

2. Convolution / frequency-domain multiplication: if $y[n] = x[n] * h[n]$, then

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}), \qquad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}. \tag{1.11.3}$$

Using the DTFT shift property, \[x[n-k] \Longleftrightarrow e^{-j\omega k}X(e^{j\omega}),\qquad y[n-k] \Longleftrightarrow e^{-j\omega k}Y(e^{j\omega}),$$ the LCCDE becomes:

$$\left(\sum_{k=0}^{N} a_k e^{-j\omega k}\right) Y(e^{j\omega}) \;=\; \left(\sum_{k=0}^{M} b_k e^{-j\omega k}\right) X(e^{j\omega}). \tag{1.11.4}$$

**Frequency response from the LCCDE:**

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}. \tag{1.11.5}$$

The factor $e^{-j\omega k}$ comes from the DTFT of the shifted sequences $x[n-k]$ and $y[n-k]$.

---

### Example: First-Order System

**Difference equation:**

\[y[n]-a\,y[n-1]=x[n], \qquad |a|<1.$$

Take DTFT of both sides:

$$Y(e^{j\omega}) - ae^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) \tag{1.11.6}$$

$$(1 - ae^{-j\omega})Y(e^{j\omega}) = X(e^{j\omega}). \tag{1.11.7}$$

**Frequency response:**

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - ae^{-j\omega}}. \tag{1.11.8}$$

**Impulse response (causal):**

$$h[n] = a^n u[n]. \tag{1.11.9}$$

### ✔ Example: Second-Order Sytem $\Longrightarrow$ Factor $\Longrightarrow$ Partial Fractions $\Longrightarrow$ $h[n]$

**Consider the difference equation:**

\[y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n].$$

**Apply the DTFT to both sides of the difference equation:**

$$\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}\right) Y(e^{j\omega}) = 2X(e^{j\omega}). \tag{1.11.10}$$

**Find the frequency response:**

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}. \tag{1.11.11}$$

**Write the frequency response using partial fraction:**

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}. \tag{1.11.12}$$

**Apply the inverse DTFT to the frequency response to find impulse response (causal):**

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \tag{1.11.13}$$

### ✔ Example: Multiply by $X(e^{j\omega})$ and Invert

**Consider the input:** $x[n] = \left(\frac{1}{4}\right)^n u[n]$ with an DTFT

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\omega}}. \tag{1.11.14}$$

If the **frequency response** is $H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$, then the **output spectrum** is

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{2}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)^2}. \tag{1.11.15}$$

Rewrite the **output spectrum** using **partial fractions**

$$Y(e^{j\omega}) = \frac{8}{1 - \frac{1}{2}e^{-j\omega}} - \frac{4}{1 - \frac{1}{4}e^{-j\omega}} - \frac{2}{\left(1 - \frac{1}{4}e^{-j\omega}\right)^2}. \tag{1.11.16}$$

Remembering the inverse DTFT pair :

\[\frac{1}{(1-ae^{-j\omega})^2} \Longleftrightarrow (n+1)a^n u[n].$$

Write the **DT output signal** by applying the inverse DTFT**:**

$$y[n] = 8\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n]. \tag{1.11.17}$$

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# 1.12: FIR Filter Design

## Learning Objectives: FIR Filter Design Basics

1. Describe the process for designing causal LTI filters with finite-length impulse responses (FIR filters).
2. Design the impulse response for a causal FIR lowpass filter with a given cutoff frequency.
3. Implement an FIR lowpass filter in MATLAB.
4. Understand which frequency-response features relate to time-domain behavior and how they are linked.

## YouTube Videos

Frequency Response Magnitude and Phase for Filters



Finite Impulse Response (FIR) Filtering Basics



Rectangular Windowed FIR Filters in MATLAB



Ideal LPF Impulse Response



## Summary

### Understanding the Fourier Transform as a Filter Process

In general, a DTFT is complex:

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| \, e^{j\angle X(e^{j\omega})}. \tag{1.12.1}$$

- $|X(e^{j\omega})|$: how much of each frequency is present (relative magnitudes).
- $\angle X(e^{j\omega})$: relative phases of those frequency components.

For an LTI system,

$$y[n] = x[n] * h[n] \quad \Longleftrightarrow \quad Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}), \tag{1.12.2}$$

where $H(e^{j\omega})$ is the **frequency response**.

**Magnitude of the Output Spectrum**:

$$\left|Y(e^{j\omega})\right| = \left|X(e^{j\omega})\right| \, \left|H(e^{j\omega})\right|. \tag{1.12.3}$$

**Phase of the Output Spectrum**:

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega}). \tag{1.12.4}$$

### Linear phase, group delay, and “no distortion” idea

A linear-phase frequency response has phase that is a linear function of $\omega$:

$$\angle H(e^{j\omega}) = a\omega + b. \tag{1.12.5}$$

**Special case of a pure delay**

Consider

$$H(e^{j\omega}) = e^{-j\omega n_0} \quad \Rightarrow \quad |H(e^{j\omega})| = 1 \text{ (no attenuation)}. \tag{1.12.6}$$

The corresponding impulse response is o

$$h[n] = \delta[n - n_0] \quad \Rightarrow \quad y[n] = x[n - n_0]. \tag{1.12.7}$$

**Definition: Group delay**

$$\tau(\omega) = -\frac{d}{d\omega}\angle H(e^{j\omega}). \tag{1.12.8}$$

For linear phase $\angle H(e^{j\omega}) = -\omega n_0$, we get $\tau(\omega) = n_0$ (constant delay).

### Ideal discrete-time lowpass filter

Ideal lowpass frequency response (gain 1, cutoff $\omega_c$):

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases} \tag{1.12.9}$$

Its impulse response is obtained via the inverse DTFT:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega. \tag{1.12.10}$$

Consequently, the ideal impulse response for the ideal lowpass filter is

$$h[0] = \frac{\omega_c}{\pi}, \qquad h[n] = \frac{\sin(\omega_c n)}{\pi n}, \quad n \neq 0. \tag{1.12.11}$$

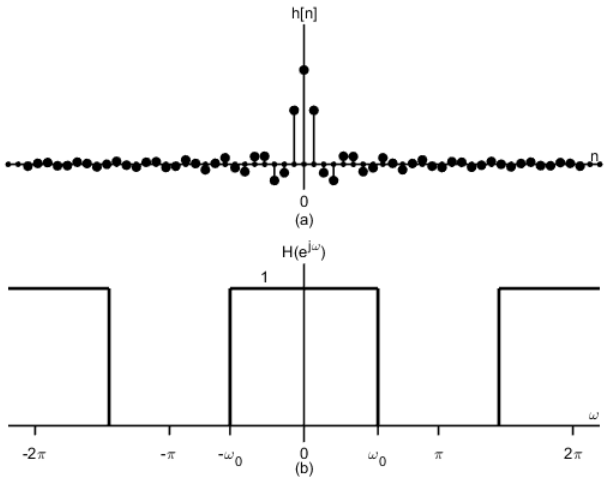



***Figure 1.12.1:*** *(a) Discrete-time impulse response of ideal lowpass filter, and (b) its corresponding frequency response.*

### Filter specifications: passband, stopband, transition band, and ripples

Practical lowpass designs are specified with:

- **Passband edge:** $\omega_p$
- **Stopband edge:** $\omega_s$
- **Transition width:** $\Delta\omega = \omega_s - \omega_p$
- **Passband ripple:** $\delta_1$ (how far $|H|$ deviates around 1)
- **Stopband ripple:** $\delta_2$ (how large the sidelobes are in the stopband)

Passband typically looks like:

$$1-\delta_1 \le \left|H(e^{j\omega})\right| \le 1+\delta_1, \qquad 0 \le \omega \le \omega_p, \tag{1.12.12}$$

and stopband like:

$$\left|H(e^{j\omega})\right| \le \delta_2, \qquad \omega_s \le \omega \le \pi. \tag{1.12.13}$$

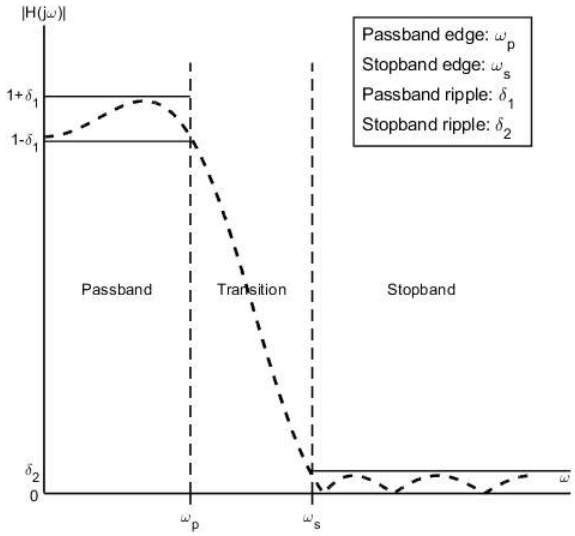



**Figure 1.12.2:** Illustration of lowpass filter specifications.

### FIR filter design procedure

1. **Design the ideal frequency response** $H_d(e^{j\omega})$ (e.g., ideal lowpass filter).
2. **Compute the ideal impulse response** $h_d[n]$ using the inverse DTFT. The impulse response is generally infinite-length and noncausal.
3. **Shift** to make it causal (introduce a delay so the sequence starts at $n = 0$).
4. **Truncate /window** to obtain a finite-length FIR filter.

> **Important note:** Flip the impulse response is *not* part of the standard FIR design steps. The key steps are: ideal → inverse DTFT → shift to causal → truncate/window.

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# 1.13: Hamming Window FIR Filters

## Learning Objectives: Hamming-Window FIR Lowpass Filters

1. Understand the tradeoff between stopband ripples and transition bandwidth when windowing an FIR lowpass filter.
2. Compare the frequency responses of rectangular and Hamming window FIR filters (same length).
3. Compare the performance of rectangular vs. Hamming window FIR filters on a data signal.
4. Learn how FIR filters can be used to create hybrid images.

## YouTube Videos

FIR Filter Design with Hamming Windows



## Summary

Reminder: causal FIR lowpass design steps

**1. Define the ideal lowpass impulse response** (infinite-length, centered at $n = 0$):

$$h_{\text{id}}[n] = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases} \tag{1.13.1}$$

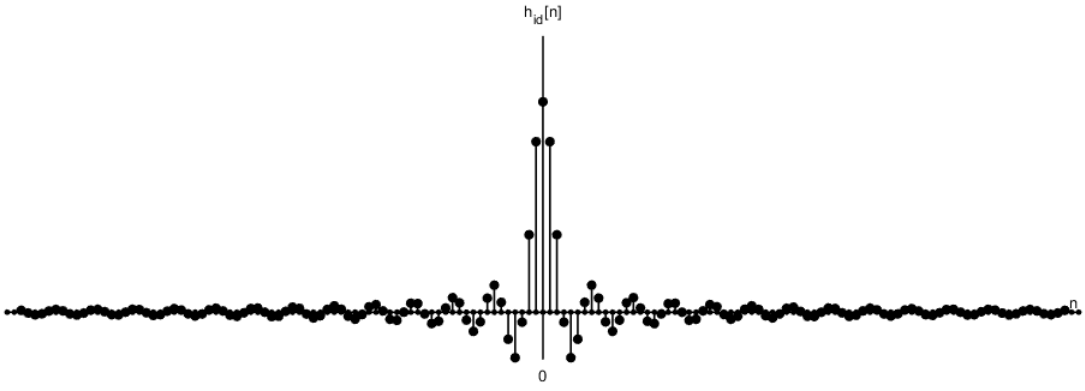



**Figure 1.13.1:** Discrete-time impulse response (i.e., sinc function) of an ideal lowpass filter.

**2. Truncate** to make it finite length (sharp truncation):

$$h[n] = \begin{cases} h_{\text{id}}[n], & -\frac{M}{2} \le n \le \frac{M}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1.13.2}$$

Sharp truncation produces ripples (Gibbs-type oscillations) in the frequency response.

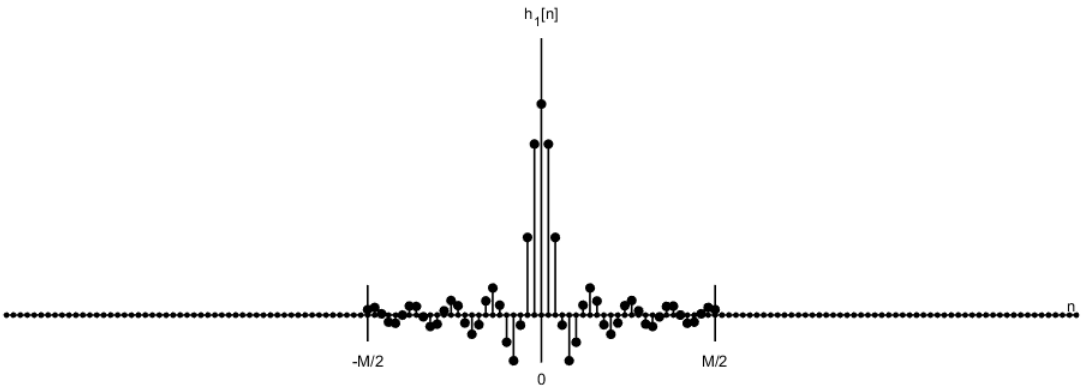



**Figure 1.13.2:** Truncated sinc signal at $n = M/2$.

**3. Shift** to make the FIR causal:

$$h_{\text{causal}}[n] = h[n - M/2], \qquad 0 \le n \le M. \tag{1.13.3}$$

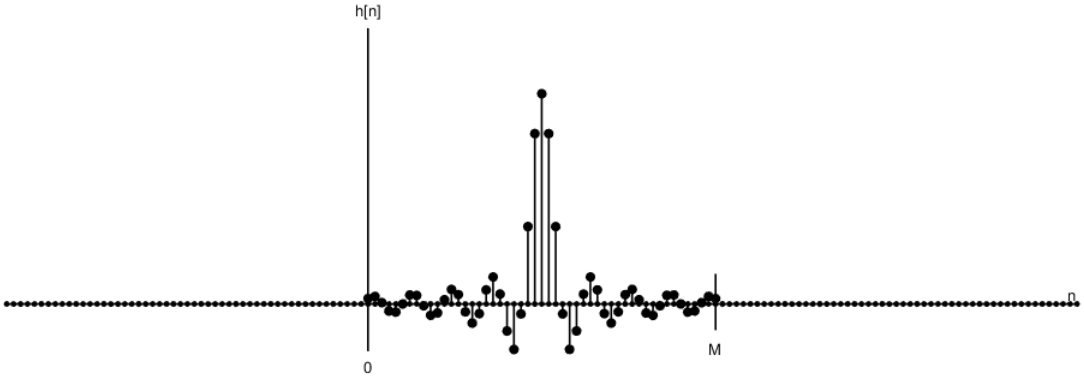



**Figure 1.13.3:** Shift the truncated sinc signal by $n = M/2$ to convert the impulse response to causal.

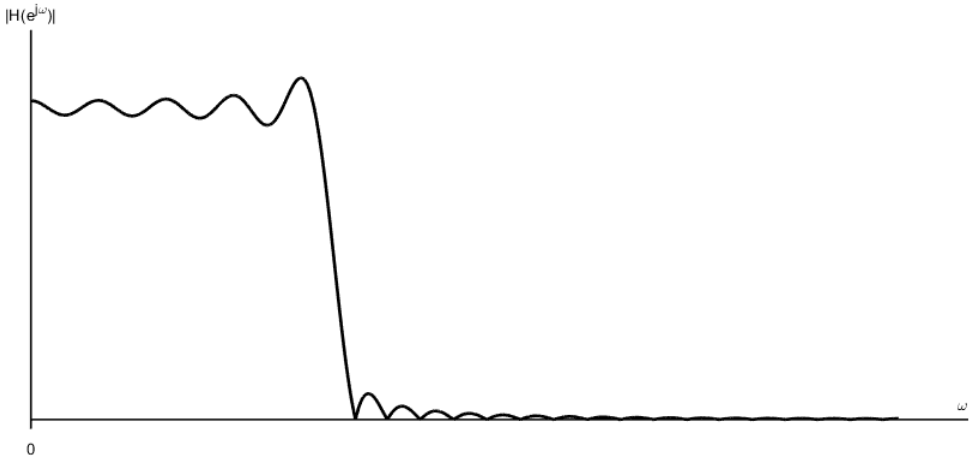


**Figure 1.13.4:** Frequency response of the causal FIR impulse response.

> **Key point:** Increasing the filter length $M$ makes the transition band narrower (sharper cutoff), but *does not substantially reduce the maximum ripple peak* when you use a rectangular (sharp) truncation.

### Windowed FIR idea

Instead of a sharp truncation, we smooth the edges by multiplying by a window:

**New Step 2 (windowing):**

$$h[n] = h_{\text{id}}[n]\, w[n]. \tag{1.13.4}$$

For a Hamming window (centered form):

$$w[n] = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{\pi n}{M/2}\right), & |n| \le \dfrac{M}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1.13.5}$$

**MATLAB Command:** `fir1(M, omega0)` returns a causal FIR lowpass of length $M+1$ using a Hamming window by default (with normalized cutoff `omega0 = wc/pi`).

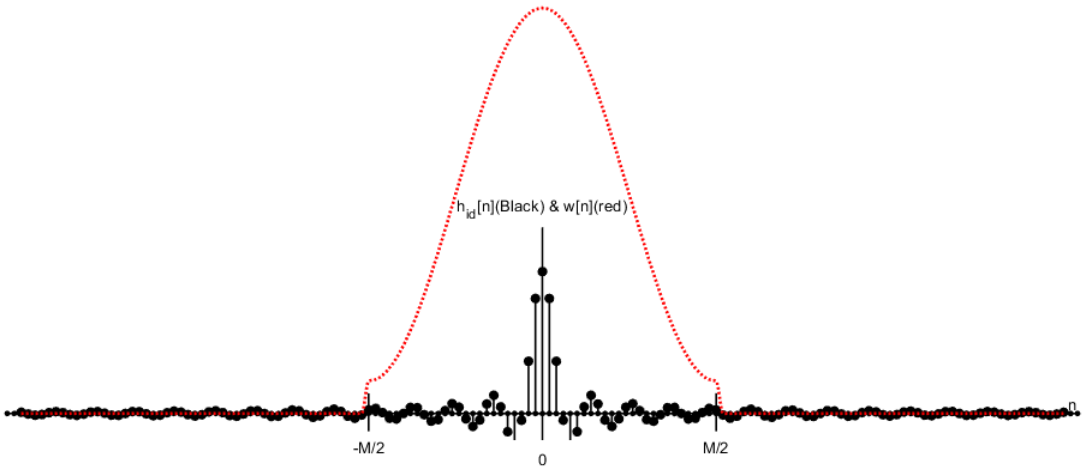



**Figure 1.13.5:** Illustration of the sinc signal (black sequence) overlapping the Hamming window (red signal).

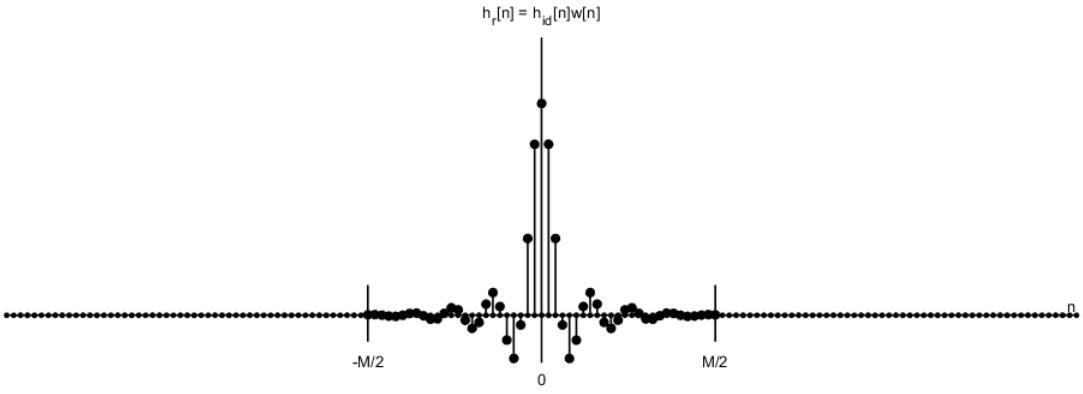



**Figure 1.13.6:** Truncated sinc signal at $n = M/2$ modified by the Hamming window.

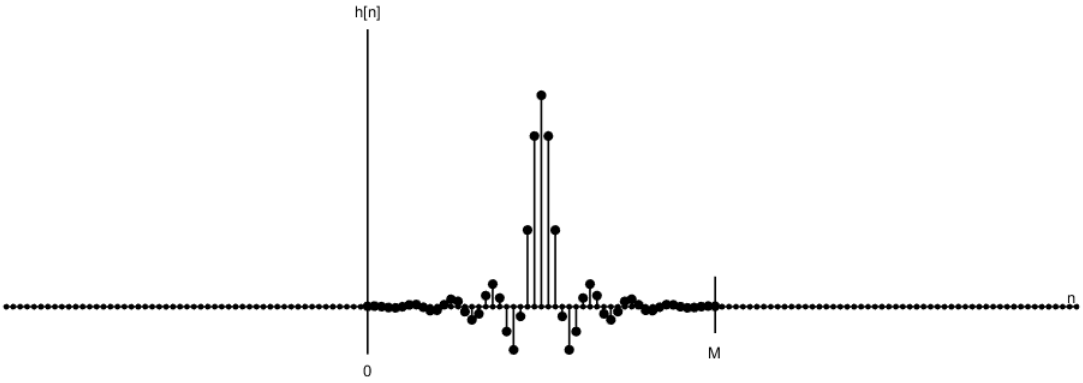



***Figure 1.13.7:*** Shift the truncated Hamming-windown sinc signal by $n = M/2$ to convert the impulse response to causal.

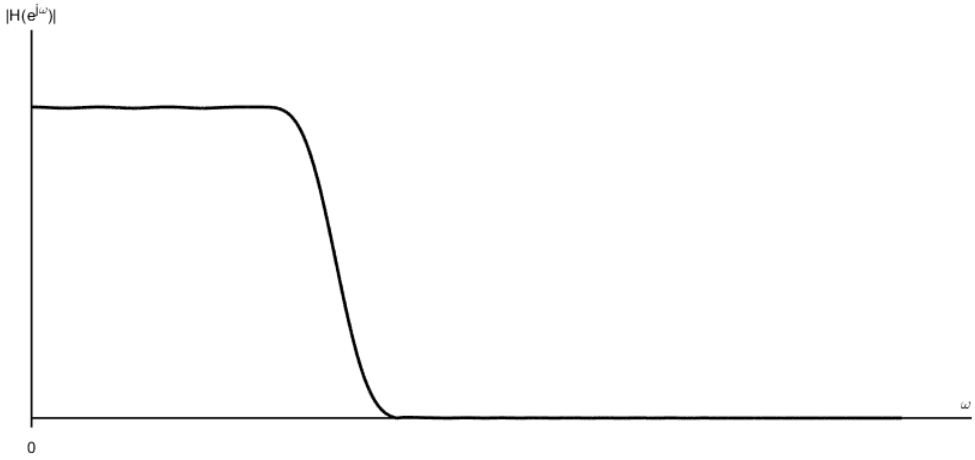



**Figure 1.13.8:** Frequency response of a causal FIR using a Hamming window.

**Tradeoff (Hamming window):**
Stopband ripples (sidelobes) decrease substantially, whereas the transition band widens.

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# 1.14: First- and Second-order DT Systems

## Learning Objectives: 1st- and 2nd-Order Discrete-Time Systems

1. Connect the time and frequency behaviors of first-order systems from graphs.
2. Understand when second-order systems are overdamped vs. underdamped (oscillatory).
3. Connect difference equations and plots of frequency-response magnitude for second-order systems.

## YouTube Videos

First-Order Discrete-Time Systems



Example of First-Order Discrete-Time Systems



Second-Order Discrete-Time Systems



Example of Frequency Responses for Second-Order Underdamped Systems



## Summary

### First-Order DT Systems

Consider a causal first-order LTI system described by the difference equation

$$y[n] - a\,y[n-1] = x[n]. \tag{1.14.1}$$

#### Frequency response

Take the DTFT to both sides:

$$Y(e^{j\omega}) - ae^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) \quad \Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-ae^{-j\omega}}. \tag{1.14.2}$$

Magnitude:

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-ae^{-j\omega})(1-ae^{+j\omega})}} = \frac{1}{\sqrt{1+a^2-2a\cos\omega}}. \tag{1.14.3}$$

**Lowpass vs. highpass behavior (rule of thumb):**

- If **a > 0** (and |a|<1), the system tends to attenuate high frequencies more (non-ideal **lowpass**).
- If **a < 0** (and |a|<1), the system tends to attenuate low frequencies more (non-ideal **highpass**).

#### Impulse response

The impulse response of the causal first-order system is

$$h[n] = a^n u[n]. \tag{1.14.4}$$

**Stability:** the system is BIBO stable if $\sum_{n=0}^{\infty} |a^n| < \infty$, i.e.

$$|a| < 1. \tag{1.14.5}$$

**Time-domain intuition:**

- If $|a|$ is small, $h[n]$ decays fast.
- If $a$ is close to 1, $h[n]$ decays slowly (long memory), and the filter is "sharper" in frequency.
- If $a < 0$, $h[n]$ alternates sign (samples switch +/−).

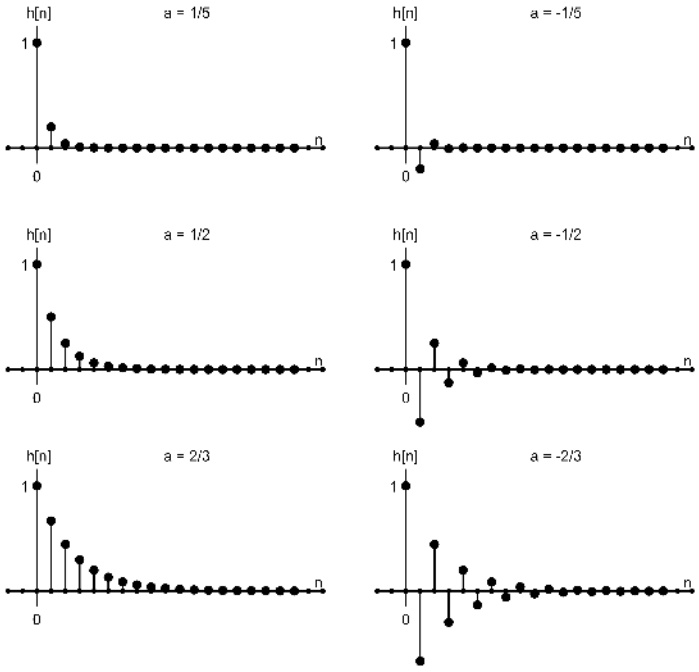



**Figure 1.14.1:** Examples of First-Order DT Systems, illustrating the effect on the sign of the coefficient

**Frequency-domain intuition for** \(|a|\>0)**:**

- If $|a|$ is small, $H(e^{j\omega}$ has a peak with low magnitude and a wide full-width at half-maximum (FWHM) at $\omega = 0$
- If $|a|$ is high (i.e., close to 1), $H(e^{j\omega}$ has a peak with high magnitude and a narrow full-width at half-maximum (FWHM) at $\omega = 0$


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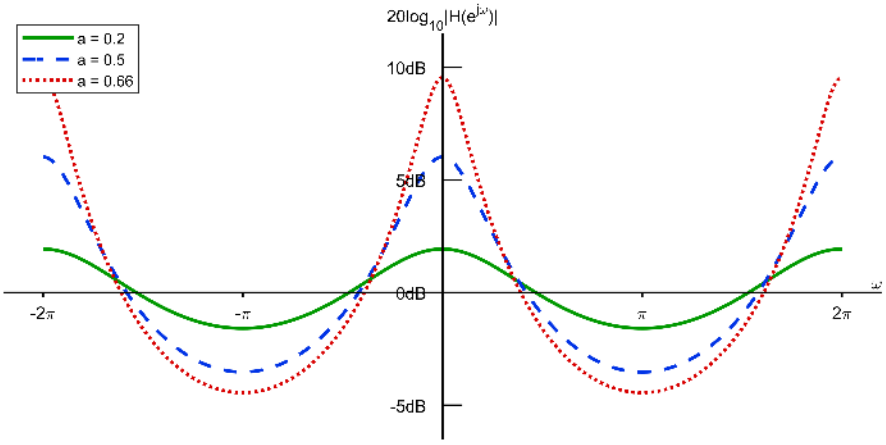



**Figure 1.14.2:** Magnitude of the frequency response in a First-Order DT Systems in terms of the positive $a$ values.

**Frequency-domain intuition for** \(|a|\<0)**:**

- If $|a|$ is small, $H(e^{j\omega}$ has a peak with low magnitude and a wide full-width at half-maximum (FWHM) at $\omega = \pi$
- If $|a|$ is high (i.e., close to 1), $H(e^{j\omega}$ has a peak with high magnitude and a narrow full-width at half-maximum (FWHM) at $\omega = \pi$

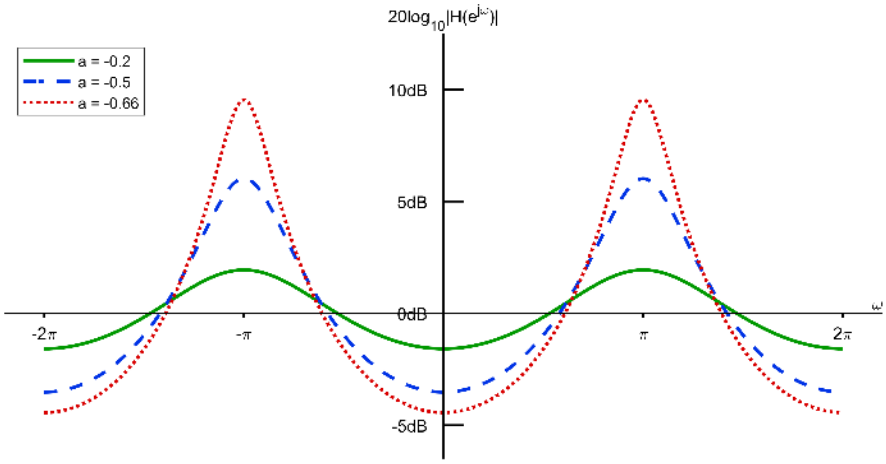



**Figure 1.14.3:** Magnitude of the frequency response in a First-Order DT Systems in terms of the negative $a$ values.

## Second-Order DT Systems (i.e., the output signal is delayed twice)

A general causal second-order LTI system can be written as

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n]. \tag{1.14.6}$$

Applying the DTFT to both sides of the difference equation, one can estimate the frequency response

$$Y(e^{j\omega})\left(1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}\right) = X(e^{j\omega}) \quad \Rightarrow \quad H(e^{j\omega}) = \frac{1}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}. \tag{1.14.7}$$

**Overdamped /underdamped/critically damped**

Consider the characteristic polynomial (in $z$):

$$z^2 + a_1 z + a_2 = 0. \tag{1.14.8}$$

being discriminant equal to $\Delta = a_1^2 - 4a_2$.

- **Overdamped:** $\Delta > 0$ so the two roots of the quadratic equation are two distinct real values → impulse response of the 2nd-order system is a sum of two 1st-order systems.
- **Critically damped:** $\Delta = 0$ so the two roots of the quadratic equation are equal and real values
- **Underdamped:** $\Delta < 0$ so the roots of the quadratic equation are two complex conjugate roots, i.e., $r^{\pm j\theta}$ → oscillation of the impulse response with exponentially decaying (or growing) envelope.

A common "underdamped form" is

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n], \tag{1.14.9}$$

where the poles are $re^{\pm j\theta}$.

For the underdamped case, the impulse response has the oscillatory form

$$h[n] = r^n \frac{\sin\big((n+1)\theta\big)}{\sin\theta}\, u[n]. \tag{1.14.10}$$

**Interpretation:**

- $r$ controls the magnitude of the frequency response $|H(e^{j\omega})|$
  - $r$ closer to 1 ⇒ sharpness of peaks in $|H(e^{j\omega})|$
- $\theta$ controls the location of peaks in $|H(e^{j\omega})|$.

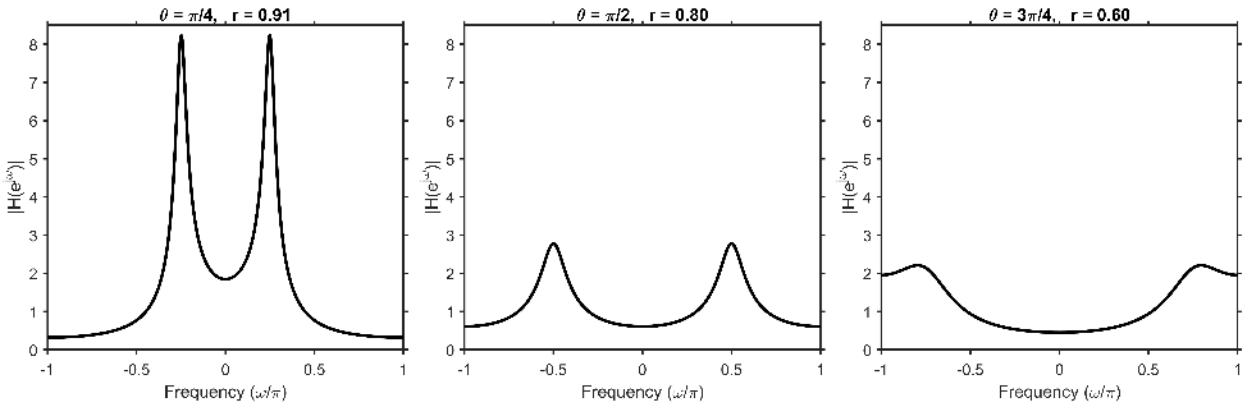



**Figure 1.14.5:** Frequency response magnitude of Second-Order DT Systems with different $a$ and $\theta$ values.

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# 1.15: Sampling and Reconstruction

## Learning Objectives: Sampling and Reconstruction of Continuous-Time (CT) Signals

1. Understand the Nyquist Theorem for preserving all information when sampling bandlimited continuous-time signals.
2. Sketch the steps for sampling in the frequency domain from a CT spectrum to the DT spectrum of the sampled signal.
3. Describe how a lowpass filter allows us to reconstruct a bandlimited CT signal from DT samples.

## YouTube Videos

Derivation of Frequency Domain Interpretation of Sampling



Sampling Theorem for Bandlimited Signals



Frequency Domain and Sampling



Reconstruction of Continuous-Time Signal from Discrete-Time Samples



## Summary

### Frequency-Domain Interpretation of Sampling

- **Sampling in time**

Sampling converts a continuous-time signal $x_c(t)$ into a discrete-time signal $x[n]$ by taking values every $T$ seconds:

$$x[n] = x_c(nT), \tag{1.15.1}$$

where $T$ is the **sampling period**.

Sampling frequency:

$$f_s = \frac{1}{T} \quad (\text{Hz}), \qquad \omega_s = \frac{2\pi}{T} \quad (\text{rad/s}). \tag{1.15.2}$$

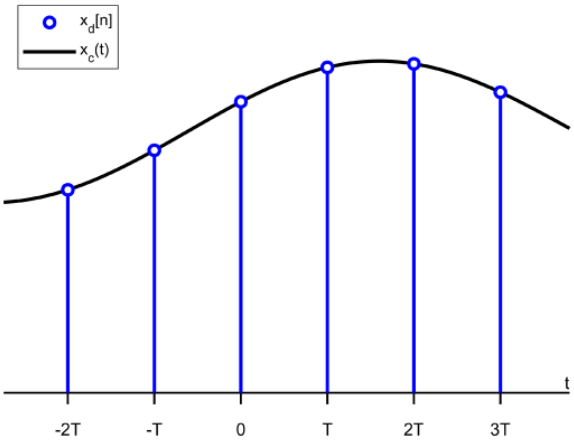



**Figure 1.15.1:** Example of sampling in time.

- **Impulse-train model of sampling**

Step 1 of sampling (in continuous time) is multiplying $x_c(t)$ by a periodic impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad x_p(t) = x_c(t)\, p(t). \tag{1.15.3}$$

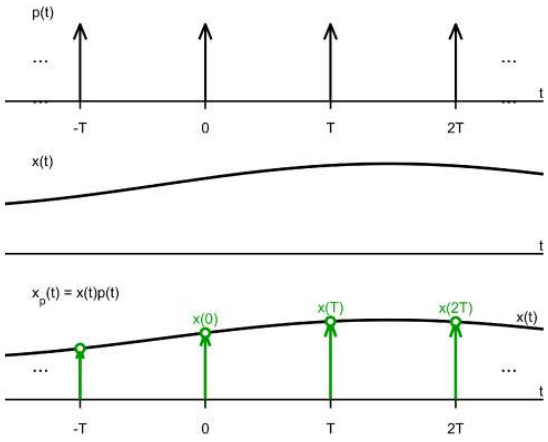



**Figure 1.15.2:** Example of the impulse-train model for sampling.

- **Fourier transform of the impulse train**

The Fourier transform of the impulse train is another impulse train in frequency:

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \qquad \omega_s = \frac{2\pi}{T}. \tag{1.15.4}$$


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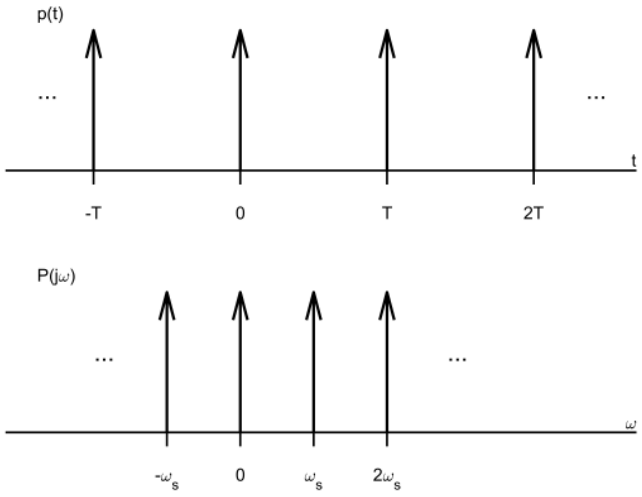



**Figure 1.15.3:** The Fourier transform of an impulse train with period T is another impulse train where the unit impulses are separated by $\omega_s = \frac{2\pi}{T}$

- **Spectrum of the sampled (impulse-sampled) signal**

Since multiplication in time corresponds to convolution in frequency,

$$X_p(j\omega) = \frac{1}{2\pi} X_c(j\omega) * P(j\omega). \tag{1.15.5}$$

Convolution with $\delta(\omega - k\omega_s)$ creates shifted copies of $X_c(j\omega)$, so the result is:

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\big(j(\omega - k\omega_s)\big). \tag{1.15.6}$$

**Interpretation:**
Sampling replicates the CT spectrum every $\omega_s$ rad/s and scales the amplitude by $1/T$.

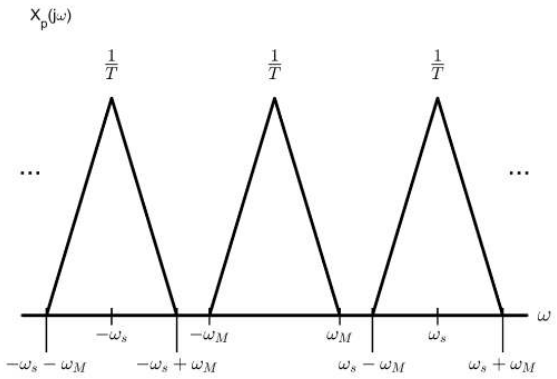



**Figure 1.15.4:** Spectrum of a sampled signal.

## Nyquist Theorem and Aliasing

Assume $x_c(t)$ is bandlimited:

$$X_c(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M. \tag{1.15.7}$$

We can recover $X_c(j\omega)$ from $X_p(j\omega)$ as long as the replicated copies do not overlap:

$$\omega_s > 2\omega_M. \tag{1.15.8}$$

In Hz units:

$$f_s > 2f_{\max}. \tag{1.15.9}$$

**Aliasing:** If $\omega_s < 2\omega_M$ where $\omega_M$ is the cutoff frequency of the CT signal, the replicated spectra overlap, and high frequencies "appear" as lower frequencies in the sampled signal.

If the Nyquist sampling theorem is not satisfied, i.e., $\omega_s < 2\omega_M$, aliasing will occur.

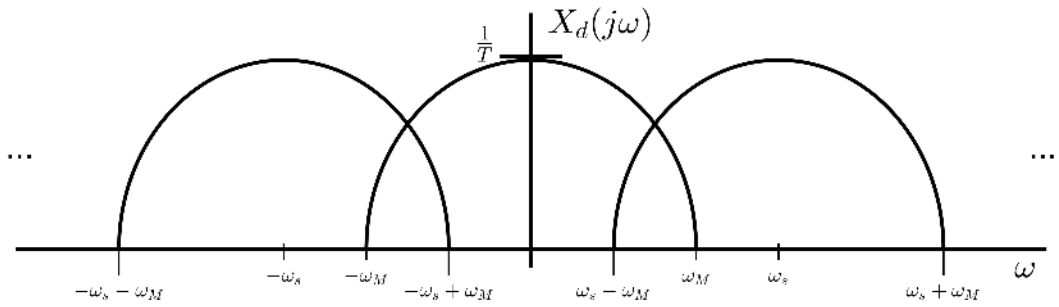



***Figure 1.15.5:*** *Example of aliasing when sampling a continuous-time signal.*

From CT frequency $\omega$ to DT frequency $\Omega$

Define the discrete-time radian frequency:

$$\Omega = \omega T. \tag{1.15.10}$$

**Three-step recipe to go from CT spectrum to DT spectrum:**

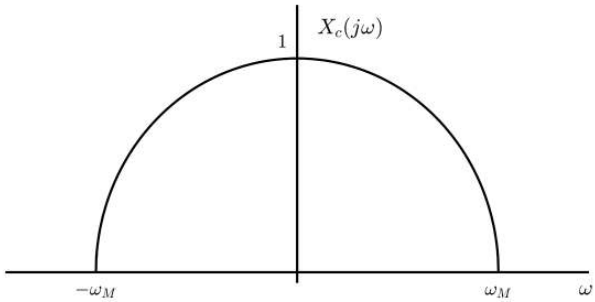



***Figure 1.15.6:*** *Example of the Fourier transform of a continuous signal.*

1. Make copies of $X_c(j\omega)$ every \(\omega_s=\frac{2\pi}{T}\) along the $\omega$-axis.

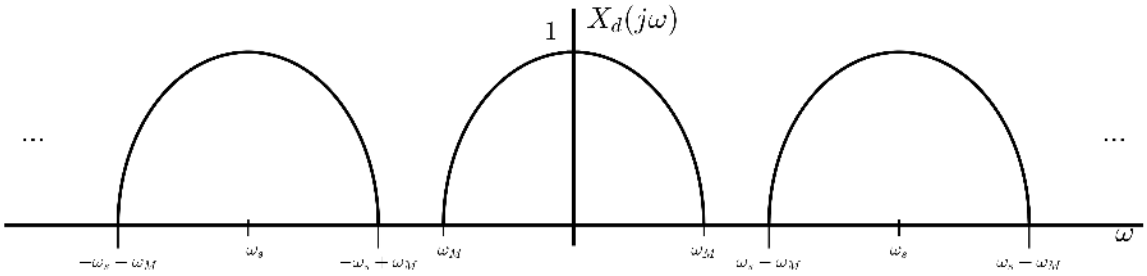



***Figure 1.15.7:*** *Illustration of how the spectrum of the CT signal is replicated through the sampling process, creating copies every \(\omega_s=\frac{2\pi}{T}\).*

2. Scale the height of each copy by $1/T$.

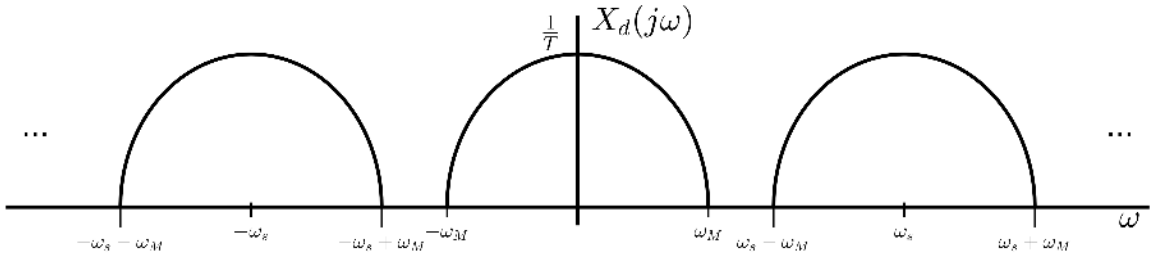



***Figure 1.15.8:*** *The magnitude of each replica is scaled by \(\frac{1}{T}\).*

3. Convert the frequency axis using $\Omega = \omega T$ ,where $\Omega$ is the discrete frequency. In discrete frequency, the spectrum is repeated every $2\pi$ in $\Omega$.)

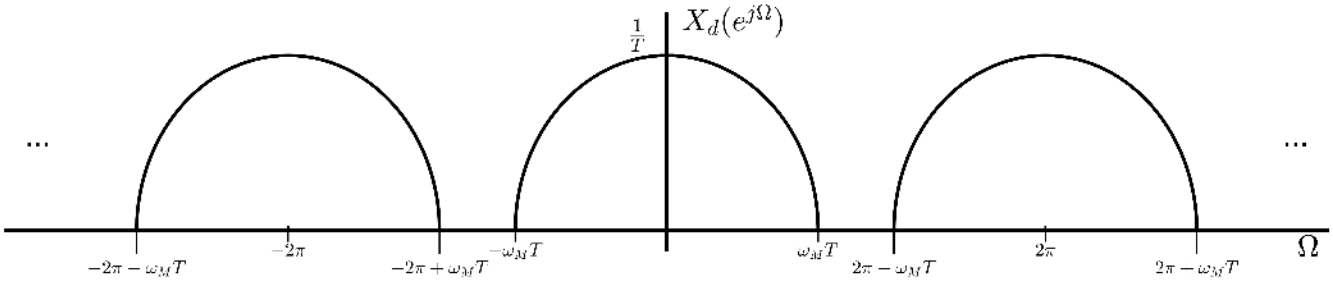



***Figure 1.15.9:*** *Conversion of the spectrum of the sampled signal in discrete frequency.*

## Reconstruction of a CT Signal from DT Samples

If $\omega_s > 2\omega_M$ (no aliasing), we can reconstruct $x_c(t)$ by passing $x_p(t)$ through an ideal CT lowpass filter that keeps only the center replica and rescales it:

$$X_r(j\omega) = H_r(j\omega)\, X_p(j\omega) = X_c(j\omega). \tag{1.15.11}$$

The magnitude of the reconstructed lowpass filter is:

$$H_r(j\omega) = \begin{cases} T, & |\omega| \le \omega_s/2 = \pi/T, \\ 0, & \text{otherwise.} \end{cases} \tag{1.15.12}$$

The corresponding impulse response is a sinc function:

$$h_r(t) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T e^{j\omega t}\, d\omega = \frac{\sin(\pi t/T)}{\pi t/T} = \operatorname{sinc}\left(\frac{t}{T}\right). \tag{1.15.13}$$

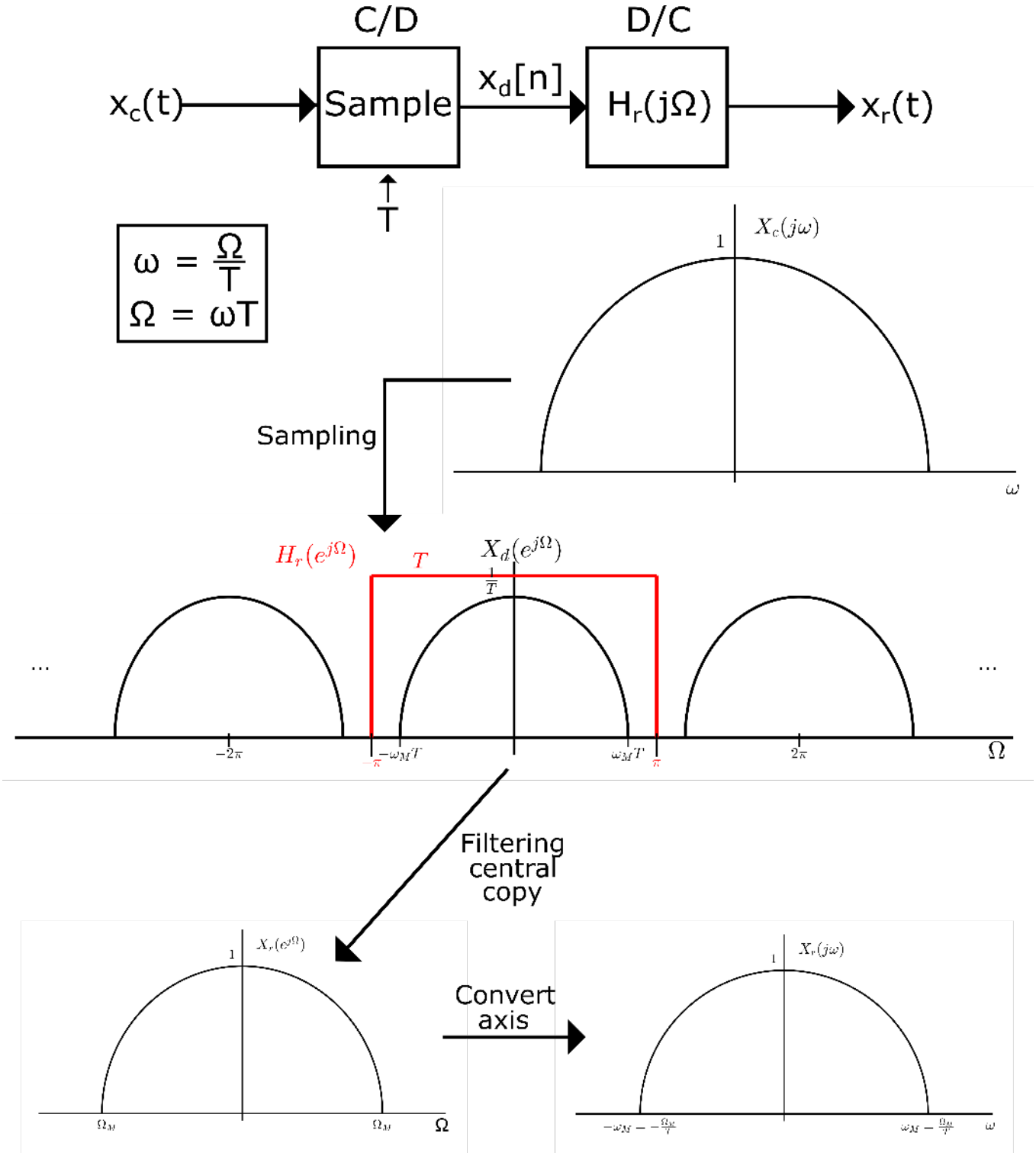



***Figure 1.15.10:*** *Reconstruction of the original continuous-time signal.*

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# 1.16: Discrete-Time Processing of Continuous-Time Signals

## Learning Objectives: Discrete-Time (DT) Processing of Continuous-Time (CT) Signals

1. Understand how to filter continuous-time signals by sampling, filtering in discrete time, and interpolating back to continuous time.
2. Sketch the Fourier transform at each stage of the CT → DT → CT process.
3. Find the equivalent continuous-time frequency response $H_c(j\omega)$ in terms of $H_d(e^{j\Omega})$ and the sampling period $T$.

## YouTube Videos

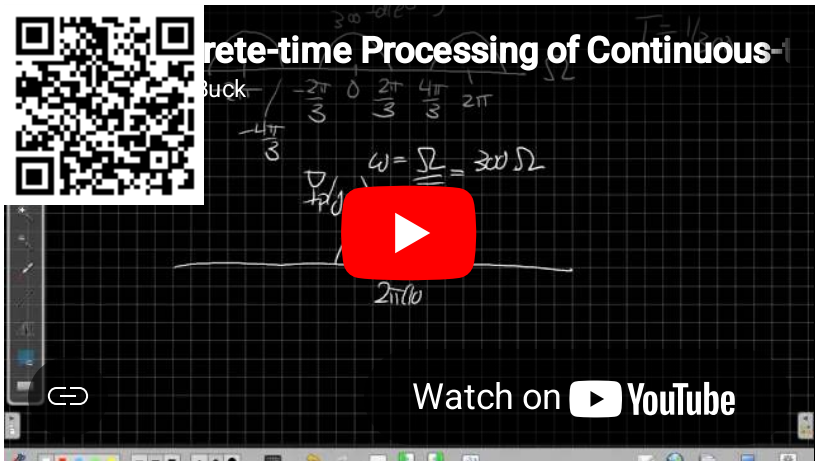
Discrete-time Processing of Continuous-time Signals: Part 1 Sampling



Example: Effect of Sampling on the Fourier Transform of a Bandlimited Signal



Discrete-time Processing of Continuous-time Signals: Part 2, Reconstruction




Discrete-time Processing of Continuous-time Signals: Equivalent Filter



## Summary

System overview (CT → DT → CT)

We process a CT input $x_c(t)$ using a DT filter by:

1. **C/D (sampling):** form DT samples $x_d[n] = x_c(nT)$
2. **DT-LTI system:** apply $H_d(e^{j\Omega})$ to get $y_d[n]$
3. **D/C (reconstruction/interpolation):** convert back to CT output $y_c(t)$

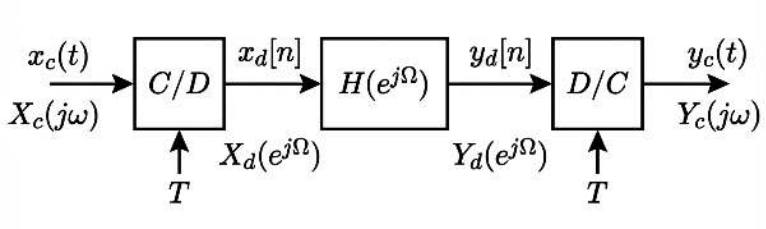



**Figure 1.16.1:** Flow diagram representing the system overview of applying a DT filter to a CT signal.

> Big goal: find an **equivalent CT frequency response** $H_c(j\omega)$ for the overall system, in terms of $H_d(e^{j\Omega})$ and $T$.

Sampling (C/D) in the frequency domain

- **Impulse-train sampling model**

Sampling can be modeled by multiplying the CT signal by an impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad x_p(t) = x_c(t)\, p(t). \tag{1.16.1}$$

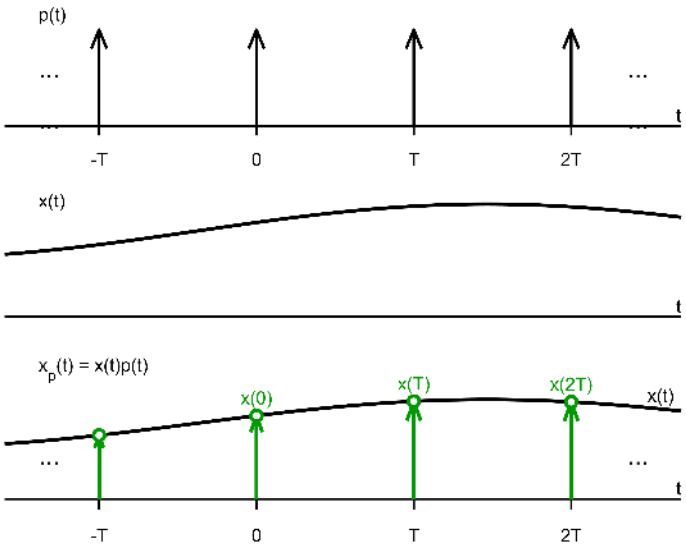



**Figure 1.16.2:** Example of the impulse-train model for sampling

- **Spectrum after sampling (replicas)**

Let $\omega_s = \dfrac{2\pi}{T}$ be the sampling radian frequency. Then the sampled spectrum is

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\big(j(\omega - k\omega_s)\big). \tag{1.16.2}$$

Interpretation: **copies** of $X_c(j\omega)$ repeat every $\omega_s$, and each copy is scaled by $1/T$.

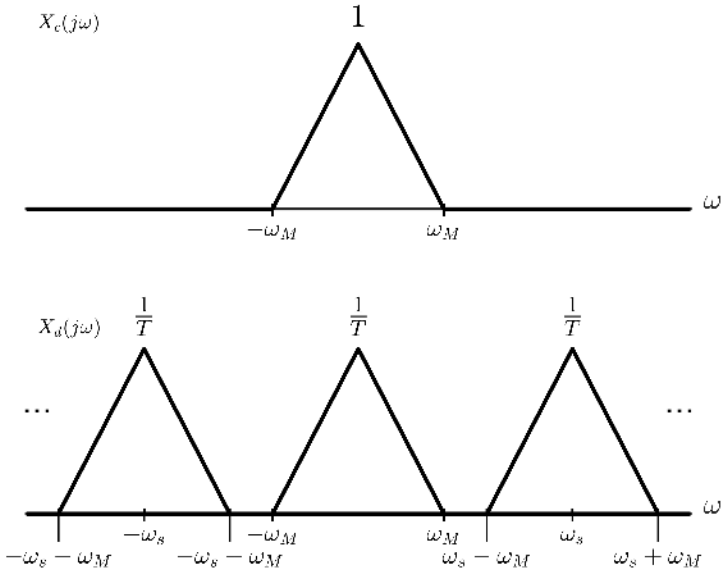



**Figure 1.16.3:** Example of the spectrum after sampling the continuous-time signal.

**No-aliasing condition:**
If $X_c(j\omega) = 0$ for $|\omega| > \omega_M$, then to avoid overlap of replicas:

$$\omega_s \geq 2\omega_M \quad \Longleftrightarrow \quad \frac{2\pi}{T} \geq 2\omega_M. \tag{1.16.3}$$

Relating CT frequency $\omega$ and DT frequency $\Omega$

The key mapping between CT and DT frequency axes is:

$$\Omega = \omega T, \qquad \text{equivalently} \qquad \omega = \frac{\Omega}{T}. \tag{1.16.4}$$

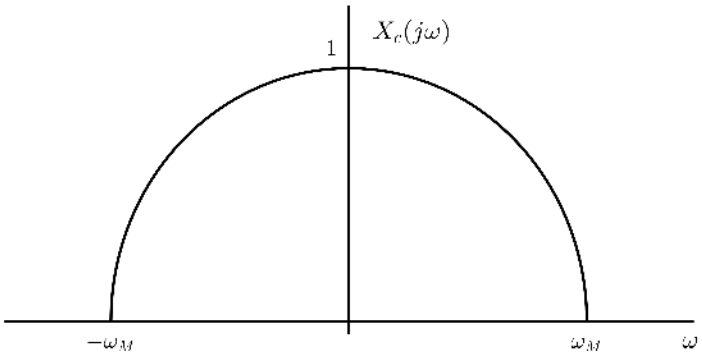



**Figure 1.16.4:** Example of the Fourier Transform of a continuous signal.

"3-step" sketch recipe (CT spectrum → DT spectrum)

1. Copy $X_c(j\omega)$ every $\omega_s = \frac{2\pi}{T}$ along the $\omega$-axis.

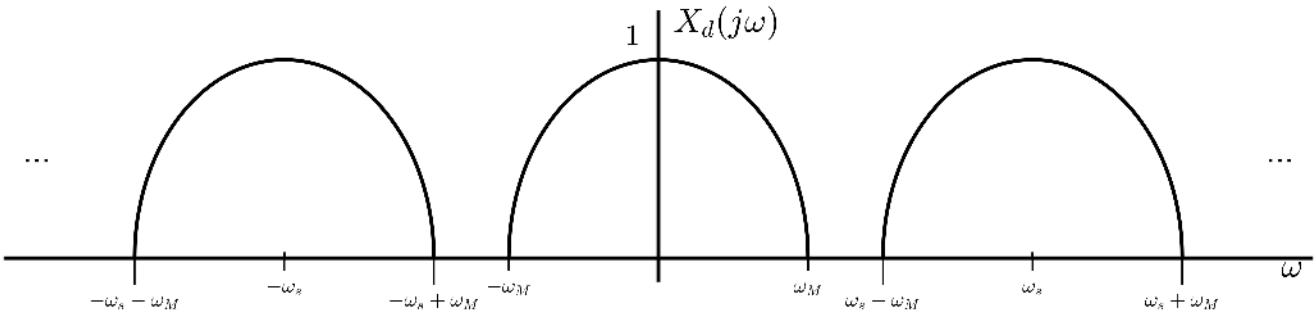



**Figure 1.16.5:** Illustration of how the spectrum of the CT signal is replicated through the sampling process, creating copies every $\frac{2\pi}{T}$.

2. Scale the height of each replica by $1/T$.

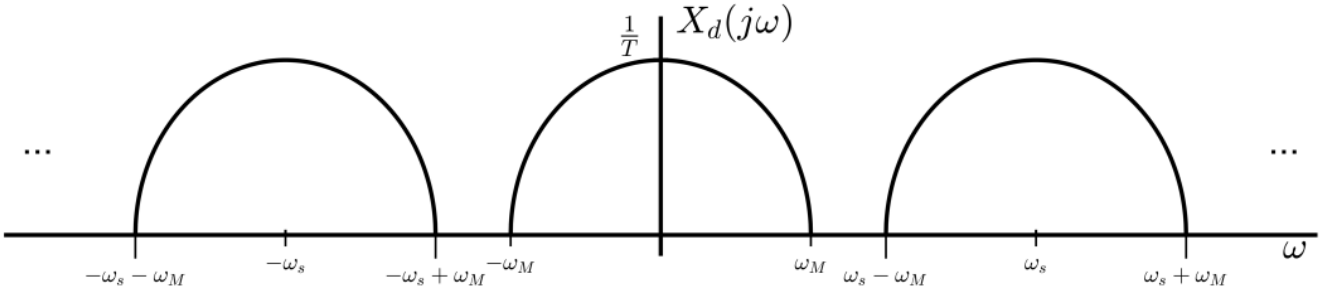



**Figure 1.16.6:** The magnitude of each replica is scaled by \(\frac{1}{T}\).

3. Convert the horizontal axis using $\Omega = \omega T$ (DT spectra repeat every $2\pi$ in $\Omega$).

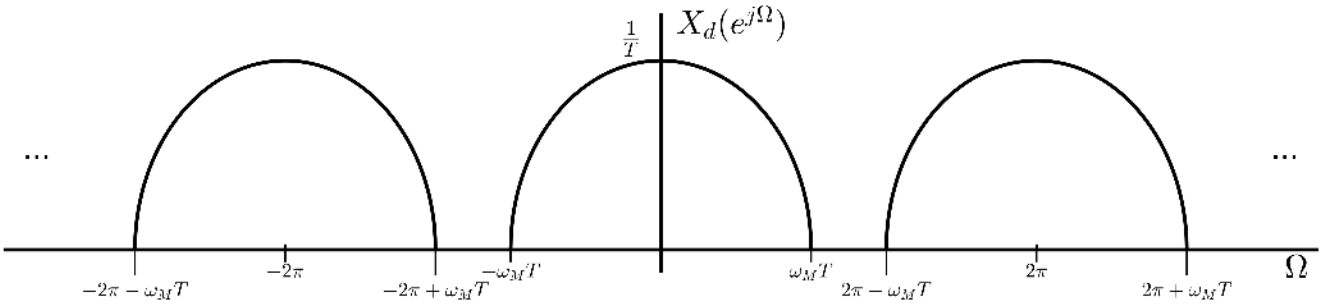



**Figure 1.16.7:** Conversion of the frequency axis from $\omega$ to $\Omega$.

## Applying discrete $H(e^{j\Omega})$ filter to the signal

By converting from continuous to discrete and then back to continuous, we gain the ability to apply discrete filters to continuous signal.

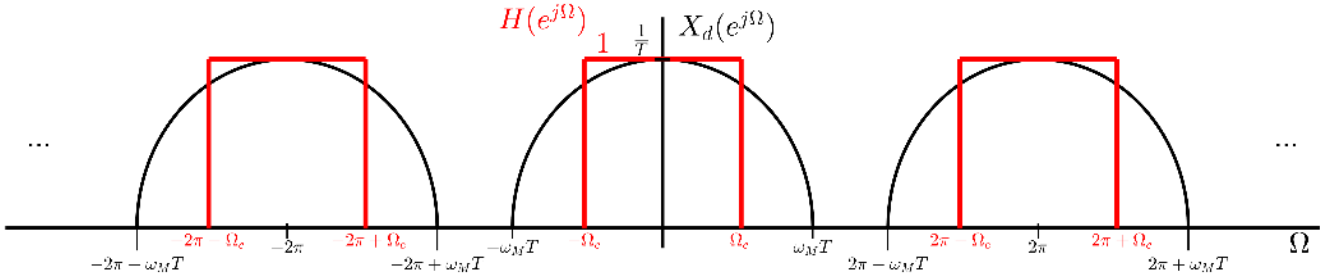



**Figure 1.16.8:** Discrete filter being applied to a signal in the frequency domain.

## Reconstruction/interpolation (D/C)

Conceptually, D/C does two things:

**1. Axis conversion:** map $\Omega$ back to $\omega = \Omega/T$.

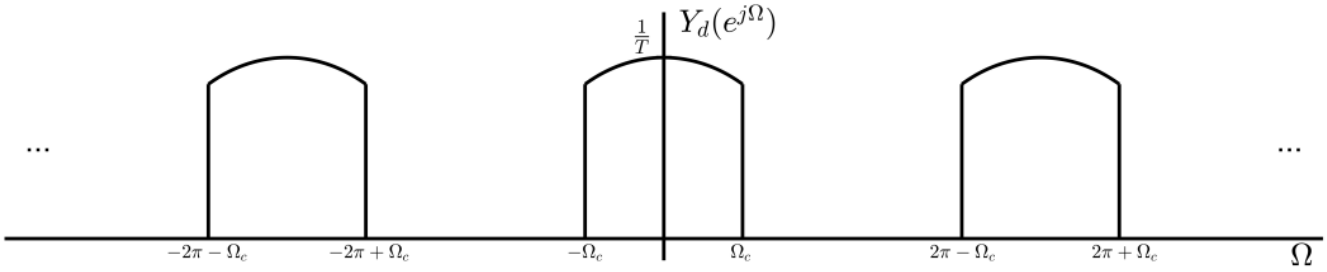
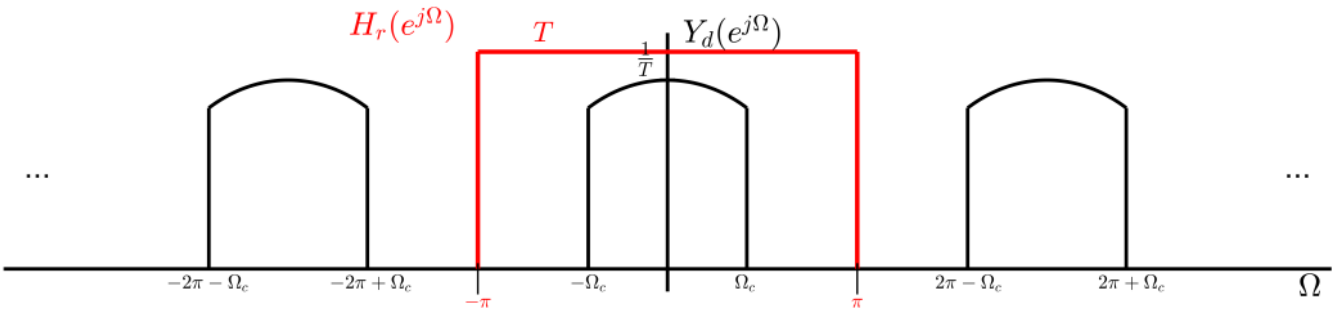



**Figure 1.16.9:** Example of the result of doing axis conversion to convert from DT to CT.

**2. Ideal LPF selection:** keep only the *central replica* in CT frequency.





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**Figure 1.16.10:** Example of applying an Ideal LPF to keep only the central replica in CT frequency.

A standard ideal CT reconstruction filter is:

$$H_r(j\omega) = \begin{cases} T, & |\omega| \le \pi/T, \\ 0, & \text{otherwise.} \end{cases} \tag{1.16.5}$$

The gain $T$ compensates the $1/T$ scaling introduced by impulse sampling (so amplitudes match in the no-aliasing case).

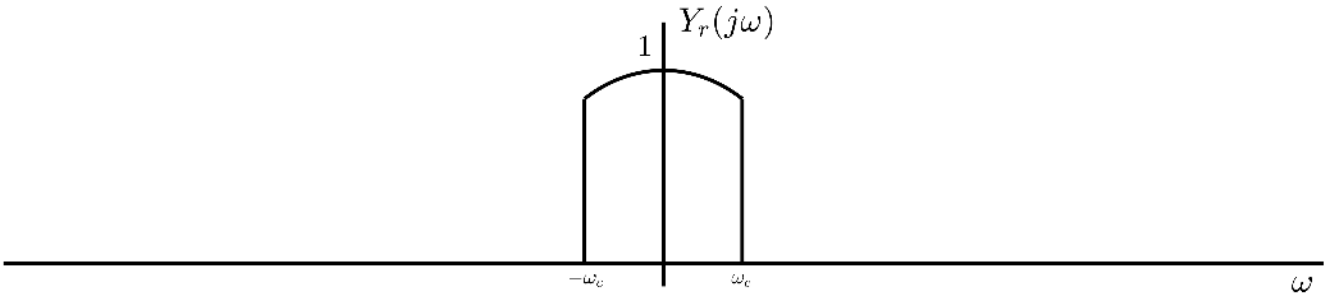



**Figure 1.16.11:** result of successfully converting from DT to CT frequency.

### Equivalent overall CT frequency response

If there is **no aliasing** and the reconstruction stage selects only the central replica, then the overall system behaves like a CT LTI filter:

$$Y_c(j\omega) = H_c(j\omega)\, X_c(j\omega). \tag{1.16.6}$$

Over the reconstructable CT band $|\omega| \le \pi/T$, the equivalent CT response is simply the DT response evaluated at $\Omega = \omega T$:

$$H_c(j\omega) = H_d\left(e^{j\Omega}\right)\Big|_{\Omega=\omega T} = H_d\left(e^{j\omega T}\right), \qquad |\omega| \le \pi/T. \tag{1.16.7}$$

Outside $|\omega| > \pi/T$, the ideal reconstruction LPF sets the output to zero, so effectively $H_c(j\omega) = 0$ there.

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# 1.17: z-Transform

## Learning Objectives: z-transform

1. Understand how the DT Fourier transform and z-transform are related.
2. Compute the z-transform and its Region-of-Convergence (ROC) for signals such as decaying exponentials, finite rectangular pulses, and simple finite-length signals.
3. Determine the poles and zeros using the z-transform of a signal
4. Explain how time-domain properties relate to the ROC properties.

## YouTube Videos

Big Picture for z-Transform



z-Transform Definition



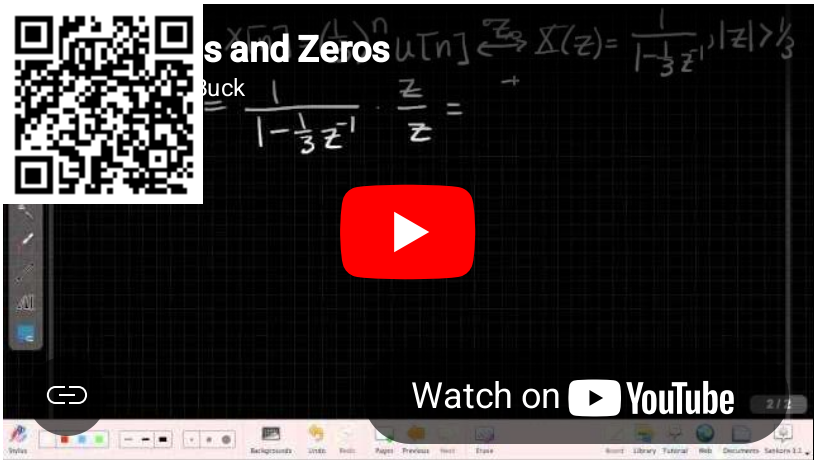
z-Transform Example 1



z-Transform Example 2



Poles and Zeros




Region of Convergence Properties



## Summary

### Why the z-transform?

The z-transform is used when a signal does not have a Fourier transform. Whereas the DTFT (i.e., the Discrete-Time Fourier Transform) exists for signals with finite energy, the z-transform is a generalization of the DTFT, working for almost all DT signals.

**Definition: z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}. \tag{1.17.1}$$

**Definition: System Function**

The system function is defined as the z-transform of the impulse response in an LTI DT system

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}. \tag{1.17.2}$$

**Connection to the DTFT:** if $z = e^{j\omega}$ (i.e., $|z| = 1$), then

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}}. \tag{1.17.3}$$

In words: **the DTFT is the z-transform evaluated on the unit circle**.

Complex-plane reminder: $z = re^{j\theta}$. The unit circle is $|z| = 1$.

**Example 1: Right-Sided Exponential**

**Signal:** $x_1[n] = \left(\frac{1}{3}\right)^n u[n]$

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{3} z^{-1}}, \tag{1.17.4}$$

using the geometric-series result $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ for $|\alpha| < 1$.

**ROC:**

$$\left|\frac{1}{3} z^{-1}\right| < 1 \quad \Longleftrightarrow \quad |z| > \frac{1}{3}. \tag{1.17.5}$$

**Unit-circle evaluation (DTFT):**

$$X_1(e^{j\omega}) = X_1(z)\big|_{z=e^{j\omega}} = \frac{1}{1-\frac{1}{3}e^{-j\omega}}. \tag{1.17.6}$$

### Example 2: Left-Sided Exponential

**Signal:** $x_2[n] = -\left(\frac{1}{3}\right)^n u[-n-1]$

(Left-sided signals typically produce an ROC *inside* a circle.)

$$X_2(z) = \sum_{n=-\infty}^{-1} -\left(\frac{1}{3}\right)^n z^{-n} = -\frac{3z}{1-3z}. \tag{1.17.7}$$

**ROC:**

$$|3z| < 1 \quad \Longleftrightarrow \quad |z| < \frac{1}{3}. \tag{1.17.8}$$

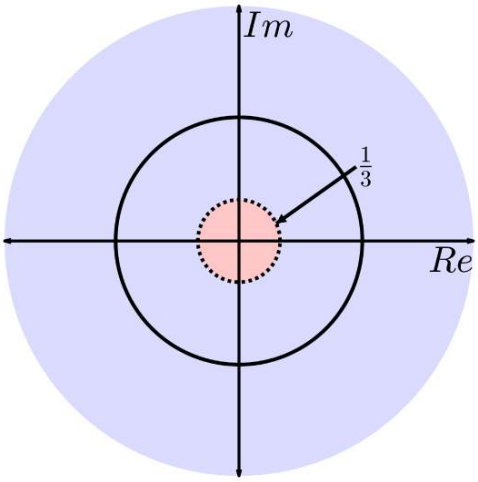



**Figure 1.17.1:** Illustration of the Region of Convergence for the signals in Example1 1 (i.e., \(z>\frac{1}{3}\), see purple shadowed circle), and Example 2 (i.e., \(z<\frac{1}{3}\), see red shadowed circle).

### How to estimate the poles and zeros using the ROC?

To find the poles and zeros, one should write the z-transform as a rational function:

$$X(z) = \frac{P(z)}{Q(z)}. \tag{1.17.9}$$

- **Zeros:** values of $z$ where $P(z) = 0$ so $X(z) = 0$.
- **Poles:** values of $z$ where $Q(z) = 0$ so $X(z) \to \infty$.

### Key ROC facts

- The ROC is **bounded by poles** and **cannot include them** (i.e., poles).
- If the ROC **includes the unit circle**, the DTFT exists, and the system is stable.
- If the ROC is **outside the outermost pole**, the system is right-sided, and it is causal.

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# 1.18: z-Transform Properties and Frequency Response

## Learning Objectives: z-transform Properties, Poles and Zeros

1. Connect causality and stability of an LTI system to properties of its region of convergence (ROC).
2. Recognize common z-transform properties.
3. Describe how pole/zero locations determine features of the frequency-response magnitude.

## YouTube Videos

Causality, Stability, and the Region of Convergence



z-Transform Properties



Poles, Zeros, and the Frequency Response



Frequency Response of a Single Pole



## Summary

### z-transform Definition

The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \tag{1.18.1}$$

If the ROC includes the unit circle $|z| = 1$, then the DT Fourier transform exists and

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}. \tag{1.18.2}$$

> The DTFT is the z-transform evaluated on the unit circle $|z| = 1$.

### First-order system review (pole location ↔ frequency peak)

For a causal first-order impulse response

$$h[n] = a^n u[n]. \tag{1.18.3}$$

The system function (i.e., z-transform of the impulse response) is

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad \text{ROC: } |z| > |a|. \tag{1.18.4}$$

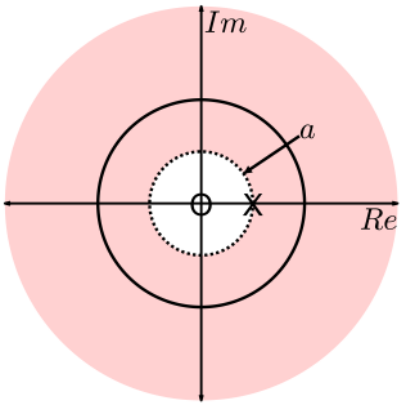



**Figure 1.18.1:** Pole-zero plot of a basic First-Order System with a pole (x) at z = a and a zero at z = 0.

Dividing the numerator and denominator by $z$

$$H(z) = \frac{1}{1 - az^{-1}} \tag{1.18.5}$$

Evaluating on the unit circle $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \tag{1.18.6}$$

. Figure 1.18.2 shows the magnitude of the frequency response for a first-order system in terms of the $a$ value.

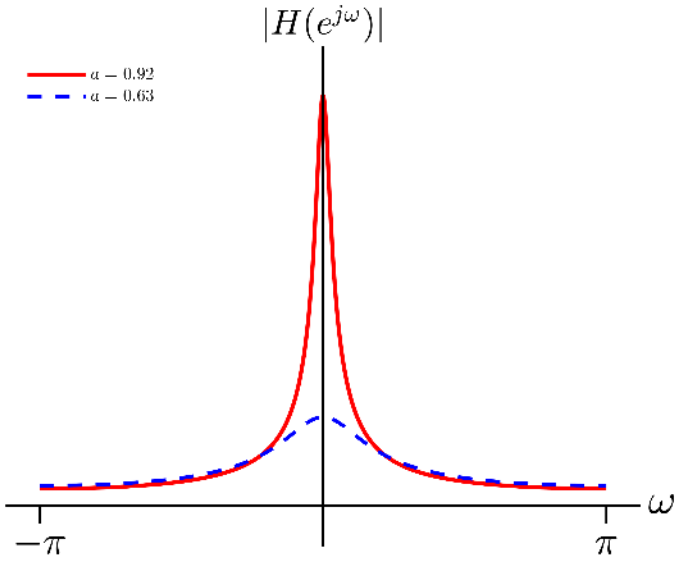



**Figure 1.18.2:** Magnitude of the frequency response for a first-order system in terms of the $a$ value.

**Key intuition:**

- The magnitude peak becomes **higher** as the pole radius $|a|$ gets closer to 1 (i.e., the unit circle).
- The peak also becomes **sharper** (i.e., narrower width) as $|a| \to 1$.

### Second-order underdamped systems

The frequency response of second-order underdamped system has two peaks at frequencies located at $\omega = \pm\theta$. In general, the frequency response of a second-order underdamped system is

$$H(z) = \frac{1}{1 - 2r\cos(\theta)\, z^{-1} + r^2 z^{-2}}, \qquad 0 < r < 1,\ 0 \le \theta \le \pi. \tag{1.18.7}$$

Poles:

$$z_1 = re^{j\theta}, \qquad z_2 = re^{-j\theta}. \tag{1.18.8}$$

Zeros: two zeros at $z = 0$ due to the $z^{-2}$ factor.

Frequency response:

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos(\theta)e^{-j\omega} + r^2 e^{-j2\omega}}. \tag{1.18.9}$$

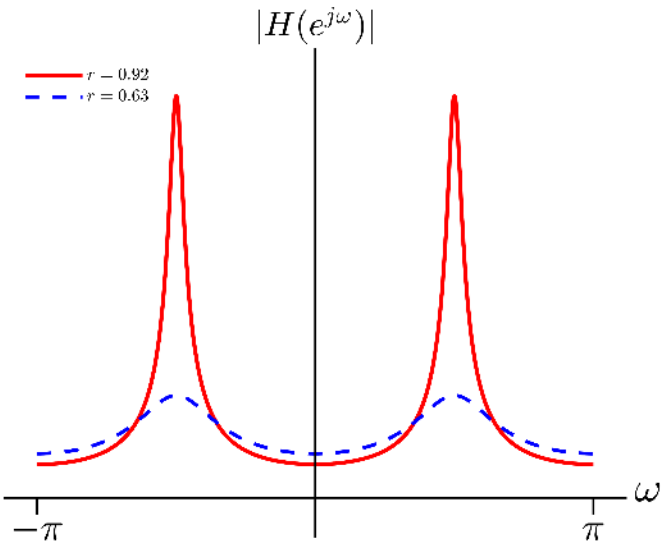



**Figure 1.18.3:** Magnitude of the frequency response for a second-order underdamped system in terms of the $r$ value.

**Peak rules:**

- $|H(e^{j\omega})|$ has peaks near $\omega = \pm\theta$.
- Peak separation is approximately $2\theta$.
- Peak amplitude increases as $r$ increases (i.e., poles closer to the unit circle).

Properties of the z-transform ($x[n] \Longleftrightarrow X(z)$)

- **Linearity**

If $x_1[n] \Longleftrightarrow X_1(z)$ with ROC $R_1$ and $x_2[n] \Longleftrightarrow X_2(z)$ with ROC $R_2$, then

$$ax_1[n] + bx_2[n] \Longleftrightarrow aX_1(z) + bX_2(z). \tag{1.18.10}$$

and the ROC contains at least $R_1 \cap R_2$ (it can be larger if pole-zero cancellations occur).

- **Time shifting**

\[x[n-n_0] \Longleftrightarrow z^{-n_0}X(z)$$
with essentially the same ROC, except possibly adding/removing $z = 0$ or $z = \infty$ (depending on the shift and sidedness).

- **Scaling/modulation (multiplying by $z_0^n$)**

$$z_0^n x[n] \Longleftrightarrow X\left(\frac{z}{z_0}\right), \qquad \text{ROC scales by } |z_0|. \tag{1.18.11}$$

In particular, if $z_0 = e^{j\omega_0}$, this corresponds to a frequency shift (modulation) on the unit circle.

- **Time reversal**

$$x[-n] \Longleftrightarrow X\left(\frac{1}{z}\right), \qquad \text{ROC maps } R \to \frac{1}{R}. \tag{1.18.12}$$

- **Time expansion (upsampling by $k$)**

Define

$$x_k[n] = \begin{cases} x[n/k], & n \text{ is a multiple of } k \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad x_k[n] \Longleftrightarrow X\left(z^k\right). \tag{1.18.13}$$

with ROC mapping roughly $R \to R^{1/k}$.

- **Conjugation**

\[x^*[n] \Longleftrightarrow X^*(z^*)$$ with the same ROC.

- **Convolution**

If $y[n] = x_1[n] * x_2[n]$, then

$$Y(z) = X_1(z)\, X_2(z), \tag{1.18.14}$$

with ROC containing at least $R_1 \cap R_2$.

- **Differentiation in the z-domain**

$$n\, x[n] \Longleftrightarrow -z\, \frac{dX(z)}{dz}. \tag{1.18.15}$$

- **Initial-value theorem (causal sequences)**

If $x[n]$ is causal, then

$$x[0] = \lim_{z \to \infty} X(z). \tag{1.18.16}$$

ROC properties

1. The ROC cannot contain any poles.
2. The unit circle $|z| = 1$ must be included in the ROC for a **stable** system.
3. The system is **causal** if the ROC is outside the outermost pole (right-sided $h[n]$), and there are no poles at $z = 0$.
4. For rational system functions, the ROC is determined by pole locations (and sidedness), not by zeros.
5. If there are more zeros than poles (relative degree), there are poles at $z = \infty$; this is inconsistent with a causal (proper) system function.

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# 1.19: z-Transforms and Block diagram

## Learning Objectives: Poles & Zeros, Frequency Response, and Block Diagrams

1. Describe how the locations of poles and zeros of the system function determine features of the frequency-response magnitude.
2. Find difference equations from block diagrams and vice versa using z-transform properties.

## YouTube Videos

z-Transforms and Difference Equations



z-Transform and Block Diagrams



## Summary

### LTI Systems and z-Transforms

For an LTI system with impulse response $h[n]$, the output is the convolution:

$$y[n] = x[n] * h[n]. \tag{1.19.1}$$

Taking the z-transform:

$$Y(z) = X(z)H(z), \tag{1.19.2}$$

where

$$H(z) = \text{system function (transfer function)}. \tag{1.19.3}$$

If the **unit circle** $|z| = 1$ is included in the ROC of $H(z)$, then the frequency response is obtained by

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}. \tag{1.19.4}$$

**Reminder**

$z^n$ is an eigenfunction of an LTI system.

Causality based on ROC and rational form of H(z)

**Definition: Causality in time-domain**

A causal LTI system satisfies:

$$h[n] = 0 \quad \text{for } n < 0. \tag{1.19.5}$$

**Definition: Causality using the z-Transform**

Consider a system function of a DT LTI system that be written in rational form by

$$H(z) = \frac{P(z)}{Q(z)}, \tag{1.19.6}$$

where $P(z)$ and $Q(z)$ are polynomials in $z$. The DT LTI system is said to be causal if

- the degree of $P(z)$ and $Q(z)$ polynomials satisfies that $\deg(P) \leq \deg(Q)$.

and

- the ROC of $H(z)$ is outside the outermost pole. This implies that if there is a pole at infinity, the system cannot be causal.

**Example**

Given

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}. \tag{1.19.7}$$

Because $\deg(P) = 3$ is higher than $\deg(Q) = 2$, the system is **non-causal,** independently of its ROC.

**Example: Poles, ROC, and Causality**

Given

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}, \qquad |z| > 2, \tag{1.19.8}$$

with ROC $|z| > 2$). Writing the system function in a common-denominator form

$$H(z) = \frac{2 - \frac{5}{2}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} = \frac{2z^2 - \frac{5}{2}z}{\left(z - \frac{1}{2}\right)(z - 2)}. \tag{1.19.9}$$

Poles are located at:

$$z = \frac{1}{2}, \quad z = 2. \tag{1.19.10}$$

.

Since the ROC is $|z| > 2$, which is outside the outermost pole, the system is **causal**.

From the system function

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \tag{1.19.11}$$

, the impulse response can be found

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[n]. \tag{1.19.12}$$

Stability

**Definition: Stability using the z-Transform**

An LTI system is stable *if and only if* the ROC of $H(z)$ includes the unit circle $|z| = 1$.

**Definition: Causality and Stability using the system function $H(z)$**

A causal LTI system with rational system function $H(z)$ is stable *if and only if* all poles lie **inside** the unit circle

LTI Systems Described by Difference Equations

General LCCDE form

In general:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \quad \Longrightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{1.19.13}$$

Equivalent time-domain difference equation:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]. \tag{1.19.14}$$

**Example**

Given the difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]. \tag{1.19.15}$$

Apply the z-transform to both sides of the equation:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z). \tag{1.19.16}$$

Group terms of $Y(Z)$ and $X(z)$:

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)\left(1 + \frac{1}{3}z^{-1}\right). \tag{1.19.17}$$

Write the system function by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}. \tag{1.19.18}$$

Expand the system function:

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}. \tag{1.19.19}$$

Since there is a pole located at $|z| < \frac{1}{2}$, there are two possible ROCs:

$$|z| > \frac{1}{2} \quad \text{or} \quad |z| < \frac{1}{2}. \tag{1.19.20}$$

If $|z| > \frac{1}{2}$:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]. \tag{1.19.21}$$

If $|z| < \frac{1}{2}$:

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n]. \tag{1.19.22}$$

## Block-Diagram Notes

- **Feedback interconnection**

Figure 1.19.1 shows a negative-feedback structure with forward block $H_1(z)$ and feedback block $H_2(z)$:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}. \tag{1.19.23}$$

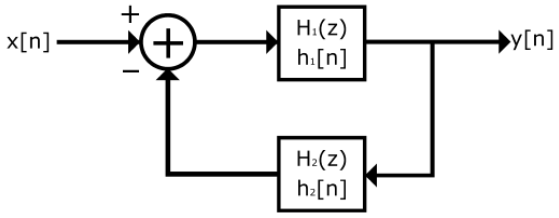



**Figure 1.19.1:** Block diagram for a negative-feedback structure.

### Example: Block diagram of a first-order system

Consider

$$H(z) = \frac{1}{1 - \frac{1}{8}z^{-1}} \quad \Longleftrightarrow \quad y[n] - \frac{1}{8}y[n-1] = x[n]. \tag{1.19.24}$$

The block diagram is

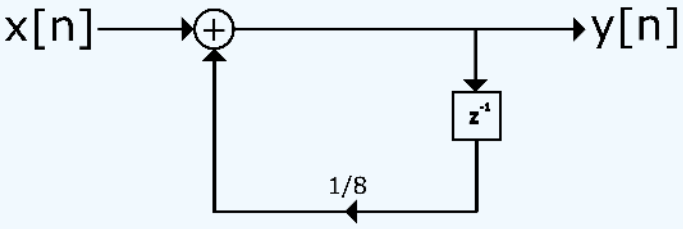



**Figure 1.19.2:** Solution of the block diagram.

### Example: Block diagram of a first-Order system using direct and cascade form

Consider

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \underbrace{\frac{1}{1 - \frac{1}{4}z^{-1}}}_{H_1(z)} \underbrace{\left(1 - 2z^{-1}\right)}_{H_2(z)}. \tag{1.19.25}$$

This system function consists of the sum of two individual systems:

$$H_1(z): \quad y[n] - \frac{1}{4}y[n-1] = x[n], \qquad H_2(z): \quad y[n] = x[n] - 2x[n-1]. \tag{1.19.26}$$

The block diagram is

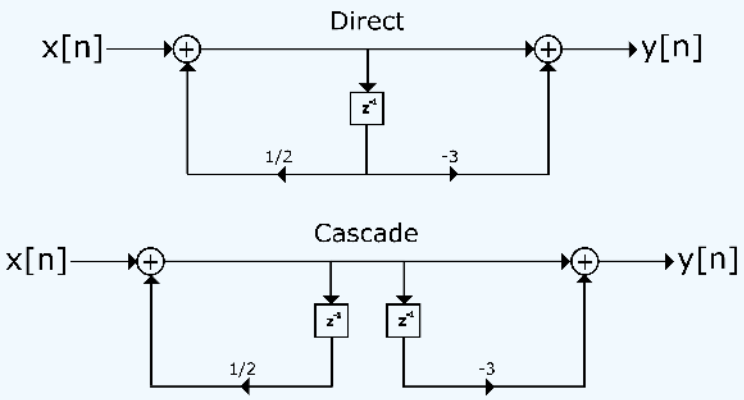



**Figure 1.19.3:** Solution of the block diagram using both direct (top panel) and cascade (bottom panel) structure.

## Example: Block diagram of a second-Order system

Consider

$$H(z) = \frac{1}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)} = \frac{1}{1+\frac{1}{8}z^{-1}-\frac{1}{32}z^{-2}}. \tag{1.19.27}$$

The difference equation is derived by expressing the system function as a rational function of $z^{(}-1)$, rearranging terms to isolate $\boldsymbol{Y}(z)$ and \(X(z)\, and applying the inverse Z-transform. After some maths, the difference equation is

$$y[n]+\frac{1}{8}y[n-1]-\frac{1}{32}y[n-2]=x[n]. \tag{1.19.28}$$

Figure 1.19.4 shows the block diagram of this difference equation using the direct form and a cascade of two first-order systems.

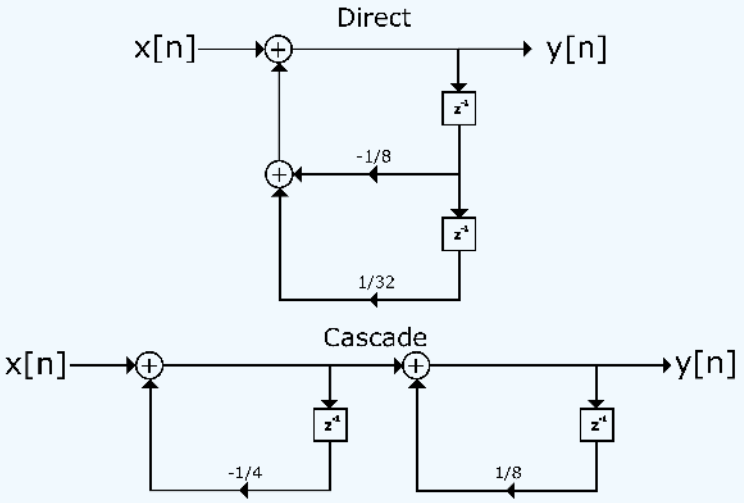



**Figure 1.19.4:** Block diagram of the second-order system using direct and cascade structures.

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# 1.20: IIR Filter Design

## Learning Objectives: IIR Filter Design

1. Understand basic properties of IIR filters, such as filter order, passband behavior and stopband behavior.
2. Design Butterworth and Chebyshev Type 2 filters in Matlab
3. Apply filters to a signal to make speech more understandable.
4. Examine how different IIR filters have characteristic patterns in the locations of poles and zeros

## YouTube Videos

Infinite Impulse Response Filter Design Overview



Butterworth Filter Design in MATLAB



Designing Chebyshev Type 2 Filters in MATLAB



## 1 Infinite-impulse response filter design

The goal of designing real causal infinite impulse response (IIR) filters is to find a linear, constant-coefficient difference equation that approximates a desired frequency response magnitude $|H_{id}(e^{j\Omega})|$. Because the required techniques involve transforming continuous-time (CT) filters to discrete-time (DT) filters, we use $\Omega$ for the DT frequency and $\omega$ for the CT frequency.

As in FIR filter design, it is not possible to obtain an ideal frequency response from a finite-order difference equation; therefore, we must approximate the desired frequency-response magnitude, as shown in Figure 1.20.1 for a DT lowpass filter. We use the same terminology to describe the filter's specifications as in the FIR case. The passband is the region $\Omega < \Omega_p$, the stopband is the region $\Omega > \Omega_s$. The transition band is still the region between the passband and stopband, $\Omega_p \leq \Omega \leq \Omega_s$, with width $\Delta\Omega = \Omega_s - \Omega_p$. Within the passband, we accept some ripple given by $\delta_1$, and similarly the stopband ripple is given by stopband ($\delta_2$). For the IIR designs, both the passband and stopband ripples are often given in dB of attenuation, which is computed as $-20\log_{10}\delta$. The IIR filters can exhibit different passband and stopband ripple, which is an important distinction from the FIR filter designs.

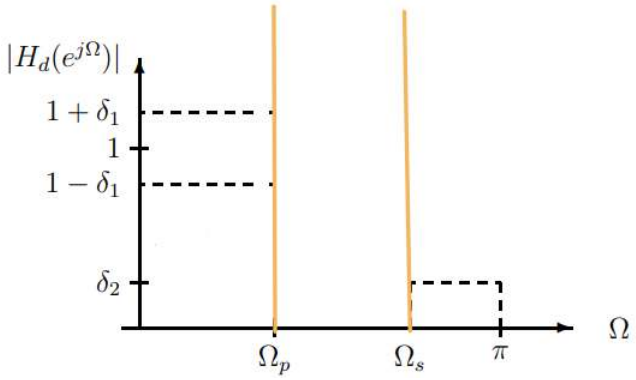



**Figure 1.20.1:** Specifications for the frequency response magnitude of a discrete-time lowpass filter.

Previously, we have described how recursive linear, constant-coefficient difference equations with delayed $y[n-k]$ terms produce infinite-impulse response (IIR) filters. To simplify, we assume that all of the coefficients are real, so that $h[n]$ is also real and $|H_d(e^{j\Omega})|$ is even symmetric. The task is then to find the order $\max(N, M)$ and coefficients $a_k$ and $b_\ell$ in

$$\sum_{k=0}^{N} a_k\, y[n-k] \;=\; \sum_{\ell=0}^{M} b_\ell\, x[n-\ell] \tag{1.20.1}$$

so that the frequency response

$$H_d(e^{j\Omega}) \;=\; \frac{\sum_{\ell=0}^{M} b_\ell e^{-j\Omega\ell}}{\sum_{k=0}^{N} a_k e^{-j\Omega k}} \tag{1.20.2}$$

satisfies the specification given in Figure 1.20.1.

## 1.1 Bilinear transform

The most common method for designing IIR filters is called the bilinear transform. This technique begins with a CT system function $H_c(s)$, and makes a change of variables for $s$ to get a DT system function $H_d(z)$. The simplest change of variables to do this is

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \tag{1.20.3}$$

Before we get into details of how the DT filter design procedures works, let's look at an example of using this change of variables to transform a CT filter into a DT filter.

### Example 1 - Bilinear Transform

Consider the CT lowpass filter with the system function

$$H_c(s) = \frac{0.1925}{s^3 + 1.1547s^2 + 0.6667s + 0.1925} \tag{1.20.4}$$

The CT frequency response magnitude for this filter is shown in Figure 1.20.2. This third-order lowpass filter was designed so that the -3 dB point in the passband was at $\omega = \frac{1}{\sqrt{3}} \approx 0.58$.

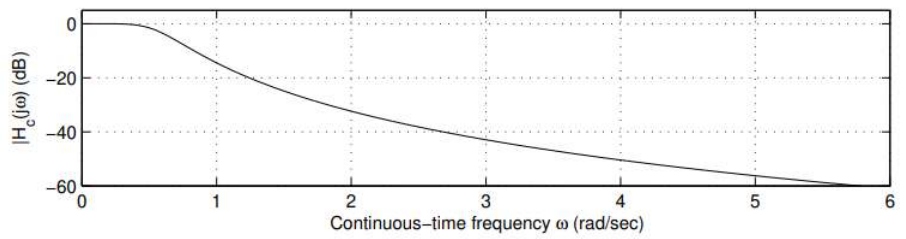



**Figure 1.20.2:** Continuous-time frequency response magnitude $|H_c(j\omega)|$ for Example 1.

The discrete-time system function is obtained by replacing each $s$ in Eq. (1.20.4) by $(1 - z^{-1})/(1 + z^{-1})$, then solving for new polynomials in $z^{-1}$. When the dust clears from all this algebra, the result is

$$H_d(z) = \frac{0.0639 + 0.1916z^{-1} + 0.1916z^{-2} + 0.0639z^{-3}}{1 - 0.9658z^{-1} + 0.5826z^{-2} - 0.1060z^{-3}} \tag{1.20.5}$$

Substituting $z = e^{j\Omega}$ into $H_d(z)$ gives the frequency response shown in Figure 1.20.3. The new DT filter is still a lowpass filter, just like the CT filter used as the prototype. The -3 dB point for this new filter falls at $\pi/3$. There is straightforward relationship between the CT and DT frequencies for this change of variables which we'll derive shortly. Using our razor-sharp z-transform skills, we can find that the linear, constant-coefficient difference equation for this system is

$$\begin{aligned} &0.0639y[n] + 0.1916y[n-1] + 0.1916y[n-2] + 0.0639y[n-3] \\ &= x[n] - 0.9658x[n-1] + 0.5826x[n-2] - 0.1060x[n-3] \end{aligned} \tag{1.20.6}$$

This filter can be implemented using a block diagram with the coefficients from Eq. (1.20.6).

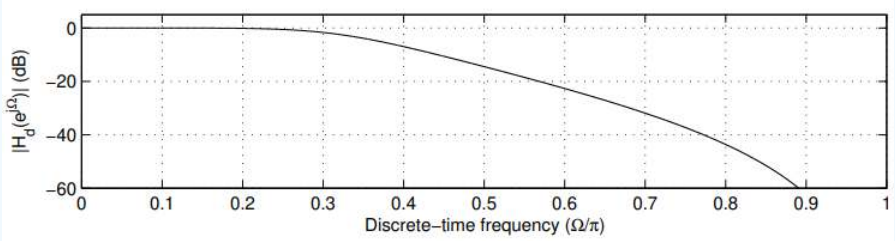



**Figure 1.20.3:** Discrete-time frequency response magnitude $|H_d(e^{j\Omega})|$ for Example 1.

This example illustrates some important properties of the bilinear transfer. First, it transforms causal, stable CT filters into causal, stable DT filters. Second, it preserves the type of frequency-selective filters, like lowpass or highpass filters. However,

it raises an important question. In this example, we began with a CT filter and changed it into a DT filter. How can we choose the right CT filter so that the DT filter which results from the change of variables satisfies our specification? To answer this question, let's look more closely at the change of variables.

The change of variables $s = (1 - z^{-1})/(1 + z^{-1})$ has two important properties. Poles in the left-half s-plane map to poles inside the unit circle of the z-plane. As mentioned above, this means that causal stable CT filters map to causal stable DT filters. The second important property is that the CT frequency axis $s = j\omega$ maps onto the DT frequency axis $z = e^{j\Omega}$. Because the frequency response is determined on the $s = j\omega$ contour for CT systems, and on the $z = e^{j\Omega}$ contour for DT systems, the mapping between the two frequencies is obtained by making these substitutions into Eq. (1.20.3).

$$\begin{aligned} s\big|_{s=j\omega} &= \left.\frac{1-z^{-1}}{1+z^{-1}}\right|_{z=e^{j\Omega}} \\ j\omega &= \frac{1-e^{-j\Omega}}{1+e^{-j\Omega}} \\ &= \left(\frac{e^{-j\Omega/2}}{e^{-j\Omega/2}}\right)\left(\frac{e^{j\Omega/2}-e^{-j\Omega/2}}{e^{j\Omega/2}+e^{-j\Omega/2}}\right) \\ &= \frac{2j\sin(\Omega/2)}{2\cos(\Omega/2)} \end{aligned}$$

$$\omega = \tan(\Omega/2) \quad \text{or} \tag{1.20.7}$$

$$\Omega = 2\arctan(\omega) \tag{1.20.8}$$

As a result, we can relate the CT and DT frequency responses for the bilinear transform according to

$$H_d(e^{j\Omega}) = H_c(j\omega)\big|_{\omega=\tan(\Omega/2)} \tag{1.20.9}$$

Figure 1.20.4 plots Eq. (1.20.8) over a wide range of $\omega$. There are two important things to notice about this picture. First, the change of variables is not linear, so there is no simple scaling factor between $\omega$ and $\Omega$ as there is in unaliased sampling with $\Omega = \omega T$. Frequency selective filters such as lowpass, highpass and bandpass filters will preserve their type in the mapping, since their frequency responses are piecewise constant. However, frequency responses which are not piecewise constant, such as a DT differentiator, will not preserve their shape through this mapping. Second, the nonlinear mapping warps the $\omega$ frequency axis so that the infinitely long $s = j\omega$ axis fits onto one circumference of the unit circle. A consequence of this is that large values of $\omega$ will be compressed very close together in $\Omega$. For example, compare how close $\omega = 15$ and $\omega = 20$ end up in $\Omega$ compared with $\omega = 5$ and $\omega = 10$.

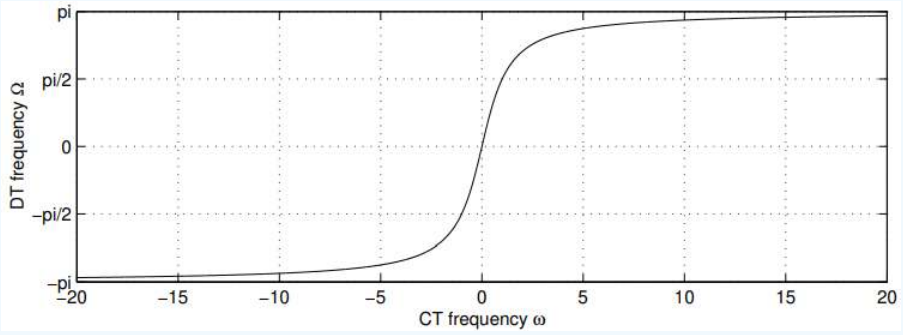



**Figure 1.20.4:** Mapping between CT frequency $\omega$ and DT frequency $\Omega$ from Eq. (1.20.8).

Figure 1.20.4 and Eqs. (1.20.7) – (1.20.9) contain the key to our design problem. With this knowledge of how the frequencies map between CT and DT for the bilinear transform, we can work backward from our desired passband edge $\Omega_p$ and stopband edge $\Omega_s$ to find the appropriate band edges $\omega_p$ and $\omega_s$ for our CT prototype filter. If we design the CT filter to meet these specifications, the DT filter that results when we make the change of variables from Eq. (1.20.3) will "automatically" satisfy our DT specifications.

**Example 2 - Bilinear transform**

Suppose that we want to design a DT lowpass filter with $\Omega_p = \pi/2$ and $\Omega_s = 3\pi/4$. From Eq. (1.20.7), we get the CT band edge frequencies as

$$\omega_p = \tan(\Omega_p/2) = \tan(\pi/4) = 1$$

$$\omega_s = \tan(\Omega_s/2) = \tan(3\pi/8) \approx 2.41$$

Note that because the bilinear transform only warps the frequency axis, and not the vertical axis, the passband and stopband ripples of $\delta_1$ and $\delta_2$ will not change between DT and CT. There are many common CT filter design techniques and programs to find a system function $H_c(s)$ satisfying these specifications. Once you obtain $H_c(s)$, making the change of variables from Eq. (1.20.3) and a lot of algebra later, you would have $H_d(z)$ satisfying the specification given.

Pulling together all of these ideas results in a **four-step procedure for designing a DT frequency-selective filter using the bilinear transform**.

**Step 1:** Write down the DT specification clearly in terms of the band edges and ripples.

**Step 2:** Pre-warp the band edge frequencies using Eq. (1.20.7) to find the appropriate CT filter specification. The ripples in each band remain unchanged.

**Step 3:** Find the system function $H_c(s)$ using a common CT filter design technique.

**Step 4:** Substitute Eq. (1.20.3) into $H_c(s)$ to find $H_d(z)$ for a DT filter satisfying the specifications from Step 1.

The CT filter design in **Step 3** is usually one of the four common CT filters: Butterworth, Chebyshev Type I, Chebyshev Type II, and Elliptic (or Cauer) filters. The following section discuss them.

## 1.2 Designing IIR filters in MATLAB

MATLAB provides helpful functions that greatly simplify the design of IIR filters. These functions compute the filter order needed to satisfy a DT specification, and then design the CT filter, perform the bilinear transform, and do all the algebra to return $H_d(z)$ for you. In this section, we introduce the functions used to define each of the four common filter types. To do this, we use the same DT specifications for all four filters, and compare the results we obtain. The DT filter we design is a lowpass filter satisfying

$$

$$0.99 \leq \left|H_d(e^{j\Omega})\right| \leq 1.01, \quad |\Omega| \leq 0.2\pi,$$
$$\left|H_d(e^{j\Omega})\right| \leq 0.001, \quad 0.4\pi \leq |\Omega| \leq \pi \tag{1.20.10}$$

\]

In the examples that follow, the Butterworth filter requires the highest order to satisfy this specification, the elliptic filter satisfies the specification with the lowest order, and the Chebyshev filters fall between them. This is generally the trend in filter order versus filter type for any frequency-selective filter designed using the bilinear transformation.

### 1.2.1 Butterworth filters

Butterworth filters have frequency responses that decrease monotonically from 1 in the passband to nearly 0 in the stopband without ripples. Figure 1.20.5 plots the frequency response (in dB) of a 12th order Butterworth filter satisfying the specification from Eq. 1.20.10. As expected, the passband is nearly flat until $0.2\pi$, then drops rapidly so that it is below $-20\log_{10} 0.001 = 60$ dB by the time it reaches $0.4\pi$. Figure 1.20.6 plots just the passband and stopband on a linear frequency scale to confirm that they satisfy the DT specification given in Eq. (1.20.10).

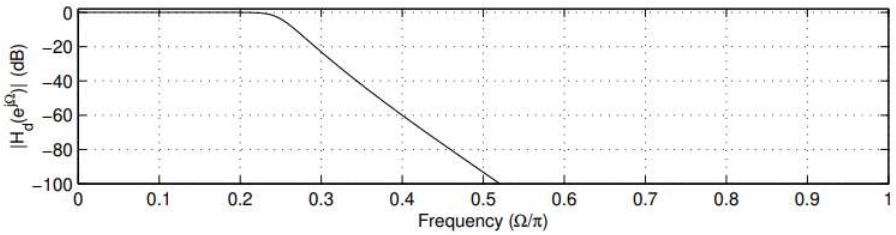



**Figure 1.20.5:** Frequency response magnitude $|H_d(e^{j\Omega})|$ for a 12th order Butterworth filter satisfying Eq. (1.20.10).

The MATLAB command $[N, \text{Omegan}] = \text{buttord}(\text{Omegap}, \text{Omegas}, Rp, Rs)$ is used to find the order $N$ and cutoff frequency Omegan necessary for a Butterworth lowpass filter to satisfy the specification given. Omegap is the passband edge normalized by $\pi$, i.e., $\Omega_p/\pi$.

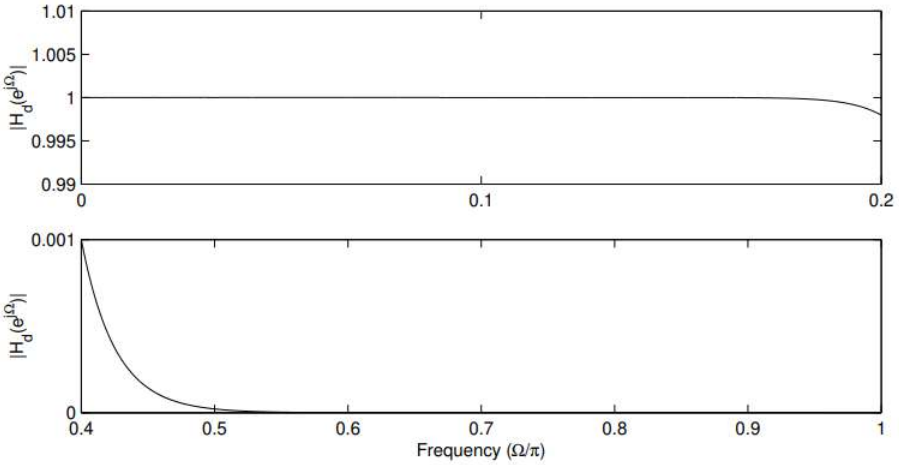



**Figure 1.20.6:** Zoom in on the passband and stopband for Butterworth filter $|H_d(e^{j\Omega})|$.

Similarly, Omegas is the stopband edge normalized by $\pi$, or $\Omega_s/\pi$. Rp is the minimum allowable passband amplitude in dB. In terms of the specification in Figure 1.20.1, this is $Rp = -20\log_{10}(1-\delta_1)$. The last parameter, Rs, is the maximum allowable stopband ripple in dB, or $-20\log_{10}(\delta_2)$. For the example, the following commands determine the order and cutoff frequency of the Butterworth filter shown in Figure 1.20.5.

```
Omegap = 0.2;
Omegas = 0.4;
delta1 = 0.01;
delta2 = 0.001;
Rp = -20*log10(1-delta1);
Rs = -20*log10(delta2);
[N,Omegan] = buttord(Omegap,Omegas,Rp,Rs);
```

Once you obtain the values of N and Omegan, the MATLAB function *butter* computes the numerator and denominator coefficients for $H_d(z)$ as shown below

```
[bbutter,abutter] = butter(N,Omegan);
```

The vectors *bbutter* and *abutter* above are the linear, constant-coefficient difference equation coefficients for the filter. Using z-transform properties, these are also the coefficients for the numerator and denominator polynomials in $z^{-1}$ in $H_d(z)$. For this 12th-order filter, these vectors are each 13 elements long, and therefore not very practical to write out for $H_d(z)$ or the linear, constant-coefficient difference equation. To find these values, MATLAB performed the algebra required to change variables in Eq. (1.20.3).


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DT Butterworth filters generally have a distinctive pole-zero pattern. Figure 1.20.7 plots the pole-zero response for the 12th-order filter whose frequency response is shown in Figure 1.20.5. The poles are in a radius around $\Omega = 0$ to raise the passband frequency response, while the zeros are clustered around $\Omega = \pi$ to bring down the stopband.

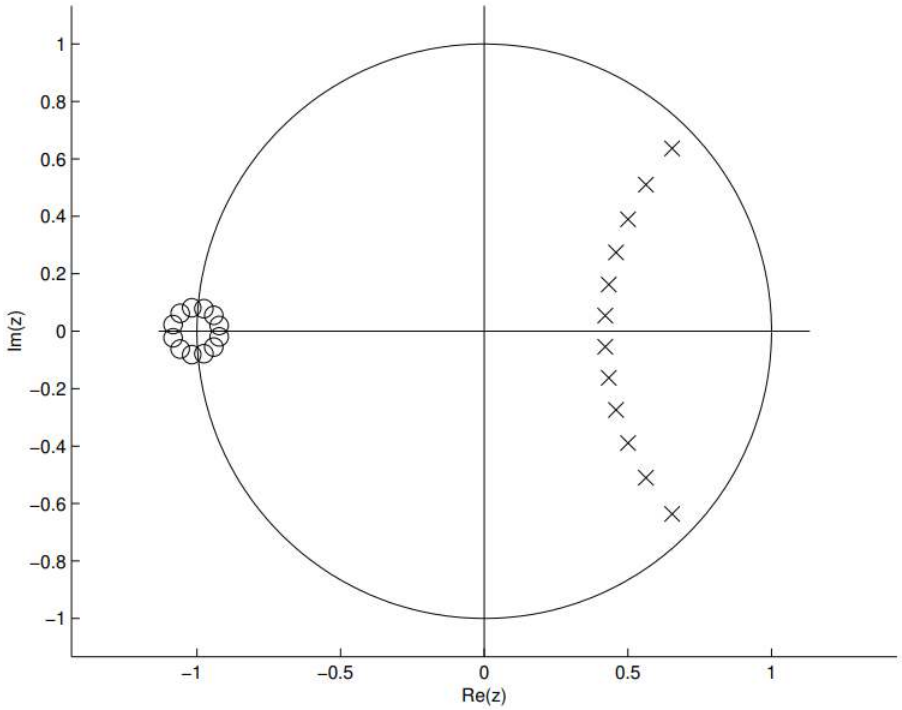



**Figure 1.20.7:** Pole-zero plot for the Butterworth filter satisfying Eq. (1.20.10).

There are other input options to *buttord* and *butter* to produce other common filters such as highpass and bandpass filters, read the MATLAB's help on these commands to learn more.

### 1.2.2 Chebyshev Type I filters

Chebyshev Type I filters have ripples in the passband but a smooth monotonic stopband. Figure 1.20.8 plots the frequency response of a 7th-order Chebyshev Type I filter satisfying Eq. (1.20.10). Because it is plotted on a dB scale, the passband ripples are not obvious. It is clear from this plot that the stopband is more than 60 dB down by $\Omega_s = 0.4\pi$. Figure 1.20.9 enlarges the passband and stopband. The passband ripples are clearly visible in the top panel, but the ripples stay within the range specified in Eq. (1.20.10). The bottom panel demonstrates that the stopband actually exceeds the specification.

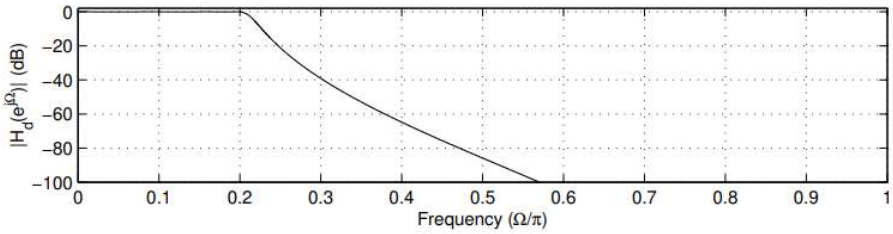



**Figure 1.20.8:** Frequency response magnitude $|H_d(e^{j\Omega})|$ for a 7th-order Chebyshev Type I filter satisfying Eq. (1.20.10).

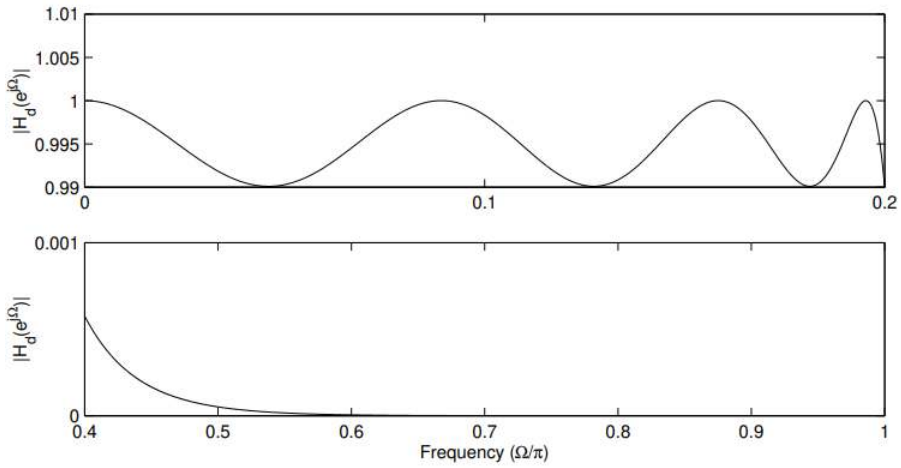



**Figure 1.20.9:** Zoom in on the passband and stopband for Chebyshev Type I filter $|H_d(e^{j\Omega})|$.

The MATLAB commands to make these filters are similar to the Butterworth filter commands. cheb1ord uses the same input arguments as buttord to find the order N and break frequency Omegan.

```
Omegap = 0.2;
Omegas = 0.4;
delta1 = 0.01;
delta2 = 0.001;
Rp = -20*log10(1-delta1);
Rs = -20*log10(delta2);
Rpb = min(abs([20*log10(1+delta1),20*log10(1-delta1)]));
[N,Omegan] = cheb1ord(Omegap,Omegas,Rp,Rs);
```

The MATLAB command *cheby1* returns the linear, constant-coefficient difference equation coefficients for the filter. *cheby1* takes both N and Omegan as input arguments. It also requires an additional argument specifying the passband ripple limits in dB. This is the smaller of the absolute values of $20\log_{10}(1+\delta_1)$ and $20\log_{10}(1-\delta_1)$.

```
Rpb = min(abs([20*log10(1+delta1),20*log10(1-delta1)]));
[bcheby1,acheby1] = cheby1(N,Rpb,Omegan);
```

Again, the output vectors *bcheby1* and *acheby1* contain the linear, constant-coefficient difference equation coefficients for the filter, or the numerator and denominator coefficients for $H_d(z)$. We can use these coefficients to find the pole-zero plot shown in Figure 1.20.10. The pattern of poles and zeros shown is typical of Chebyshev Type I filters. The collection of poles near $\Omega = 0$ raises the passband, but the nearly linear arrangement results in ripples shown in Figure 1.20.9. This change in the pole locations from the arc of the Butterworth filter in Figure 1.20.7 produces the passband ripples, in contrast to the smooth passband of the Butterworth filter. The collection of zeros at $\Omega = \pi$ reduces the stopband frequency response.

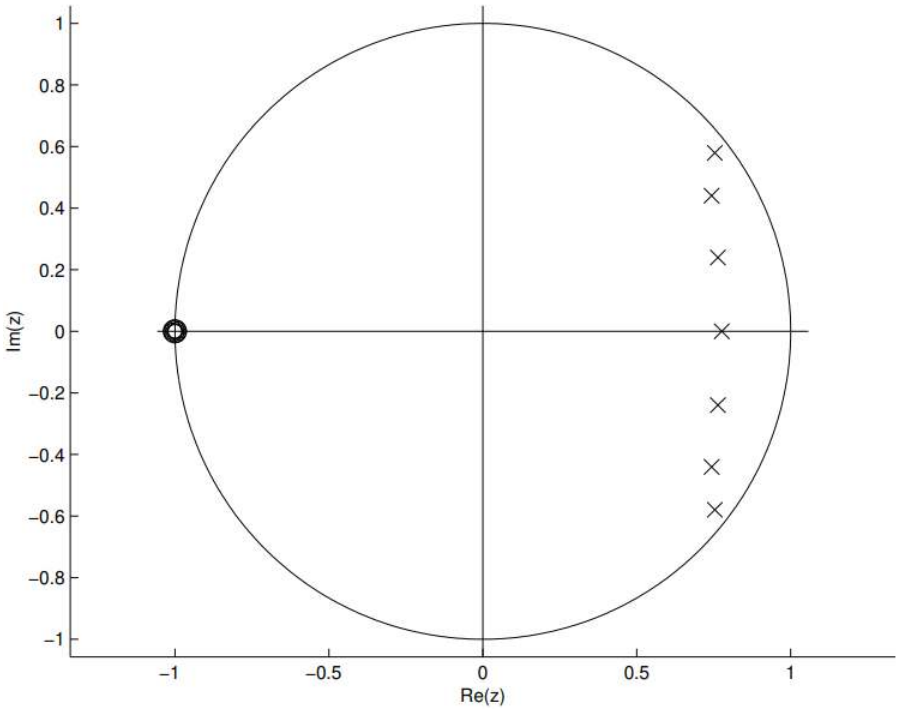



**Figure 1.20.10:** Pole-zero plot for the Chebyshev Type I filter satisfying Eq. (1.20.10).

### 1.2.3 Chebyshev Type II filters

Chebyshev Type II filters have a monotonic passband and ripples in the stopband. Figure 1.20.11 plots the frequency response of a 7th-order Chebyshev Type II filter satisfying Eq. (1.20.10). The passband is basically flat, and the ripples in the stopband all fall below -60 dB.

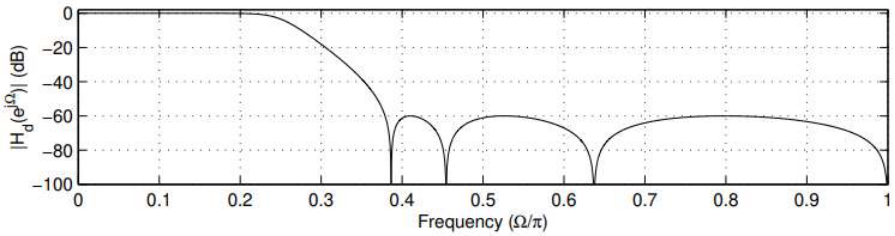



**Figure 1.20.11:** Frequency response magnitude $|H_d(e^{j\Omega})|$ for a 7th-order Chebyshev Type II filter satisfying Eq. (1.20.10).

Figure 1.20.12 enlarges the passband and stopband. The passband just meets the specification at $\Omega_p = 0.2\pi$, while the stopband ripples go up to 0.001 but no higher.


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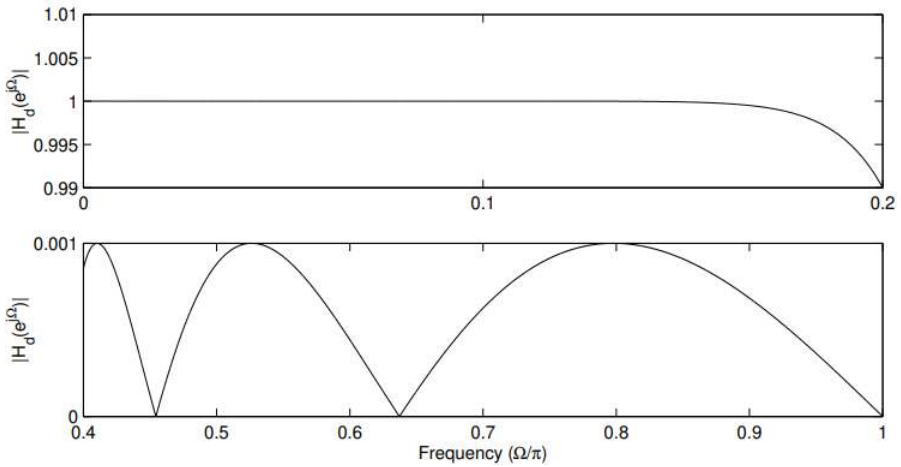



**Figure 1.20.12:** Zoom in on the passband and stopband for Chebyshev Type II filter $|H_d(e^{j\Omega})|$.

The MATLAB command *cheb2ord* has the same inputs and outputs as *cheb1ord*. The linear, constant-coefficient difference equation coefficients are computed by *cheby2*, which is similar to *cheby1* in input arguments, with one change. The second input argument to *cheby2* is the maximum stopband ripple in dB, which is just Rs. The following commands define the filter coefficients for the filter whose frequency response appears in Figure 1.20.11.

```
Omegap = 0.2;
Omegas = 0.4;
delta1 = 0.01;
delta2 = 0.001;
Rp = -20*log10(1-delta1);
Rs = -20*log10(delta2);
[N,Omegan] = cheb2ord(Omegap,Omegas,Rp,Rs);
[bcheby2,acheby2] = cheby2(N,Rs,Omegan);
```

Figure 1.20.13 plots the poles and zeros for this filter. This pattern is typical of Chebyshev Type II filters. The poles which produce the smooth passband are in an arc similar to that seen in the Butterworth filter. The zeros for the stopband lie on the unit circle, causing the deep notches observed in Figures 1.20.11 and 1.20.12. The angles of the zeros on the unit circle correspond to the frequencies $\Omega$ with $|H_d(e^{j\Omega})| = 0$ seen in the bottom panel of Figure 1.20.12.

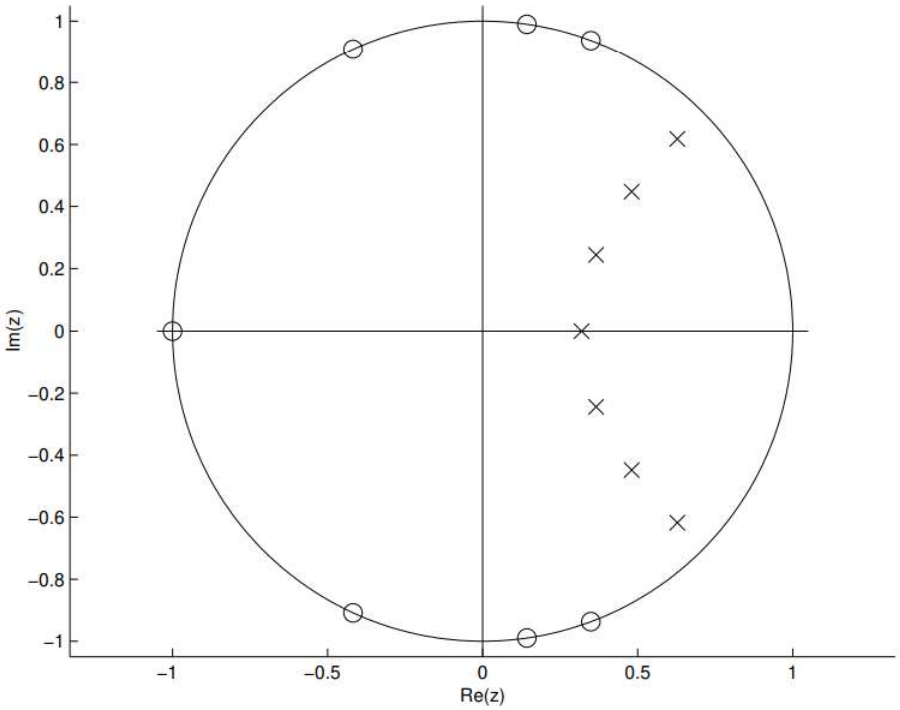



**Figure 1.20.13:** Pole-zero plot for the Chebyshev Type II filter satisfying Eq. (10).

### 1.2.4 Elliptic filters

Elliptic, or Cauer, filters have ripples in both the passband and stopband. Figure 1.20.14 plots the frequency response on a dB scale for a 5th order elliptic filter satisfying Eq. (1.20.10). On this scale, the passband ripples are not apparent, but they can been seen in the top panel of Figure 1.20.15. Figure 1.20.15 illustrates clearly that the passband exactly meets the specification at $\boldsymbol{\Omega_p}$ and that the maximum stopband ripples are just up to 0.001.

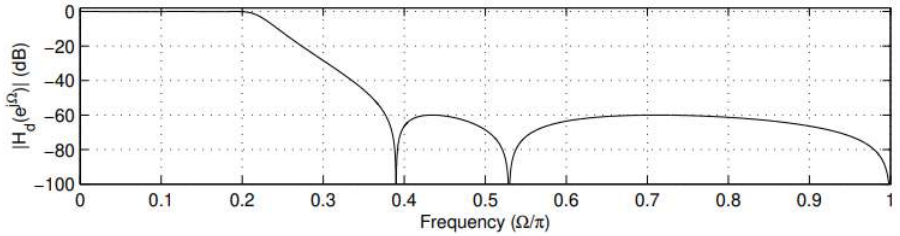
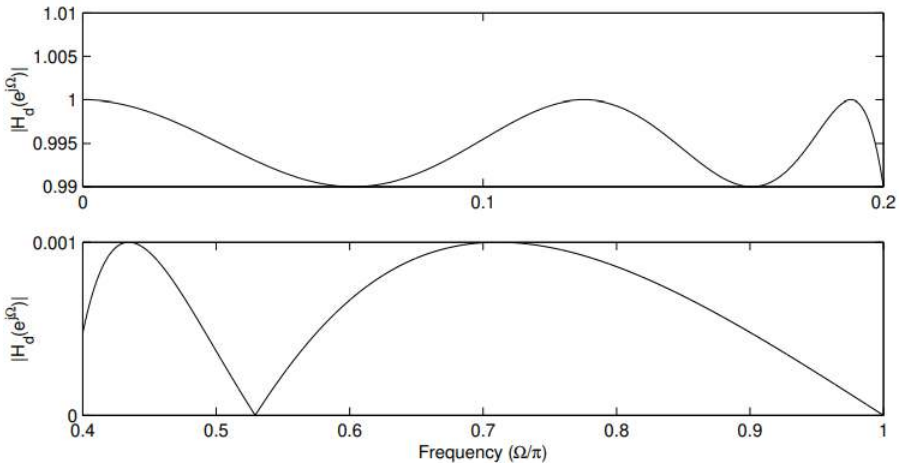



**Figure 1.20.14:** Frequency response magnitude $|\boldsymbol{H_d}(e^{j\Omega})|$ for a 5th-order Elliptic filter satisfying Eq. (1.20.10).





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**Figure 1.20.15:** Zoom in on the passband and stopband for the Elliptic filter $|H_d(e^{j\Omega})|$.

The MATLAB commands for elliptic filters combine the features seen for the three previous filter types. The command *ellipord* returns the filter order N and break frequency Omegan needed to satisfy the specification given by the band edges $\Omega_p$ and $\Omega_s$, and the associated ripples Rp and Rs. The MATLAB command *ellip* computes the linear, constant-coefficient difference equation coefficients given N and Omegan, as well as both the passband ripple, as defined for the Chebyshev Type I filter, and the stopband ripple, as defined for the Chebyshev Type II filter. The following commands produce the coefficients specifying the elliptic filter.

```
Omegap = 0.2;
Omegas = 0.4;
delta1 = 0.01;
delta2 = 0.001;
Rp = -20*log10(1-delta1);
Rs = -20*log10(delta2);
Rpb = min(abs([20*log10(1+delta1),20*log10(1-delta1)]));
[N,Omegan] = ellipord(Omegap,Omegas,Rp,Rs);
[bellip,aellip] = ellip(N,Rp,Rs,Omegan);
```

Figure 1.20.16 plots the pole-zero plot for the elliptic filter. The pattern of poles and zeros combines features we have seen already in the Chebyshev filters. The poles generating the passband are arranged in a very shallow arc, nearly in a line, similar to the arrangement for the Chebyshev Type I filter, which also exhibits passband ripples. The zeros fall directly on the unit circle, as seen in the Chebyshev Type II filter. This fits with the stopband ripples seen in both the elliptic and Chebyshev Type II filter. Again, the angles of these poles match the frequencies of the zeros seen in the bottom panel of Figure 1.20.15.

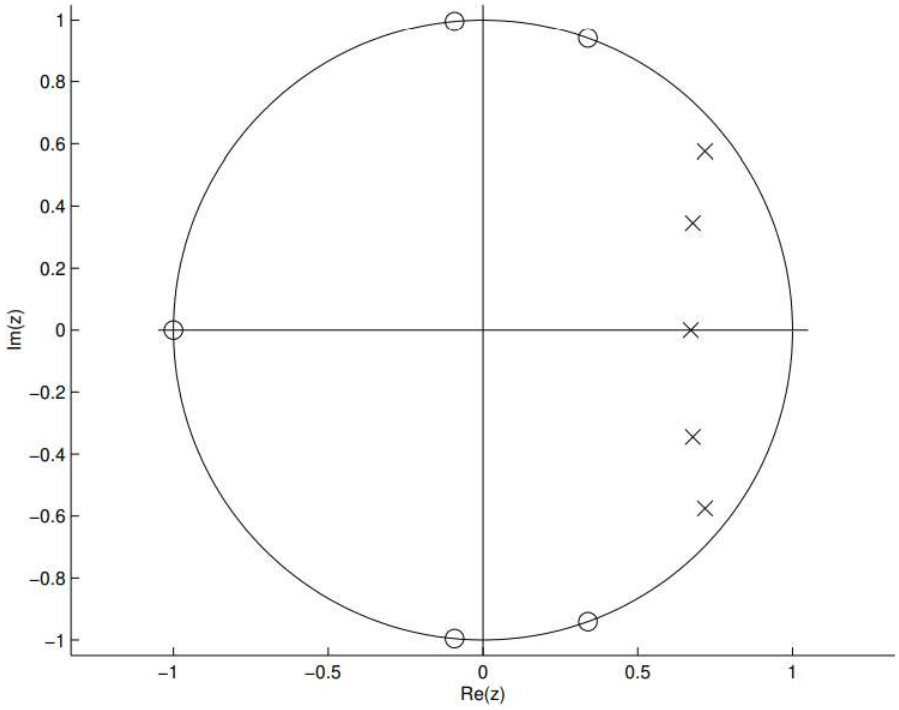



**Figure 1.20.16:** Pole-zero plot for the elliptic filter satisfying Eq. (1.20.10).

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# 1.21: Discrete Fourier Transform (DFT)

## Learning Objectives:

1. Understand how evaluating the DT Fourier Transform at evenly spaced frequencies in omega is a form of sampling, and implicitly creates periodic signals in the time domain.
2. Explain the effect of time-domain aliasing created by undersampling in frequency
3. Predict how many samples in omega are required to represent finite length signals without aliasing.
4. Apply the Discrete Fourier Transform (DFT) to approximate the full DT Fourier transform for signals
5. Understand how multiplying the DFTs of two finite length signals results in circular convolution
6. Compute the results of circular convolution using linear convolution followed by aliasing

## YouTube Videos

Overview of the Discrete Fourier Transform (DFT)



Discrete Fourier Transform: Equations



DFT Example: Rectangular Pulse



DFT Example: Approximating the DT Fourier Transform



Circular Convolution with the DFT



## 1 The Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is a computational technique especially useful for processing finite-length signals in the frequency domain. The computational difficulty with using the Discrete-Time Fourier Transform (DTFT) $X(e^{j\omega})$ is that the frequency $\omega$ is a continuous variable, and the computer can only work with discrete values. A natural approach to address this difficulty is to evaluate $X(e^{j\omega})$ every $\Delta\omega$ in frequency. If the values in frequency are close enough together, i.e., $\Delta\omega$ is small enough, it seems reasonable that this should represent all the important information in $X(e^{j\omega})$. This is exactly what the DFT does. However, one must be careful when applying DFT as this approach can yield unexpected consequences.

An important decision in computing the DFT is how many frequencies to use in evaluating $X(e^{j\omega})$? If we use $N$ frequencies, the most straightforward thing to do is to space them evenly in frequency over one period of $X(e^{j\omega})$, say for $0 \leq \omega < 2\pi$. The frequency interval between DFT values is then $\Delta\omega = 2\pi/N$ and the DFT is

$$X[k] = X(e^{j\omega})\big|_{\omega=(2\pi/N)k} \tag{1.21.1}$$

Substituting the definition of the DTFT into Eq. (1.21.1) gives the following

$$X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\Big|_{\omega=(2\pi/N)k} \tag{1.21.2}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j(2\pi/N)kn} \tag{1.21.3}$$

Given the $N$-point DFT values $X[k]$ from Eq. (1.21.3), the inverse DFT equation can be used to compute the time domain signal for those samples

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn} \tag{1.21.4}$$

Note that Eqs. (1.21.4) and (1.21.3) should be familiar, as they are nearly identical to the discrete-time Fourier series. In computing the DFT values $X[k]$ every $\Delta\omega = 2\pi/N$ in frequency, we are effectively sampling $X(e^{j\omega})$. Sampling a signal in one domain means making the signal periodic in the other domain. Whereas sampling $x_c(t)$ in time generates a periodic spectrum in frequency, sampling in frequency generates a periodic signal in time. The only difference between the DFT [i.e., Eqs. (1.20.3) and (1.20.4)] and the DT Fourier series equations is which equation has the (1/N) term in front. For the DT Fourier series, the (1/N) is in the analysis equation, while for the DFT the (1/N) is in the synthesis equation. This discrepancy does not impact the main concepts relating to the time and frequency domains.

The most common application of the DFT is for finite-length signals. It is important to understand this relationship between sampling and periodicity for the case when $x[n]$is an $M$-point finite-length signal, i.e., $x[n] = 0$ for $n < 0$ or $n > M-1$. Specifically, we need to understand when $X[k]$ contains all of the important information in the signal. The process of evaluating $X(e^{j\omega})$ every $\Delta\omega$ in frequency is, in fact, sampling, but in the frequency domain instead of the time domain. Recall that when we sample a continuous-time signal $x_c(t)$ every $T$ seconds in the time domain, we make copies of its Fourier transform every $2\pi/T$ in frequency. In order to preserve all of the information in $x_c(t)$, we must make $T$ small enough so that the copies of the Fourier transform do not overlap, or alias, in frequency. This also requires the assumption that the Fourier transform of $x(t)$ is bandlimited in frequency.

Understanding how the frequency domain sampling in computing the DFT affects the signal requires the mental agility to use duality to its fullest, and exchange time and frequency in the discussion above. Instead of sampling every $T$ seconds, we are sampling every $\Delta\omega$ radians. Instead of sampling in time, causing us to make copies of the CT Fourier transform every $2\pi/T$ in frequency, we are sampling in frequency, causing us to make copies of the signal $x[n]$ every $2\pi/\Delta\omega$ in time. Substituting $\Delta\omega = 2\pi/N$ into this equation shows that the copies of $x[n]$ are placed every $2\pi/(2\pi/N) = N$ samples in time. As expected from duality, sampling in one domain renders the signal periodic in the other. In order for this operation to preserve all the information in the signal, these copies of $x[n]$ must not overlap, which would be aliasing in time. To avoid this time domain aliasing, the signal $x[n]$ must be finite length, and at most $N$ samples long from $0 \leq n \leq N-1$. This means that for an $M$-point signal, $X[k]$ will contain all of the information in the signal so long as the DFT size $N \geq M$. Let's see how this works in some examples.

### ✔ Example 1: The DFT and Time Domain Aliasing

Figure 1.21.1 shows a finite-length discrete-time signal $x[n]$with length $M = 12$. For this example, we look at the time domain effect of computing the DFT with three different sizes. For the first case, we will use $N = 12$ to compute the DFT

$$X_1[k] = X(e^{j\omega})\big|_{\omega=(2\pi/12)k}$$

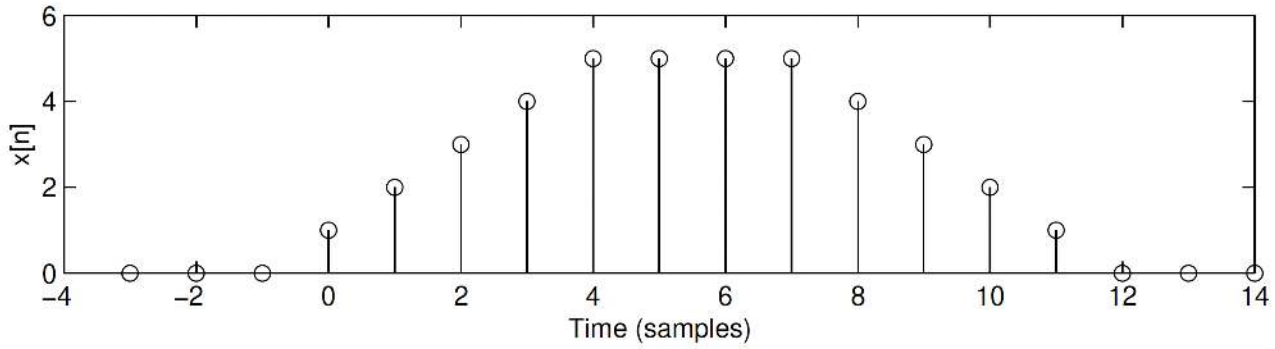



**Figure 1.21.1:** M = 12 point finite-length signal x[n] for Example 1

Substituting the values of $X_1[k]$ into Eq. (1.21.4) gives the periodic time-domain signal $x_1[n]$ generated by the sampling every $2\pi/12$ in frequency. Figure 1.21.2 shows the resulting signal. Comparing Figs. 1.21.1 and 1.21.2 shows that the copies of $x[n]$ do not overlap, and so $X[k]$ contains all of the information in $x[n]$.

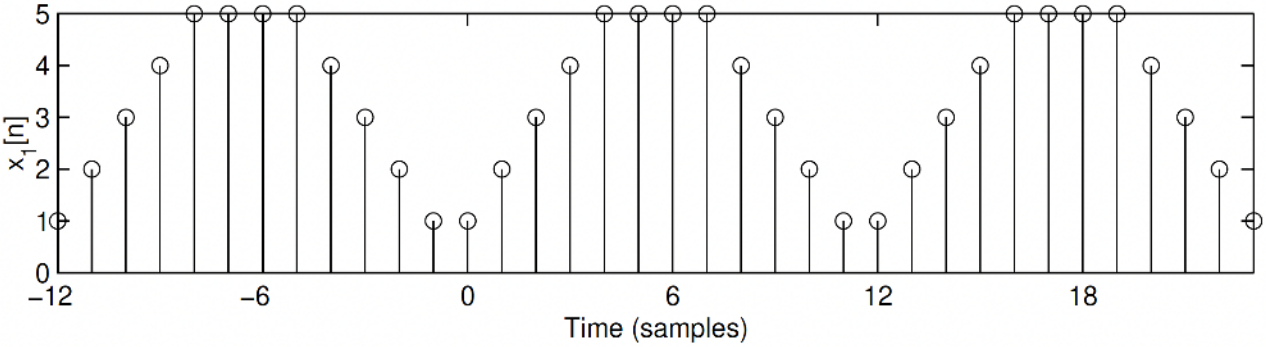



**Figure 1.21.2:** Periodic signal $x_1[n]$ generated by using a N = 12 point DFT.

If we only compute $N = 10$ samples of $X(e^{j\omega})$, we expect to see time-domain aliasing in the signal. If we define

$$X_2[k] = X(e^{j\omega})\big|_{\omega=(2\pi/10)k}$$

,

then use these values in Eq. (1.21.4) to find the inverse DFT, we get a different periodic signal $x_2[n]$, shown in Figure 1.21.3. From this plot, it is clear that the expected aliasing has occurred, information has been lost, and that we cannot recover $x[n]$ from $x_2[n]$.

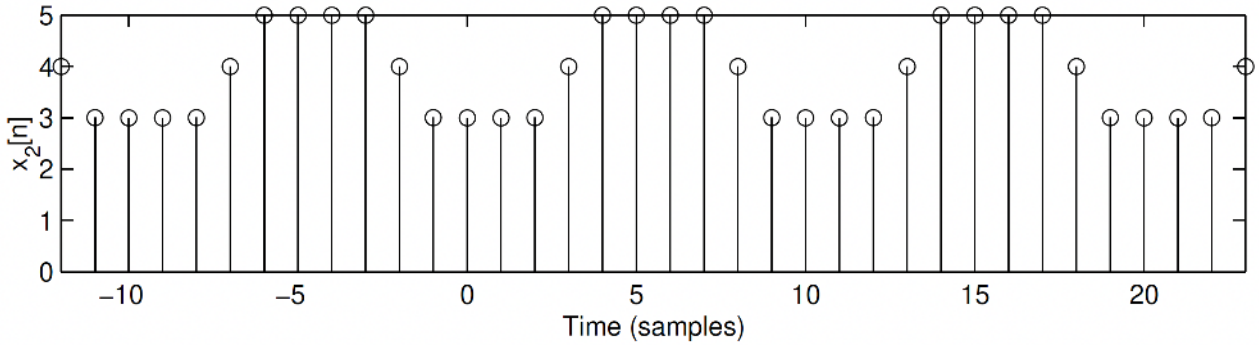



**Figure 1.21.3:** Periodic signal $x_2[n]$ generated by using a N = 10 point DFT.

Finally, a common technique is to compute more DFT samples than are needed. This technique is known as zero padding, since it is equivalent to adding a string of zeros onto the end of $x[n]$, then pretending the signal is longer than it really is. For this example, we can use $N = 15$ to zero pad this signal, defining

$$X_3[k] = X(e^{j\omega})\big|_{\omega=(2\pi/15)k}$$

,

then again using Eq. (1.21.4) to find the resulting periodic time signal $x_3[n]$. As shown in Figure 1.21.4, $x_3[n]$ preserves the original signal $x[n]$, and even has space between the copies of $x[n]$ generated by the frequency domain sampling. In pages 1.15 and 1.16, we saw examples where if we sampled even faster than required by the Nyquist sampling theorem, the spectrum $X(e^{j\omega})$ had gaps with amplitude zero. The $N = 15$ case is the DFT dual of this phenomenom: by taking more samples in frequency than were immediately required, the periodic time signal has gaps between copies.

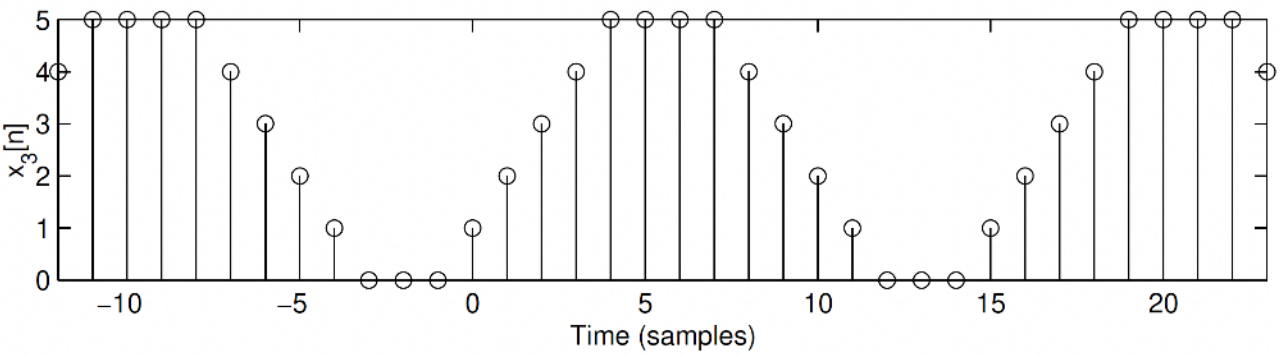



**Figure 1.21.4:** Periodic signal $x_3[n]$ generated by using a N = 15 point DFT.

A common application of the DFT is to examine the spectrum of a finite-length signal. The FFT, or Fast Fourier transform, is an efficient way to compute the DFT values, $X[k]$[1]. The MATLAB command $X = \mathrm{fft}(x, N)$ computes the $N$-point DFT of the finite-length signal $x[n]$. The FFT turns out to be especially efficient when $N$ is a power of 2, i.e., $N = 2^v$ for some integer $v$. Choosing a large value of $N$, or zero padding as we saw in the example above, means that the samples of $X(e^{j\omega})$ in X are very close together, and can be plotted as a good approximation to the DTFT. This oversampling is often important to get a good picture of $X(e^{j\omega})$ - and sometimes a value of $N$ large enough to avoid aliasing will provide a misleading picture of $X(e^{j\omega})$. The following example demonstrates why it is often a good idea to use a large $N$ when using the DFT $X[k]$ to approximate the DTFT $X(e^{j\omega})$.


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## Example 2: Using the DFT to examine the spectrum $X(e^{j\omega})$

Figure 1.21.5 shows a rectangular pulse with length $M = 5$. Applying the time shift property, the DTFT of this signal can be found as

$$X(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}, \text{ for } |\omega| \leq \pi. \tag{1.21.5}$$

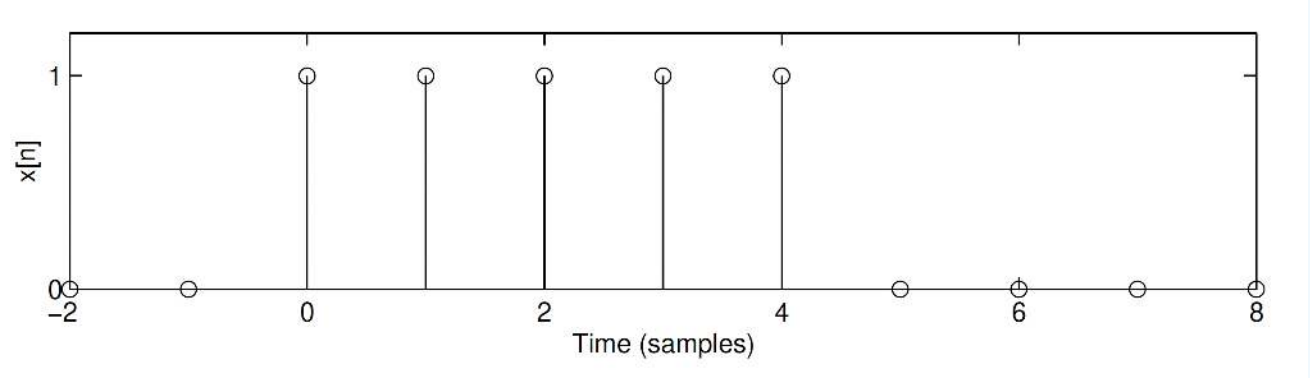



**Figure 1.21.5:** 5-point rectangular pulse $x[n]$ for Example 2.

From our discussion of sampling above, we know that if the DFT has $N \geq 5$, there will be no time domain aliasing, and the samples $X[k]$ will preserve all of the information in $x[n]$. However, if $N = 5$, the values of $X[k]$ may not give a good sense of the underlying DT sinc for $X(e^{j\omega})$. Figure 1.21.6 shows the values obtained for $X[k]$ when $N = 5$ in Eq. (1.21.3).

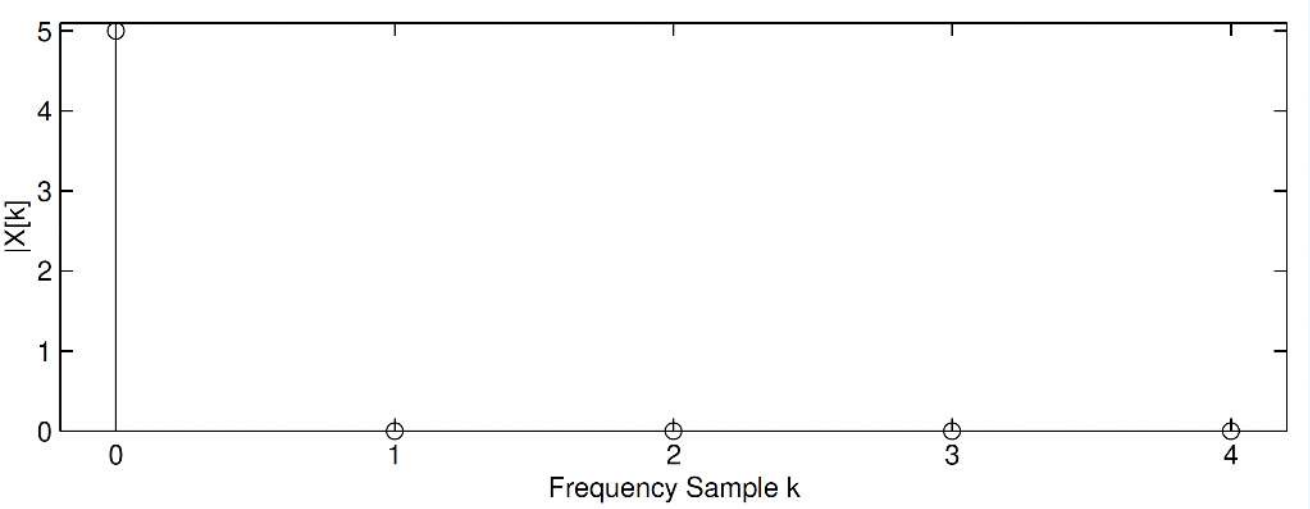



**Figure 1.21.6:** $N = 5$ point DFT for the signal in Fig. 1.21.5.

At first glance, we might think this signal has an impulse in its spectrum at $k = 0$, and is zero at all other frequencies. This would be incorrect, though. When $N = 5$, the DFT samples $X(e^{j\omega})$ right on the zero-crossings of $\sin(5\omega/2)/\sin(\omega/2)$. These crossing occur when $\sin(5\omega/2) = 0$, which is when $5\omega/2 = m\pi$ for a non-zero integer $m$. This gives us zeros in $X(e^{j\omega})$ for $\omega = 2\pi m/5$. (When $\omega = 0$, both numerator and denominator are zero, and we must use L'Hopital's rule to figure out that $X(e^{j\omega})|_{\omega=0} = 5$.) Figure 1.21.7 plots the DFT samples $X[k]$ overlaid with the DTFT $X(e^{j\omega})$, demonstrating how this happens. If we keep in mind the periodic copies of $x[n]$ caused by the sampling, this result also makes sense. If we repeat $x[n]$ every 5 samples, the resulting periodic signal will be a constant, which will have only one nonzero discrete-time Fourier series coefficient, at $k = 0$.

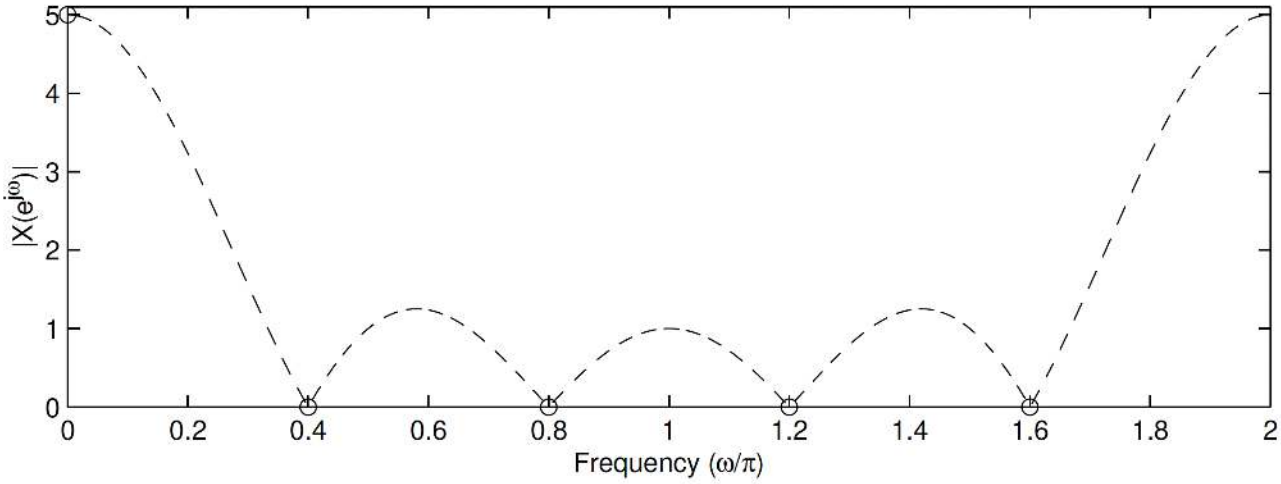



**Figure 1.21.7:** DTFT $X(e^{j\omega})$ with DFT samples $X[k]$ for $N = 5$ overlaid.

If we want a more informative picture of $X(e^{j\omega})$, we need to zero-pad $x[n]$ and compute more samples in frequency. If we increase $N$ to 20, and define those signals as

$$X_2[k] = X(e^{j\omega})\big|_{\omega=(2\pi/20)k}$$

we obtain the samples shown in Figure 1.21.8, again with $X(e^{j\omega})$ overlaid. This plot gives a better sense of the underlying DT sinc function $X(e^{j\omega})$. Making the DFT size $N$ even larger will give still better pictures of $X(e^{j\omega})$.

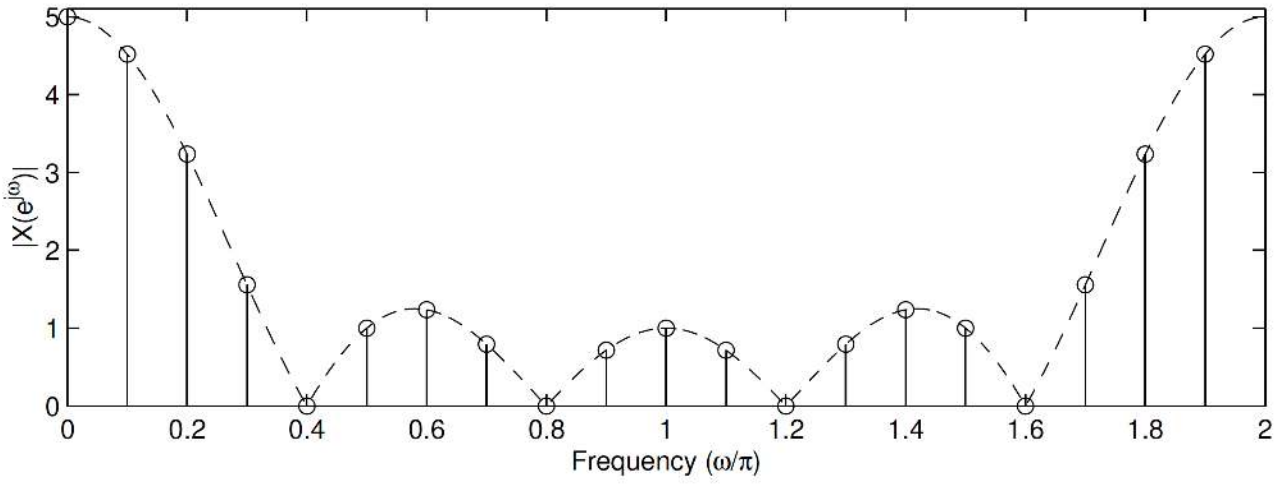



**Figure 1.21.8:** DTFT $X(e^{j\omega})$ with DFT samples $X_2[k]$ for $N = 20$ overlaid.

In practice there are relatively few truly periodic DT signals of interest. However, numerous studies examine finite-length signals. In fact, every practical experiment or measurement produces a finite-length signal, since we can only record and sample for a finite amount of time. The underlying strategy in all examples using DFT to process finite-length signals is to model the signal as one period of a periodic signal with period $N$. So long as $N$ is larger than the length of the periodic signal, we can always recover the finite-length signal by discarding all of the periods except $0 \le n \le N-1$. A good approach to understand what is happening to the signals in the time domain is to follow the three-step procedure below:

**Step 1:** When computing the $N$-point DFT of a finite-length signal $x[n]$, make a periodic signal $\tilde{x}[n]$ by adding copies of $x[n]$ every $N$ samples in time.

**Step 2:** Modify $\tilde{x}[n]$ at each step of the problem or algorithm as appropriate for the operations given.

**Step 3:** As the final step, truncate the $\tilde{x}[n]$ to obtain the new finite-length signal that results from the inverse DFT.

If we use these steps to remind us of the implied periodicity while working with the DFT, it helps us to explain some strange and unanticipated results that sometimes occur.

### Example 3: Convolving signals using the DFT

This example compares two approaches to computing the convolution. The first approach is the standard use of the convolution sum. For an $M$-point input $x[n]$ and $P$-point impulse response $h[n]$, the output $y_{lin}[n] = x[n] * h[n]$ can be computed using the convolution sum as

$$y_{lin}[n] = \sum_{\ell=0}^{N-1} x[\ell]h[n-\ell]$$

From our knowledge of convolution, we know that the output $y[n]$ is at most $M + P - 1$ points long.

The other approach to computing the convolution is to compute the $N$-point DFT $X[k]$, define $Y[k] = H[k]X[k]$, then compute the inverse DFT $y[n]$. The essential question is how large $N$ must be such that the DFT-based procedure gives the same result as $y_{lin}[n]$. Multiplying two DFTs implements convolution in the time domain, but it is a form of convolution called **circular convolution**, which convolves the periodic versions of the signals obtained by sampling the frequency domain to obtain the DFT. Because the convolution $y[n] = x[n] * h[n]$ makes an output which can be longer than either input, if we make a poor choice of $N$, it is possible to alias the output $y[n]$ even if neither $x[n]$ or $h[n]$ is aliased. To get the circular convolution result to match the standard convolution result, the DFT size must be larger than the largest possible length for $y[n]$, so $N \geq M + P - 1$.

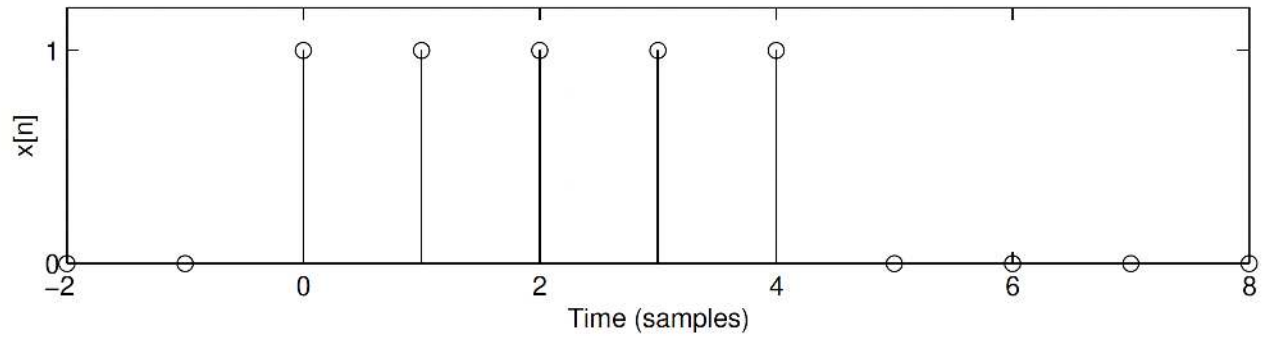



**Figure 1.21.9:** 5-point rectangular pulse $x[n]$ for Example 3.

To illustrate these issues, we look at the convolution of a rectangular pulse $x[n]$ (shown in Figure 1.21.9) with itself, so $y_{lin}[n] = x[n] * x[n]$. Because $x[n]$ is a rectangular pulse 5 points long, the convolution sum $y_{lin}[n] = x[n] * x[n]$ will be a triangle which is $5 + 5 - 1 = 9$ points long, and with a peak height of 5. Figure 1.21.10 shows the signal obtained using MATLAB $y_{lin} = \text{conv}(x, x)$ command to implement the convolution sum.

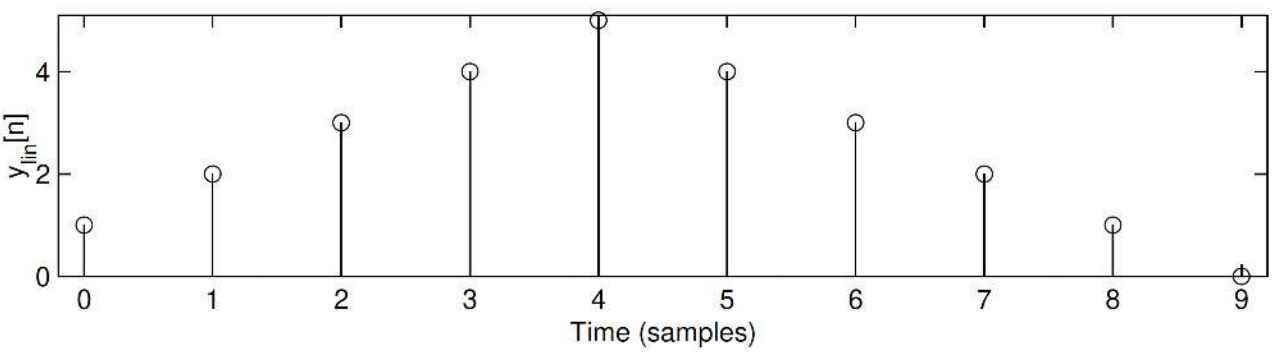



**Figure 1.21.10:** Triangle output signal $y_{lin}[n]$ for Example 3.

Now let us compute the time-domain convolution by multiplying DFTs. If we only take the length of $x[n]$ into account when choosing the DFT size $N$, and do not consider the length of the output $y[n]$, we would choose $N = 5$. Figure 1.21.11 shows the aliased output that results. In fact, the triangles in $y_{lin}[n]$ have overlapped in the periodic copies so that they add up to a constant for all n. Clearly, the output $y[n]$ in Figure 1.21.11 and the output $y_{lin}[n]$ in Figure 1.21.10 are not the same.

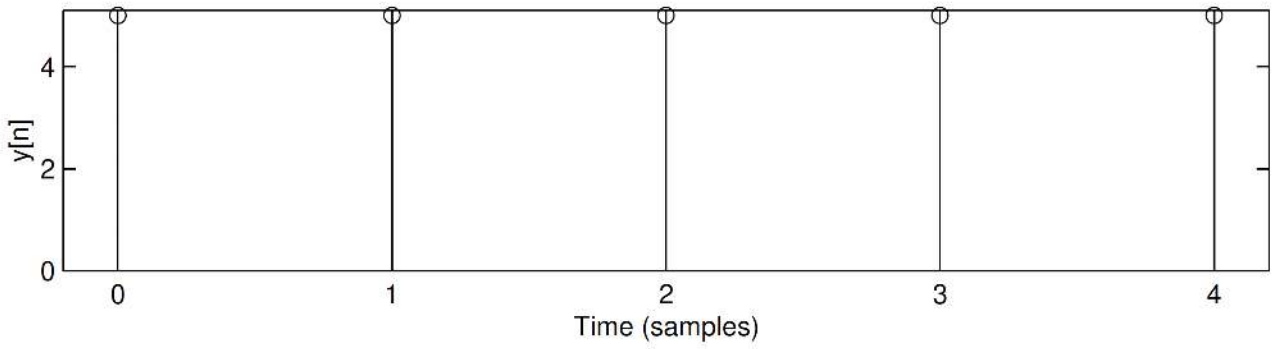



**Figure 1.21.11:** Aliased output signal $y[n]$ caused by DFT size $N = 5$ in Example 3.

If we increase the DFT size to $N = 8$, the aliasing is reduced because $y_{lin}[n]$ is 9 points long, there is still one point of aliasing, where the last point of the triangle aliases on top of the first point. Figure 1.21.12 illustrates the output $y[n]$ resulting for this DFT size. The signal in Figure 1.21.13 is closer to $y_{lin}[n]$, but is missing the point at $n = 8$ and has the wrong value at $n = 0$ due to aliasing.

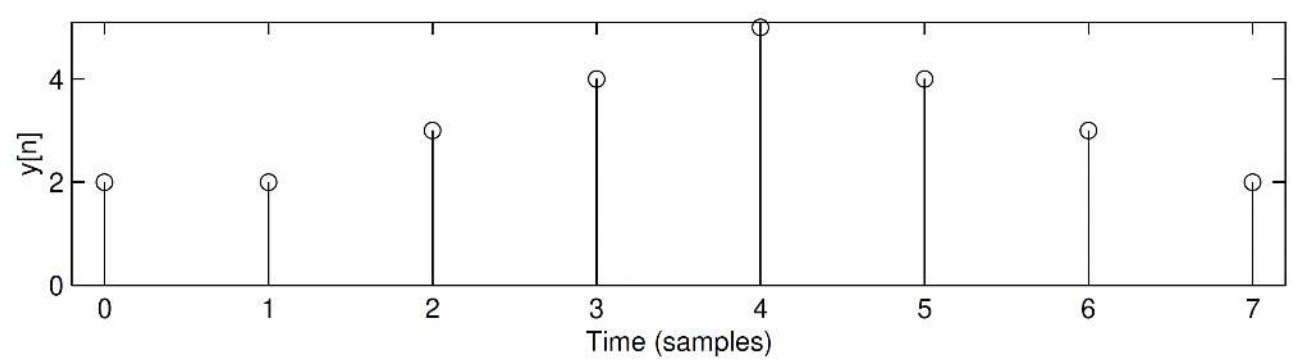



**Figure 1.21.12:** Aliased output signal $y[n]$ caused by DFT size $N = 8$ in Example 3.

Finally, if we increase the DFT size to $N = 10$, the result of computing the convolution with the DFT is the same as with the convolution sum. The DFT size is larger than the length of $y_{lin}[n]$, so the output is no longer aliased. Figure 1.21.13 plots the signal that results for this case. This matches $y_{lin}[n]$ in Figure 1.21.10 exactly. This example illustrated the most important point when using the DFT to implement time-domain convolution. The DFT size must be at least as big as the longest possible output signal $y[n]$.


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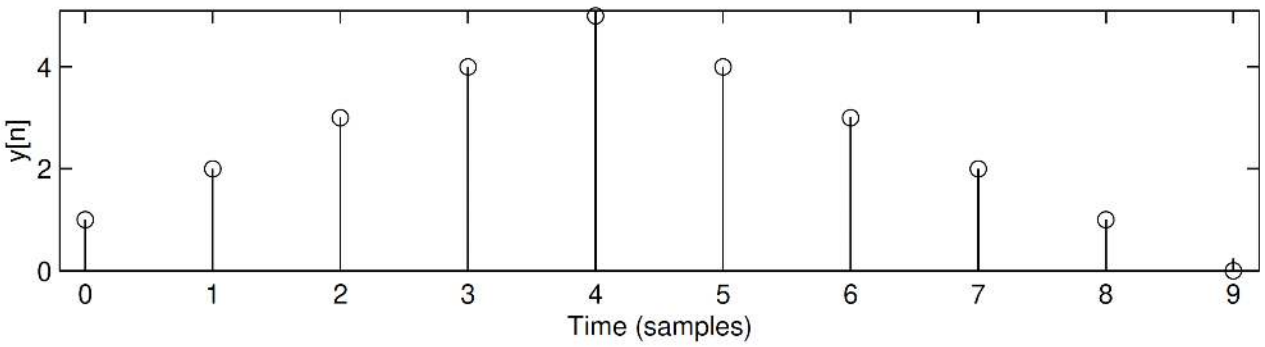



**Figure 1.21.13:** Unaliased output signal $y[n]$ produced by DFT size $N = 10$ in Example 3.

The circular convolution between two finite-length signals is found by:

**Step 1:** Find the linear convolution $y_{linear}[n] = x[n] * h[n]$ using ticker-tape method or similar.

**Step 2:** Shift the linear convolution to $y_{linear}[n - N]$ and $y_{linear}[n + N]$ where $N$ is the number of samples in frequency domain

**Step 3:** The circular convolution is equal to $y_{circular}[n] = y_{linear}[n + N] + y_{linear}[n] + y_{linear}[n - N]$ within $n = 0$ to $n = N - 1$

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# CHAPTER OVERVIEW

## 2: Continuous-Time Linear Systems

### Course Overview

This course introduces students to continuous-time signal analysis and linear systems. Topics include classification of signals and systems, basic signal manipulation, system properties, time domain analysis of continuous-time linear time-invariant (LTI) systems, Laplace transform and its use in LTI system analysis, transfer functions and feedback, frequency response and analog filters, Fourier series representation and properties, continuous-time Fourier transform, spectral analysis and AM modulation, and simulation. Students learn to use signal analysis tools.

### Course Learning Objectives

The objectives are to analyze existing continuous-time linear systems and to modify or design new ones to satisfy specific requirements. By the end of this course, students should be able to:

1. Classify signals and systems based upon their properties and, in particular, understand the implications when a system is linear, time-invariant, stable, and causal.
2. Graph mathematical descriptions of continuous-time signals and write mathematical descriptions for graphed functions.
3. Develop an accurate mathematical description for a physical linear system, realize a mathematical description in a block diagram, and analyze a block diagram realization to determine the mathematical description.
4. Represent an LTI system as an impulse response and find the response of that system to an arbitrary input using convolution.
5. Compute the Fourier series components for periodic signals and sketch their amplitude and phase spectra.
6. Recognize the various properties of transforms (e.g., symmetry, duality, time shift, frequency shift) and apply them appropriately to find transforms.
7. Use the spectrum of an input signal and the frequency response of a system to determine the spectrum of the output signal.
8. Use Laplace transforms to determine continuous-time system transfer functions; solve for a response given the input, system description, and initial conditions; and answer questions related to system stability.
9. Compute and use the continuous-time FT to analyze system performance.
10. Understand the relationships among the ordinary differential equation, impulse response, frequency response, and transfer function descriptions of a system.
11. Use computer tools, such as MATLAB, to simulate and verify analytically obtained solutions, and to analyze continuous-time signals and systems.


2.1: Continuous-Time (CT) Signals
2.2: Continuous-Time (CT) Systems
2.3: Convolution
2.4: LTI Systems Properties
2.5: Differential Equations and Block Diagrams
2.6: Eigenfunctions and CT Fourier Series
2.7: Fiiltering with the CTFS
2.8: Differential Equations and Convergence
2.9: CT Fourier Transform
2.10: CT FT Properties
2.11: Modulation and Differential Equation
2.12: Frequency Response and Filtering
2.13: Bode Plots - First-order systems
2.14: Bode Plots - Second-order system
2.15: Amplitude Modulation

2.16: Frequency Modulation
2.17: Time-dependent Fourier Transform
2.18: Laplace Transform
2.19: Laplace Transform Properties
2.20: System Functions and Block Diagrams
2.21: Feedback

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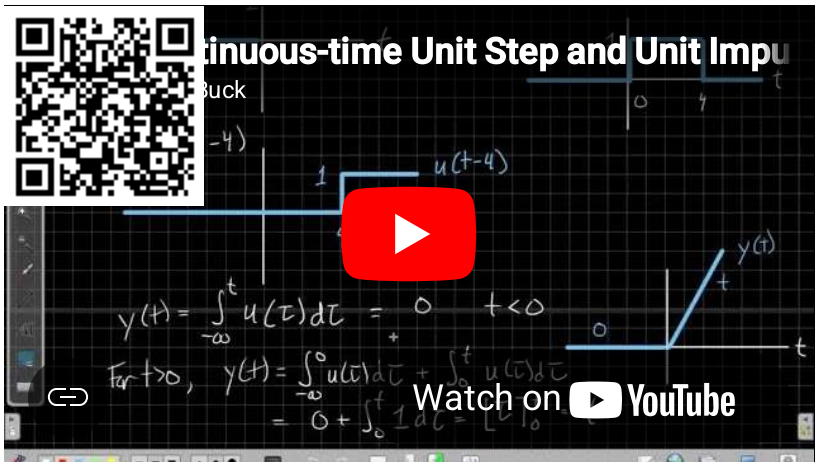
# 2.1: Continuous-Time (CT) Signals

## Learning Objectives: CT Signal Basics

1. Graph simple transforms of CT signals
2. Identify periodic CT signals and compute their periods.
3. Match analytic expressions and graphs for basic signals.

## YouTube Videos

Continuous-time Unit Step and Unit Impulse




Continuous-time Sinusoidal and Exponential Signals



## Summary

Signals used to describe physical phenomena → pattern of variations of some form

Signals $=$ $f(x,t)$ functions of one or more independent variables $(x,t)$ → CT signals => 1D function (for simplification) $f(t)$

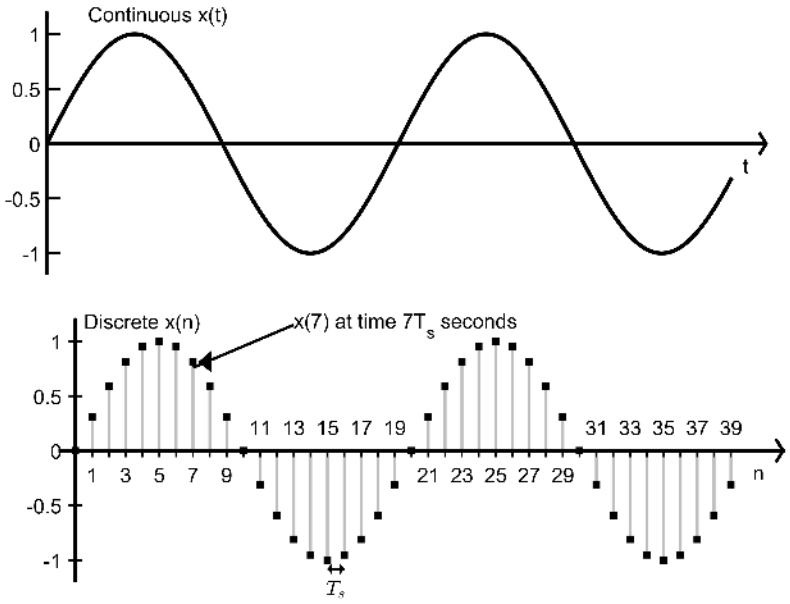



**Figure 2.1.1:** Illustration of the difference between continuous (top panel) and discrete (bottom panel) signals.

| | |
|---|---|
| Continuous-time signal: $x(t)$<br>Continuous range of values: $t$<br>$x(t)$ exists for all values of $t$. | Discrete-time signals: $x[n]$<br>$x[t]$ only exists at discrete values of $n$<br>$n \in$ integer values |

## Elemental Signal Transformations

**Time shift:** \(x(t-t_0)\) represents a delayed ($t_0 > 0$) or advanced ($t_0 < 0$) version of $x(t)$

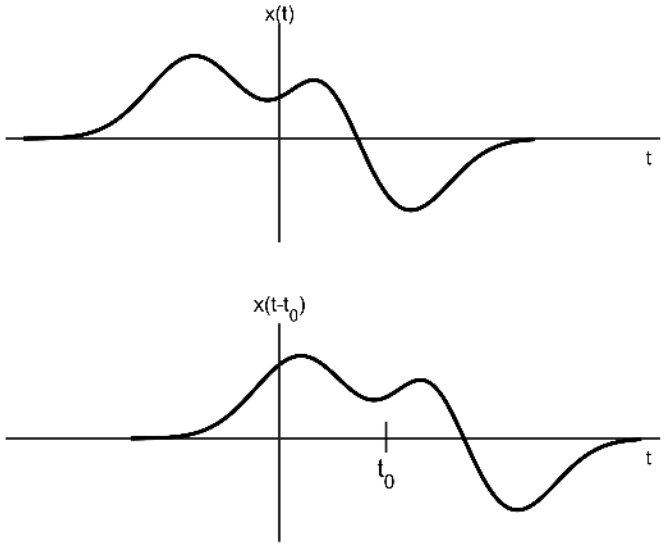



**Figure 2.1.2:** Example of a signal being shifted in time by $t_0$ samples.

**Time reversal:** \(x(t)\rightarrow x(-t)\) reflection of $x(t)$ about $t = 0$ (i.e., mirror reflection around the vertical axis)

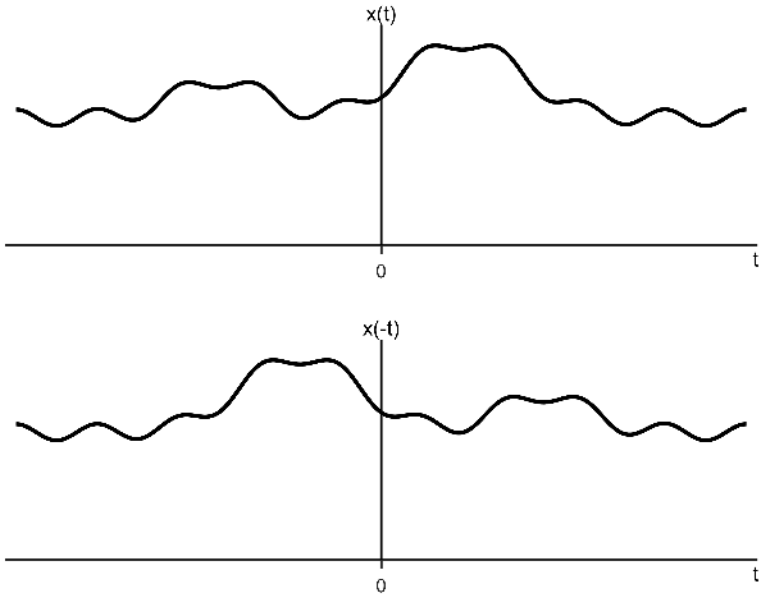



**Figure 2.1.3:** Illustration of the time reversal property in a continuous signal.

**General transformation:** \(x(t)\longrightarrow x(\alpha t+\beta)\) where $\alpha$ also controls time reversal if $\alpha < 0$ and $\beta$ controls time delay

$\alpha$ controls the stretching of the signal. If $|\alpha| > 1$ → the shape of $x(\alpha t)$ is reduced. If $|\alpha| < 1$, the shape of $x(\alpha t)$ is expanded

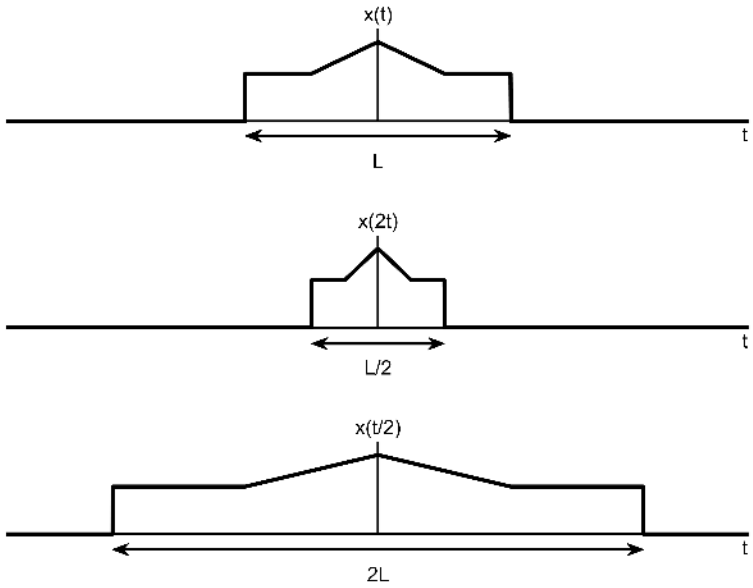



**Figure 2.1.4:** Time compression and expansion of a continuous signal: $x(t)$, its compressed version $x(2t)$, and its expanded version $x(t/2)$.

## Periodic Signals

**Periodic signals** $x(t) = x(t+T)$ for all values of $t$

$x(t)$ is periodic if its values are the same by a time shift of $T$

$T$ is the minimum positive value that meets $x(t) = x(t+T)$

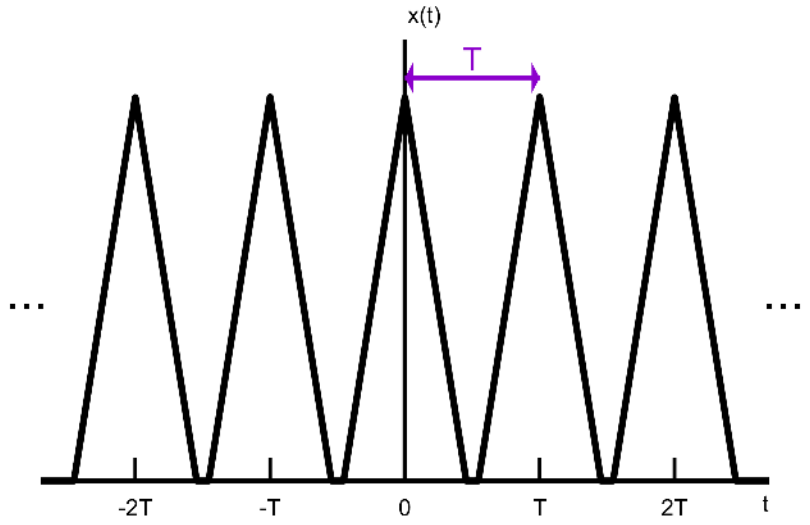



**Figure 2.1.5:** Illustration of a periodic continuous signal with fundamental period = T.

if $x(t)$ is not periodic, thus $x(t)$ is aperiodic

Example

$$x(t) = \begin{cases} \cos(\frac{t}{2}) & \text{if } t < 0 \\ \sin(\frac{t}{2}) & \text{if } t \geq 0 \end{cases} \tag{2.1.1}$$

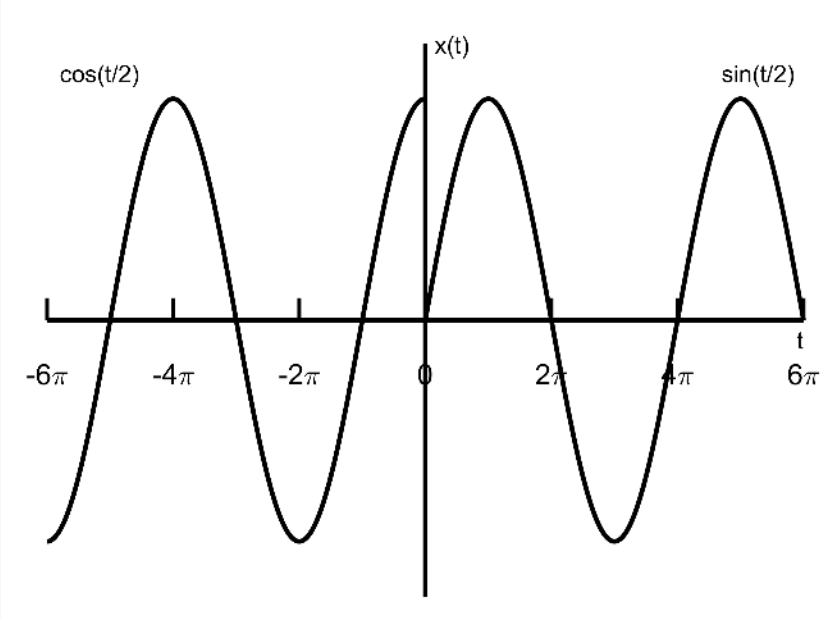



**Figure 2.1.6:** Illustration of an aperiodic signal.

$\cos(t) = \cos(t + 2\pi)$ and $\sin(t) = \sin(t + 2\pi)$, so each part of $x(t)$ is periodic. However, $x(t)$ does not repeat itself because there is a discontinuity at $t = 0$, which does not occur at any other time. $x(t)$ is periodic if and only if each feature in the shape of $x(t)$ is recurrent/repeated → $x(t)$ is aperiodic

Even vs. Odd Signals

**odd**: $x(t) = -x(-t)$ reflection around the vertical axis between $t > 0$


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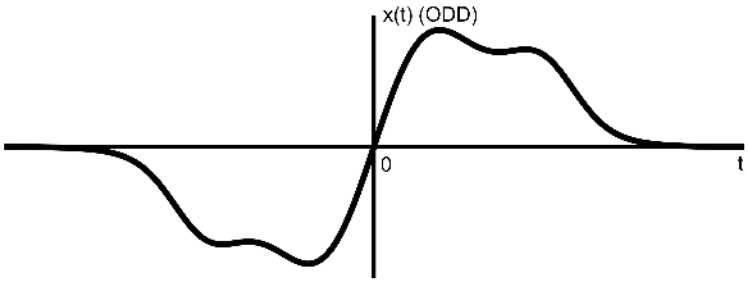



**Figure 2.1.7:** Illustration of an odd continuous signal.

**even** $x(t) = x(-t)$ same signal after time reversal

In general: $x(t) = x_e(t) + x_o(t)$ Any signal can be decomposed by an even and odd signals

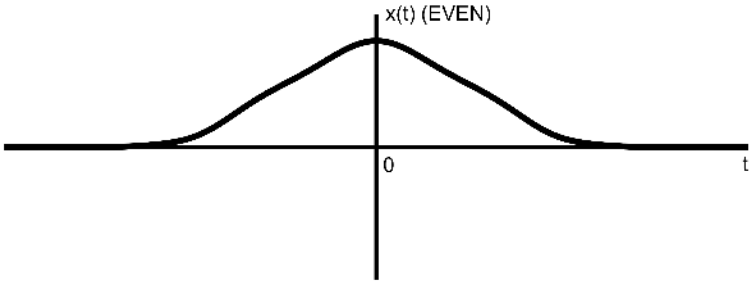



**Figure 2.1.8:** Illustration of an even continuous signal.

## Exponential Signal

**Exponential signal** $x(t) = Ce^{\alpha t}$ being $C$ and $\alpha$ are complex numbers

$C$ and $\alpha$ are real values

$\alpha > 0$ $x(t) \uparrow$ when $t \uparrow$

$\alpha < 0$ $x(t) \downarrow$ when $t \uparrow$

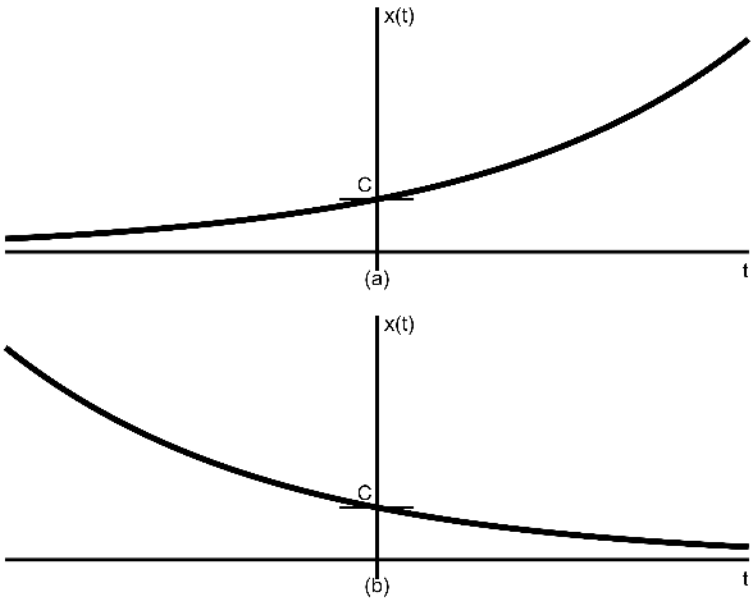


**Figure 2.1.9:** Illustration of continuous exponentials with (a) $\alpha > 0$ and (b) $\alpha < 0$

$x(t) = C$ if $\alpha = 0$ constant

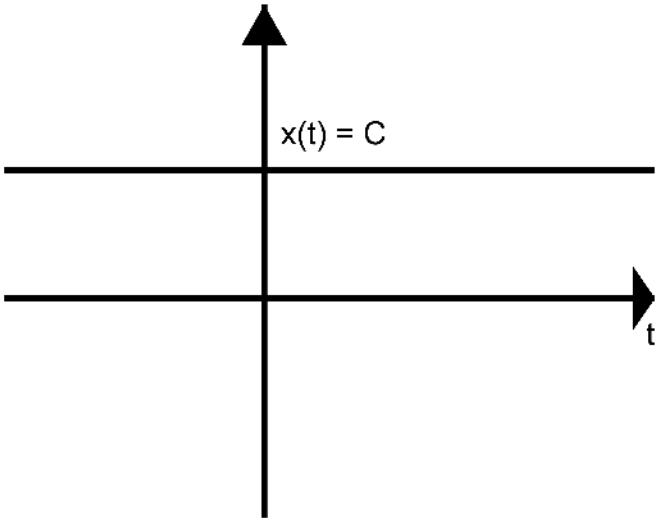



**Figure 2.1.10:** Illustration of a constant signal (i.e., exponential where $\alpha = 0$)

$x(t) = Ce^{j\omega t}$ $\alpha = j\omega$ is a pure imaginary value

$$x(t+T) = Ce^{j\omega(t+T)} = Ce^{j\omega t}\, e^{j\omega T} \tag{2.1.2}$$

If $e^{j\omega T} = 1$, then x(t) is periodic, i.e., $e^{j\omega T} = e^{j2\pi k}$ where k is any integer (i.e, k = 0,1,2,3,...). Since $e^{j\omega T} = e^{j2\pi k} \Rightarrow \omega T = 2\pi k \Rightarrow$ $\boxed{T = \frac{2\pi k}{\omega}}$

General Complex Exponential

$x(t) = Ce^{\alpha t}$

$C = |C|\, e^{j\theta}$

$\alpha = r + j\omega$

$$x(t) = |C|\, e^{j\theta}\, e^{rt}\, e^{j\omega t} = |C|\, e^{rt}\, e^{j(\omega t+\theta)} \tag{2.1.3}$$

*Use Euler properties*

$$= |C|\, e^{rt} \cos(\omega t + \theta) + j\,|C|\, e^{rt} \sin(\omega t + \theta) \tag{2.1.4}$$

*magnitude of the complex exponential:* \(|C|e^{rt}\)

For $r = 0 \rightarrow x(t) = |C| \cos(\omega t + \theta) + j\,|C| \sin(\omega t + \theta)$ real and imaginary part of $x(t)$ are sinusoidal signals.

For $r > 0 \rightarrow$ the amplitude of the real and imaginary sinusoidal signals increase with the time following an exponential, $e^{rt}$.

For $r < 0 \rightarrow$ the amplitude of the real and imaginary sinusoids decrease with the time, $e^{-rt}$.

Continuous-Time Unit Step

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{2.1.5}$$

Remember that the unit step function is discontinuous at $t = 0$

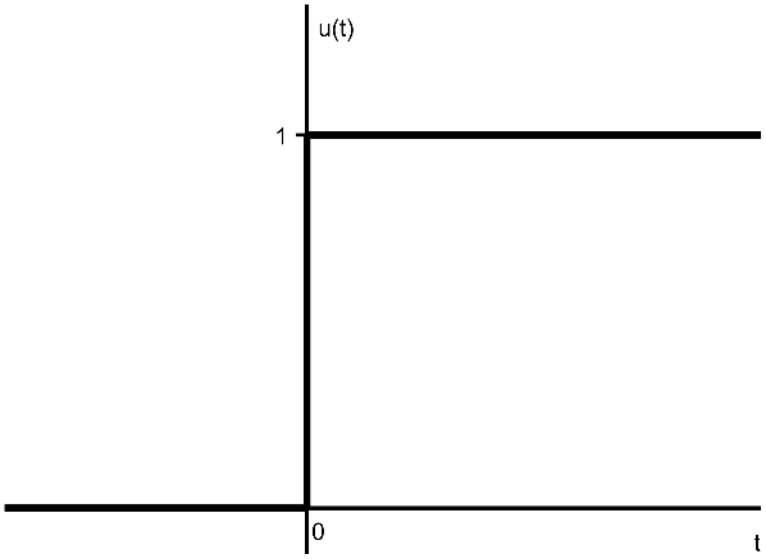



**Figure 2.1.11:** Illustration of a Continuous-Time Unit Step signal.

Continuous Unit Impulse

$$\delta(t) = \frac{du(t)}{dt} = \begin{cases} 1 & t = 0 \\ 0 & \text{elsewhere} \end{cases} \tag{2.1.6}$$

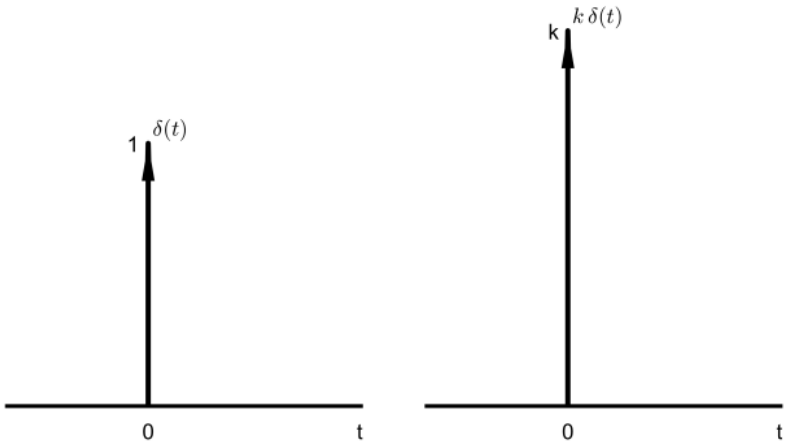



**Figure 2.1.12:** Illustration of Continuous-Time Unit Impulses.

**Properties of the unit impulse**

$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$

$x(t) \cdot \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$

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# 2.2: Continuous-Time (CT) Systems

## Learning Objectives: CT System Basics

1. Convert verbal descriptions of systems into equations
2. Recognize the four major properties for systems: linear, time invariant, causal and stable
3. Practice proving systems are linear and time-invariant

## YouTube Videos

Continuous-Time System Fundamentals



CT System Properties: Time-Invariance



CT System Properties: Linearity




CT System Properties: Causality



## System Properties: Stability



## Summary

**Continuous-Time Systems:** Systems = processes that transform input signals into new outputs

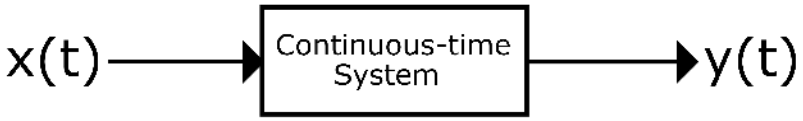



**Figure 2.2.1:** Basic flow diagram of a Continuous-Time system.

## Interconnections of Systems

Many real systems are composed of interconnected subsystems. It is important to understand how the subsystems are interconnected for analyzing the operation and behaviour of the overall system

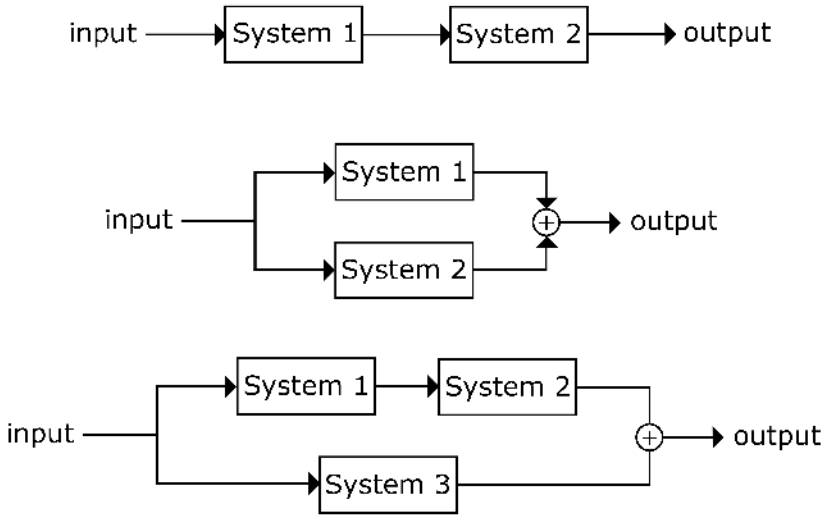



**Figure 2.2.2:** Basic flow diagrams of cascade (top), parallel (middle), and a combination of cascade and parallel (bottom) systems.

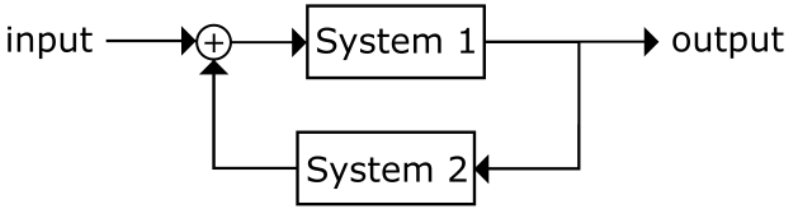



**Figure 2.2.3:** Basic flow diagram of a feedback system.

## Basic System Properties

**Systems with and without memory**

A system is memoryless if its output for each value of the independent variable at a given time only depends on the input at that same time.

Example: $y(t) = Rx(t)$
Identity system: $y(t) = x(t)$

A system has memory if it retains information about input values at times other than the current time.

**Example**

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau \tag{2.2.1}$$

Memory is associated with the storage of energy

The concept of memory of a system refers to (1) storing past input and output values, as well as (2) the current output depends on future input and output values.

**Invertibility and Inverse Systems**

A system is invertible if distinct inputs lead to distinct output

When a system is invertible, then there is an inverted system that converts the output of the system to the original input.

$x(t) \xrightarrow{S} y(t) = 2x(t) \xrightarrow{S^{-1}} w(t) = \frac{1}{2}y(t) = x(t)$

$x(t) \xrightarrow{S} y(t) = x^2(t) \xrightarrow{S^{-1}} w(t) = \pm\sqrt{y(t)} = \pm x(t)$

The second system is not invertible because we cannot determine the sign of the input based on output $y(t)$

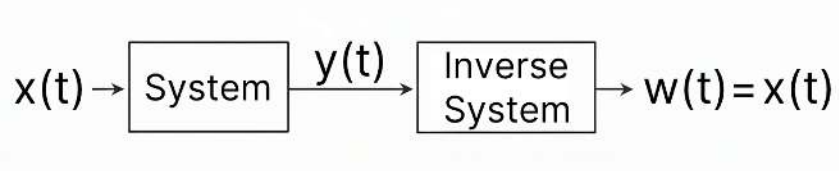



**Figure 2.2.4:** Basic flow diagram of an inverse system being applied to a system.

**Causality**

A system is causal if the output at any time depends only on values of the input at the present time and in the past. A causal system is called nonanticipative if its output does not anticipate future input values.

| $Causal$ | $Non-causal$ |
|---|---|
| $x(t) \rightarrow y(t) = 2x(t)$<br>$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$ | $y(t) = x(t+1)$<br>The output @ $t$ depends on a future value of $x$. |

All memoryless systems are causal since the output responds only to the current value of the input

**Example**

$$y(t) = x(t)\cos(t+1) = x(t)\, g(t) \tag{2.2.2}$$

the current value of input $x(t)$ influences the current value of the output $y(t)$ → system is casual.

**Stability**

A system is stable if inputs lead to outputs that do not diverge. In other words, the system's response cannot diverge

**Time Invariance**

A system is time-invariant if the behavior and characteristics of the system are fixed over time → same output values are obtained if we measure the response today or tomorrow.

A system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

$$x(t) \xrightarrow{S} y(t)$$

$$x(t-t_0) \longrightarrow y(t-t_0)$$

**Example**

$y(t) = \sin[x(t)]$ system time invariant?

$x_1(t) \longrightarrow y_1(t) = \sin[x_1(t)]$

$x_2(t) = x_1(t-t_0) \longrightarrow y_2(t) = \sin[x_2(t)] = \sin[x_1(t-t_0)]$

$y(t-t_0) = \sin[x_1(t-t_0)]$

Both eqs. are equal → so system is TI

**Linearity**

A system is linear if it possesses the property of scaling and superposition

Consider an input consisting of the weighted sum of several inputs, i.e., $x(t) = ax_1(t) + bx_2(t)$. A system is linear if the output is the superposition (weighted sum) of the individual responses of each signal

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$
$$a\,x_1(t) + b\,x_2(t) \to a\,y_1(t) + b\,y_2(t)$$

being $a, b$ complex constant numbers

**Example**

$y(t) = t\,x(t)$

$x_1(t) \to y_1(t) = t\,x_1(t)$
$x_2(t) \to y_2(t) = t\,x_2(t)$

$x_3(t) = a\,x_1(t) + b\,x_2(t) \to y_3(t) = t\,x_3(t)$

$$y_3(t) = t\big(a\,x_1(t) + b\,x_2(t)\big) = t\,a\,x_1(t) + t\,b\,x_2(t) = a\,y_1(t) + b\,y_2(t) \tag{2.2.3}$$

The system passes both scaling and superposition, thus the system is linear.

A system is linear only and only if the system satisfies both the additivity and homogeneity properties, and the signal and constants are allowed to be complex.

**Example**

$y(t) = x^2(t)$ Is the system linear?

$x_1(t) \xrightarrow{S} y_1(t) = x_1^2(t)$
$x_2(t) \xrightarrow{S} y_2(t) = x_2^2(t)$

$x_3(t) = a\,x_1(t) + b\,x_2(t) \xrightarrow{S} y_3(t) = x_3^2(t)$

$$y_3(t) = \big(a\,x_1(t) + b\,x_2(t)\big)^2 = a^2x_1^2(t) + 2ab\,x_1(t)x_2(t) + b^2x_2^2(t) \tag{2.2.4}$$

$$\neq a\,y_1(t) + b\,y_2(t) = a\,x_1^2(t) + b\,x_2^2(t) \tag{2.2.5}$$

The system does not pass both scaling and superposition; thus, the system is not linear.

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# 2.3: Convolution

## Learning Objectives: CT Convolution

1. Understand how the convolution integral is a consequence of the linearity and time-invariance properties
2. Interpret the convolution integral as three steps: flip and shift, then multiply with $x(\tau)$ to form a product signal, then compute the area under the product signal
3. Practice identifying the different regions for the convolution integral expression.

## YouTube Videos

CT Convolution Integral



CT Convolution Example



(Optional) CT Convolution with Barker Code Sequences. *Credit to Prof. Kathleen E. Wage*



## Summary

Any arbitrary continuous-time signal can be represented as idealized pulses with vanishingly small duration. Let's consider $\hat{x}(t)$ the pulse or staircase approximation of the continuous-time signal $x(t)$, see Figure 2.3.1.

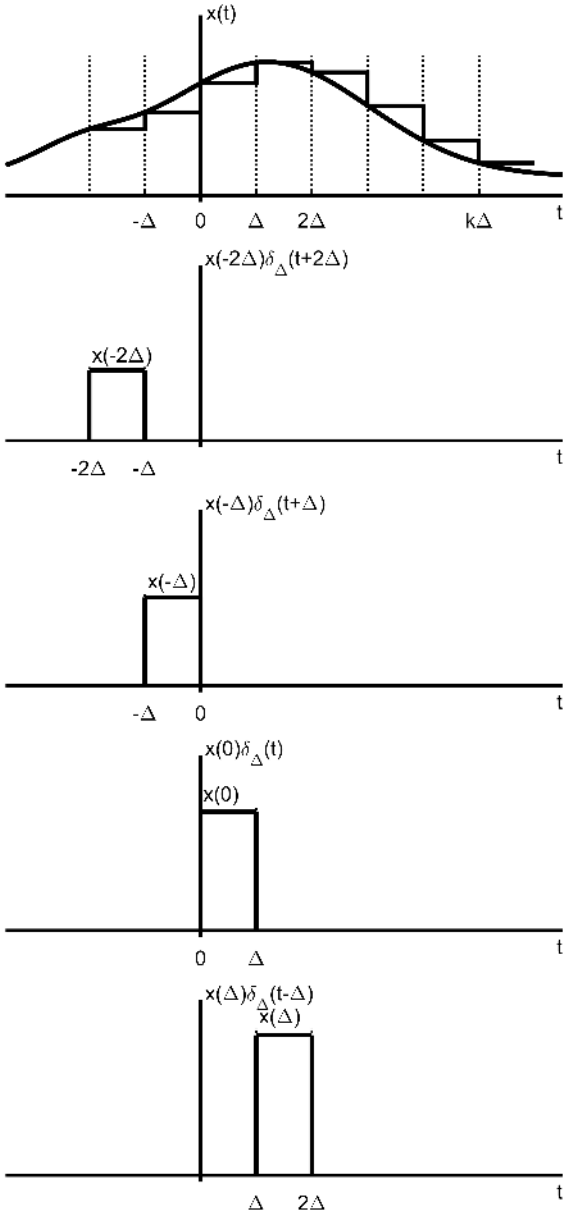



**Figure 2.3.1:** Illustration of the compensation of an arbitrary continuous signal following the staircase approximation.

Thus

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_{\Delta}(t - k\Delta)\,\Delta \tag{2.3.1}$$

where

$$\delta_{\Delta}(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{otherwise} \end{cases} \tag{2.3.2}$$

If $\Delta \to 0$,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t - \tau)\,d\tau \tag{2.3.3}$$

*shifting property of a CT impulse*

**Example:**

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\,\delta(t-\tau)\,d\tau \tag{2.3.4}$$

$$u(\tau) = 0 \text{ for } \tau < 0$$
$$u(\tau) = 1 \text{ for } \tau \geq 0$$

Note that

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} x(t)\,\delta(t-\tau)\,d\tau = x(t)\int_{-\infty}^{+\infty} \delta(t-\tau)\,d\tau = x(t) \tag{2.3.5}$$

$$x(\tau)\,\delta(t-\tau) = x(t)\,\delta(t-\tau)$$

The response/output $\hat{y}(t)$ of a linear system is the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\,\hat{h}_{k\Delta}(t)\,\Delta \tag{2.3.6}$$

when $\Delta \to 0$ and the summation becomes an integral

**CONVOLUTION INTEGRAL**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h_\tau(t)\,d\tau \tag{2.3.7}$$

**Response of a linear-time invariant system in continuous time**

$h_\tau(t) = h_0(t-\tau)$ response of the LTI system to a shifted unit impulse $\delta(t-\tau)$

Unit response $h(t) = h_0(t)$ "centered" unit impulse

$$y(t) = x(t) * h(t) \tag{2.3.8}$$

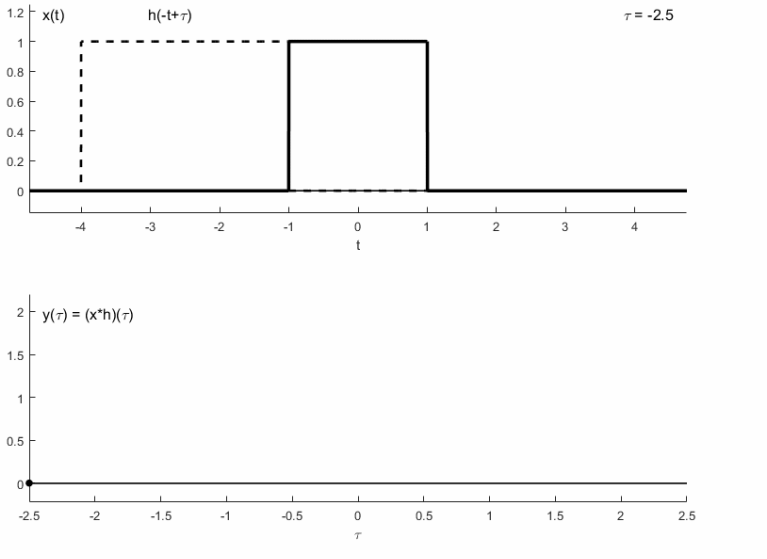
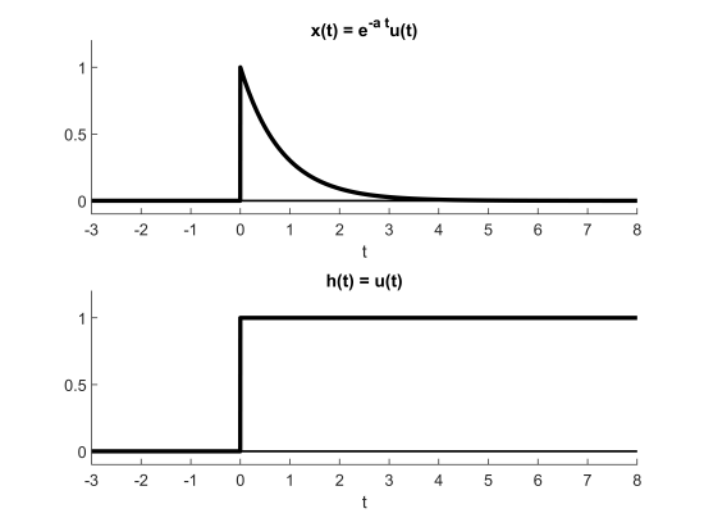



**Figure 2.3.2:** Animation showing the convolution between two rectangular functions.

A continuous-time LTI system is completely characterized by its unit response (i.e., the system's response to a single elementary signal $\delta(t)$)

Example

$x(t) = e^{-at}u(t)$ for $a > 0$ and $h(t) = u(t)$



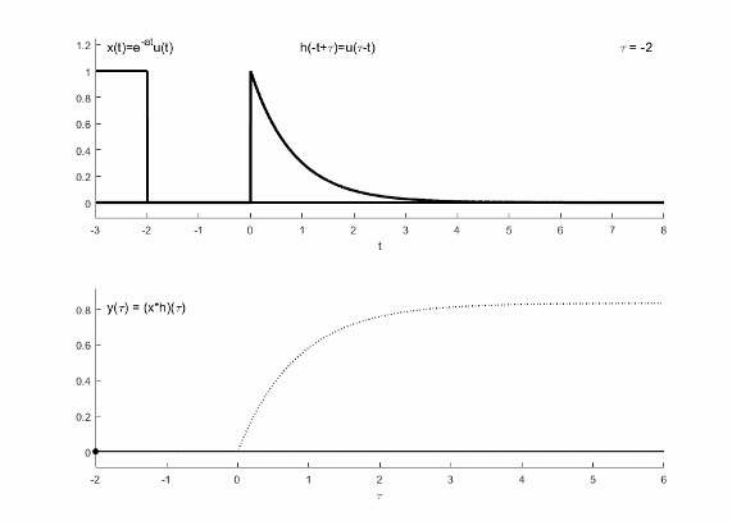




**Figure 2.3.3:** Animation showing the convolution (i.e., y(t)) between a decaying exponential signal (i.e., x(t)) and a unit step signal (i.e., h(t)).

For $t < 0$, $x(\tau)\,h(t-\tau) = 0$ since there is no an overlapping region $\Rightarrow$ $y(t) = 0$

For $t > 0$,

$$x(\tau)\,h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases} \tag{2.3.9}$$

$$y(t) = \int_0^t e^{-a\tau}\,d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a} - \frac{1}{a}e^{-at} = \frac{1}{a}(1 - e^{-at}) \tag{2.3.10}$$

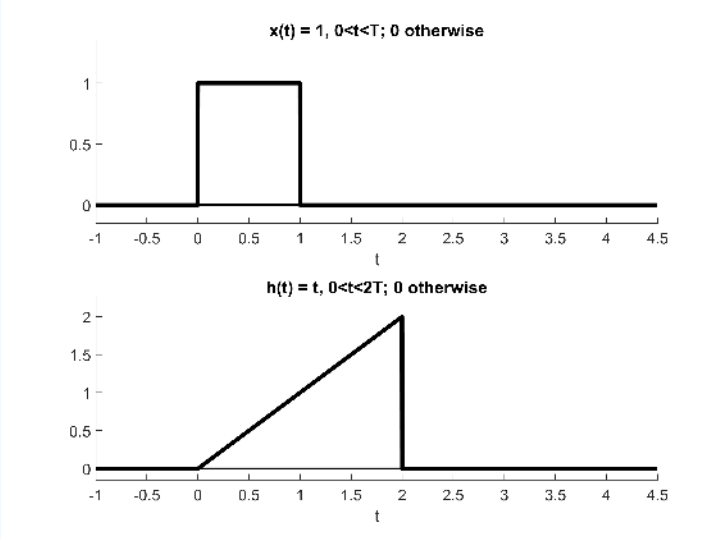
In general:

$$y(t) = \frac{1}{a}(1 - e^{-at})\,u(t) \tag{2.3.11}$$

Example

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \tag{2.3.12}$$

$$h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & \text{otherwise} \end{cases} \tag{2.3.13}$$



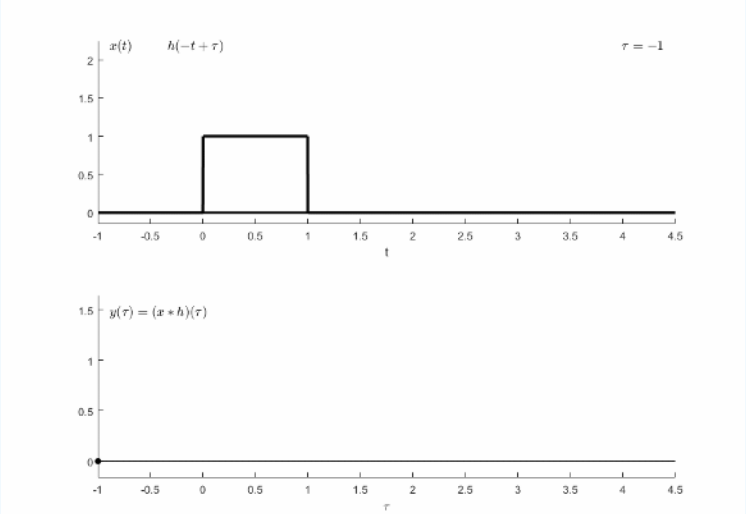




**Figure 2.3.4:** Animation showing the convolution (i.e., y(t)) between a rectangular signal (i.e., x(t)) and a linear ramp (i.e., h(t))).

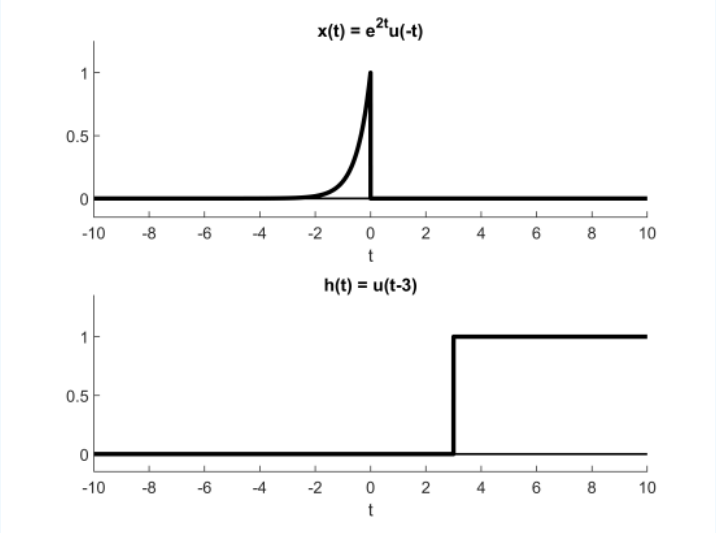
for $t < 0$, $y(t) = 0$

for $t > 3T$, $y(t) = 0$

$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \quad \text{I} \\ Tt - \frac{1}{2}T^2, & T < t < 2T \quad \text{II} \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & t > 3T \end{cases} \tag{2.3.14}$$

Example

$x(t) = e^{2t}u(-t)$ and $h(t) = u(t-3)$



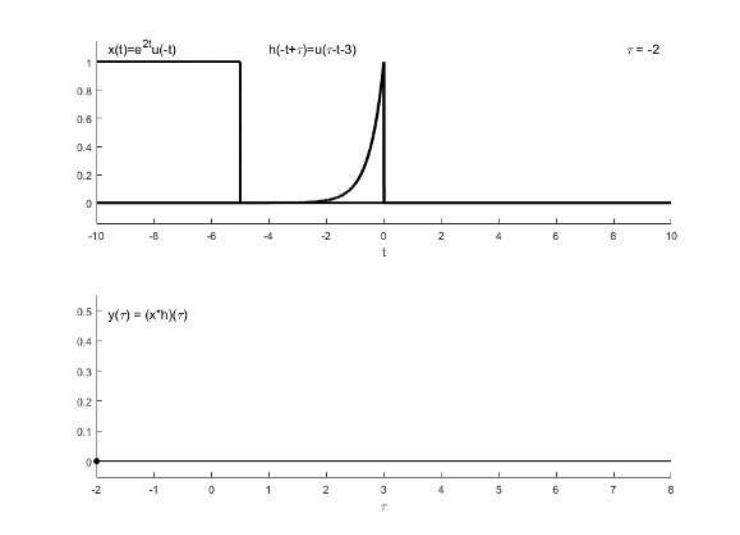




**Figure 2.3.5:** Animation showing the convolution (i.e., y(t)) between a growing exponential signal (i.e., x(t)) and a laterally-shifted unit step signal (i.e., h(t))).

$x(t)$ and $h(t)$ always overlap regardless the $t$ value

When $t-3 \leq 0$ (i.e., $t \leq 3$),

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2} e^{2(t-3)} \tag{2.3.15}$$

For $t-3 \geq 0$ (i.e., $t \geq 3$),

$$y(t) = \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2} \tag{2.3.16}$$

*total overlap between $x(t)$ and $h(t)$*

Unit impulse

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{+\infty} x(t-\tau)\, \delta(\tau)\, d\tau$$

$\delta(t) = \delta(t) * \delta(t)$

### Properties of the unit impulse

1) $x(t) = x(t) * \delta(t) = \delta(t) * x(t) = \int_{-\infty}^{+\infty} \delta(\tau)\, x(t-\tau)\, d\tau$

$$1 = \int_{-\infty}^{+\infty} \delta(\tau)\, d\tau \tag{2.3.17}$$

-> the unit impulse has unit area, i.e., area of 1.

2) $g(-t) = g(-t) * \delta(t) = \delta(t) * g(-t) = \int_{-\infty}^{+\infty} \delta(\tau)\, g(\tau - t)\, d\tau$

@ $t = 0$ $g(0) = \int_{-\infty}^{+\infty} \delta(\tau)\, g(\tau)\, d\tau$

3) $f(t) \cdot \delta(t) = f(0) \cdot \delta(t)$

---

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# 2.4: LTI Systems Properties

**Learning Objectives: LTI System Properties**

1. Identify LTI systems as causal or not from the impulse response $h(t)$
2. Identify LTI systems as stable or not from the impulse response $h(t)$
3. Identify whether two LTI systems are inverses of each other from their impulse responses $h(t)$ and $g(t)$
4. Exploit convolution properties to solve problems more efficiently

## YouTube Videos

CT LTI Systems and Stability



CT LTI Systems and Causality



CT LTI Systems and Invertibility



Convolution Properties



## Summary

*Remember that $h(t)$ completely determines the behavior of an LTI system*

**Output as the result of a convolution integral**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

*The impulse response holds all characteristics of the system*

### Convolution Properties

**Commutative Property**

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \tag{2.4.1}$$

This means that the output of an LTI system with input $x(t)$ and unit impulse response $h(t)$ is identical to the output of an LTI system with input $h(t)$ and unit impulse response $x(t)$.

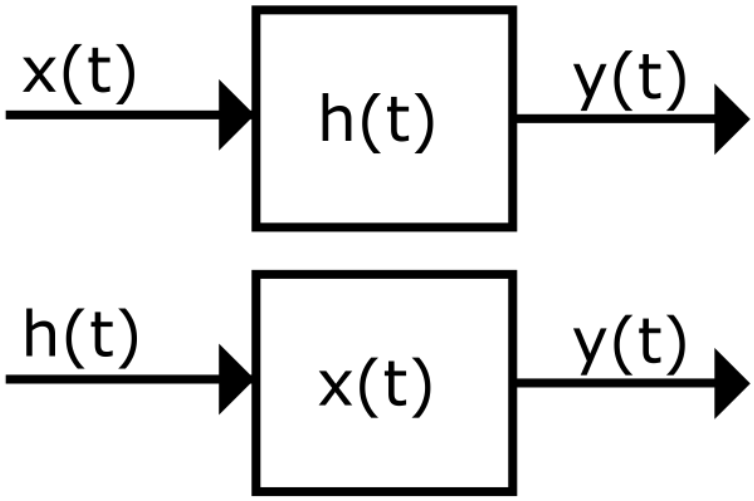



**Figure 2.4.1:** Basic flow diagram of the commutative property.

**Distributive Property**

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t) \tag{2.4.2}$$

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t) \tag{2.4.3}$$

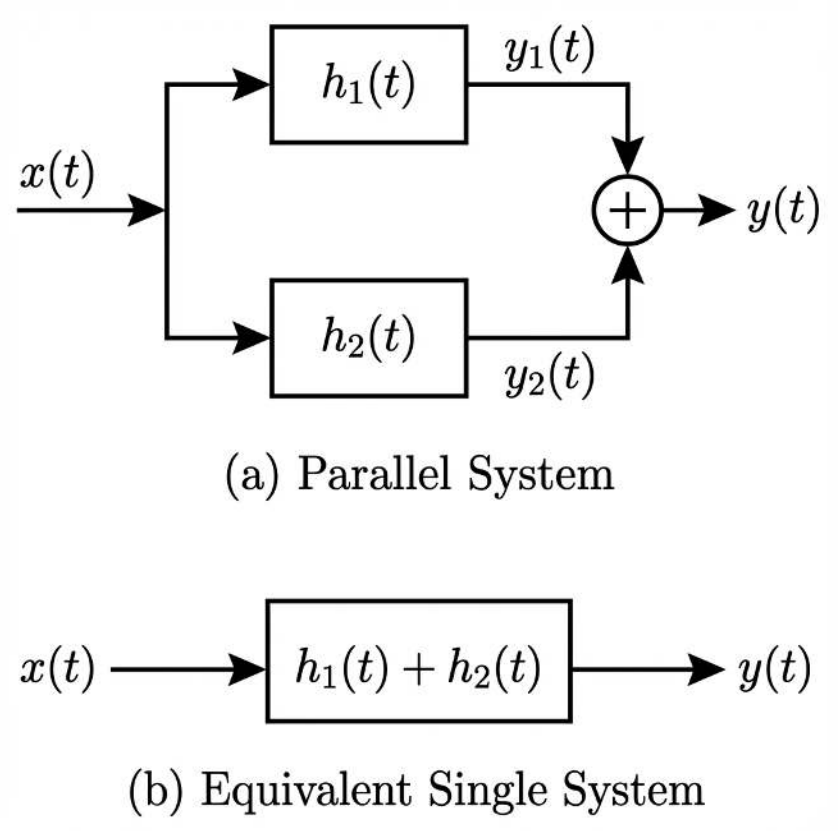



**Figure 2.4.2:** Basic flow diagram illustrating the distributive property within a parallel system

**Associative Property**

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t) \tag{2.4.4}$$

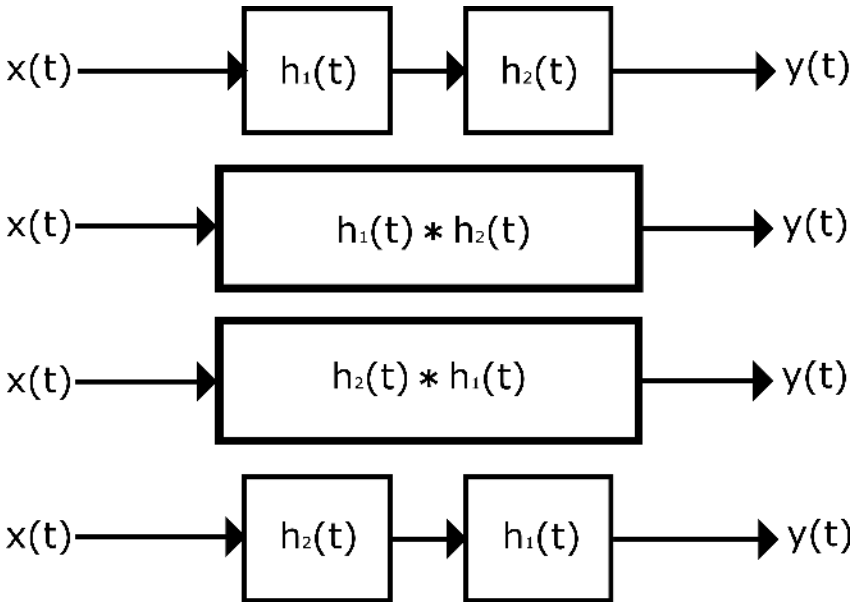



**Figure 2.4.3:** Basic flow diagram of the associative property in a cascade system.

*The output of LTI systems in cascade does not depend on the order of the individual systems.*

### LTI Systems with and without Memory

$$y(t) = x(t) * h(t) \text{ where } h(t) = k\,\delta(t)$$

**Definition:** A system is memoryless if its output at any time depends only on the value of the input at the same time. → $y(t) = k \cdot x(t)$ where $k$ is constant and $h(t) = k\,\delta(t)$

If $k = 1$ (i.e., output = input) →

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \tag{2.4.5}$$

### Invertibility of LTI Systems

$$h(t) * h_1(t) = \delta(t) \tag{2.4.6}$$

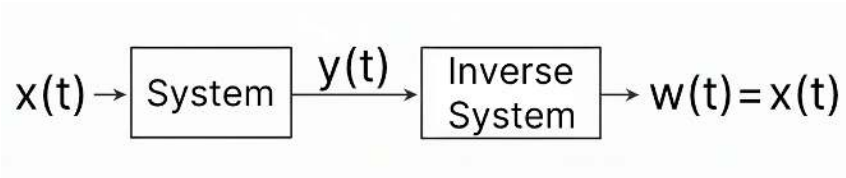



**Figure 2.4.4:** Basic flow diagram to demonstrate an inverse LTI system.

**Example - Pure Time-Shift LTI System**

LTI system consisting of a pure time shift:

$$y(t) = x(t - t_0) \tag{2.4.7}$$

Such a system is a delay if $t_0 > 0$ and an advance if $t_0 < 0$. If $t_0 > 0$, the output at time $t$ is equal to the value of the input at an earlier time $t - t_0$. If $t_0 = 0$, the system is the identity system and memoryless. For any other value of $t_0$, the system has memory as its response depends on the input at a time different than the current one.

If

$$y(t) = x(t - t_0) = x(t) * \delta(t - t_0), h(t) = \delta(t - t_0) \tag{2.4.8}$$

,

then one can recover the input from the output by shifting back the output →

$$h_1(t) = \delta(t + t_0). \tag{2.4.9}$$

Thus

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t) \tag{2.4.10}$$

### Causality for LTI Systems

**Definition:** The output of a causal system depends only on the present and past values of the input to the system. This means that the impulse response of a causal LTI system is $h(t) = 0$ for $t < 0$.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, h(t - \tau)\, d\tau \tag{2.4.11}$$

Note that a pure-shift LTI system with $h(t) = \delta(t - t_0)$ is causal for $t_0 \geq 0$ (i.e., when the time shift is a delay) but is noncausal for $t_0 < 0$ because the output anticipates future values of the input.

### Stability for LTI Systems

A system is stable if every bounded input produces a bounded output.

Let's consider an input $x(t)$ that is bounded in magnitude, $|x(t)| < B$ for any $t$ value, thus the magnitude of the output

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)\, x(t - \tau)\, d\tau \right| \leq \int_{-\infty}^{+\infty} |h(\tau)|\, |x(t - \tau)|\, d\tau \leq B \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau \tag{2.4.12}$$

Thus, a system is stable if its impulse response is absolutely integrable:

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty \tag{2.4.13}$$

**Example**

A system that is a pure time shift, $h(t) = \delta(t - t_0)$.

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} |\delta(t - t_0)|\, dt = 1 \tag{2.4.14}$$

so system is stable.

**Example**

An accumulator system with $h(t) = u(t)$

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} u(t)\, dt = \int_{0}^{\infty} dt = \infty \tag{2.4.15}$$

so system is unstable!

### Unit Step Response of an LTI System

System's output for a unit step input $x(t) = u(t)$, i.e.,

$$s(t) = u(t) * h(t) \tag{2.4.16}$$

in integral form:

$$s(t) = \int_{-\infty}^{+\infty} u(t - \tau)\, h(\tau)\, d\tau = \int_{-\infty}^{t} h(\tau)\, d\tau \tag{2.4.17}$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau \implies h(t) = \frac{ds(t)}{dt} \tag{2.4.18}$$

The unit impulse response $h(t)$ is the first derivative of the unit step response.

The unit step response can also be used to characterize an LTI system.

---

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# 2.5: Differential Equations and Block Diagrams

## Learning Objectives: Differential Equations and Block Diagrams

1. Solve for the output of simple differential equations representing causal LTI systems
2. Identify the order of a differential equation and recognize whether it is a recursive or non-recursive system implementation.
3. Draw block diagrams to represent systems implementing differential equations.

## YouTube Videos

First-order Differential Equations Example



Block Diagrams for CT Systems



## Summary

A wide variety of systems and physical phenomena are represented by continuous-time systems that for which the input and output are related through a linear constant-coefficient differential equation.

### Linear Constant-Coefficient Differential Eqs.

**1st order:**

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{2.5.1}$$

$y(t)$: system's output
$x(t)$: system's input

A linear constant-coefficient differential equation describes a relationship between the input and output, rather than an explicit expression for the system output as a function of the input.

To solve a linear constant-coefficient differential equation, we must specify 1 or more auxiliary conditions. Different choices for the auxiliary conditions lead to different relationships between the input and output.

**Example**

$$\frac{dy(t)}{dt} + 3y(t) = x(t) \qquad \text{when } x(t) = e^{-5t}u(t) \tag{2.5.2}$$

if system is LTI and causal → initial rest condition is met, and the solution of the output signal is composed of the particular solution $y_p(t)$ and the homogeneous solution $y_h(t)$:

$$y(t) = y_p(t) + y_h(t) \tag{2.5.3}$$

**RECIPE**

1. Solve for $y_h(t)$
2. Solve for $y_p(t)$
3. Combine $y_h + y_p$
4. Apply the initial rest condition
5. Find the final solution

**Step 1) Solve for $y_h(t)$** since $\frac{dy_h(t)}{dt} + 3y_h(t) = 0$

Guess form $y_h(t) = Ae^{st}$ →

$$sAe^{st} + 3Ae^{st} = 0 \tag{2.5.4}$$

$$(s+3)Ae^{st} = 0 \;\Rightarrow\; s = -3 \tag{2.5.5}$$

$$y_h(t) = Ae^{-3t}, \qquad t > 0 \quad \text{causal} \tag{2.5.6}$$

**Step 2) Solve for $y_p(t)$** since $\frac{dy_p(t)}{dt} + 3y_p(t) = x(t) = e^{-5t},\ t > 0$

Guess $y_p(t) = Ke^{-5t}$ (i.e., same form of input) →

$$-5Ke^{-5t} + 3Ke^{-5t} = e^{-5t} \tag{2.5.7}$$

$$-5K + 3K = 1 \;\Rightarrow\; -2K = 1 \;\Rightarrow\; K = -\frac{1}{2} \tag{2.5.8}$$

$$y_p(t) = -\frac{1}{2}e^{-5t}, \qquad t > 0 \tag{2.5.9}$$

**Step 3) Combine $y_h$ and $y_p$**

$$y(t) = y_p(t) + y_h(t) = -\frac{1}{2}e^{-5t} + Ae^{-3t}, \qquad t > 0 \tag{2.5.10}$$

**Step 4) Find $A$ value using the initial condition**

$$y(t=0) = -\frac{1}{2} + A = 0 \;\Rightarrow\; A = \frac{1}{2} \tag{2.5.11}$$

**Step 5) Final solution**

$$y(t) = -\frac{1}{2}e^{-5t} + \frac{1}{2}e^{-3t} = \frac{1}{2}\left(e^{-3t} - e^{-5t}\right), \qquad t > 0 \tag{2.5.12}$$

$$y(t) = \frac{1}{2}\left(e^{-3t} - e^{-5t}\right)u(t) \tag{2.5.13}$$

Example

$$x(t) = k\,e^{3t}u(t) \qquad \text{where } k \text{ is a real number} \tag{2.5.14}$$

The solution can be composed by summing a particular solution and a homogeneous solution

$$y(t) = y_p(t) + y_h(t) \tag{2.5.15}$$

Thus

$$\frac{dy_h(t)}{dt} + 2y_h(t) = 0 \tag{2.5.16}$$

$$\frac{dy_p(t)}{dt} + 2y_p(t) = x(t) \tag{2.5.17}$$

Because $x(t)$ is an exponential signal $\rightarrow$ $y_p(t) = Ye^{3t}$ for $t > 0$

Since $\frac{dy(t)}{dt} + 2y(t) = x(t)$, replace $y_p$ and $x$:

$$3Ye^{3t} + 2Ye^{3t} = ke^{3t} \tag{2.5.18}$$

$$3Y + 2Y = k \;\Rightarrow\; 5Y = k \;\Rightarrow\; Y = \frac{k}{5} \tag{2.5.19}$$

$$y_p(t) = \frac{k}{5}e^{3t} \qquad \text{for } t > 0 \tag{2.5.20}$$

Let's assume $y_h(t) = Ae^{st}$. Since $\frac{dy_h(t)}{dt} + 2y_h(t) = 0$,

$$sAe^{st} + 2Ae^{st} = 0 \tag{2.5.21}$$

$$(2+s)Ae^{st} = 0 \;\Rightarrow\; s = -2 \tag{2.5.22}$$

$$y_h(t) = Ae^{-2t} \qquad t > 0 \tag{2.5.23}$$

$$y(t) = y_p(t) + y_h(t) = Ae^{-2t} + \frac{k}{5}e^{3t}, \qquad t > 0 \tag{2.5.24}$$

Because $x(t) = 0$ for $t < 0$ then $y(t) = 0$ for $t < 0$.

Use initial rest (auxiliary condition) at $t = 0$:

$$0 = A + \frac{k}{5} \;\Rightarrow\; A = -\frac{k}{5} \tag{2.5.25}$$

$$y(t) = \frac{k}{5}\left(e^{3t} - e^{-2t}\right)u(t) \tag{2.5.26}$$

General Nth-order linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.5.27}$$

if $N = 0$ $\rightarrow$

$$y(t) = \frac{1}{a_0}\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.5.28}$$

output is an explicit function of input $x(t)$ and its derivatives

$$y(t) = y_p(t) + y_h(t) \tag{2.5.29}$$

where

$y_p(t)$ is solved by

$$\sum_{k=0}^{N} a_k \frac{d^k y_p(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.5.30}$$

and $y_h(t)$ by

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0 \tag{2.5.31}$$

Our auxiliary condition is the **initial rest**: if $x(t) = 0$ for $t \le t_0$, then $y(t) = 0$ for $t \le t_0$

The initial rest condition tells us that if $x(t) = 0$ then $y(t) = 0$.

## Block Diagram for CT system (LCCDE)

Consider

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) \tag{2.5.32}$$

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t) \tag{2.5.33}$$

multiplication by a constant ($-\frac{1}{a}$) differentiation ($\frac{dy(t)}{dt}$) addition ($+\frac{b}{a}x(t)$)

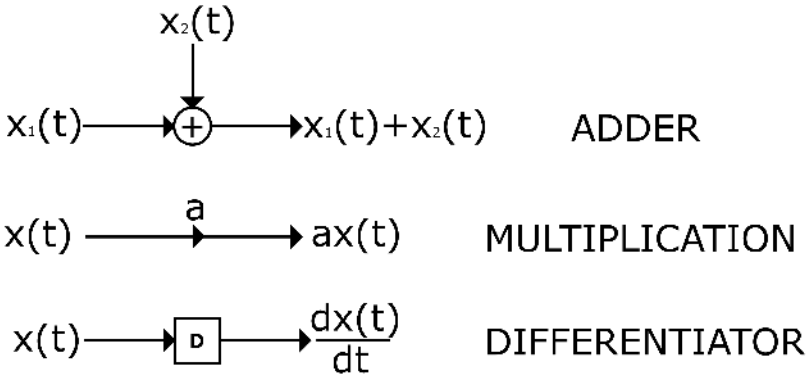



**Figure 2.5.1:** Basic block diagram operations.

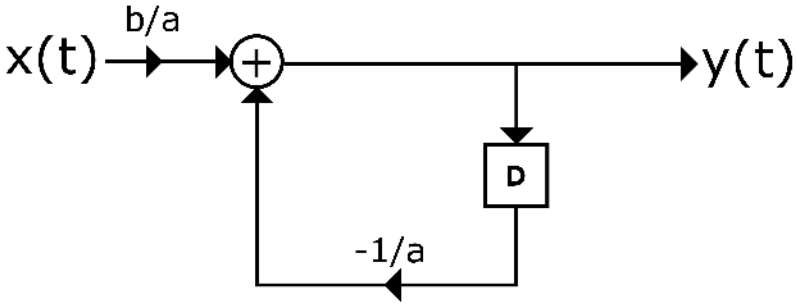
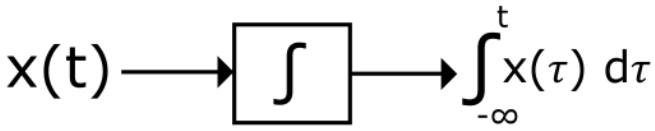



**Figure 2.5.2:** Basic block diagram using an adder, multiplier, and differentiator.

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t) \implies \frac{dy(t)}{dt} = b\,x(t) - a\,y(t) \tag{2.5.34}$$

$$y(t) = \int_{-\infty}^{t} \left[b\,x(\tau) - a\,y(\tau)\right] d\tau \tag{2.5.35}$$

1st order → 1 unknown constant



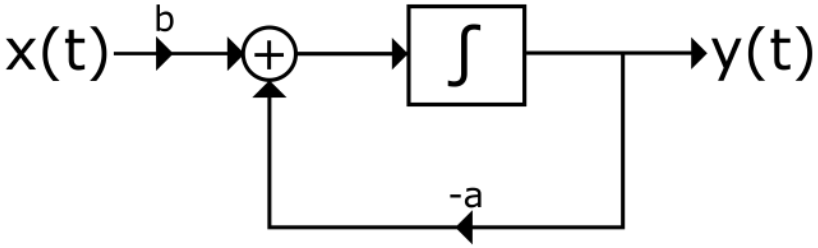




**Figure 2.5.3:** Basic block diagram using integrators.

Derivatives are replaced by integrators in practical solutions to avoid the noise amplification of derivatives

## Example: Causal, LTI System

A 1st order equation

$$\frac{dy(t)}{dt} + 3y(t) = 2x(t) + 4\frac{dx(t)}{dt} \tag{2.5.36}$$

$$\frac{dy(t)}{dt} = -3y(t) + 2x(t) + 4\frac{dx(t)}{dt} \tag{2.5.37}$$

*integral*

$$y(t) = \int_{-\infty}^{t} \left[-3y(\tau) + 2x(\tau)\right] d\tau + 4x(t) \tag{2.5.38}$$

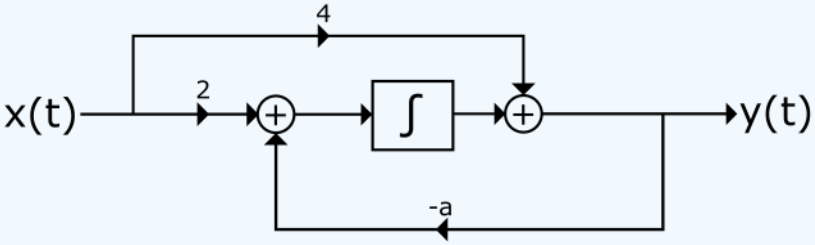



**Figure 2.5.4:** Example of a causal LTI system using block diagram.

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# 2.6: Eigenfunctions and CT Fourier Series

## Learning Objectives: CT Fourier Series

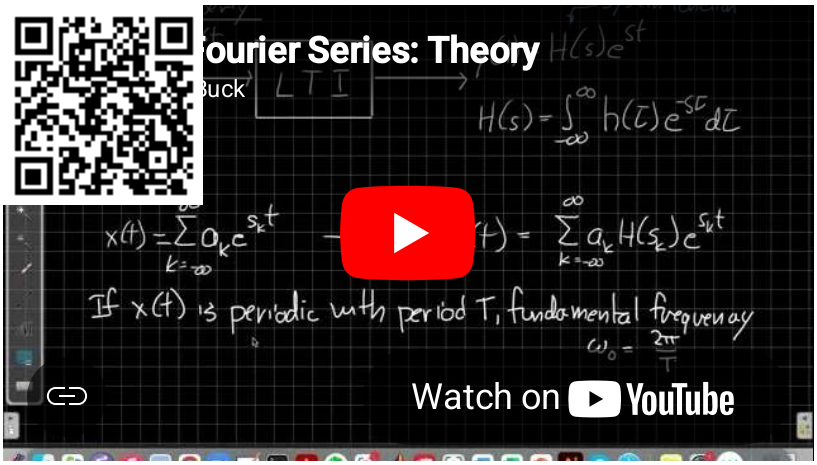
1. Applying the eigenfunction property of LTI systems to exponential inputs to find the corresponding output
2. Computing the Fourier series coefficients $a_k$ from a periodic signal $x(t)$
3. Computing a periodic signal $x(t)$ from its Fourier series coefficients $a_k$
4. Concluding whether a system could be LTI based on whether inputs and outputs satisfy the eigenfunction relationship

## YouTube Videos

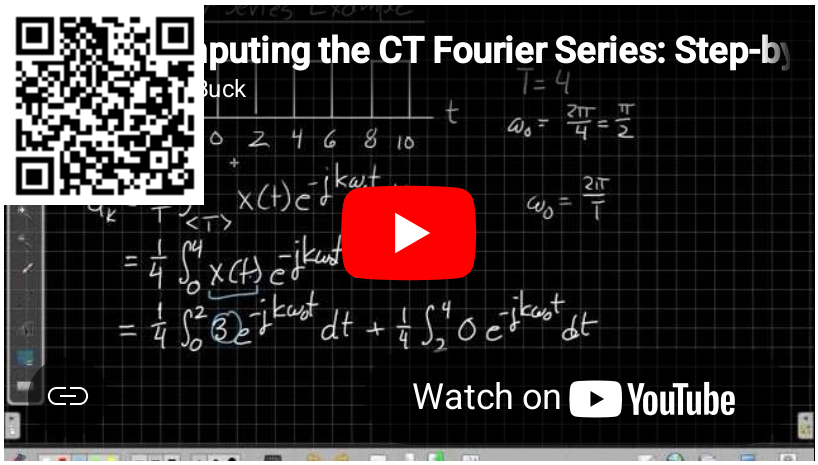
Eigenfuntions of CT Linear Time-Invariant Systems



CT Fourier Series: Theory




Computing the CT Fourier Series: Step-by-Step Example




## Summary

It is advantageous in the study of LTI systems to represent signals as linear combinations of basic signals that possess the following 2 properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

These two properties are satisfied by CT complex exponential signals, i.e., $e^{st}$ being $s$ a complex number.

**Why a complex exponential is so important in a LTI system?**

The response of an LTI system to a complex exponential input is the same complex exponential with only a change of magnitude.

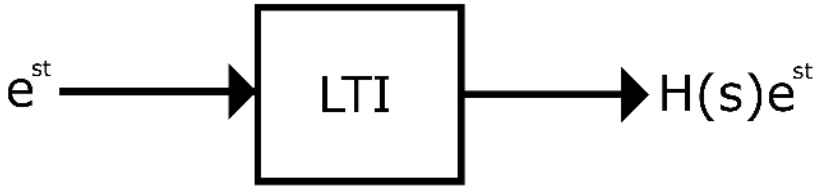



**Figure 2.6.1:** Illustration that complex exponentials are the eigenfunctions of LTI systems. In other words, the output is the same complex exponentials with a change in its magnitude.

### Proof - Eigenfunction Property

**Convolution integral:**

$$y(t) = \int_{-\infty}^{+\infty} h(c)\, x(t-c)\, dc = \int_{-\infty}^{+\infty} h(c)\, e^{s(t-c)}\, dc \tag{2.6.1}$$

$$= e^{st} \int_{-\infty}^{+\infty} h(c)\, e^{-sc}\, dc = H(s)\, e^{st} \tag{2.6.2}$$

In Eq. (2.6.2), $e^{st}$ is outside the integral because it does not depend on $c$. Remember that the Fourier transform of $h(c) = H(s)$

Complex exponentials are eigenfunctions of CT LTI system.

## Linear Combination of Complex Exponentials

Consider an input $x(t)$ expressed as the linear combination of 3 complex exponentials, i.e.,

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \tag{2.6.3}$$

$$\xrightarrow{\text{LTI}} \quad y(t) = a_1 H_1 e^{s_1 t} + a_2 H_2 e^{s_2 t} + a_3 H_3 e^{s_3 t} \tag{2.6.4}$$

where $H_1 = H(s_1)$, $H_2 = H(s_2)$ and $H_3 = H(s_3)$

In general,

$$x(t) = \sum_k a_k e^{s_k t} \xrightarrow{\text{LTI}} y(t) = \sum_k a_k H(s_k) e^{s_k t} \tag{2.6.5}$$

> If the input to an LTI system is represented as a linear combination of complex exponentials, then the output can also be represented as a linear combination of the same complex exponential signals. Each coefficient of the output is obtained as the product of $a_k$ and $H(s_k)$.

In this course, $s$ is a pure imaginary value, i.e., $s = j\omega$ so $e^{st} = e^{j\omega t}$.

**Example**

$y(t) = x(t-3)$, i.e., the input and output are related by a time shift of 3.

If

$$x(t) = e^{j2t} \tag{2.6.6}$$

then

$$y(t) = e^{j2(t-3)} = e^{-j6} e^{j2t} \equiv H(s)\, e^{jst} \tag{2.6.7}$$

$H(s) = e^{-j6}$ (eigenvalue)

**Example**

$x(t) = \cos(4t) + \cos(7t)$

$$y(t) = x(t-3) = \cos\big(4(t-3)\big) + \cos\big(7(t-3)\big) \tag{2.6.8}$$

using Euler

$$= \frac{1}{2}\left[e^{j4(t-3)} + e^{-j4(t-3)}\right] + \frac{1}{2}\left[e^{j7(t-3)} + e^{-j7(t-3)}\right] \tag{2.6.9}$$

$$= \frac{1}{2} e^{-j12} e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j21} e^{j7t} + \frac{1}{2} e^{j21} e^{-j7t} \tag{2.6.10}$$

$H(j4) = e^{-j12}$, $H(-j4) = e^{j12}$, $H(j7) = e^{-j21}$, $H(-j7) = e^{j21}$

## Periodic Signals and Harmonics

A signal is periodic if $x(t) = x(t+T)$ for all $t$ values and $T > 0$.

$T$ is the fundamental period of $x(t)$ if it is the minimum positive, nonzero value.

$$\omega_0 = \frac{2\pi}{T} \quad \text{fundamental frequency} \tag{2.6.11}$$

Harmonically-related complex exponential

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk\frac{2\pi}{T}t} \qquad k = 0, \pm 1, \pm 2, \ldots \tag{2.6.12}$$

Each of these $\phi_k(t)$ signals has a fundamental frequency that is a multiple of $\omega_0$ and, therefore, each is periodic with period $T$. Although for $|k| \geq 2$, the fundamental period of $\phi_k(t)$ is a fraction of $T$.

$\Rightarrow$ a linear combination of harmonically-related complex exponentials is also periodic with $T$

$$x(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} e^{jk\frac{2\pi}{T}t} \tag{2.6.13}$$

$k = 0 \rightarrow x(t) = 1$
$k = \pm 1 \rightarrow$ first-order harmonic, $T$
$k = \pm 2 \rightarrow$ second-order harmonic, $T/2$

**Example**

Consider a periodic signal $x(t)$ with fundamental frequency $2\pi$

$$x(t) = \sum_{k=-3}^{3} a_k e^{j2\pi kt} \tag{2.6.14}$$

where $a_0 = 1,\ a_1 = a_{-1} = \frac{1}{4},\ a_2 = a_{-2} = \frac{1}{2},\ a_3 = a_{-3} = \frac{1}{3}$

$$x(t) = \frac{1}{3}e^{-j6\pi t} + \frac{1}{2}e^{-j4\pi t} + \frac{1}{4}e^{-j2\pi t} + 1 + \frac{1}{4}e^{j2\pi t} + \frac{1}{2}e^{j4\pi t} + \frac{1}{3}e^{j6\pi t} \tag{2.6.15}$$

$$= \frac{1}{3}\,2\cos(6\pi t) + \frac{1}{2}\,2\cos(4\pi t) + \frac{1}{4}\,2\cos(2\pi t) + 1 \tag{2.6.16}$$

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t) \tag{2.6.17}$$

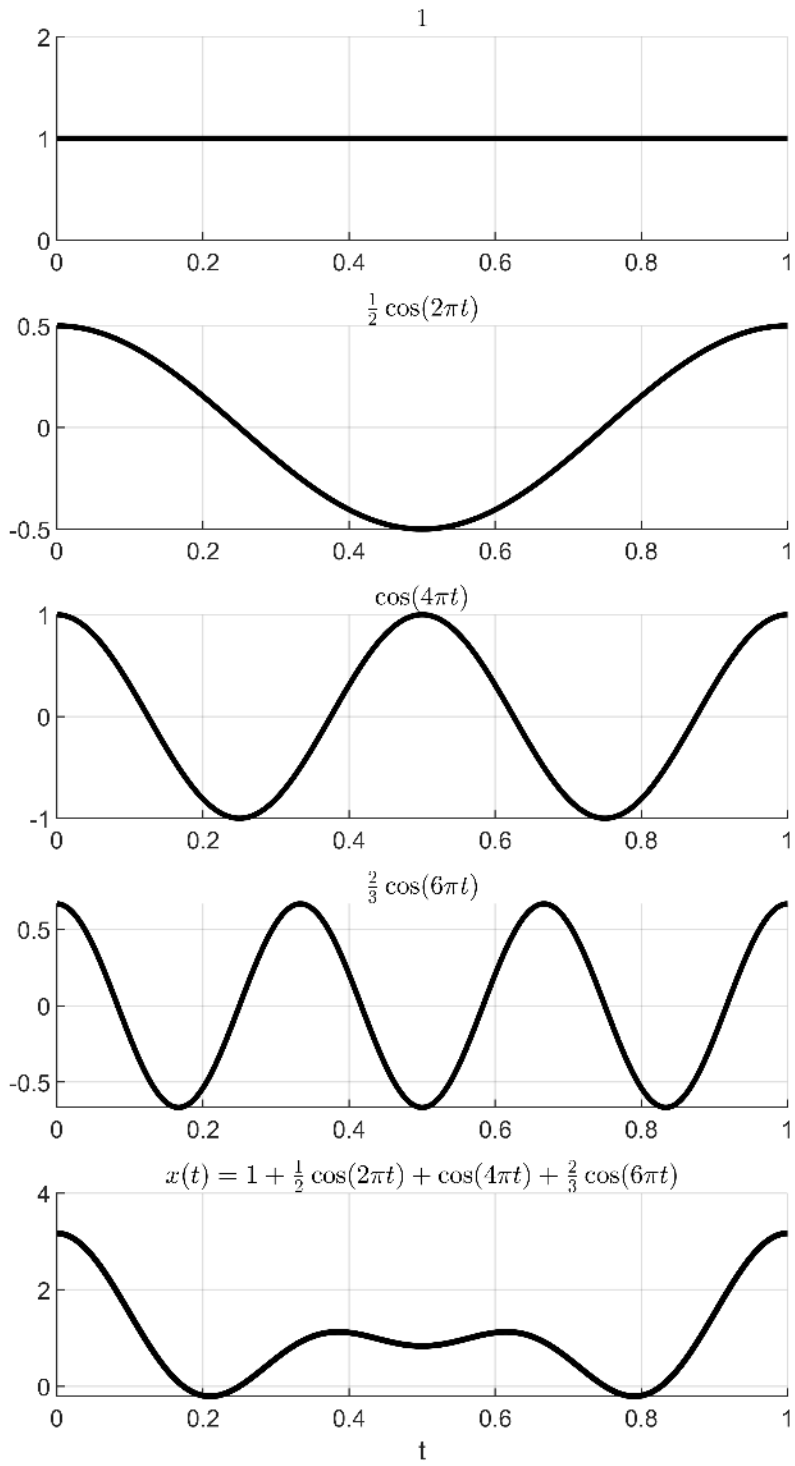



**Figure 2.6.2:** Individual components of a periodic signal and their combination.

Fourier Series (Synthesis and Analysis Equations)

**Synthesis equation**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{2.6.18}$$

$x(t)$ represented by a Fourier series

**Analysis equation**

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}\,dt \tag{2.6.19}$$

$\{a_k\}$: the set of $\{a_k\}$ coefficients are called the Fourier series coefficients or the spectral coefficients of $x(t)$

$\Rightarrow a_k$ provides a measure of $x(t)$ present in each harmonic

**Example**

$$x(t) = \sin(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} - \frac{1}{2}e^{-j\omega_0 t} \tag{2.6.20}$$

*Euler*

so $a_1 = \frac{1}{2}$, $a_{-1} = -\frac{1}{2}$ and $a_k = 0$ else

**Example**

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right) \tag{2.6.21}$$

*Euler*

$$= 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + \frac{2}{2}e^{j\omega_0 t} + \frac{2}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t}e^{j\pi/4} + \frac{1}{2}e^{-j2\omega_0 t}e^{-j\pi/4} \tag{2.6.22}$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{e^{j\pi/4}}{2}e^{j2\omega_0 t} + \frac{e^{-j\pi/4}}{2}e^{-j2\omega_0 t} \tag{2.6.23}$$

$$a_0 = 0, \quad a_1 = 1 + \frac{1}{2j} = 1 - \frac{j}{2}, \quad a_{-1} = 1 - \frac{1}{2j} = 1 + \frac{j}{2} \tag{2.6.24}$$

$$a_2 = \frac{e^{j\pi/4}}{2}, \qquad a_{-2} = \frac{e^{-j\pi/4}}{2} \tag{2.6.25}$$

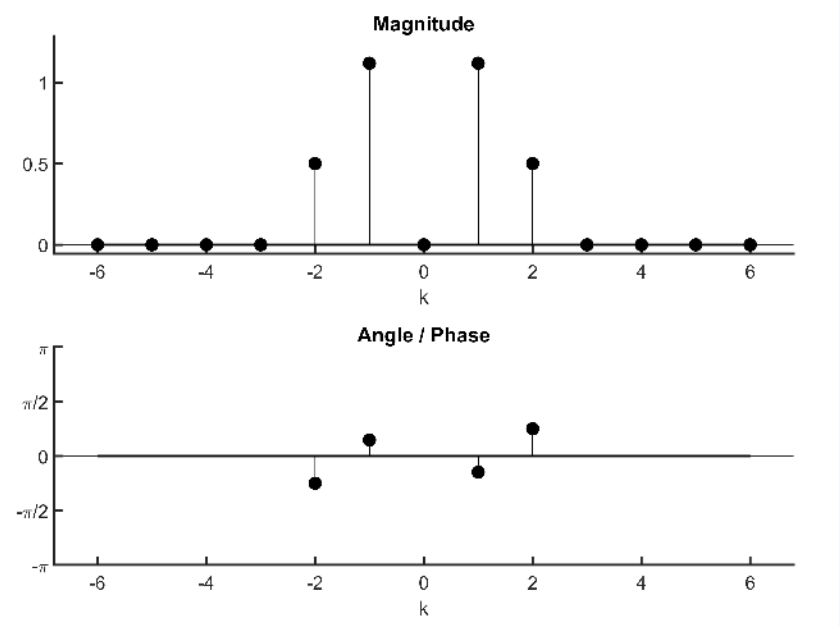



**Figure 2.6.3:** Plots of the magnitude (top) and angle/phase (bottom) of the Fourier coefficients from this example.


https://eng.libretexts.org/@go/page/126954

**Remember:**

$$e^{j\pi/4} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \tag{2.6.26}$$

Example

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases} \tag{2.6.27}$$

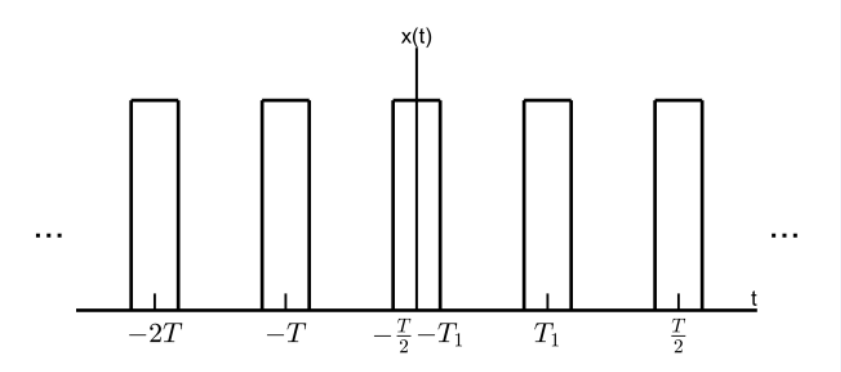



**Figure 2.6.4:** Illustration of the given periodic square wave signal.

Note that the symmetry of $x(t)$ is around $t = 0$ so

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt \tag{2.6.28}$$

for $k = 0 \rightarrow$

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{1}{T}\,t\Big|_{-T_1}^{T_1} = \frac{2T_1}{T} \tag{2.6.29}$$

$k \neq 0 \rightarrow$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\frac{1}{-k\omega_0 j}\left(e^{-jk\omega_0 t}\right)\Big|_{-T_1}^{T_1} \tag{2.6.30}$$

$$a_k = -\frac{1}{T}\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{jk\omega_0} = \frac{1}{T}\frac{2}{k\omega_0}\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j} = \frac{2}{Tk\omega_0}\sin(k\omega_0 T_1) \tag{2.6.31}$$

considering $T \cdot \omega_0 = T \cdot \frac{2\pi}{T} = 2\pi \rightarrow$

$$a_k = \frac{1}{k\pi}\sin(k\omega_0 T_1) \tag{2.6.32}$$

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# 2.7: Fiiltering with the CTFS

## Learning Objectives: Filtering

1. Identify continuous-time frequency-selective filters such as lowpass, highpass, and bandpass filters
2. Solve for the output of LTI filters using the Fourier series for the periodic input signal $x(t)$

## YouTube Videos

CT Fourier Series and Filtering: Overview



CT Fourier Series and Filtering Example: Part 1



CT Fourier Series and Filtering Example: Part 2



## Summary

# Fourier Series and LTI System

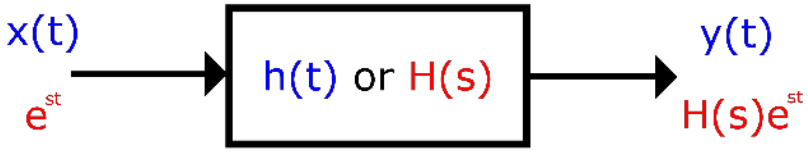



**Figure 2.7.1:** Basic flow diagram of an LTI system with an exponential signal input.

$$H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt \tag{2.7.1}$$

*system function impulse response*

In CT, $s = j\omega$ so

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt \tag{2.7.2}$$

*frequency response*

Consider $x(t)$ is periodic so $x(t)$ can be written by the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \Longrightarrow \quad y(t) = \sum_{k=-\infty}^{\infty} a_k\, H(jk\omega_0)\, e^{jk\omega_0 t} \tag{2.7.3}$$

Whereas $a_k$ are the Fourier coefficients of $x(t)$, the effect of the LTI system is the modification of the Fourier coefficients of $y(t)$ by $H(jk\omega_0)$.

**Example**

$$h(t) = e^{-t}u(t) \tag{2.7.4}$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(1+j\omega)t}\, dt \tag{2.7.5}$$

$$= \left[ -\frac{1}{1+j\omega} e^{-(1+j\omega)t} \right]_0^{\infty} = \frac{1}{1+j\omega} \tag{2.7.6}$$

Then

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \tag{2.7.7}$$

Define $b_k = a_k H(jk\omega_0)$

## Filtering

**Filtering** is the process of modifying the relative amplitudes of frequency components in a signal, or of eliminating certain frequency components entirely.

LTI systems modify the frequency spectrum of the input $x(t)$.

Systems that pass certain frequencies without distortion/attenuation and significantly attenuate or even suppress other frequencies are known as **frequency-selective filters**.

**Ideal lowpass filter:**

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \tag{2.7.8}$$

A **lowpass filter** is a filter that passes low frequencies (i.e., frequencies around $\omega = 0$) and attenuates or suppress higher frequencies.

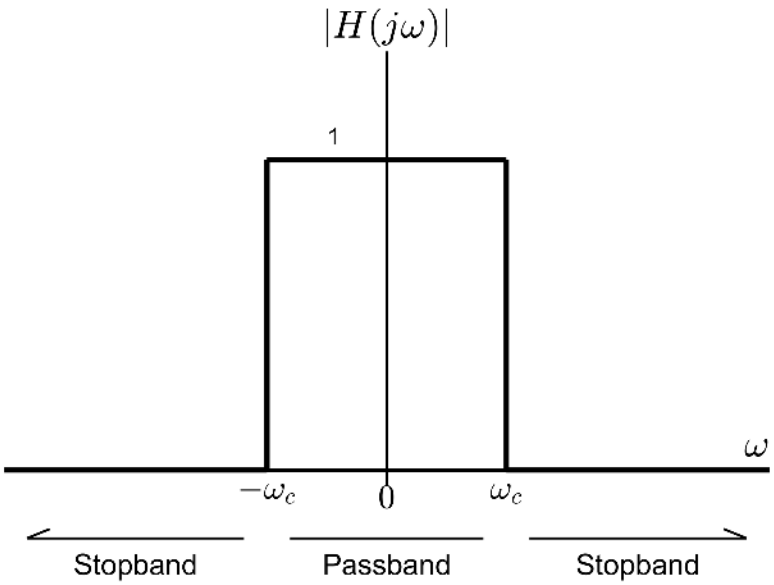



**Figure 2.7.2:** Magnitude of the frequency response of an ideal lowpass filter with a cutoff at $\omega = \omega_c$.

**Ideal highpass filter:**

$$H(j\omega) = \begin{cases} 0, & |\omega| \le \omega_c \\ 1, & |\omega| > \omega_c \end{cases} \tag{2.7.9}$$

A **highpass filter** transmits high frequencies and blocks or attenuates medium and low frequencies.

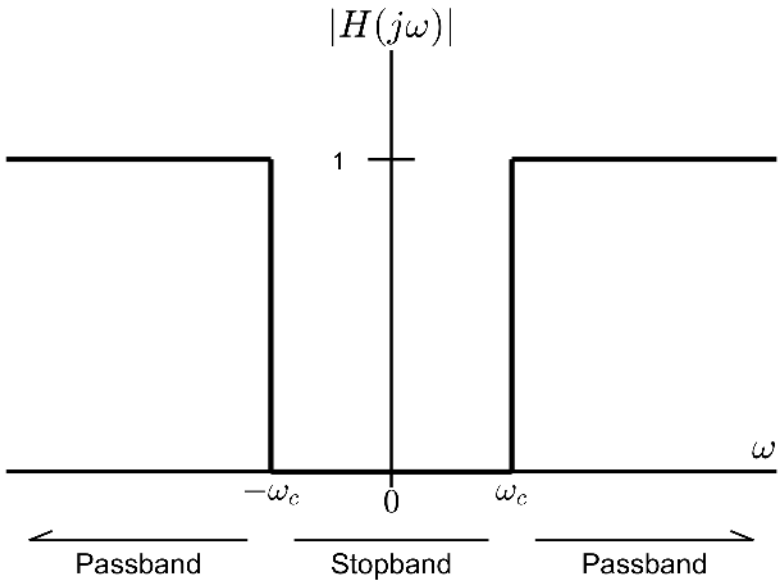



**Figure 2.7.3:** Magnitude of the frequency response of an ideal highpass filter with a cutoff at $\omega = \omega_c$.

**Ideal bandpass filter (we need two cutoff frequencies):**

$$H(j\omega) = \begin{cases} 0, & |\omega| < \omega_{c1} \\ 1, & \omega_{c1} \le |\omega| \le \omega_{c2} \\ 0, & |\omega| > \omega_{c2} \end{cases} \tag{2.7.10}$$

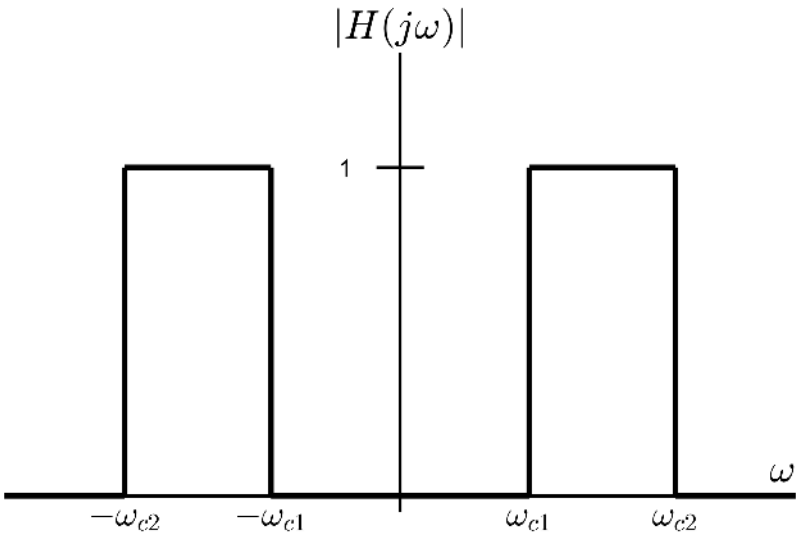



**Figure 2.7.4:** Magnitude of the frequency response of an ideal bandpass filter being the bandpass frequency within $\omega_{c1} \le \omega \le \omega_{c2}$.

**Definition: Cutoff frequency**

The **cutoff frequency** is the frequency that separates the blocked and passed regions.

Recipe of CTFS Filtering

1. Find the Fourier coefficients $a_k$ from input $x(t)$ using CTFS analytic equation

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt \tag{2.7.11}$$

2. Implement the filtering by multiplying input CTFS coefficients $a_k$ with $H(jk\omega_0)$ to obtain output CTFS coefficients $b_k$

$$b_k = a_k \cdot H(jk\omega_0) \tag{2.7.12}$$

3. Use the CTFS synthesis equation to build output $y(t)$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \tag{2.7.13}$$

*remember to use Euler relationship to simplify $y(t)$*

**Example**

Consider $x(t)$ a periodic pulse train where there is one pulse per period

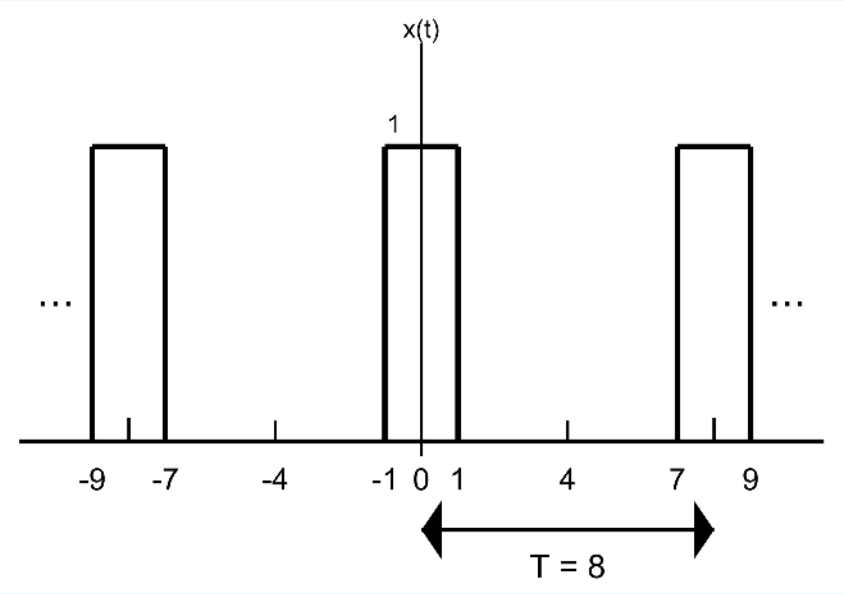



**Figure 2.7.5:** Input periodic square wave signal with a fundamental frequency of $T = 8$.

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4} \tag{2.7.14}$$

*Key: choose symmetric limits if the signal is symmetric*

$$a_k = \frac{1}{8}\int_{-4}^{4} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{8}\int_{-1}^{1} e^{-jk\omega_0 t}\,dt = \frac{1}{8}\left[\frac{-1}{jk\omega_0}e^{-jk\omega_0 t}\right]_{-1}^{1} \tag{2.7.15}$$

$$= \frac{1}{4k\omega_0}\sin(k\omega_0) = \frac{1}{k\pi}\sin\left(\frac{k\pi}{4}\right) \tag{2.7.16}$$

$$a_k = \begin{cases} \frac{1}{4}, & k = 0 \\ \frac{1}{k\pi}\sin\left(\frac{k\pi}{4}\right), & k \neq 0 \end{cases} \tag{2.7.17}$$

Sine is an even function so $a_k$ has the same value for $\pm k$

Step 2: Apply the Filter

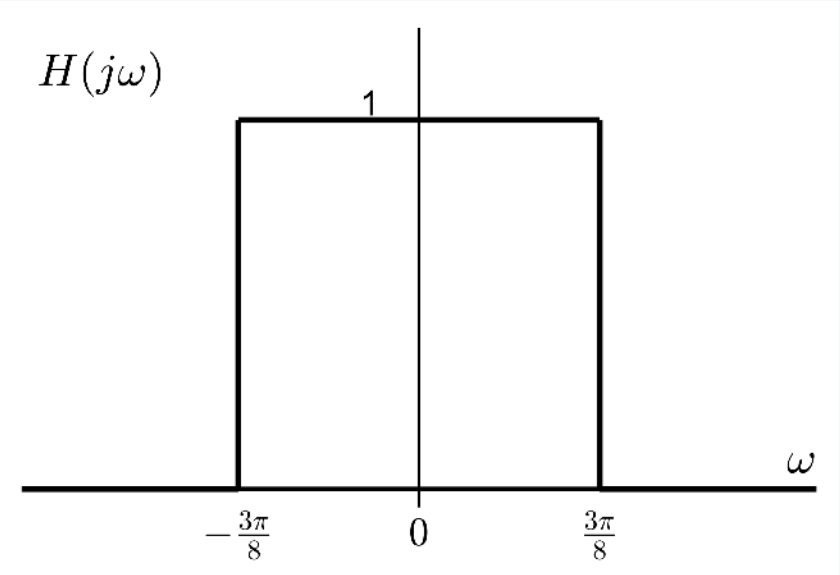



**Figure 2.7.6:** Magnitude of the frequency response of an ideal lowpass filter with a cutoff at $\omega = \frac{3\pi}{8}$.

$$b_k = a_k\, H(jk\omega_0) \tag{2.7.18}$$

$$k = 0 \rightarrow b_0 = a_0\, H(0) = \tfrac{1}{4}$$

$$k = 1 \rightarrow b_1 = a_1\, H(j\omega_0) = \tfrac{\sqrt{2}}{2\pi} \text{ (since } \omega_0 = \pi/4 = 2\pi/8 \text{ is in the passband)}$$

$$k = -1 \rightarrow b_{-1} = a_{-1} = \tfrac{\sqrt{2}}{2\pi}$$

$$k = 2 \rightarrow b_2 = a_2\, H(j2\omega_0) = 0 \text{ (since } 2\omega_0 = \pi/2 = 4\pi/8 \text{ is in the stopband)}$$

This means that $b_k = 0$ for $|k| \geq 2$

$$b_k = \begin{cases} \frac{1}{4}, & k = 0 \\ \frac{\sqrt{2}}{2\pi}, & k = \pm 1 \\ 0, & |k| \geq 2 \end{cases} \tag{2.7.19}$$

Step 3: Build the Output $y(t)$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = b_{-1}e^{-j\omega_0 t} + b_1 e^{j\omega_0 t} + b_0 \tag{2.7.20}$$

$$y(t) = \frac{\sqrt{2}}{2\pi}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{4} = \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos(\omega_0 t) \tag{2.7.21}$$

with $\omega_0 = \dfrac{\pi}{4}$, so

$$y(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right) \tag{2.7.22}$$

$$y(t) = \frac{1}{4} + \frac{\sqrt{2}}{\pi}\cos\left(\frac{\pi}{4}t\right) \approx 0.25 + 0.45\cos\left(\frac{\pi}{4}t\right) \tag{2.7.23}$$

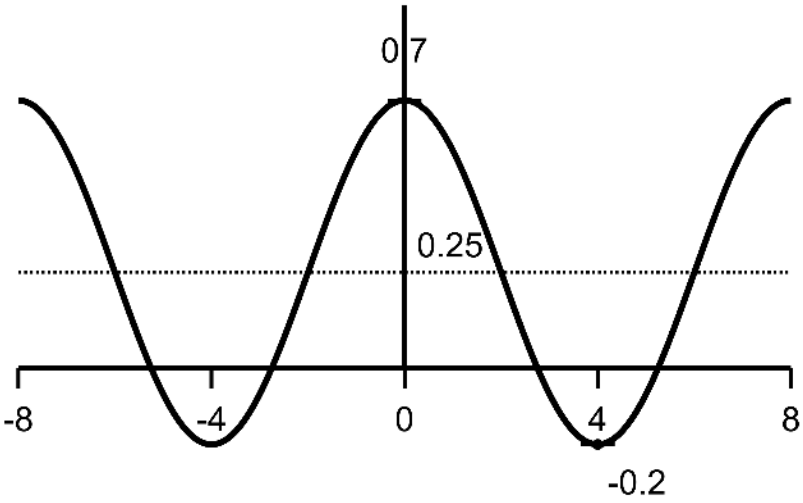



**Figure 2.7.7:** Output periodic wave signal from the given input (Fig. 2.7.5) and frequency response (Fig. 2.7.6).

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# 2.8: Differential Equations and Convergence

## Learning Objectives: Frequency Response of Differential Equations

1. Solve for the frequency response of simple differential equations
2. Find the output of these systems for sinusoidal inputs.

## YouTube Videos

Frequency Response for Differential Equation: Part 1



Frequency Response for Differential Equations: Part 2



## Summary

### Convergence of the Fourier Series

Fourier says that any periodic signal could be represented by a Fourier series. Although this is not quite true, it is true that the Fourier series can represent an extremely large class of periodic signals (i.e., all signals in this course).

Consider $x(t)$ as a linear combination of a finite number of harmonically related complex exponential

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt \qquad x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \tag{2.8.1}$$

Then, the approximate error

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \tag{2.8.2}$$

The energy in the error provides a quantitative measure of the approximation error:

$$E_N = \int_T |e_N(t)|^2 \, dt \tag{2.8.3}$$

Assuming

$$a_k = \frac{1}{T} \int_T x(t) \, e^{-jk\omega_0 t} \, dt, \tag{2.8.4}$$

if $x(t)$ has a Fourier series representation, the best approximation using only a finite number of harmonically related complex exponentials is obtained by truncating the Fourier series to the desired number of terms. As $N$ increases, new terms are added and $E_N$ decreases. In fact, if $x(t)$ has a Fourier series representation, then $E_N \to 0$ if $N \to \infty$. This statement is true for many discontinuous signals.

There are **2** different classes of conditions that a periodic signal can satisfy to guarantee that it can be represented by a Fourier series

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period, i.e., signals for which

**DIRICHLET'S CONDITION 1:**

$$\int_T |x(t)|^2 \, dt < \infty \quad \longrightarrow \quad \textbf{this condition guarantees that } a_k \textbf{ is finite.} \qquad |a_k| < \infty \tag{2.8.5}$$

**DIRICHLET'S CONDITION 2:** $x(t)$ is of bounded variation in any finite interval of time - i.e., there are no more than a finite number of maxima and minima during a single period of the signal.

**DIRICHLET'S CONDITION 3:** There are only a finite number of discontinuities.

> Signals that do not satisfy these Dirichlet conditions are not found in practical cases

For a periodic signal without discontinuities, the Fourier series representation converges and equals the original signal at every value of $t$.

For a periodic signal with a finite number of discontinuities in each period, the Fourier series representation equals the signal everywhere except at the isolated points of discontinuity, at which the series converges to the average value of the signal on either side of the discontinuity. In this case the difference between the original signal and its Fourier series representation contains no energy, and consequently, the two signals can be thought of as being the same for all practical purposes. Specifically, since the signals differ only at isolated points, the integrals of both signals over any interval are identical.

→ $x(t)$ and $x_N(t)$ behave identically under convolution and LTI systems

In the vicinity of the discontinuity, the frequency response exhibits ripples, and slow spectral Fourier amplitudes of the representation cannot be decreased with increasing values of $N$.

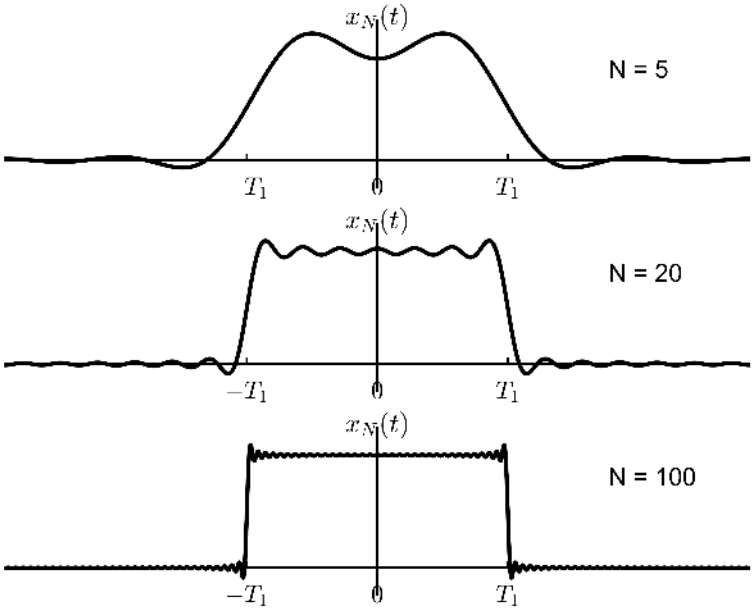



**Figure 2.8.1:** Representation of the Gibbs phenomenon for different values of N.

Gibbs phenomenon → as $N$ increases, the ripples in the partial sums become compressed towards the discontinuity.

The truncated Fourier series approx. $x_N(t)$ of a discontinuities signal $x(t)$ exhibits high frequency ripples and overshoot $x(t)$ near the discontinuities.

$N$ large → total energy in these ripples is insignificant

@ discontinuity, the partial sum converges to $\frac{1}{2}$ the sum of the values of the signal on either side of the discontinuity.

The closer $t$ is chosen to the point of discontinuity, the larger $N$, the smaller the error

### CT Filters described by Differential Equations

- Filters described by differential or difference equations can be easily implemented using either analog or digital hardware.
- Systems described by differential/difference equations offer an extremely broad and flexible range of designs

#### Example: NonIdeal Lowpass Filter

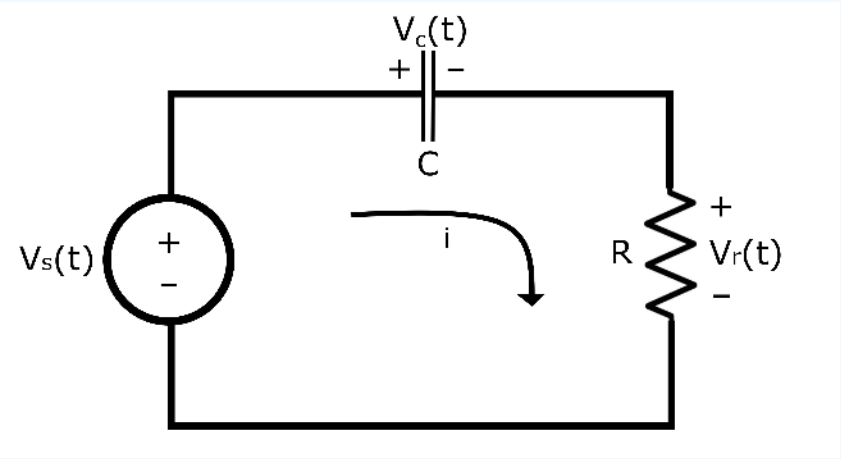



**Figure 2.8.2:** Example of a signal produced by an RC Circuit.

$V_s(t)$: system input
$V_c(t)$: capacitor voltage is the system output

$$RC\frac{dV_c(t)}{dt}+V_c(t)=V_s(t) \tag{2.8.6}$$

LTI system when the initial rest conditions are considered. Considering $V_s(t)=e^{j\omega t}$ and $V_c(t)=H(j\omega)e^{j\omega t}$

$$RC\,j\omega\,H(j\omega)e^{j\omega t}+H(j\omega)e^{j\omega t}=e^{j\omega t} \implies (1+j\omega RC)\,H(j\omega)=1 \tag{2.8.7}$$

$$H(j\omega)=\frac{1}{1+j\omega RC} \tag{2.8.8}$$

$$@\ \omega\approx 0 \rightarrow |H(j\omega)|\approx 1$$
$$|\omega| \text{ large} \rightarrow |H(j\omega)|\ll 1$$

$$h(t)=\frac{1}{RC}e^{-t/RC}u(t) \qquad \text{System impulse response} \tag{2.8.9}$$

$$s(t)=\left[1-e^{-t/RC}\right]u(t) \qquad \text{Step response} \tag{2.8.10}$$

If RC is large, frequencies other than the low ones are attenuated sufficiently.

If RC is large, the step response takes considerable time to reach its long-term value of 1.

RC is a small value; the filter passes higher frequencies.

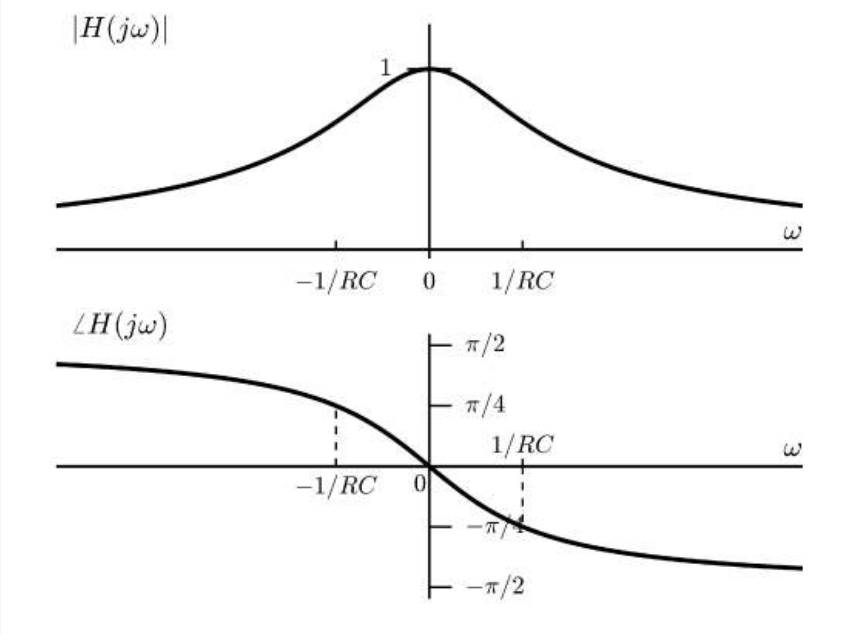



**Figure 2.8.3:** Plots of the magnitude and phase response for the RC Circuit's output $v_c(t)$ from Fig. 2.8.2.

## Example: RC Highpass Filter

→ capacitor voltage as the system output

$$RC\frac{dV_r(t)}{dt}+V_r(t)=RC\frac{dV_s(t)}{dt} \tag{2.8.11}$$

$$V_s(t)=e^{j\omega t}$$
$$V_r(t)=G(j\omega)e^{j\omega t}$$

$$G(j\omega)=\frac{j\omega RC}{1+j\omega RC} \tag{2.8.12}$$

This system $G(j\omega)$ attenuates lower frequencies and passes higher frequencies

@ $|\omega| \gg \frac{1}{RC}$ , minimal attenuation

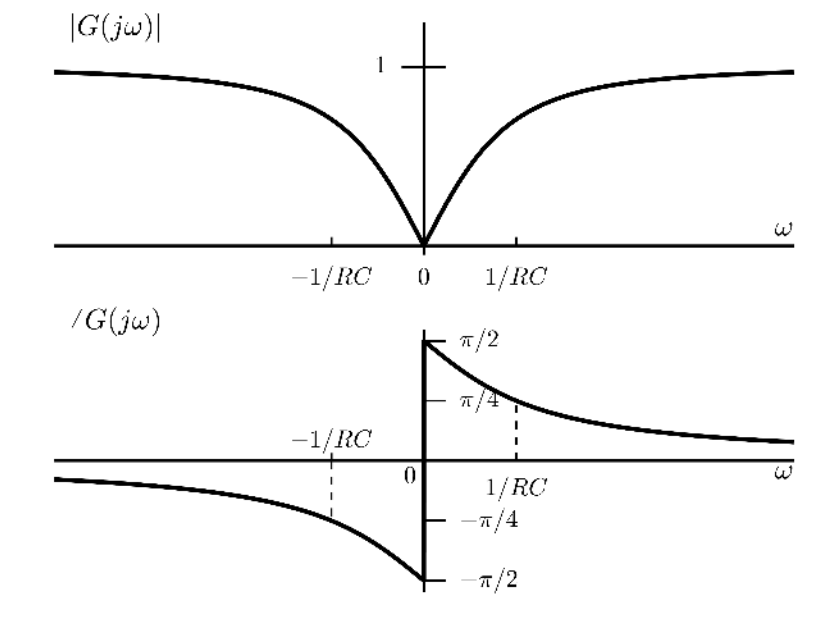
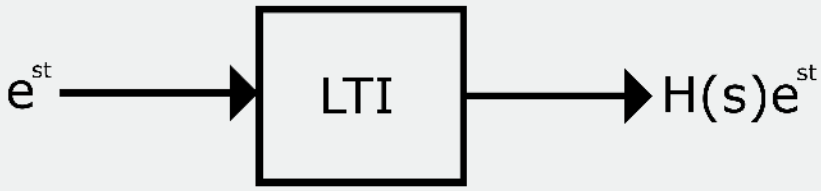



**Figure 2.8.4:** Plot of the magnitude and phase response for the RC Circuit's output $v_r(t)$ from Figure 2.8.2.

Remember: Complex Exponentials are Eigenfunctions of CT LTI systems




$$x(t) = e^{j\omega t} \rightarrow (\text{LTI}) \rightarrow y(t) = H(j\omega)e^{j\omega t}$$

$$\frac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t} \tag{2.8.13}$$

Example

1st-order differential equation

$$\frac{dy(t)}{dt} + ay(t) = b\frac{dx(t)}{dt} \tag{2.8.14}$$

To find the frequency response $H(j\omega)$, let's assume that $x(t)$ is the eigenfunction

$x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega)e^{j\omega t}$

$$\frac{dx}{dt} = j\omega e^{j\omega t} \qquad \frac{dy(t)}{dt} = j\omega H(j\omega)e^{j\omega t} \tag{2.8.15}$$

Insert in the differential eq

$$j\omega H(j\omega)e^{j\omega t} + a\, e^{j\omega t}H(j\omega) = j\omega b\, e^{j\omega t} \implies H(j\omega)\,[j\omega + a] = j\omega b \tag{2.8.16}$$

$$H(j\omega) = \frac{j\omega b}{j\omega + a} \tag{2.8.17}$$

Example

$$\frac{dy(t)}{dt} + ay(t) = b\frac{dx(t)}{dt} \qquad \Longrightarrow \qquad H(j\omega) = \frac{jb\omega}{j\omega + a} \tag{2.8.18}$$

Input 1: $x(t) = \cos(100at)\ \omega = 100a \ \rightarrow \ H(j100a) \simeq b$
high frequency $\gg a$

Applying Euler

$$x(t) = \frac{1}{2}e^{j100at} + \frac{1}{2}e^{-j100at} \tag{2.8.19}$$

$$\begin{aligned} y(t) &= \frac{1}{2}H(j100a)e^{j100at} + \frac{1}{2}H(-j100a)e^{-j100at} \\ &= \frac{1}{2}be^{j100at} + \frac{1}{2}be^{-j100at} = b\cos(100at) \end{aligned} \tag{2.8.20}$$

$$H(j100a) = \frac{jb(100a)}{j100a + a} = \frac{j100b}{j100 + 1} \simeq \frac{j100b}{j100} \simeq b \tag{2.8.21}$$

ignore Re because Re $\ll 100$

$$H(-j100a) \simeq b \tag{2.8.22}$$

output with same frequency as input but scaled by a factor of $b$.

Input 2: $x_2(t) = \cos\left(\frac{a}{100}t\right)$ and $H(j\omega) = \frac{j\omega b}{j\omega + a}$

$\omega = \frac{a}{100}$ Intuition $H\left(j\frac{a}{100}\right) \simeq 0$

$$x_2(t) = \frac{1}{2}e^{j\frac{a}{100}t} + \frac{1}{2}e^{-j\frac{a}{100}t} \tag{2.8.23}$$

$$y_2(t) = \frac{1}{2}H\left(j\frac{a}{100}\right)e^{j\frac{a}{100}t} + \frac{1}{2}H\left(-j\frac{a}{100}\right)e^{-j\frac{a}{100}t} \tag{2.8.24}$$

$$H\left(j\frac{a}{100}\right) = \frac{jb\left(\frac{a}{100}\right)}{j\left(\frac{a}{100}\right) + a} = \frac{jb/100}{j/100 + 1} \simeq \frac{jb/100}{1} = \frac{jb}{100} \tag{2.8.25}$$

Im $\ll$ Re

$$H\left(-j\frac{a}{100}\right) = -\frac{jb}{100} \tag{2.8.26}$$

$$y_2(t) = \frac{1}{2}\frac{jb}{100}e^{j\frac{a}{100}t} + \frac{1}{2}\left(-\frac{jb}{100}\right)e^{-j\frac{a}{100}t} = \frac{b}{100}j\left[\frac{e^{j\frac{a}{100}t} - e^{-j\frac{a}{100}t}}{2}\right] \tag{2.8.27}$$

$$y_2(t) = \frac{b}{100}\,j \cdot j\,\sin\left(\frac{a}{100}t\right) = -\frac{b}{100}\sin\left(\frac{a}{100}t\right) \tag{2.8.28}$$

still output oscillating with the frequency of the input, but with a phase delay since cos() changes to sin()

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# 2.9: CT Fourier Transform

## Learning Objectives: CT Fourier Transform

1. Compute the CT Fourier Transform of signals such as decaying exponentials, finite rectangular pulses, and simple finite-length signals.
2. Explain how different time domain properties of the signal impact the magnitude of the Fourier Transform

## YouTube Videos

Continuous-Time Fourier Transform Overview



CT Fourier Transform: Rectangular Pulse Example



(Optional) MITOpenCourseware Lecture 8 Continuous-Time Fourer Transform (Credit to Prof. Alan V. Oppenheim)



## Summary

In a previous class, we showed that truncated signals can be represented as a combination of complex exponentials.For aperiodic signals that are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum. → **FOURIER TRANSFORMS**

**FOURIER VISION** → An aperiodic signal is a periodic signal with an infinite period. (I.e., Non-periodic)

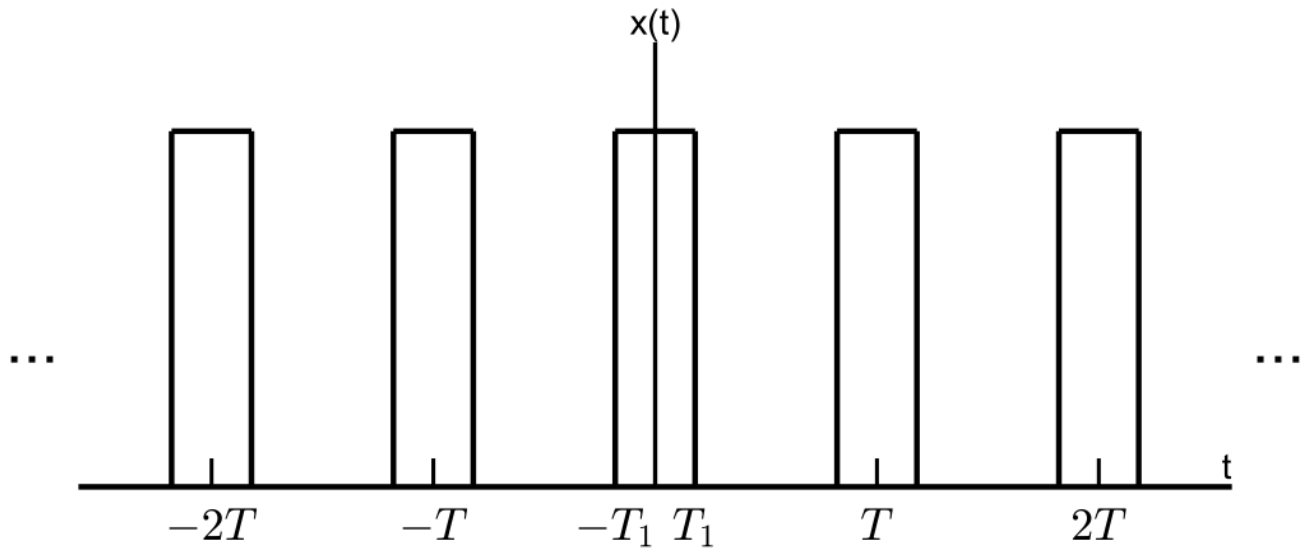



**Figure 2.9.1:** Periodic square wave signal with a fundamental frequency of T.

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases} \tag{2.9.1}$$

$$x(t) = \sum_k a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \tag{2.9.2}$$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \tag{2.9.3}$$

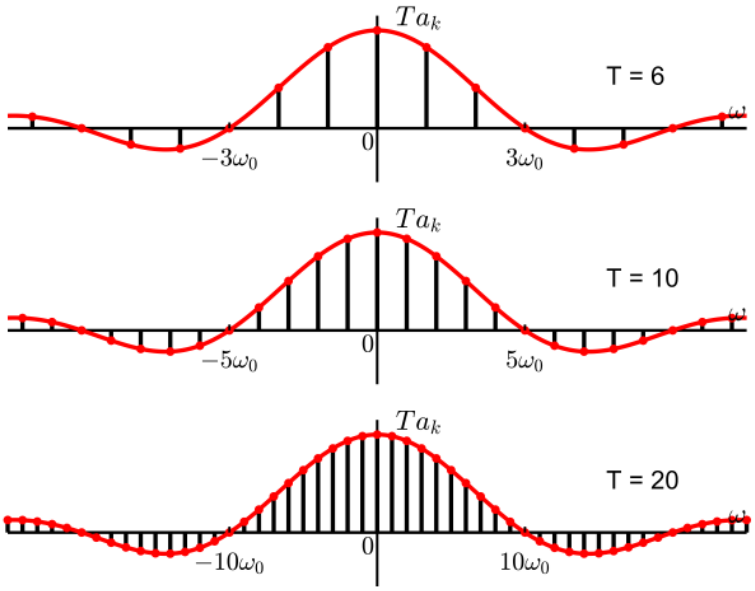



**Figure 2.9.2:** Representation of Figure 2.9.1's Fourier series coefficients and envelope for different values of T.

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \tag{2.9.4}$$

$$Ta_k = \frac{2\sin(\omega T_1)}{\omega} \qquad \omega = k\omega_0 \tag{2.9.5}$$

$$\omega_0 = \frac{2\pi}{T} \quad \rightarrow \quad \text{the envelope is sampled with closer and closer spacing} \tag{2.9.6}$$

$x(t)$ → Fourier Transform → $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$ → Inverse Fourier Transform → $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad Ta_k = X(j\omega) \tag{2.9.7}$$

$$a_k = \left.\frac{X(j\omega)}{T}\right|_{\omega = k\omega_0} \tag{2.9.8}$$

Relationship between the $a_k$ coefficients and the Fourier transform.

$X(j\omega)$ is the spectrum of $x(t)$

## Convergence of Fourier Transforms

**Definition:** $X(j\omega)$ is finite (i.e., the integral converges) if the signal $x(t)$ has finite energy, i.e.,

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt < \infty \tag{2.9.9}$$

**DIRICHLET CONDITIONS**

1. $x(t)$ is absolutely integrable

$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt < \infty \tag{2.9.10}$$

2. $x(t)$ has a finite number of minima and maxima within a finite interval
3. $x(t)$ has a finite number of discontinuities within a finite interval

**Example**

$x(t) = e^{-at}u(t)\ a > 0$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{+\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt \tag{2.9.11}$$
$$= \int_{0}^{+\infty} e^{-at} e^{-j\omega t}\, dt = \int_{0}^{+\infty} e^{-(j\omega + a)t}\, dt$$

$$= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_0^{\infty} = -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)\infty} - e^{-(a+j\omega)0}\right] = \frac{1}{a+j\omega} \qquad (a>0) \tag{2.9.12}$$

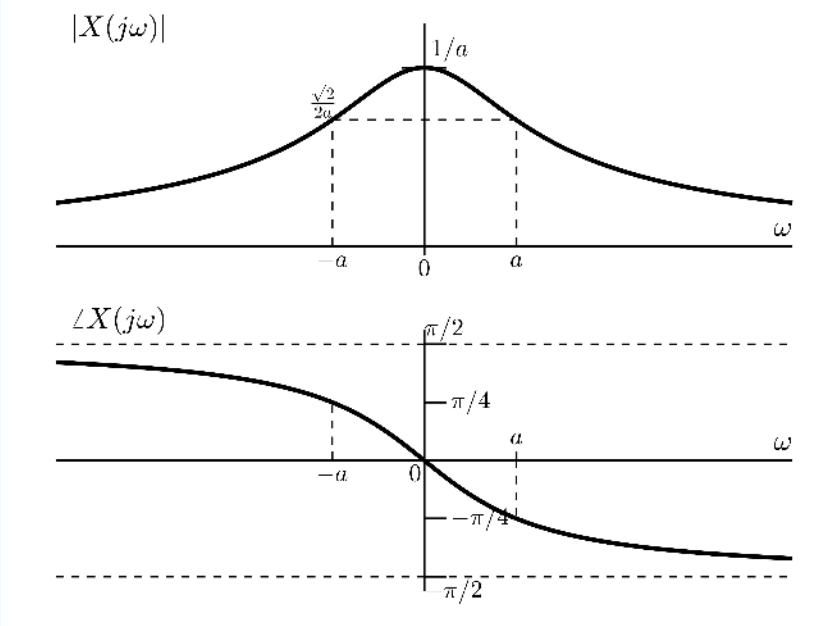



**Figure 2.9.3:** Plots of the magnitude and phase of the Fourier Transform.

$x(t) = e^{-at}u(t)\ a > 0$

$$X(j\omega) = \frac{1}{a+j\omega} \tag{2.9.13}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \text{Magnitude} \tag{2.9.14}$$

$$\angle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right) \tag{2.9.15}$$

**Example**

$x(t) = e^{-a|t|}\ ,\ a > 0$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-a|t|}\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{+\infty} e^{-at} e^{-j\omega t}\, dt \tag{2.9.16}$$

$$= \frac{e^{(a-j\omega)t}}{a-j\omega}\Big|_{-\infty}^{0} + \left(-\frac{e^{-(a+j\omega)t}}{a+j\omega}\right)\Big|_{0}^{+\infty} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \tag{2.9.17}$$

$$= \frac{(a+j\omega)+(a-j\omega)}{(a-j\omega)(a+j\omega)} = \frac{2a}{a^2+\omega^2} = X(j\omega) \tag{2.9.18}$$

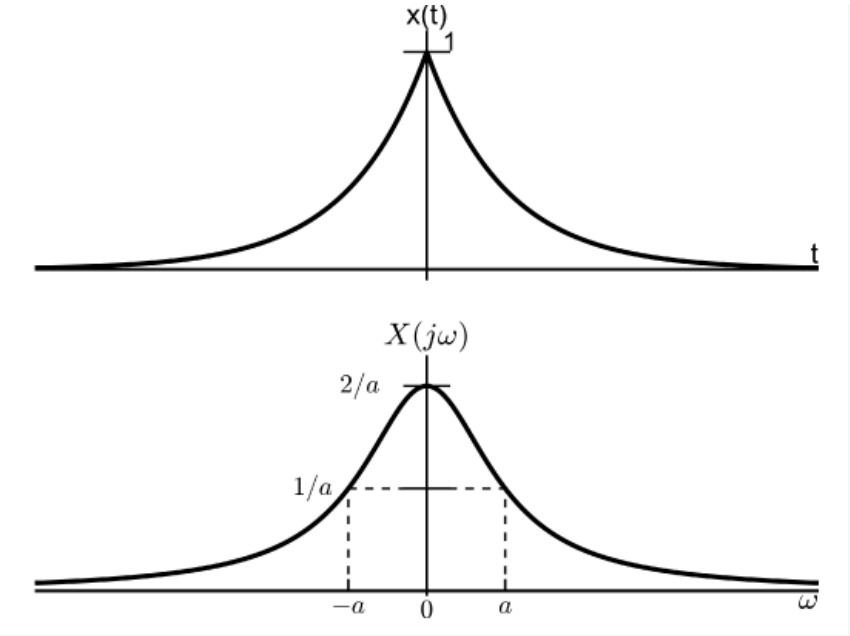



**Figure 2.9.4:** Illustration of the paired relationship between a continuous-time signal and its Fourier Transform.

## Example

$x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}\,dt = 1 \tag{2.9.19}$$

$e^{-j\omega 0} = e^{0} = 1$

The unit impulses have a Fourier Transform consisting of equal contributions at all frequencies

## Example

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \tag{2.9.20}$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt \tag{2.9.21}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right] = \frac{2\sin(\omega T_1)}{\omega} \tag{2.9.22}$$

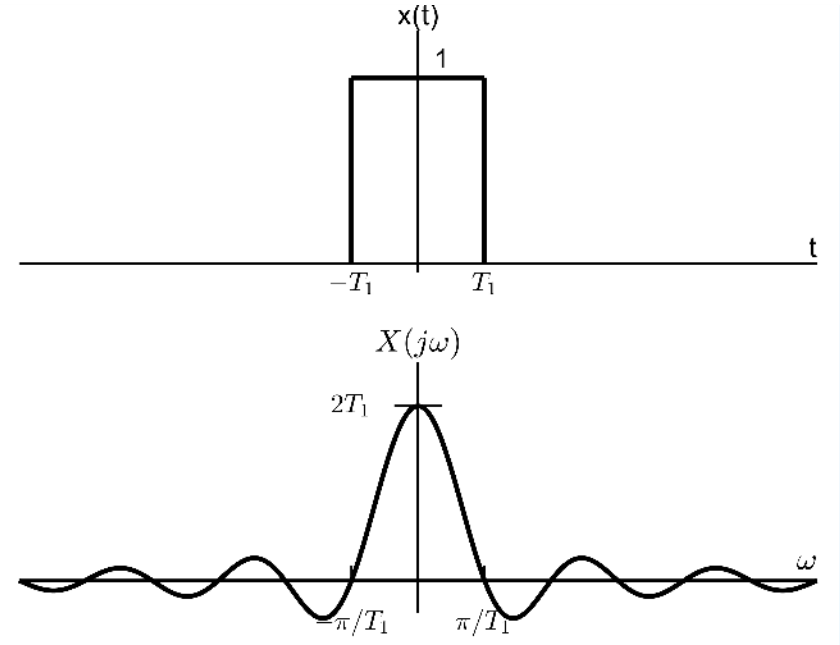



**Figure 2.9.5:** Illustration of the Fourier Transform of a rectangular signal.

Example

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \tag{2.9.23}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{1}{2\pi}\,\frac{e^{j\omega t}}{jt}\Big|_{-W}^{W} \tag{2.9.24}$$

$$= \frac{1}{2\pi jt}\left(e^{jWt} - e^{-jWt}\right) = \frac{\sin(Wt)}{\pi t} \tag{2.9.25}$$

$$\operatorname{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta} \tag{2.9.26}$$

$$\frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) \qquad \theta = \frac{Wt}{\pi} \tag{2.9.27}$$

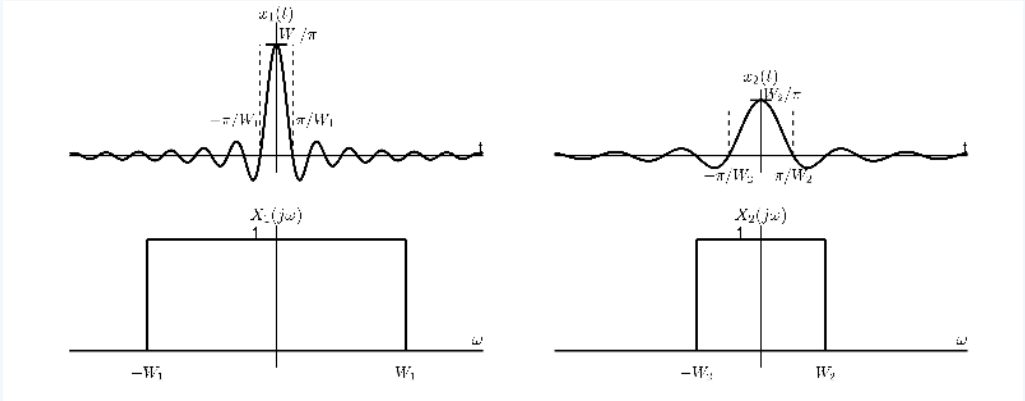



**Figure 2.9.6:** Illustration of the Fourier transform of a sinc function, demonstrating the impact on the width W. This figure highlights the inverse relationship between the width of the signal in both time and spectral domains. If the sinc function is narrow, its Fourier transform has a wider width, and vice versa.

Inverse duality between the time vs frequency/Fourier domain:

$x(t)$ narrow ——————————→ $X(j\omega)$ wide

$x(t)$ wide ——————————→ $X(j\omega)$ narrow

**Example**

$$X(j\omega) = 2\pi\,\delta(\omega-\omega_0) \quad \xrightarrow{\text{IFT}} x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \tag{2.9.28}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega-\omega_0)\,e^{j\omega t}\,d\omega = e^{j\omega_0 t} \tag{2.9.29}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\,\delta(\omega-k\omega_0) \tag{2.9.30}$$

linear combination of impulses equally spaced in frequency

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\sum_k 2\pi a_k\,\delta(\omega-k\omega_0)\right\}e^{j\omega t}\,d\omega \tag{2.9.31}$$

$$= \sum_k a_k\left\{\int_{-\infty}^{+\infty}\delta(\omega-k\omega_0)e^{j\omega t}\,d\omega\right\} = \sum_k a_k e^{jk\omega_0 t}$$

Fourier series of a periodic signal

The Fourier Transform of a periodic signal with Fourier series coefficients $\{a_k\}$ is a train of impulses located at harmonically-spaced frequencies

**Example**

$$x(t) = \sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{j2} \qquad \text{(Euler)} \tag{2.9.32}$$

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \tag{2.9.33}$$

$$X(j\omega) = \sum_k 2\pi a_k\,\delta(\omega-k\omega_0) = \frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0) \tag{2.9.34}$$

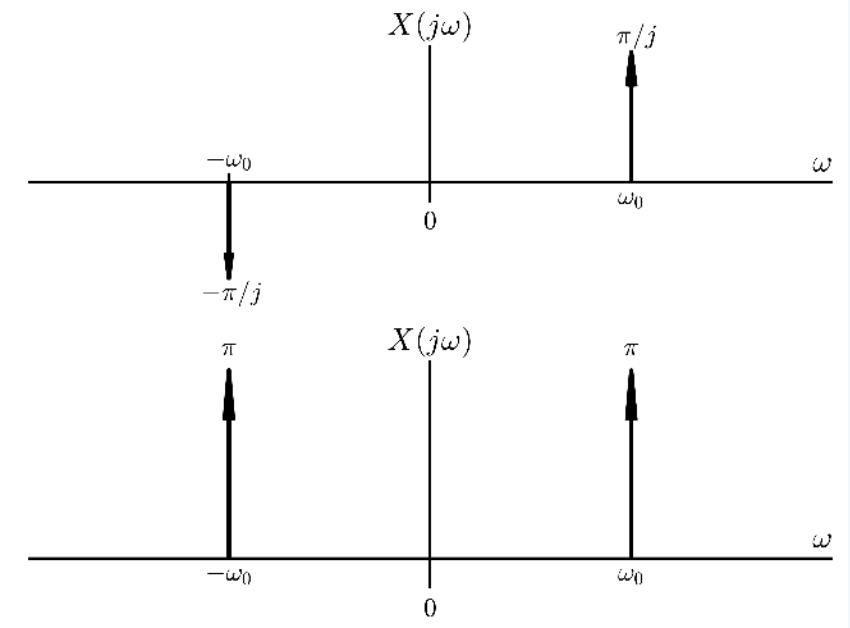



**Figure 2.9.7:** Fourier Transform of $\sin(\omega)$ (top) and $\cos(\omega)$ (bottom) signals.

$$x(t) = \cos(\omega_0 t) \quad \Longrightarrow \quad X(j\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) + \frac{\pi}{j}\delta(\omega + \omega_0) \tag{2.9.35}$$

Example

$$x(t) = \sum_k \delta(t - kT) \quad \Longrightarrow \quad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}e^0 = \frac{1}{T} \tag{2.9.36}$$

** all unit impulses have the same amplitude*

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right) \tag{2.9.37}$$

**SUPER IMPORTANT:** The Fourier transform of a periodic impulse train in the time domain with period $T$ is a periodic impulse train in the frequency domain with period $\frac{2\pi}{T}$. So, if $T$ increases (i.e., the impulses in the time domain get more spread out), then the impulses in the frequency/Fourier domain get closer.

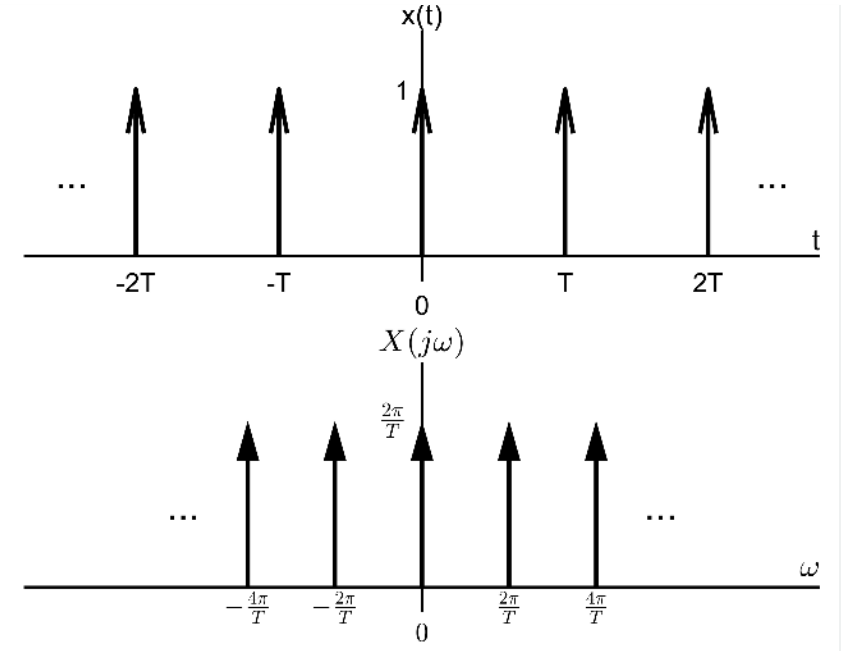



**Figure 2.9.8:** Representation of a continuous-time periodic impulse train and its periodic Fourier transform.

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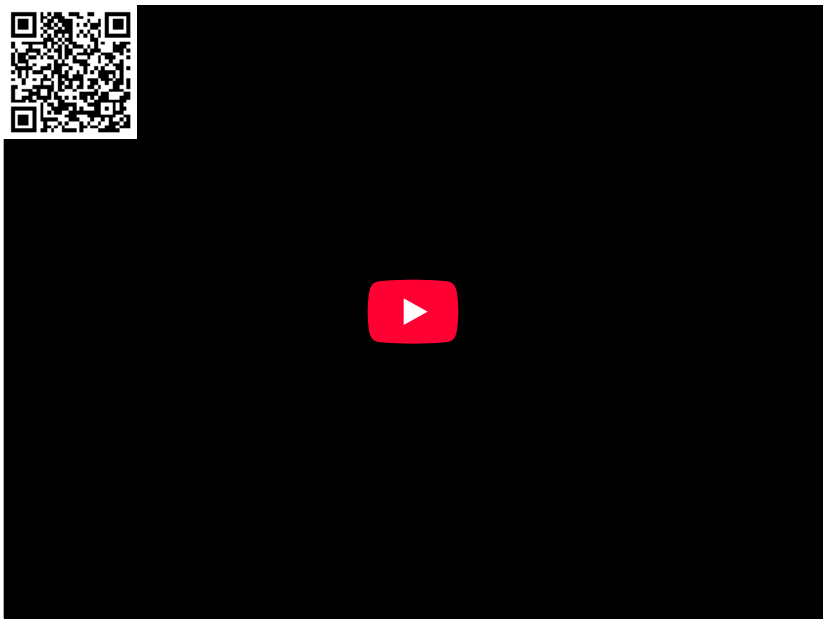
## 2.10: CT FT Properties

### Learning Objectives: CT Fourier Transform (FT) Properties

1. Practice with Fourier Transform properties

### YouTube Videos

MIT OpenCourseware Lecture 9 Fourier Transform Properties (*Credit to Prof. Alan W. Oppenheim*)



Symmetry property appears at 3:00 min

Parseval's theorem appears at 16:30 min

Delay/Time Shift property appears at 19 min

Convolution/Multiplication appears at 23 min

### Summary

Here, we discuss the FT properties, providing significant insight into the transform and the time- and frequency-domain descriptions of a signal.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \quad \overset{\text{FT}}{\longleftrightarrow} \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \tag{2.10.1}$$

$$x(t) = \text{IFT}\{X(j\omega)\} = \text{FT}^{-1}[X(j\omega)] \tag{2.10.2}$$

$$X(j\omega) = \text{FT}[x(t)] \tag{2.10.3}$$

**Remember:**

$$x(t) = e^{-at}u(t) \quad \longleftrightarrow \quad X(j\omega) = \frac{1}{a+j\omega} \tag{2.10.4}$$

Property - Linearity

If

$$x(t) \longleftrightarrow X(j\omega), \qquad y(t) \longleftrightarrow Y(j\omega), \$\$then\$\$ a\,x(t) + b\,y(t) \longleftrightarrow a\,X(j\omega) + b\,Y(j\omega). \tag{2.10.5}$$

Property - Time shifting

Consider

$$x(t) \longleftrightarrow X(j\omega), \;\; \text{thus} \;\; x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(j\omega). \tag{2.10.6}$$

**Proof:**

$$\begin{aligned} x(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(X(j\omega)e^{-j\omega t_0}\right)e^{j\omega t}\,d\omega \end{aligned} \tag{2.10.7}$$

$$X'(j\omega) = X(j\omega)e^{-j\omega t_0} \tag{2.10.8}$$

$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(j\omega) \tag{2.10.9}$$

a signal which is shifted in time does not have the magnitude of its Fourier Transform altered. The time shift only introduces a phase shift of $(-\omega t_0)$ in the phase of its Fourier Transform

**Previous class** $\rightarrow$ $x(t) = \text{rect}\left(\frac{t}{T}\right)$

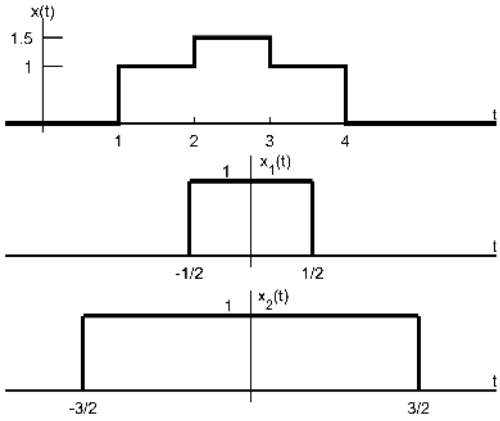



**Figure 2.10.1:** Representation of separating a more complicated signal (top) into two simpler signals (middle and bottom).

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5) \tag{2.10.10}$$

$$x_1(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases} \qquad x_2(t) = \begin{cases} 1, & |t| \le \frac{3}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{2.10.11}$$

$$\begin{aligned} X(j\omega) &= \frac{1}{2}X_1(j\omega)e^{-j\omega 2.5} + X_2(j\omega)e^{-j2.5\omega} \\ &= e^{-j2.5\omega}\left[\frac{1}{2}X_1(j\omega) + X_2(j\omega)\right] \end{aligned} \tag{2.10.12}$$

$$X_1(j\omega) = \int_{-\infty}^{+\infty} x_1(t)e^{-j\omega t}\,dt = \int_{-1/2}^{1/2} e^{-j\omega t}\,dt = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right) \tag{2.10.13}$$

$$X_2(j\omega) = \int_{-3/2}^{3/2} e^{-j\omega t}\,dt = \frac{2}{\omega}\sin\left(\frac{3\omega}{2}\right) \tag{2.10.14}$$

Thus

$$X(j\omega) = e^{-j5\omega/2}\left[\frac{1}{\omega}\sin\left(\frac{\omega}{2}\right) + \frac{2}{\omega}\sin\left(\frac{3\omega}{2}\right)\right] \tag{2.10.15}$$

Property - Conjugate and Conjugate Symmetry

Consider

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega), \text{ then } x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega) \tag{2.10.16}$$

$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt\right]^* = \int_{-\infty}^{+\infty} x^*(t)e^{j\omega t}\,dt \\ \text{change } \omega \to -\omega: \quad X^*(-j\omega) &= \int_{-\infty}^{+\infty} x^*(t)e^{-j\omega t}\,dt \end{aligned} \tag{2.10.17}$$

Definition of the FT of $x^*(t)$

Consider $x(t)$ is real so $x(t) = x^*(t)$ because there is no imaginary parts

$$X(j\omega) = X^*(-j\omega) \quad \Longleftrightarrow \quad X(-j\omega) = X^*(j\omega) \tag{2.10.18}$$

*(change $\omega$ to $-\omega$)*

In general,

$$X(j\omega) = \text{Re}[X] + j\,\text{Im}[X] = |X(j\omega)|\,e^{j\angle X(j\omega)} \tag{2.10.19}$$

if $x(t) =$ real $\Rightarrow$

$$\text{Re}[X(j\omega)] = \text{Re}[X(-j\omega)] \quad \text{and} \quad \text{Im}[X(j\omega)] = -\text{Im}[X(-j\omega)] \tag{2.10.20}$$

FT of a real function is even frequency function FT of a real function is a odd frequency function

$\Rightarrow$ $|X(j\omega)|$ is an even function of $\omega$ and $\angle X(j\omega)$ is an even function of $\omega$

If $x(t)$ is a real and even function $\Rightarrow$ $X(j\omega)$ is an even function

$$\begin{aligned} &X(j\omega) = X^*(j\omega) \text{ is an even function} \quad \Rightarrow \quad X(j\omega) \text{ is} \\ &\text{also a real and even function} \end{aligned} \tag{2.10.21}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt \\ X(-j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{j\omega t}\,dt \overset{t=-\tau}{=} \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}\,d\tau = X(j\omega) \quad (\text{even } x(\tau) = x(-\tau)) \end{aligned} \tag{2.10.22}$$

if $x(t)$ is real and odd function $\Rightarrow$ $X(j\omega)$ is purely imaginary and odd frequency function
$x(t) = -x(-t)$

In general

$$x(t) = x_e(t) + x_o(t) \quad \Longrightarrow \quad X(j\omega) = X_e(j\omega) + X_o(j\omega) \tag{2.10.23}$$

$x_e(t)$: *even* $x_o(t)$: *odd* $X_e(j\omega)$: *real function* $X_o(j\omega)$: *pure imaginary function*

**Example**

FT of $e^{-a|t|}$, $a>0$ ?

$$x(t)=e^{-a|t|}=e^{-at}u(t)+e^{at}u(-t)=2\frac{e^{-at}u(t)+e^{at}u(-t)}{2}=2\,\text{Even}[e^{-at}u(t)] \tag{2.10.24}$$

From Class 10:

$$e^{-at}u(t)\overset{\mathcal{F}}{\longleftrightarrow}\frac{1}{a+j\omega} \tag{2.10.25}$$

$$\text{Even}[e^{-at}u(t)]\longleftrightarrow\text{Re}\left[\frac{1}{a+j\omega}\right] \tag{2.10.26}$$

$$e^{-a|t|}\longleftrightarrow 2\,\text{Re}\left[\frac{1}{a+j\omega}\right]=2\frac{2a}{a^2+\omega^2} \tag{2.10.27}$$

Property: Differentiation and Integration

Consider $x(t)\longleftrightarrow X(j\omega)$,

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}\,d\omega \tag{2.10.28}$$

$$\frac{d}{dt}x(t)=\frac{d}{dt}\left[\frac{1}{2\pi}\int(\cdots)\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}j\omega\,X(j\omega)e^{j\omega t}\,d\omega \tag{2.10.29}$$

$$\boxed{\frac{dx(t)}{dt}\overset{\text{FT}}{\longleftrightarrow}j\omega\,X(j\omega)} \tag{2.10.30}$$

The operation of differentiation in time domain becomes to the product of $j\omega$ in the frequency domain

$$\boxed{\int_{-\infty}^{t}x(\tau)\,d\tau\overset{\text{FT}}{\longleftrightarrow}\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)} \tag{2.10.31}$$

The integration in the time domain becomes to the division of $j\omega$ plus a dc/average value

**Example**

FT of $u(t)$?

$$g(t)=\delta(t)\overset{\text{FT}}{\longleftrightarrow}G(j\omega)=1 \tag{2.10.32}$$

If

$$x(t)=\int_{-\infty}^{t}g(\tau)\,d\tau\longleftrightarrow X(j\omega) \tag{2.10.33}$$

$$=\frac{1}{j\omega}G(j\omega)+\pi G(0)\delta(\omega)\$\$then\$\$X(j\omega)=\frac{1}{j\omega}+\pi\delta(\omega)$$

$$\delta(t)=\frac{du(t)}{dt}\longleftrightarrow j\omega\left[\frac{1}{j\omega}+\pi\delta(\omega)\right]=1\quad\text{because}\quad\omega\delta(\omega)=0 \tag{2.10.34}$$

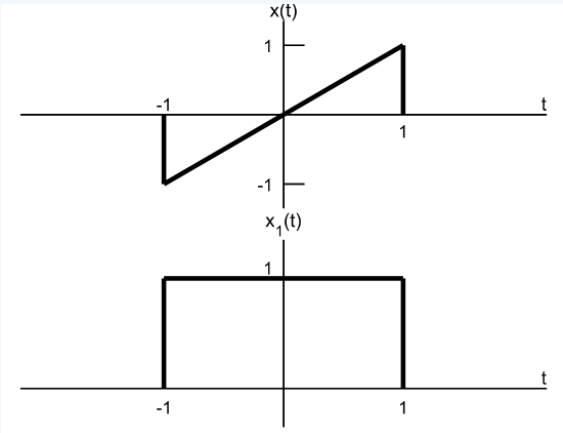



**Figure 2.10.2:** A continuous-time signal (top) and its derivative (bottom).

$g(t)=$ rectangle pulse + 2 impulses

$$G(j\omega)=\frac{2\sin(\omega)}{\omega}-e^{j\omega}-e^{-j\omega} \tag{2.10.35}$$

$$g(t)=\frac{dx(t)}{dt}\Rightarrow x(t)=\int g(t)\,dt \tag{2.10.36}$$

$$X(j\omega)=\frac{1}{j\omega}G(j\omega)+\pi G(0)\delta(\omega)\qquad(G(0)=0) \tag{2.10.37}$$

$$X(j\omega)=\frac{1}{j\omega}\left[\frac{2\sin(\omega)}{\omega}-e^{j\omega}-e^{-j\omega}\right] \tag{2.10.38}$$

$$X(j\omega)=\frac{2\sin(\omega)}{j\omega^2}-\frac{2\cos(\omega)}{j\omega} \tag{2.10.39}$$

Property - Time and Frequency scaling

$$x(t) \longleftrightarrow X(j\omega) \tag{2.10.40}$$

$$\int_{-\infty}^{+\infty} x(at)e^{-j\omega t}\,dt = \frac{1}{a}\int_{-\infty}^{+\infty} x(t')e^{-j\frac{\omega}{a}t'}\,dt' \tag{2.10.41}$$

$at = t'$

$$x(at) \longleftrightarrow \frac{1}{|a|}X\left(j\frac{\omega}{a}\right) \tag{2.10.42}$$

$$x(-t) \longleftrightarrow X(-j\omega) \tag{2.10.43}$$

Property - Duality

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \quad \longleftrightarrow \quad X_1(j\omega) = \frac{2\sin(\omega T_1)}{\omega} \tag{2.10.44}$$

$$x_2(t) = \frac{\sin(Wt)}{W} \quad \longleftrightarrow \quad X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \tag{2.10.45}$$

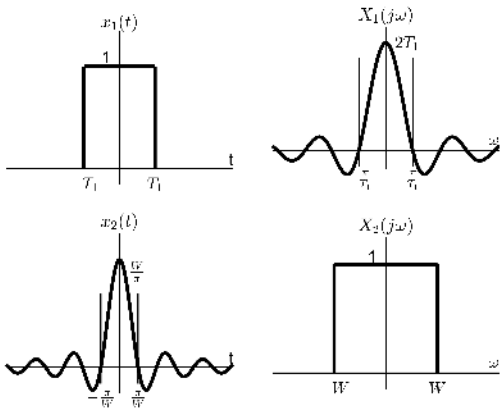



**Figure 2.10.3:** Representation of the duality property between the time and frequency domain.

$$-jtx(t) \longleftrightarrow \frac{dX(j\omega)}{d\omega} \tag{2.10.46}$$

$$e^{j\omega_0 t}x(t) \longleftrightarrow X[j(\omega - \omega_0)] \tag{2.10.47}$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \longleftrightarrow \int_{-\infty}^{\omega} X(j\omega)\,d\omega \tag{2.10.48}$$

Property - Parseval's relation

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\,d\omega \tag{2.10.49}$$

The total energy in the signal in time domain is equal to $\frac{1}{2\pi}$ times the total energy in the frequency domain

## Convolution Property

$$y(t)=x(t)*h(t)\Longleftrightarrow Y(j\omega)=X(j\omega)\cdot H(j\omega)$$

A LTI system is stable if

$$\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty \quad \longleftrightarrow \quad H(j\omega) \text{ exists} \tag{2.10.50}$$

**Example**

$h(t) = \delta(t - t_0) \longleftrightarrow H(j\omega) = e^{-j\omega t_0}$

$$Y(j\omega) = H(j\omega)\cdot X(j\omega) = e^{-j\omega t_0}X(j\omega) \longleftrightarrow y(t) = x(t - t_0) \tag{2.10.51}$$

**Example**

$y(t) = \frac{dx(t)}{dt} \longleftrightarrow Y(j\omega) = (j\omega)X(j\omega)$

$$H(j\omega) = j\omega \quad \text{Frequency Response of a differentiator} \tag{2.10.52}$$

**Example**

$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

$$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \tag{2.10.53}$$

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega) \tag{2.10.54}$$

Example

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases} \qquad h(t) = \frac{\sin(\omega_c t)}{\pi t} \tag{2.10.55}$$

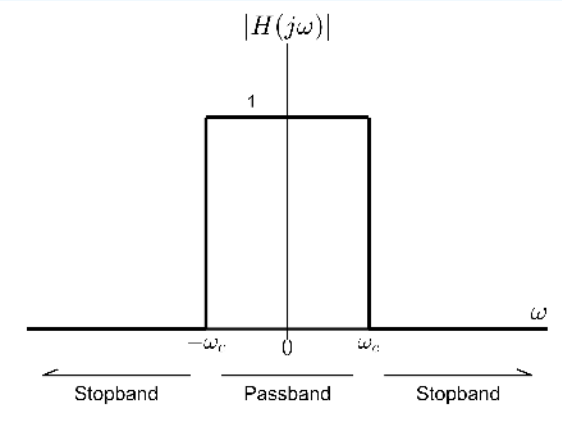



**Figure 2.10.4:** Representation of an ideal LPF's frequency response.

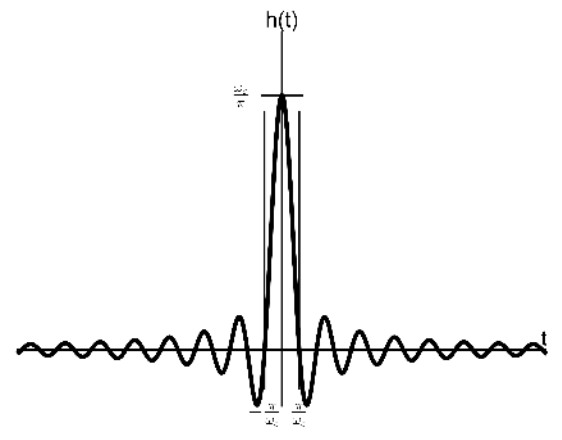



**Figure 2.10.5:** Representation of an ideal LPF's impulse response.

Example

$$h(t) = e^{-t}u(t) \leftrightarrow H(j\omega) = \frac{1}{1+j\omega} \tag{2.10.56}$$

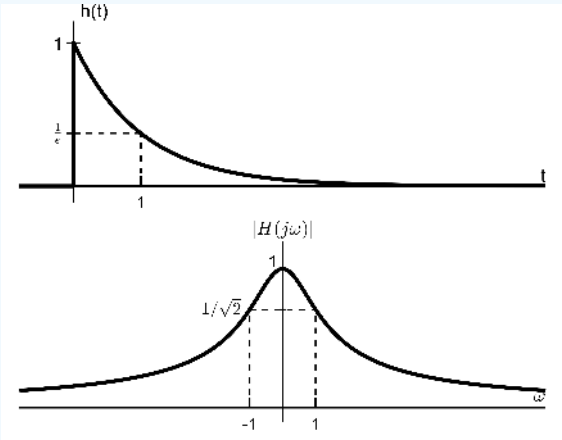



**Figure 2.10.6:** Representation of this impulse response (top) and its Fourier Transform (bottom).

Example

$$x(t) = e^{-bt}u(t) \leftrightarrow X(j\omega) = \frac{1}{b+j\omega} \tag{2.10.57}$$

$$h(t) = e^{-at}u(t) \leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \tag{2.10.58}$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) \tag{2.10.59}$$

$$Y(j\omega) = \frac{1}{b + j\omega}\frac{1}{a + j\omega} = \frac{A}{a + j\omega} + \frac{B}{b + j\omega} \tag{2.10.60}$$

$$\frac{1}{(a + j\omega)(b + j\omega)} = \frac{Ab + Aj\omega + Ba + Bj\omega}{(a + j\omega)(b + j\omega)} \tag{2.10.61}$$

$$Ab + Ba = 1 \qquad B(a - b) = 1 \;\Rightarrow\; B = \frac{1}{a - b} \tag{2.10.62}$$

$$Aj\omega + Bj\omega = 0 \;\Rightarrow\; A = -B \tag{2.10.63}$$

$$Y(j\omega) = \frac{1}{a - b}\left[\frac{1}{b + j\omega} - \frac{1}{a + j\omega}\right] \overset{\text{IFT}}{\leftrightarrow} y(t) = \frac{1}{a - b}\left[e^{-bt}u(t) - e^{-at}u(t)\right] \tag{2.10.64}$$

| Aperiodic signal | Fourier transform |
|---|---|
| $x(t),\ y(t)$ | $X(j\omega),\ Y(j\omega)$ |
| Parseval's Relation for Aperiodic Signals | |
| $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2\, dt$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2\, d\omega$ |
| Linearity | |
| $a\,x(t) + b\,y(t)$ | $a\,X(j\omega) + b\,Y(j\omega)$ |
| Time Shifting | |
| $x(t - t_0)$ | $e^{-j\omega t_0}\,X(j\omega)$ |
| Frequency Shifting | |
| $e^{j\omega_0 t}\,x(t)$ | $X\big(j(\omega - \omega_0)\big)$ |
| Conjugation | |
| $x^*(t)$ | $X^*(-j\omega)$ |
| Time Reversal | |
| $x(-t)$ | $X(-j\omega)$ |
| Time and Frequency Scaling | |
| $x(at)$ | $\frac{1}{\lvert a\rvert}\,X\left(j\frac{\omega}{a}\right)$ |
| Convolution | |
| $x(t) * y(t)$ | $X(j\omega)\,Y(j\omega)$ |
| Multiplication | |
| $x(t)\,y(t)$ | $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,Y\big(j(\omega - \theta)\big)\, d\theta$ |
| Differentiation in Time | |
| $\frac{d}{dt}x(t)$ | $j\omega\,X(j\omega)$ |
| Integration | |
| $\int_{-\infty}^{t} x(\tau)\, d\tau$ | $\frac{1}{j\omega}\,X(j\omega) + \pi X(0)\delta(\omega)$ |
| Differentiation in Frequency | |
| $t\,x(t)$ | $j\,\frac{d}{d\omega}X(j\omega)$ |
| Conjugate Symmetry for Real Signals | |
| $x(t)$ real | $X(j\omega) = X^*(-j\omega)$<br>$\Re\{X(j\omega)\} = \Re\{X(-j\omega)\}$<br>$\Im\{X(j\omega)\} = -\Im\{X(-j\omega)\}$<br>$\lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert$<br>$\angle X(j\omega) = -\angle X(-j\omega)$ |
| Symmetry for Real and Even Signals | |
| $x(t)$ real and even | $X(j\omega)$ real and even |
| Symmetry for Real and Odd Signals | |
| $x(t)$ real and odd | $X(j\omega)$ purely imaginary and odd |
| Even-Odd Decomposition for Real Signals | |
| $x_e(t) = \mathcal{E}\{x(t)\}$ $[x(t)$ real$]$<br>$x_o(t) = \mathcal{O}\{x(t)\}$ $[x(t)$ real$]$ | $\Re\{X(j\omega)\}$<br>$j\Im\{X(j\omega)\}$ |

| Signal | Fourier transform | Fourier series coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ | $2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$ | $a_k$ |
| $e^{j\omega_0 t}$ | $2\pi \delta(\omega - \omega_0)$ | $a_1 = 1$<br>$a_k = \{0, \quad \text{otherwise}$ |
| $\cos(\omega_0 t)$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ | $a_1 = a_{-1} = \frac{1}{2}$<br>$a_k = \{0, \quad \text{otherwise}$ |
| $\sin(\omega_0 t)$ | $\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ | $a_1 = -a_{-1} = \frac{1}{2j}$<br>$a_k = \{0, \quad \text{otherwise}$ |
| $x(t) = 1$ | $2\pi \delta(\omega)$ | $a_0 = 1, \quad a_k = 0, \ k \neq 0$<br>(this is the Fourier series representation for any choice of $T > 0$) |
| **Periodic square wave**<br>$x(t) = \begin{cases} 1, \lvert t \rvert \leq T_1 \\ 0, T_1 < \lvert t \rvert \leq \frac{T}{2} \end{cases}$<br>and<br>$x(t + T) = x(t)$ | $\sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k} \delta(\omega - k\omega_0)$ | $\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin(k\omega_0 T_1)}{k\pi}$ |
| $\sum_{n=-\infty}^{\infty} \delta(t - nT)$ | $\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$ | $a_k = \frac{1}{T}$ for all $k$ |
| $x(t) = \begin{cases} 1, \lvert t \rvert \leq T_1 \\ 0, \lvert t \rvert > T_1 \end{cases}$ | $\frac{2\sin(\omega T_1)}{\omega}$ | — |
| $\frac{\sin(Wt)}{\pi t}$ | $X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$ | — |
| $\delta(t)$ | $1$ | — |
| $u(t)$ | $\frac{1}{j\omega} + \pi\delta(\omega)$ | — |
| $\delta(t - t_0)$ | $e^{-j\omega t_0}$ | — |
| $e^{-at}u(t), \ \Re\{a\} > 0$ | $\frac{1}{a + j\omega}$ | — |

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# 2.11: Modulation and Differential Equation

## Learning Objectives: Modulation and Differential Equations

1. Exploit the filtering and modulation properties to find outputs of LTI systems through the frequency domain.
2. Solve for the frequency response of a differential equation using the derivative property.
3. Solve for the impulse response of a differential equation via the frequency domain using partial fractions

### YouTube Videos

Finding the frequency response and impulse response from a differential equation



### Summary

Multiplication Property

$$r(t) = s(t) \cdot p(t) \quad \text{(product)}$$

$$r(t) = s(t) \cdot p(t) \quad \leftrightarrow \quad R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\omega_1)\, P\big(j(\omega - \omega_1)\big)\, d\omega_1 \tag{2.11.1}$$

Because the duality between the time and frequency domains

$$R(j\omega) = \frac{1}{2\pi}\, S(j\omega) * P(j\omega) \tag{2.11.2}$$

$r(t) = s(t) \cdot p(t) \;\Rightarrow\; p(t)$ modulates/scales the $s(t)$ signal

OR

**AMPLITUDE MODULATION** $s(t)$ modulates/scales $p(t)$ signal

**Example**

Consider the signal s(t) shown in Figure 2.11.1.

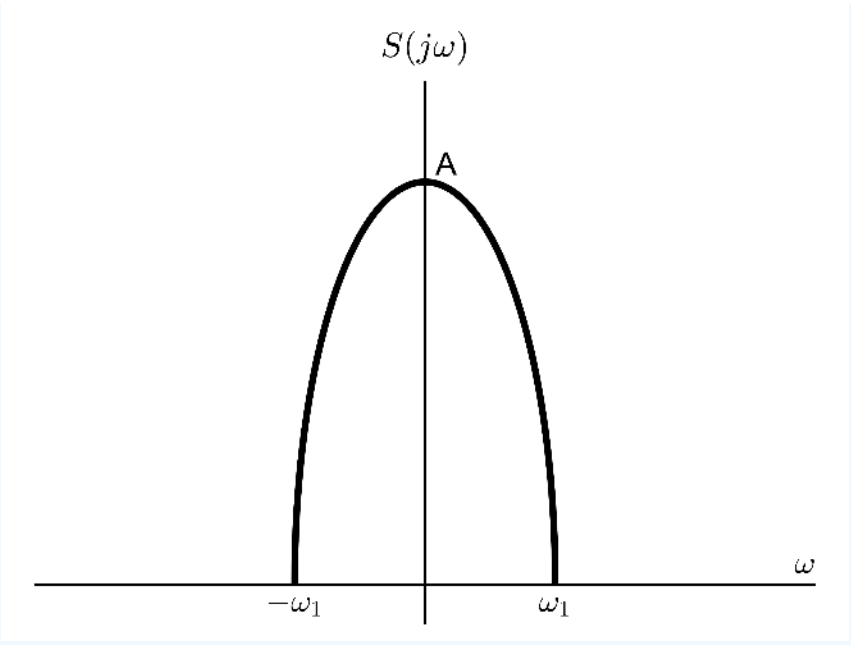



**Figure 2.11.1:** Example of a signal in the frequency domain.

Also consider the signal

$$p(t) = \cos(\omega_0 t) \tag{2.11.3}$$

Then

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \tag{2.11.4}$$

$$r(t) = s(t) \cdot p(t) \quad \xrightarrow{\text{FT}} \quad R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega) \tag{2.11.5}$$

$$R(j\omega) = \frac{1}{2\pi} S(j\omega) * \pi\delta(\omega - \omega_0) + \frac{1}{2\pi} S(j\omega) * \pi\delta(\omega + \omega_0) \tag{2.11.6}$$

$$R(j\omega) = \frac{1}{2} S\big(j(\omega - \omega_0)\big) + \frac{1}{2} S\big(j(\omega + \omega_0)\big) \tag{2.11.7}$$

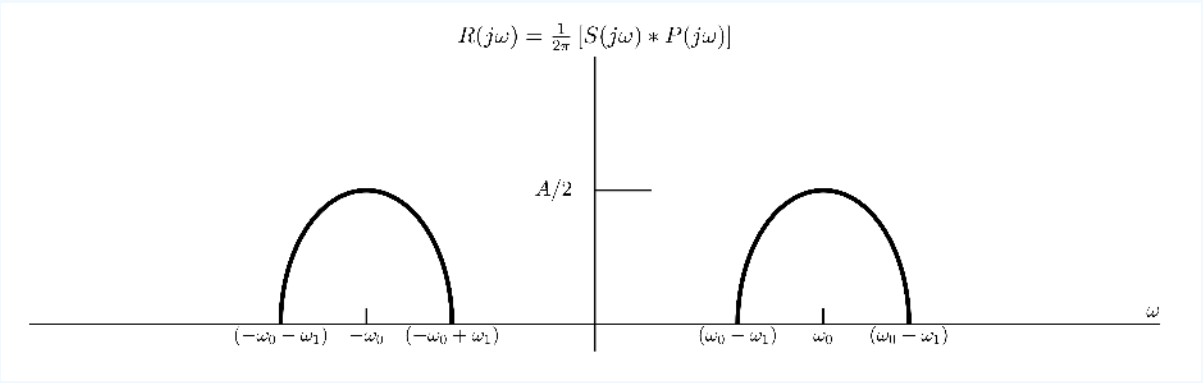



**Figure 2.11.2:** Spectrum of $R(j\omega)$ as the convolution in the frequency domain between the signal's spectrum from Figure 2.11.1 and the Fourier transform of $\cos(\omega_0)$.

The spectrum of $R(j\omega)$ consists of two shifted replicas of $S(j\omega)$ scaled by $1/2$.


https://eng.libretexts.org/@go/page/126959

**Example**

Consider $r(t)$, obtained in the previous example, and let

$$g(t) = r(t)p(t) \quad \Rightarrow \quad G(j\omega) = \frac{1}{2\pi} R(j\omega) * P(j\omega) \tag{2.11.8}$$

where $p(t) = \cos(\omega_0 t)$, and $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are seen in Figure 2.11.3.

From **Figure 2.11.3** combined with the linearity of the Fourier transform, we can deduce that $g(t)$ is the sum of $(1/2)s(t)$, our original signal from the previous example, and two separate $(1/4)s(t)$ centered at $\omega_0$ and $-\omega_0$.

**Key difference** → $R(j\omega)$ is form by two replicas

$$P(j\omega) = \pi\omega_0$$

unit impulses with amplitude $\pi$

centered at $\omega = -\omega_0$ and $\omega = +\omega_0$

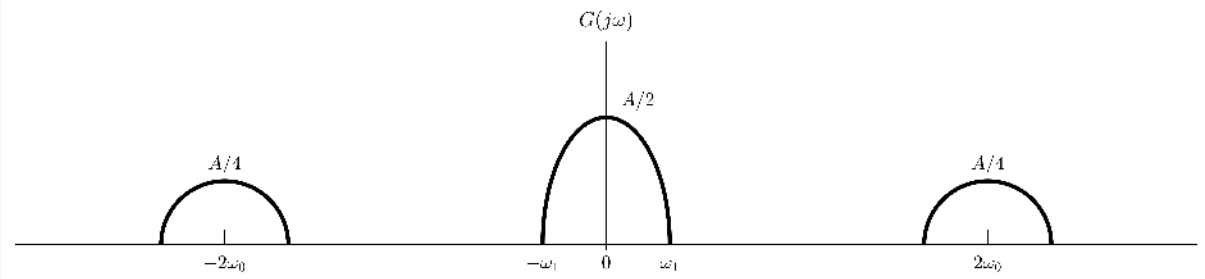



**Figure 2.11.3:** Spectrum of $G(j\omega)$ as the convolution in the frequency domain between $R(j\omega)$ from Figure 2.11.2 and the Fourier transform of $cos(\omega_0)$.

**Example**

$$x(t) = \frac{\sin(t)\,\sin(t/2)}{\pi^2 t^2} = \left[\frac{\sin(t)}{\pi t}\right]\left[\frac{\sin(t/2)}{\pi t}\right] \tag{2.11.9}$$

$$X(j\omega) = \frac{1}{2\pi}\,\mathrm{rect}\left(\frac{\omega}{2}\right) * \mathrm{rect}(\omega) \tag{2.11.10}$$

Convolution of two rectangulars of different size

Fourier pair:

$$\frac{\sin(at)}{\pi t} \quad \leftrightarrow \quad \mathrm{rect}\left(\frac{\omega}{2a}\right) \tag{2.11.11}$$

$$\mathrm{rect}\left(\frac{\omega}{2}\right) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & |\omega| > 1 \end{cases} \tag{2.11.12}$$

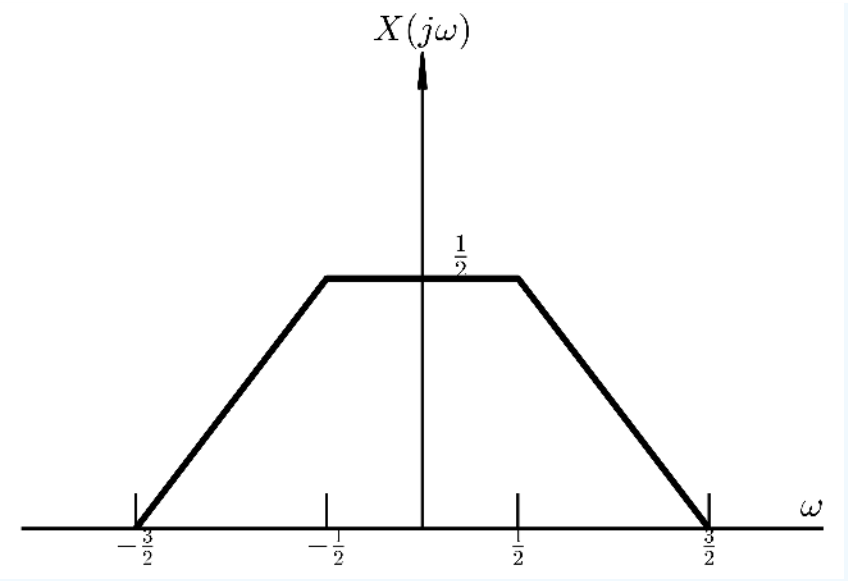



**Figure 2.11.4:** The Fourier transform of $x(t)$ from this example.

## Systems Characterized by Linear Constant-Coefficient Differential Eq.

Systems whose input and output satisfy a linear constant-coefficient differential eq:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \rightarrow \quad \text{how to determine } H(j\omega)? \tag{2.11.13}$$

**Method #1: Eigenfunctions of LTI systems**

If $x(t) = e^{j\omega t}$, the output $y(t) = H(j\omega)e^{j\omega t}$

Using these relationships into the differential equation, we can solve for $H(j\omega)$

**Method #2: Convolution property**

$$Y(j\omega) = H(j\omega) \cdot X(j\omega) \;\Rightarrow\; H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{2.11.14}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.11.15}$$

Order $= \max\{M, N\}$

$$\mathrm{FT}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathrm{FT}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

From the linearity property

$$\sum a_k \,\mathrm{FT}\left[\frac{d^k y(t)}{dt^k}\right] = \sum b_k \,\mathrm{FT}\left[\frac{d^k x(t)}{dt^k}\right] \tag{2.11.16}$$

using the differentiation property:

$$\mathrm{FT}\left[\frac{d^k y(t)}{dt^k}\right] = (j\omega)^k Y(j\omega) \tag{2.11.17}$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \tag{2.11.18}$$

$Y(j\omega)$ and $X(j\omega)$ can be moved outside $\sum$ because it does not depend on $k$

$$Y(j\omega)\sum_{k=0}^{N} a_k(j\omega)^k = X(j\omega)\sum_{k=0}^{M} b_k(j\omega)^k \tag{2.11.19}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k} \tag{2.11.20}$$

$b_k$: coefficients of input
$a_k$: coefficient of output

$H(j\omega)$ is a rational function (i.e., ratio of polynomials in $j\omega$)

**Example**

$$\frac{dy(t)}{dt} + a\,y(t) = x(t) \tag{2.11.21}$$

Step 1 - Identification of coefficients $a_1 = 1,\ a_0 = a,\ b_0 = 1$

Step 2 -

$$H(j\omega) = \frac{\sum b_k(j\omega)^k}{\sum a_k(j\omega)^k} = \frac{b_0(j\omega)^0}{a_0(j\omega)^0 + a_1(j\omega)^1} = \frac{1}{a + j\omega} \tag{2.11.22}$$

Step 3 - Perform the Inverse Fourier Transform of $H(j\omega)$

$$h(t) = \text{IFT}[H(j\omega)] = \text{IFT}\left[\frac{1}{a + j\omega}\right] = e^{-at}u(t) \tag{2.11.23}$$

**Example**

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t) \tag{2.11.24}$$

Step 1 - Identification of coefficients
$a_2 = 1,\ a_1 = 4,\ a_0 = 3$
$b_1 = 1,\ b_0 = 2$

Step 2 - Build $H(j\omega)$ →

$$H(j\omega) = \frac{2 + j\omega}{3 + 4j\omega + (j\omega)^2} = \frac{2 + j\omega}{3 - \omega^2 + 4j\omega} \tag{2.11.25}$$

Calculate the roots of denominator:

$$3 + 4j\omega + (j\omega)^2 = 0 \tag{2.11.26}$$

$j\omega = x$

$$3 + 4x + x^2 = 0 \quad \Rightarrow \quad x = \frac{-4 \pm \sqrt{16 - 4\cdot 3}}{2} = \frac{-4 \pm 2}{2} \Rightarrow x = -1, -3 \tag{2.11.27}$$

$$3 + 4x + x^2 = (x+1)(x+3) \ \Rightarrow\ \text{replace } x \text{ by } j\omega \ \Rightarrow\ (j\omega + 1)(j\omega + 3) \tag{2.11.28}$$

$$H(j\omega) = \frac{2 + j\omega}{(j\omega + 1)(j\omega + 3)} = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 3} \tag{2.11.29}$$

Step 3 - Solve for $A$ and $B$ →

$$A(j\omega + 3) + B(j\omega + 1) = 2 + j\omega \tag{2.11.30}$$

$A + B = 1$
$3A + B = 2$

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$
$B = 1 - A = \frac{1}{2}$

$$H(j\omega) = \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3} \tag{2.11.31}$$

$$\xrightarrow{\text{IFT}} \quad h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t) \tag{2.11.32}$$

$\text{IFT}\left[\frac{1}{j\omega+1}\right] = e^{-t}u(t)$
$\text{IFT}\left[\frac{1}{j\omega+3}\right] = e^{-3t}u(t)$

If

$$x(t) = e^{-t}u(t) \quad \rightarrow \quad X(j\omega) = \frac{1}{j\omega + 1} \tag{2.11.33}$$

Thus

$$Y(j\omega) = H(j\omega) \cdot X(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)} \cdot \frac{1}{j\omega + 1} = \frac{(j\omega + 2)}{(j\omega + 1)^2(j\omega + 3)} \tag{2.11.34}$$

$$Y(j\omega) = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)} = \frac{D}{j\omega + 1} + \frac{E}{(j\omega + 1)^2} + \frac{F}{j\omega + 3} \tag{2.11.35}$$

partial fraction expansion

$$D(j\omega + 1)(j\omega + 3) + E(j\omega + 3) + F(j\omega + 1)^2 = j\omega + 2 \tag{2.11.36}$$

$$D(-\omega^2 + 4j\omega + 3) + E(j\omega + 3) + F(-\omega^2 + 2j\omega + 1) = j\omega + 2 \tag{2.11.37}$$

no terms with $\omega^2$ → $D + F = 0 \Rightarrow D = -F$

Imag:

$$4\omega D + E\omega + 2\omega F = \omega \;\Rightarrow\; 4D + E + 2F = 1 \tag{2.11.38}$$

$$\Rightarrow\; 4D + E - 2D = 1 \Rightarrow 2D + E = 1 \Rightarrow E = 1 - 2D \tag{2.11.39}$$

Re (without $\omega$ term):

$$3D + 3E + F = 2 \tag{2.11.40}$$

substitute $F = -D$ and $E = 1 - 2D$:

$$3D + 3(1 - 2D) - D = 2 \Rightarrow -4D + 3 = 2 \Rightarrow D = \frac{1}{4} \tag{2.11.41}$$

$$F = -D = -\frac{1}{4} \qquad E = 1 - 2D = 1 - \frac{1}{2} = \frac{1}{2} \tag{2.11.42}$$

$$Y(j\omega) = \frac{D}{j\omega + 1} + \frac{E}{(j\omega + 1)^2} + \frac{F}{j\omega + 3} \xrightarrow{\text{IFT}} y(t) \tag{2.11.43}$$
$$= De^{-t}u(t) + Ete^{-t}u(t) + Fe^{-3t}u(t)$$

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# 2.12: Frequency Response and Filtering

## Learning Objectives: Filter Magnitude and Phase Response, and Filter Design

1. Understand how the magnitude and phase of a frequency response effect each frequency in the input Fourier transform.
2. Interpret the specifications for practical (nonideal) filter designs
3. Design a CT Butterworth lowpass filter in MATLAB

## YouTube Videos

Interpreting the CT Frequency Response with Magnitude and Phase



Specifications for Practical CT Filters



Designing CT Butterworth Filters in MATLAB



## Summary

Remember that, in LTI systems, difference equations and convolution operators in time domain becomes algebraic operations in frequency domain

### Magnitude-Phase Representation of FT

$$X(j\omega) = |X(j\omega)|\, e^{j\angle X(j\omega)} \tag{2.12.1}$$

In general, \(X(j\omega)\) is complex. $X(j\omega)$ decomposes $x(t)$ into a sum of complex exponentials at different frequencies where $|X(j\omega)|^2$ is the energy-density spectrum of $x(t)$. In fact, $|X(j\omega)|^2 \frac{d\omega}{2\pi}$ is the amount of energy in the signal $x(t)$ that lies in the infinitesimal frequency band between $\omega$ and $\omega + d\omega$.

The phase angle $\angle X(j\omega)$ does not affect the amplitudes of the individual frequency components, but it provides information regarding the relative phases of the complex exponentials. The $\angle X(j\omega)$ term contains a significant information on $x(t)$ and its nature.

### Time-Domain Properties of Ideal Frequency-Selective Filters

Selective filters: LTI systems with frequency responses chosen so as to pass one or several bands of frequencies with little or no attenuation and to stop or significantly attenuate frequencies outside those bands

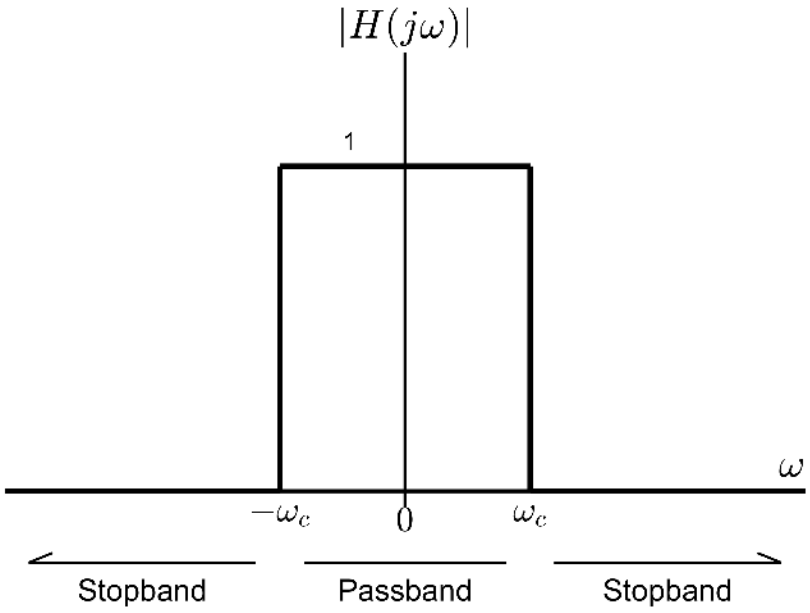



**Figure 2.12.1:** Ideal lowpass filter with a cutoff at $\omega = \omega_c$.

$$H_c(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases} \tag{2.12.2}$$

all input frequencies $\leq \omega_c$ pass without attenuation
input frequencies $> \omega_c$ are blocked

Nonideal Filter Specifications

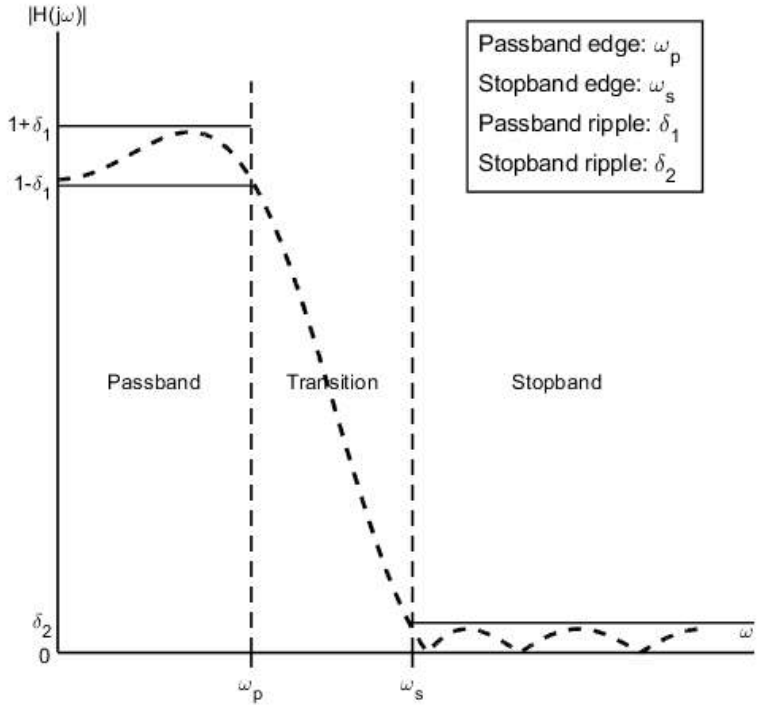



**Figure 2.12.2:** Illustration of the magnitude of the frequency response for a non-ideal filter $H(j\omega)$.

$\Rightarrow$ Magnitude of $H_c(j\omega)$ where the magnitude can be over the unity within the passband. A deviation of $\delta_2$ from 0 is allowed in the stopband

$\delta_1$: passband ripple $\rightarrow$ amount by which $|H_c(j\omega)|$ differs from 1.

$\delta_2$: stopband ripple $\rightarrow$ amount by which $|H_c(j\omega)|$ differs from 0.

$\omega_p$: passband edge
$\omega_s$: stopband edge
$\Delta\omega = \omega_s - \omega_p$ transition band

> A narrow transition band creates more ripples in the step response.

Example

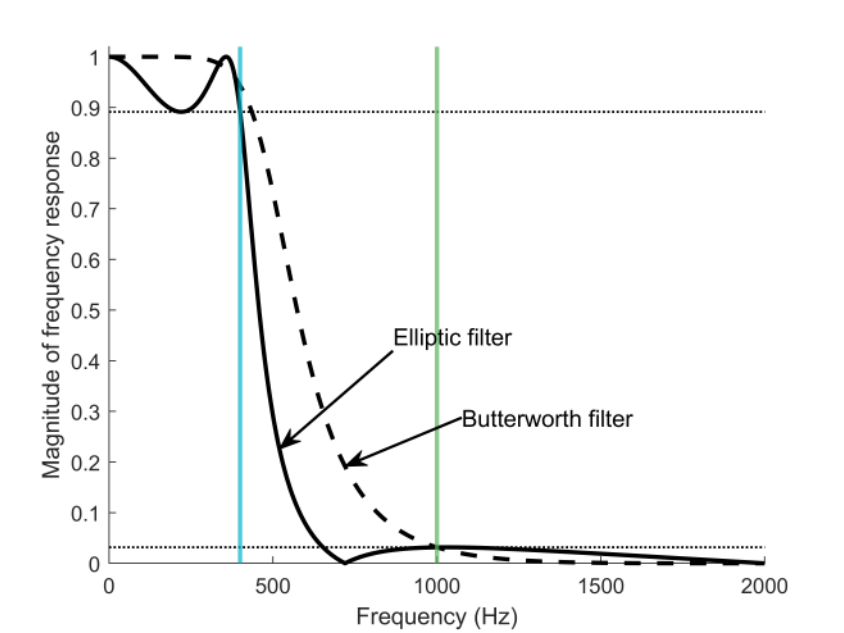



**Figure 2.12.3:** Plot of the magnitude of the frequency response for a non-ideal elliptic and Butterworth filter.

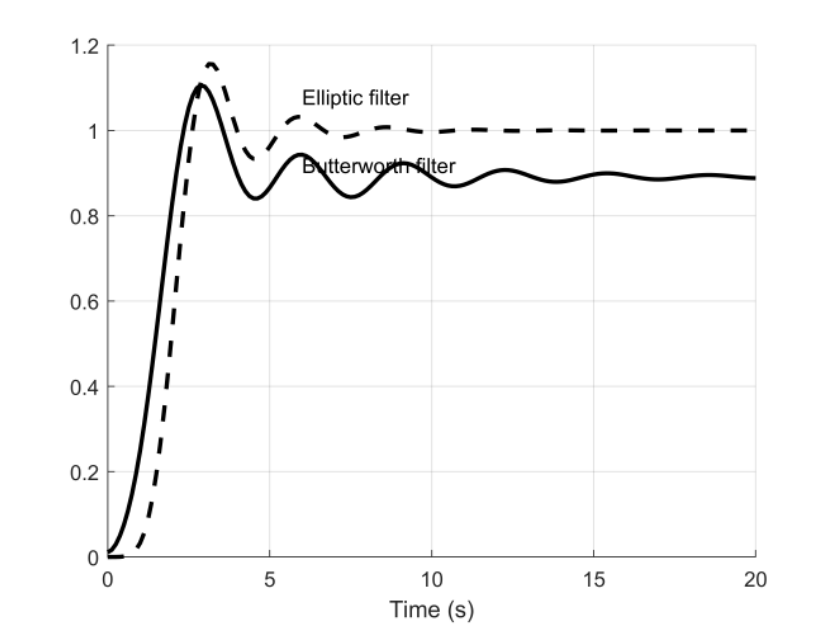



***Figure 2.12.4:*** Plot of the step response in time for a non-ideal elliptic and Butterworth filter.

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# 2.13: Bode Plots - First-order systems

## Learning Objectives: Bode-plots for first-order systems

1. Sketch Bode plots for first-order factors, correctly indicating the slope and the breakpoint frequency
2. Combine multiple first-order Bode plots to approximate the frequency response magnitude for higher-order systems
3. Discuss the phase for Bode plots.

### YouTube Videos

First-order Bode plots: Theory



First-order Bode plots: Example



### Summary

Bode plot ~ approximation of the $|H(j\omega)|$ magnitude frequency response in a log-log scale

Therefore, the Bode plot is a tool for designing and analyzing systems

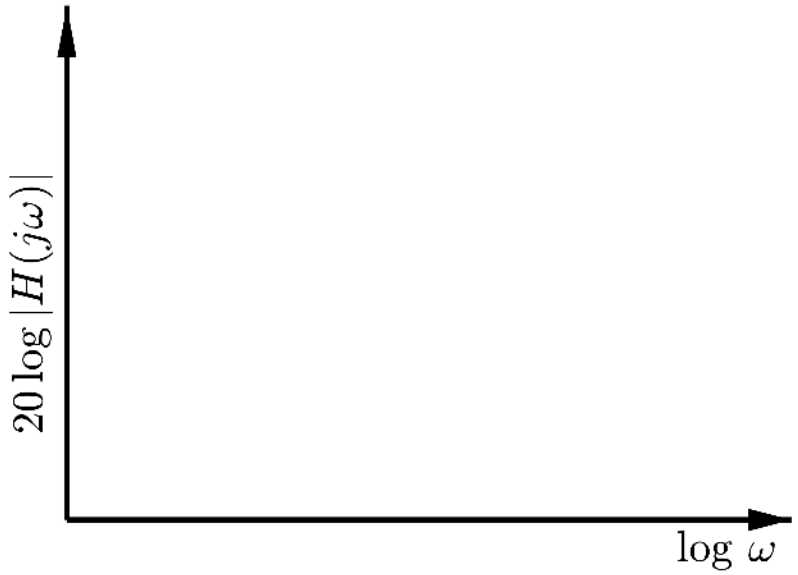



**Figure 2.13.1:** Pot of the magnitude of the frequence response in dB ($20 \log |H(j\omega)|$) versus the frequency ($\log(\omega)$).

For Bode plot for first-order systems → plots consist of two straight lines

Any complex system with real coefficients can be factored by the product of 1st-order and 2nd-order systems with both real coefficients.

**Standard 1st-order differential equation**:

$$\tau \frac{dy(t)}{dt} + y(t) = a\, x(t) \tag{2.13.1}$$

Step 1: Take derivative of the Fourier Transform:

$$\tau\, j\omega\, Y(j\omega) + Y(j\omega) = a\, X(j\omega) \tag{2.13.2}$$

Step 2: Group $Y(j\omega)$ term

$$(j\tau\omega + 1)\, Y(j\omega) = a\, X(j\omega) \tag{2.13.3}$$

Step 3: Find frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \tag{2.13.4}$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{a}{1 + j\tau\omega} = H(j\omega) \tag{2.13.5}$$

$$H(j\omega) = \frac{a}{1 + j\omega\tau} \qquad \Longrightarrow \qquad |H(j\omega)| = \frac{|a|}{|1 + j\tau\omega|} \tag{2.13.6}$$

Bode plots = behavior of $|H(j\omega)|$ for $\omega \to 0$ (small) and $\omega \to \infty$ (large)

$$|H(j\omega)| = \frac{|a|}{|1 + j\omega\tau|} \xrightarrow[\omega \to 0]{} |H(j\omega)| = \frac{|a|}{|1|} = |a| \tag{2.13.7}$$

$$20 \log_{10} |H(j\omega)| = 20 \log_{10} |a| \tag{2.13.8}$$

As $\omega \to \infty$:

$$|H(j\omega)| = \frac{|a|}{|j\omega\tau|} = \frac{|a|}{|\omega\tau|} \qquad (j\omega \gg 1) \tag{2.13.9}$$

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left(\frac{|a|}{|\omega\tau|}\right) = 20\log_{10}|a| - 20\log_{10}(|\omega\tau|) \tag{2.13.10}$$

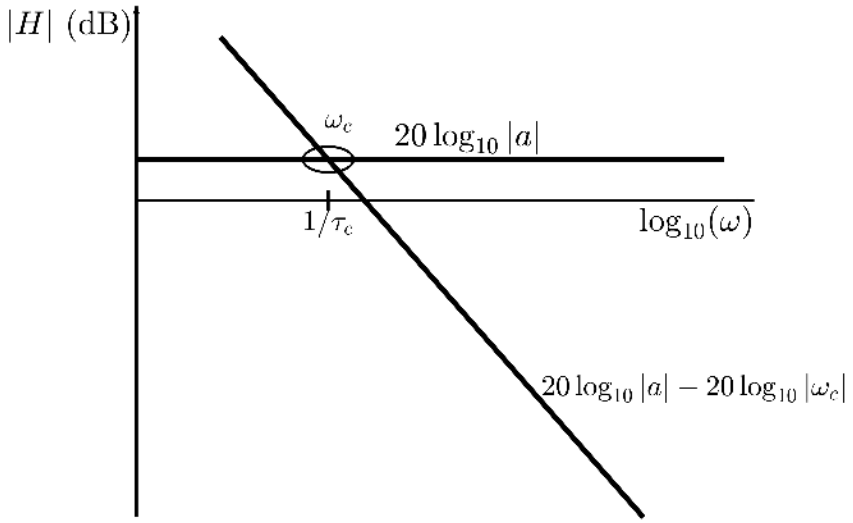



**Figure 2.13.2:** 1st-order lowpass Bode magnitude approximation.

1st-order lowpass filter since low frequencies pass and high frequencies are attenuated

$$20\log_{10}|a| \qquad \text{(flat line)} \tag{2.13.11}$$

$$20\log_{10}|a| - 20\log_{10}(|\omega\tau|) \qquad \text{(line with slope)} \tag{2.13.12}$$

slope = $-20$ dB / decade

Decade = a factor of 10 in frequency

Cutoff frequency $\omega_c$ = frequency where two lines meet

$$20\log_{10}|a| = 20\log_{10}|a| - 20\log_{10}(|\omega_c\tau|) \tag{2.13.13}$$

$$20\log_{10}(|\omega_c\tau|) = 0 \Rightarrow \log_{10}(\omega_c\tau) = 0 \Rightarrow \omega_c\tau = 1 \Rightarrow \boxed{\omega_c = \frac{1}{\tau}} \tag{2.13.14}$$

Another approach to find $\omega_c$:

$$|H(j\omega)| = \frac{|a|}{|1 + j\tau\omega|} \tag{2.13.15}$$

Set Re and Im equal in the denominator to find the cutoff frequency $\omega_c$

$$|1 + j\omega\tau| \Rightarrow 1 = \omega_c\tau \Rightarrow \omega_c = \frac{1}{\tau} \tag{2.13.16}$$

For $\omega \to \infty$, then

$$20\log_{10}|H(j\omega)| = 20\log_{10}a - 20\log_{10}(\omega\tau) = 20\log_{10}a - 20\log_{10}\omega - 20\log_{10}\tau \tag{2.13.17}$$

Consider $\omega_1 = 10\omega$,

$$20\log_{10}|H(j\omega_1)| = 20\log_{10}a - 20\log_{10}\omega_1 - 20\log_{10}\tau \tag{2.13.18}$$

$$20\log_{10}(10\omega) = 20\log_{10}\omega + 20\log_{10}10 = 20\log_{10}\omega + 20 \tag{2.13.19}$$

$$20\log_{10}|H(j\omega_1)| = 20\log_{10}|H(j\omega)| - 20 \tag{2.13.20}$$

Increasing the frequency by a factor of 10, reducing the amplitude response by 20 dB

If $\omega_2 = 100\omega$ → decay of 40 dB

In general

$$|H(j\omega)| = \frac{|H_1(j\omega)|}{|H_2(j\omega) \cdot H_3(j\omega)|} \tag{2.13.21}$$

$$20\log|H| = 20\log|H_1| - 20\log|H_2 \cdot H_3| \tag{2.13.22}$$

$$20\log|H| = 20\log|H_1| - 20\log|H_2| - 20\log|H_3| \tag{2.13.23}$$

**Example - First-order highpass filter**

$$H(j\omega) = \frac{j\omega\tau}{1+j\omega\tau} = (j\omega\tau)\left(\frac{1}{1+j\omega\tau}\right) \tag{2.13.24}$$

$$20\log|H(j\omega)| = 20\log\left|(j\omega\tau)\left(\frac{1}{1+j\omega\tau}\right)\right| = 20\log|j\omega\tau| - 20\log|1+j\omega\tau| \tag{2.13.25}$$

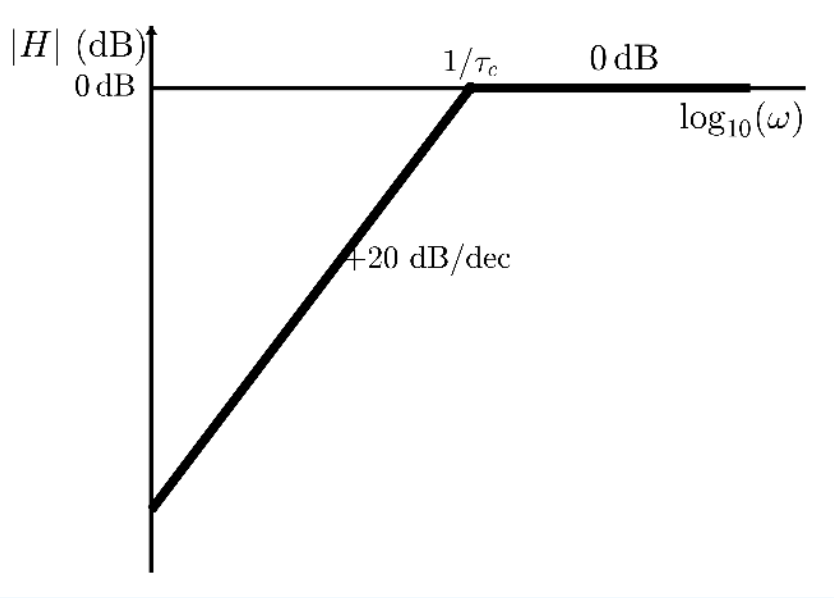



**Figure 2.13.3:** Highpass Bode magnitude plot (+20 dB/dec up to $\omega = 1/\tau$, then 0 dB).

## Example: First-order Bode Plot Magnitude

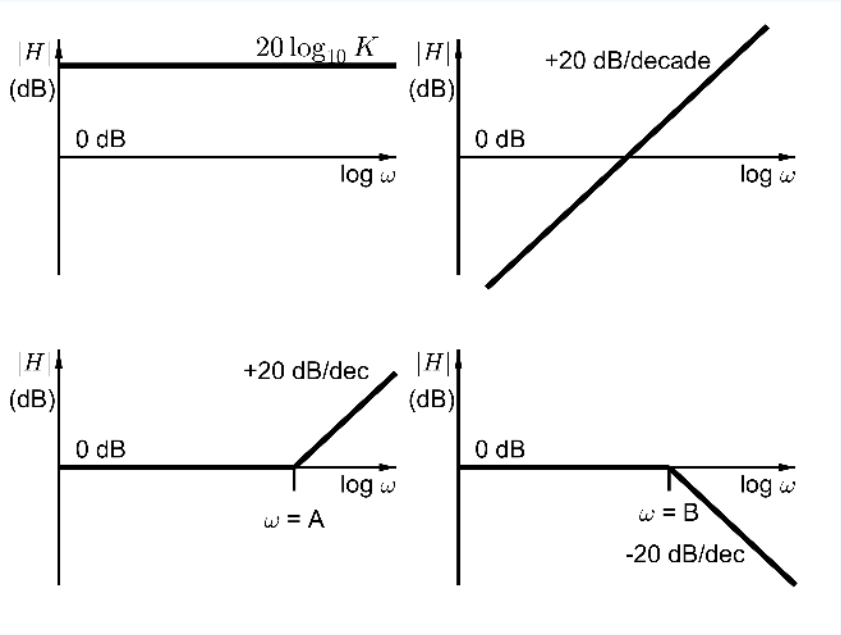



**Figure 2.13.4:** Reference sketches of first-order factor Bode magnitude plots.

$$H(j\omega) = K \quad \Rightarrow \quad 20\log|H| = 20\log K \tag{2.13.26}$$

$$H(j\omega) = j\omega \quad \Rightarrow \quad 20\log|j\omega| = 20\log\omega \tag{2.13.27}$$

(+20 dB/decade line)

$$H(j\omega) = 1 + \frac{j\omega}{A} \tag{2.13.28}$$

$$20\log\left|1 + \frac{j\omega}{A}\right| \xrightarrow[\omega\to 0]{} 20\log 1 = 0 \text{ dB} \tag{2.13.29}$$

$$20\log\left|1 + \frac{j\omega}{A}\right| \xrightarrow[\omega\to\infty]{} 20\log\left|\frac{j\omega}{A}\right| = 20\log\omega - 20\log A \tag{2.13.30}$$

(break at $\omega = A$, then +20 dB/dec)

$$H(j\omega) = \frac{1}{1 + \frac{j\omega}{B}} \tag{2.13.31}$$

(break at $\omega = B$, then -20 dB/dec)

## Example

$$H(j\omega) = \frac{2 + j4\omega}{1 + j\omega\,200} \tag{2.13.32}$$

Step 1: Apply $20\log_{10}$ to both sides of the equation

$$20\log_{10}|H(j\omega)| = 20\log_{10}|2 + j4\omega| - 20\log_{10}|1 + j\omega\,200| \tag{2.13.33}$$

$$20\log_{10}|2 + j4\omega| = 20\log_{10}|2(1 + j2\omega)| = 20\log_{10}|2| + 20\log_{10}|1 + j2\omega| \tag{2.13.34}$$

$$20\log|H(j\omega)| = 20\log 2 + 20\log|1 + j2\omega| - 20\log|1 + j\omega\,200| \tag{2.13.35}$$

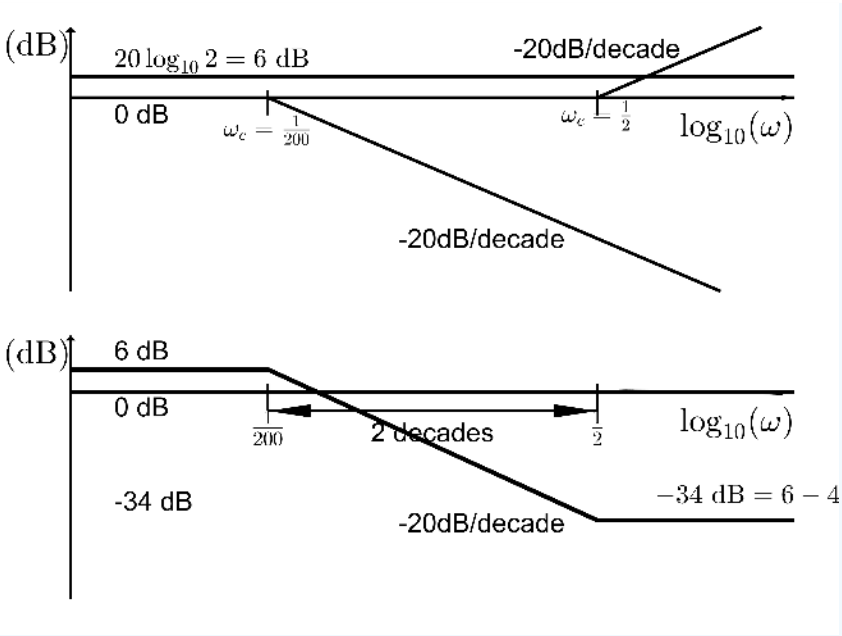



**Figure 2.13.5:** *Component Bode lines (*$20\log 2 \approx 6$ *dB, +20 dB/dec at* $\omega_c = 1/2$*, -20 dB/dec at* $\omega_c = 1/200$*) (top) and combined result (bottom).*

$$\omega_c \text{ @ } 1 = \omega 200 \quad \Rightarrow \quad \omega = \frac{1}{200} \tag{2.13.36}$$

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# 2.14: Bode Plots - Second-order system

## Learning Objectives: Bode Plots for Second-order systems.

1. Put a second-order system into standard form, and solve for the natural frequency, damping ratio and gain.
2. Determine whether a second-order system is underdamped, overdamped or critically damped
3. Sketch the Bode plot magnitude of a second-order system
4. Sketch the Bode plot approximation for the phase of first-order systems

## YouTube Videos

Second-order Bode plots: Theory



Example for Second-order Bode plots



## Summary

Second-order systems

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \tag{2.14.1}$$

$\omega_n$: undamped/natural frequency → ~ frequency of system's oscillations

$\zeta$: damping ratio → behavior of the system depending on $\zeta$ value

if $0 < \zeta < 1$, the system is underdamped - an impulse response in time-domain that oscillates

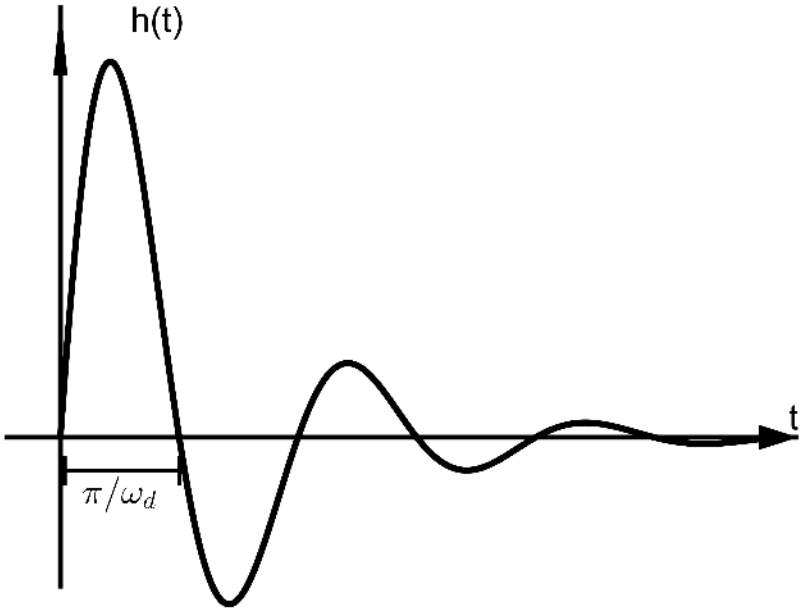



**Figure 2.14.1:** Underdamped Second-order system.

if $\zeta > 1$, the system is overdamped - 2nd-order system = a cascade of two 1st-order systems

If $\zeta = 1$, the system is critically damped → special case of 2nd-order system which are represented by the cascade of 2 identical 1st-order system (i.e., denominator in $H(j\omega)$ is a perfect square)

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2} \tag{2.14.2}$$

1) Experimentally, critically-damped systems do not occur a lot. So, we will not study them

2) Second-order overdamped systems can be analyzed by the strategies shown in the previous class/page

3) This class/page focuses on 2nd-order underdamped systems.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \tag{2.14.3}$$

If $\omega \to 0$, then $(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2 \simeq \omega_n^2$

$$H(j\omega) \simeq \frac{\omega_n^2}{\omega_n^2} = 1 \tag{2.14.4}$$

$$20\log_{10}|H(j\omega)| = 20\log_{10}(1) = 0 \text{ dB} \tag{2.14.5}$$

If $\omega \to \infty$, then $(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2 \simeq (j\omega)^2$ We can ignore the terms of $(j\omega)$ and $\omega_n^2$ since are smaller than $\infty^2$

$$H(j\omega) \simeq \frac{\omega_n^2}{(j\omega)^2} = \frac{\omega_n^2}{-\omega^2} \tag{2.14.6}$$

$$|H(j\omega)| = \frac{\omega_n^2}{\omega^2} \Rightarrow 20\log_{10}|H(j\omega)| \simeq 20\log_{10}\left(\frac{\omega_n^2}{\omega^2}\right) \tag{2.14.7}$$

$$= 20\log_{10}\omega_n^2 - 20\log_{10}\omega^2 = 40\log_{10}\omega_n - 40\log_{10}\omega \tag{2.14.8}$$

The $\omega_n$ term is a constant, while the $\omega$ term has a slope of $-40$ dB/decade

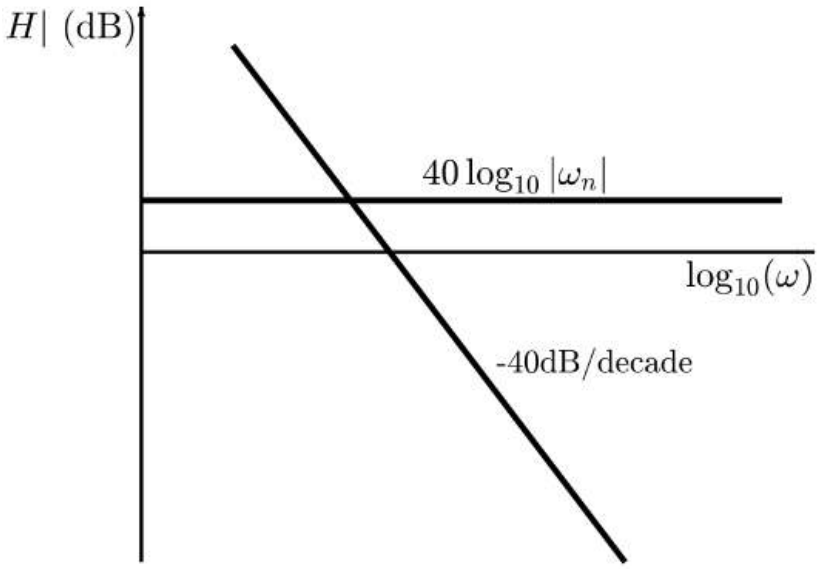



**Figure 2.14.2:** Second-order LPF system.

A second-order system is a lowpass filter that attenuates faster than a 1st-order system ($-40$ dB/decade vs. $-20$ dB/decade)

## Bode Plot Approximation

**Intercept frequency when**

$$0 = 40\log\omega_n - 40\log(\omega) \tag{2.14.9}$$

$$40\log\omega_n = 40\log\omega \Rightarrow \log\omega_n = \log\omega \Rightarrow \boxed{\omega_n = \omega} \tag{2.14.10}$$

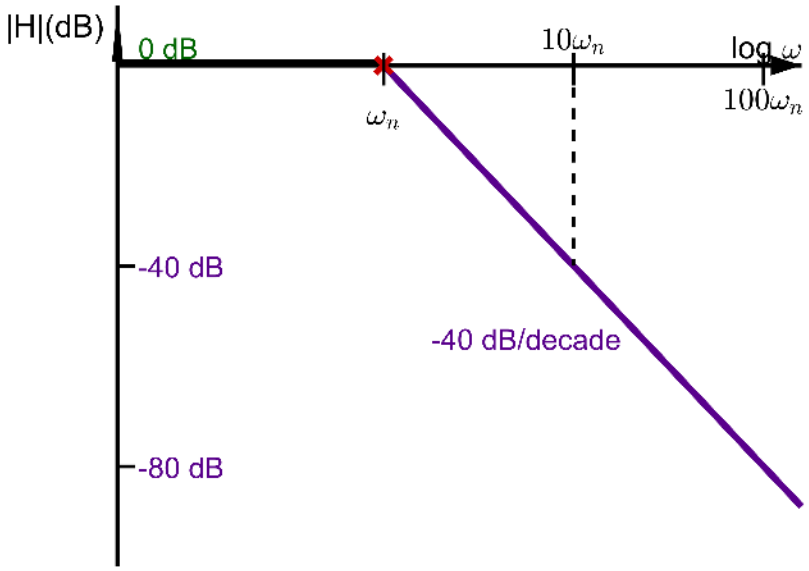



**Figure 2.14.3:** Second-order LPF system with a cutoff frequency $\omega = \omega_n$.

If $\zeta < \frac{1}{\sqrt{2}} \approx 0.7$, then there is a peak around $\omega = \omega_n$ with a magnitude equal to

$$\frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.14.11}$$

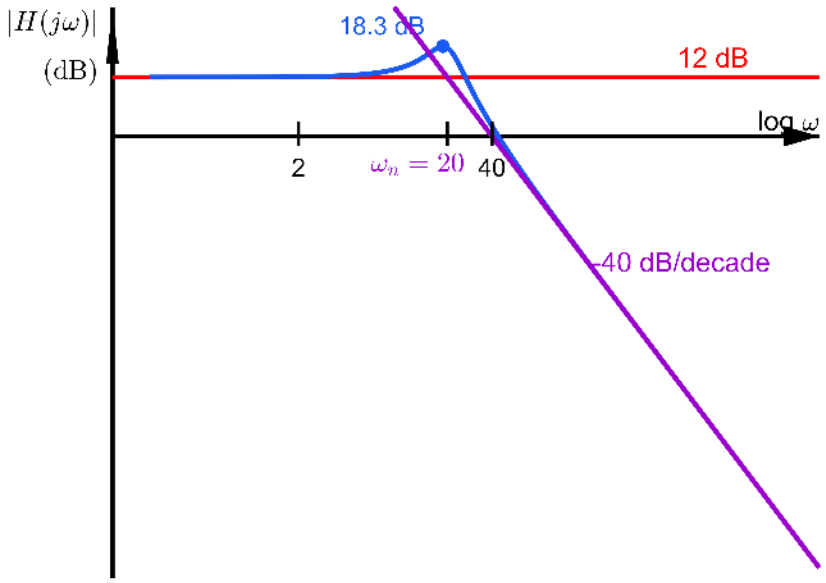



**Figure 2.14.4:** Second-order LPF system with a cutoff frequency $\omega = \omega_n = 20$ and $\zeta < \frac{1}{\sqrt{2}} \approx 0.7$.

The smaller $\zeta$ is, the taller and sharper the resonance peak is

If $\frac{1}{\sqrt{2}} < \zeta < 1$ (i.e., barely underdamped), then $|H(j\omega)|$ does not have a strong peak (close to a 1st-order system)

Details to estimate the magnitude of the peak for underdamped 2nd-order systems:
For $0 < \zeta < \frac{1}{\sqrt{2}}$,

$$\omega_{\max} \simeq \omega_n \sqrt{1-\zeta^2} \tag{2.14.12}$$

if $\zeta = 0 \rightarrow \omega_{\max} \simeq \omega_n$

if $\zeta = \frac{1}{\sqrt{2}} \rightarrow 1-\zeta^2 = 1-\frac{1}{2} = \frac{1}{2} \rightarrow \omega_{\max} \simeq \omega_n \sqrt{\frac{1}{2}} = \frac{\omega_n}{\sqrt{2}}$ $\omega_{\max} \simeq 0.7\omega_n$

$0.7\omega_n < \omega_{\max} < \omega_n$

$$|H(j\omega_{\max})| \simeq \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{2.14.13}$$

Range of $|H(j\omega_{\max})|$:

if $\zeta = \frac{1}{\sqrt{2}} \rightarrow$

$$2\zeta\sqrt{1-\zeta^2} = 2\left(\frac{1}{\sqrt{2}}\right)\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} \tag{2.14.14}$$

$$|H_{\max}| = \frac{1}{1/\sqrt{2}} \simeq \sqrt{2} \tag{2.14.15}$$

if $\zeta = 0 \rightarrow$

$$H(j\omega_{\max}) \propto \frac{1}{\zeta} \rightarrow \infty \quad \text{really sharp!} \tag{2.14.16}$$

**Example**

$$H(j\omega) = \frac{1600}{(j\omega)^2 + 16(j\omega) + 400} = \frac{A\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \Leftarrow \mathit{StandardForm} \tag{2.14.17}$$

$$\omega_n^2 = 400 \Rightarrow \omega_n = \sqrt{400} = 20 \tag{2.14.18}$$

$$2\zeta\omega_n = 16 \Rightarrow \zeta = \frac{16}{2\omega_n} = \frac{16}{2\cdot 20} = \frac{16}{40} = \frac{2}{5} = 0.4 \tag{2.14.19}$$

since $\zeta < 1$ underdamped 2nd-order system!

$$1600 = A\omega_n^2 \Rightarrow A = \frac{1600}{\omega_n^2} = \frac{1600}{400} = 4 \Rightarrow gain \tag{2.14.20}$$

$$H(j\omega) = 4 \cdot \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \tag{2.14.21}$$

$$\begin{aligned} 20\log|H(j\omega)| &= 20\log 4 + 20\log\left|\frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}\right| \\ &\approx 12 \text{ dB} + 20\log\left|\frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}\right| \end{aligned} \tag{2.14.22}$$

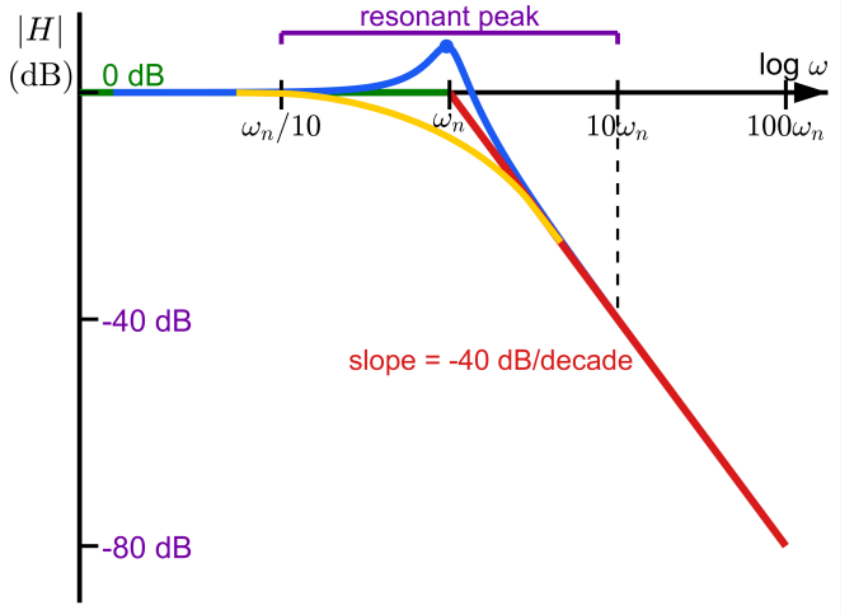



**Figure 2.14.5:** Second-order LPF system with a cutoff frequency $\omega = \omega_n = 20$.

$$\zeta = \frac{2}{5} \Rightarrow \omega_{\max} = \omega_n\sqrt{1-\zeta^2} = 20\sqrt{1-\left(\frac{2}{5}\right)^2} = 20\sqrt{\frac{21}{25}} \approx 18 \tag{2.14.23}$$

$$H(j\omega_{\max}) = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{2\cdot\frac{2}{5}\cdot\sqrt{\frac{21}{25}}} \approx 1.36 \Rightarrow 20\log|H| \approx 2.67 \text{ dB} \tag{2.14.24}$$

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# 2.15: Amplitude Modulation

## Learning Objectives: Amplitude Modulation

1. Explain what amplitude modulation is and how it works in time and frequency
2. Explain basic demodulation in frequency
3. Understand how signal bandwidth determines the spacing of carrier frequencies in AM systems

## YouTube Videos

Amplitude Modulation



Demodulating AM Signals



## Summary

### Amplitude Modulation (AM)

$$y(t) = x(t)\cos(\omega_c t) \tag{2.15.1}$$

\(\cos(\omega_c t)\): carrier signal with a higher frequency
$x(t)$: modulating signal containing information

**How high should the modulation frequency be?**

$$y(t) = x(t)\cdot\cos(\omega_c t) \quad \leftrightarrow \quad Y(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * (\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c))\right] \tag{2.15.2}$$

$$Y(j\omega) = \frac{1}{2}X\big(j(\omega-\omega_c)\big) + \frac{1}{2}X\big(j(\omega+\omega_c)\big) \tag{2.15.3}$$

$\omega_c$ large enough so the 2 replicas of $X(j\omega)$ do not overlap.

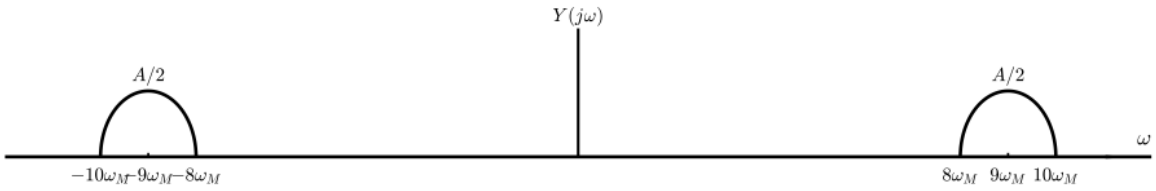



**Figure 2.15.1:** Spectrum of the output signal when x(t) is multiplied by a carrier signal $\cos(9\omega_M)$

Figure 2.15.1 shows that a whole copy of the object spectrum is displaced at a higher frequency, i.e., $\omega_c = \pm 9\,\omega_M$
We need a demodulated system to bring back the replicas @ 0 frequency axis

### Frequency Division Multiplexing (FDM)

**Definition:** Frequency Division Multiplexing (FDM) is a technique for transmitting multiple signals simultaneously by assigning each signal a unique carrier frequency.

In theory, we can retrieve (i.e., demodulate) each signal

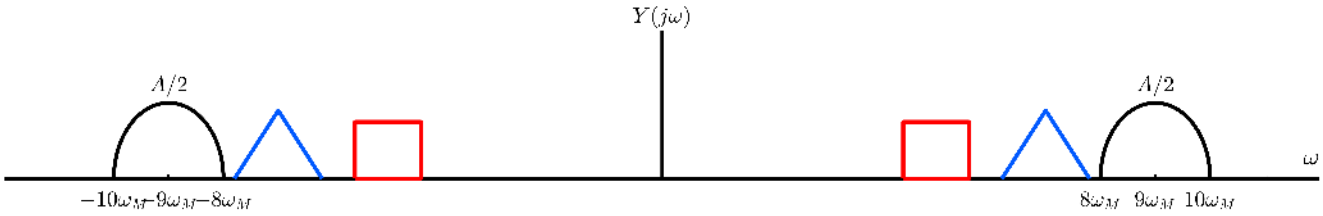



**Figure 2.15.2:** Output spectrum when a complex signal is multiplied by a carrier signal $\cos(9\omega_M)$. The individual components of the combined signal can be transmitted simultaneously without overlap.

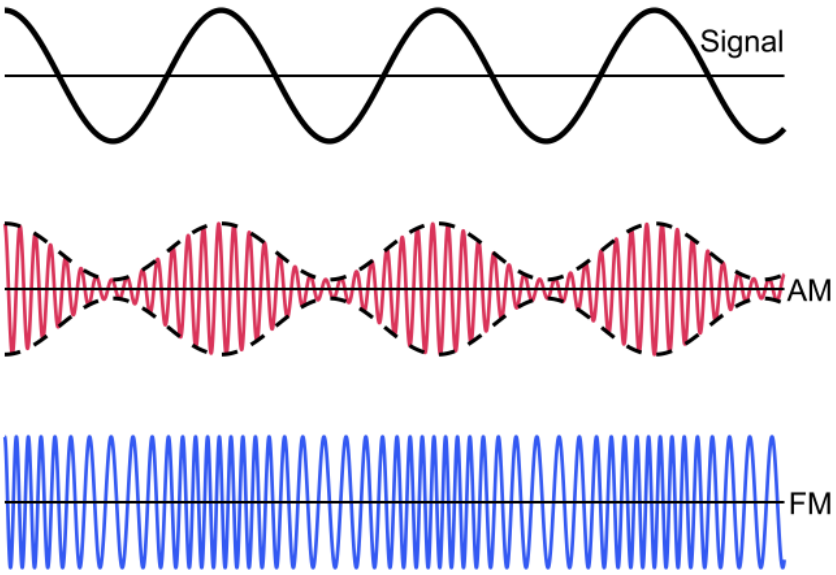



**Figure 2.15.3:** Visual representation of AM V.S. FM.

$$y(t) = x(t)\cos(\omega_c t) \tag{2.15.4}$$

AM = Amplitude Modulation → $x(t)$ is modulated with a cosine with high frequency, modulating its amplitude.

FM = Frequency Modulation → next class

Demodulation

**How can we recover the original signal $x(t)$ from the modulated signal $y(t)$?**

**Step 1** to recover the signal is

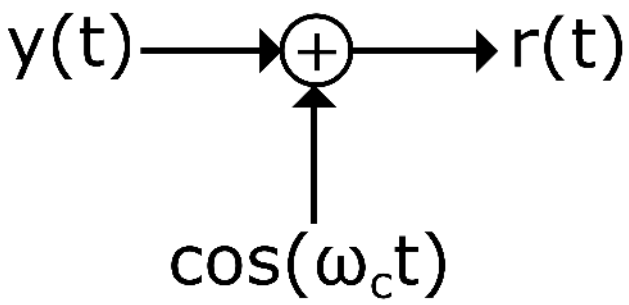



**Figure 2.15.4:** Flow diagram of the first step to demodulation.

$$r(t) = y(t)\cos(\omega_c t) \tag{2.15.5}$$

$$R(j\omega) = \frac{1}{2}Y\big(j(\omega-\omega_c)\big) + \frac{1}{2}Y\big(j(\omega+\omega_c)\big) \tag{2.15.6}$$

scaled version of the input spectrum

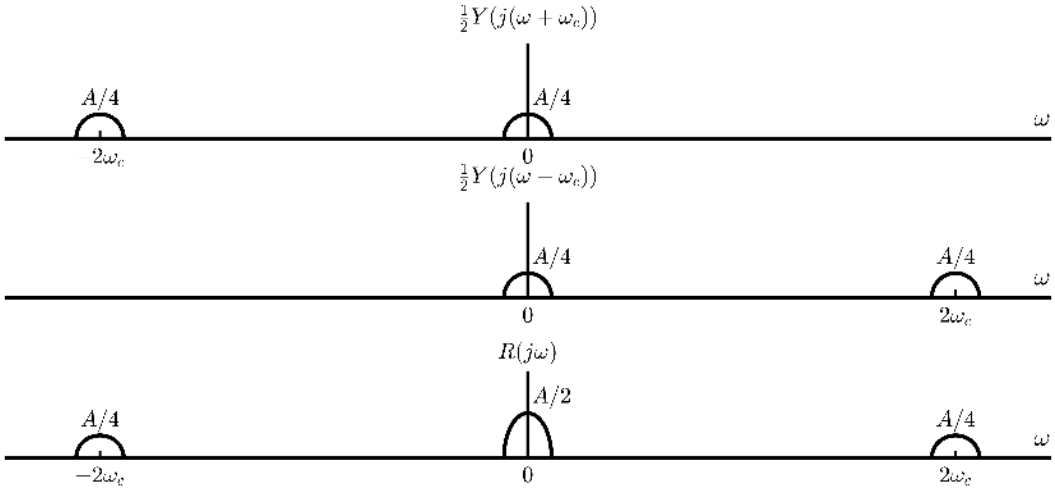



**Figure 2.15.5:** Representation of step 1 to demodulation being applied to the signal from Figure 2.15.1.

**Step 2:** Apply lowpass filter

$$H_D(j\omega) = \begin{cases} 1, & |\omega| \le \omega_M \\ 0, & |\omega| > \omega_M \end{cases} \tag{2.15.7}$$

$$y(t) \xrightarrow{\times\cos(\omega_c t)} r(t) \xrightarrow{H_D(j\omega)} s(t)$$

$$S(j\omega) = R(j\omega)\cdot H_D(j\omega) \tag{2.15.8}$$

The demodulated cosine should have exactly the same frequency and phase as the carrier signal.

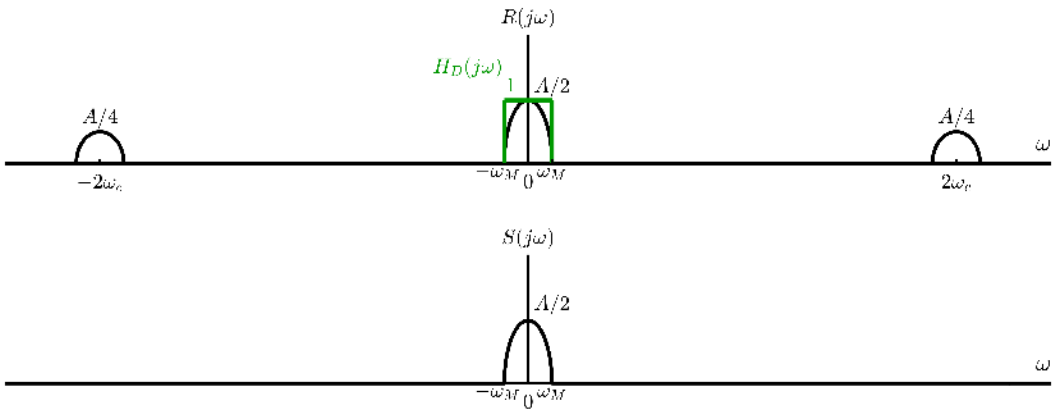



**Figure 2.15.6:** Representation of step 2 to demodulation being applied to the signals from Figure 2.15.1 & 2.15.5.

$S'(j\omega)$ has the same spectrum as the original spectrum $X(j\omega)$ but with a different gain, i.e.,

$$S'(j\omega) = \frac{1}{2}X(j\omega) \tag{2.15.9}$$

→ by changing the volume (i.e., gain) then we can match

$$X(j\omega) = 2S'(j\omega) \tag{2.15.10}$$

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# 2.16: Frequency Modulation

## Learning Objectives: Frequency Modulation

1. Understand how instantaneous phase and instantaneous frequency are related for FM signals.
2. Recognize the differences between narrowband and broadband FM spectra.

## YouTube Videos

MIT Courseware Lecture 24: Modulation, Part 2 (*Credit to Prof. Alan V. Oppenheim*)



## Summary

### Frequency Modulation (FM)

**Definition**: Frequency Modulation (FM) → The modulating signal is used to control the frequency of a sinusoidal carrier

- With FM, the envelope of the carrier is constant
- FM systems are highly non-linear
- FM reception is typically better than AM reception

Consider a sinusoidal carrier:

$$c(t) = A\cos(\omega_c t + \theta_c) = \cos(\theta(t)) \tag{2.16.1}$$

$\omega_c$: carrier's frequency
$\theta_c$: carrier's phase

Angle Modulation: $\theta(t) = \omega_c t + \theta_c$ corresponds to using the modulating signal to change/vary the phase $\theta(t)$

$$y(t) = A\cos\left[\omega_c t + \theta_c(t)\right] \quad \text{where} \quad \theta_c(t) = \theta_0 + k_p x(t) \tag{2.16.2}$$

If $x(t)$ is constant, the phase of $y(t)$, i.e., $\theta_c(t)$, is constant and proportional to the amplitude of $x(t)$

$$y(t) = A\cos[\theta(t)] \tag{2.16.3}$$

$$\frac{d\theta(t)}{dt} = \omega_c + k_f x(t) \qquad \text{if } x(t) = \text{constant} \tag{2.16.4}$$

## Frequency Modulation

$$y(t) = A\cos\left[(\omega_c + k_f x)t\right] \tag{2.16.5}$$

$y(t)$ follows a sinusoid with a frequency that is offset from the carrier frequency $\omega_c$ by an amount proportional to the amplitude of $x(t)$.

Note that phase modulation and frequency modulation are different forms of angle modulation

For phase modulation:

$$\frac{d\theta(t)}{dt} = \omega_c + k_p\frac{dx(t)}{dt} \tag{2.16.6}$$

Frequency modulation with a step corresponds to the frequency of the sinusoidal carrier changing instantaneously from one value to another when $x(t)$ changes value at $t = 0$.

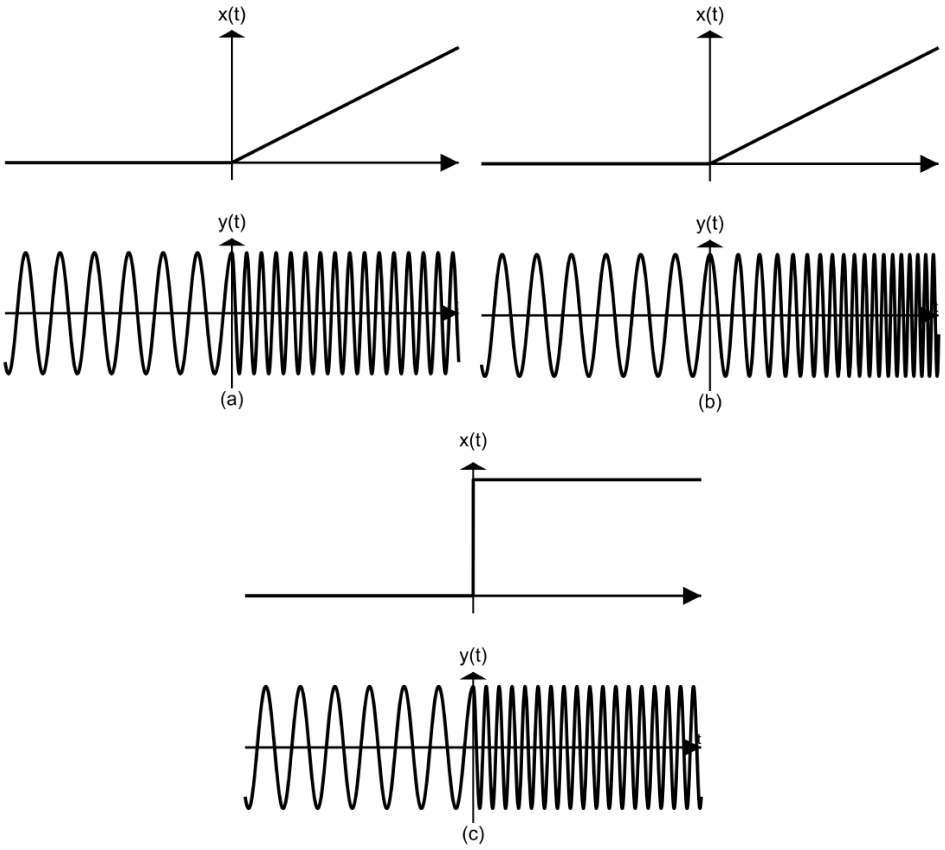



**Figure 2.16.1:** (a) Phase modulation if x(t) is a ramp signal; (b) Frequency modulation if x(t) is a ramp signal; and (c) Frequency modulation if x(t) is a unit step.

### Instantaneous Frequency

When the frequency modulation is a ramp, the frequency changes linearly

$$y(t) = A\cos\theta(t) \qquad \omega_i = \frac{d\theta(t)}{dt} \tag{2.16.7}$$

If

$$y(t) = A\cos[\omega_c t + \theta_0] \;\Rightarrow\; \omega_i = \frac{d}{dt}[\omega_c t + \theta_0] = \omega_c \tag{2.16.8}$$

For FM,

$$\theta(t) = (\omega_c + k_f x(t))t \;\Rightarrow\; \omega_i = \omega_c + k_f x(t) \tag{2.16.9}$$

For phase modulation

$$\omega_i = \omega_c + k_p \frac{dx(t)}{dt} \tag{2.16.10}$$

Narrowband Frequency Modulation

$$x(t) = A\cos(\omega_m t) \tag{2.16.11}$$

$$\omega_i = \omega_c + k_f A\cos(\omega_m t) \tag{2.16.12}$$

The instantaneous frequency $\omega_i$ ranges from $\omega_c - k_f A$ to $\omega_c + k_f A$

$$\Delta\omega = \frac{\max - \min}{2} = k_f A \tag{2.16.13}$$

Thus

$$\omega_i = \omega_c + \Delta\omega\cos(\omega_m t) \tag{2.16.14}$$

$$y(t) = \cos\left[\omega_c t + \int x(t)\,dt\right] = \cos\left[\omega_c t + \frac{\Delta\omega}{\omega_m}\sin(\omega_m t) + \theta_0\right] \tag{2.16.15}$$

m: modulating index

$$y(t) = \cos\left[\omega_c t + m\sin\omega_m t + \theta_0\right] \tag{2.16.16}$$

if $m$ is small → narrowband FM

if $\theta_0 = 0$

$$y(t) = \cos(\omega_c t)\cos(m\sin\omega_m t) - \sin(\omega_c t)\sin(m\sin\omega_m t) \tag{2.16.17}$$

If $m \ll$, then $\cos(m\sin\omega_m t) \approx 1$ and $\sin(m\sin\omega_m t) \approx m\sin\omega_m t$ since $\sin\theta \approx \theta$. Thus

$$y(t) \approx \cos(\omega_c t) - m\sin(\omega_c t)\sin(\omega_m t) \tag{2.16.18}$$

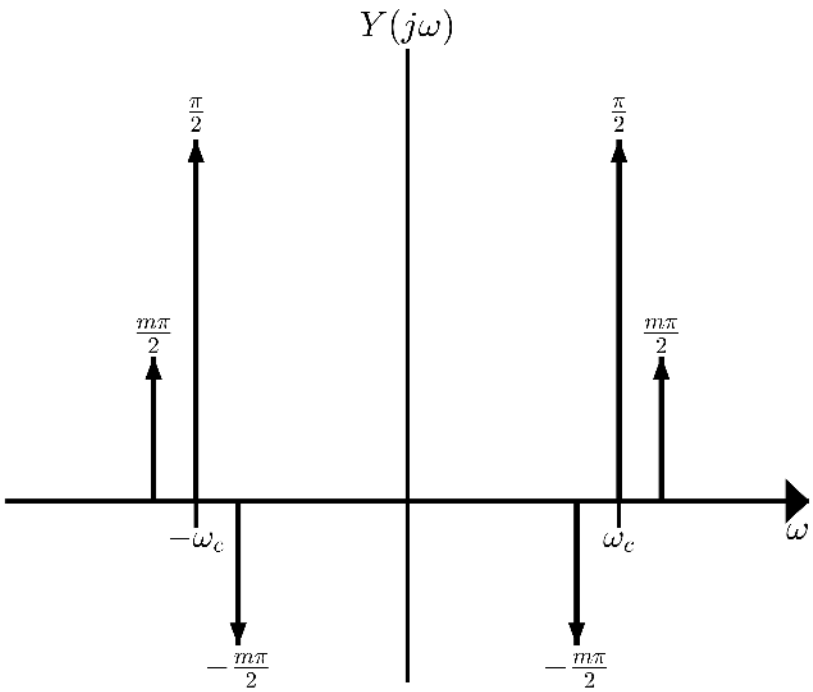



**Figure 2.16.2:** Narrowband FM spectrum approximation.


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$$y(t) = \cos\left[\omega_c t + m \sin \omega_m t + \theta_0\right] \tag{2.16.19}$$

if $m$ is large → wideband FM

$$y(t) = \cos(\omega_c t)\cos(m \sin \omega_m t) - \sin(\omega_c t)\sin(m \sin \omega_m t) \tag{2.16.20}$$

$(\cos(\omega_c t)\cos(m \sin \omega_m t))$: $\cos(\omega_c t)$ carrier modulating by $\cos(m \sin \omega_m t)$
$(\cos(m \sin \omega_m t))$ and $(\sin(m \sin \omega_m t))$: periodic signals with frequency $\omega_m$
$(\sin(\omega_c t)\sin(m \sin \omega_m t))$: $\sin(\omega_c t)$ carrier modulated by $\sin(m \sin \omega_m t)$

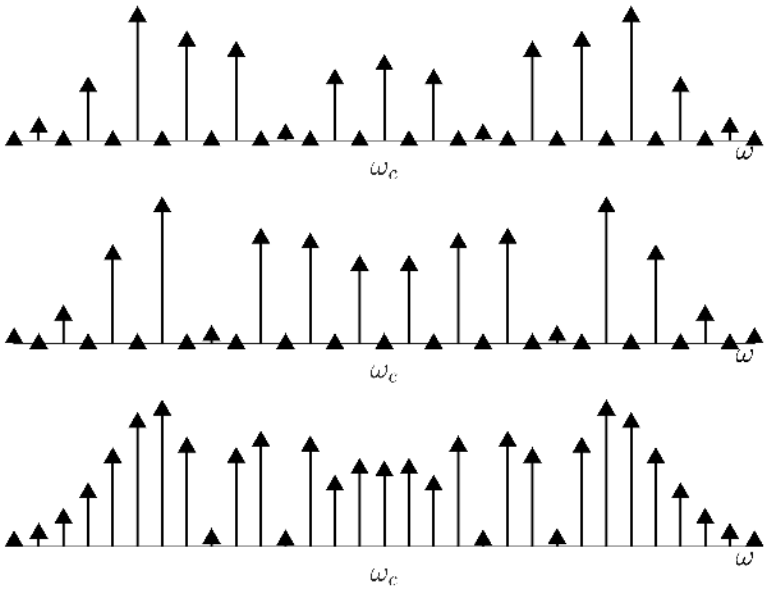



**Figure 2.16.3:** Wideband frequency modulation spectrum magnitude with $m = 11$: $\cos(\omega_c t)\cos(m \sin \omega_m t)$ (top); $\sin(\omega_c t)\sin(m \sin \omega_m t)$ (middle); $\cos(\omega_c t + m \sin \omega_m t)$ (bottom).

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# 2.17: Time-dependent Fourier Transform

## Learning Objectives: Time-Dependent Fourier Transform

1. Compute the spectrogram for a simple FM signal
2. Examine how changing the window length and overlap impact the spectrogram

## YouTube Videos

Time Dependent Fourier Transform Basics



MATLAB Spectrogram Tutorial



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# 2.18: Laplace Transform

## Learning Objectives: Laplace Transform

1. Compute the Laplace transform, including region of convergence, for signals such as decaying exponentials, finite rectangular pulses, and simple finite length signals.
2. Find simple inverse Laplace transforms
3. Understand how the CT Fourier transform and Laplace transform are related

### YouTube Videos

Laplace Transform Definition



Laplace Transform Example 1



Laplace Transform Example 2



## Summary

Continuous-time Fourier Transform → Laplace Transformation
Discrete-time Fourier Transform → z-transform

useful to analyze many unstable systems
Relevant to analyse the stability or instability of systems

By defining $s = j\omega$, then $e^{j\omega t} = e^{st}$ and $y(t) = H(s)e^{st}$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}\,dt \tag{2.18.1}$$

**Laplace Transform**

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \tag{2.18.2}$$

if $s =$ imaginary, Laplace Transform = Fourier Transform

if $s = \sigma + j\omega$ complex number with Re and Im parts

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}\,dt = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt \tag{2.18.3}$$

Laplace Transform of $x(t)$ is the FT of $x(t)$ after multiplication by a real exponential signal $\mathbf{FT}\left[x(t)e^{-\sigma t}\right]$

**Example 1**

$$x(t) = e^{-at}u(t) \tag{2.18.4}$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \frac{1}{j\omega + a}, \quad a > 0 \tag{2.18.5}$$

$$X(s) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+a)t}\,dt = \frac{1}{s+a} \tag{2.18.6}$$

$s = \sigma + j\omega$

$$= \frac{1}{(\sigma + a) + j\omega} \tag{2.18.7}$$

$\sigma + a > 0$

$\mathbf{Re}\{s\} = \sigma > -a$

Whereas the FT does not converge for all signals, the Laplace Transform may converge for some values of $\mathrm{Re}\{s\}$ and not for others.

**Example 2**

$$x(t) = -e^{-at}u(-t) \tag{2.18.8}$$

$$X(s) = \int_{-\infty}^{+\infty} -e^{-at}u(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(s+a)t}\,dt = \frac{1}{s+a} \tag{2.18.9}$$

$$\mathrm{Re}\{s+a\} < 0 \qquad \Rightarrow \qquad \mathrm{Re}\{s\} < -a \tag{2.18.10}$$

**Region of Convergence (ROC):** range of values of $s$ for which the Laplace Transform converges

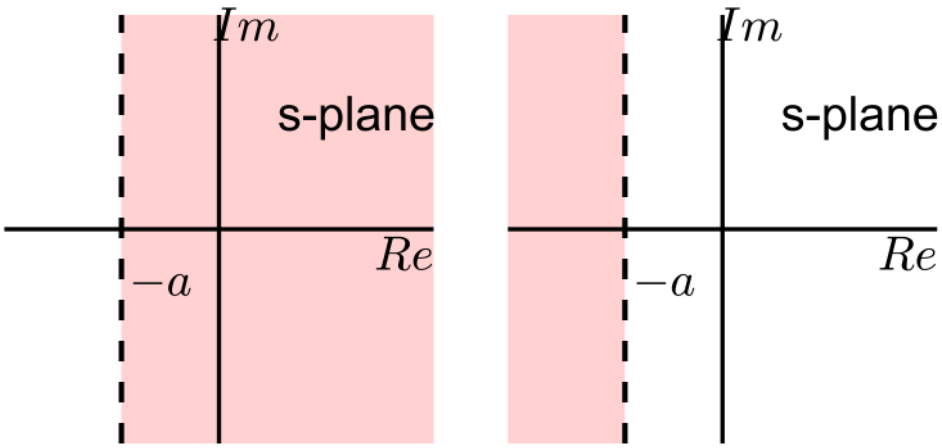



**Figure 2.18.1:** Illustration of the ROC for Examples 1 and 2.

**Example 3**

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t) \tag{2.18.11}$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt = 3\int_{0}^{\infty} e^{-(2+s)t}\,dt - 2\int_{0}^{\infty} e^{-(1+s)t}\,dt \tag{2.18.12}$$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2} \tag{2.18.13}$$

$$\mathrm{Re}\{s\} > -1 \tag{2.18.14}$$

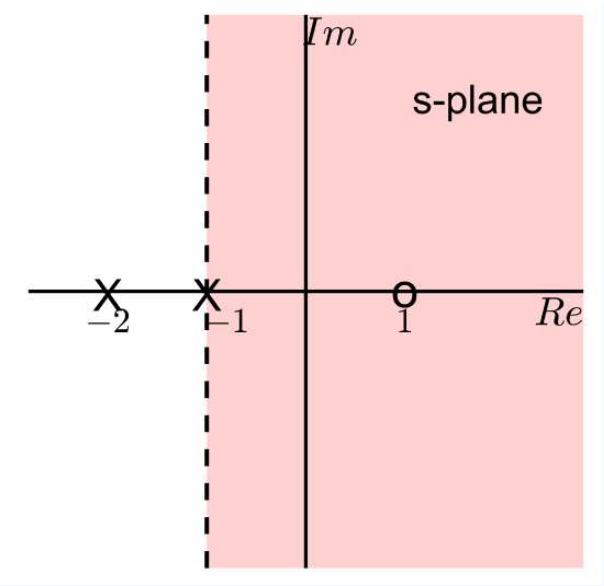



**Figure 2.18.2:** Illustration of the ROC for Example 3.

## Example 4

$$x(t) = e^{-2t}u(t) + e^{-t}\big(\cos 3t\big)u(t) \tag{2.18.15}$$

Euler

$$e^{-t}\cos(3t)u(t) = e^{-t}\frac{e^{j3t}}{2}u(t) + e^{-t}\frac{e^{-j3t}}{2}u(t) = \frac{1}{2}e^{-(1-3j)t}u(t) + \frac{1}{2}e^{-(1+3j)t}u(t) \tag{2.18.16}$$

$$X(s) = \int_0^\infty e^{-(2+s)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(1+s-3j)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(1+s+3j)t}\,dt \tag{2.18.17}$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+1-3j} + \frac{1}{2}\frac{1}{s+1+3j} = \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)} \tag{2.18.18}$$

$$\mathrm{Re}\{s\} > -1 \tag{2.18.19}$$

$$X(s) = \frac{N(s)}{D(s)} \tag{2.18.20}$$

Roots of $D(s)$ are poles: x
Roots of $N(s)$ are zeros: o

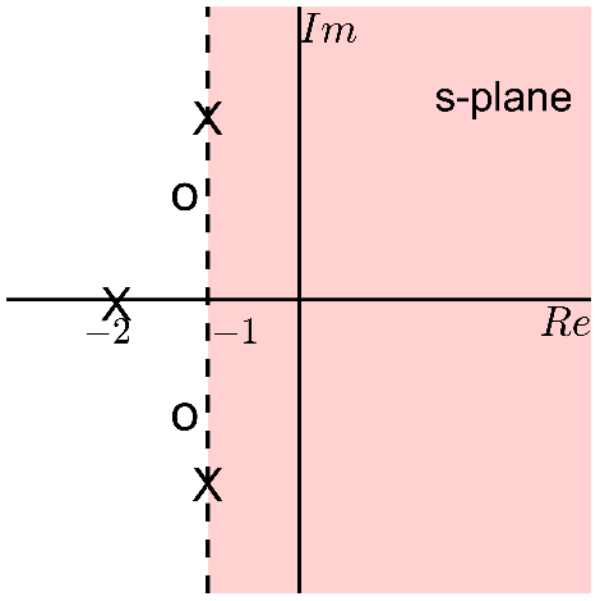



**Figure 2.18.3:** Illustration of the ROC for Example 4.

If order of $D(s)$ > order of $N(s)$ → $X(s) = 0$ as $s \to \infty$

If order of $N(s)$ > order of $D(s)$, then $X(s)$ is unbounded as $s \to \infty$

In general, if the order of $D(s)$ exceeds the order of the numerator by $k$, $X(s)$ will have $k$ zeros at infinity. Similarly, if the order of the numerator exceeds the order of the denominator by $k$, $X(s)$ will have $k$ poles at infinity.

## Example 5

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t) \tag{2.18.21}$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \tag{2.18.22}$$

$$= \underbrace{\int_{-\infty}^{+\infty} \delta(t)e^{-st}\,dt}_{1} - \frac{4}{3}\int_{0}^{\infty} e^{-(1+s)t}\,dt + \frac{1}{3}\int_{0}^{\infty} e^{-(s-2)t}\,dt$$

$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} \tag{2.18.23}$$

Remember that the ROC is the set of values of $s$ for which the Laplace transforms of all three terms in $x(t)$ converge.

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)} \qquad \mathrm{Re}\{s\} > 2 \tag{2.18.24}$$

Since the order of $N(s)$ and $D(s)$ are equal, $X(s)$ has neither poles nor zeros at infinity.

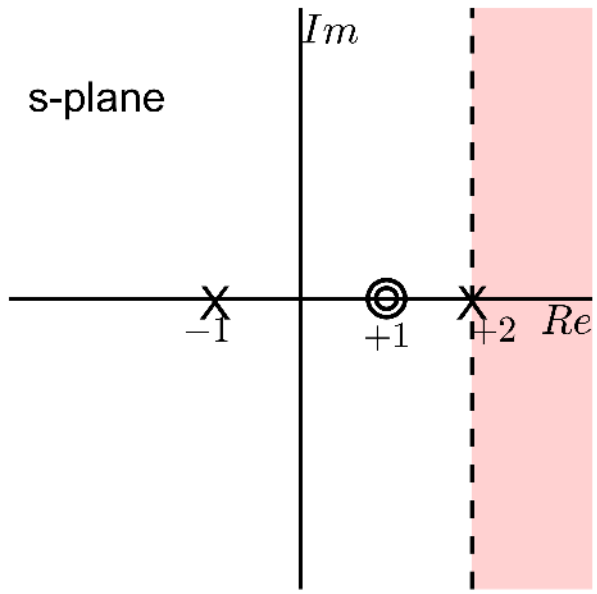



**Figure 2.18.4:** Illustration of the ROC for Example 5.

## Properties of the ROC

**#1** – The ROC of $X(s)$ consists of strips parallel to the $j\omega$-axis in the $s$-plane

ROC of $X(s)$ consists of those values of $s = \sigma + j\omega$ for which the FT of $x(t)e^{-\sigma t}$ converges. This means that

$$\int_{-\infty}^{+\infty} |x(t)|\, e^{-\sigma t}\, dt < \infty \tag{2.18.25}$$

**#2** – For rational Laplace Transforms, the ROC does not contain any poles

**#3** – If $x(t)$ is of finite duration and is absolutely integrable, the ROC is the entire $s$-plane

$$\begin{aligned} x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \quad &\Longrightarrow \quad X(s) \\ = \int_0^T e^{-at} e^{-st}\, dt &= \frac{1}{s+a}\left[1 - e^{-(s+a)T}\right] \end{aligned} \tag{2.18.26}$$

Note that $X(-a) = T$ when we use L'Hôpital rule

**#4** – If $x(t)$ is right sided and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of $s$ for which $\mathrm{Re}\{s\} > \sigma_0$ will also be in the ROC

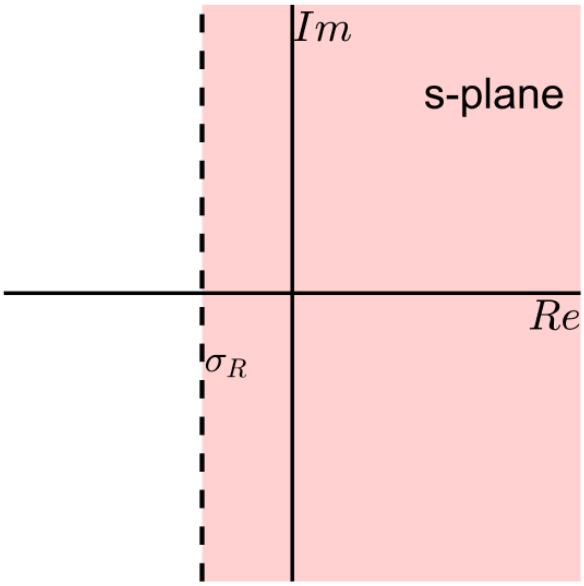



**Figure 2.18.5:** Illustration of a right-sided ROC.

**#5** – If $x(t)$ is left sided and if the line $\mathbf{Re}\{s\} = \sigma_0$ is in the ROC, then all values of $s$ for which $\mathbf{Re}\{s\} < \sigma_0$ will also be in the ROC

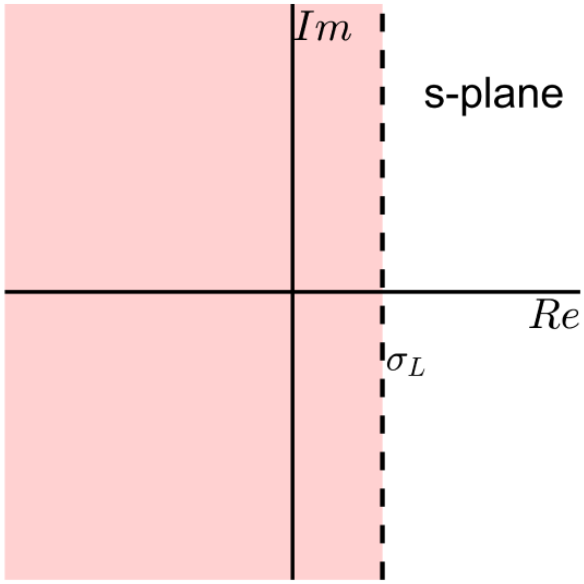



**Figure 2.18.6:** Illustration of a left-sided ROC.

**#6** – If $x(t)$ is two sided, and if the line $\mathbf{Re}\{s\} = \sigma_0$ is in the ROC, then ROC will consist of a strip in the $s$-plane that includes the line $\mathbf{Re}\{s\} = \sigma_0$

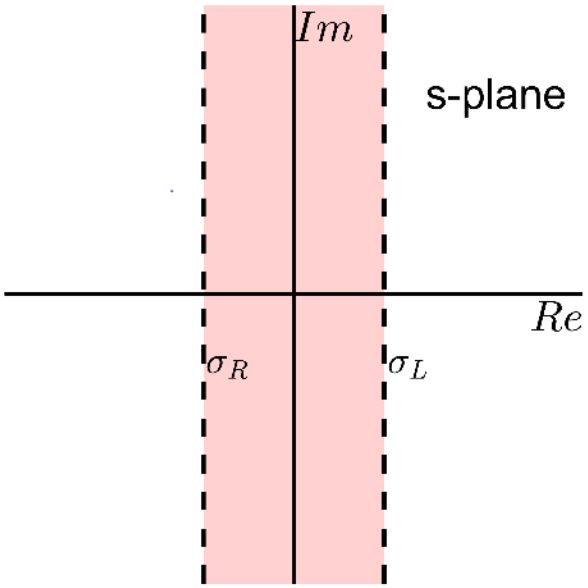



**Figure 2.18.7:** Illustration of a two-sided ROC.

**#7** – If the Laplace Transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

**#8** – If $X(s)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the $s$-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the $s$-plane to the left of the leftmost pole.

### The inverse Laplace Transform

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds \tag{2.18.27}$$

If

$$X(s) = \sum_{i=1}^{m}\frac{A_i}{s+a_i} \quad \Longrightarrow \quad x(t) = \sum_{i=1}^{m} A_i e^{-a_i t}u(t) \tag{2.18.28}$$

### Causality

For a causal LTI system, $h(t) = 0$ for $t < 0$ → $h(t)$ is right sided
→ The ROC associated with a causal system is right-half plane

For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole

$h(t) = e^{-t}u(t)$ so $h(t) = 0$ for $t < 0$ → causal

$$H(s) = \frac{1}{s+1} \qquad \mathrm{Re}\{s\} > -1 \tag{2.18.29}$$

$H(s)$ is rational and ROC is to the right of the rightmost pole → causal

$h(t) = e^{-|t|}$ since $h(t) \neq 0$ for $t < 0$ → no causal

$$H(s) = -\frac{2}{s^2-1} \qquad -1 < \mathrm{Re}\{s\} < 1 \tag{2.18.30}$$

$H(s)$ is rational and has a ROC that is not the right of the rightmost pole, consistent with the fact that the system is not causal

## Stability

The stability of an LTI system is equivalent to the absolute integrability of its impulse response.

An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis

$$[\text{i.e. } \text{Re}\{s\} = 0] \tag{2.18.31}$$

A causal system with rational system function $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left-half of the $s$-plane → i.e., all poles have negative real parts.

---

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# 2.19: Laplace Transform Properties

## Learning Objectives: Laplace Properties

1. Connect the causality and stability of an LTI system to properties of its region of convergence.
2. Recognize some of the most common Laplace transform properties
3. Describe how the locations of poles and zeros of the system function determine features of the frequency response magnitude.

## YouTube Videos

Geometry of the Frequency Response



Laplace Transform Properties



## Summary

### Geometry Interpretation of the Frequency Space

Location of the poles and zeros of the Laplace transform and understand where there are the high/low values of the frequency response and where we should have sharp versus broad peaks in the frequency response

**Step 1:** Factor $H(s)$ to get poles and zeros

$$H(s) = \frac{N(s)}{D(s)} = \frac{A(s-\beta_1)(s-\beta_2)\cdots(s-\beta_K)}{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_L)} \tag{2.19.1}$$

$\beta_k$ zeros $\alpha_l$ poles

$$H(s) = A \frac{\prod_{k=1}^{K} (s - \beta_k)}{\prod_{l=1}^{L} (s - \alpha_l)} \tag{2.19.2}$$

$$

$$|H(j\omega)| = |H(s)||_{s=j\omega} = \left| A \frac{\prod_{k=1}^{K} (j\omega - \beta_k)}{\prod_{l=1}^{L} (j\omega - \alpha_l)} \right| = |A| \frac{\prod_{k=1}^{K} |j\omega - \beta_k|}{\prod_{l=1}^{L} |j\omega - \alpha_l|}. \tag{2.19.3}$$

\]

Remember that the magnitude of a product is the product of the magnitudes

## First-order system

$$H(s) = \frac{1}{s - \alpha} \tag{2.19.4}$$

$$|H(j\omega)| = \frac{1}{|j\omega - \alpha|} \tag{2.19.5}$$

length of vector $|j\omega - \alpha|$

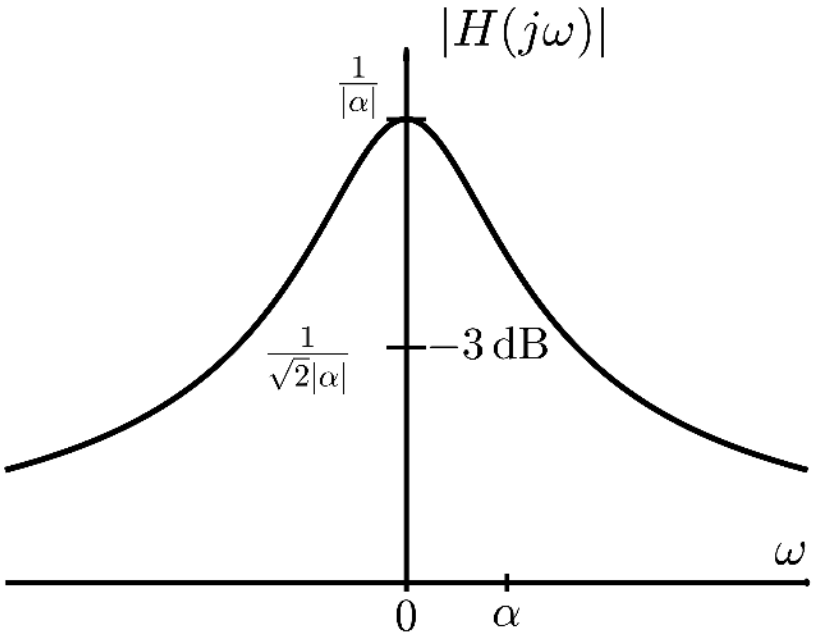



**Figure 2.19.1:** First-order system magnitude (dB).

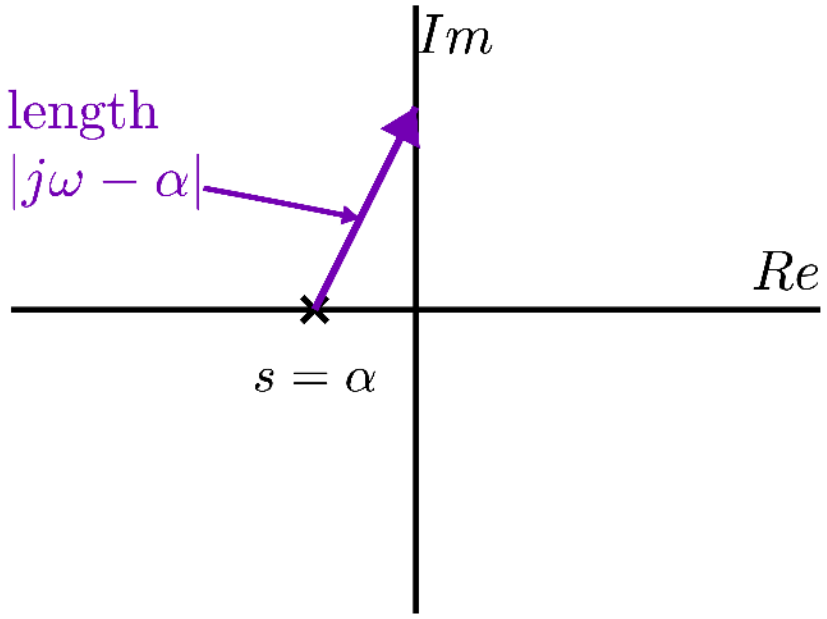



**Figure 2.19.2:** First-order system's poles.

As $\omega$ increases, the length of the vector increases, thus the denominator increases and, as a consequence $|H(j\omega)|$ gets smaller

When $\omega = \sigma_e$ is the $-3$ dB point

## Second-order poles

**Standard form:**

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2.19.6}$$

complex conjugate poles → underdamped $0 < \zeta < 1$

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \tag{2.19.7}$$

$\omega_n$: natural frequency

As $\zeta \to 0$, poles move closer to the imaginary axis and $\omega_n\sqrt{1-\zeta^2} \to \omega_n$

$$|H(j\omega)| \propto \frac{1}{\text{length } |j\omega - \alpha_l|} \tag{2.19.8}$$

As $\omega \to \infty$, we get a peak on $\omega = \sqrt{1-\zeta^2}\,\omega_n$

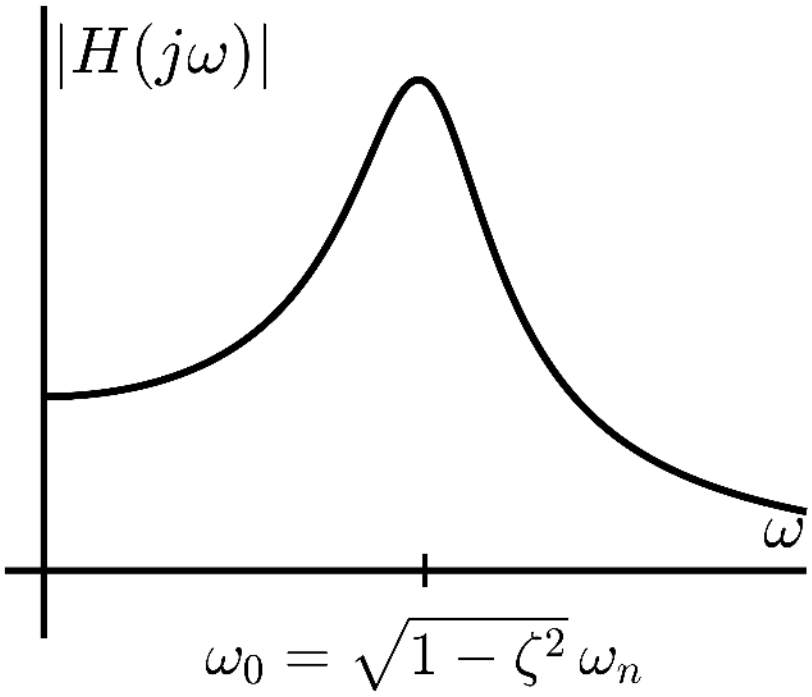



**Figure 2.19.3:** Second-order system magnitude (dB).

The peak gets sharper as $\zeta \to 0$ because the poles move closer to the imaginary axis

### Laplace Transform Properties

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \tag{2.19.9}$$

| Time | Laplace |
|---|---|
| $y(t) = x(t - t_0)$ | $Y(s) = e^{-st_0} X(s)$ same ROC |
| $y(t) = e^{s_0 t}\, x(t)$ | $Y(s) = X(s - s_0)$ shift ROC by $s_0$ |
| $y(t) = x(t) * h(t)$ | $Y(s) = X(s) \cdot H(s)$ ROC of $Y$ includes $ROC_X \cap ROC_H$ (intersection) |
| $y(t) = \frac{dx(t)}{dt}$ | $Y(s) = s\, H(s)$ |
| $y(t) = -t\, x(t)$ | $Y(s) = \frac{d}{ds} X(s)$ |
| $y(t) = \int_{-\infty}^{t} x(z)\, dz$ | $Y(s) = \frac{1}{s} X(s)$ |

If $x(t) = 0$ for $t < 0$, $x(0^+) = \lim_{s \to \infty} sX(s)$ Initial Value Theorem

If $x(t) = 0$ for $t < 0$, $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

Laplace Transforms of Elementary Functions

| Signal | Transform | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | All $s$ |
| $u(t)$ | $\frac{1}{s}$ | $\mathrm{Re}\{s\} > 0$ |
| $-u(-t)$ | $\frac{1}{s}$ | $\mathrm{Re}\{s\} < 0$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} > -a$ |
| $-e^{-at}u(-t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} < -a$ |
| $\delta(t-T)$ | $e^{-sT}$ | All $s$ |
| $[\cos \omega_0 t]u(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $[\sin \omega_0 t]u(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $[e^{-at}\cos \omega_0 t]u(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > -a$ |
| $[e^{-at}\sin \omega_0 t]u(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > -a$ |
| $u_n(t) = \frac{d^n\delta(t)}{dt^n}$ | $s^n$ | All $s$ |
| $u_{-n}(t) = u(t) * \cdots * u(t)$ (n times) | $\frac{1}{s^n}$ | $\mathrm{Re}\{s\} > 0$ |

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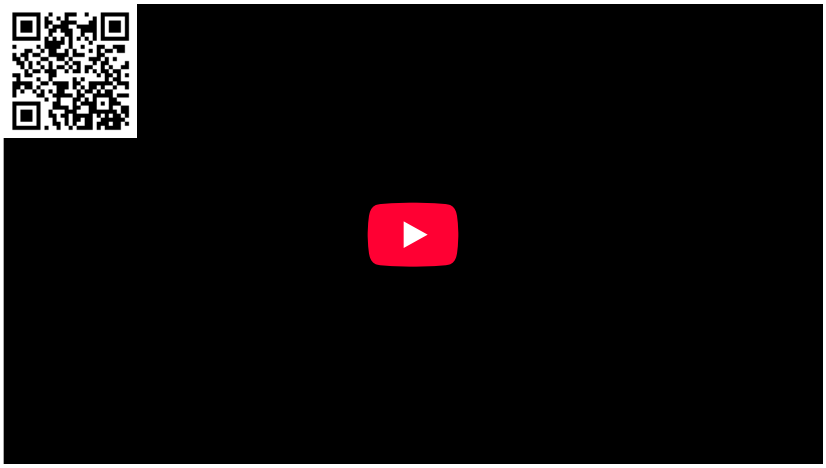
# 2.20: System Functions and Block Diagrams

## Learning Objectives: System Functions and Block Diagrams

1. Use Laplace transforms to simplify translations between differential equations and block diagrams
2. Find transfer functions for series, parallel and feedback connections of systems using Laplace properties.
3. Review how Laplace transform ROCs relate to the stability and causality of LTI systems

## YouTube Videos

Laplace Transform and Block Diagrams Fundamentals



Example from Differential Equation to Block Diagram



Block Diagram to Differential Equation



## Summary

**Analysis and characterization of LTI systems using the Laplace Transform**

LTI system:

$$x(t) * h(t) = y(t) \quad \Longrightarrow \quad Y(s) = X(s) \cdot H(s) \tag{2.20.1}$$

if $x(t) = e^{st}$ (eigenfunction) $\longrightarrow y(t) = H(s)\, e^{st}$ i.e., eigenfunction of the system with $H(s)$ eigenvalue equal to the Laplace transform of $h(t)$ being $s$ within the ROC of $H(s)$.

If the ROC of $H(s)$ includes the imaginary axis $s = j\omega$, $H(s)$ is the frequency response of the LTI system. Otherwise, $H(s)$ is known as the system function or transfer function

### Causality

For a causal LTI system, $h(t) = 0$ for $t < 0$ and thus is right sided.

From last class, the ROC associated with $H(s)$ for a causal system is a right-half plane (i.e., ROC is to the right of the rightmost pole).

**Example**

$$h(t) = e^{-t}u(t) \quad \Longrightarrow \quad H(s) = \frac{1}{s+1} \tag{2.20.2}$$

pole @ $s = -1$. Two possible ROC: $\mathrm{Re}(s) < -1$, $\mathrm{Re}(s) > -1$ $\mathrm{Re}(s) > -1$ is the correct choice (because system is causal).

**Example**

$$h(t) = e^{-|t|} \tag{2.20.3}$$

since $h(t) \neq 0$ for $t < 0$, the system is not causal.

$$H(s) = \frac{-2}{s^2 - 1} = \frac{-2}{(s-1)(s+1)} \tag{2.20.4}$$

poles @ $s = 1$ and $s = -1$ ROC: $-1 < \mathrm{Re}(s) < \mathrm{Re}(1)$

**Example**

$$H(s) = \frac{e^s}{s+1}, \qquad \mathrm{Re}(s) > -1 \tag{2.20.5}$$

Because ROC is to right of the rightmost pole, the system is causal and $h(t)$ must be right side

$$e^{-t}u(t) \implies \frac{1}{s+1}, \qquad \mathrm{Re}(s) > -1 \tag{2.20.6}$$

$$e^{-(t+1)}u(t+1) \implies \frac{e^s}{s+1}, \qquad \mathrm{Re}(s) > -1 \tag{2.20.7}$$

$$h(t) = e^{-(t+1)}u(t+1) \tag{2.20.8}$$

which is noncausal for $-1 < t < 0$

Hence, the system is not causal. This example shows that causality implies that ROC is to the right of the rightmost pole, but the inverse is not generally true unless the system function is rational

$$H(s) = \frac{N(s)}{D(s)} \tag{2.20.9}$$

For an anticausal system (i.e., $h(t) = 0$ for $t > 0$), then $h(t)$ is left-sided and the ROC of $H(s)$ has to be left-half plane.

If the ROC of $H(s)$ is a left-half plane (to the left of the leftmost pole) then $h(t)$ is left-sided.

If and only if $H(s)$ is rational, having an ROC to the left of the leftmost pole is equivalent to the system being anticausal.

### Stability

The stability of a LTI system is equivalent to its $h(t)$ being absolutely integrable. An LTI system is stable if and only if the ROC of its $H(s)$ function includes the $j\omega$-axis [i.e., $\mathrm{Re}(s) = 0$].

$$H(s) = \frac{s-1}{(s+1)(s-2)} \qquad \text{poles @ } s = -1 \text{ and } s = 2 \tag{2.20.10}$$

- If $\mathrm{Re}(s) < -1$, $H(s)$ is left-sided $\Longrightarrow$ $h(t)$ is anticausal and non-stable
- If $\mathrm{Re}(s) > 2$, $H(s)$ is right-sided $\Longrightarrow$ $h(t)$ is causal and non-stable
- If $-1 < \mathrm{Re}(s) < 2$, $h(t)$ is non-causal and stable because ROC includes the $j\omega$-axis.

Using partial fractions,

$$\frac{s-1}{(s+1)(s-2)} = \frac{2/3}{s+1} + \frac{1/3}{s-2} \tag{2.20.11}$$

For $\mathrm{Re}(s) > 2$:

$$h(t) = \left[\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right]u(t) \tag{2.20.12}$$

For $-1 < \mathrm{Re}(s) < 2$:

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t) \tag{2.20.13}$$

is absolutely integrable

A causal system with rational system function $H(s)$ is stable if and only if all poles of $H(s)$ lie in the left-half of the $s$-plane, i.e., all poles have negative real parts

### LTI systems characterized by LCCDE

Consider an LTI system for which $x(t)$ and $y(t)$ satisfy a LCCDE

$$\begin{aligned} \frac{dy(t)}{dt} + 3y(t) = x(t) \quad &\Longrightarrow \quad sY(s) + 3Y(s) \\ = X(s) \quad &\Longrightarrow \quad (s+3)Y(s) = X(s) \end{aligned} \tag{2.20.14}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3} \tag{2.20.15}$$

Note that a LCCDE is not a complete specification of a LTI system, since there are several $h(t)$ that are consistent with the differential equation

Assuming that $H(s)$ is causal $\Longrightarrow$

$$H(s) = \frac{1}{s+3}, \quad \text{Re}(s) > -3 \quad \Longrightarrow \quad h(t) = e^{-3t}u(t) \tag{2.20.16}$$

If $H(s)$ is anticausal $\Longrightarrow$

$$H(s) = \frac{1}{s+3}, \quad \text{Re}(s) < -3 \quad \Longrightarrow \quad h(t) = -e^{-3t}u(-t) \tag{2.20.17}$$

### System Function Algebra and Block Diagram Representation

The Laplace Transform allows us to replace time-domain operations such as differentiation, convolution, time shifting, with algebraic operations.

Consider 2 systems in parallel

$$h(t) = h_1(t) + h_2(t) \quad \Longrightarrow \quad H(s) = H_1(s) + H_2(s) \tag{2.20.18}$$

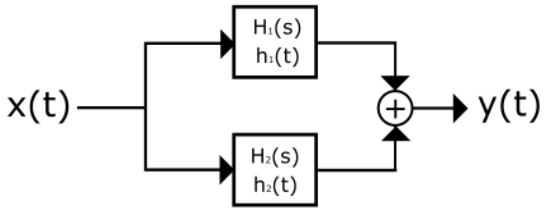



**Figure 2.20.1:** Basic block diagram of a parallel system.

Consider 2 LTI systems in series (cascade)

$$h(t) = h_1(t) * h_2(t) \quad \Longrightarrow \quad H(s) = H_1(s) \cdot H_2(s) \tag{2.20.19}$$

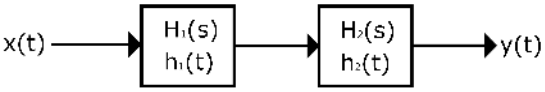



**Figure 2.20.2:** Basic block diagram of a cascade system.

Consider the feedback interconnection of 2 systems

$$Y(s) = H_1(s)\,E(s) \quad \text{where} \quad E(s) = X(s) - Z(s) \tag{2.20.20}$$

and

$$Z(s) = H_2(s)\,Y(s) \tag{2.20.21}$$

$$Y(s) = H_1(s)\,[X(s) - H_2(s)Y(s)] \tag{2.20.22}$$

$$[1 + H_1(s)H_2(s)]\,Y(s) = H_1(s)\,X(s) \tag{2.20.23}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \tag{2.20.24}$$

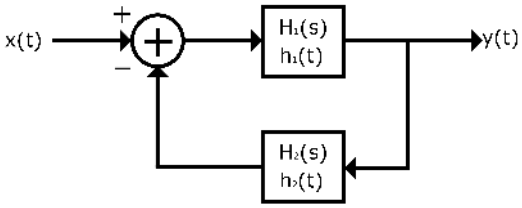



**Figure 2.20.3:** Basic block diagram of a feedback system.

### Block Diagram for CAUSAL LTI systems DESCRIBED BY LCCDE AND RATIONAL $H(s)$

**Example**

$$H(s) = \frac{1}{s+3} \quad \Longrightarrow \quad \frac{Y(s)}{X(s)} = \frac{1}{s+3} \quad \Longrightarrow \quad sY(s) + 3Y(s) = X(s) \tag{2.20.25}$$

$$\frac{dy(t)}{dt} + 3y(t) = x(t) \tag{2.20.26}$$

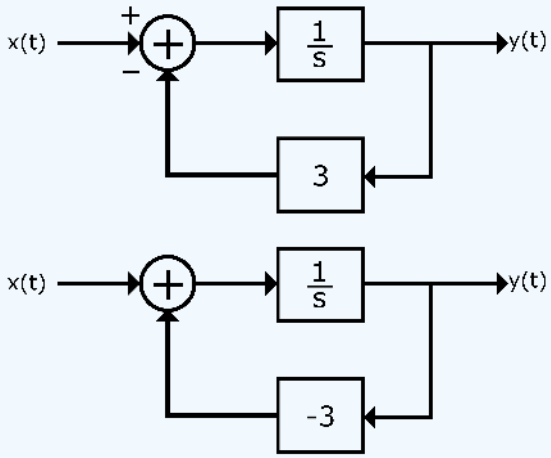



**Figure 2.20.4:** Two equal block diagrams of a feedback system representing this example.

$\frac{1}{s}$ is system function of an integrator

**Example**

causal LTI system

$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3}\right)(s+2) \tag{2.20.27}$$

$$\Longrightarrow H(s) \text{ cascade of } H_1 = \frac{1}{s+3} \text{ and } H_2 = s+2 \tag{2.20.28}$$

$$y(t) = \frac{dz(t)}{dt} + 2z(t) \tag{2.20.29}$$

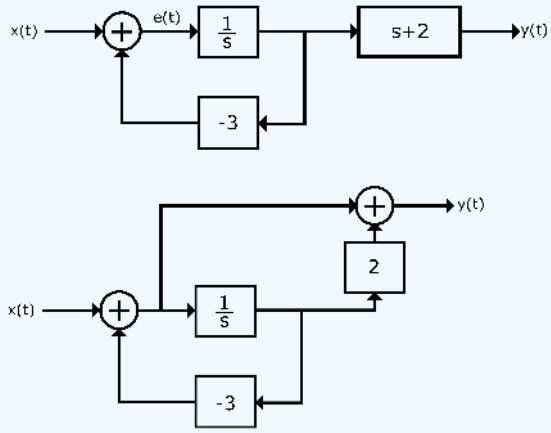



**Figure 2.20.5:** Two equal block diagrams of a system representing this example.

The input of the integrator $e(t)$ is exactly the derivative of $z(t)$

$$y(t) = \frac{dz(t)}{dt} + 2z(t) = e(t) + 2z(t) \tag{2.20.30}$$

**Example**

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2} \tag{2.20.31}$$

LCCDE

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) \tag{2.20.32}$$

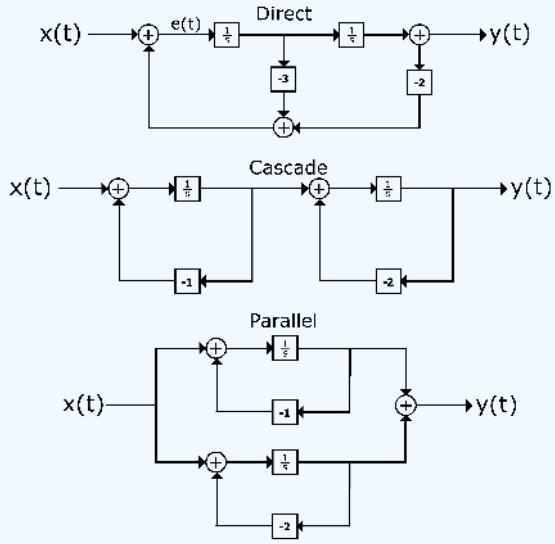
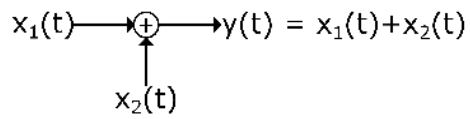
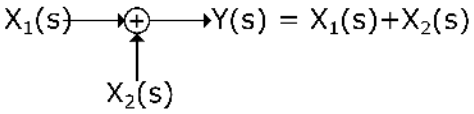
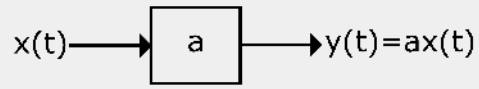
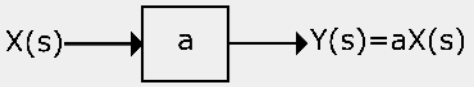
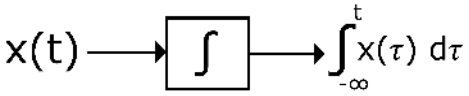
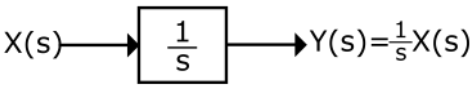
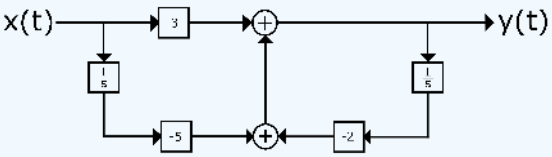



**Figure 2.20.6:** Three equal block diagrams representing the direct, cascade, and parallel forms of this example.

**3 elements for Block Diagrams**

| Time | Laplace (Frequency) |
|---|---|
| (1) Add<br>$x_1(t)$ and $x_2(t) \longrightarrow y(t) = x_1(t) + x_2(t)$<br>  | $Y(s) = X_1(s) + X_2(s)$<br>  |
| (2) Scaling<br>$x(t) \longrightarrow \alpha \longrightarrow y(t) = \alpha x(t)$<br>  | $Y(s) = \alpha X(s)$<br>  |
| (3) Integrators<br>$x(t) \longrightarrow \int \longrightarrow y(t) = \int_0^t x(\tau)\, d\tau$<br>  | $Y(s) = \frac{1}{s} X(s)$<br>  |

Example




$$Y(s) = 3X(s) - 5\frac{1}{s}X(s) - 2\frac{1}{s}Y(s) \tag{2.20.33}$$

$$Y(s) + \frac{2}{s}Y(s) = -\frac{5}{s}X(s) + 3X(s) \tag{2.20.34}$$

multiply both sides by $s$

$$sY(s) + 2Y(s) = -5X(s) + 3sX(s) \tag{2.20.35}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3s-5}{s+2} \tag{2.20.36}$$

if $H(s)$ is causal $\Longrightarrow$ $Re(s) > -2$ (ROC of $H(s)$ right-side)

LCCDE

$$\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} - 5x(t) \tag{2.20.37}$$

Example

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = 2\frac{d^2x(t)}{dt^2} + 14\frac{dx(t)}{dt} + 20x(t) \tag{2.20.38}$$

$$s^2Y(s) + 5sY(s) + 4Y(s) = 2s^2X(s) + 14sX(s) + 20X(s) \tag{2.20.39}$$

Find for $H(s) = \frac{Y(s)}{X(s)}$

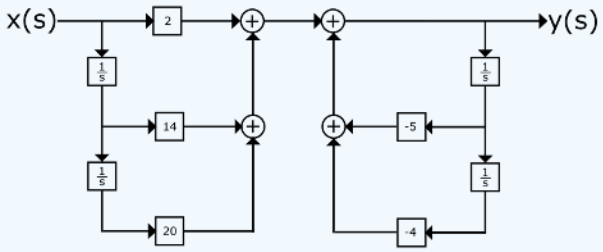
$$H(s) = \frac{2s^2 + 14s + 20}{s^2 + 5s + 4} \tag{2.20.40}$$

Find for block diagram:

$$\frac{1}{s^2}\left[s^2Y(s) + 5sY(s) + 4Y(s)\right] = \frac{1}{s^2}\left[2s^2X(s) + 14sX(s) + 20X(s)\right] \tag{2.20.41}$$

$$Y(s) + \frac{5}{s}Y(s) + \frac{4}{s^2}Y(s) = 2X(s) + \frac{14}{s}X(s) + \frac{20}{s^2}X(s) \tag{2.20.42}$$

$$Y(s) = -\frac{5}{s}Y(s) - \frac{4}{s^2}Y(s) + 2X(s) + \frac{14}{s}X(s) + \frac{20}{s^2}X(s) \tag{2.20.43}$$




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# 2.21: Feedback

## Learning Objectives: Feedback

1. Understand how the closed-loop gain relates to the feedforward and feedback system functions
2. Implement stabilization using negative feedback
3. Design systems to compensate for nonideal amplifiers

## YouTube Videos

Feedback



Feedback Stabilization Example



## Summary

**Feedback** → using the output of a system to control or modify your input

└ Adaptive Optics

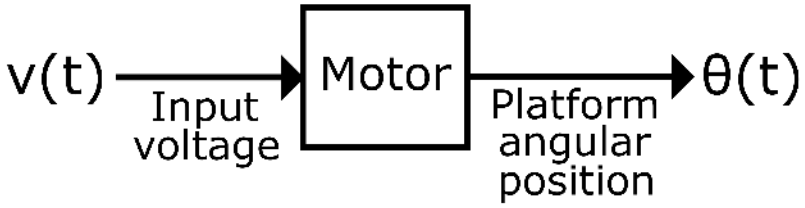


**Figure 2.21.1:** Example of an open-loop system.

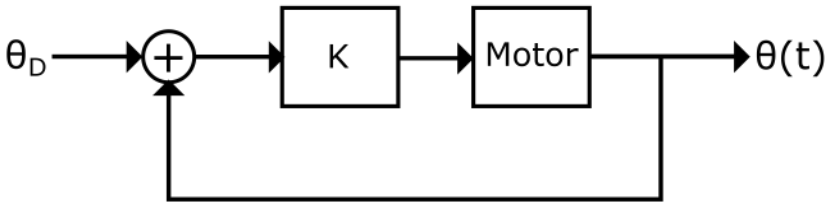



**Figure 2.21.2:** Example of a closed-loop system.

Whereas the open-loop system does not allow us to make corrections, a closed-loop system allows us to have

- Better control of system components
- Being able to detect any misalignment

## Linear Feedback Systems

Assumption: $H(s)$ is either unilateral or bilateral transform → $H(s)$ causal and $ROC_H$ to the right of the rightmost pole

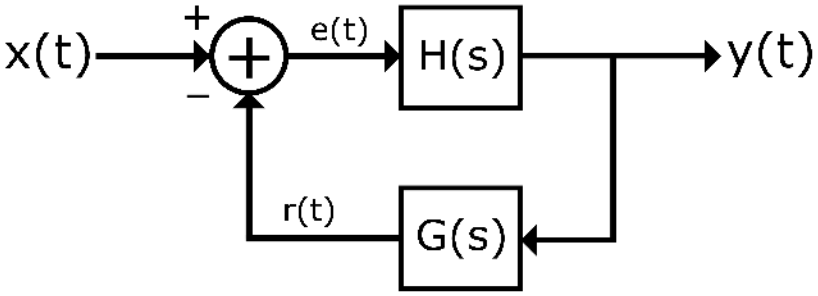



**Figure 2.21.3:** Example of a linear feedback system design.

$r(t)$: signal fed back

$e(t) = x(t) - r(t)$: error between input signal and actual response

$H(s)$: system function of the forward path

$G(s)$: system function of the feedback path

$Q(s)$: closed-loop system function

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)} \tag{2.21.1}$$

include both forward and feedback paths

## Inverse System Design

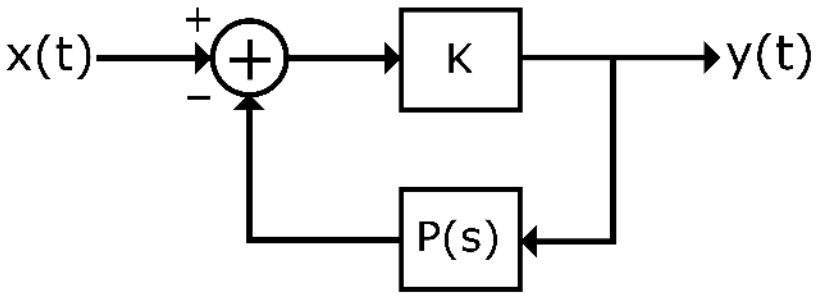



**Figure 2.21.3:** Example of an inverse system design.

$H(s) = k$

$P(s)$: system function of feedback path $= G(s)$

$$Q(s) = \frac{H(s)}{1 + H(s)P(s)} = \frac{k}{1 + kP(s)} \approx \frac{k}{kP(s)} = \frac{1}{P(s)} \tag{2.21.2}$$

$k \gg$ large and $1 \ll kP(s)$

$Q(s) = k$ is the inverse of the feedback system

### Compensation for NonIdeal Elements

Use of feedback systems to correct some non-ideal properties/measurements of the open-loop system

$G(s) = K$ →

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)} = \frac{H(s)}{1 + K\,H(s)} \approx \frac{H(s)}{K\,H(s)} = \frac{1}{K} = Q(s) \tag{2.21.3}$$

CONDITION: the closed loop gain $\frac{1}{K}$ is substantially less than the open-loop gain $|H(j\omega)|$

**KEY POINT**: If $K \ll 1$, $|H(j\omega)|$ is an amplifier but without erratic gain

**WATCH OUT:** Find right $K$ so it is small for high $|Q|$ but big enough so $|KH| \gg 1$.

### Stabilization of Unstable Systems

Use of feedback systems to stabilize systems that are a-priori unstable

**Example**

(1) control trajectory of a rocket; (2) regularization of nuclear reactions in a nuclear power plant; (3) stabilization of an aircraft; (4) natural and regulatory control of animal populations

$$H(s) = \frac{b}{s - a} \tag{2.21.4}$$

1st-order system $\quad a > 0 \rightarrow h(t)$ unstable

$$G(s) = k \tag{2.21.5}$$

constant gain

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)} = \frac{\frac{b}{s-a}}{1 + k\frac{b}{s-a}} = \frac{b}{s - a + kb}.$$

pole @ $s = a - kb$

The close-loop system is stable if the pole $s < 0$, so $kb > a$

proportional feedback system

$$H(s) = \frac{b}{s^2 + a} \tag{2.21.6}$$

$a > 0$ → 2nd-order system → $H(s)$ has imaginary poles and $h(t)$ is sinusoidal

$H(s) = \frac{b}{s^2 + a}$, $a < 0$ → $H(s)$ has one pole in the left-side plane and one in the right-half plane → $H(s)$ is unstable.

$$G(s) = k \tag{2.21.7}$$

$$Q(s) = \frac{H(s)}{1 + G(s)H(s)} = \frac{\frac{b}{s^2+a}}{1 + k\frac{b}{s^2+a}} = \frac{b}{s^2 + a + kb} \tag{2.21.8}$$

$$\frac{b}{s^2+a+kb} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{2.21.9}$$

We cannot stabilize the system because we do not have any damping $\zeta$.

$$G(s) = k_1 + k_2 s \quad \text{(proportional-plus-derivative)} \tag{2.21.10}$$

$$Q(s) = \frac{b}{s^2 + bk_2 s + (a + k_1 b)} \tag{2.21.11}$$

where $bk_2 > 0$ and $a + k_1 b > 0$

### Sampled-data feedback systems

The output of a CT signal is sampled

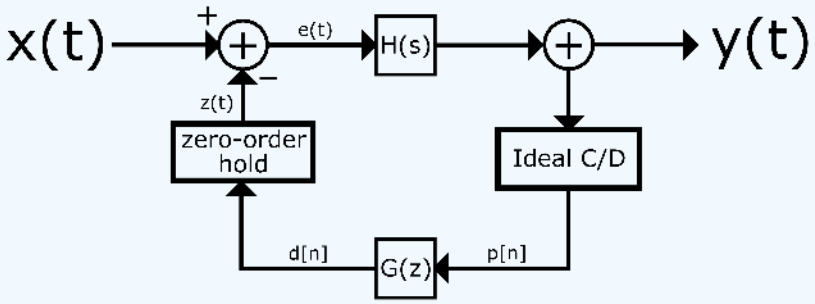



**Figure 2.21.4:** Representation of the stabilization system for the given example.

$$z(t) = d[n] \quad \text{for } nT \le t < (n+1)T \quad \text{only a period} \tag{2.21.12}$$

$$d[n] = p[n] * g[n] \quad \text{(in TL)} \tag{2.21.13}$$

$$p[n] = y(nT) \tag{2.21.14}$$

### Tracking systems

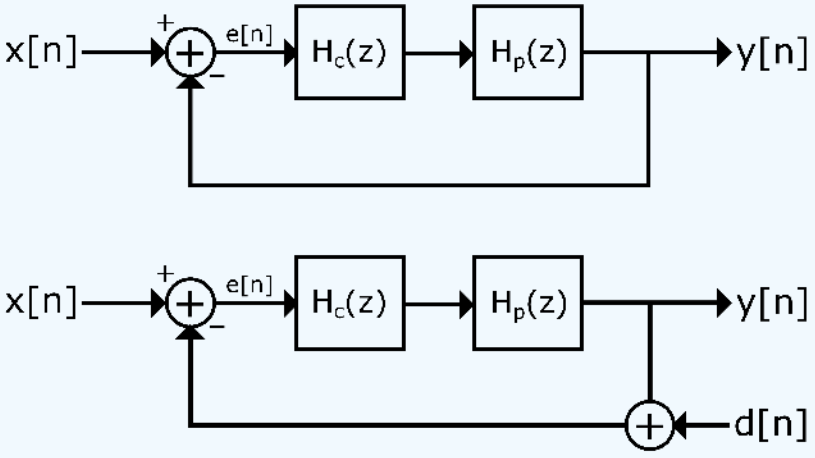



**Figure 2.21.5:** Representation of a tracking system without (top) and with (bottom) accounting for disturbance $d[n]$.

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# Index

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# Glossary

## Important Discrete Formulas & Tables:

Discrete-Time Convolution Equation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] \qquad (1)$$

**DTFS Synthesis (build $x[n]$ from $a_k$)**

\[x[n] = \sum_{k=0}^{N-1} a_k\,e^{j\frac{2\pi}{N}kn}$$

**DTFS Analysis (compute $a_k$ from $x[n]$)**

\[a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\,e^{-j\frac{2\pi}{N}kn}$$

**Table 1.7.1** Properties of the Discrete-Time Fourier Series (DTFS)

| Periodic Signal | Fourier Series Coefficients |
|---|---|
| $x[n]$, $y[n]$ periodic with period $N$ and fundamental frequency $\omega_0 = \frac{2\pi}{N}$ | $a_k$, $b_k$ periodic with period $N$ |
| $x[-n]$ | $a_{-k}$ |
| $x[n-n_0]$ | $a_k e^{-j\frac{2\pi}{N}kn_0}$ |
| $e^{j\frac{2\pi}{N}mn}x[n]$ | $a_{k-m}$ |
| $x^*[n]$ | $a_{-k}^*$ |
| $x[n]*y[n]$ | $N a_k b_k$ |
| $A\,x[n] + B\,y[n]$ | $A\,a_k + B\,b_k$ |
| $x[n]\,y[n]$ | $\sum_{\ell=\langle N\rangle} a_\ell\, b_{k-\ell}$ |
| $\sum_{r=\langle N\rangle} x[r]\,y[n-r]$ | $N a_k b_k$ |
| $\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$ | Parseval's Relation for Periodic Signals |

**Table 1.10.2** Basic Discrete-Time Fourier Transform (DTFT) Pairs

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{N}kn}$ | $2\pi\sum_{k=-\infty}^{\infty} a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi\sum_{\ell=-\infty}^{\infty}\delta(\omega-\omega_0-2\pi\ell)$ | (a) $\omega_0 = \frac{2\pi m}{N}$: $a_k = \begin{cases}1, & k = m, m\pm N, \ldots\\ 0, & \text{otherwise}\end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational → The signal is aperiodic |
| $\sin(\omega_0 n)$ | $\frac{\pi}{j}\sum_{\ell=-\infty}^{\infty}\left[\delta(\omega-\omega_0-2\pi\ell) - \delta(\omega+\omega_0-2\pi\ell)\right]$ | (a) $\omega_0 = \frac{2\pi r}{N}$: $a_k = \begin{cases}-\frac{j}{2}, & k = r, r\pm N, \ldots\\ \frac{j}{2}, & k = -r, -r\pm N, \ldots\\ 0, & \text{otherwise}\end{cases}$ <br> (b) $\frac{\omega_0}{2\pi}$ irrational → The signal is aperiodic |
| $x[n] = 1$ | $2\pi\sum_{\ell=-\infty}^{\infty}\delta(\omega-2\pi\ell)$ | $a_k = \begin{cases}1, & k = 0, \pm N, \ldots\\ 0, & \text{otherwise}\end{cases}$ <br> (viewed as periodic with period $N$) |
| **Periodic square wave** <br> $x[n] = \begin{cases}1, & \lvert n\rvert \le N_1\\ 0, & N_1 < \lvert n\rvert \le N/2\end{cases}$ <br> $x[n+N] = x[n]$ | $2\pi\sum_{k=-\infty}^{\infty} a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \begin{cases}\frac{\sin\left(\frac{2\pi k}{N}\left(N_1+\frac{1}{2}\right)\right)}{N\sin\left(\frac{\pi k}{N}\right)}, & k \neq 0, \pm N, \ldots\\ \frac{2N_1+1}{N}, & k = 0, \pm N, \ldots\end{cases}$ <br> $a_k = a_{-k}$ <br> $\Re\{a_k\} = \Re\{a_{-k}\}$ <br> $\Im\{a_k\} = -\Im\{a_{-k}\}$ <br> $\lvert a_k\rvert = \lvert a_{-k}\rvert$ <br> $\angle a_k = -\angle a_{-k}$ |
| $\sum_{k=-\infty}^{\infty}\delta[n-kN]$ | $\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{1-ae^{-j\omega}}$ | $a_k$ real and even |
| $x[n] = \begin{cases}1, & \lvert n\rvert \le N_1\\ 0, & \lvert n\rvert > N_1\end{cases}$ | $\frac{\sin\left(\omega\left(N_1+\frac{1}{2}\right)\right)}{\sin(\omega/2)}$ | purely imaginary and odd |
| $\frac{\sin(\omega_m n)}{\pi n} = \frac{\omega_m}{\pi}\,\mathrm{sinc}\left(\frac{\omega_m}{\pi}n\right)$ <br> $0 < \omega_m < \pi$ | $X(\omega) = \begin{cases}1, & 0 \le \lvert\omega\rvert \le \omega_m\\ 0, & \omega_m < \lvert\omega\rvert \le \pi\end{cases}$ <br> $X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1-e^{-j\omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$ | — |
| $\delta[n-n_0]$ | $e^{-j\omega n_0}$ | Fourier Transform |

DTFT Analysis Equation (Definition)

**DTFT pair (Analysis and Synthesis):**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega. \qquad (2)$$

**Table 1.10.1** Properties of the Discrete-Time Fourier Transform (DTFT)

| Discrete-Time Signal | Fourier Transform |
|---|---|
| periodic with period | |
| Parseval's Relation for Aperiodic Signals | |

**Definition: z-Transform**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}. \tag{8}$$

## Important Continuous Formulas & Tables:

**Definition: Convolution Integral**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h_\tau(t)\, d\tau \tag{9}$$

$$y(t) = x(t) * h(t) \tag{10}$$

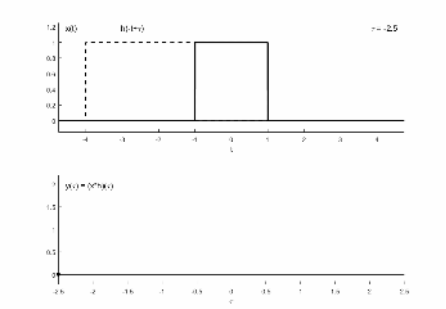


**Figure 2.3.2:** Animation showing the convolution between two rectangular functions.

Fourier Series (Synthesis and Analysis Equations)

**Synthesis equation**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \tag{11}$$

**Analysis equation**

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t) e^{-jk\frac{2\pi}{T}t}\, dt \tag{12}$$

Fourier Transform (Analysis and Synthesis Equations)

$x(t)$ → Fourier Transform → $X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$ → Inverse Fourier Transform → $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

**Laplace Transform**

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st}\, dt \tag{13}$$

Laplace Transform Properties

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \tag{14}$$

| Time | Laplace |
|---|---|
| $y(t) = x(t - t_0)$ | $Y(s) = e^{-st_0} X(s)$ same ROC |
| $y(t) = e^{s_0 t} x(t)$ | $Y(s) = X(s - s_0)$ shift ROC by $s_0$ |
| $y(t) = x(t) * h(t)$ | $Y(s) = X(s) \cdot H(s)$ ROC of $Y$ includes $ROC_X \cap ROC_H$ (intersection) |
| $y(t) = \frac{dx(t)}{dt}$ | $Y(s) = s\,H(s)$ |
| $y(t) = -t\, x(t)$ | $Y(s) = \frac{d}{ds} X(s)$ |
| $y(t) = \int_{-\infty}^{t} x(z)\, dz$ | $Y(s) = \frac{1}{s} X(s)$ |

If $x(t) = 0$ for $t < 0$, $x(0^+) = \lim_{s\to\infty} sX(s)$ Initial Value Theorem

If $x(t) = 0$ for $t < 0$, $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Laplace Transforms of Elementary Functions

| Signal | Transform | ROC |
|---|---|---|
| $\delta(t)$ | $1$ | All $s$ |
| $u(t)$ | $\frac{1}{s}$ | $\mathrm{Re}\{s\} > 0$ |
| $-u(-t)$ | $\frac{1}{s}$ | $\mathrm{Re}\{s\} < 0$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} > -a$ |
| $-e^{-at}u(-t)$ | $\frac{1}{s+a}$ | $\mathrm{Re}\{s\} < -a$ |
| $\delta(t-T)$ | $e^{-sT}$ | All $s$ |
| $[\cos\omega_0 t]u(t)$ | $\frac{s}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $[\sin\omega_0 t]u(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > 0$ |
| $[e^{-at}\cos\omega_0 t]u(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > -a$ |
| $[e^{-at}\sin\omega_0 t]u(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ | $\mathrm{Re}\{s\} > -a$ |
| $u_n(t) = \frac{d^n\delta(t)}{dt^n}$ | $s^n$ | All $s$ |
| $u_{-n}(t) = u(t) * \cdots * u(t)$ (n times) | $\frac{1}{s^n}$ | $\mathrm{Re}\{s\} > 0$ |

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