

STATICS (NWTC)



Northeast Wisconsin Technical College

Statics (NWTC)

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This text was compiled on 05/15/2026

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CHAPTER OVERVIEW

1: Principles of Statics

Introduction

Statics is a branch of **Mechanics**, which is a subset of **Physical Sciences**. It involves the study of bodies that are at rest or move with constant velocity. Unlike Dynamics, which deals with bodies in motion, Statics focuses on mechanical systems in equilibrium.

The fundamental concept in Statics is the **equilibrium of forces**. This means that for a system to be in equilibrium, the net force and the net torque (moment of force) acting on it must be zero. This is based on **Newton's First Law of Motion**, which states that an object at rest tends to stay at rest, and an object in motion tends to stay in motion, unless acted upon by an external force.

Statics is used in many fields of engineering, especially in the design of structures and machines. Engineers use the principles of statics to calculate forces in structures like bridges, buildings, and mechanical devices, ensuring they can withstand the forces they encounter.

Statics is the foundation for future courses like **Strength of Materials**. In Statics, when we solve problems, all bodies are considered rigid and do not deform. When we get to Strength of Materials, then we will study the relationships between external forces acting on bodies at rest and the internal responses generated. In Strength of Materials, bodies are assumed to be deformable (real world deformation).

In summary, Statics is a crucial area of study in physics and engineering that helps us understand and predict the behavior of stationary objects and structures. It provides the foundation for designing safe and efficient structures and machines. Understanding statics can also give us insights into the natural world, helping us understand how objects and structures, from the smallest machine to the largest skyscraper, withstand the forces they experience.

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1.1: Right Triangles

Learning Objectives

- Find the six trigonometric function values of an angle in a right triangle.

Sine, cosine, tangent, and other ratios of sides of a right triangle.

Sine, Cosine, and Tangent

Trigonometry is the study of the relationships between the sides and angles of right triangles. The legs are called *adjacent* or *opposite* depending on which **acute angle** is being used.

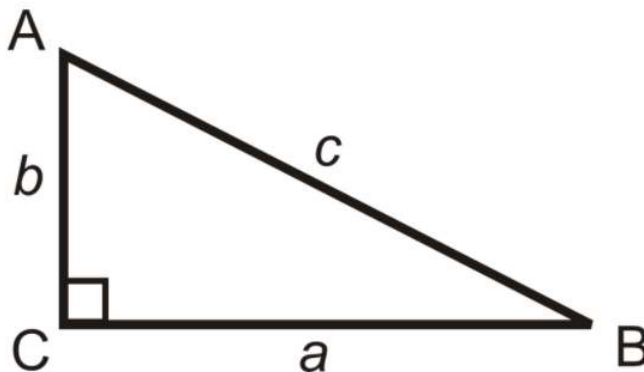


Figure 1.1.1

a is adjacent to $\angle B$ a is opposite $\angle A$
 b is adjacent to $\angle A$ b is opposite $\angle B$
 c is the hypotenuse

The three basic **trigonometric ratios** are called sine, cosine and tangent. For right triangle $\triangle ABC$, we have:

$$\begin{aligned} \text{sine Ratio: } \frac{\text{opposite leg}}{\text{hypotenuse}} & \quad \sin A = \frac{a}{c} \text{ or } \sin B = \frac{b}{c} \\ \text{cosine Ratio: } \frac{\text{adjacent leg}}{\text{hypotenuse}} & \quad \cos A = \frac{b}{c} \text{ or } \cos B = \frac{a}{c} \\ \text{Tangent Ratio: } \frac{\text{opposite leg}}{\text{adjacent leg}} & \quad \tan A = \frac{a}{b} \text{ or } \tan B = \frac{b}{a} \end{aligned}$$

An easy way to remember ratios is to use SOH-CAH-TOA.

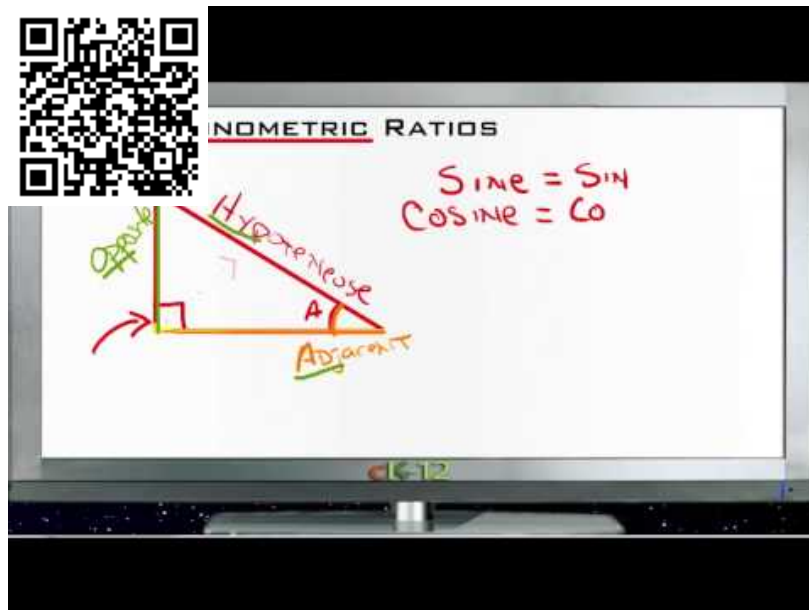
$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{Cosine} = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{Tangent} = \frac{\text{Opposite}}{\text{Adjacent}}$$

Figure 1.1.2

A few important points:

- Always **reduce ratios** (fractions) when you can.
- Use the **Pythagorean Theorem** to find the missing side (if there is one).
- If there is a **radical** in the denominator, **rationalize the denominator**.

What if you were given a right triangle and told that its sides measure 3, 4, and 5 inches? How could you find the sine, cosine, and tangent of one of the triangle's non-right angles?



[Video Transcript](#)

✓ Example 1.1.1

Find the sine, cosine and tangent ratios of $\angle A$.

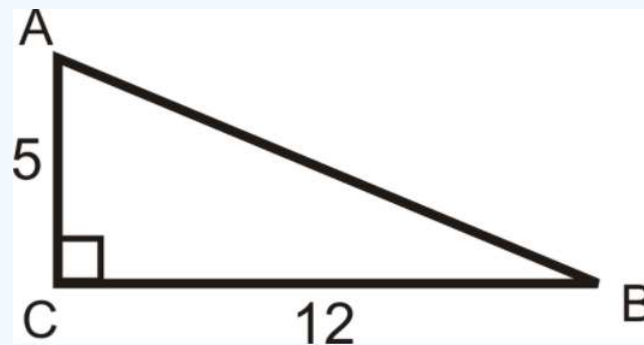


Figure 1.1.3

Solution

First, we need to use the Pythagorean Theorem to find the length of the **hypotenuse**.

$$5^2 + 12^2 = c^2$$

$$13 = c$$

$$\sin A = \frac{\text{leg opposite } \angle A}{\text{hypotenuse}} = \frac{12}{13}$$

$$\tan A = \frac{\text{leg opposite } \angle A}{\text{leg adjacent to } \angle A} = \frac{12}{5}$$

$$\cos A = \frac{\text{leg adjacent to } \angle A}{\text{hypotenuse}} = \frac{5}{13},$$

✓ Example 1.1.2

Find the sine, cosine, and tangent of $\angle B$.

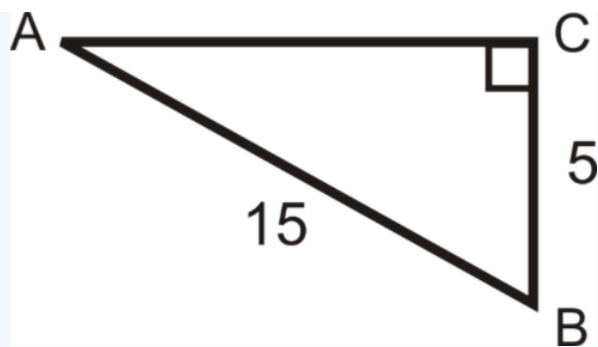


Figure 1.1.4

Find the length of the missing side.

Solution

$$\begin{aligned} AC^2 + 5^2 &= 15^2 \\ AC^2 &= 200 \\ AC &= 10\sqrt{2} \\ \sin B &= \frac{10\sqrt{2}}{15} = \frac{2\sqrt{2}}{3} \quad \cos B = \frac{5}{15} = \frac{1}{3} \quad \tan B = \frac{10\sqrt{2}}{5} = 2\sqrt{2} \end{aligned}$$

✓ Example 1.1.3

Find the sine, cosine and tangent of 30° .

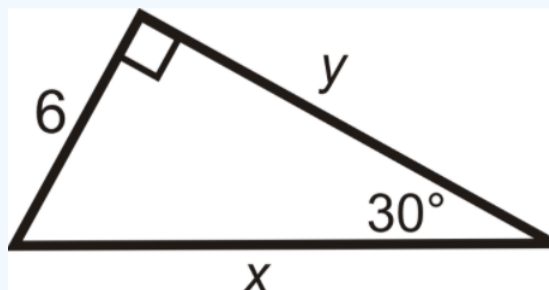


Figure 1.1.5

Solution

This is a 30-60-90 triangle. The short leg is 6, $y = 6\sqrt{3}$ and $x = 12$.

$$\sin 30^\circ = \frac{6}{12} = \frac{1}{2} \quad \cos 30^\circ = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2} \quad \tan 30^\circ = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

✓ Example 1.1.4

Answer the questions about the following image. Reduce all fractions.

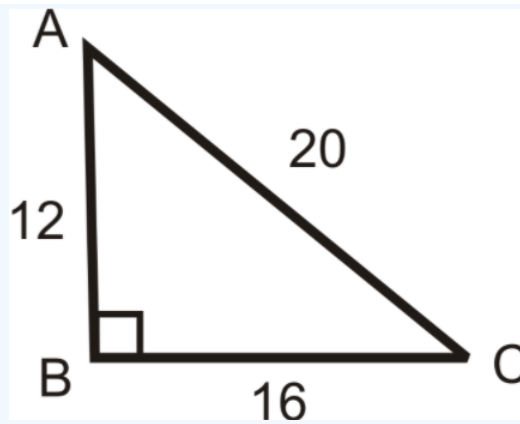


Figure 1.1.6

What is $\sin A$, $\cos A$, and $\tan A$?

Solution

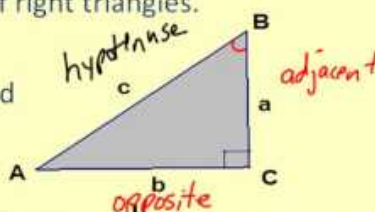
$$\begin{aligned}\sin A &= \frac{16}{20} = \frac{4}{5} \\ \cos A &= \frac{12}{20} = \frac{3}{5} \\ \tan A &= \frac{16}{12} = \frac{4}{3}\end{aligned}$$

Resources



metric functions can be defined in terms of ratios of sides of right triangles.

Here are the sine, cosine, and tangent functions.



$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$	$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}$	$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$
$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$	$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{c}$	$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{b}{a}$

SOH CAH TOA

[Video Transcript 2](#)

Vocabulary

Term	Definition
Acute Angle	An acute angle is an angle with a measure of less than 90 degrees.
Adjacent Angles	Two angles are adjacent if they share a side and vertex. The word 'adjacent' means 'beside' or 'next-to'.
Hypotenuse	The hypotenuse of a right triangle is the longest side of the right triangle. It is across from the right angle.
Legs of a Right Triangle	The legs of a right triangle are the two shorter sides of the right triangle. Legs are adjacent to the right angle.
opposite	The opposite of a number x is $-x$. A number and its opposite always sum to zero.
Pythagorean Theorem	The Pythagorean Theorem is a mathematical relationship between the sides of a right triangle, given by $a^2 + b^2 = c^2$, where a and b are legs of the triangle and c is the hypotenuse of the triangle.
Radical	The $\sqrt{\quad}$, or square root, sign.
sine	The sine of an angle in a right triangle is a value found by dividing the length of the side opposite the given angle by the length of the hypotenuse.
Trigonometric Ratios	Ratios that help us to understand the relationships between sides and angles of right triangles.

Additional Resources

Video: Introduction to Trigonometric Functions Using Triangles

Activities: Sine, Cosine, Tangent Discussion Questions

Study Aids: Trigonometric Ratios Study Guide

Practice: Right Triangle Trigonometry

Real World: Sine Cosine Tangent

The Pythagorean Theorem

[Pythagoras](#) was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Croton, whose members eventually became known as the Pythagoreans. The inner circle at the school, the Mathematikoi, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the Pythagorean Theorem.

Prior to revealing the contents of the Pythagorean Theorem, we pause to provide the definition of a right triangle and its constituent parts.

Right Triangle

A triangle with one right angle (90°) is called a **right triangle**. In the figure below, the right angle is marked with a little square.

The side of the triangle that is directly opposite the right angle is called the **hypotenuse**. The sides of the triangle that include the right angle are called the *legs* of the right triangle.

Now we can state one of the most ancient theorems of mathematics, the *Pythagorean Theorem*.

Pythagorean Theorem

The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2,$$

is called the **Pythagorean Theorem**.

Here are two important observations.

Observations Regarding the Hypotenuse

Two important facts regarding the hypotenuse of the right triangle are:

1. The hypotenuse is the longest side of the triangle and lies directly opposite the right angle.
2. In the Pythagorean equation $a^2 + b^2 = c^2$, the hypotenuse lies by itself on one side of the equation.

*The Pythagorean Theorem can **only** be applied to right triangles.*

Let's look at a simple application of the Pythagorean Theorem.

✓ Example 1

The legs of a right triangle measure 3 and 4 meters, respectively. Find the length of the hypotenuse.

Solution

Let's follow the *Requirements for Word Problem Solutions*.

1. *Set up a Variable Dictionary.* Let c represent the length of the hypotenuse, as pictured in the following sketch.
2. *Set up an Equation.* The Pythagorean Theorem says that

$$a^2 + b^2 = c^2.$$

In this example, the legs are known. Substitute 4 for a and 3 for b (3 for a and 4 for b works equally well) into the Pythagorean equation.

$$4^2 + 3^2 = c^2$$

3. *Solve the Equation.*

$$\begin{aligned} 4^2 + 3^2 &= c^2 && \text{The Pythagorean equation.} \\ 16 + 9 &= c^2 && \text{Exponents first: } 4^2 = 16 \text{ and } 3^2 = 9. \\ 25 &= c^2 && \text{Add: } 16 + 9 = 25. \\ 5 &= c && \text{Take the nonnegative square root.} \end{aligned}$$

Technically, there are two answers to $c^2 = 25$, i.e., $c = -5$ or $c = 5$. However, c represents the hypotenuse of the right triangle and must be nonnegative. Hence, we must choose $c = 5$.

4. *Answer the Question.* The hypotenuse has length 5 meters.
5. *Look Back.* Do the numbers satisfy the Pythagorean Theorem? The sum of the squares of the legs should equal the square of the hypotenuse. Let's check.

$$\begin{aligned} 4^2 + 3^2 &= 5^2 \\ 16 + 9 &= 25 \\ 25 &= 25 \end{aligned}$$

All is well!

Sometimes an approximate answer is desired, particularly in applications.

✓ Example 2

Ginny want to create a vegetable garden in the corner of her yard in the shape of a right triangle. She cuts two boards of length 8 feet which will form the legs of her garden. Find the length of board she should cut to form the hypotenuse of her garden, correct to the nearest tenth of a foot.

Solution

1. *Set Up an Equation.* The hypotenuse is isolated on one side of the Pythagorean equation.

$$x^2 = 8^2 + 8^2$$

2. *Solve the Equation.*

$$\begin{aligned}x^2 &= 8^2 + 8^2 && \text{The Pythagorean equation.} \\x^2 &= 64 + 64 && \text{Exponents first: } 8^2 = 64 \text{ and } 8^2 = 64. \\x^2 &= 128 && \text{Add: } 64 + 64 = 128. \\x &= \sqrt{128} && \text{Take the nonnegative square root.}\end{aligned}$$

3. *Answer the Question.* The exact length of the hypotenuse is $\sqrt{128}$ feet, but we're asked to find the hypotenuse to the nearest tenth of a foot. Using a calculator, we find an approximation for $\sqrt{128}$.

$$\sqrt{128} \approx 11.313708499$$

To round to the nearest tenth, first identify the rounding and test digits.

The test digit is less than five. So we leave the rounding digit alone and truncate. Therefore, correct to the nearest tenth of a foot, the length of the hypotenuse is approximately 11.3 feet.

4. *Look Back.* The sum of the squares of the legs is

$$\begin{aligned}8^2 + 8^2 &= 64 + 64 \\&= 128.\end{aligned}$$

The square of the hypotenuse is

$$(11.3)^2 = 127.69$$

These are almost the same, the discrepancy due to the fact that we rounded to find an approximation for the hypotenuse.

Exercises

In Exercises 1-16, your solutions should include a well-labeled sketch.

1. The length of one leg of a right triangle is 15 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.
2. The length of one leg of a right triangle is 7 meters, and the length of the hypotenuse is 25 meters. Find the exact length of the other leg.
3. The lengths of two legs of a right triangle are 12 meters and 16 meters. Find the exact length of the hypotenuse.
4. The lengths of two legs of a right triangle are 9 meters and 12 meters. Find the exact length of the hypotenuse.
5. The length of one leg of a right triangle is 13 meters, and the length of the hypotenuse is 22 meters. Find the exact length of the other leg.

6. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.
7. The lengths of two legs of a right triangle are 2 meters and 21 meters. Find the exact length of the hypotenuse.
8. The lengths of two legs of a right triangle are 7 meters and 8 meters. Find the exact length of the hypotenuse.
9. The length of one leg of a right triangle is 12 meters, and the length of the hypotenuse is 19 meters. Find the exact length of the other leg.
10. The length of one leg of a right triangle is 5 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.
11. The lengths of two legs of a right triangle are 6 meters and 8 meters. Find the exact length of the hypotenuse.
12. The lengths of two legs of a right triangle are 5 meters and 12 meters. Find the exact length of the hypotenuse.
13. The length of one leg of a right triangle is 6 meters, and the length of the hypotenuse is 10 meters. Find the exact length of the other leg.
14. The length of one leg of a right triangle is 9 meters, and the length of the hypotenuse is 15 meters. Find the exact length of the other leg.
15. The lengths of two legs of a right triangle are 6 meters and 22 meters. Find the exact length of the hypotenuse.
16. The lengths of two legs of a right triangle are 9 meters and 19 meters. Find the exact length of the hypotenuse.

In Exercises 17-24, your solutions should include a well-labeled sketch.

17. The lengths of two legs of a right triangle are 3 meters and 18 meters. Find the length of the hypotenuse. Round your answer to the nearest hundredth.
 18. The lengths of two legs of a right triangle are 10 feet and 16 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.
 19. The length of one leg of a right triangle is 2 meters, and the length of the hypotenuse is 17 meters. Find the length of the other leg. Round your answer to the nearest tenth.
 20. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 12 meters. Find the length of the other leg. Round your answer to the nearest hundredth.
 21. The lengths of two legs of a right triangle are 15 feet and 18 feet. Find the length of the hypotenuse. Round your answer to the nearest hundredth.
 22. The lengths of two legs of a right triangle are 6 feet and 13 feet. Find the length of the hypotenuse. Round your answer to the nearest tenth.
 23. The length of one leg of a right triangle is 4 meters, and the length of the hypotenuse is 8 meters. Find the length of the other leg. Round your answer to the nearest hundredth.
 24. The length of one leg of a right triangle is 3 meters, and the length of the hypotenuse is 15 meters. Find the length of the other leg. Round your answer to the nearest tenth.
-
25. Greta and Fritz are planting a 13-meter by 18-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.
 26. Markos and Angelina are planting an 11-meter by 19-meter rectangular garden, and are laying it out using string. They would like to know the length of a diagonal to make sure that right angles are formed. Find the length of a diagonal. Round your answer to the nearest tenth. Your solution should include a well-labeled sketch.
 27. The base of a 24-meter long guy wire is located 10 meters from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-

labeled sketch.

28. The base of a 30-foot long guy wire is located 9 feet from the base of the telephone pole that it is anchoring. How high up the pole does the guy wire reach? Round your answer to the nearest hundredth. Your solution should include a well-labeled sketch.

29. **Hiking Trail.** A hiking trail runs due south for 8 kilometers, then turns west for about 15 kilometers, and then heads northeast on a direct path to the starting point. How long is the entire trail?

30. **Animal Trail.** An animal trail runs due east from a watering hole for 12 kilometers, then goes north for 5 kilometers. Then the trail turns southwest on a direct path back to the watering hole. How long is the entire trail?

31. **Upper Window.** A 10-foot ladder leans against the wall of a house. How close to the wall must the bottom of the ladder be in order to reach a window 8 feet above the ground?

32. **How high?** A 10-foot ladder leans against the wall of a house. How high will the ladder be if the bottom of the ladder is 4 feet from the wall? Round your answer to the nearest tenth.

Answers

1. 20 meters
3. 20 meters
5. $\sqrt{315}$ meters
7. $\sqrt{445}$ meters
9. $\sqrt{217}$ meters
11. 10 meters
13. 8 meters
15. $\sqrt{520}$ meters
17. 18.25 meters
19. 16.9 meters
21. 23.43 feet
23. 6.93 meters
25. 22.20 meters
27. 21.82 meters
29. 40 kilometers
31. 6 ft.

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1.2: Oblique Triangles

Law of Sines

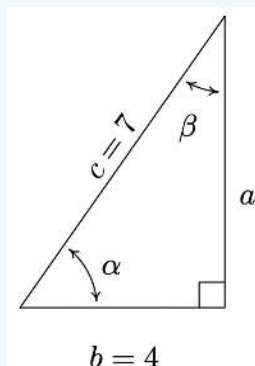
Trigonometry literally means ‘measuring triangles’ and with section 1 under our belts, we are more than prepared to do just that. The main goal of this section and the next is to develop theorems which allow us to ‘solve’ triangles – that is, find the length of each side of a triangle and the measure of each of its angles. We’ve had some experience solving right triangles. The following example reviews what we know.

✓ Example 1.2.1

Given a right triangle with a hypotenuse of length 7 units and one leg of length 4 units, find the length of the remaining side and the measures of the remaining angles. Express the angles in decimal degrees, rounded to the nearest hundredth of a degree.

Solution

For definitiveness, we label the triangle below.



To find the length of the missing side a , we use the Pythagorean Theorem to get $a^2 + 4^2 = 7^2$ which then yields $a = \sqrt{33}$ units. Now that all three sides of the triangle are known, there are several ways we can find α using the inverse trigonometric functions. To decrease the chances of propagating error, however, we stick to using the data given to us in the problem. In this case, the lengths 4 and 7 were given, so we want to relate these to α . According to Theorem 10.4, $\cos(\alpha) = \frac{4}{7}$. Since α is an acute angle, $\alpha = \arccos\left(\frac{4}{7}\right)$ radians. Converting to degrees, we find $\alpha \approx 55.15^\circ$. Now that we have the measure of angle α , we could find the measure of angle β using the fact that α and β are complements so $\alpha + \beta = 90^\circ$. Once again, we opt to use the data given to us in the problem. According to Theorem 10.4, we have that $\sin(\beta) = \frac{4}{7}$ so $\beta = \arcsin\left(\frac{4}{7}\right)$ radians and we have $\beta \approx 34.85^\circ$.

A few remarks about Example 1.2.1 are in order. First, we adhere to the convention that a lower case Greek letter denotes an angle¹ and the corresponding lowercase English letter represents the side² opposite that angle. Thus, a is the side opposite α , b is the side opposite β and c is the side opposite γ . Taken together, the pairs (α, a) , (β, b) and (γ, c) are called angle-side opposite pairs. Second, as mentioned earlier, we will strive to solve for quantities using the original data given in the problem whenever possible. While this is not always the easiest or fastest way to proceed, it minimizes the chances of propagated error.³ Third, since many of the applications which require solving triangles ‘in the wild’ rely on degree measure, we shall adopt this convention for the time being.⁴ The Pythagorean Theorem along with Theorems 10.4 and 10.10 allow us to easily handle any given right triangle problem, but what if the triangle isn’t a right triangle? In certain cases, we can use the **Law of Sines** to help.

Theorem 1.2. The Law of Sines

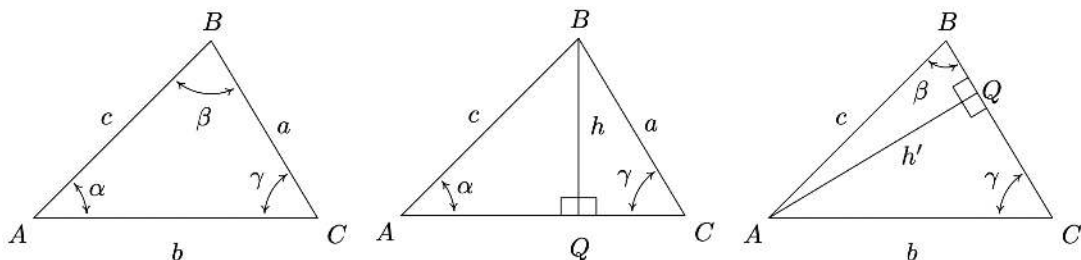
Given a triangle with angle-side opposite pairs (α, a) , (β, b) and (γ, c) , the following ratios hold

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

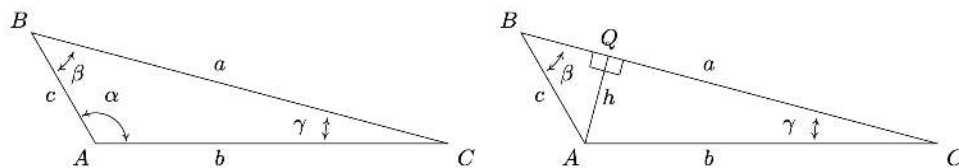
or, equivalently,

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

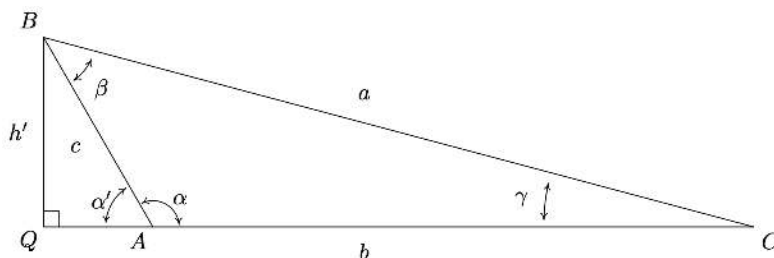
The proof of the Law of Sines can be broken into three cases. For our first case, consider the triangle $\triangle ABC$ below, all of whose angles are acute, with angle-side opposite pairs (α, a) , (β, b) and (γ, c) . If we drop an altitude from vertex B , we divide the triangle into two right triangles: $\triangle ABQ$ and $\triangle BCQ$. If we call the length of the altitude h (for height), we get from Theorem 10.4 that $\sin(\alpha) = \frac{h}{c}$ and $\sin(\gamma) = \frac{h}{a}$ so that $h = c \sin(\alpha) = a \sin(\gamma)$. After some rearrangement of the last equation, we get $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$. If we drop an altitude from vertex A , we can proceed as above using the triangles $\triangle ABQ$ and $\triangle ACQ$ to get $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$, completing the proof for this case.



For our next case consider the triangle $\triangle ABC$ below with obtuse angle α . Extending an altitude from vertex A gives two right triangles, as in the previous case: $\triangle ABQ$ and $\triangle ACQ$. Proceeding as before, we get $h = b \sin(\gamma)$ and $h = c \sin(\beta)$ so that $\frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$.



Dropping an altitude from vertex B also generates two right triangles, $\triangle ABQ$ and $\triangle BCQ$. We know that $\sin(\alpha') = \frac{h'}{c}$ so that $h' = c \sin(\alpha')$. Since $\alpha' = 180^\circ - \alpha$, $\sin(\alpha') = \sin(\alpha)$, so in fact, we have $h' = c \sin(\alpha)$. Proceeding to $\triangle BCQ$, we get $\sin(\gamma) = \frac{h'}{a}$ so $h' = a \sin(\gamma)$. Putting this together with the previous equation, we get $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$, and we are finished with this case.



The remaining case is when $\triangle ABC$ is a right triangle. In this case, the Law of Sines reduces to the formulas given in Theorem 10.4 and is left to the reader. In order to use the Law of Sines to solve a triangle, we need at least one angle-side opposite pair. The next example showcases some of the power, and the pitfalls, of the Law of Sines.

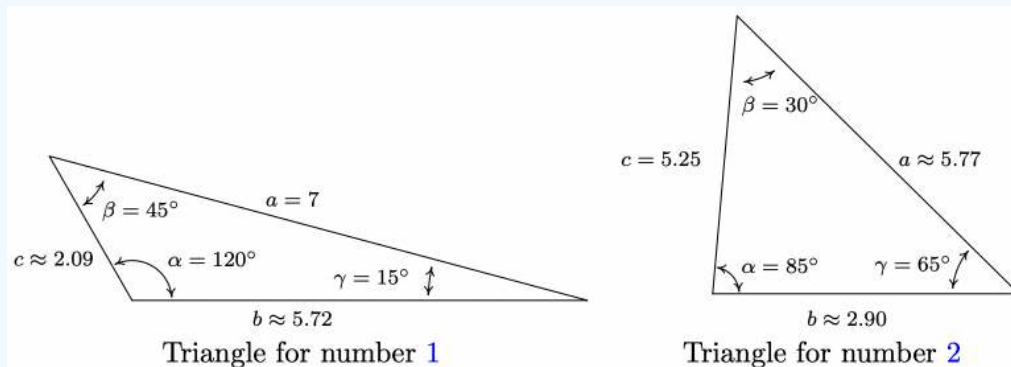
✓ Example 1.2.2

Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

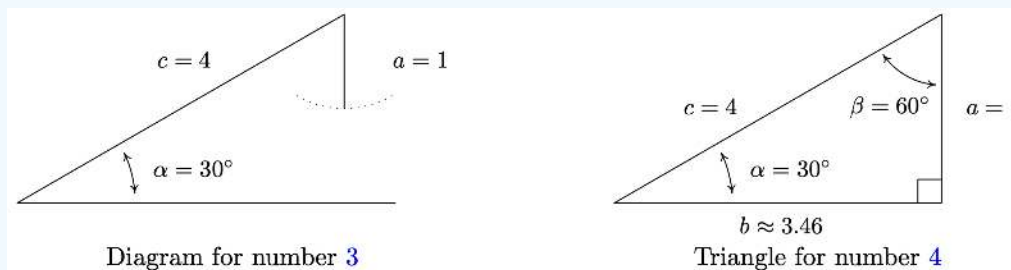
1. $\alpha = 120^\circ$, $a = 7$ units, $\beta = 45^\circ$
2. $\alpha = 85^\circ$, $\beta = 30^\circ$, $c = 5.25$ units
3. $\alpha = 30^\circ$, $a = 1$ units, $c = 4$ units
4. $\alpha = 30^\circ$, $a = 2$ units, $c = 4$ units
5. $\alpha = 30^\circ$, $a = 3$ units, $c = 4$ units
6. $\alpha = 30^\circ$, $a = 4$ units, $c = 4$ units

Solution

- Knowing an angle-side opposite pair, namely α and a , we may proceed in using the Law of Sines. Since $\beta = 45^\circ$, we use $\frac{b}{\sin(45^\circ)} = \frac{7}{\sin(120^\circ)}$ so $b = \frac{7 \sin(45^\circ)}{\sin(120^\circ)} = \frac{7\sqrt{6}}{3} \approx 5.72$ units. Now that we have two angle-side pairs, it is time to find the third. To find γ , we use the fact that the sum of the measures of the angles in a triangle is 180° . Hence, $\gamma = 180^\circ - 120^\circ - 45^\circ = 15^\circ$. To find c , we have no choice but to use the derived value $\gamma = 15^\circ$, yet we can minimize the propagation of error here by using the given angle-side opposite pair (α, a) . The Law of Sines gives us $\frac{c}{\sin(15^\circ)} = \frac{7}{\sin(120^\circ)}$ so that $c = \frac{7 \sin(15^\circ)}{\sin(120^\circ)} \approx 2.09$ units.⁵
- In this example, we are not immediately given an angle-side opposite pair, but as we have the measures of α and β , we can solve for γ since $\gamma = 180^\circ - 85^\circ - 30^\circ = 65^\circ$. As in the previous example, we are forced to use a derived value in our computations since the only angle-side pair available is (γ, c) . The Law of Sines gives $\frac{a}{\sin(85^\circ)} = \frac{5.25}{\sin(65^\circ)}$. After the usual rearrangement, we get $a = \frac{5.25 \sin(85^\circ)}{\sin(65^\circ)} \approx 5.77$ units. To find b we use the angle-side pair (γ, c) which yields $\frac{b}{\sin(30^\circ)} = \frac{5.25}{\sin(65^\circ)}$ hence $b = \frac{5.25 \sin(30^\circ)}{\sin(65^\circ)} \approx 2.90$ units.

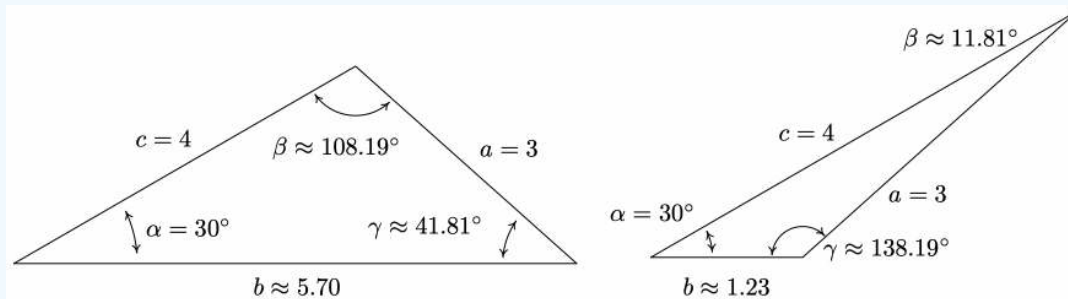


- Since we are given (α, a) and c , we use the Law of Sines to find the measure of γ . We start with $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{1}$ and get $\sin(\gamma) = 4 \sin(30^\circ) = 2$. Since the range of the sine function is $[-1, 1]$, there is no real number with $\sin(\gamma) = 2$. Geometrically, we see that side a is just too short to make a triangle. The next three examples keep the same values for the measure of α and the length of c while varying the length of a . We will discuss this case in more detail after we see what happens in those examples.
- In this case, we have the measure of $\alpha = 30^\circ$, $a = 2$ and $c = 4$. Using the Law of Sines, we get $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{2}$ so $\sin(\gamma) = 2 \sin(30^\circ) = 1$. Now γ is an angle in a triangle which also contains $\alpha = 30^\circ$. This means that γ must measure between 0° and 150° in order to fit inside the triangle with α . The only angle that satisfies this requirement and has $\sin(\gamma) = 1$ is $\gamma = 90^\circ$. In other words, we have a right triangle. We find the measure of β to be $\beta = 180^\circ - 30^\circ - 90^\circ = 60^\circ$ and then determine b using the Law of Sines. We find $b = \frac{2 \sin(60^\circ)}{\sin(30^\circ)} = 2\sqrt{3} \approx 3.46$ units. In this case, the side a is precisely long enough to form a unique right triangle.

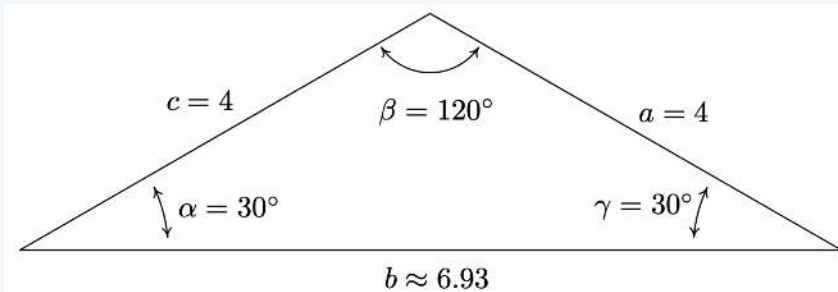


- Proceeding as we have in the previous two examples, we use the Law of Sines to find γ . In this case, we have $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{3}$ or $\sin(\gamma) = \frac{4 \sin(30^\circ)}{3} = \frac{2}{3}$. Since γ lies in a triangle with $\alpha = 30^\circ$, we must have that $0^\circ < \gamma < 150^\circ$. There are two angles γ that fall in this range and have $\sin(\gamma) = \frac{2}{3}$: $\gamma = \arcsin(\frac{2}{3})$ radians $\approx 41.81^\circ$ and $\gamma = \pi - \arcsin(\frac{2}{3})$ radians $\approx 138.19^\circ$. At this point, we pause to see if it makes sense that we actually have two viable cases to consider. As we have discussed, both candidates for γ are 'compatible' with the given angle-side pair $(\alpha, a) = (30^\circ, 3)$ in that both choices for γ can fit in a triangle with α and both have a sine of $\frac{2}{3}$. The only other given piece of information is that $c = 4$ units. Since $c > a$, it must be true that γ , which is opposite c , has greater measure than α which is opposite a . In both case, $\gamma > \alpha$, which is opposite c , has greater measure than α which is opposite a . In both case, $\gamma > \alpha$, so both candidates for γ are compatible with

this last piece of given information as well. Thus have two triangles on our hands. In the case $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians $\approx 41.81^\circ$, we find⁶ $\beta \approx 180^\circ - 30^\circ - 41.81^\circ = 108.19^\circ$. Using the Law of Sines with the angle-side opposite pair (α, a) and β , we find $b \approx \frac{3 \sin(108.19^\circ)}{\sin(30^\circ)} \approx 5.70$ units. In the case $\gamma = \pi - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 138.19^\circ$, we repeat the exact same steps and find $\beta \approx 11.81^\circ$ and $b \approx 1.23$ units.⁷ Both triangles are drawn below.



6. For this last problem, we repeat the usual Law of Sines routine to find $\frac{\sin(\gamma)}{4} = \frac{\sin(30^\circ)}{4}$ so that $\sin(\gamma) = \frac{1}{2}$. Since γ must inhabit a triangle $\alpha = 30^\circ$, we must have $0^\circ < \gamma < 150^\circ$. Since the measure of γ must be strictly less than 150° , there is just one angle which satisfies both required conditions, namely $\gamma = 30^\circ$. So $\beta = 180^\circ - 30^\circ - 30^\circ = 120^\circ$ and, using the Law of Sines one last time, $b = \frac{4 \sin(120^\circ)}{\sin(30^\circ)} = 4\sqrt{3} \approx 6.93$ units.



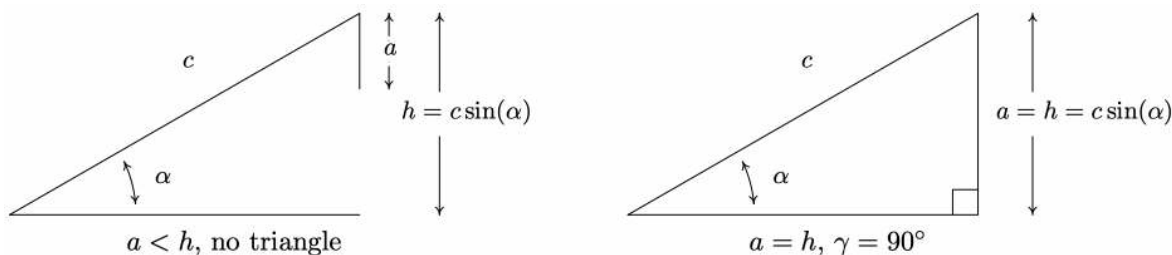
Some remarks about Example 1.2.2 are in order. We first note that if we are given the measures of two of the angles in a triangle, say α and β , the measure of the third angle γ uniquely determined using the equation $\gamma = 180^\circ - \alpha - \beta$. Knowing the measures of all three angles of a triangle completely determines its shape. If in addition we are given the length of one of the sides of the triangle, we can then use the Law of Sines to find the lengths of the remaining two sides to determine the size of the triangle. Such is the case in numbers 1 and 2 above. In number 1, the given side is adjacent to just one of the angles – this is called the ‘Angle-Angle-Side’ (AAS) case.⁸ In number 2, the given side is adjacent to both angles which means we are in the so-called ‘Angle-Side-Angle’ (ASA) case. If, on the other hand, we are given the measure of just one of the angles in the triangle along with the length of two sides, only one of which is adjacent to the given angle, we are in the ‘Angle-Side-Side’ (ASS) case.⁹ In number 3, the length of the one given side a was too short to even form a triangle; in number 4, the length of a was just long enough to form a right triangle; in 5, a was long enough, but not too long, so that two triangles were possible; and in number 6, side a was long enough to form a triangle but too long to swing back and form two. These four cases exemplify all of the possibilities in the Angle-Side-Side case which are summarized in the following theorem.

Theorem 1.3

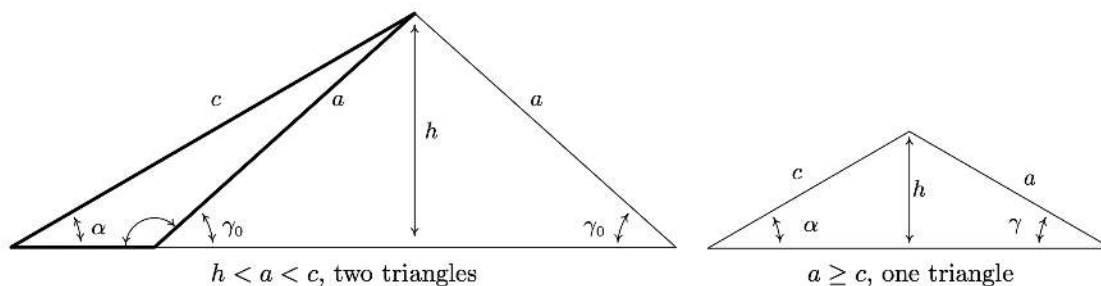
Suppose (α, a) and (γ, c) are intended to be angle-side pairs in a triangle where α , a and c are given. Let $h = c \sin(\alpha)$.

- If $a < h$, then no triangle exists which satisfies the given criteria.
- If $a = h$, then $\gamma = 90^\circ$ so exactly one (right) triangle exists which satisfies the criteria.
- If $h < a < c$, then two distinct triangles exist which satisfy the given criteria.
- If $a \geq c$, then γ is acute and exactly one triangle exists which satisfies the given criteria

Theorem 1.3 is proved on a case-by-case basis. If $a < h$, then $a < c \sin(\alpha)$. If a triangle were to exist, the Law of Sines would have $\frac{\sin(\gamma)}{c} = \frac{\sin(\alpha)}{a}$ so that $\sin(\gamma) = \frac{c \sin(\alpha)}{a} > \frac{a}{a} = 1$, which is impossible. In the figure below, we see geometrically why this is the case.



Simply put, if $a < h$ the side a is too short to connect to form a triangle. This means if $a \geq h$, we are always guaranteed to have at least one triangle, and the remaining parts of the theorem tell us what kind and how many triangles to expect in each case. If $a = h$, then $a = c \sin(\alpha)$ and the Law of Sines gives $\frac{\sin(\alpha)}{a} = \frac{\sin(\gamma)}{c}$ so that $\sin(\gamma) = \frac{c \sin(\alpha)}{a} = \frac{a}{a} = 1$. Here, $\gamma = 90^\circ$ as required. Moving along, now suppose $h < a < c$. As before, the Law of Sines¹⁰ gives $\sin(\gamma) = \frac{c \sin(\alpha)}{a}$. Since $h < a$, $c \sin(\alpha) < a$ or $\frac{c \sin(\alpha)}{a} < 1$ which means there are two solutions to $\sin(\gamma) = \frac{c \sin(\alpha)}{a}$: an acute angle which we'll call γ_0 , and its supplement, $180^\circ - \gamma_0$. We need to argue that each of these angles 'fit' into a triangle with α . Since (α, a) and (γ_0, c) are angle-side opposite pairs, the assumption $c > a$ in this case gives us $\gamma_0 > \alpha$. Since γ_0 is acute, we must have that α is acute as well. This means one triangle can contain both α and γ_0 , giving us one of the triangles promised in the theorem. If we manipulate the inequality $\gamma_0 > \alpha$ a bit, we have $180^\circ - \gamma_0 < 180^\circ - \alpha$ which gives $(180^\circ - \gamma_0) + \alpha < 180^\circ$. This proves a triangle can contain both of the angles α and $(180^\circ - \gamma_0)$, giving us the second triangle predicted in the theorem. To prove the last case in the theorem, we assume $a \geq c$. Then $\alpha \geq \gamma$, which forces γ to be an acute angle. Hence, we get only one triangle in this case, completing the proof.



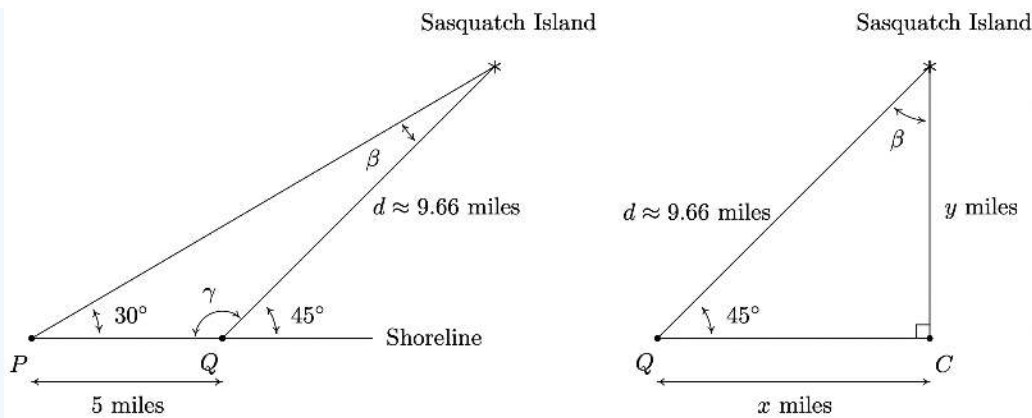
✓ Example 1.2.3

Sasquatch Island lies off the coast of Ippizuti Lake. Two sightings, taken 5 miles apart, are made to the island. The angle between the shore and the island at the first observation point is 30° and at the second point the angle is 45° . Assuming a straight coastline, find the distance from the second observation point to the island. What point on the shore is closest to the island? How far is the island from this point?

Solution

We sketch the problem below with the first observation point labeled as P and the second as Q . In order to use the Law of Sines to find the distance d from Q to the island, we first need to find the measure of β which is the angle opposite the side of length 5 miles. To that end, we note that the angles γ and 45° are supplemental, so that $\gamma = 180^\circ - 45^\circ = 135^\circ$. We can now find $\beta = 180^\circ - 30^\circ - \gamma = 180^\circ - 30^\circ - 135^\circ = 15^\circ$. By the Law of Sines, we have $\frac{d}{\sin(30^\circ)} = \frac{5}{\sin(15^\circ)}$ which gives $d = \frac{5 \sin(30^\circ)}{\sin(15^\circ)} \approx 9.66$ miles. Next, to find the point on the coast closest to the island, which we've labeled as C , we need to find the perpendicular distance from the island to the coast.¹¹

Let x denote the distance from the second observation point Q to the point C and let y denote the distance from C to the island. Using Theorem 10.4, we get $\sin(45^\circ) = \frac{y}{d}$. After some rearranging, we find $y = d \sin(45^\circ) \approx 9.66 \left(\frac{\sqrt{2}}{2} \right) \approx 6.83$ miles. Hence, the island is approximately 6.83 miles from the coast. To find the distance from Q to C , we note that $\beta = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ so by symmetry,¹² we get $x = y \approx 6.83$ miles. Hence, the point on the shore closest to the island is approximately 6.83 miles down the coast from the second observation point.



We close this section with a new formula to compute the area enclosed by a triangle. Its proof uses the same cases and diagrams as the proof of the Law of Sines and is left as an exercise.

Theorem 1.4

Suppose (α, a) , (β, b) and (γ, c) are the angle-side opposite pairs of a triangle. Then the area A enclosed by the triangle is given by

$$A = \frac{1}{2}bc \sin(\alpha) = \frac{1}{2}ac \sin(\beta) = \frac{1}{2}ab \sin(\gamma)$$

Example 1.2.4

Find the area of the triangle in Example 1.2.2 number 1.

Solution

From our work in Example 1.2.2 number 1, we have all three angles and all three sides to work with. However, to minimize propagated error, we choose $A = \frac{1}{2}ac \sin(\beta)$ from Theorem 1.4 because it uses the most pieces of given information. We are given $a = 7$ and $\beta = 45^\circ$, and we calculated $c = \frac{7 \sin(15^\circ)}{\sin(120^\circ)}$. Using these values, we find $A = \frac{1}{2}(7) \left(\frac{7 \sin(15^\circ)}{\sin(120^\circ)} \right) \sin(45^\circ) \approx 5.18$ square units. The reader is encouraged to check this answer against the results obtained using the other formulas in Theorem 1.4.

1.2.1 Exercises

In Exercises 1 - 20, solve for the remaining side(s) and angle(s) if possible. As in the text, (α, a) , (β, b) and (γ, c) are angle-side opposite pairs.

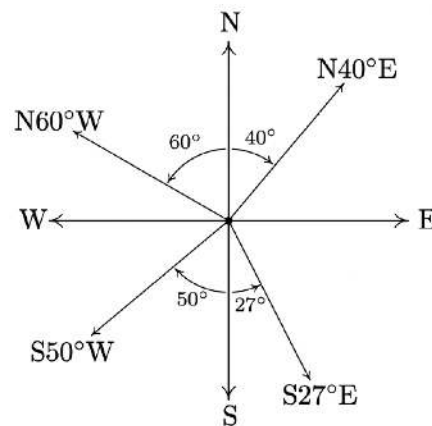
1. $\alpha = 13^\circ, \beta = 17^\circ, a = 5$
2. $\alpha = 73.2^\circ, \beta = 54.1^\circ, a = 117$
3. $\alpha = 95^\circ, \beta = 85^\circ, a = 33.33$
4. $\alpha = 95^\circ, \beta = 62^\circ, a = 33.33$
5. $\alpha = 117^\circ, a = 35, b = 42$
6. $\alpha = 117^\circ, a = 45, b = 42$
7. $\alpha = 68.7^\circ, a = 88, b = 92$
8. $\alpha = 42^\circ, a = 17, b = 23.5$
9. $\alpha = 68.7^\circ, a = 70, b = 90$
10. $\alpha = 30^\circ, a = 7, b = 14$
11. $\alpha = 42^\circ, a = 39, b = 23.5$
12. $\gamma = 53^\circ, \alpha = 53^\circ, c = 28.01$
13. $\alpha = 6^\circ, a = 57, b = 100$
14. $\gamma = 74.6^\circ, c = 3, a = 3.05$
15. $\beta = 102^\circ, b = 16.75, c = 13$

16. $\beta = 102^\circ, b = 16.75, c = 18$
17. $\beta = 102^\circ, \gamma = 35^\circ, b = 16.75$
18. $\beta = 29.13^\circ, \gamma = 83.95^\circ, b = 314.15$
19. $\gamma = 120^\circ, \beta = 61^\circ, c = 4$
20. $\alpha = 50^\circ, a = 25, b = 12.5$
21. Find the area of the triangles given in Exercises 1, 12 and 20 above.

(Another Classic Application: Grade of a Road) The grade of a road is much like the pitch of a roof (See [Example 10.6.6](#)) in that it expresses the ratio of rise/run. In the case of a road, this ratio is always positive because it is measured going uphill and it is usually given as a percentage. For example, a road which rises 7 feet for every 100 feet of (horizontal) forward progress is said to have a 7% grade. However, if we want to apply any Trigonometry to a story problem involving roads going uphill or downhill, we need to view the grade as an angle with respect to the horizontal. In Exercises 22 - 24, we first have you change road grades into angles and then use the Law of Sines in an application.

22. Using a right triangle with a horizontal leg of length 100 and vertical leg with length 7, show that a 7% grade means that the road (hypotenuse) makes about a 4° angle with the horizontal. (It will not be exactly 4° , but it's pretty close.)
23. What grade is given by a 9.65° angle made by the road and the horizontal?¹³
24. Along a long, straight stretch of mountain road with a 7% grade, you see a tall tree standing perfectly plumb alongside the road.¹⁴ From a point 500 feet downhill from the tree, the angle of inclination from the road to the top of the tree is 6° . Use the Law of Sines to find the height of the tree. (Hint: First show that the tree makes a 94° angle with the road.)

(Another Classic Application: Bearings) In the next several exercises we introduce and work with the navigation tool known as bearings. Simply put, a bearing is the direction you are heading according to a compass. The classic nomenclature for bearings, however, is not given as an angle in standard position, so we must first understand the notation. A bearing is given as an acute angle of rotation (to the east or to the west) away from the north-south (up and down) line of a compass rose. For example, $N40^\circ E$ (read "40° east of north") is a bearing which is rotated clockwise 40° from due north. If we imagine standing at the origin in the Cartesian Plane, this bearing would have us heading into Quadrant I along the terminal side of $\theta = 50^\circ$. Similarly, $S50^\circ W$ would point into Quadrant III along the terminal side of $\theta = 220^\circ$ because we started out pointing due south (along $\theta = 270^\circ$) and rotated clockwise 50° back to 220° . Counter-clockwise rotations would be found in the bearings $N60^\circ W$ (which is on the terminal side of $\theta = 150^\circ$) and $S27^\circ E$ (which lies along the terminal side of $\theta = 297^\circ$). These four bearings are drawn in the plane below.



The cardinal directions north, south, east and west are usually not given as bearings in the fashion described above, but rather, one just refers to them as 'due north', 'due south', 'due east' and 'due west', respectively, and it is assumed that you know which quadrantal angle goes with each cardinal direction. (Hint: Look at the diagram above.)

25. Find the angle θ in standard position with $0^\circ \leq \theta < 360^\circ$ which corresponds to each of the bearings given below.
 - a. due west
 - b. $S83^\circ E$
 - c. $N5.5^\circ E$
 - d. due south

- e. $N31.25^\circ W$
- f. $S72^\circ 41' 12'' W^{15}$
- g. $N45^\circ E$
- h. $S45^\circ W$

26. The Colonel spots a campfire at a bearing of $N42^\circ E$ from his current position. Sarge, who is positioned 3000 feet due east of the Colonel, reckons the bearing to the fire to be $N20^\circ W$ from his current position. Determine the distance from the campfire to each man, rounded to the nearest foot.
27. A hiker starts walking due west from Sasquatch Point and gets to the Chupacabra Trailhead before she realizes that she hasn't reset her pedometer. From the Chupacabra Trailhead she hikes for 5 miles along a bearing of $N53^\circ W$ which brings her to the Muffin Ridge Observatory. From there, she knows a bearing of $S65^\circ E$ will take her straight back to Sasquatch Point. How far will she have to walk to get from the Muffin Ridge Observatory to Sasquatch Point? What is the distance between Sasquatch Point and the Chupacabra Trailhead?
28. The captain of the SS Bigfoot sees a signal flare at a bearing of $N15^\circ E$ from her current location. From his position, the captain of the HMS Sasquatch finds the signal flare to be at a bearing of $N75^\circ W$. If the SS Bigfoot is 5 miles from the HMS Sasquatch and the bearing from the SS Bigfoot to the HMS Sasquatch is $N50^\circ E$, find the distances from the flare to each vessel, rounded to the nearest tenth of a mile.
29. Carl spies a potential Sasquatch nest at a bearing of $N10^\circ E$ and radios Jeff, who is at a bearing of $N50^\circ E$ from Carl's position. From Jeff's position, the nest is at a bearing of $S70^\circ W$. If Jeff and Carl are 500 feet apart, how far is Jeff from the Sasquatch nest? Round your answer to the nearest foot.
30. A hiker determines the bearing to a lodge from her current position is $S40^\circ W$. She proceeds to hike 2 miles at a bearing of $S20^\circ E$ at which point she determines the bearing to the lodge is $S75^\circ W$. How far is she from the lodge at this point? Round your answer to the nearest hundredth of a mile.
31. A watchtower spots a ship off shore at a bearing of $N70^\circ E$. A second tower, which is 50 miles from the first at a bearing of $S80^\circ E$ from the first tower, determines the bearing to the ship to be $N25^\circ W$. How far is the boat from the second tower? Round your answer to the nearest tenth of a mile.
32. Skippy and Sally decide to hunt UFOs. One night, they position themselves 2 miles apart on an abandoned stretch of desert runway. An hour into their investigation, Skippy spies a UFO hovering over a spot on the runway directly between him and Sally. He records the angle of inclination from the ground to the craft to be 75° and radios Sally immediately to find the angle of inclination from her position to the craft is 50° . How high off the ground is the UFO at this point? Round your answer to the nearest foot. (Recall: 1 mile is 5280 feet.)
33. The angle of depression from an observer in an apartment complex to a gargoyle on the building next door is 55° . From a point five stories below the original observer, the angle of inclination to the gargoyle is 20° . Find the distance from each observer to the gargoyle and the distance from the gargoyle to the apartment complex. Round your answers to the nearest foot. (Use the rule of thumb that one story of a building is 9 feet.)
34. Prove that the Law of Sines holds when $\triangle ABC$ is a right triangle.
35. Discuss with your classmates why knowing only the three angles of a triangle is not enough to determine any of the sides.
36. Discuss with your classmates why the Law of Sines cannot be used to find the angles in the triangle when only the three sides are given. Also discuss what happens if only two sides and the angle between them are given. (Said another way, explain why the Law of Sines cannot be used in the SSS and SAS cases.)
37. Given $\alpha = 30^\circ$ and $b = 10$, choose four different values for a so that
 - a. the information yields no triangle
 - b. the information yields exactly one right triangle
 - c. the information yields two distinct triangles
 - d. the information yields exactly one obtuse triangle

Explain why you cannot choose a in such a way as to have $\alpha = 30^\circ$, $b = 10$ and your choice of a yield only one triangle where that unique triangle has three acute angles.

38. Use the cases and diagrams in the proof of the Law of Sines (Theorem 1.2) to prove the area formulas given in Theorem 1.4. Why do those formulas yield square units when four quantities are being multiplied together?

1.2.2 Answers

1. $\alpha = 13^\circ$ $\beta = 17^\circ$ $\gamma = 150^\circ$
 $a = 5$ $b \approx 6.50$ $c \approx 11.11$
2. $\alpha = 73.2^\circ$ $\beta = 54.1^\circ$ $\gamma = 52.7^\circ$
 $a = 117$ $b \approx 99.00$ $c \approx 97.22$
3. Information does not produce a triangle
4. $\alpha = 95^\circ$ $\beta = 62^\circ$ $\gamma = 23^\circ$
 $a = 33.33$ $b \approx 29.54$ $c \approx 13.07$
5. Information does not produce a triangle
6. $\alpha = 117^\circ$ $\beta \approx 56.3^\circ$ $\gamma \approx 6.7^\circ$
 $a = 45$ $b = 42$ $c \approx 5.89$
7. $\alpha = 68.7^\circ$ $\beta \approx 76.9^\circ$ $\gamma \approx 34.4^\circ$
 $a = 88$ $b = 92$ $c \approx 53.36$
8. $\alpha = 42^\circ$ $\beta \approx 67.66^\circ$ $\gamma \approx 70.34^\circ$
 $a = 17$ $b = 23.5$ $c \approx 23.93$
9. Information does not produce a triangle
10. $\alpha = 30^\circ$ $\beta = 90^\circ$ $\gamma = 60^\circ$
 $a = 7$ $b = 14$ $c = 7\sqrt{3}$
11. $\alpha = 42^\circ$ $\beta \approx 23.78^\circ$ $\gamma \approx 114.22^\circ$
 $a = 39$ $b = 23.5$ $c \approx 53.15$
12. $\alpha = 53^\circ$ $\beta = 74^\circ$ $\gamma = 53^\circ$
 $a = 28.01$ $b \approx 33.71$ $c = 28.01$
13. $\alpha = 6^\circ$ $\beta \approx 169.43^\circ$ $\gamma \approx 4.57^\circ$
 $a = 57$ $b = 100$ $c \approx 43.45$
14. $\alpha \approx 78.59^\circ$ $\beta \approx 26.81^\circ$ $\gamma = 74.6^\circ$
 $a = 3.05$ $b \approx 1.40$ $c = 3$
15. $\alpha \approx 28.61^\circ$ $\beta = 102^\circ$ $\gamma \approx 49.39^\circ$
 $a \approx 8.20$ $b = 16.75$ $c = 13$
16. Information does not produce a triangle
17. $\alpha = 43^\circ$ $\beta = 102^\circ$ $\gamma = 35^\circ$
 $a \approx 11.68$ $b = 16.75$ $c \approx 9.82$
18. $\alpha = 66.92^\circ$ $\beta = 29.13^\circ$ $\gamma = 83.95^\circ$
 $a \approx 593.69$ $b = 314.15$ $c \approx 641.75$
19. Information does not produce a triangle
20. $\alpha = 50^\circ$ $\beta \approx 22.52^\circ$ $\gamma \approx 107.48^\circ$
 $a = 25$ $b = 12.5$ $c \approx 31.13$
21. The area of the triangle from Exercise 1 is about 8.1 square units.

The area of the triangle from Exercise 12 is about 377.1 square units.

The area of the triangle from Exercise 20 is about 149 square units.

22. $\arctan\left(\frac{7}{100}\right) \approx 0.699$ radians, which is equivalent to $\approx 4.004^\circ$

23. About 17%

24. About 53 feet

25. a. $\theta = 180^\circ$
b. $\theta = 353^\circ$
c. $\theta = 84.5^\circ$
d. $\theta = 270^\circ$
e. $\theta = 121.25^\circ$
f. $\theta = 197^\circ 18' 48''$
g. $\theta = 45^\circ$
h. $\theta = 225^\circ$

26. The Colonel is about 3193 feet from the campfire.

Sarge is about 2525 feet to the campfire.

27. The distance from the Muffin Ridge Observatory to Sasquach Point is about 7.12 miles.

The distance from Sasquach Point to the Chupacabra Trailhead is about 2.46 miles.

28. The SS Bigfoot is about 4.1 miles from the flare.

The HMS Sasquatch is about 2.9 miles from the flare.

29. Jeff is about 371 feet from the nest.

30. She is about 3.02 miles from the lodge

31. The boat is about 25.1 miles from the second tower.

32. The UFO is hovering about 9539 feet above the ground.

33. The gargoye is about 44 feet from the observer on the upper floor.

The gargoye is about 27 feet from the observer on the lower floor.

The gargoye is about 25 feet from the other building.

Law of Cosines

We developed the Law of Sines (Theorem 1.2) to enable us to solve triangles in the ‘Angle-Angle-Side’ (AAS), the ‘Angle-Side-Angle’ (ASA) and the ambiguous ‘Angle-Side-Side’ (ASS) cases. In this section, we develop the Law of Cosines which handles solving triangles in the ‘Side-Angle-Side’ (SAS) and ‘Side-Side-Side’ (SSS) cases.¹⁶ We state and prove the theorem below.

Theorem 1.5. Law of Cosines

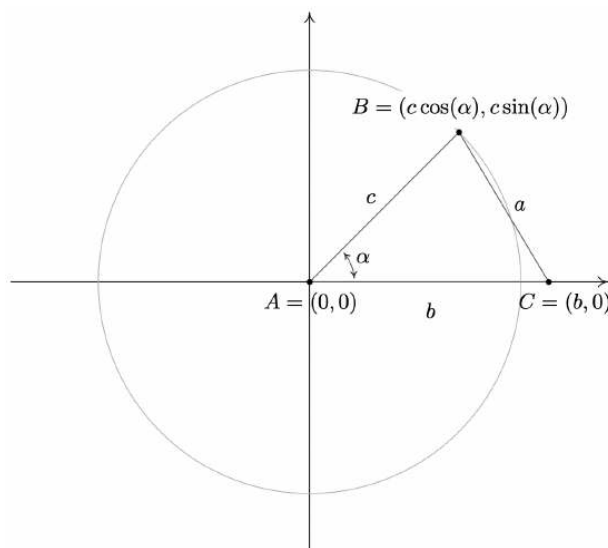
Given a triangle with angle-side opposite pairs (α, a) , (β, b) and (γ, c) , the following equations hold

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha) \quad b^2 = a^2 + c^2 - 2ac \cos(\beta) \quad c^2 = a^2 + b^2 - 2ab \cos(\gamma)$$

or, solving for the cosine in each equation, we have

$$\cos(\alpha) = \frac{b^2 + c^2 - a^2}{2bc} \quad \cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} \quad \cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$$

To prove the theorem, we consider a generic triangle with the vertex of angle α at the origin with side b positioned along the positive x -axis.



From this set-up, we immediately find that the coordinates of A and C are $A(0,0)$ and $C(b,0)$. From Theorem 10.3, we know that since the point $B(x,y)$ lies on a circle of radius c , the coordinates of B are $B(x,y) = B(c \cos(\alpha), c \sin(\alpha))$. (This would be true even if α were an obtuse or right angle so although we have drawn the case when α is acute, the following computations hold for any angle

α drawn in standard position where $0 < \alpha < 180^\circ$.) We note that the distance between the points B and C is none other than the length of side a . Using the distance formula, Equation 1.1, we get

$$\begin{aligned} a &= \sqrt{(c \cos(\alpha) - b)^2 + (c \sin(\alpha) - 0)^2} \\ a^2 &= \left(\sqrt{(c \cos(\alpha) - b)^2 + c^2 \sin^2(\alpha)} \right)^2 \\ a^2 &= (c \cos(\alpha) - b)^2 + c^2 \sin^2(\alpha) \\ a^2 &= c^2 \cos^2(\alpha) - 2bc \cos(\alpha) + b^2 + c^2 \sin^2(\alpha) \\ a^2 &= c^2 (\cos^2(\alpha) + \sin^2(\alpha)) + b^2 - 2bc \cos(\alpha) \\ a^2 &= c^2(1) + b^2 - 2bc \cos(\alpha) && \text{Since } \cos^2(\alpha) + \sin^2(\alpha) = 1 \\ a^2 &= c^2 + b^2 - 2bc \cos(\alpha) \end{aligned}$$

The remaining formulas given in Theorem 1.5 can be shown by simply reorienting the triangle to place a different vertex at the origin. We leave these details to the reader. What's important about a and α in the above proof is that (α, a) is an angle-side opposite pair and b and c are the sides adjacent to α – the same can be said of any other angle-side opposite pair in the triangle. Notice that the proof of the Law of Cosines relies on the distance formula which has its roots in the Pythagorean Theorem. That being said, the Law of Cosines can be thought of as a generalization of the Pythagorean Theorem. If we have a triangle in which $\gamma = 90^\circ$, then $\cos(\gamma) = \cos(90^\circ) = 0$ so we get the familiar relationship $c^2 = a^2 + b^2$. What this means is that in the larger mathematical sense, the Law of Cosines and the Pythagorean Theorem amount to pretty much the same thing.¹⁷

✓ Example 1.2.5

Solve the following triangles. Give exact answers and decimal approximations (rounded to hundredths) and sketch the triangle.

- $\beta = 50^\circ$, $a = 7$ units, $c = 2$ units
- $a = 4$ units, $b = 7$ units, $c = 5$ units

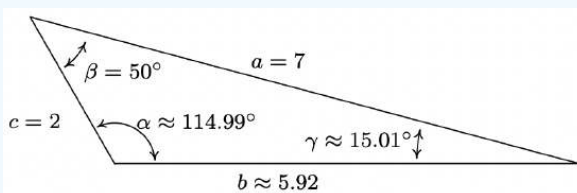
Solution

$$1. \cos(\alpha) = \frac{2 - 7 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}$$

Since α is an angle in a triangle, we know the radian measure of α must lie between 0 and π radians. This matches the range of the arccosine function, so we have

$$\alpha = \arccos\left(\frac{2 - 7 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}\right) \text{ radians} \approx 114.99^\circ$$

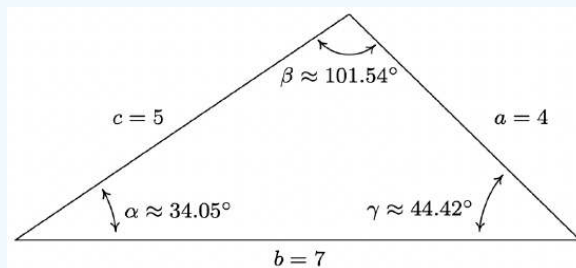
At this point, we could find γ using $\gamma = 180^\circ - \alpha - \beta \approx 180^\circ - 114.99^\circ - 50^\circ = 15.01^\circ$, that is if we trust our approximation for α . To minimize propagation of error, however, we could use the Law of Cosines again,⁴ in this case using $\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$. Plugging in $a = 7$, $b = \sqrt{53 - 28 \cos(50^\circ)}$ and $c = 2$, we get $\gamma = \arccos\left(\frac{7 - 2 \cos(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}\right)$ radians $\approx 15.01^\circ$. We sketch the triangle below.



As we mentioned earlier, once we've determined b it is possible to use the Law of Sines to find the remaining angles. Here, however, we must proceed with caution as we are in the ambiguous (ASS) case. It is advisable to first find the smallest of the unknown angles, since we are guaranteed it will be acute.⁵ In this case, we would find γ since the side opposite γ is smaller than the side opposite the other unknown angle, α . Using the angle-side opposite pair (β, b) , we get $\frac{\sin(\gamma)}{2} = \frac{\sin(50^\circ)}{\sqrt{53 - 28 \cos(50^\circ)}}$. The usual calculations produces $\gamma \approx 15.01^\circ$ and $\alpha = 180^\circ - \beta - \gamma \approx 180^\circ - 50^\circ - 15.01^\circ = 114.99^\circ$.

- Since all three sides and no angles are given, we are forced to use the Law of Cosines. Following our discussion in the previous problem, we find β first, since it is opposite the longest side, b . We get $\cos(\beta) = \frac{a^2 + c^2 - b^2}{2ac} = -\frac{1}{5}$, so we get $\beta = \arccos\left(-\frac{1}{5}\right)$ radians $\approx 101.54^\circ$. As in the previous problem, now that we have obtained an angle-side opposite pair (β, b) ,

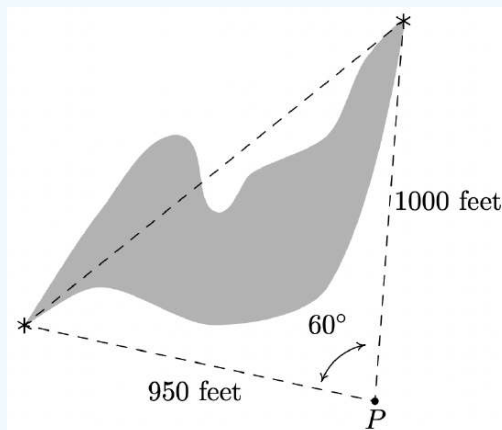
we could proceed using the Law of Sines. The Law of Cosines, however, offers us a rare opportunity to find the remaining angles using only the data given to us in the statement of the problem. Using this, we get $\gamma = \arccos\left(\frac{5}{7}\right)$ radians $\approx 44.42^\circ$ and $\alpha = \arccos\left(\frac{29}{35}\right)$ radians $\approx 34.05^\circ$.



We note that, depending on how many decimal places are carried through successive calculations, and depending on which approach is used to solve the problem, the approximate answers you obtain may differ slightly from those the authors obtain in the Examples and the Exercises. A great example of this is number 2 in Example 1.2.5, where the approximate values we record for the measures of the angles sum to 180.01° , which is geometrically impossible. Next, we have an application of the Law of Cosines

✓ Example 1.2.6

A researcher wishes to determine the width of a vernal pond as drawn below. From a point P , he finds the distance to the eastern-most point of the pond to be 950 feet, while the distance to the western-most point of the pond from P is 1000 feet. If the angle between the two lines of sight is 60° , find the width of the pond.



Solution

We are given the lengths of two sides and the measure of an included angle, so we may apply the Law of Cosines to find the length of the missing side opposite the given angle. Calling this length w (for width), we get $w^2 = 950^2 + 1000^2 - 2(950)(1000)\cos(60^\circ) = 952500$ from which we get $w = \sqrt{952500} \approx 976$ feet.

We used the proof of the Law of Sines to develop Theorem 1.4 as an alternate formula for the area enclosed by a triangle. In this section, we use the Law of Cosines to derive another such formula - Heron's Formula.

🔗 Theorem 1.6. Heron's Formula

Suppose a , b and c denote the lengths of the three sides of a triangle. Let s be the semiperimeter of the triangle, that is, let $s = \frac{1}{2}(a + b + c)$. Then the area A enclosed by the triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

We prove Theorem 1.6 using Theorem 1.4. Using the convention that the angle γ is opposite the side c , we have $A = \frac{1}{2}ab\sin(\gamma)$ from Theorem 1.4. In order to simplify computations, we start by manipulating the expression for A^2 .

$$\begin{aligned}
 A^2 &= \left(\frac{1}{2} ab \sin(\gamma) \right)^2 \\
 &= \frac{1}{4} a^2 b^2 \sin^2(\gamma) \\
 &= \frac{a^2 b^2}{4} (1 - \cos^2(\gamma)) \quad \text{since } \sin^2(\gamma) = 1 - \cos^2(\gamma).
 \end{aligned}$$

The Law of Cosines tells us $\cos(\gamma) = \frac{a^2 + b^2 - c^2}{2ab}$, so substituting this into our equation for A^2 gives

$$\begin{aligned}
 A^2 &= \frac{a^2 b^2}{4} (1 - \cos^2(\gamma)) \\
 &= \frac{a^2 b^2}{4} \left[1 - \left(\frac{a^2 + b^2 - c^2}{2ab} \right)^2 \right] \\
 &= \frac{a^2 b^2}{4} \left[1 - \frac{(a^2 + b^2 - c^2)^2}{4a^2 b^2} \right] \\
 &= \frac{a^2 b^2}{4} \left[\frac{4a^2 b^2 - (a^2 + b^2 - c^2)^2}{4a^2 b^2} \right] \\
 &= \frac{4a^2 b^2 - (a^2 + b^2 - c^2)^2}{16} \\
 &= \frac{(2ab)^2 - (a^2 + b^2 - c^2)^2}{16} \quad \text{difference of squares.} \\
 &= \frac{(2ab - [a^2 + b^2 - c^2]) (2ab + [a^2 + b^2 - c^2])}{16} \\
 &= \frac{(c^2 - a^2 + 2ab - b^2) (a^2 + 2ab + b^2 - c^2)}{16} \quad \text{os} \\
 A^2 &= \frac{(c^2 - [a^2 - 2ab + b^2]) ([a^2 + 2ab + b^2] - c^2)}{16} \\
 &= \frac{(c^2 - (a - b)^2) ((a + b)^2 - c^2)}{16} \quad \text{perfect square trinomials.} \\
 &= \frac{(c - (a - b))(c + (a - b))(a + b - c)(a + b + c)}{16} \quad \text{difference of squares.} \\
 &= \frac{(b + c - a)(a + c - b)(a + b - c)(a + b + c)}{16} \\
 &= \frac{(b + c - a)}{2} \cdot \frac{(a + c - b)}{2} \cdot \frac{(a + b - c)}{2} \cdot \frac{(a + b + c)}{2}
 \end{aligned}$$

At this stage, we recognize the last factor as the semiperimeter, $s = \frac{1}{2}(a + b + c) = \frac{a+b+c}{2}$. To complete the proof, we note that

$$(s - a) = \frac{a+b+c}{2} - a = \frac{a+b+c-2a}{2} = \frac{b+c-a}{2}$$

Similarly, we find $(s - b) = \frac{a+b+c}{2} - b = \frac{a+b+c-2b}{2} = \frac{a+c-b}{2}$. Hence, we get

$$\begin{aligned}
 A^2 &= \frac{(b + c - a)}{2} \cdot \frac{(a + c - b)}{2} \cdot \frac{(a + b - c)}{2} \cdot \frac{(a + b + c)}{2} \\
 &= (s - a)(s - b)(s - c)s
 \end{aligned}$$

so that $A = \sqrt{s(s-a)(s-b)(s-c)}$ as required.

We close with an example of Heron's Formula.

✓ Example 1.2.7

Find the area enclosed of the triangle in Example 1.2.5 number 2.

Solution

We are given $a = 4$, $b = 7$ and $c = 5$. Using these values, we find $s = \frac{1}{2}(4 + 7 + 5) = 8$, $(s - a) = 8 - 4 = 4$, $(s - b) = 8 - 7 = 1$ and $(s - c) = 8 - 5 = 3$. Using Heron's Formula, we get $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(8)(4)(1)(3)} = \sqrt{96} = 4\sqrt{6} \approx 9.80$ square units.

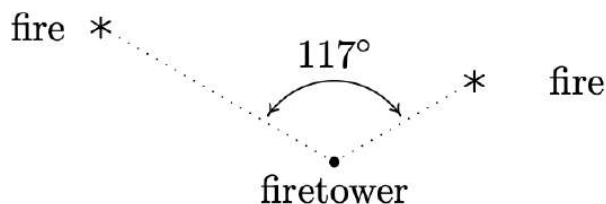
1.2.3 Exercises

In Exercises 1 - 10, use the Law of Cosines to find the remaining side(s) and angle(s) if possible.

1. $a = 7, b = 12, \gamma = 59.3^\circ$
2. $\alpha = 104^\circ, b = 25, c = 37$
3. $\alpha = 120^\circ, b = 3, c = 4$
4. $a = 3, b = 4, \gamma = 90^\circ$
5. $\alpha = 120^\circ, b = 3, c = 4$
6. $a = 7, b = 10, c = 13$
7. $a = 1, b = 2, c = 5$
8. $a = 300, b = 302, c = 48$
9. $a = 5, b = 5, c = 5$
10. $a = 5, b = 12, c = 13$

In Exercises 11 - 16, solve for the remaining side(s) and angle(s), if possible, using any appropriate technique.

11. $a = 18, \alpha = 63^\circ, b = 20$
12. $a = 37, b = 45, c = 26$
13. $a = 16, \alpha = 63^\circ, b = 20$
14. $a = 22, \alpha = 63^\circ, b = 20$
15. $\alpha = 42^\circ, b = 117, c = 88$
16. $\beta = 7^\circ, \gamma = 170^\circ, c = 98.6$
17. Find the area of the triangles given in Exercises 6, 8 and 10 above.
18. The hour hand on my antique Seth Thomas schoolhouse clock is 4 inches long and the minute hand is 5.5 inches long. Find the distance between the ends of the hands when the clock reads four o'clock. Round your answer to the nearest hundredth of an inch.
19. A geologist wants to measure the diameter of a crater. From her camp, it is 4 miles to the northern-most point of the crater and 2 miles to the southern-most point. If the angle between the two lines of sight is 117° , what is the diameter of the crater? Round your answer to the nearest hundredth of a mile.
20. From the Piedmont International Airport a tour helicopter can fly to Cliffs of Insanity Point by following a bearing of $N8.2^\circ E$ for 192 miles and it can fly to Bigfoot Falls by following a bearing of $S68.5^\circ E$ for 207 miles. Find the distance between Cliffs of Insanity Point and Bigfoot Falls. Round your answer to the nearest mile.
21. Cliffs of Insanity Point and Bigfoot Falls from Exercise 20 above both lie on a straight stretch of the Great Sasquatch Canyon. What bearing would the tour helicopter need to follow to go directly from Bigfoot Falls to Cliffs of Insanity Point? Round your angle to the nearest tenth of a degree.
22. A naturalist sets off on a hike from a lodge on a bearing of $S80^\circ W$. After 1.5 miles, she changes her bearing to $S17^\circ W$ and continues hiking for 3 miles. Find her distance from the lodge at this point. Round your answer to the nearest hundredth of a mile. What bearing should she follow to return to the lodge? Round your angle to the nearest degree.
23. The HMS Sasquatch leaves port on a bearing of $N23^\circ E$ and travels for 5 miles. It then changes course and follows a heading of $S41^\circ E$ for 2 miles. How far is it from port? Round your answer to the nearest hundredth of a mile. What is its bearing to port? Round your angle to the nearest degree.
24. The SS Bigfoot leaves a harbor bound for Nessie Island which is 300 miles away at a bearing of $N32^\circ E$. A storm moves in and after 100 miles, the captain of the Bigfoot finds he has drifted off course. If his bearing to the harbor is now $S70^\circ W$, how far is the SS Bigfoot from Nessie Island? Round your answer to the nearest hundredth of a mile. What course should the captain set to head to the island? Round your angle to the nearest tenth of a degree.
25. From a point 300 feet above level ground in a firetower, a ranger spots two fires in the Yeti National Forest. The angle of depression²¹ made by the line of sight from the ranger to the first fire is 2.5° and the angle of depression made by line of sight from the ranger to the second fire is 1.3° . The angle formed by the two lines of sight is 117° . Find the distance between the two fires. Round your answer to the nearest foot. (Hint: In order to use the 117° angle between the lines of sight, you will first need to use right angle Trigonometry to find the lengths of the lines of sight. This will give you a Side-Angle-Side case in which to apply the Law of Cosines.)



26. If you apply the Law of Cosines to the ambiguous Angle-Side-Side (ASS) case, the result is a quadratic equation whose variable is that of the missing side. If the equation has no positive real zeros then the information given does not yield a triangle. If the equation has only one positive real zero then exactly one triangle is formed and if the equation has two distinct positive real zeros then two distinct triangles are formed. Apply the Law of Cosines to Exercises 11, 13 and 14 above in order to demonstrate this result.
27. Discuss with your classmates why Heron's Formula yields an area in square units even though four lengths are being multiplied together.

1.2.4 Answers

1. $\alpha \approx 35.54^\circ$ $\beta \approx 85.16^\circ$ $\gamma = 59.3^\circ$
 $a = 7$ $b = 12$ $c \approx 10.36$
2. $\alpha = 104^\circ$ $\beta \approx 29.40^\circ$ $\gamma \approx 46.60^\circ$
 $a \approx 49.41$ $b = 25$ $c = 37$
3. $\alpha \approx 85.90^\circ$ $\beta = 8.2^\circ$ $\gamma \approx 85.90^\circ$
 $a = 153$ $b \approx 21.88$ $c = 153$
4. $\alpha \approx 36.87^\circ$ $\beta \approx 53.13^\circ$ $\gamma = 90^\circ$
 $a = 3$ $b = 4$ $c = 5$
5. $\alpha = 120^\circ$ $\beta \approx 25.28^\circ$ $\gamma \approx 34.72^\circ$
 $a = \sqrt{37}$ $b = 3$ $c = 4$
6. $\alpha \approx 32.31^\circ$ $\beta \approx 49.58^\circ$ $\gamma \approx 98.21^\circ$
 $a = 7$ $b = 10$ $c = 13$
7. Information does not produce a triangle
8. $\alpha \approx 83.05^\circ$ $\beta \approx 87.81^\circ$ $\gamma \approx 9.14^\circ$
 $a = 300$ $b = 302$ $c = 48$
9. $\alpha = 60^\circ$ $\beta = 60^\circ$ $\gamma = 60^\circ$
 $a = 5$ $b = 5$ $c = 5$
10. $\alpha \approx 22.62^\circ$ $\beta \approx 67.38^\circ$ $\gamma = 90^\circ$
 $a = 5$ $b = 12$ $c = 13$
 $\alpha = 63^\circ$ $\beta \approx 98.11^\circ$ $\gamma \approx 18.89^\circ$
 $a = 18$ $b = 20$ $c \approx 6.54$
11. $\alpha = 63^\circ$ $\beta \approx 81.89^\circ$ $\gamma \approx 35.11^\circ$
 $a = 18$ $b = 20$ $c \approx 11.62$
12. $\alpha \approx 55.30^\circ$ $\beta \approx 89.40^\circ$ $\gamma \approx 35.30^\circ$
 $a = 37$ $b = 45$ $c = 26$
13. Information does not produce a triangle
14. $\alpha = 63^\circ$ $\beta \approx 54.1^\circ$ $\gamma \approx 62.9^\circ$
 $a = 22$ $b = 20$ $c \approx 21.98$
15. $\alpha = 42^\circ$ $\beta \approx 89.23^\circ$ $\gamma \approx 48.77^\circ$
 $a \approx 78.30$ $b = 117$ $c = 88$
16. $\alpha \approx 3^\circ$ $\beta = 7^\circ$ $\gamma = 170^\circ$
 $a \approx 29.72$ $b \approx 69.2$ $c = 98.6$
17. The area of the triangle given in Exercise 6 is $\sqrt{1200} = 20\sqrt{3} \approx 34.64$ square units.
 The area of the triangle given in Exercise 8 is $\sqrt{51764375} \approx 7194.75$ square units.
 The area of the triangle given in Exercise 10 is exactly 30 square units.
18. The distance between the ends of the hands at four o'clock is about 8.26 inches.
19. The diameter of the crater is about 5.22 miles.
20. About 313 miles

21. N31.8°W
22. She is about 3.92 miles from the lodge and her bearing to the lodge is N37°E.
23. It is about 4.50 miles from port and its heading to port is S47°W.
24. It is about 229.61 miles from the island and the captain should set a course of N16.4°E to reach the island.
25. The fires are about 17456 feet apart. (Try to avoid rounding errors.)

Reference

¹ as well as the measure of said angle

² as well as the length of said side

³ Your Science teachers should thank us for this.

⁴ Don't worry! Radians will be back before you know it!

⁵ The exact value of $\sin(15^\circ)$ could be found using the difference identity for sine or a half-angle formula, but that becomes unnecessarily messy for the discussion at hand. Thus "exact" here means $\frac{7\sin(15^\circ)}{\sin(120^\circ)}$.

⁶ To find an exact expression for β , we convert everything back to radians: $\alpha = 30^\circ = \frac{\pi}{6}$ radians, $\gamma = \arcsin\left(\frac{2}{3}\right)$ radians and $180^\circ = \pi$ radians. Hence, $\beta = \pi - \frac{\pi}{6} - \arcsin\left(\frac{2}{3}\right) = \frac{5\pi}{6} - \arcsin\left(\frac{2}{3}\right)$ radians $\approx 108.19^\circ$.

⁷ An exact answer for β in this case is $\beta = \arcsin\left(\frac{2}{3}\right) - \frac{\pi}{6}$ radians $\approx 11.81^\circ$.

⁸ If this sounds familiar, it should. From high school Geometry, we know there are four congruence conditions for triangles: Angle-Angle-Side (AAS), Angle-Side-Angle (ASA), Side-Angle-Side (SAS) and Side-Side-Side (SSS). If we are given information about a triangle that meets one of these four criteria, then we are guaranteed that exactly one triangle exists which satisfies the given criteria.

⁹ In more reputable books, this is called the 'Side-Side-Angle' or SSA case.

¹⁰ Remember, we have already argued that a triangle exists in this case!

¹¹ Do you see why C must lie to the right of Q ?

¹² Or by [Theorem 10.4](#) again . . .

¹³ I have friends who live in Pacifica, CA and their road is actually this steep. It's not a nice road to drive.

¹⁴ The word 'plumb' here means that the tree is perpendicular to the horizontal.

¹⁵ See [Example 10.1.1](#) in [Section 10.1](#) for a review of the DMS system.

¹⁶ Here, 'Side-Angle-Side' means that we are given two sides and the 'included' angle - that is, the given angle is adjacent to both of the given sides.

¹⁷ This shouldn't come as too much of a shock. All of the theorems in Trigonometry can ultimately be traced back to the definition of the circular functions along with the distance formula and hence, the Pythagorean Theorem.

¹⁸ after simplifying . . .

¹⁹ Your instructor will let you know which procedure to use. It all boils down to how much you trust your calculator.

²⁰ There can only be one obtuse angle in the triangle, and if there is one, it must be the largest.

²¹ See [Exercise 78](#) in [Section 10.3](#) for the definition of this angle.

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1.3: Calculations, Units, and Unit Conversion

SI Base Units

Quantity	Unit	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Angle	radian	rad
Temperature	kelvin	K

SI Derived Units

Quantity	Symbol	Units/Formula
Acceleration	-	m/s^2
Area	-	m^2
Density	-	kg/m^3
Force	N	$\text{kg}\cdot\text{m/s}^2$
Pressure/Stress	Pa	N/m^2
Pressure/Stress	MPa	N/mm^2
Weight	N	$\text{kg}\cdot\text{m/s}^2$
Work	J	$\text{N}\cdot\text{m}$

SI Prefixes

Prefix	SI Symbol	Factor
giga	G	$1,000,000,000 = 10^9$
mega	M	$1,000,000 = 10^6$
kilo	k	$1,000 = 10^3$
milli	m	$0.001 = 10^{-3}$
micro	μ	$0.000\ 001 = 10^{-6}$

Unit Conversions

Interactive File

The table below shows how to convert units. Open this [Excel file](#) or look for the file in your course to access the interactive version that allows you to enter inputs to calculate outputs.

Left to right	These units multiplied by...-->	..this factor will...-->	..convert to these units.
Reverse, right to left...	Convert to these units...<--	<---this factor will...	<---These units divided by...
<u>Length</u>			
	inches	2.54	centimeters
	feet	0.3048	meters
	yards	0.9144	meters
	miles	1.609	kilometers
	miles	5,280	feet
<u>Area</u>			
	square inches	645.2	square millimeters
	square feet	0.0929	square meters
	square yards	0.8361	square meters
<u>Volume</u>			
	cubic inches	16,387	cubic millimeters
	cubic feet	0.02832	cubic meters
	cubic yards	0.7646	cubic meters
	gallons	0.003785	cubic meters
<u>Force</u>			
	pounds	4.448	newtons
	kips	4,448	newtons
	kips	1,000	pounds
	Tons (short)	2,000	pounds
<u>Force per unit length</u>			
	pounds per foot	14.594	newtons per meter
	kips per foot	14,594	newtons per meter
<u>Load per unit volume</u>			
	pounds per cubic foot	0.15714	kilonewtons per cubic meter
<u>Bending moment (torque)</u>			
	inch-pounds	0.113	newton meters
	foot-pounds	1.356	newton meters
<u>Stress, pressure, loading</u> <u>Force per unit area</u>			
	pounds per square inch	6,895	pascals
	pounds per square inch	6.895	kilopascals

Left to right	These units multiplied by...-->	..this factor will...-->	..convert to these units.
	pounds per square inch	0.006895	megapascals
<u>Mass</u>			
	pounds	0.4536	kilograms
<u>Moment of Inertia</u>			
	inches ⁴	416,231	millimeters ⁴

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1.4: Forces

A force is any influence that causes a body to accelerate. Forces on a body can also cause stress in that body, which can result in the body deforming or breaking. Though forces can come from a variety of sources, there are three distinguishing features to every force. These features are the **magnitude** of the force, the **direction** of the force, and the **point of application** of the force. Forces are often represented as **vectors** (as in the diagram to the right) and each of these features can be determined from a vector representation of the forces on the body.

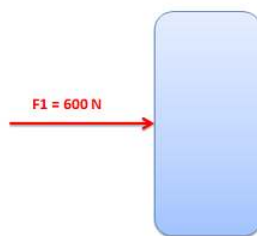


Figure 1.4.1: A basic point force acting on a body.

Magnitude:

The magnitude of a force is the degree to which the force will accelerate the body it is acting on; it is represented by a scalar (a single number). The magnitude can also be thought of as the strength of the force. When forces are represented as vectors, the magnitude of the force is usually explicitly labeled. The length of the vector also often corresponds to the relative magnitude of the vector, with longer vectors indicating larger magnitudes.

The magnitude of force is measured in units of mass times length over time squared. In metric units the most common unit is the Newton (N), where one Newton is one kilogram times one meter over one second squared. This means that a force of one Newton would cause a one-kilogram object to accelerate at a rate of one meter per second squared. In English units, the most common unit is the pound (lb), where one pound is equal to one slug times a foot over a second squared. This means that a one-pound force would cause an object with a mass of one slug to accelerate at a rate of one foot per second squared.

$$Force = \frac{(mass)(distance)}{(time)^2} \quad (1.4.1)$$

$$1 \text{ Newton (N)} = \frac{(kg)(m)}{s^2} \quad (1.4.2)$$

$$1 \text{ pound (lb)} = \frac{(slug)(ft)}{s^2} \quad (1.4.3)$$

Direction:

In addition to having magnitudes, forces also have directions. As we said before, a force is any influence that causes a body to accelerate. Since acceleration has a specific direction, force also has a specific direction that matches this acceleration. The direction of the force is indicated in diagrams by the direction of the vector representing the force.

Direction has no units, but it is usually given by reporting angles between the vector representing the force and coordinate axes, or by reporting the X, Y, and Z components of the vector. Often times vectors that have the same direction as one of the coordinate axes will not have any angles or components listed. If this is true, it is usually safe to assume that the direction does match the direction of one of the coordinate axes.

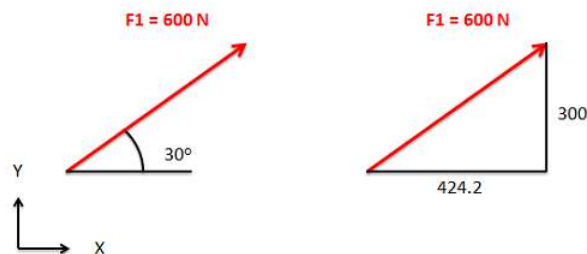


Figure 1.4.2: The magnitude and direction of a vector can be given as a magnitude and an angle of the vector, or by giving magnitude of the vector components in each of the coordinate axes.

Point of Application:

The point, or points, at which a force is applied to a body is important for understanding how the body will react. For particles, there is only a single point for the forces to act on, but for rigid bodies there are an infinite number of possible points of application. Some points of application will lead to the body undergoing simple linear acceleration; some will exert a moment on the body which will cause the body to undergo rotational acceleration as well as linear acceleration.

Depending on the nature of the point of application of a force, there are three general types of forces. These are **point forces**, **surface forces**, and **body forces**. Below is a diagram of a box being pulled by a rope across a frictionless surface. The box has three forces acting on it. The first is the force from the rope. This is a force applied to a single point on the box, and is therefore modeled as point force. Point forces are represented by a single vector. Second is the normal force from the ground that is supporting the box. Because this force is applied evenly to the bottom surface of the box, it is best modeled as a surface force. Surface forces are indicated by a number of vectors drawn side by side with a profile line to indicate the magnitude of the force at any point. The last force is the gravitational force pulling the box downward. Because this force is applied evenly to the entire volume of the box, it is best modeled as a body force. Body forces are sometimes shown as a field of vectors as shown, though they are often not drawn out at all because they end up cluttering the free body diagram.

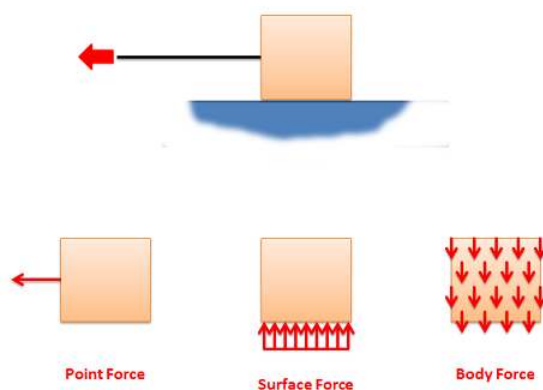


Figure 1.4.3: The box being pulled along a frictionless surface as shown above has three types of forces acting on it. The tension in the cable is best represented by a point force, the normal force supporting the box is best represented by a surface force, and the gravitational force on the box is best represented as a body force.

We will also sometimes talk about **distributed forces**. A distributed force is simply another name for either a surface or a body force.

The exact point or surface that the force is acting on can be drawn as either the head or the tail of the force vector in the free body diagram. Because of the **principle of transmissibility**, both options are known to represent the same physical system.

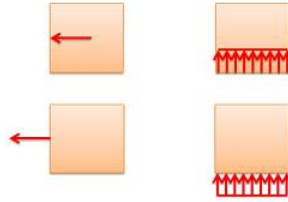


Figure 1.4.4: The diagrams on the left represent two equivalent free body diagrams for the same physical system with a single point force. The diagrams on the right represent two equivalent free body diagrams for the same physical system with a single surface force.

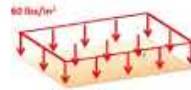


Point of Application

We can model forces as either having a single point of application or being distributed over some area or volume.

- Point forces have a single point of application and are modeled as a single vector.
- Distributed forces have a point of application that is distributed over a line, surface, or volume. They are modeled as fields of vectors.





Video 1.4.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/8MR_w3ZOOiM.
[Video Transcript](#).

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1.5: Moments

A **moment** (also sometimes called a torque) is defined as the "tendency of a force to rotate a body". Where forces cause linear accelerations, moments cause **angular accelerations**. In this way moments, can be thought of as twisting forces.

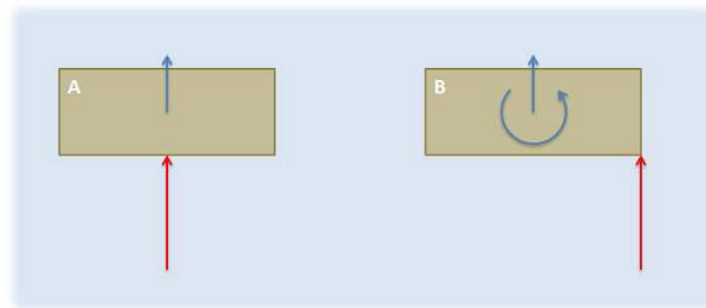


Figure 1.5.1: Imagine two boxes on an icy surface. The force on box A would simply cause the box to begin accelerating, but the force on box B would cause the box to both accelerate and to begin to rotate. The force on box B is exerting a moment, where the force on box A is not.

The Vector Representation of a Moment:

Moments, like forces, can be represented as vectors and have a magnitude, a direction, and a "point of application". For moments however a better name for the point of application is the **axis of rotation**. This will be the point or axis about which we will determine all the moments.

Magnitude:

The magnitude of a moment is the degree to which the moment will cause angular acceleration in the body it is acting on. It is represented by a scalar (a single number). The magnitude of the moment can be thought of as the strength of the twisting force exerted on the body. When a moment is represented as a vector, the magnitude of the moment is usually explicitly labeled, though the length of the moment vector also often corresponds to the relative magnitude of the moment.

The magnitude of the moment is measured in units of force times distance. The standard metric units for the magnitude of moments are Newton-meters, and the standard English units for a moment are foot-pounds.

$$M = F * d \quad (1.5.1)$$

$$\text{Metric: } N * m \quad (1.5.2)$$

$$\text{English: } lb * ft \quad (1.5.3)$$

Direction:

In a two-dimensional problem, the direction can be thought of as a scalar quantity corresponding to the direction of rotation the moment would cause. A moment that would cause a counterclockwise rotation is a positive moment, and a moment that would cause a clockwise rotation is a negative moment.

In a three-dimensional problem, however, a body can rotate about an axis in any direction. If this is the case, we need a vector to represent the direction of the moment. The direction of the moment vector will line up with the axis of rotation that moment would cause, but to determine which of the two directions we can use along that axis we have available we use the right hand rule. To use the right hand rule, align your right hand as shown in Figure 1.5.2 so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. If you do this, your thumb will be pointing in the direction of the moment vector.

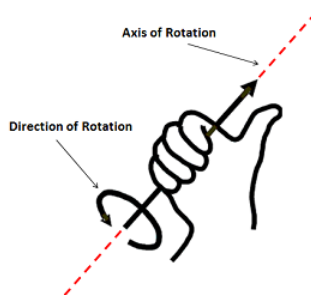


Figure 1.5.2: To use the right hand rule, align your right hand so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. This means your thumb will be pointing in the direction of the moment vector.

If we look back to two-dimensional problems, all rotations occur about an axis pointing directly into or out of the page (the z -axis). Using the right hand rule, counterclockwise rotations are represented by a vector in the positive z direction and clockwise rotations are represented by a vector in the negative z direction.

Axis of Rotation:

In engineering statics problems we can choose any point/axis as the axis of rotation. However, the choice of this point will affect the magnitude and direction of the resulting moment, and the moment is only valid about that point.

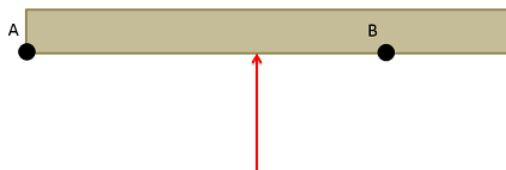


Figure 1.5.3: The magnitude and direction of a moment depends upon the chosen axis of rotation. For example, the single force above would cause different moments about Point A and Point B, because it would cause different rotations depending on the point we fix in place.

Though we can take the moment about any point in a statics problem, if we are **adding** together the moments from multiple forces, all the moments must be taken about a **common axis of rotation**. Moments taken about different points cannot be added together to find a "net moment."

Additionally, if we move into the subject of dynamics, where bodies are moving, we will want to relate moments to angular accelerations. For this to work, either we will need to take the moments about a single point that does not move (such as the hinge on a door) or we will need to take the moments about the center of mass of the body. Summing moments about other axes of rotation will not result in valid calculations.

Calculating Moments:

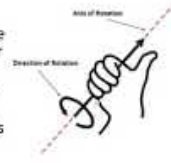
To calculate the moment that a force exerts on a body, we will have two main options: **scalar methods** and **vector methods**. Scalar methods are generally faster for two-dimensional problems where a body can only rotate clockwise or counterclockwise, while vector methods are generally faster for three-dimensional problems where the axis of rotation is more complex.



Moments as Vectors

Just as forces have both a magnitude and a direction, moments have both a **magnitude** and a **direction**.

- The magnitude of the moment is related to the degree to which the force would cause angular accelerations about a given point or axis.
 - Measured in units of force x distance (Nm or ft lbs)
- The direction of the moment vector represents the axis the force would cause the body to rotate around.
 - With positive and negative moments representing counter-clockwise and clockwise (respectively) potential rotation about that axis



Video 1.5.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/RyOwVvYEFHU>. [Video Transcript](#)

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1.6: Free Body Diagrams

A free body diagram is a tool used to solve engineering mechanics problems. As the name suggests, the purpose of the diagram is to "free" the body from all other objects and surfaces around it so that it can be studied in isolation. We will also draw in any forces or moments acting on the body, including those forces and moments exerted by the surrounding bodies and surfaces that we removed.

The diagram below shows a ladder supporting a person and the free body diagram of that ladder. As you can see, the ladder is separated from all other objects and all forces acting on the ladder are drawn in with the key dimensions and angles shown.

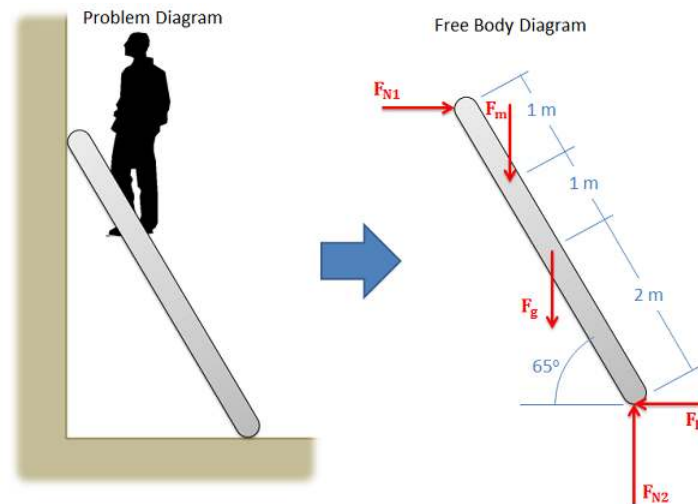


Figure 1.6.1: A ladder with a man standing on it is shown on the left. Assuming friction only at the base, a free body diagram of the ladder is shown on the right.

Constructing the Free Body Diagram:

The first step in solving most mechanics problems will be to construct a free body diagram. This simplified diagram will allow us to more easily write out the equilibrium equations for statics or strengths of materials problems, or the equations of motion for dynamics problems.

To construct the diagram we will use the following process:

1. First draw the body being analyzed, separated from all other surrounding bodies and surfaces. Pay close attention to the boundary, identifying what is part of the body, and what is part of the surroundings.
2. Second, draw in all **external** forces and moments acting directly on the body. Do not include any forces or moments that do not directly act on the body being analyzed. Do not include any forces that are **internal** to the body being analyzed.
3. Once the forces are identified and added to the free body diagram, the last step is to label any key dimensions and angles on the diagram.

Some common types of forces seen in mechanics problems are:

- **Gravitational Forces:** Unless otherwise noted, the mass of an object will result in a gravitational weight force applied to that body. This weight is usually given in pounds in the English system, and is modeled as $9.81 (g)$ times the mass of the body in kilograms for the metric system (resulting in a weight in Newtons). This force will always point down towards the center of the earth and act on the center of mass of the body.

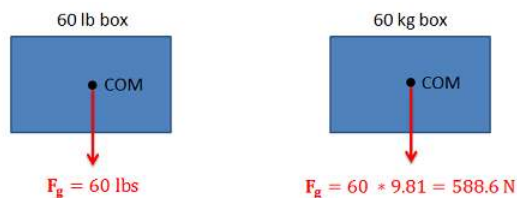


Figure 1.6.2: Gravitational forces always act downward on the center of mass.

- **Normal Forces (or Reaction Forces):** Every object in direct contact with the body will exert a normal force on that body which prevents the two objects from occupying the same space at the same time. Note that only objects in direct contact can exert normal forces on the body.
 - An object in contact with another object or surface will experience a normal force that is perpendicular (normal) to the surfaces in contact.
 - Joints or connections between bodies can also cause reaction forces or moments, and we will have one force or moment for each type of motion or rotation the connection prevents.

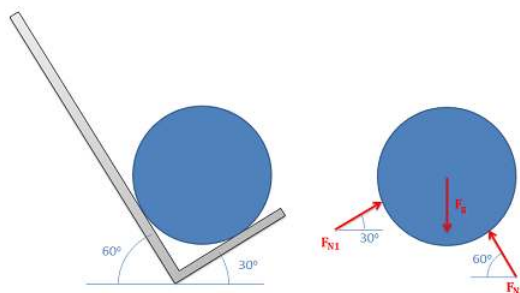


Figure 1.6.3: Normal forces always act perpendicular to the surfaces in contact. The barrel in the hand truck shown on the left has a normal force at each contact point.

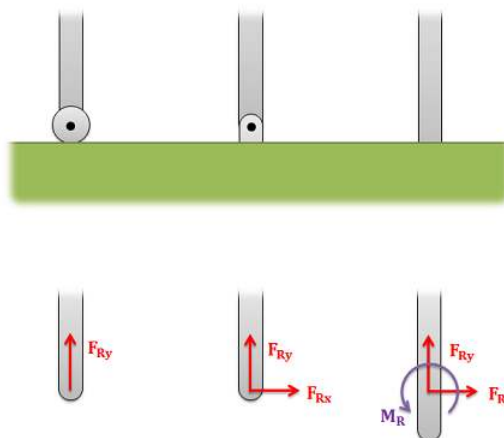


Figure 1.6.4: The roller on the left allows for rotation and movement along the surface, but a normal force in the y direction prevents motion vertically. The pin joint in the center allows for rotation, but normal forces in the x and y directions prevent motion in all directions. The fixed connection on the right has a normal forces preventing motion in all directions and a reaction moment preventing rotation.

- **Friction Forces:** Objects in direct contact with the body can also exert friction forces, which will resist the two bodies sliding against one another, on the body. These forces will always be perpendicular to the surfaces in contact. Friction is the subject of an entire chapter in this book, but for simple scenarios we usually assume rough or smooth surfaces.
 - For smooth surfaces we assume that there is no friction force.
 - For rough surfaces we assume that the bodies will not slide relative to one another, no matter what. In this case, the friction force is always just large enough to prevent this sliding.

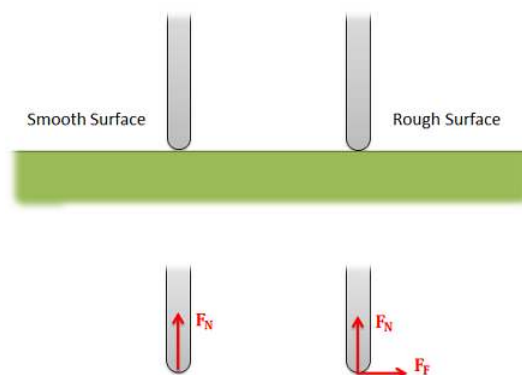


Figure 1.6.5: For a smooth surface we assume only a normal force perpendicular to the surface. For a rough surface we assume normal and friction forces are present.

- **Tension in Cables:** Cables, wires or ropes attached to the body will exert a tension force on the body in the direction of the cable. These forces will always pull on the body, as ropes, cables and other flexible tethers cannot be used for pushing.

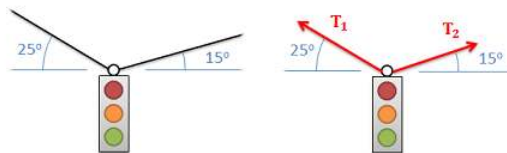
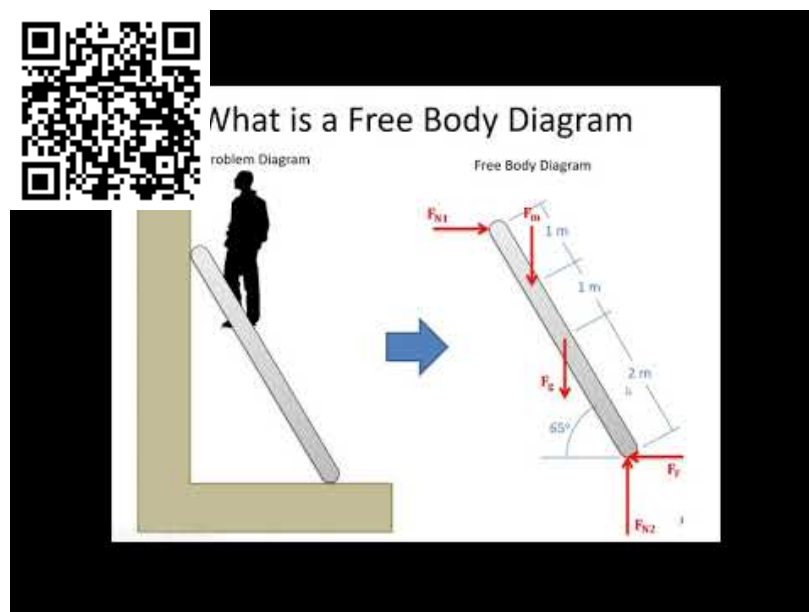


Figure 1.6.6: The tension force in cables always acts along the direction of the cable and will always be a pulling force.

The above forces are the most common, but other forces such as pressure from fluids, spring forces and magnetic forces exist and may act on the body.



Video 1.6.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/Kr7obGR68-Y>. [Video Transcript](#)

Worked Problems:

✓ Example 1.6.1

The drawing below shows two boxes sitting on a table. Draw a free body diagram of box A and box B.

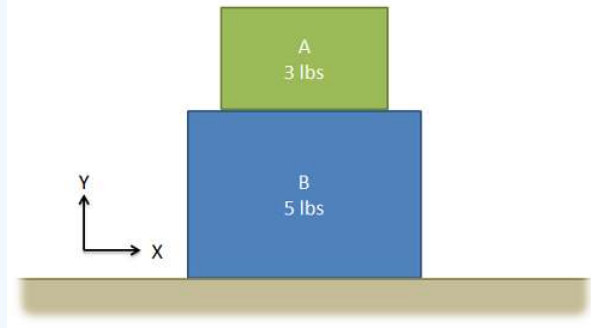
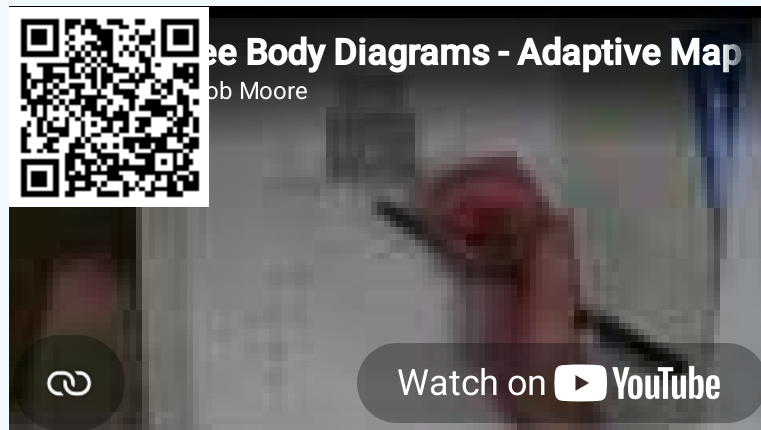


Figure 1.6.7: problem diagram for Example 1.6.1; two boxes are stacked on a flat surface, with one weighing 3 lbs on top of another weighing 5 lbs.

Solution



Video 1.6.2: Worked solution to example problem 1.6.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/RMVa9kioALs>. [Video Transcript 2](#)

✓ Example 1.6.2

Two equally sized barrels are being transported in a handtruck as shown below. Draw a free body diagram of each of the two barrels.

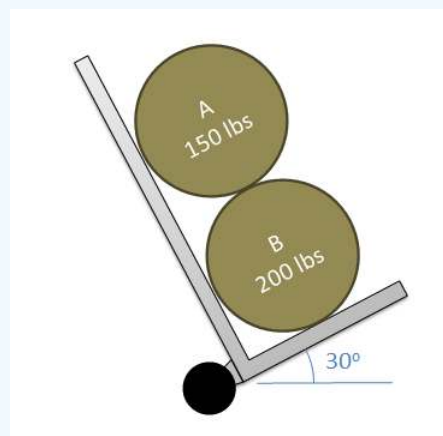
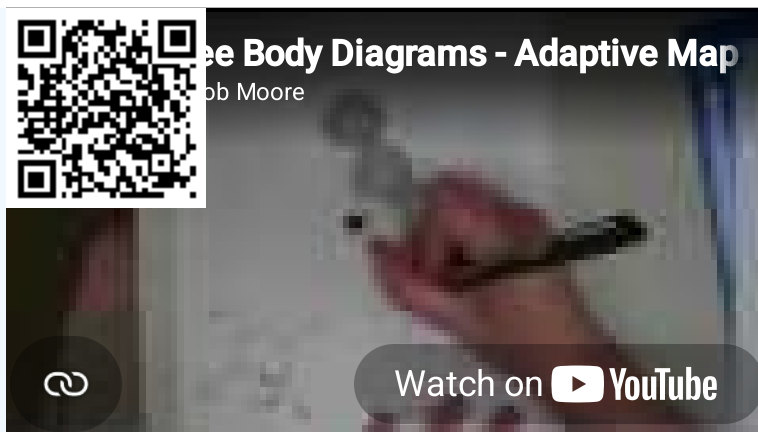


Figure 1.6.8: problem diagram for Example 1.6.2; two barrels are stacked horizontally, on a handcart tilted so the bottom is 30° above the horizontal.

Solution



Video 1.6.3: Worked solution to example problem 1.6.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/1a9gjFOIpK8>. [Video Transcript 3](#)

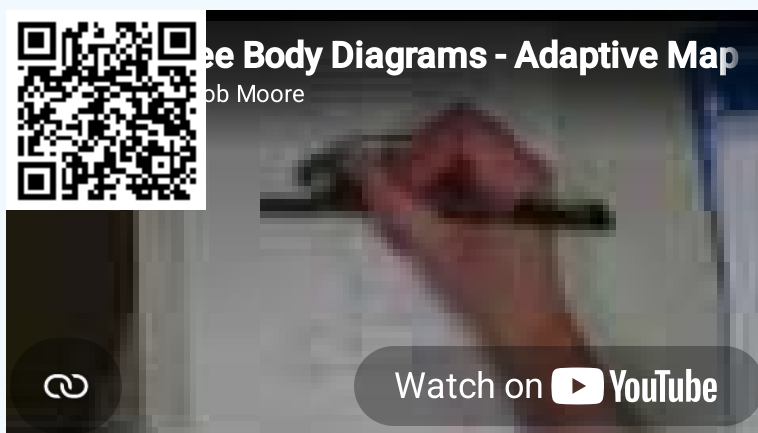
✓ Example 1.6.3

The car shown below is moving and then slams on the brakes locking up all four wheels. The distance between the two wheels is 8 feet and the center of mass is 3 feet behind and 2.5 feet above the point of contact between the front wheel and the ground. Draw a free body diagram of the car as it comes to a stop.



Figure 1.6.9: Car traveling on a level surface, facing left. Public domain image, no author listed.

Solution



Video 1.6.4: Worked solution to example problem 1.6.3, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/GiB3_fSIJBA. [Video Transcript 4](#)

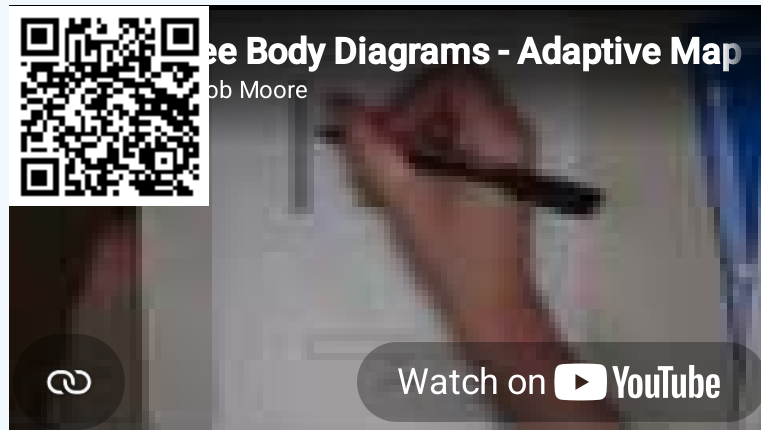
✓ Example 1.6.4

A 600-pound load is supported by a 5 meter long, 100-pound cantilever beam. Assume the beam is firmly anchored to the wall. Draw a free body diagram of the beam.



Figure 1.6.10: problem diagram for Example 1.6.4; a 600-lb load hangs from the free end of a horizontal beam whose other end is attached to a wall.

Solution



Video 1.6.5: Worked solution to example problem 1.6.4, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/8xSKVzT8c_s. Video Transcript 5

✓ Example 1.6.5

The main arm of a crane has a mass of 400 kg (assume the center of mass is at the midpoint of the arm), and supports a 200 kg load and a 600 kg counterweight. The arm is connected to the vertical support via a pin joint and two flexible cables. Draw a free body diagram of the arm.

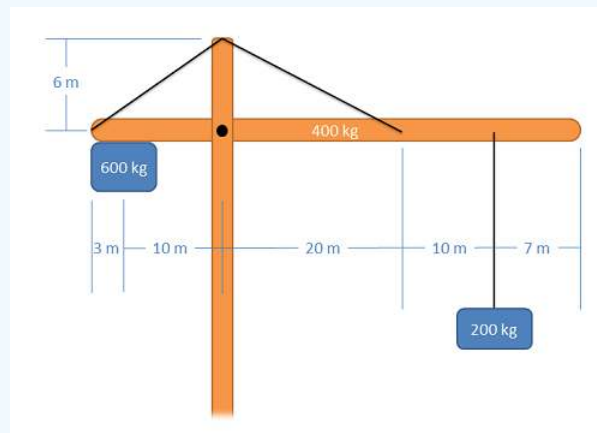
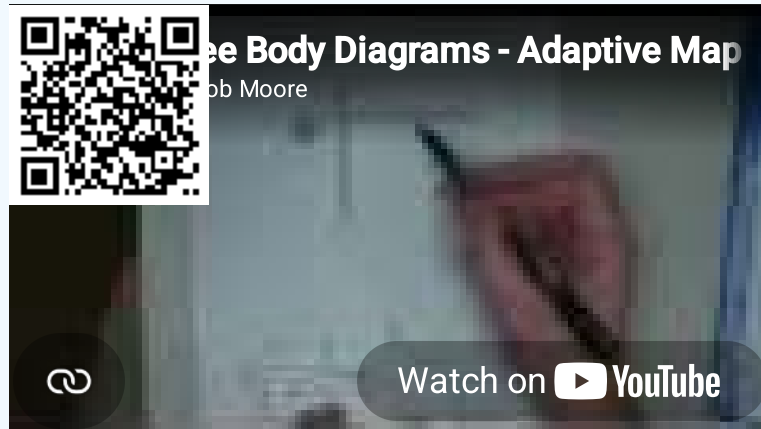


Figure 1.6.11: problem diagram for Example 1.6.5; a crane's arm, currently in the horizontal position, holds a load and a counterweight on opposite ends.

Solution



Video 1.6.6: Worked solution to example problem 1.6.5, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/0V7ULRnnhmA>. [Video Transcript 6](#)

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1.7: Chapter 1 Homework Problems

? Exercise 1.7.1

A pulley system is being used to hoist a 50 kg engine block as shown below. If distance d is currently 1 meter and we assume the pulleys are all frictionless, draw a free body diagram of the engine block with the attached pulley. Include all forces and important angles.

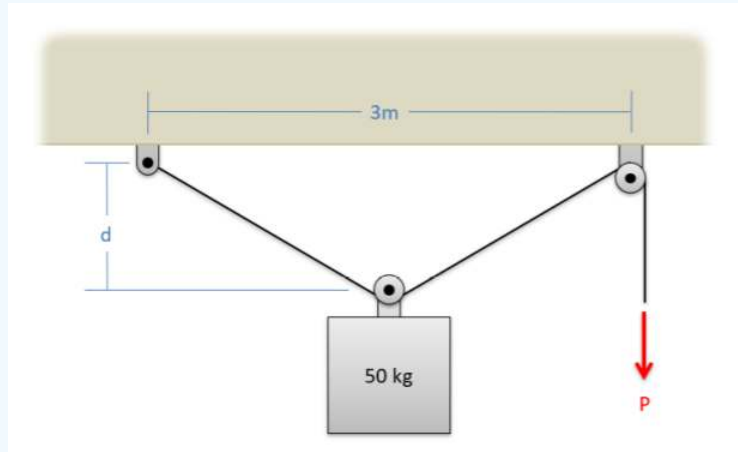


Figure 1.7.1: problem diagram for Exercise 1.7.1; an engine block is suspended by a cable with one end attached to an anchor point and the other end passing over a pulley.

? Exercise 1.7.2

The car shown below has a weight of 4500 lbs and a center of mass at point G. Assuming the car is not moving and is sitting on a level surface, draw a free body diagram of the car. Include all forces and important distances.

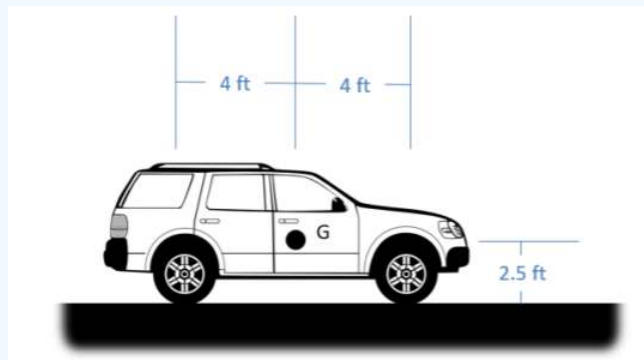


Figure 1.7.2: problem diagram for Exercise 1.7.2; a car sitting on a level surface is marked with a point G indicating the location of its center of mass.

? Exercise 1.7.3

A telephone pole sits on a rough surface. A cable attached to an excavator is then used to pull the pole along the surface as shown below. Assume the telephone pole has a mass of 350 kg and a length of 12 meters. Draw a free body diagram of the telephone pole. Include all forces, important distances, and important angles.

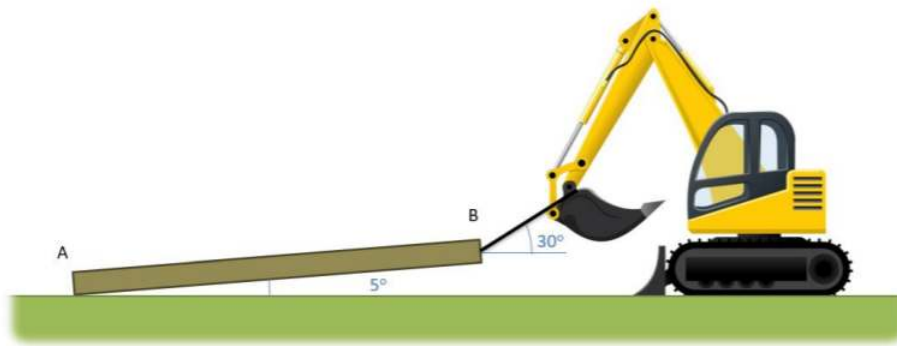


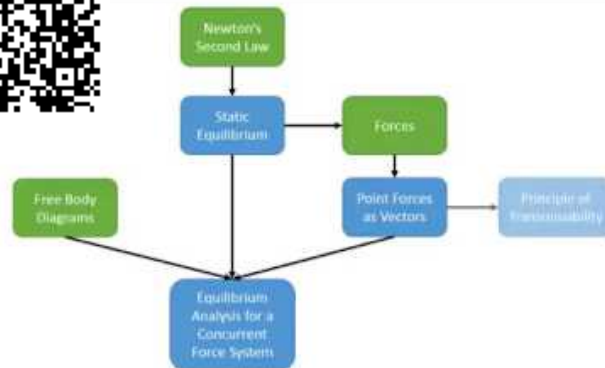
Figure 1.7.3: problem diagram for Exercise 1.7.3; an excavator pulls a telephone pole across the ground at an angle, through the use of a cable.

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CHAPTER OVERVIEW

2: Static Equilibrium in Concurrent Force Systems



Video introduction to Chapter 2, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/VXhvhP8VBMU>. [Video Transcript](#)

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2.1: Static Equilibrium

Objects in static equilibrium are objects that are not accelerating (either linear acceleration or angular acceleration). These objects may be stationary, such as a building or a bridge, or they may have a constant velocity, such as a car or truck moving at a constant speed on a straight patch of road.



Figure 2.1.1: (left) Because this high rise building is stationary with no acceleration, the members and overall structure are in equilibrium. Image by Jakem Bradford CC-BY-SA 4.0. (right): Assuming that this truck is maintaining a constant speed and direction, this truck is in equilibrium because its velocity is not changing over time. Public Domain image by Klever.

Newton's Second Law states that the force exerted on an object is equal to the mass of the object times the acceleration it experiences. Therefore, if we know that the acceleration of an object is equal to zero, then we can assume that the sum of all forces acting on the object is zero. Individual forces acting on the object, represented by force vectors, may not have zero magnitude but the sum of all the force vectors will always be equal to zero for objects in equilibrium. Engineering statics is the study of objects in static equilibrium, and the simple assumption of all forces adding up to zero is the basis for the subject area of engineering statics.

$$\sum \vec{F} = m\vec{a} \quad (2.1.1)$$

$$a = 0; \sum \vec{F} = 0 \quad (2.1.2)$$

Equilibrium follows a similar pattern for angular accelerations. The rotational equivalent of Newton's Second Law states that the moment exerted on an object is equal to the moment of inertia of that object times the angular acceleration of the object. If we know the angular acceleration of an object is equal to zero, then we know the sum of all moments acting on the object is equal to zero.

$$\sum \vec{M} = I\vec{\alpha} \quad (2.1.3)$$

$$\vec{\alpha} = 0; \sum \vec{M} = 0 \quad (2.1.4)$$



Analyzing Bodies in Equilibrium



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/2Ekp8MkgUYI>. [Video Transcript](#)

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2.2: Point Forces as Vectors

A point force is any force where the **point of application** is considered to be a **single point**. In reality, most forces are technically surface forces, where the force is applied over an area, but when the area is small enough (in comparison to the bodies being analyzed) it can often be approximated as a point force. Because point forces can be represented as a single vector (rather than a field of vectors for distributed forces), they are much easier to work with in engineering analysis. For this reason, point forces are used in place of distributed forces in engineering analysis whenever possible. Below are some examples of where it is appropriate to use point forces.



Figure 2.2.1: The tensions in the cables supporting this container can be treated as point forces pulling in the direction of the cables. Adapted from Image by maxronnersjo CC-BY-SA 3.0.



Figure 2.2.2: The friction force between the bow and string on this cello can be treated as a point force. Adapted from Public Domain image by Levi.



Figure 2.2.3: Though gravitational forces are technically body forces, they are often approximated as a single point force acting on the center of gravity of the object. Adapted from Public Domain image, no author listed.

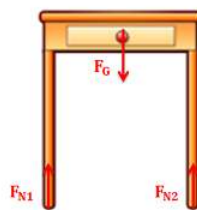


Figure 2.2.4: The gravitational force and the normal forces acting on each leg of this table can all be approximated as point forces. Adapted from Public Domain image by Seahen.

In addition to the magnitude, direction, and point of application of the point force, another important term to understand is the **line of action** of the force. The line of action of a force is the line along which the force acts. Given the direction and point of application, one can find the line of action, but this term will be important in discussing concurrent forces and in the principle of transmissibility.

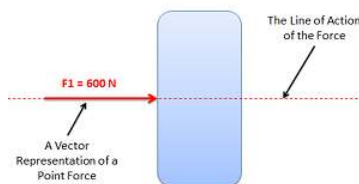


Figure 2.2.5: The line of action of a point force is the line along which the force acts.

Force Vector Representation:

When vectors are drawn to form free body diagrams, the magnitude and direction are usually given in one of two formats:

- Overall magnitude and angle(s) to indicate direction (often called **magnitude and direction form**).
- Magnitudes in each of the coordinate directions (often called **component form**).

In either format we will need two values to fully define a force vector in a 2D system (either a magnitude and a single angle or a magnitude in each of the two coordinate axes), and three values to fully define a force vector in a 3D system (either a magnitude and two angles or a magnitude in each of the three coordinate axes). Below are some examples of force vectors in both representations.

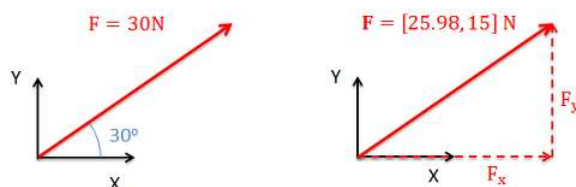


Figure 2.2.6: The same force can be represented with a magnitude and an angle, as shown in the left, or with magnitudes in relation to each of the coordinate axes as shown on the right.

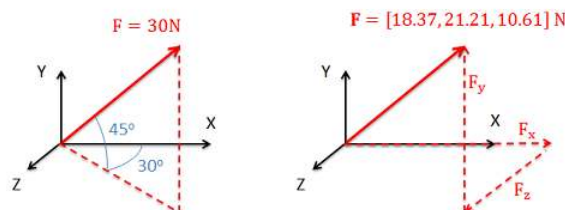


Figure 2.2.7: In three dimensions forces are represented with either a magnitude and two directions, as shown on the left, or with magnitudes in relation to each of the three coordinate axes as shown on the right.

Changing Force Vector Forms:

Because the two different forms of the vector are equivalent, we can switch between representations without changing the problem. Often in engineering problems, it will initially be easier to write the force in magnitude and angle form, but later, analysis will be easier if forces are written in component form. To switch from magnitude and direction form to component form you will use right triangles and trigonometry to determine the component of the overall magnitude in each direction. This is a simple vector decomposition, and more information on this process can be seen on the vector decomposition page. To switch back from component form into magnitude and direction form you simply use the reverse of this initial process.



Point Forces

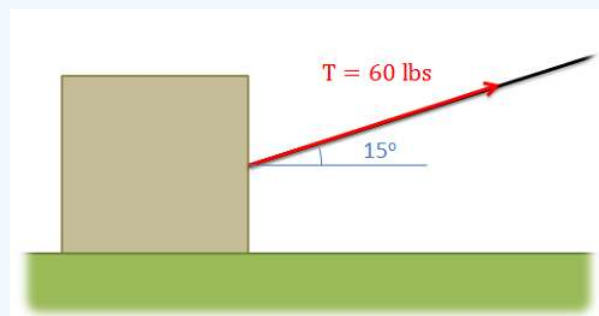
Although almost every force is actually applied over some area or volume, we approximate many forces as point forces.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/kq8vqOKhKeA>. [Video Transcript](#)

? Example 2.2.1

The tension force on the box below is given in magnitude and direction form. Redraw the diagram with the tension force given in component form.



Solution



Point Forces - Adaptive Map Worked

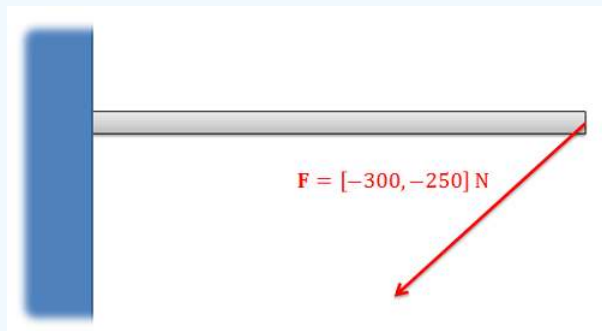
by Jacob Moore

Watch on  YouTube

Video 2.2.2: Worked solution to example problem 2.2.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/ZERURXWnlDg>. [Video Transcript 2](#)

? Example 2.2.2

The force acting on the cantilever beam shown below is given in component form. Redraw the diagram with the force given in magnitude and direction form.



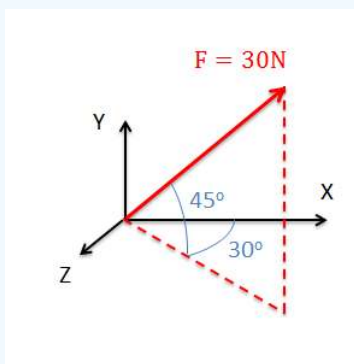
Solution



Video 2.2.3: Worked solution to example problem 2.2.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/M3UjDfZzRHY>. [Video Transcript 3](#)

? Example 2.2.3

The force shown below is given in magnitude and direction form. Redraw the diagram with the force vector given in component form.



Solution



Video 2.2.4: Worked solution to example problem 2.2.3, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/DroNv0TxnyA>. [Video Transcript 4](#)

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2.3: Principle of Transmissibility

The principle of transmissibility states that the point of application of a force can be moved anywhere along its line of action without changing the **external reaction forces** on a **rigid body**. Any force that has the same magnitude and direction, and which has a point of application somewhere along the same line of action will cause the same acceleration and will result in the same moment. Therefore, the points of application of forces may be moved along the line of action to simplify the analysis of rigid bodies.

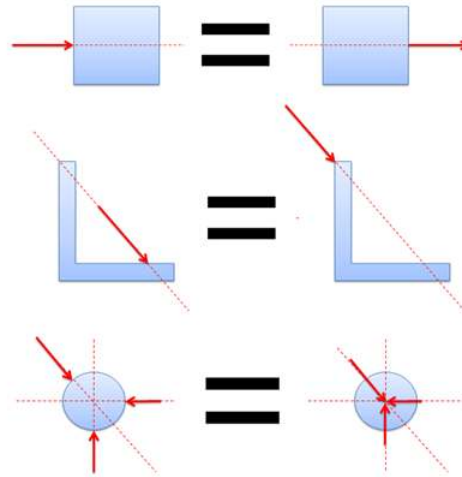


Figure 2.3.1: Because of the principle of transmissibility, each of the above pairs is equivalent.

When analyzing the internal forces (stress) in a rigid body, the exact point of application does matter. This difference in stresses may also result in changes in geometry which will in turn affect reaction forces. For this reason, the principle of transmissibility should only be used when examining external forces on bodies that are assumed to be rigid.

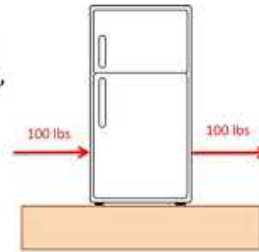


Figure 2.3.2: The exact point of application of a force will impact how internal forces (stresses) are distributed, so the principle of transmissibility cannot be applied when examining internal forces.



Principle of Transmissibility

- When forces of the same magnitude and direction **do not share the same line of action**, they will **always produce dissimilar results**, and they are not interchangeable.



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2.4: Concurrent Forces

A set of point forces is considered concurrent if all the **lines of action of those forces all come together at a single point**.

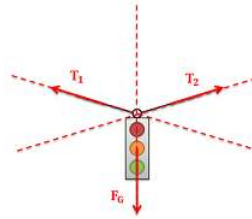


Figure 2.4.1: Because the lines of action for the gravitational force and the two tension forces line up at a single point, these forces are considered concurrent.

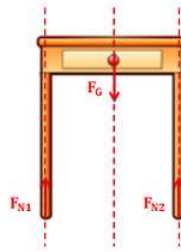
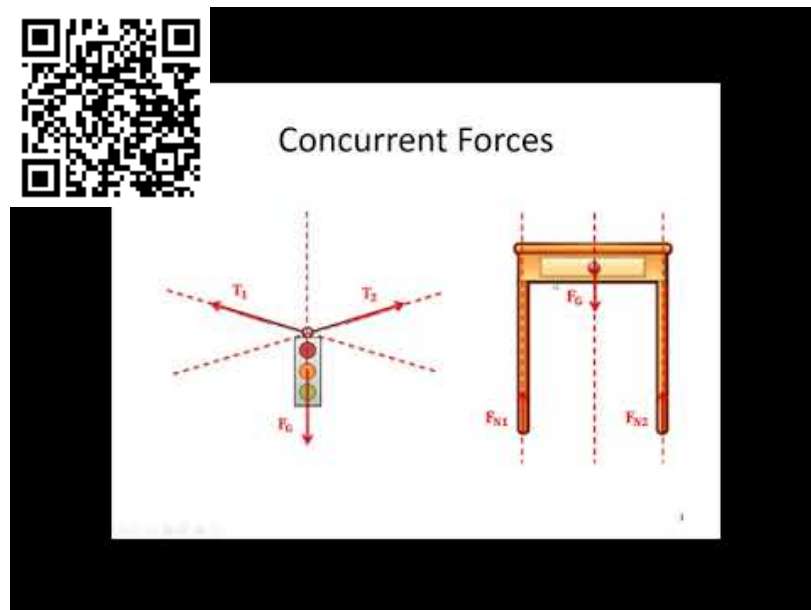


Figure 2.4.2: Because the lines of action of the gravitational force and the two normal forces do not intersect at a single point, these forces are not considered concurrent. Adapted from Public Domain image by Seahen.

Because the forces all act through a single point, there are no moments about this point. Because no moments exist, we can treat this body as a **particle**. In fact, because real particles only exist in theory, most particle analysis is actually applied to extended bodies with concurrent forces acting on them.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/puLnyApKfuc>. [Video Transcript](#)

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2.5: Equilibrium Analysis for Concurrent Force Systems

If a body is in static equilibrium, then by definition that body is not accelerating. If we know that the body is not accelerating then we know that **the sum of the forces acting on that body must be equal to zero**. This is the basis for equilibrium analysis for a particle.

In order to solve for any unknowns in our sum of forces equation, we actually need to turn the one vector equation into a set of scalar equations. For two dimensional problems, we will split our one vector equation down into two scalar equations. We do this by summing up all the x components of the force vectors and setting them equal to zero in our first equation, and summing up all the y components of the force vectors and setting them equal to zero in our second equation.

$$\sum \vec{F} = 0 \quad (2.5.1)$$

$$\sum F_x = 0; \sum F_y = 0 \quad (2.5.2)$$

Once we have written out the equilibrium equations, we can solve the equations for any unknown forces.

Finding the Equilibrium Equations:

The first step in finding the equilibrium equations is to **draw a free body diagram** of the body being analyzed. This diagram should show all the known and unknown force vectors acting on the body. In the free body diagram, provide values for any of the known magnitudes or directions for the force vectors and provide variable names for any unknowns (either magnitudes or directions).

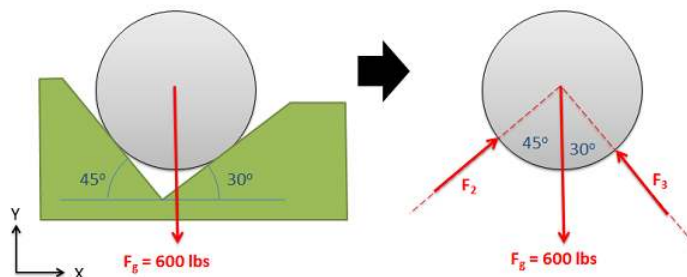
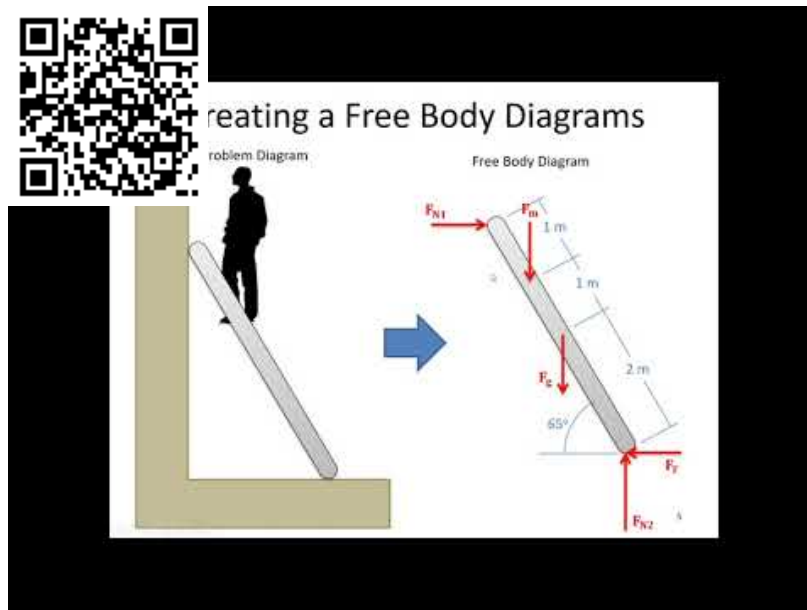


Figure 2.5.1: The first step in equilibrium analysis is drawing a free body diagram. This is done by removing everything but the body and drawing in all forces acting on the body. It is also useful to label all forces, key dimensions, and angles.

Next you will need to choose the x , y , and z axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the x , y and z directions (see the vectors page in Appendix 1 if you need more guidance on this). Your first equation will be the sum of the magnitudes of the components in the x direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the y direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the z direction being equal to zero. Collectively these are known as the **equilibrium equations**.

Once you have your equilibrium equations, you can solve them for unknowns using algebra. The number of unknowns that you will be able to solve for will be the number of equilibrium equations that you have. In instances where you have more unknowns than equations, the problem is known as a **statically indeterminate problem** and you will need additional information to solve for the given unknowns.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/Dbd9Svdfn8>. [Video Transcript](#)

? Example 2.5.1

The diagram below shows a 3-lb box (Box A) sitting on top of a 5-lb box (box B). Determine the magnitude and direction of all the forces acting on box B.

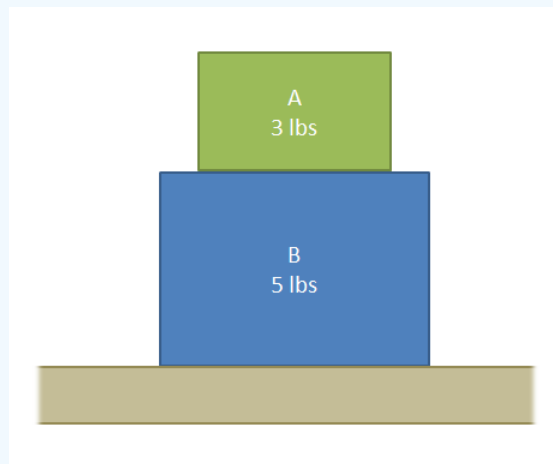


Figure \(\PageIndex{2}\): problem diagram for Example \(\PageIndex{1}\); two stacked boxes sitting on a flat surface.

Solution



Video 2.5.2: Worked solution to example problem 2.5.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/J54OZSitzzM>. [Video Transcript 2](#)

? Example 2.5.2

A 600-lb barrel rests in a trough as shown below. The barrel is supported by two normal forces (F_2 and F_3). Determine the magnitude of both of these normal forces.

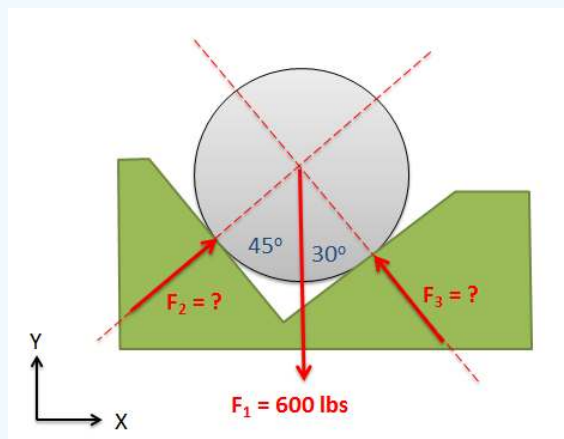


Figure 2.5.3: problem diagram for Example 2.5.2; a barrel resting in a trough with straight, angled sides.

Solution



Video 2.5.3: Worked solution to example problem 2.5.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/qKhZvf55Bc0>. [Video Transcript 3](#)

? Example 2.5.3

A 6-kg traffic light is supported by two cables as shown below. Find the tension in each of the cables supporting the traffic light.

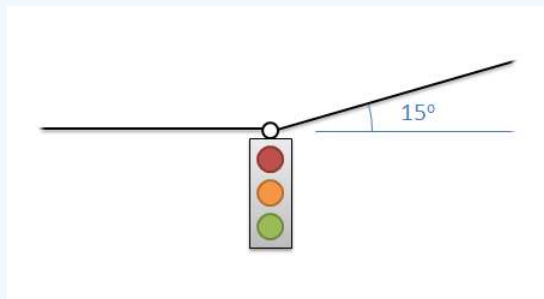


Figure 2.5.4: problem diagram for Example 2.5.3; a traffic light is held in midair by two cables, one horizontal and one angled..

Solution



Video 2.5.4: Worked solution to example problem 2.5.3, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/Oi2yDg1SmrI>. [Video Transcript 4](#)

? Example 2.5.4

A 400-kg wrecking ball rests against a surface as shown below. Assuming the wrecking ball is currently in equilibrium, determine the tension force in the cable supporting the wrecking ball and the normal force that exists between the wrecking ball and the surface.

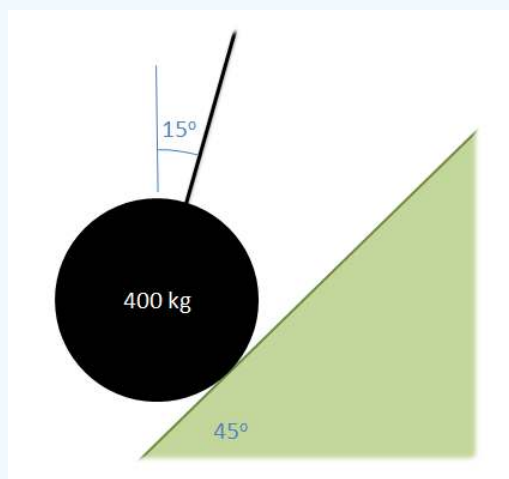


Figure 2.5.5: problem diagram for Example 2.5.4; a wrecking ball on a cable is resting against an angled surface.

Solution



Video 2.5.5: Worked solution to example problem 2.5.4, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/gETMTfy5Sew>. [Video Transcript 5](#)

? Example 2.5.5

Barrels A and B are supported in a foot truck as seen below. Assuming the barrels are in equilibrium, determine all forces acting on barrel B.

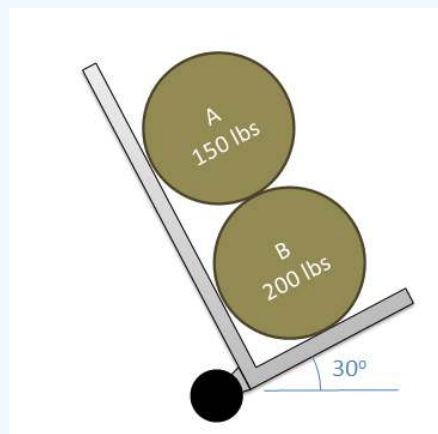


Figure 2.5.6: problem diagram for Example 2.5.5; two barrels stacked on their sides are on a handcart, whose bottom is tilted upwards.

Solution



Video 2.5.6: Worked solution to example problem 2.5.5, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/8DgrClhT4AM>. [Video Transcript 6](#)

? Example 2.5.6

Three soda cans, each weighing 0.75 lbs and having a diameter of 4 inches, are stacked in a formation as shown below. Assuming no friction forces, determine the normal forces acting on can B.

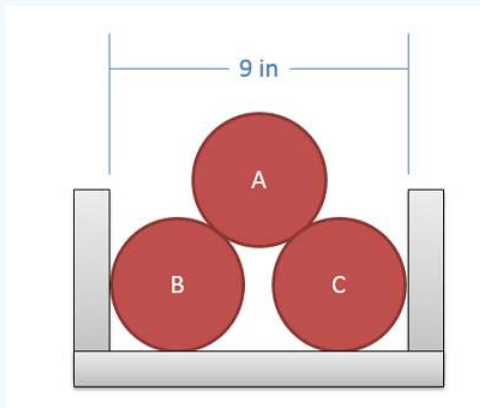
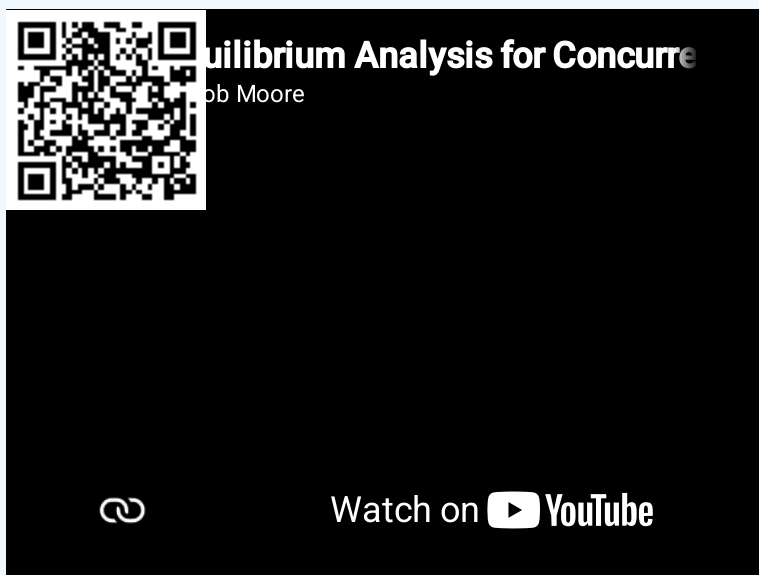


Figure 2.5.7: problem diagram for Example 2.5.6; three soda cans are stacked lying on their sides, in a flat area bounded on two sides by walls.

Solution



Video 2.5.7: Worked solution to example problem 2.5.6, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/1AUahV7Mml4>. [Video Transcript 7](#)

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2.6: Chapter 2 Homework Problems

? Exercise 2.6.1

A 30 kg barrel is sitting on a handcart as shown below. Determine the normal forces at A and B.

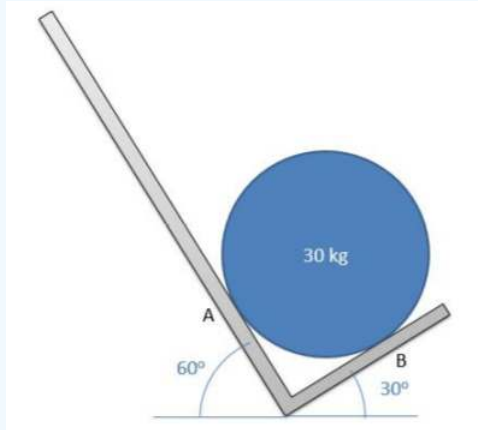


Figure 2.6.1: problem diagram for Exercise 2.6.1; a barrel sitting in a tilted handcart.

Answer

$$F_A = 147.15 \text{ N}; F_B = 254.87 \text{ N}.$$

? Exercise 2.6.2

A 0.25kg ball rolls into a corner as shown below. Assuming the surfaces are smooth (no friction), determine the normal forces at A and B.

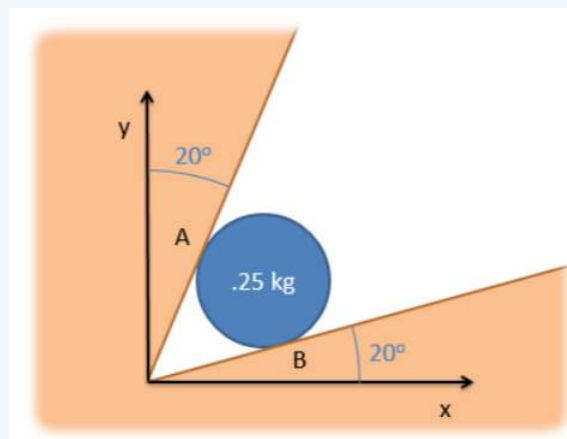


Figure 2.6.2: problem diagram for Exercise 2.6.2; a ball wedged into a narrow corner.

Answer

$$F_A = 1.09 \text{ N}; F_B = 3.01 \text{ N}$$

? Exercise 2.6.3

A traffic light is supported by two cables as shown below. The tension in cable one is measured to be 294.8 N. What is the tension in cable two? What is the mass of the traffic light?

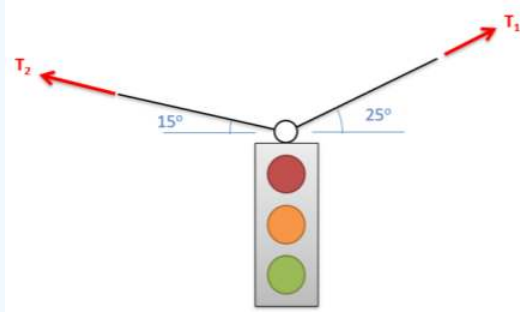


Figure 2.6.3: problem diagram for Exercise 2.6.3; a traffic light suspended by two angled cables.

Answer

$$T_2 = 276.6 \text{ N}; m = 20 \text{ kg}$$

? Exercise 2.6.4

A 50 kg truck engine is lifted using the setup shown below. Assuming that the pulleys shown in the diagram are frictionless, what force P must be applied to the cable to hold the engine in the position shown below with $d = 1$ meter? (Hint: Draw a free body diagram of the pulley supporting the engine block)

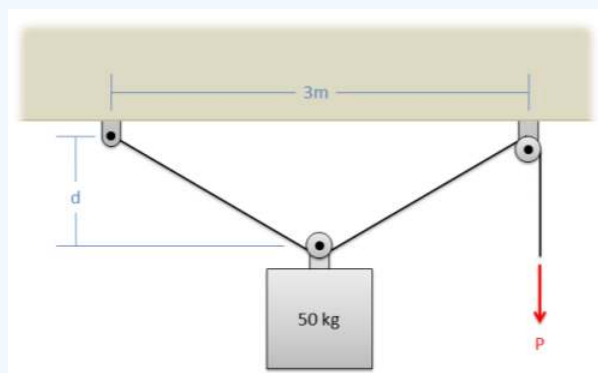


Figure 2.6.4: problem diagram for Exercise 2.6.4; an engine block suspended by a cable running through one anchor point and one pulley.

Answer

$$P = 442.1 \text{ N}$$

? Exercise 2.6.5

Two weights are supported via cables as shown below. If body B has a weight of 60 pounds, what is the expected weight of body A based on the angles of the cables?

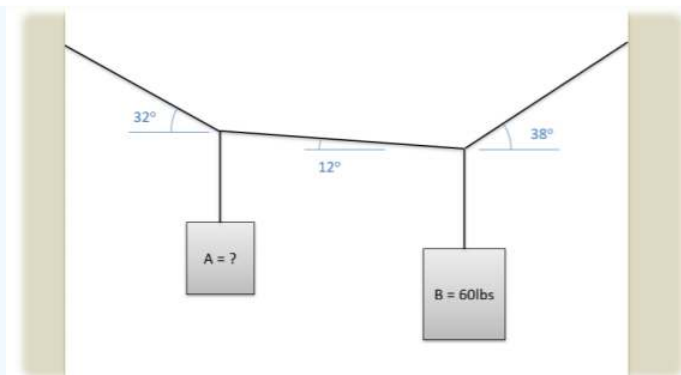


Figure 2.6.5: problem diagram for Exercise 2.6.5; two weights hanging from a single cable with fixed ends.

Answer

$$F_{gA} = 24.89 \text{ lbs}$$

? Exercise 2.6.6

Three equally sized cylinders, each with mass 100 kg, are stacked in a groove as shown below. Determine all forces acting on cylinder C and show them in a diagram.

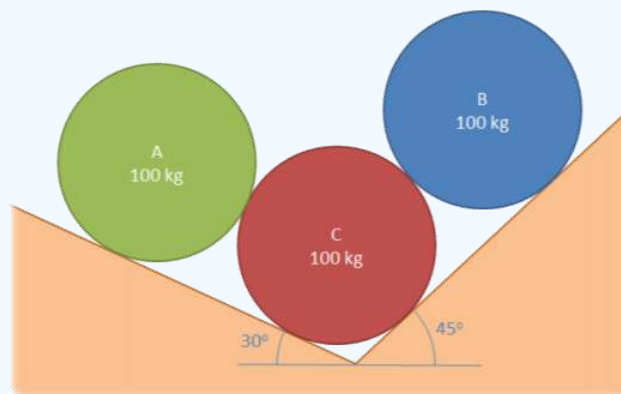


Figure 2.6.6: problem diagram for Exercise 2.6.6; three balls wedged in a groove with angled sides.

Answer

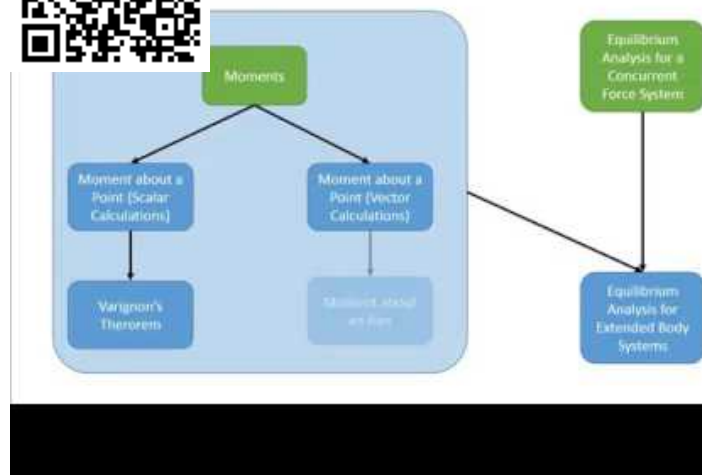
$$F_{AC} = 490.5 \text{ N}; F_{BC} = 693.7 \text{ N}; F_{C1} = 1304.6 \text{ N}; F_{C2} = 829.7 \text{ N}; F_g = 981 \text{ N}$$

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CHAPTER OVERVIEW

3: Static Equilibrium of Rigid Bodies



Video introduction to the topics covered in Chapter 3, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/x_L6S6ohu-k. [Video Transcript](#)

[3.1: Moment of a Force \(Scalar\)](#)

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3.1: Moment of a Force (Scalar)

The **moment of a force** is the tendency of some forces to cause rotation. Any easy way to visualize the concept is set a box on smooth surface. If you were to apply a force to the center of the box, it would simply slide across the surface without rotating. If you were instead to push on one side of the box, it will start rotating as it moves. Even though the forces have the same magnitude and the same direction, they cause different reactions. This is because the off-center force has a different point of application, and exerts a moment about the center of the box, whereas the force on the center of the box does not exert a moment about the box's center point.

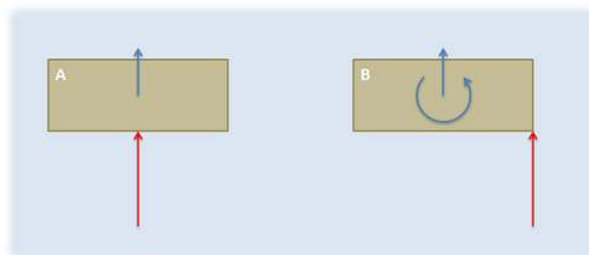


Figure 3.1.1: If we push a box in the center, it will simply begin sliding. If we push a box off-center, we will exert a moment and the box will rotate in addition to sliding.

Just like forces, moments have a magnitude (the degree of rotation it would cause) and a direction (the axis the body would rotate about). Determining the magnitude and direction of these moments about a given point is an important step in the analysis of rigid body systems (bodies that are both rigid and not experiencing concurrent forces). The scalar method below is the easiest way to do this in simple two-dimensional problems, while the alternative vector methods, which will be covered later, work best for more complex three-dimensional systems.

The Scalar Method in 2 Dimensions

In discussing how to calculate the moment of a force about a point via scalar quantities, we will begin with the example of a force on a simple lever as shown below. In this simple lever there is a force on the end of the lever, distance d away from the center of rotation for the lever (point A) where the force has a magnitude F .

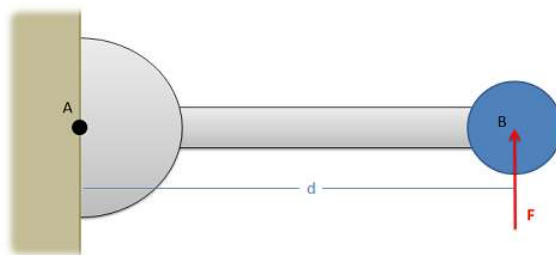


Figure 3.1.2: The magnitude of the moment that force F exerts about point A on this lever will be equal to the magnitude of the force times distance d .

When using scalar quantities, the magnitude of the moment will be equal to the perpendicular distance between the line of action of the force and the point we are taking the moment about.

$$M = F * d \quad (3.1.1)$$

To determine the sign of the moment, we determine what type of rotation the force would cause. In this case, we can see that the force would cause the lever to rotate counterclockwise about point A. Counterclockwise rotations are caused by positive moments while clockwise rotations are caused by negative moments.

Another important factor to remember is that the value d is the perpendicular distance from the force to the point we are taking the moment about. We could measure the distance from point A to the head of the force vector, or the tail of the force vector, or really any point along the line of action of force F . The distance we need to use for the scalar moment calculation, however, is the

shortest distance between the point and the line of action of the force. This will always be a line perpendicular to the line of action of the force, going to the point about which we are taking the moment.

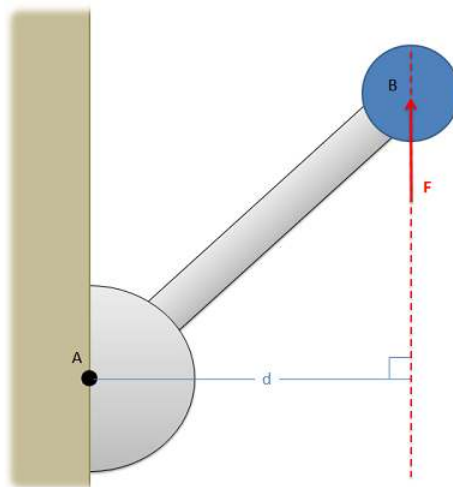


Figure 3.1.3: Distance d always needs to be the shortest length between the line of action of the force and the point we are taking the moment about. This distance will be perpendicular to the line of action of the force.

The Scalar Model in 3 Dimensions

For three-dimensional scalar calculations, we will still find the magnitude of the moment in the same way, multiplying the magnitude of the force by the perpendicular distance between the point and the line of action of the force. This perpendicular distance again is the minimum distance between the point and the line of action of the force. In some cases, finding this distance may be very difficult.

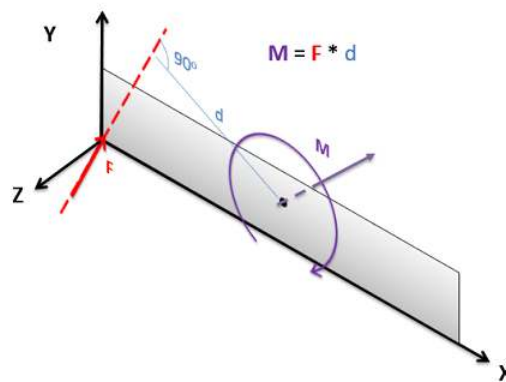


Figure 3.1.4: For moments in three dimensions, the moment vector will always be perpendicular to both the force vector $\vec{F}$ and the distance vector $\vec{d}$.

Another difficult factor in three dimensional scalar problems is finding the axis of rotation, as this is now more complex than just "clockwise or counterclockwise". The axis of rotation will be a line traveling through the point about which we are taking the moment, and perpendicular to both the force vector and the perpendicular displacement vector (the vector going from the point about which the moment was taken to the point of application of the force). While this is possible in any situation, it becomes very difficult if the force or displacement vectors do not lie in one of the three coordinate directions.

To further find the direction of the moment vector (which will act along the established line for axis of rotation), we will use the right-hand rule in a modified form. Wrap the fingers of your right hand around the axis of rotation line with your fingertips curling in the direction the body would rotate. If you do this, your thumb should point out along the line in the direction of the moment vector. This is an important last step, because we can rotate clockwise or counterclockwise in about any given axis of rotation. With the final moment vector, we know not only the axis of rotation, but which way the body would rotate about that axis.

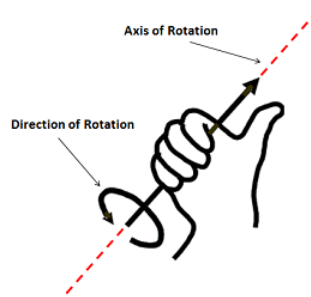
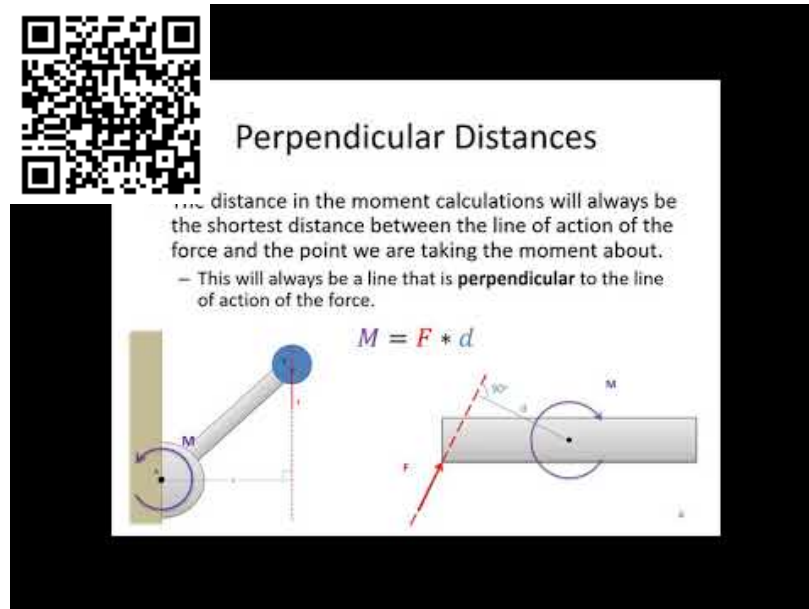


Figure 3.1.5: To use the right-hand rule, align your right hand as shown so that your thumb lines up with the axis of rotation for the moment and your curled fingers point in the direction of rotation for your moment. If you do this, your thumb will be pointing in the direction of the moment vector. Adapted from Public Domain image by Schorschi2.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/15h8bIQDjGE>. [Video Transcript](#)

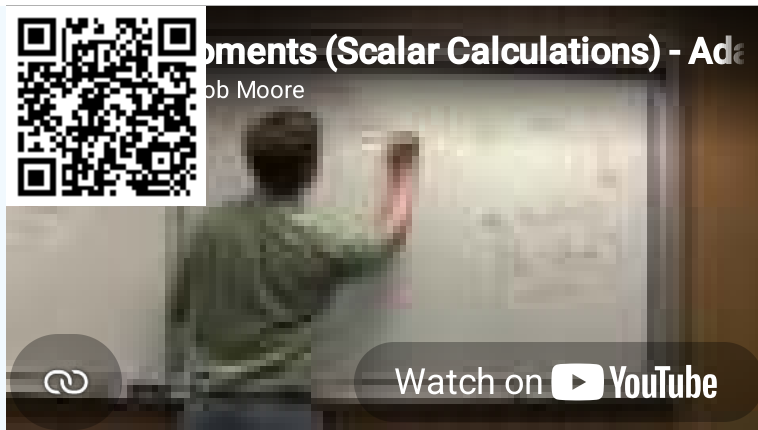
? Example 3.1.1

What is the moment that Force A exerts about point A? What is the moment that Force B exerts about Point A?



Figure 3.1.6: problem diagram for Example 3.1.1; a lever is attached to a wall with two forces exerted upon the lever's free end.

Solution



Video 3.1.7: Worked solution to example problem 3.1.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/E9Xq1fXdcyE>. [Video Transcript 2](#)

? Example 3.1.2

What is the moment that this force exerts about point A? What is the moment this force exerts about point B?

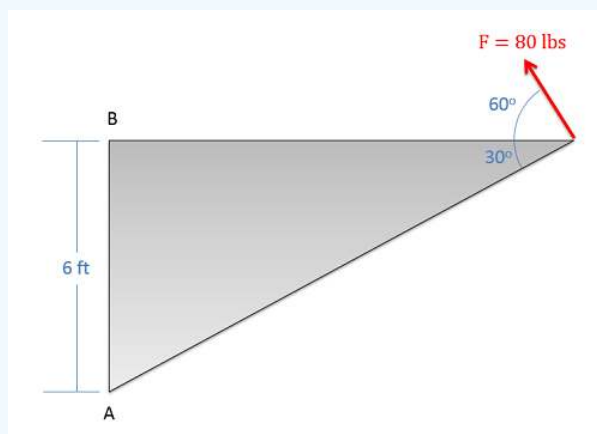
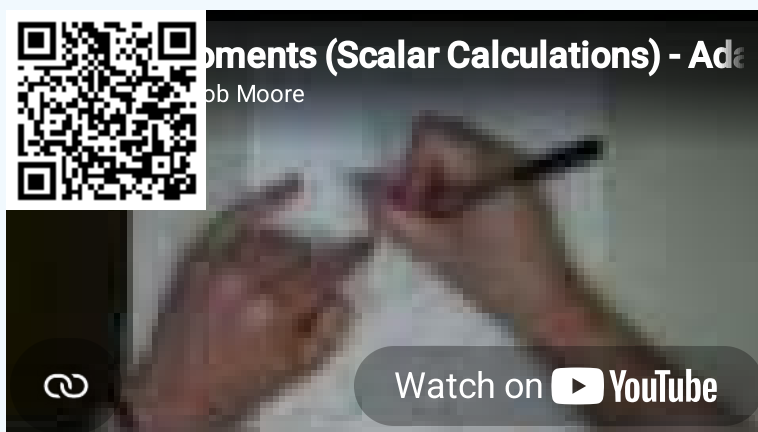


Figure 3.1.2: problem diagram for Example 3.1.2; a force is exerted upon one corner of a right triangle.

Solution



Video 3.1.3: Worked solution to example problem 3.1.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/9nb2q3EN5gs>. [Video Transcript 3](#)

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3.2: Varignon's Theorem

Varignon's Theorem, also often called the **principle of moments**, is a very useful tool in scalar moment calculations. In cases where the perpendicular distance is hard to determine, Varignon's Theorem offers an alternative to finding that distance.

In its basic form, Varignon's Theorem states that if we have two or more **concurrent forces**, the sum of the moments that each force creates about a single point will be equal to the moment created by the sum of those forces about the same point.

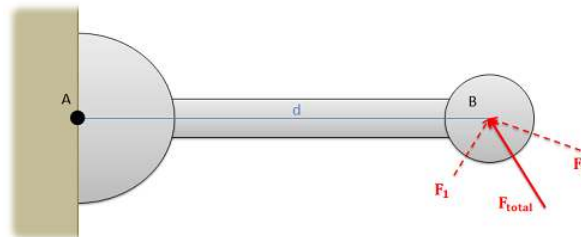


Figure 3.2.1: If the sum of $\vec{F}_1$ and $\vec{F}_2$ is $\vec{F}_{total}$, then we can assume that the sum of the moments about point A exerted by $\vec{F}_1$ and $\vec{F}_2$ will be equal to the moment exerted about point A by $\vec{F}_{total}$.

On its surface this doesn't seem that useful, but in practice we will often use this theorem in reverse by breaking down a force into components (the components being a set of concurrent forces). We can solve for the moment exerted by each component (where perpendicular distance d is easier to find) and then simply add together the moments from each component to find the moment from the original force.

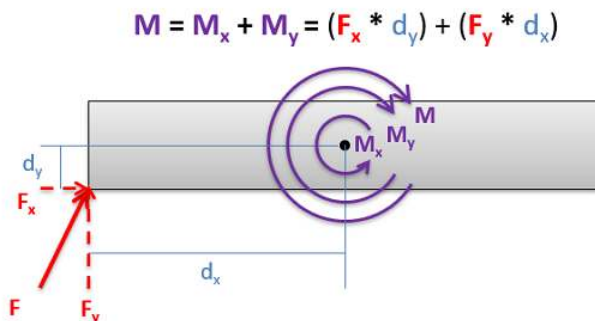


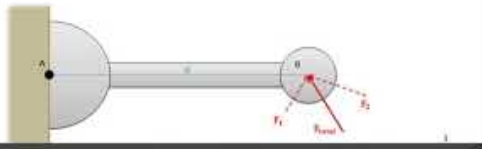
Figure 3.2.2: When finding the moment of force $\vec{F}$ about the center point, it will be easier to break down the force into components and find the moments of each component rather than trying to find the perpendicular distance directly using complex geometric relationships.



Varignon's Theorem

In the case below, $F_1 + F_2 = F_{\text{total}}$, and we are taking the moment about point A.

- We have two alternate paths to find the overall moment about point A.
 - Add F_1 and F_2 to find F_{total} , then find the moment F_{total} exerts about point A.
 - Find the moment from F_1 , the moment from F_2 , and add those two moments together.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/XcxXyPv7Wp4>. [Video Transcript](#)

? Example 3.2.1

Use Varignon's Theorem to find the moment that the forces in the diagram below exert about point A.

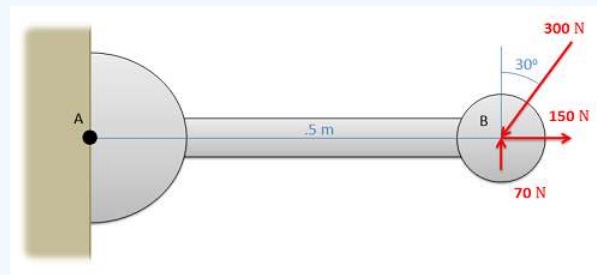


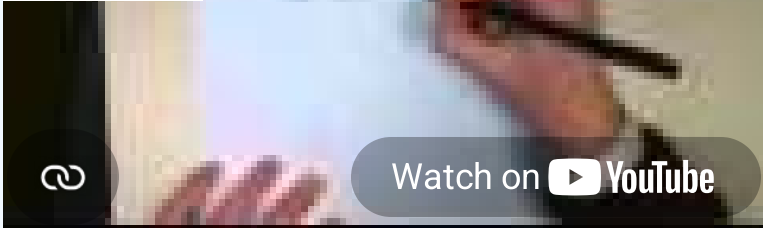
Figure 3.2.3: problem diagram for Example 3.2.1. A lever is attached to a wall at one end, and 3 different forces are applied to the lever's free end, 0.5 m from the base's point of contact with the wall (point A).

Solution



Varignon's Theorem - Adaptive Map

Jacob Moore



Watch on  YouTube

Video 3.2.2: Worked solution to example problem 3.2.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/JlpgZGxJWu4>. [Video Transcript 2](#)

? Example 3.2.2

Use Varignon's Theorem to find the moment that the force in the diagram below exerts about point B.

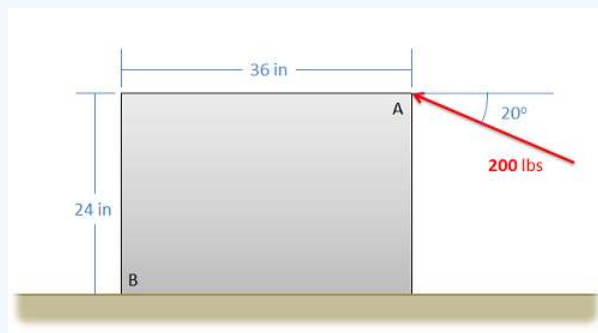
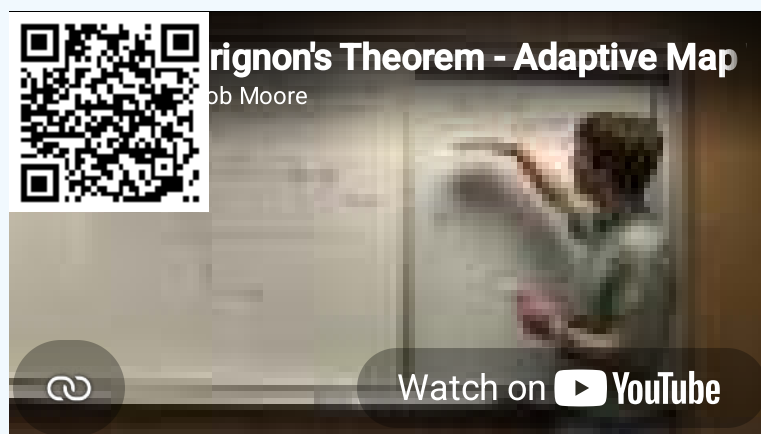


Figure 3.2.4: problem diagram for Example 3.2.2. A rectangular box, which can be considered a two-dimensional shape, sits on a flat surface and experiences a force applied to its upper right corner.

Solution



Video 3.2.3: Worked solution to example problem 3.2.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/jpaLEprFndA>. [Video Transcript 3](#)

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3.3: Couples

A **couple** is a set of equal and opposite forces that exerts a net moment on an object but no net force. Because the couple exerts a net moment without exerting a net force, couples are also sometimes called **pure moments**.



Figure 3.3.1: The two equal and opposite forces exerted on this lug wrench are a couple. They exert a moment on the lug nut on this wheel without exerting any net force on the wheel. Adapted from image by Steffen Heinz Caronna CC-BY-SA 3.0.

The moment exerted by a couple also differs from the moment exerted by a single force in that it is independent of the location you are taking the moment about. In the example below we have a couple acting on a beam. Each force has a magnitude F and the distance between the two forces is d .

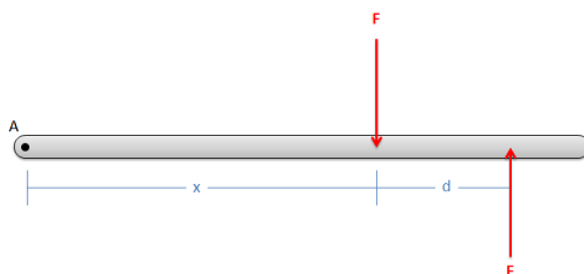


Figure 3.3.2: The moment exerted by this couple is independent of the of the distance x .

Now we have some point A, which is distance x from the first of the two forces. If we take the moment of each force about point A, and then add these moments together for the net moment about A we are left with the following formula.

$$M = -(F * x) + (F * (x + d)) \quad (3.3.1)$$

If we rearrange and simplify the formula above, we can see that the variable x actually disappears from the equation, leaving the net moment equal to the magnitude of the forces (F) times the distance between the two forces (d).

$$M = -(F * x) + (F * x) + (F * d) \quad (3.3.2)$$

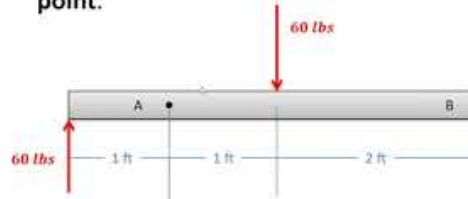
$$M = (F * d) \quad (3.3.3)$$

This means that no matter what value of x we have, the magnitude of the moment exerted by the couple will be the same. The magnitude of the moment due to the couple is independent of the location we are taking the moment about. This will work in two or three dimensions as well. The magnitude of the moment due to a couple will always be equal to the magnitude of the forces times the perpendicular distance between the two forces.



Couples

- in addition to exerting no net force, couples also have another interesting characteristic, **they will exert the same moment about every point.**



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/2U4APUz__Gk. [Video Transcript](#)

? Example 3.3.1

What is the moment that the couple below exerts about point A?

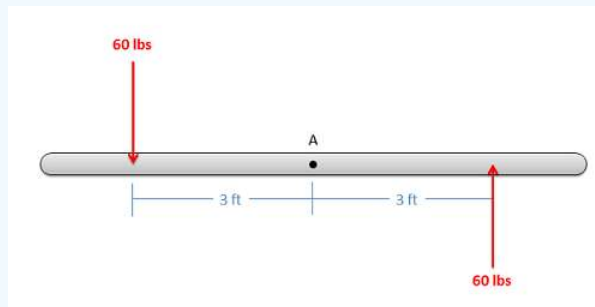


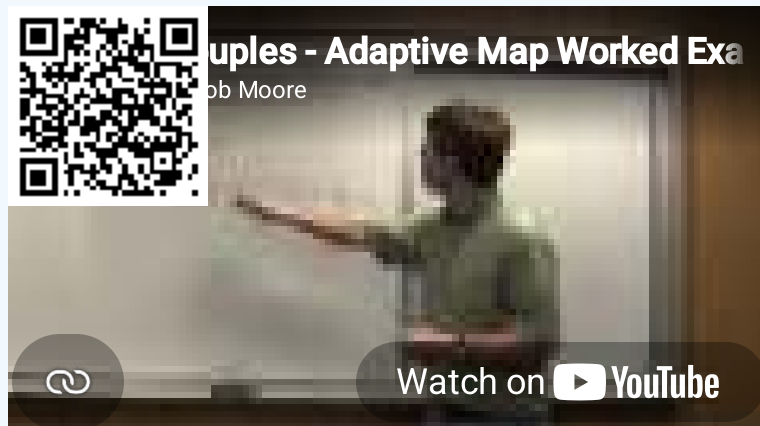
Figure 3.3.3: problem diagram for Example 3.3.1. A couple is applied to a rod, with each force being equidistant from point A on the rod.

Solution



Couples - Adaptive Map Worked Exa

ob Moore



Video 3.3.2: Worked solution to example problem 3.3.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/rfD-b6V5qNY>. [Video Transcript 2](#)

? Example 3.3.2

What is the moment that the couple below exerts about point A?

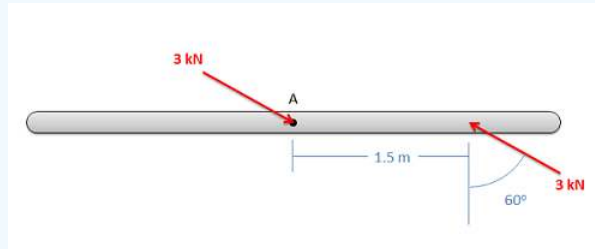
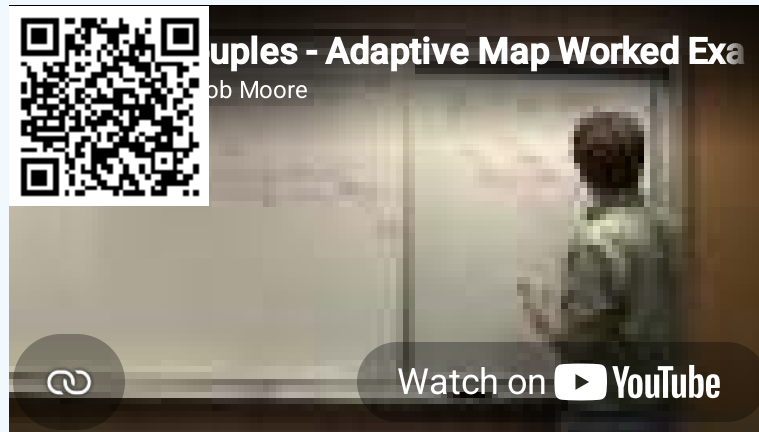


Figure 3.3.4: problem diagram for Example 3.3.2. A couple is applied to a rod, with one force applied directly at point A and the other being applied 1.5 meters to the right of A.

Solution



Video 3.3.3: Worked solution to example problem 3.3.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/s3-RJwKkH3Y>. [Video Transcript 3](#)

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3.4: Equilibrium Analysis for a Rigid Body

For an **rigid body** in static equilibrium—that is, a non-deformable body where forces are not concurrent—the sum of both the **forces** and the **moments** acting on the body must be equal to zero. The addition of moments (as opposed to particles, where we only looked at the forces) adds another set of possible equilibrium equations, allowing us to solve for more unknowns as compared to particle problems.

Moments, like forces, are vectors. This means that our vector equation needs to be broken down into scalar components before we can solve the equilibrium equations. In a two-dimensional problem, the body can only have clockwise or counterclockwise rotation (corresponding to rotations about the z axis). This means that a rigid body in a two-dimensional problem has three possible equilibrium equations; that is, the sum of force components in the x and y directions, and the moments about the z axis. The sum of each of these will be equal to zero.

For a two-dimensional problem, we break our one vector force equation into two scalar component equations.

$$\sum \vec{F} = 0 \quad (3.4.1)$$

$$\sum F_x = 0; \sum F_y = 0 \quad (3.4.2)$$

The one moment vector equation becomes a single moment scalar equation.

$$\sum \vec{M} = 0 \quad (3.4.3)$$

$$\sum M_z = 0 \quad (3.4.4)$$

Finding the Equilibrium Equations

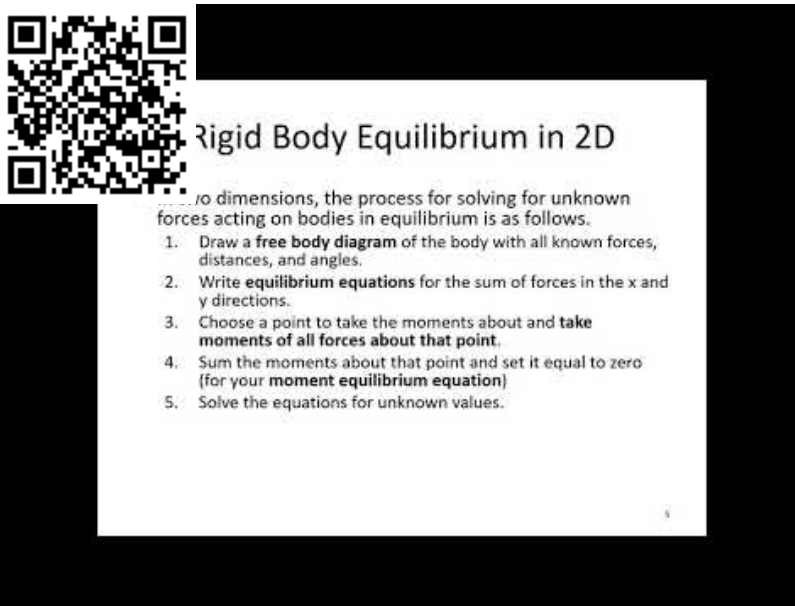
As with particles, the first step in finding the equilibrium equations is to draw a free body diagram of the body being analyzed. This diagram should show all the force vectors acting on the body. In the free body diagram, provide values for any of the known magnitudes, directions, and points of application for the force vectors and provide variable names for any unknowns (either magnitudes, directions, or distances).

Next you will need to choose the x , y , and z axes. These axes do need to be perpendicular to one another, but they do not necessarily have to be horizontal or vertical. If you choose coordinate axes that line up with some of your force vectors you will simplify later analysis.

Once you have chosen axes, you need to break down all of the force vectors into components along the x , y and z directions (see the vectors page in Appendix 1 page for more details on this process). Your first equation will be the sum of the magnitudes of the components in the x direction being equal to zero, the second equation will be the sum of the magnitudes of the components in the y direction being equal to zero, and the third (if you have a 3D problem) will be the sum of the magnitudes in the z direction being equal to zero.

Next you will need to come up with the the moment equations. To do this you will need to choose a point to take the moments about. Any point should work, but it is usually advantageous to choose a point that will decrease the number of unknowns in the equation. Remember that any force vector that travels through a given point will exert no moment about that point. To write out the moment equations, simply sum the moments exerted by each force (adding in pure moments shown in the diagram) about the given point and the given axis, and set that sum equal to zero. All moments will be about the z axis for two-dimensional problems.

Once you have your equilibrium equations, you can solve these formulas for unknowns. The number of unknowns that you will be able to solve for will again be the number of equations that you have.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/OiJ2xbMIxY>. [Video Transcript](#)

? Example 3.4.1

The car below has a weight of 1500 lbs with the center of mass 4 ft behind the front wheels of the car. What are the normal forces on the front and the back wheels of the car?

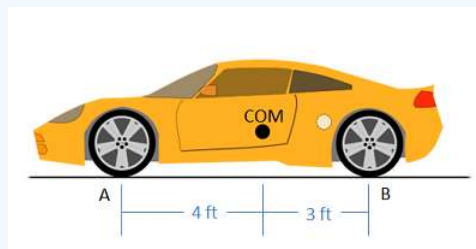


Figure 3.4.1: problem diagram for Example 3.4.1. Adapted from the public domain image by Ebaychatter0.

Solution



Video 3.4.2: Worked solution to example problem 3.4.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/1LD5QW-70PA>. [Video Transcript 2](#)

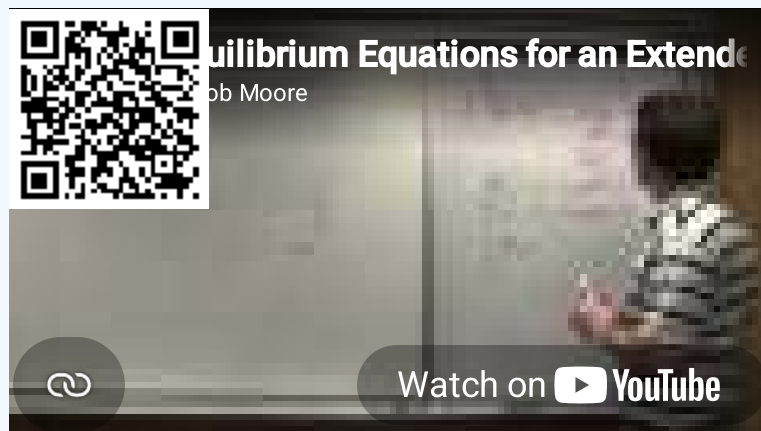
? Example 3.4.2

A 5-meter-long beam has a fixed connection to a wall at point A and a force acting as shown at point B. What are the reaction forces acting on the beam at point A?



Figure 3.4.2: problem diagram for Example 3.4.2. A horizontal beam attached to the wall at one end experiences a force applied at its free end.

Solution



Video 3.4.3: Worked solution to example problem 3.4.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/JrVV7k1aQEk>. [Video Transcript 3](#)

? Example 3.4.3

A ladder with negligible mass is supporting a 120-lb person as shown below. If the contact point at A is frictionless, and the contact point at B is a rough connection, determine the forces acting at contact points A and B.

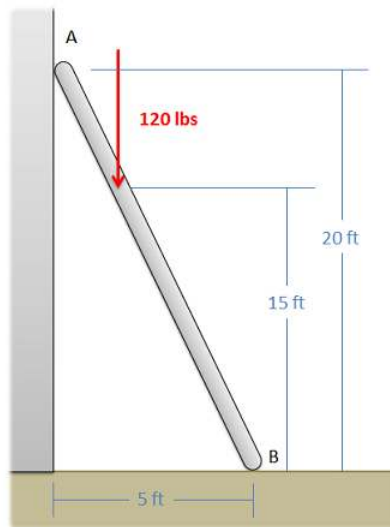
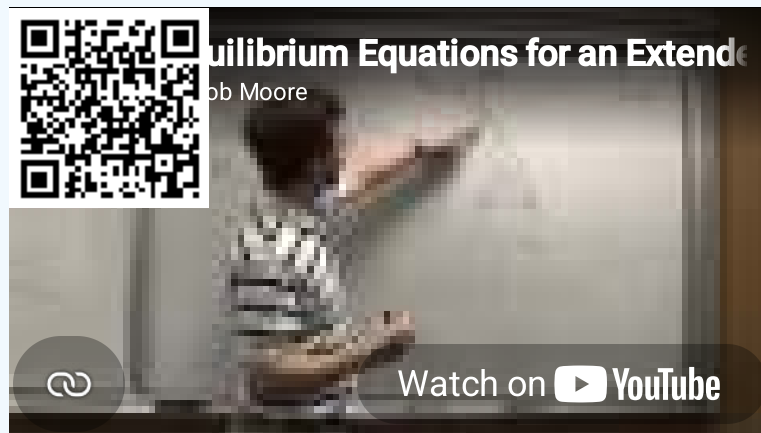


Figure 3.4.3: problem diagram for Example 3.4.3. A ladder with its base on a rough floor leans against a frictionless wall, with a person standing on the ladder partway up.

Solution



Video 3.4.4: Worked solution to example problem 3.4.3, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/WzkAnPdha04>. [Video Transcript 4](#)

? Example 3.4.4

Member ABC is 6 meters long, with point B being at its midpoint. Determine all forces acting on member ABC.

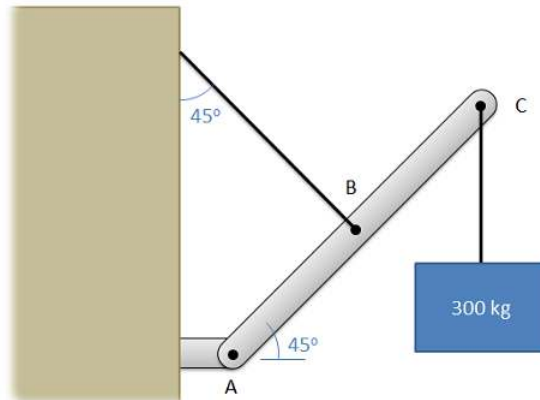
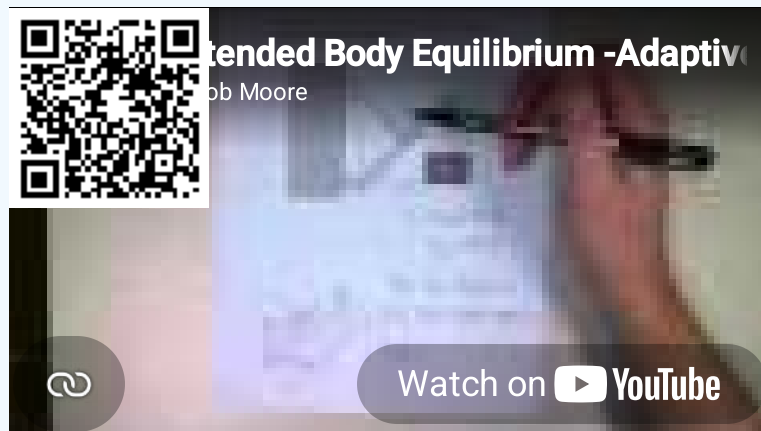


Figure 3.4.4: problem diagram for Example 3.4.4. A diagonal structural member is attached to the wall at one end, connected to the wall via cable at its midpoint, and holding up a 300-kg load at its free end.

Solution



Video 3.4.5: Worked solution to example problem 3.4.4, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/sMQrjwUMpSQ>. [Video Transcript 5](#)

? Example 3.4.5

While sitting in a chair, a person exerts the forces in the diagram below. Determine all forces acting on the chair at points A and B. (Assume A is frictionless and B is a rough surface).

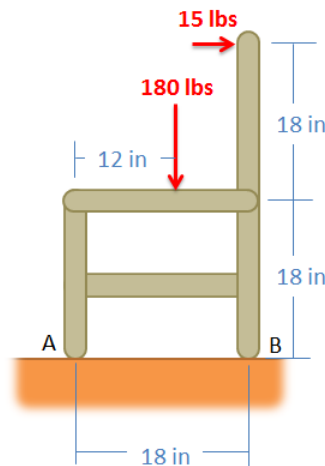
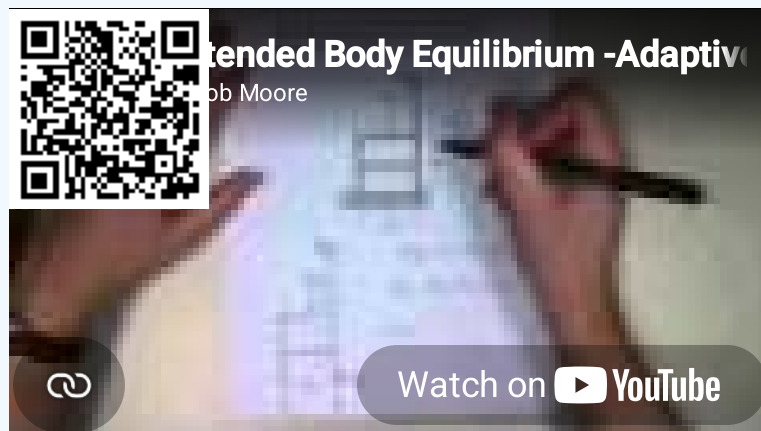


Figure 3.4.5: problem diagram for Example 3.4.5. A chair is placed on a flat surface, with that surface assumed to be frictionless where it contacts the chair's front leg (point A) and exerting friction where it contacts the back leg (point B).

Solution



Video 3.4.6: Worked solution to example problem 3.4.5, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/nSOxK1ZMggA>. [Video Transcript 6](#)

? Example 3.4.6

The trailer shown below consists of a deck with a weight of 250 lbs on an axle with wheels with a weight of 350 lbs. Assume the weight forces act in the center of each component. If we wish the tongue weight (F_T) of the unloaded trailer to be 50 lbs, what is the distance d from the front where we must place the axle?

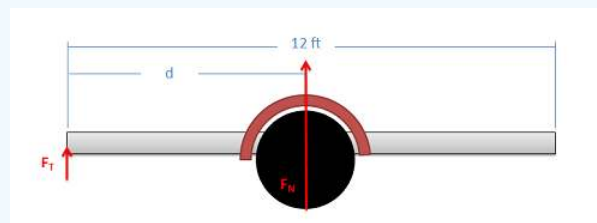
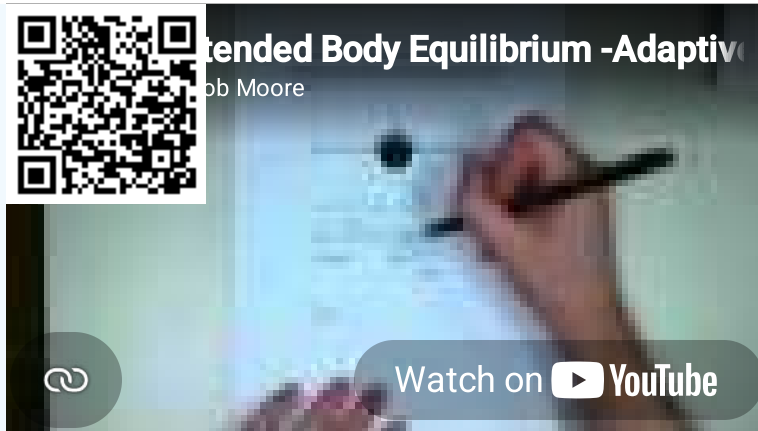


Figure 3.4.6: problem diagram for Example 3.4.6. A trailer consists of a flat rectangular deck on top of two wheels on an axle.

Solution



Video 3.4.7: Worked solution to example problem 3.4.6, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/wpEBuitLD5s>. [Video Transcript 7](#)

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3.5: Chapter 3 Homework Problems

? Exercise 3.5.1

An 18-inch shelf is supported by a pin joint at point A, and a cable at point B. The shelf itself has a weight of 60 lbs. If we want the net moment about point A to be zero, what should the tension in the cable be?

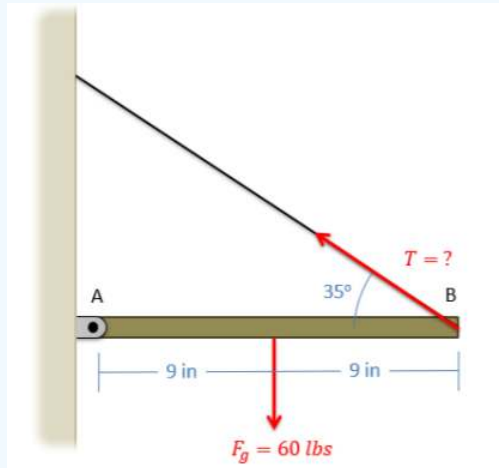


Figure 3.5.1: problem diagram for Exercise 3.5.1; a horizontal shelf is attached to a wall with a pin joint at one end and a cable at the other end.

Solution

$$T = 52.30 \text{ lbs}$$

? Exercise 3.5.2

What is the moment the force shown below exerts about Point A? About Point B?

Hint: use Varignon's Theorem.

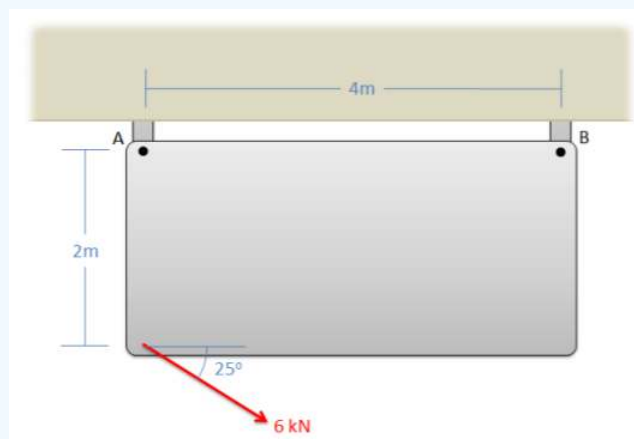


Figure 3.5.2: problem diagram for Exercises 3.5.2; a rectangular slab is bolted to a wall at two points, A and B, with a force is exerted on one of the rectangle's unattached corners.

Solution

$$M_A = 10.88 \text{ kNm}$$

$$M_B = 21.02 \text{ kNm}$$

? Exercise 3.5.3

You are attempting to rotate a heavy table about the base of one leg at point O and are going to exert a 100-lb force at the opposite end. Person A recommends pulling straight up, while person B recommends pulling up at 30° from vertical. What would be the moment about point O, in inch-pounds, in either case?

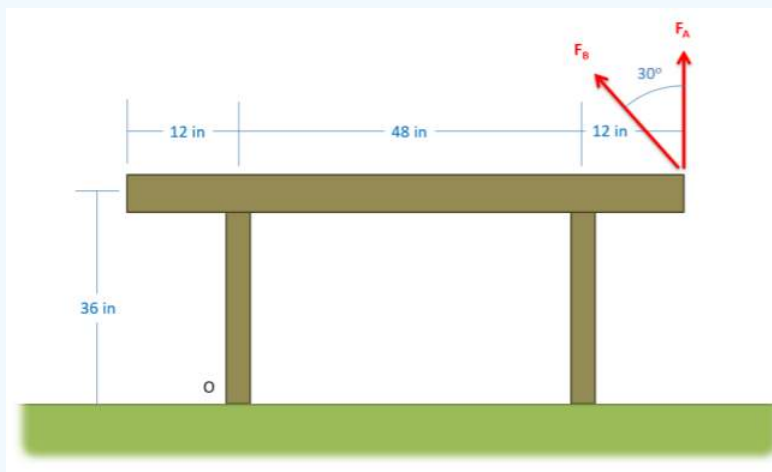


Figure 3.5.3: problem diagram for Exercise 3.5.3; a table is rotated about the base of one leg, with one of the two proposed forces applied at the tabletop at the opposite end.

Solution

$$M_{AO} = 6000 \text{ in-lbs}$$

$$M_{BO} = 6996.15 \text{ in-lbs}$$

? Exercise 3.5.4

A space station consists of a large ring that spins in order to provide an artificial gravity for the astronauts in the station. To start the station spinning, a pair of thrusters is attached to the outside of the ring, each pointing in opposite directions as shown below.

- If we want to exert a 10 kN-m moment with the thrusters, and the ring has a diameter of 45 meters, what thrust force should each thruster produce?
- If we were to use the same thrusters on a 60-meter diameter ring, what moment would they exert?

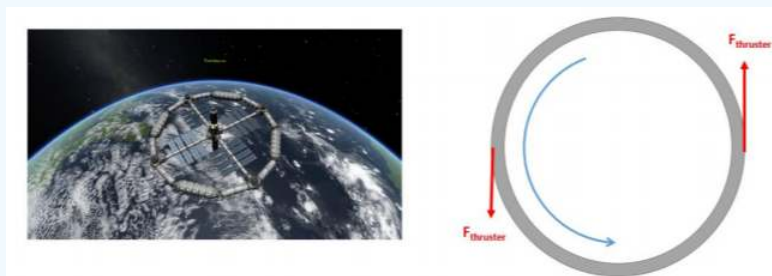


Figure 3.5.4: problem diagram for Exercise 3.5.4. An example of the spinning ring-shaped space station as described in the problem (left); a diagram of the locations and directions of the thruster forces on the ring (right).

Solution

$$\text{a) } F_{thruster} = 222.22 \text{ N}$$

$$\text{b) } M_{thruster} = 13.33 \text{ kNm}$$

? Exercise 3.5.5

The diving board shown below is supported by a pin joint at A and frictionless support at B. A 150-lb diver is standing at the end of the board. Determine the reaction forces acting on the diving board at points A and B.

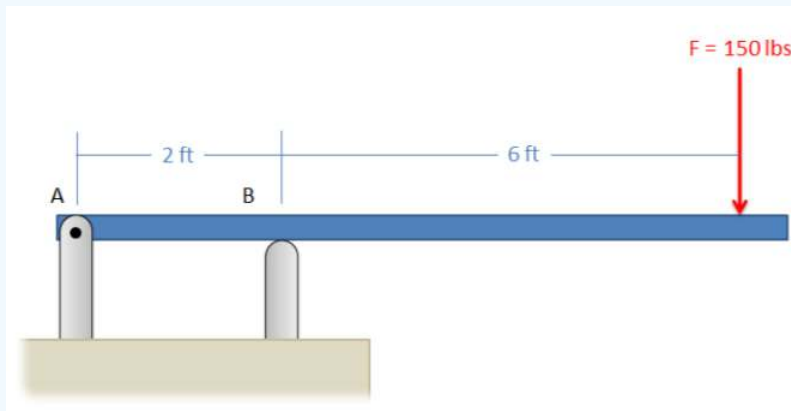


Figure 3.5.5: problem diagram for Exercise 3.5.5; a diving board is supported by a pin joint at its leftmost end (point A) and a frictionless support two feet to the right (point B), with a 150-lb diver standing 6 feet to the right of B.

Solution

$$F_{AX} = 0$$

$$F_{AY} = -450 \text{ lbs}$$

$$F_{BY} = 600 \text{ lbs}$$

? Exercise 3.5.6

A simplified crane is shown lifting a 400-kg load. The crane is supported by a pin joint at A, and a cable at B. Assuming the crane arm is in equilibrium, what are the reaction forces at A and the tension at B?

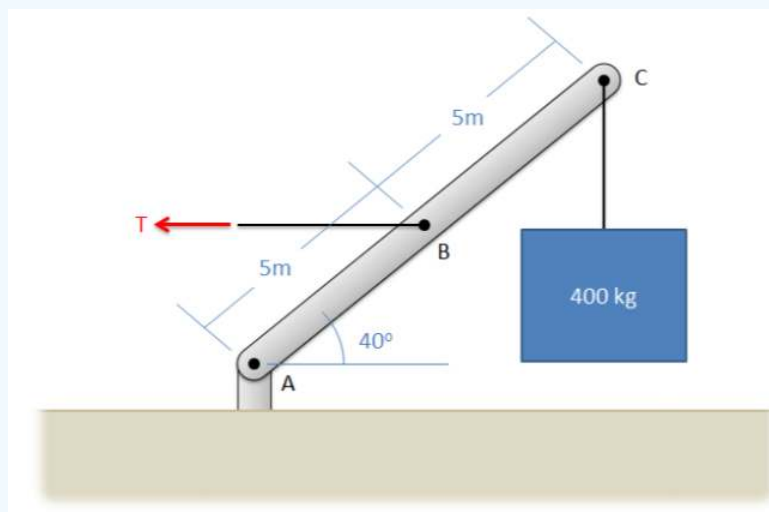


Figure 3.5.6: problem diagram for Exercise 3.5.6; a crane arm is attached to a vertical support at one end (A), is connected to a horizontal cable at its midpoint (B), and lifts a 400-kg load at the other end.

Solution

$$F_{AX} = 9352.9 \text{ N}$$

$$F_{AY} = 3924 \text{ N}$$

$$T_B = 9352.9 \text{ N}$$

? Exercise 3.5.7

An 8-foot ladder sits propped up against a wall at a 60-degree angle as shown below. It has a weight of 50 lbs acting at its center point and supports a 120-lb woman 6 feet from the bottom. Assume that friction acts at the bottom of the ladder, but not the top. What are the normal forces acting at the bottom and top of the ladder, and what is the friction force acting at the bottom of the ladder?

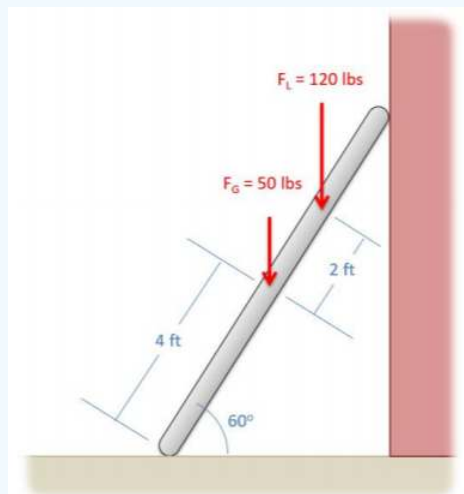


Figure 3.5.7: problem diagram for Exercise 3.5.7; an 8-ft ladder weighing 50 lbs and supporting a 120-lb woman who has climbed 75% of the way up leans, at 60° above the horizontal, against a wall.

Solution

$$F_{N \text{ Top}} = 66.4 \text{ lbs}$$

$$F_{N \text{ Bottom}} = 170 \text{ lbs}$$

$$F_f = 66.4 \text{ lbs}$$

? Exercise 3.5.8

An SUV with a weight of 4200 lbs and a center of mass located as shown below is parked pointed downhill on a 10-degree incline. The parking is engaged, locking up the back wheels but not the front wheels. What is the expected normal force at the front wheels, the expected normal force at the back wheels, and the expected friction force at the back wheels assuming the SUV does not slip?

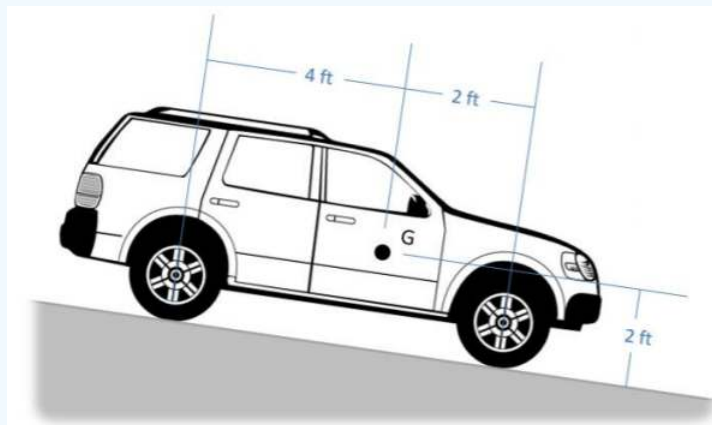


Figure 3.5.8: problem diagram for Exercise 3.5.8; an SUV with front and rear wheels 6 feet apart, and a center of mass 2 feet behind the front wheel, is parked pointing downhill on a 10° incline.

Solution

$$F_f = 729.3 \text{ lbs}$$

$$F_{N \text{ front}} = 3000.6 \text{ lbs}$$

$$F_{N \text{ back}} = 1135.6 \text{ lbs}$$

? Exercise 3.5.9

A cart with a mass of 3500 kg sits on an inclined surface as shown below. Determine the reaction forces acting on each wheel of the cart as well as the tension in the cable supporting the cart.

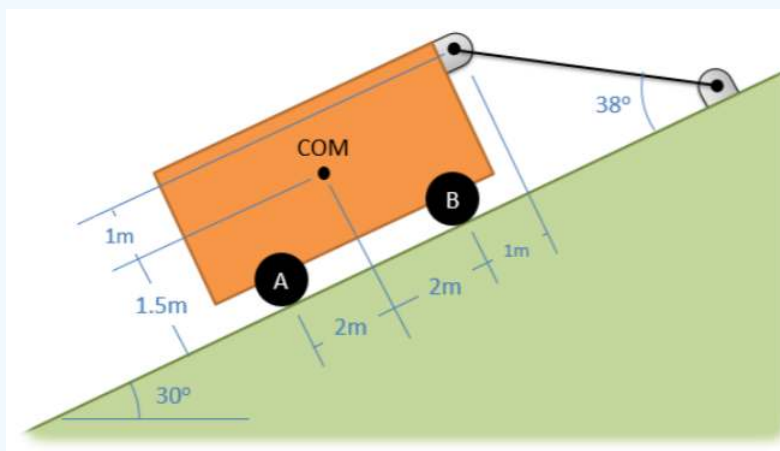


Figure 3.5.9: a two-wheeled cart is parked facing uphill on a 30° slope. A cable stretches from the front of the cart to a support on the incline, making a 38° angle with the plane of the incline.

Solution

$$T = 21786 \text{ N}$$

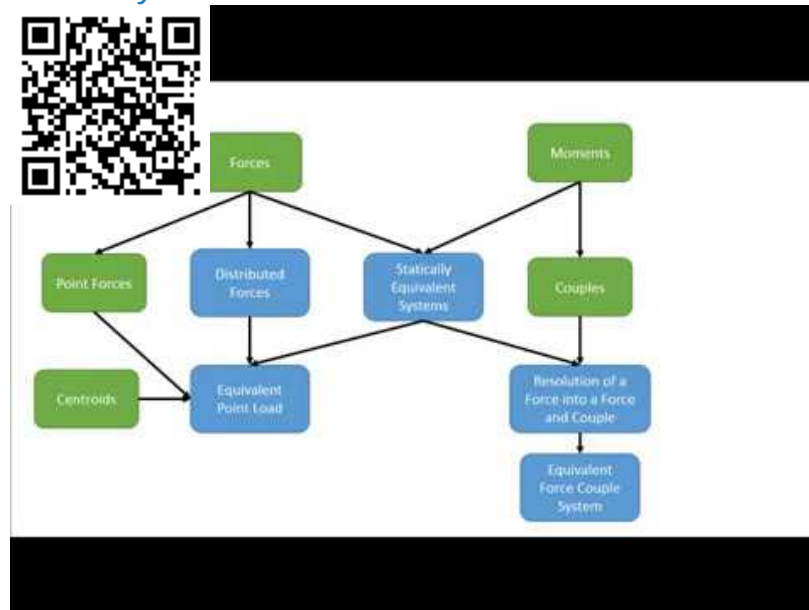
$$F_A = 7222 \text{ N}; F_B = 35925 \text{ N}$$

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CHAPTER OVERVIEW

4: Statically Equivalent Systems



Video introduction to the topics covered in Chapter 4, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/jXY48L2sRR0>. [Video Transcript](#)

- [4.1: Statically Equivalent Systems](#)
- [4.2: Resolution of a Force into a Force and a Couple](#)
- [4.3: Equivalent Force Couple System](#)
- [4.4: Distributed Forces](#)
- [4.5: Equivalent Point Load](#)
- [4.6: Centroids and Centers of Mass via Method of Composite Parts](#)
- [4.7: Centroids and Area Moments of Inertia for 2D Shapes](#)
- [4.8: Center of Mass and Mass Moments of Inertia for Homogeneous 3D Bodies](#)
- [4.9: Chapter 4 Homework Problems](#)

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4.1: Statically Equivalent Systems

Two sets of forces are considered **statically equivalent** if they **cause the same set of reaction forces** on a body. Because this is true, the two statically equivalent sets of forces are **interchangeable** in statics analysis.

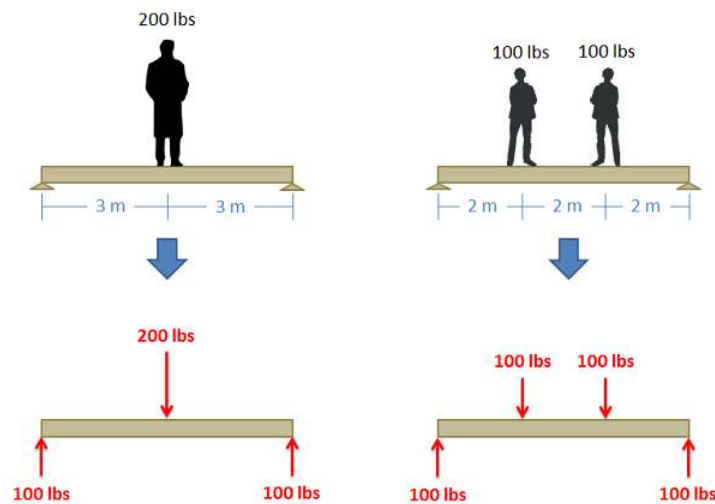


Figure 4.1.1: A single 200-lb man standing at the center of a beam causes the same reaction forces as two 100-lb children standing evenly along the beam as shown above. The weight force of the man is therefore statically equivalent to the weight forces of the two children.

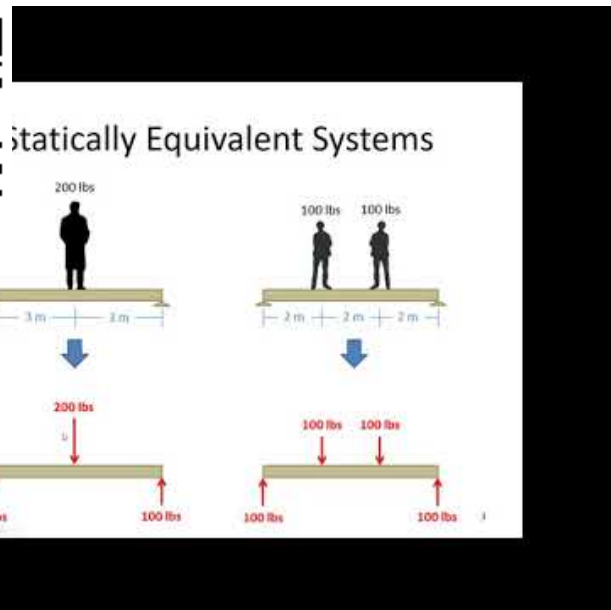
Determining if Forces are Statically Equivalent:

To determine if two sets of forces are statically equivalent, you must solve for the reaction forces in both cases. If the reaction forces are the same then the two sets of forces must be statically equivalent. For any one set of forces, there are an infinite number of sets of forces that are statically equivalent to original set of forces.

Finding a Single Equivalent Point Force:

In statics analysis, we are usually looking to simplify a problem by turning multiple forces into a single, statically equivalent force. To find a single point force that is equivalent to multiple point forces you can use the following procedure.

1. Solve for the reaction forces in the original scenario.
2. Draw a new free body diagram with these reaction forces. You will also add one force with an unknown magnitude, direction, and point of application to your diagram. This is the single point load that will be equivalent to your original set of forces.
3. Write out the equations of equilibrium for this scenario, including the known values for the reaction forces.
4. First, solve the force equations to find the x and y components of this unknown force (or x , y and z components for a 3D problem). This can be used to find the magnitude and direction of the statically equivalent point force.
5. Next, use the moment equation (or equations, for 3D problems) to determine the location of the statically equivalent point force.



Video 4.1.1: Lecture video covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/r0pwzQ_Ge00. [Video Transcript](#)

? Example 4.1.1

Determine if the two sets of forces shown below are statically equivalent.

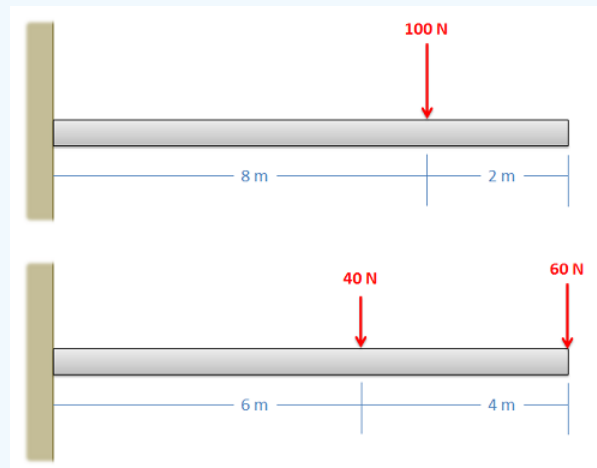
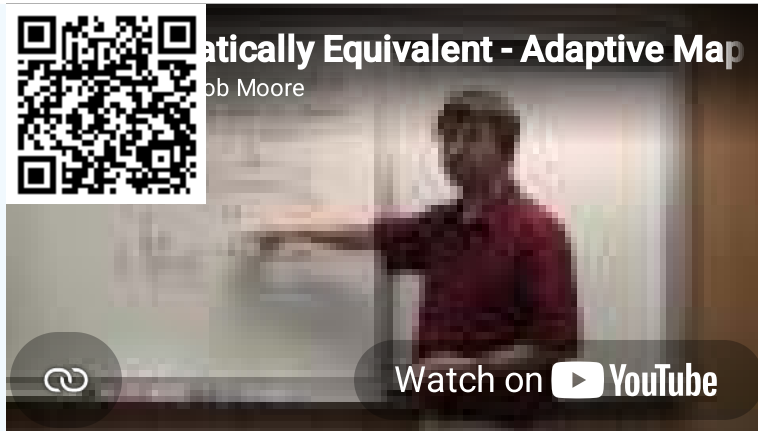


Figure 4.1.2: problem diagram for Example 4.1.1; two identical bars experience the same total magnitude and direction of external force, applied in different locations and distributions.

Solution



Video 4.1.2: worked solution to example problem 4.1.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/Xv5DyvQD29E>. [Video Transcript 2](#)

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4.2: Resolution of a Force into a Force and a Couple

As discussed on the sections on moments, a force can have a tendency to cause both a linear and angular acceleration. For example, below is diagram of a force acting on an extended body. If we were to think about everything relative to the center of mass of this body, there would be some force acting on the center of mass that would cause the same linear acceleration and some pure moment (couple) that would cause the same angular acceleration. This would be a force and couple that is **statically equivalent** to the original force, though the point of application of the force changes.

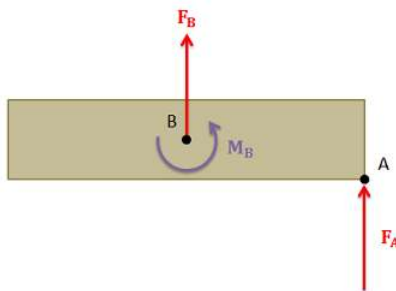


Figure 4.2.1: The original force acting at A would cause both linear and angular acceleration. The force at B would cause the same linear acceleration and the moment at B would cause the same angular acceleration. The force and the moment at B are statically equivalent to the original force at A.

The process of transforming one force applied at one point, into a force and a couple at some other point is known as **resolving a force into a force and a couple**. There are a few reasons that we may want to do this, but one primary reason is to find the **equivalent force couple system** for a complex set of forces and moments. The equivalent force couple system is used to simplify more complex analysis, and consists of a single force and a single pure moment (couple) that are statically equivalent to some more complex combination of forces and moments. An important first step in finding the equivalent force couple system is to resolve all the forces so that everything is acting at the same point.

In order to visualize the process of resolving a force into a force and a couple, you can use the process outlined in the diagram below. Imagine we have a body with a force acting at some point A. We want to resolve the force into a force and a couple about some other point B. To do this we will first add two forces to the diagram at point B. One will have the same magnitude and direction as the original force and the other will be equal and opposite to the original force. Because these two forces are equal, opposite, and collinear, this will not change the situation (it's the equivalent to adding zero to an equation). Now, with these three forces acting on the diagram, we can break it down into two sets. The first is a force acting at point B with the same magnitude and direction as the original force. The other two forces act as a couple, exerting a pure moment about point B. Finally, we can redraw the system as a force acting at point B and the pure moment acting about point B.

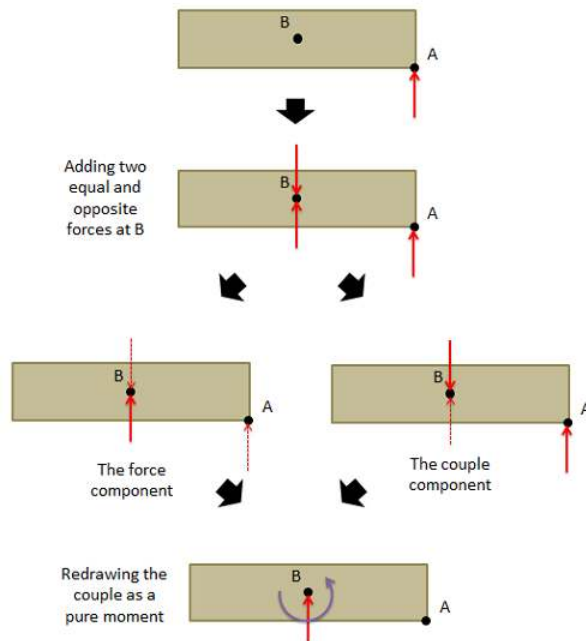
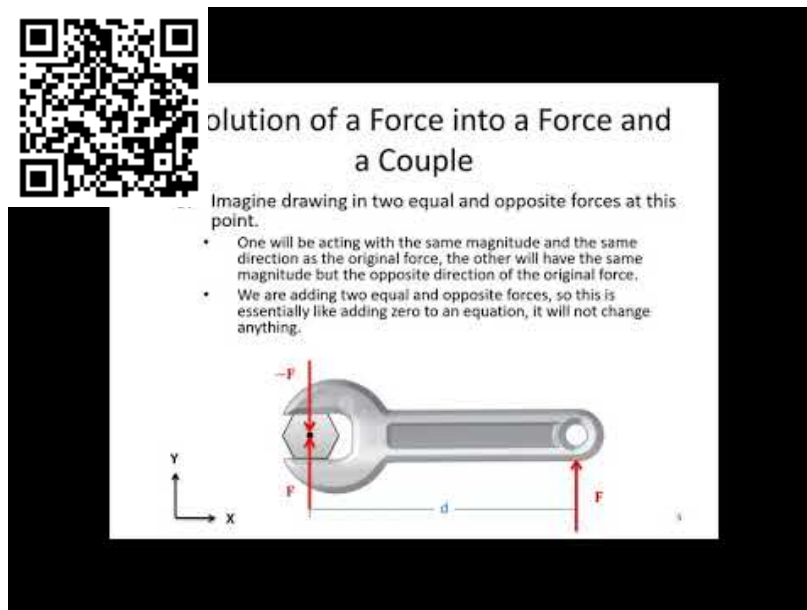


Figure 4.2.2: The process of resolving a force about some point A into a force and a couple about some point B.

As a shortcut to the process described above, we can also see that the force in the equivalent force couple system will always have a magnitude and direction equal to the original force and the couple will be equal to the moment exerted by the original force about the new point of application.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/sLixBOWBCuY>. [Video Transcript](#)

? Example 4.2.1

Resolve the force shown below to a force and a couple acting at point A.

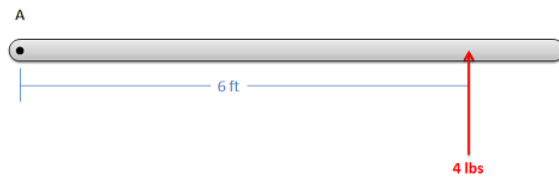
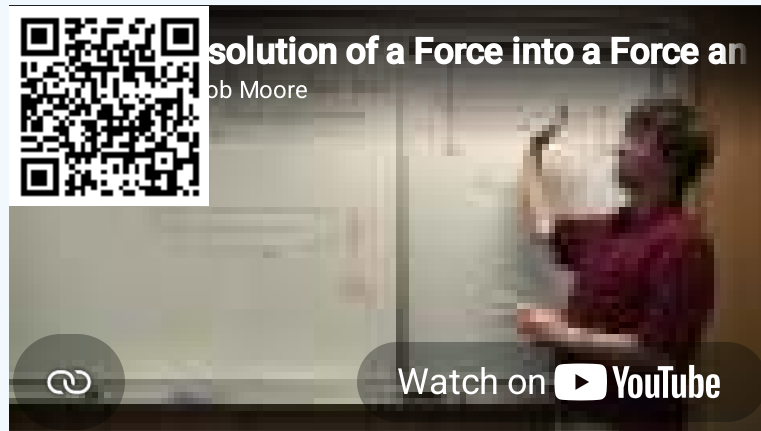


Figure 4.2.3: problem diagram for Example 4.2.1; a rod experiences a single, upwards force applied 6 feet from the left end of the rod (point A).

Solution



Video 4.2.2: Worked solution to example problem 4.2.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/Wf13S_5FD0Q. [Video Transcript 2](#)

? Example 4.2.2

Resolve the force shown below to a force and a couple acting at point A.

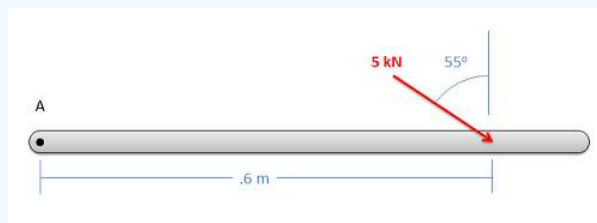
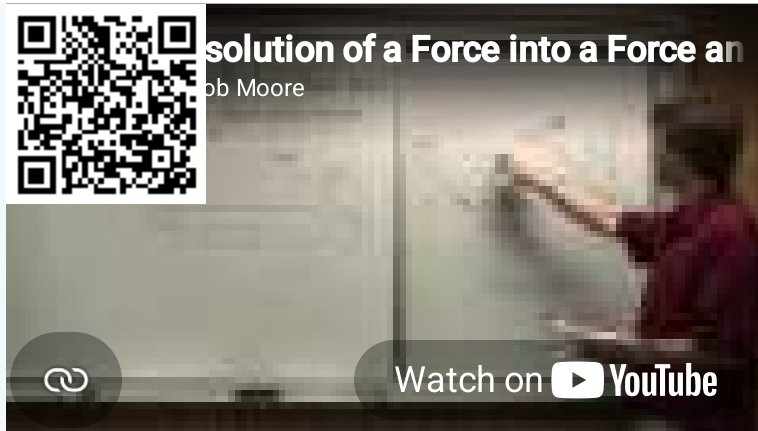


Figure 4.2.4: problem diagram for Example 4.2.2; a rod experiences a single force, pointing downwards and to the right, applied 0.6 meters from the left end of the rod (point A).

Solution



Video 4.2.3: Worked solution to example problem 4.2.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/1hRrKhWzf98>. [Video Transcript 3](#)

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4.3: Equivalent Force Couple System

Every set of forces and moments has an **equivalent force couple system**. This is a single force and pure moment (couple) acting at a single point that is **statically equivalent** to the original set of forces and moments.

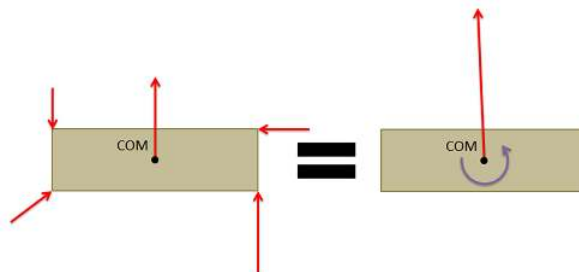


Figure 4.3.1: Any set of forces on a body can be replaced by a single force and a single couple acting that is statically equivalent to the original set of forces and moments. This set of an equivalent force and a couple is known as the equivalent force couple system.

To find the equivalent force couple system, you simply need to follow the steps below.

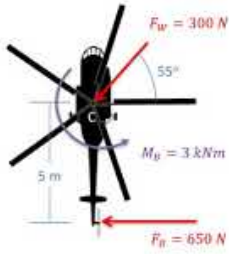
1. First, choose a point to take the equivalent force couple system about. Any point will work, but the point you choose will affect the final values you find for the equivalent force couple system. Traditionally, this point will either be the center of mass of the body or some connection point for the body.
2. Next, resolve all the forces not acting through that point to a force and a couple acting at the point you chose.
3. To find the "force" part of the equivalent force couple system, add together all the force vectors. This will give you the magnitude and the direction of the force in the equivalent force couple system.
4. To find the "couple" part of the equivalent force couple system, add together any moment vectors (this could be moments originally acting on the body, or moments from the resolution of the forces into forces and couples). This will give you the magnitude and direction of the pure moment (couple) in the equivalent force couple system.



Equivalent Force Couple System

By finding the equivalent force couple system, we can...

- Gain clearer insight into what will happen to the body.
- Simplify later analysis, particularly if we are designing something and will need to repeatedly model the impact of a set of forces.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/bs6Tnlje3IU>. [Video Transcript](#)

? Example 4.3.1

Find the equivalent force couple system for the forces shown below about point A.

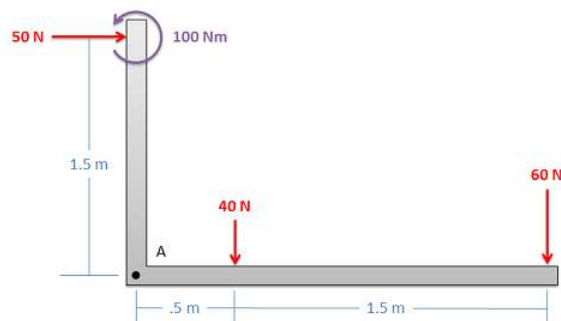
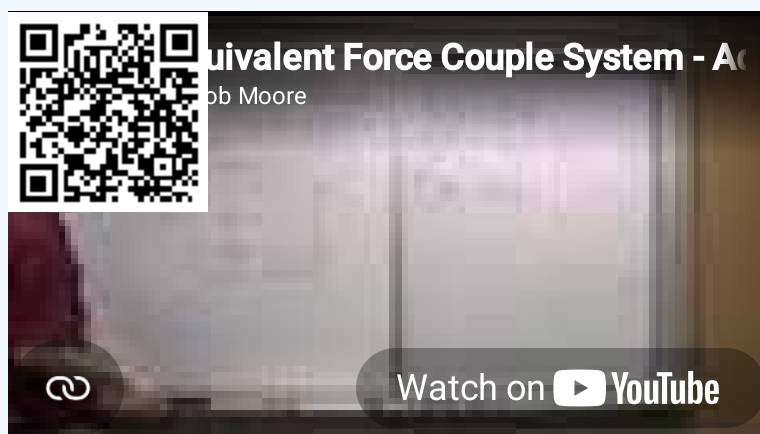


Figure 4.3.2: problem diagram for Example 4.3.1. An L-shaped part whose two arms connect at point A experiences a rightwards force and a counterclockwise moment about the end of its vertical arm, as well as two downwards forces along its horizontal arm.

Solution



Video 4.3.2: Worked solution to example problem 4.3.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/cw-rwBZef5w>. [Video Transcript 2](#)

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4.4: Distributed Forces

A distributed force is any force where the **point of application** of the force is an **area** or a **volume**. This means that the "point of application" is not really a point at all. Though distributed forces are more difficult to analyze than point forces, distributed forces are quite common in real world systems so it is important to understand how to model them.

Distributed forces can be broken down into **surface forces** and **body forces**. Surface forces are distributed forces where the point of application is an area (a surface on the body). Body forces are forces where the point of application is a volume (the force is exerted on all molecules throughout the body). Below are some examples of surface and body forces.

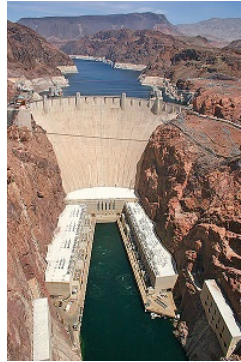


Figure 4.4.1: The water pressure pushing on the surface of this dam is an example of a surface force. Image by Curimedia CC-BY-SA 2.0.

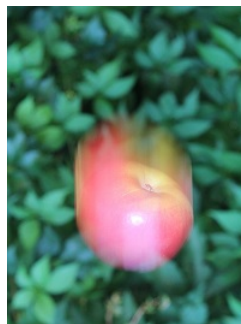


Figure 4.4.2: The gravitational force on this apple is distributed over the entire volume of the fruit. Gravitational forces are an example of body forces. Image by Zátanyi Sándor CC-BY 3.0.

Representing Distributed Forces:

Distributed forces are represented as a field of vectors. This is drawn as a number of discrete vectors along a line, over a surface, or over a volume, that are connected with a line or a surface as shown below.

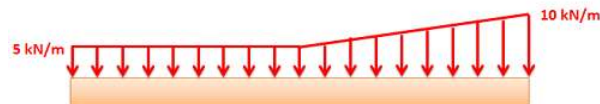


Figure 4.4.3: This is a representation of a surface force in a 2D problem (a force distributed over a line). The magnitude is given in units of force per unit distance.

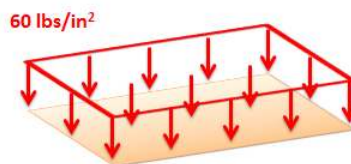


Figure 4.4.4: This is a representation of a surface force in a 3D problem (a force distributed over an area). The magnitude is given in units of force per unit area (also called a pressure).

Though these representations show a discrete number of individual vectors, there is actually a magnitude and direction at all points along the line, surface, or body. The individual vectors represent a sampling of these magnitudes and directions.

It is also important to realize that the magnitudes of distributed forces are given in force per unit distance, area, or volume. We must integrate the distributed force over its entire range to convert the force into the usual units of force.

Analyzing Distributed Forces:

For analysis purposes in statics and dynamics, we will usually substitute in a single point force that is statically equivalent to the distributed force in the problem. This single point force is called the **equivalent point load**, and it will cause the same accelerations or reaction forces as the distributed force while simplifying the math. However, in analysis that focuses on the strength of materials where the bodies are not rigid, this substitution will not work as the distributed forces will not cause the same deformations and stresses as the point force.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/VSlopUTg9kA>. [Video Transcript](#)

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4.5: Equivalent Point Load

An equivalent point load is a **single point force** that will have the **same effect** on a body as the original loading condition, which is usually a distributed force. The equivalent point load should always cause the same linear acceleration and angular acceleration as the original force it is equivalent to (or cause the same reaction forces if the body is constrained). Finding the equivalent point load for a distributed force often helps simplify the analysis of a system by removing the integrals from the equations of equilibrium or equations of motion in later analysis.

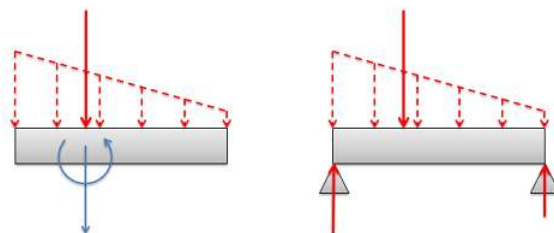


Figure 4.5.1: If the body is unconstrained as shown on the left, the equivalent point load (shown as a solid vector) will cause the same linear and angular acceleration as the original distributed load (shown with dashed vectors). If the body is constrained as on shown on the right, the equivalent point load (shown as a solid vector) will cause the same reaction forces as the original distributed force (shown with dashed vectors).

Finding the Equivalent Point Load

When finding the equivalent point load, we need to find the magnitude, direction, and point of application of a single force that is equivalent to the distributed force we are given. In this course we will only deal with distributed forces with a uniform direction, in which case the direction of the equivalent point load will match the uniform direction of the distributed force. This leaves the magnitude and the point of application to be found. There are two options available to find these values:

1. We can find the magnitude and the point of application of the equivalent point load **via integration** of the force functions.
2. We can use the **area/volume and the centroid/center of volume** of the area or volume under the force function.

The first method is more flexible, allowing us to find the equivalent point load for any force function that we can make a mathematical formula for (assuming we have the skill in calculus to integrate that function). **The second method is usually faster, assuming that we can look up the values for the area or volume under the force curve and the values for the centroid or center of volume for the area under the curve.** In this course, we will use the second method.

Using the Area and Centroid in 2D Surface Force Problems:

As an alternative to using integration, we can use the area under the force curve and the centroid of the area under the force curve to find the equivalent point load's magnitude and point of application respectively.

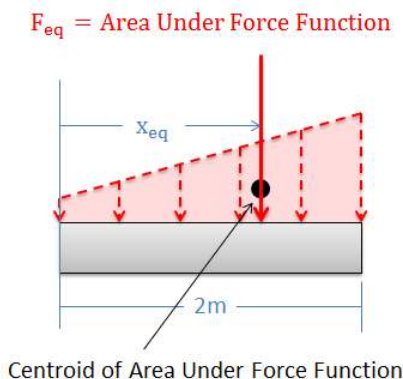


Figure 4.5.4: The magnitude of the equivalent point load is equal to the area under the force function. Also, the equivalent point load will travel through the centroid of the area under the force function.

The **magnitude** (F_{eq}) of the equivalent point load will be equal to the **area under the force function**. We can find this area using calculus, but there are often easier geometry-based ways of finding the area under the force function.

The equivalent point load will also **travel through centroid of the area under the force function**. This allows us to find the value for x_{eq} . The centroid for many common shapes can be looked up in tables, and the parallel axis theorem can be used to determine the centroid of more complex shapes (see the Appendix page on centroids for more details).

Using Volume and Center of Volume in 3D Surface Force Problems:

Just as in the 2D problems, there are some available shortcuts to finding the equivalent point load in 3D surface force problems. For a force spread over an area, the **magnitude** (F_{eq}) of the equivalent point load will be equal to the **volume under the force function**. The equivalent point load will also **travel through the center of volume of the volume under the force function**. This should allow you to determine both x_{eq} and y_{eq} .

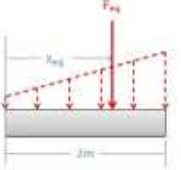
The center of volume for a shape will be the same as the center of mass for a shape if the shape is assumed to have uniform density. It should be possible to look these values up for common shapes in a table. Again, the parallel axis theorem can be used to find the center of volume for more complex shapes (See the Center of Mass page in Appendix 2 for more details).



The Equivalent Point Load via Integration

We can imagine a distributed force as an infinite number of very small forces.

- Rather than summing all of these forces as we did before to find the **magnitude of the equivalent point load**, we will take the **integral** to find the **area under the force function**.

$$F_{eq} = \int_{x_{min}}^{x_{max}} F(x) dx$$


Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/eikERyUNIOo>. [Video Transcript](#)

? Example 4.5.1

Determine the magnitude and the point of application for the equivalent point load of the distributed force shown below.

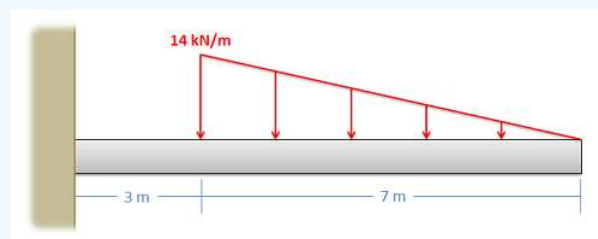
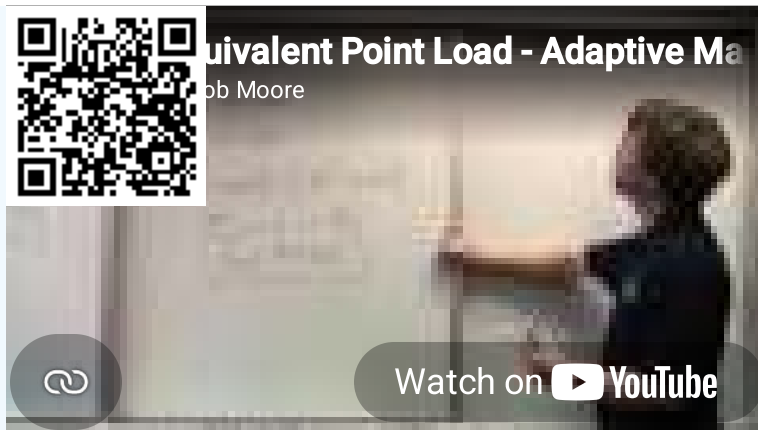


Figure 4.5.6: problem diagram for Example 4.5.1; a bar attached to a wall experiences a distributed force whose magnitude varies linearly over part of its length.

Solution



Video 4.5.2: Worked solution to example problem 4.5.1, provided by Dr. Jacob Moore. YouTube source: https://youtu.be/h_E0XjIJaiI. [Video Transcript 2](#)

? Example 4.5.2

Determine the magnitude and the point of application for the equivalent point load of the distributed force shown below.

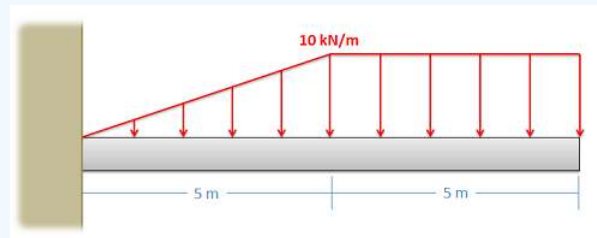


Figure 4.5.2: problem diagram for Example 4.5.2; a bar attached to a wall experiences a distributed force whose magnitude varies linearly over part of its length and remains constant for the remainder.

Solution



Video 4.5.2: Worked solution to example problem 4.5.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/UbnfNAQctyg>. [Video Transcript 3](#)

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4.6: Centroids and Centers of Mass via Method of Composite Parts

As an alternative to the use of moment integrals, we can use the **Method of Composite Parts** to find the **centroid** of an area or volume or the **center of mass** of a body. This method is often easier and faster than the integration method; however, it will be limited by the table of centroids you have available. The method works by breaking the shape or volume down into a number of more basic shapes, identifying the centroids or centers of masses of each part via a table of values, and then combining the results to find the overall centroid or center of mass.

A key aspect of the method is the use of these centroid tables. This is a set of tables that lists the centroids (and usually also moments of inertia) for a number of common areas and/or volumes. Some centroid tables can be found [here for 2D shapes](#), and [here for 3D shapes](#). The method of composite parts is limited in that we will need to be able to break our complex shape down entirely into shapes found in the centroid table we have available; otherwise, the method will not work without us also doing some moment integrals.

Finding the Centroid via the Method of Composite Parts

Start the process by labeling an origin point and axes on your shape. It will be important to measure all locations from the same point. Next, we must break our complex shape down into several simpler shapes. This may include areas or volumes (which we will count as positive areas or volumes) or holes (which we will count as negative areas or volumes). Each of these shapes will have a centroid (C) or center of mass (G) listed on the diagram.

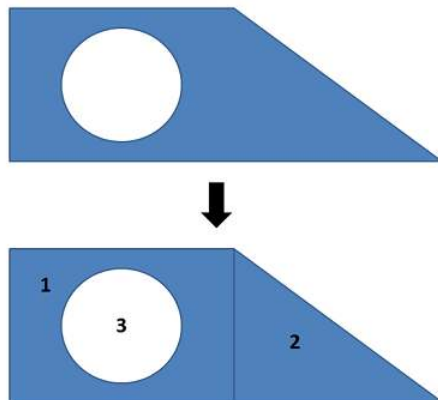


Figure 4.6.1: For the shape shown at the top, we can break it down into a rectangle (1), a right triangle (2), and a circular hole (3). Each of these simple shapes is something we have listed in the centroid table to the right.

Once we have identified the different parts, we will create a table listing the area or volume of each piece, and the x and y centroid coordinates (or x , y , and z coordinates in 3D). It is important to remember that each coordinate you list should be **relative to the same base origin point** that you drew in earlier. You may need to mentally adjust diagrams in the centroid tables so that the shape is oriented in the right direction, and account for the placement of the shape relative to the axes in your diagram.

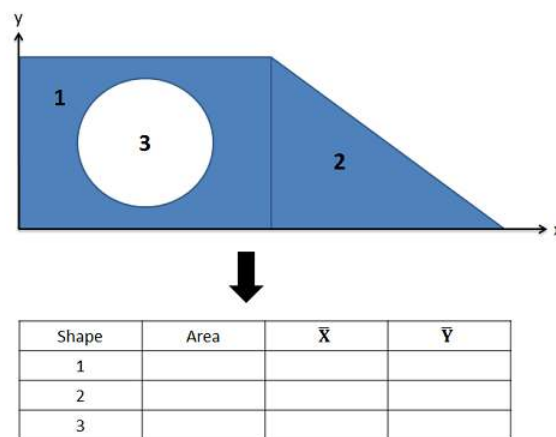


Figure 4.6.2: For each of the shapes, we need to find the area and the x and y coordinates of the centroid. Remember to find the centroid coordinates relative to a single set of axes that is the same for all the shapes.

Once you have the areas and centroid coordinates for each shape relative to your origin point, you can find the x and y coordinate of the centroid for the overall shape with the following formulas. Remember that areas or volumes for any shape that is a hole or cutout in the design will be a negative area in your formula.

$$\bar{x}_{total} = \frac{\sum A_i \bar{x}_i}{A_{total}} \quad \bar{y}_{total} = \frac{\sum A_i \bar{y}_i}{A_{total}} \quad (4.6.1)$$

This generalized formula to find the centroid's x -location is simply Area 1 times $\bar{x}_1$, plus Area 2 times $\bar{x}_2$, plus Area 3 times $\bar{x}_3$, adding up as many shapes as you have in this fashion and then dividing by the overall area of your combined shape. The equations are the same for the y -location of the overall centroid, except you will instead be using $\bar{y}$ values in your equations.

For centroids in three dimensions we will simply use volumes in place of areas, and we will have a z coordinate for our centroid as well as the x and y coordinates.

Finding the Center of Mass via the Method of Composite Parts

To use the method of composite parts to find the center of mass, we simply need to adjust the process slightly. First, center of mass calculations will always be in three dimensions. Draw an origin point and some axes on your diagram we did for the centroid. We will measure all locations relative to this origin point. We will then need to break the complex shape down into simple volumes, with each simple volume being something in the centroid table we have available. Remember that when we have a part with a uniform material, the centroid and center of mass are the same point, so we will often talk about these interchangeably.

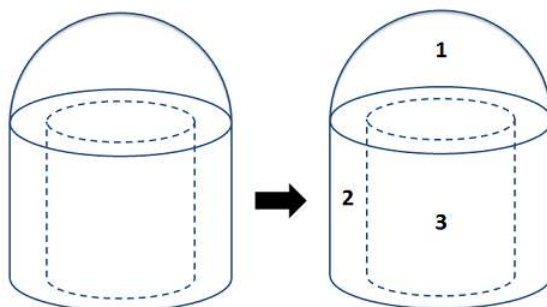


Figure 4.6.3: When finding the center of mass via composite parts, we will break the shape up into several simpler shapes. The figure on the left can be thought of as a hemisphere (1), on top of a cylinder (2) with another smaller cylinder cut out of it (3). Each of these simple volumes are listed in our centroid table.

Once we have identified the different parts, we will create a table indicating the **mass** of each part, and the x , y , and z coordinate of the center of mass for each individual part. It is important to remember that each coordinate you list should be relative to the same base origin point, so you will need to mentally rotate and position the parts in the table on your axes.

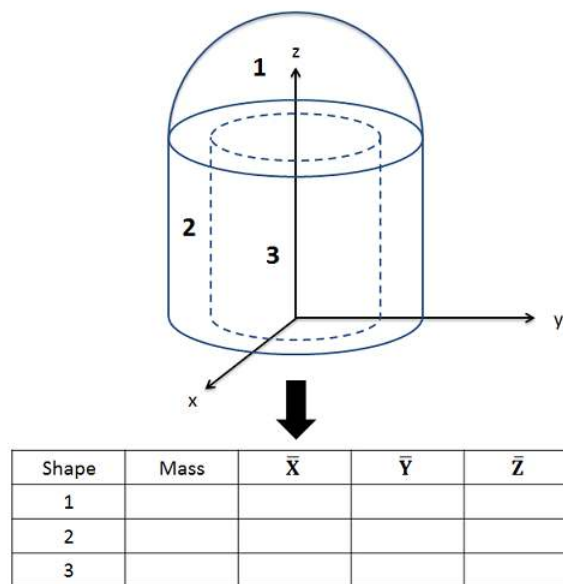


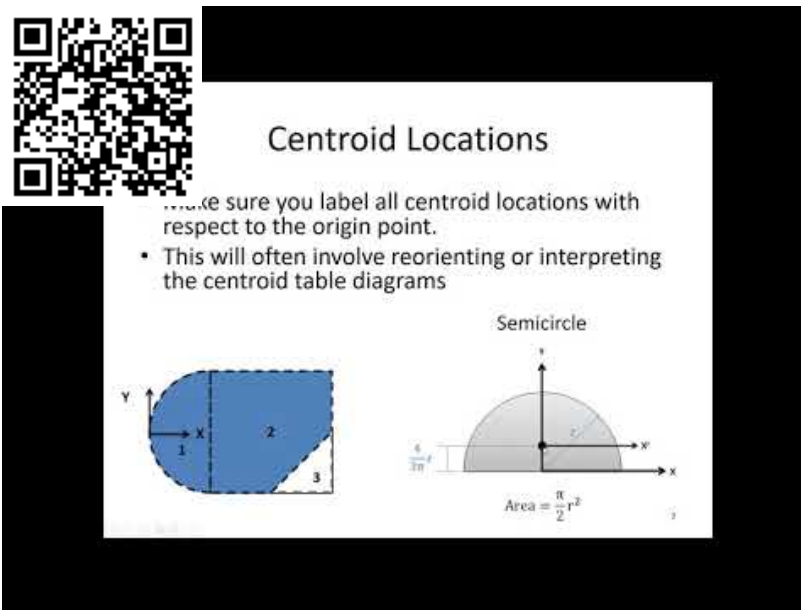
Figure 4.6.4: Create a table with the mass of each piece of the total shape, as well as the center of mass location (x , y , and z coordinates) for each piece.

One complicating factor with mass can be measuring the mass of the pieces separately. If we have a scale, we may simply know the overall mass without knowing the mass of the individual pieces. In these cases, you may need work backwards to calculate the density of the material (by dividing the overall mass by overall volume), and then use density times piece volume to find the mass of each piece individually. When doing this, remember to count cutouts as negative mass in your calculations. For example, for the hollow cylinder in the shape above, you would find the mass of a solid cylinder for Shape 2, then have a negative mass for the cylindrical cutout for Shape 3.

Finally, once you have the mass and center of mass coordinates for each shape, you can find the coordinates of the center of mass for the overall volume with the following formulas.

$$x_G = \frac{\sum m_i \bar{x}_i}{m_{total}} \quad y_G = \frac{\sum m_i \bar{y}_i}{m_{total}} \quad z_G = \frac{\sum m_i \bar{z}_i}{m_{total}} \quad (4.6.2)$$

Similar to the centroid equations, the x -equation is simply the mass of Shape 1 times $\bar{x}_1$, plus the mass of Shape 2 times $\bar{x}_2$, and so on for each part. After you have summed up these products for all the shapes, just divide by the total mass.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/wfjLNSfPXAI>. [Video Transcript](#)

? Example 4.6.1

Find the x and y coordinates of the centroid of the shape shown below.

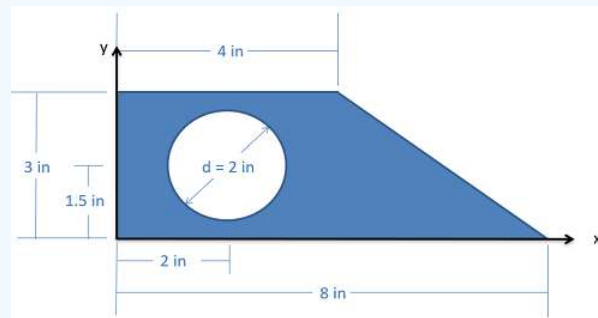
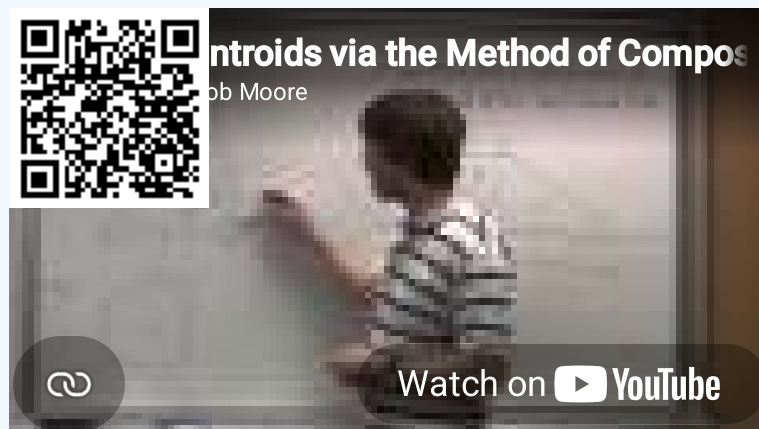


Figure 4.6.5: problem diagram for Example 4.6.1. A trapezoid with two perpendicular sides and a hole through its middle lies along the axes of the first quadrant of a Cartesian coordinate plane.

Solution:



Video 4.6.2: Worked solution to example problem 4.6.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/QIe6Hk4Bofs>. [Video Transcript 1](#)

? Example 4.6.2

Find the x and y coordinates of the centroid of the shape shown below.

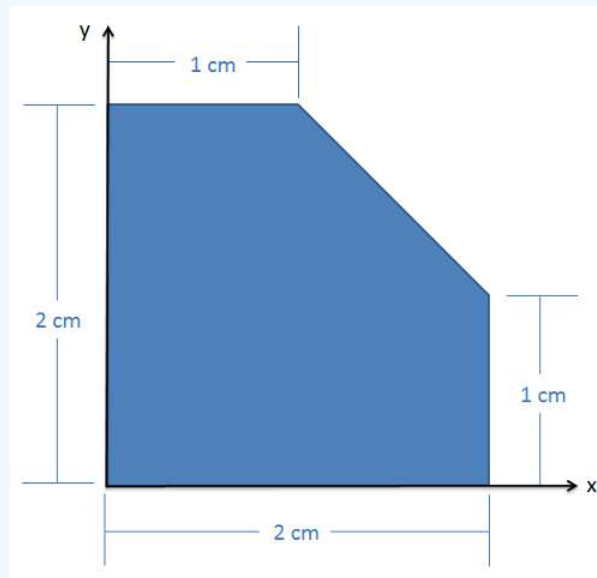


Figure 4.6.6: problem diagram for Example 4.6.2. A pentagon with two perpendicular sides lies along the axes of the first quadrant of a Cartesian coordinate plane.

Solution:

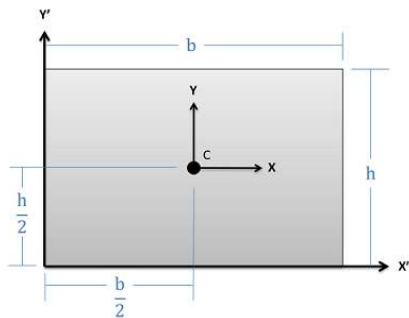
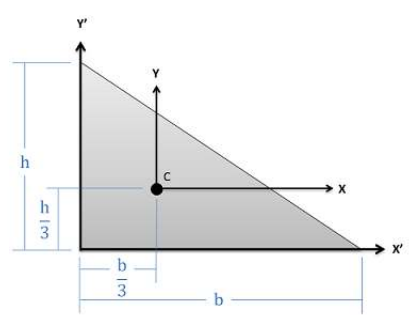
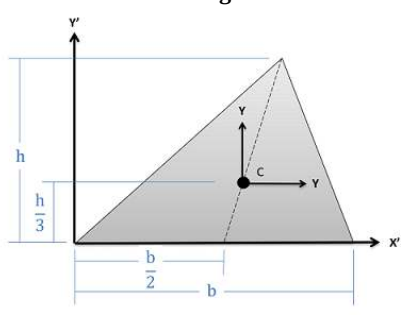


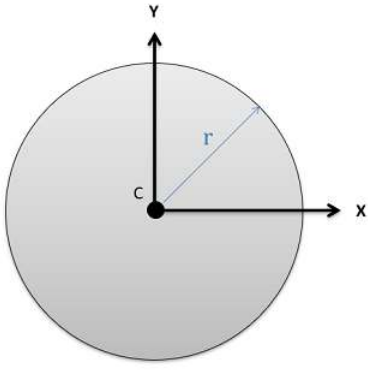
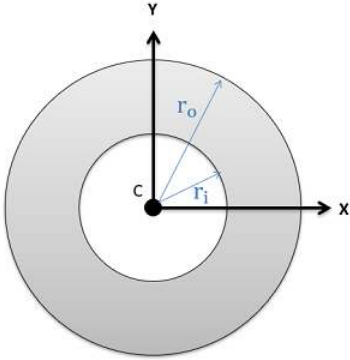
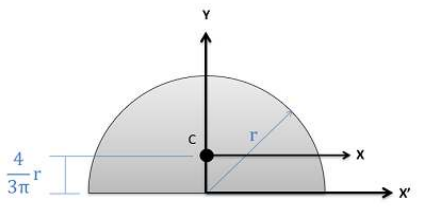
Video 4.6.3: Worked solution to example problem 4.6.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/F1rlzboPlZM>. [Video Transcript 3](#)

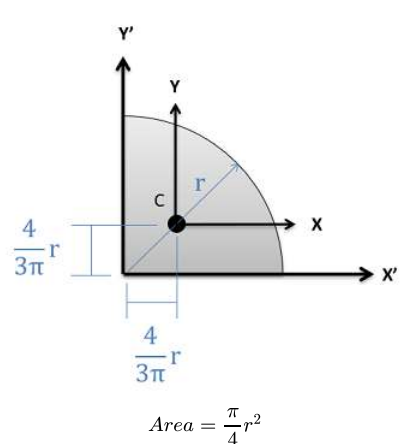
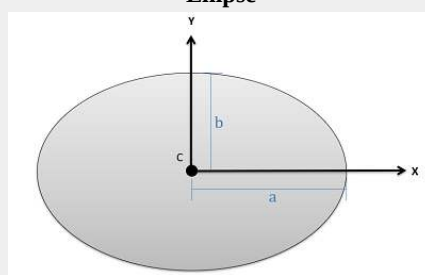
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4.7: Centroids and Area Moments of Inertia for 2D Shapes

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
Rectangle  Area = bh	$I_x = \frac{1}{12}bh^3$ $I_y = \frac{1}{12}b^3h$	$J_z = \frac{1}{12}bh(b^2 + h^2)$
Right Triangle  Area = $\frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$ $I_y = \frac{1}{36}b^3h$ $I_{x'} = \frac{1}{12}bh^3$ $I_{y'} = \frac{1}{12}b^3h$	
Triangle  Area = $\frac{1}{2}bh$	$I_x = \frac{1}{36}bh^3$ $I_{x'} = \frac{1}{12}bh^3$	
Circle	$I_x = \frac{\pi}{4}r^4$ $I_y = \frac{\pi}{4}r^4$	$J_z = \frac{\pi}{2}r^4$

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
 $Area = \pi r^2$		
Circular Annulus  $Area = \pi(r_o^2 - r_i^2)$	$I_x = \frac{\pi}{4}(r_o^4 - r_i^4)$ $I_y = \frac{\pi}{4}(r_o^4 - r_i^4)$	$J_z = \frac{\pi}{2}(r_o^4 - r_i^4)$
Semicircle  $Area = \frac{\pi}{2}r^2$	$I_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$ $I_y = \frac{\pi}{8}r^4$ $I_{x'} = \frac{8}{\pi}r^4$	$J_z = \left(\frac{\pi}{4} - \frac{8}{9\pi}\right)r^4$
Quarter Circle	$I_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$ $I_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)r^4$ $I_{x'} = \frac{\pi}{16}r^4$ $I_{y'} = \frac{\pi}{16}r^4$	$J_z = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right)r^4$

Shape with Area and Centroid Location Shown	Rectangular Area Moments of Inertia	Polar Area Moments of Inertia
 Area = $\frac{\pi r^2}{4}$		
Ellipse  Area = πab	$I_x = \frac{\pi}{4} ab^3$ $I_y = \frac{\pi}{4} a^3 b$	

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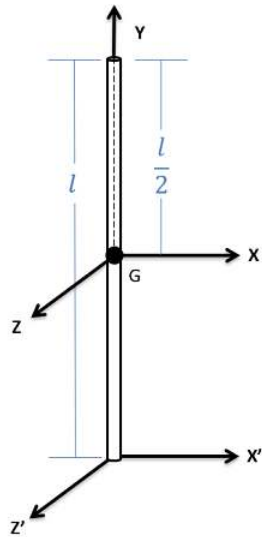
- [Centroids and Area Moments of Inertia for 2D Shapes](#) has no license indicated.

4.8: Center of Mass and Mass Moments of Inertia for Homogeneous 3D Bodies

Shape with Volume and Center of Mass Location Shown

Mass Moments of Inertia

Slender Rod

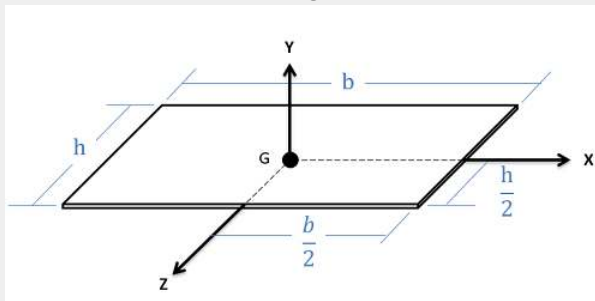


$$I_{xx} = I_{zz} = \frac{1}{12}ml^2$$

$$I_{yy} = 0$$

$$I_{xx'} = I_{zz'} = \frac{1}{3}ml^2$$

Flat Rectangular Plate

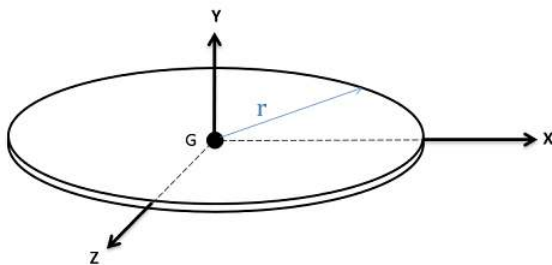


$$I_{xx} = \frac{1}{12}mh^2$$

$$I_{yy} = \frac{1}{12}m(h^2 + b^2)$$

$$I_{zz} = \frac{1}{12}mb^2$$

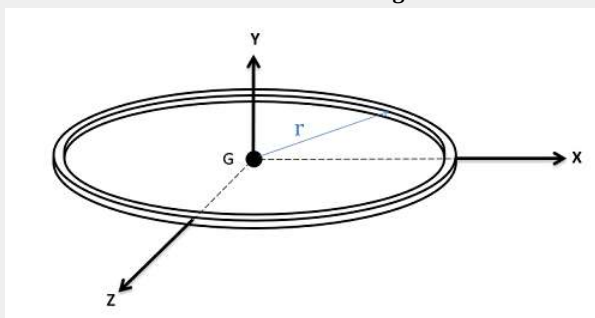
Flat Circular Plate



$$I_{xx} = I_{zz} = \frac{1}{4}mr^2$$

$$I_{yy} = \frac{1}{2}mr^2$$

Thin Circular Ring



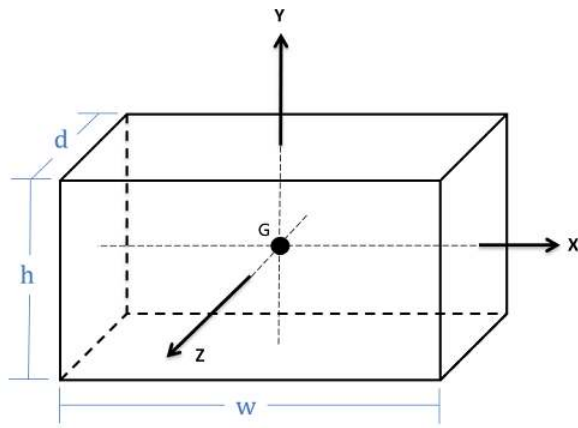
$$I_{xx} = I_{zz} = \frac{1}{2}mr^2$$

$$I_{yy} = mr^2$$

Shape with Volume and Center of Mass Location Shown

Mass Moments of Inertia

Rectangular Prism



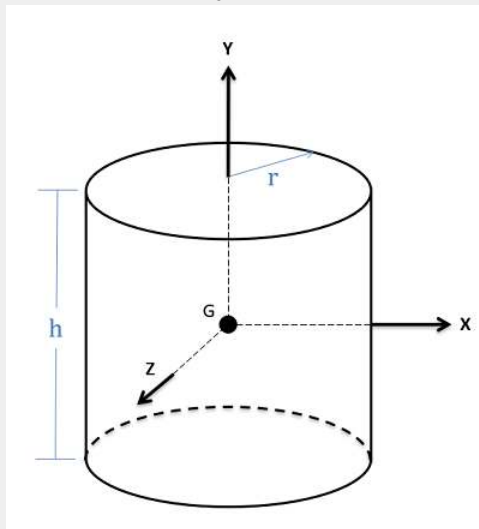
$$\text{Volume} = dwh$$

$$I_{xx} = \frac{1}{12}m(h^2 + d^2)$$

$$I_{yy} = \frac{1}{12}m(d^2 + w^2)$$

$$I_{zz} = \frac{1}{12}m(h^2 + w^2)$$

Cylinder

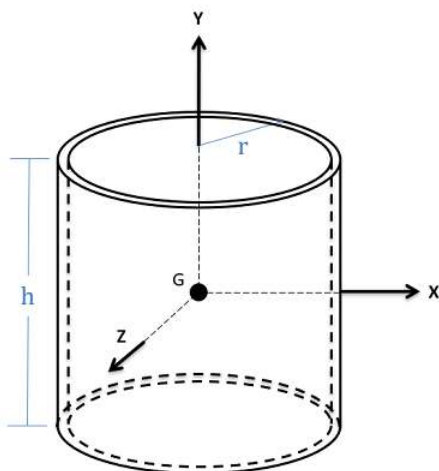


$$\text{Volume} = \pi r^2 h$$

$$I_{xx} = I_{zz} = \frac{1}{12}m(3r^2 + h^2)$$

$$I_{yy} = \frac{1}{2}mr^2$$

Thin Cylindrical Shell



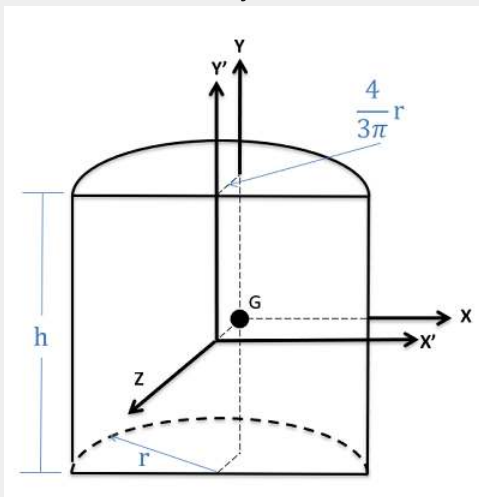
$$I_{xx} = I_{zz} = \frac{1}{6}m(3r^2 + h^2)$$

$$I_{yy} = mr^2$$

Shape with Volume and Center of Mass Location Shown

Mass Moments of Inertia

Half Cylinder



$$Volume = \frac{1}{2}\pi r^2 h$$

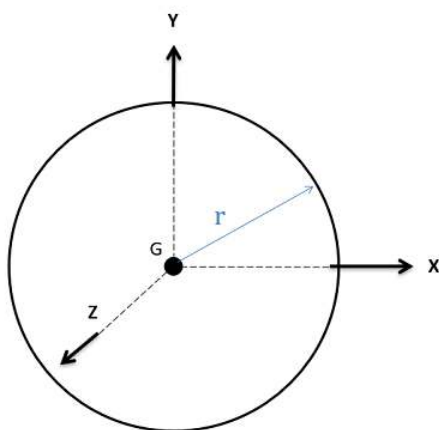
$$I_{xx} = I_{zz} = \left(\frac{1}{4} - \frac{16}{9\pi^2} \right) mr^2 + \frac{1}{12} mh^2$$

$$I_{yy} = \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) mr^2$$

$$I_{xx'} = I_{zz'} = \frac{1}{12} m(3r^2 + h^2)$$

$$I_{yy'} = \frac{1}{2} mr^2$$

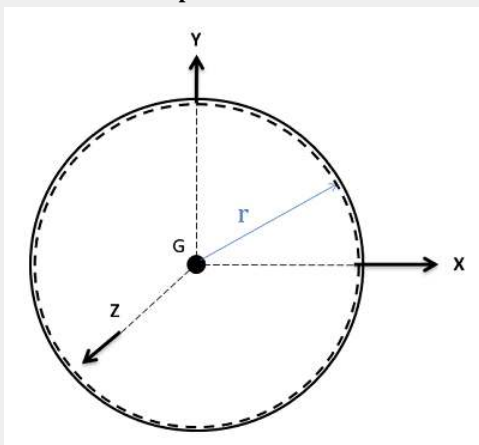
Sphere



$$Volume = \frac{4}{3}\pi r^3$$

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$$

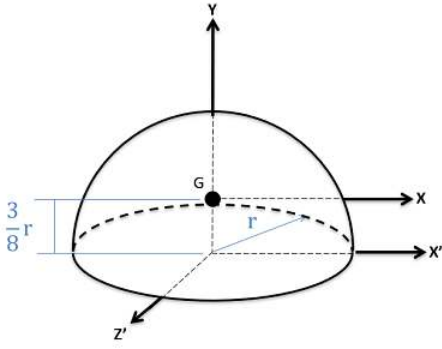
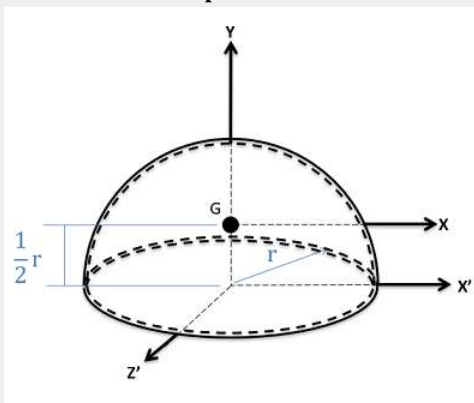
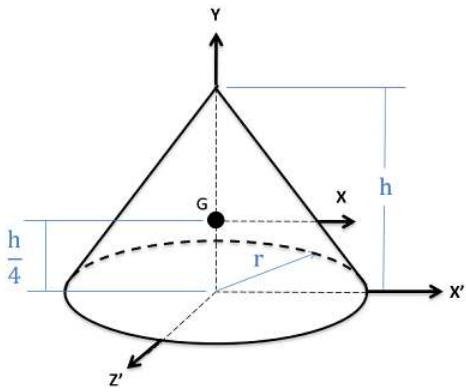
Spherical Shell



$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} mr^2$$

Hemisphere

$$I_{xx} = I_{zz} = \frac{83}{320} mr^2$$

Shape with Volume and Center of Mass Location Shown	Mass Moments of Inertia
 $Volume = \frac{2}{3}\pi r^3$	$I_{yy} = \frac{2}{5}mr^2$ $I_{xx'} = I_{zz'} = \frac{2}{5}mr^2$
Hemispherical Shell 	$I_{xx} = I_{zz} = \frac{5}{12}mr^2$ $I_{yy} = \frac{2}{3}mr^2$ $I_{xx'} = I_{zz'} = \frac{2}{3}mr^2$
Right Circular Cone  $Volume = \frac{1}{3}\pi r^2 h$	$I_{xx} = I_{zz} = \frac{3}{80}m(4r^2 + h^2)$ $I_{yy} = \frac{3}{10}mr^2$ $I_{xx'} = I_{zz'} = \frac{1}{20}m(3r^2 + 2h^2)$

4.8: Center of Mass and Mass Moments of Inertia for Homogeneous 3D Bodies is shared under a [not declared](#) license and was authored, remixed, and/or curated by LibreTexts.

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4.9: Chapter 4 Homework Problems

? Exercise 4.9.1

Determine if the two systems below are statically equivalent.

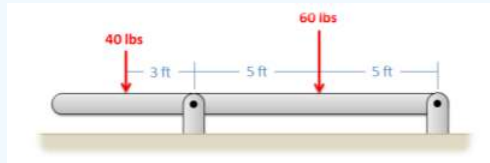


Figure 4.9.1: part 1 of the problem diagram for Exercise 4.9.1. A horizontal bar held off the ground by two supports experiences two point forces at different points along its length.

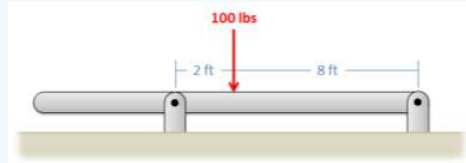


Figure 4.9.2: part 2 of the problem diagram for Exercise 4.9.1. A horizontal bar held off the ground by two supports experiences a single point force partway along its length.

Solution

No, they are not equivalent.

? Exercise 4.9.2

Determine if the set of forces in A is statically equivalent to the set of forces and moments in B.

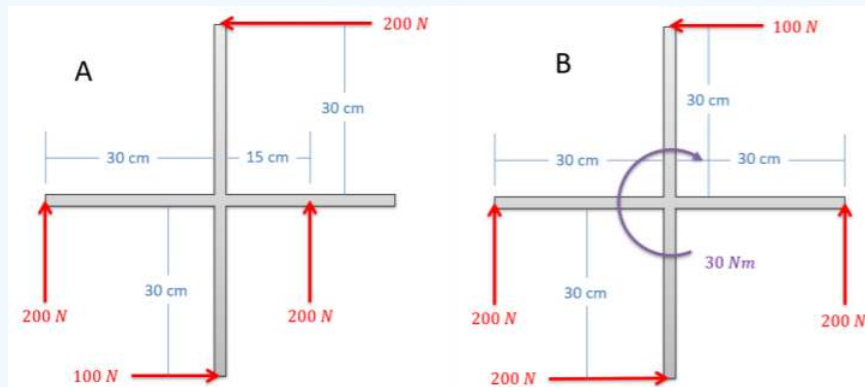


Figure 4.9.3: problem diagram for Exercise 4.9.2. Two uniform crosses of equal dimensions experience a set of forces (system A, left) and a set of forces and moments (system B, right).

Solution

No, they are not equivalent.

? Exercise 4.9.3

Determine the equivalent point load (magnitude and location) for the distributed force shown below.

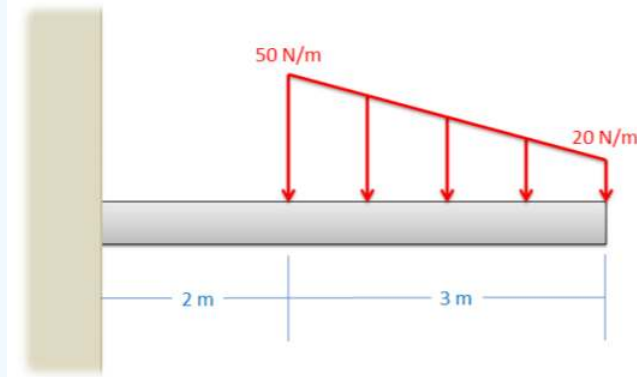


Figure 4.9.3: problem diagram for Exercise 4.9.3. A horizontal bar attached to a wall at one end experiences a distributed force, which varies linearly, over part of its length.

Solution

$$F_{eq} = 105 \text{ N}$$

$$x_{eq} = 3.29 \text{ m (measured from wall)}$$

? Exercise 4.9.4

Determine the equivalent point load (magnitude and location) for the distributed force shown below.

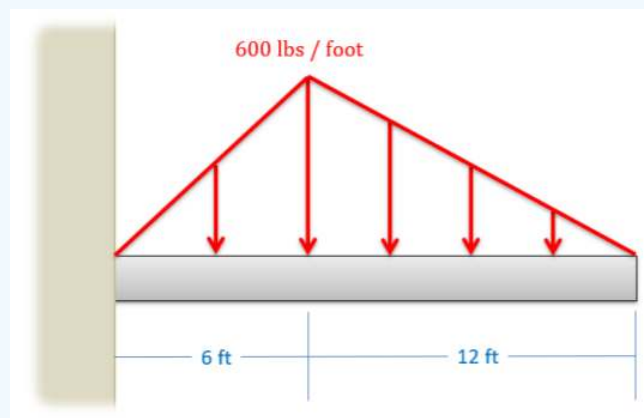


Figure 4.9.4: problem diagram for Exercise 4.9.4. A horizontal bar attached to a wall experiences a distributed force over its length, with magnitude varying linearly according to a piecewise force function.

Solution

$$F_{eq} = 5400 \text{ lbs}$$

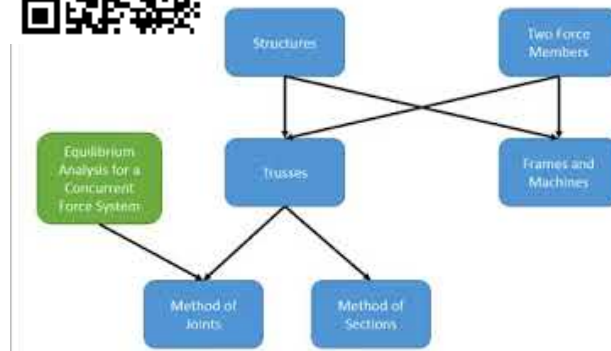
$$x_{eq} = 8 \text{ ft (measured from wall)}$$

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CHAPTER OVERVIEW

5: Engineering Structures



Video introduction to the topics to be covered in Chapter 5, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/BJ8SdEUnt2U>. [Video Transcript](#)

- [5.1: Structures](#)
- [5.2: Two-Force Members](#)
- [5.3: Trusses](#)
- [5.4: Method of Joints](#)
- [5.5: Method of Sections](#)
- [5.6: Frames and Machines](#)
- [5.7: Analysis of Frames and Machines](#)
- [5.8: Chapter 5 Homework Problems](#)

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5.1: Structures

An **engineering structure** is a term used to describe any **set of interconnected bodies**. The different bodies in the structure can move relative to one another (such as the blades in a pair of scissors) or they can be fixed relative to one another (such as the different beams connected to form a bridge).

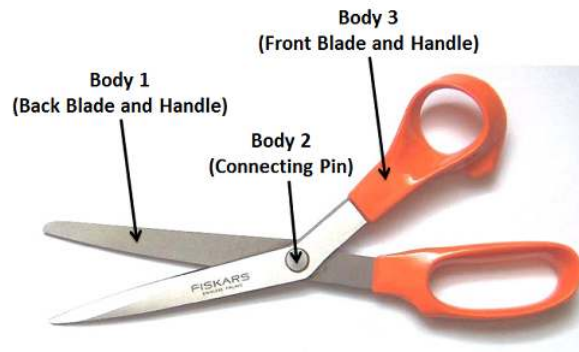


Figure 5.1.1: This pair of scissors is an example of an engineering structure. The structure consists of three different bodies that are interconnected. Adapted from Public Domain image by Comet27.

When analyzing engineering structures, we will sometimes analyze the structure as a whole, and we will sometimes break the structure down into individual bodies that are analyzed separately. The exact methods used depend upon what unknown forces we are looking for and what type of structure we are analyzing.

Internal and External Forces:

When examining a single body, we would find the forces that this body exerted on surrounding bodies, and the forces that these surrounding bodies would exert on the body we are analyzing. These forces are all considered **external forces** because they are forces between the body and the external environment.

In an engineering structure, we still have external forces where the structure is interacting with bodies external to the structure, but we also can think about the forces that different parts of the structure exert on one another (the force between the pin and the blade in Figure 5.1.1, for example). Since both of these bodies are part of the structure we are analyzing, these forces are considered **internal forces**.

If we only wish to determine the external forces acting on a structure, then we can treat the whole structure as a single body (assuming the structure is rigid as a whole). If we wish to determine the internal forces acting between components in the structure, then we will need to disassemble the structure into separate bodies in our analysis.

Types of Structures:

Another important consideration when analyzing structures is the type of structure that is being analyzed. All structures fall into one of three categories: **trusses**, **frames**, or **machines**. Frames and machines are analyzed in the same way so distinction between them is less important, but the analysis methods used for trusses vary greatly from the analysis methods used for frames and machines, so determining if a structure is a truss or not is an important first step in structure analysis.

Trusses:

A **truss** is a structure that consists **entirely** of **two-force members**. If any one body in the structure is not a two-force member, then the structure is either a frame or a machine. Also, in order to be a **statically determinate** truss (a truss where we can actually solve for all the unknowns), the truss must be independently rigid as a whole. If different parts of the truss could move relative to one another then the truss separated is not independently rigid.



Figure 5.1.2: This bridge is an example of a truss. It consists of a number of members connected at only two points (the ends of the beams). Public domain image by Leonard G.

A two-force member is a body where forces are applied at only two locations. If forces are applied at more than two locations, or if any moments are applied, then the body is not a two-force member (see the two-force member page for more details). Because of the unique assumptions we can make with two-force members, we can apply two unique methods to the analysis of trusses (the method of joints and the method of sections) that we cannot apply to frames and machines (where we cannot assume we have two-force members).

Frames and Machines:

A frame or a machine is structure where at least one component of the structure is not a two-force member. This component will be a body in the structure that has forces acting at three or more points on it. The difference between a frame and a machine is that a frame is rigid as a whole, while a machine is not rigid as a whole.



Figure 5.1.3: The legs of this stool have forces applied to them in three locations (the top, the cross beams, and the floor). The stool is also independently rigid, so this is a frame. Image by Besceh31 CC-BY-SA 2.5



Figure 5.1.4: Many pieces within this pair of locking pliers have forces applied to them at more than two locations. The pieces can also move relative to one another, so this is an example of a machine. Image by Duk CC-BY-SA 3.0

Because frames and machines do not consist entirely of two-force members, we cannot make the assumptions that allow us to use the method of joints and the method of sections. For this reason, we need to use a different analysis method (simply called the analysis of frames and machines here).



Newton's Third Law

- Though you analyze each body separately, **Third Law pairs** of forces will exist wherever two separate members in the structure are connected to one another.

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/5Hz8h7qEVEM>. [Video Transcript](#)

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5.2: Two-Force Members

A **two-force member** is a body that has forces (and only forces, no moments) acting on it in only two locations. In order to have a two-force member in static equilibrium, the net force at each location must be equal, opposite, and collinear. This will result in all two-force members being in either tension or compression, as shown in the diagram below.

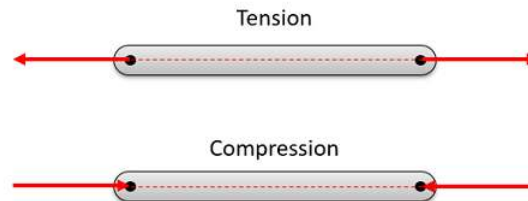


Figure 5.2.1: The forces acting on two-force members need to be equal, opposite, and collinear for the body to be in equilibrium.

Why the Forces Must Be Equal, Opposite and Collinear:

Imagine a beam where forces are only exerted at each end of the beam (a two-force member). The body has some non-zero force acting at one end of the beam, which we can draw as a force vector. If this body is in equilibrium, then we know two things:

1. the sum of the forces must be equal to zero, and
2. the sum of the moments must be equal to zero.

In order to have the sum of the forces equal to zero, the force vector on the other side of the beam must be equal in magnitude and opposite in direction. This is the only way to ensure that the sum of the forces is equal to zero with only two forces.

In order to have the sum of the moments equal to zero, the forces must be collinear. If the forces were not collinear, then the two equal and opposite forces would form a couple. This couple would exert a moment on the beam when there are no other moments to counteract the couple. Because the moment exerted by the two forces must be equal to zero, the perpendicular distance between the forces (d) must be equal to zero. The only way to achieve this is to have the forces be collinear.

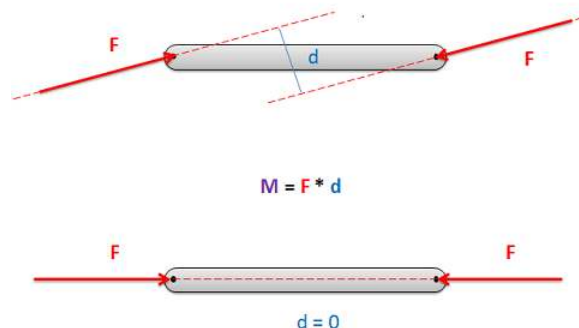


Figure 5.2.2: In order to have the sum of the moments be equal to zero, the forces acting on two-force members must always be collinear, acting along the line connecting the two points where forces are applied.

Why Two-Force Members Are Important:

By identifying two-force members, we greatly reduce the number of unknowns in our problem. In two-force members, we know that the forces must act along the line between the two connection points on the body. This means that the direction of the force vectors is known on either side of the body. Additionally, we know the forces are equal and opposite, so if we determine the magnitude and direction of the force acting on one side of the body, we automatically know the magnitude and direction of the force acting on the other side of the body.

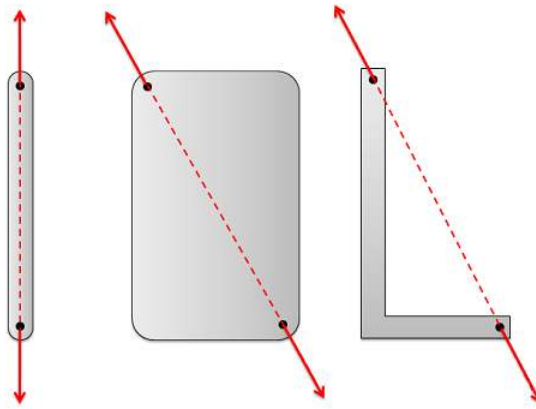


Figure 5.2.3: The forces in two-force members will always act along the line connecting the two points where forces are applied.

Two-force members are also important in distinguishing between trusses, and frames and machines. When we analyze trusses using either the method of joints or the method of sections, we will assume everything is a two-force member. If this assumption is incorrect, this will cause serious problems in the analysis. By making this assumption, though, we can use some shortcuts that will make truss analysis easier and faster than the analysis of frames and machines.



Two Force Members

The line of action of the two forces will always be the line directly between the two connection points.

- The shape or size of the member does not affect the direction of the forces.



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5.3: Trusses

A **truss** is an engineering structure that is made entirely of **two-force members**. In addition, statically determinate trusses (trusses that can be analyzed completely using the equilibrium equations) must be **independently rigid**. This means that if the truss was separated from its connection points, no one part would be able to move independently with respect to the rest of the truss.

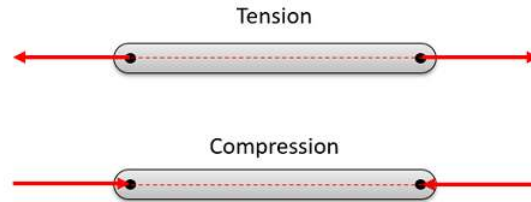


Figure 5.3.1: Trusses are made entirely of two-force members. This means that each member will either be in tension or compression, as shown here.

Trusses can be broken down further into **plane trusses** and **space trusses**. A plane truss is a truss where all members lie in a single plane. This means that plane trusses can essentially be treated as two-dimensional systems. Space trusses, on the other hand, have members that are not limited to a single plane. This means that space trusses need to be analyzed as a three-dimensional system.



Figure 5.3.2: The members of these trusses all lie in a single plane. These roof trusses are an example of a plane truss. Image by Riisipuuro CC-BY-SA 3.0.

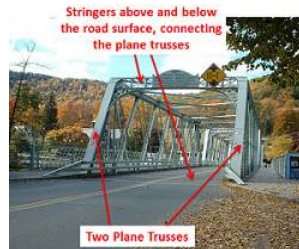


Figure 5.3.3: This bridge consists of two plane trusses connected by members called stringers. Adapted from image by ToddC4176 CC-BY-SA 3.0.

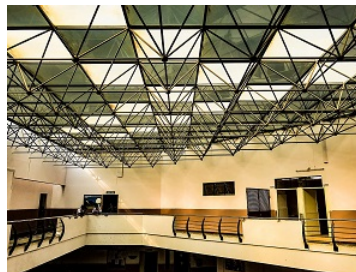


Figure 5.3.4: This roof supporting truss does not lie in a single plane. This is an example of a space truss. Image by IM3847 CC-BY-SA 4.0.



Figure 5.3.5: The power line tower also does not lie in a single plane and is therefore a space truss. Image by Anders Lagerås CC-BY-SA 2.5.

Analyzing Trusses:

When we talk about analyzing a truss, we are usually looking to identify not only the external forces acting on the truss structure, but also the forces acting on each member internally in the truss. Because each member of the truss is a two force member, we simply need to identify the magnitude of the force on each member, and determine if each member is in tension or compression.

To determine these unknowns, we have two methods available: the **method of joints**, and the **method of sections**. Both will give the same results, but each through a different process.

The method of joints focuses on the joints, or the connection points where the members come together. We assume we have a pin at each of these points that we model as a particle, we draw out the free body diagram for each pin, and then we write out the equilibrium equations for each pin. This will result in a large number of equilibrium equations that we can use to solve for a large number of unknown forces.

The method of sections involves pretending to split the truss into two or more different sections and then analyzing each section as a separate rigid body in equilibrium. In this method we determine the appropriate sections, draw free body diagrams for each section, and then write out the equilibrium equations for each section.

The method of joints is usually the easiest and fastest method for solving for all the unknown forces in a truss. The method of sections, on the other hand, is better suited to targeting and solving for the forces in just a few members without having to solve for all the unknowns. In addition, these methods can be combined if needed to best suit the goals of the problem solver.



Plane and Space Trusses

Trusses can further be broken down into **plane trusses** and **space trusses**.

- A **plane truss** is a truss that exists entirely in one plane.
 - This means that we will analyze these trusses as 2D systems.
 - Sometimes we can approximate parts of 3D truss structures as plane trusses.




Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/jn8sR-IS4tw>. [Video Transcript](#)

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5.4: Method of Joints

The **method of joints** is a process used to solve for the unknown forces acting on members of a **truss**. The method centers on the joints or connection points between the members, and it is usually the fastest and easiest way to solve for all the unknown forces in a truss structure.

Using the Method of Joints:

The process used in the method of joints is outlined below.

In the beginning, it is usually useful to label the members and the joints in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters and the joints will be labeled with numbers.

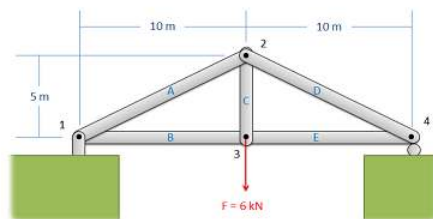


Figure 5.4.1: The first step in the method of joints is to label each joint and each member.

Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.

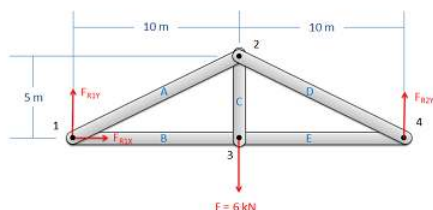


Figure 5.4.2: Treat the entire truss as a rigid body and solve for the reaction forces supporting the truss structure.

Assume there is a pin or some other small amount of material at each of the connection points between the members. Next you will draw a free body diagram for each connection point. Remember to include:

- Any external reaction or load forces that may be acting at that joint.
- A normal force for each two force member connected to that joint. Remember that for a two force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now though will simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile, then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.
- Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.

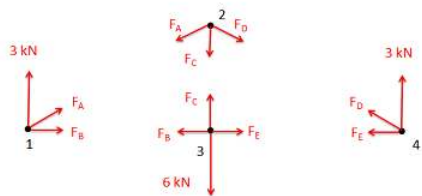


Figure 5.4.3: Drawing a free body diagram of each joint, we draw in the known forces as well as tensile forces from each two-force member.

- Write out the equilibrium equations for **each of the joints**. You should treat the joints as particles, so there will be force equations but no moment equations. With either two (for 2D problems) or three (for 3D problems) equations for each joint; this should give you a large number of equations.
 - In planar trusses, the sum of the forces in the x direction will be zero and the sum of the forces in the y direction will be zero for each of the joints.

$$\sum \vec{F} = 0 \quad (5.4.1)$$

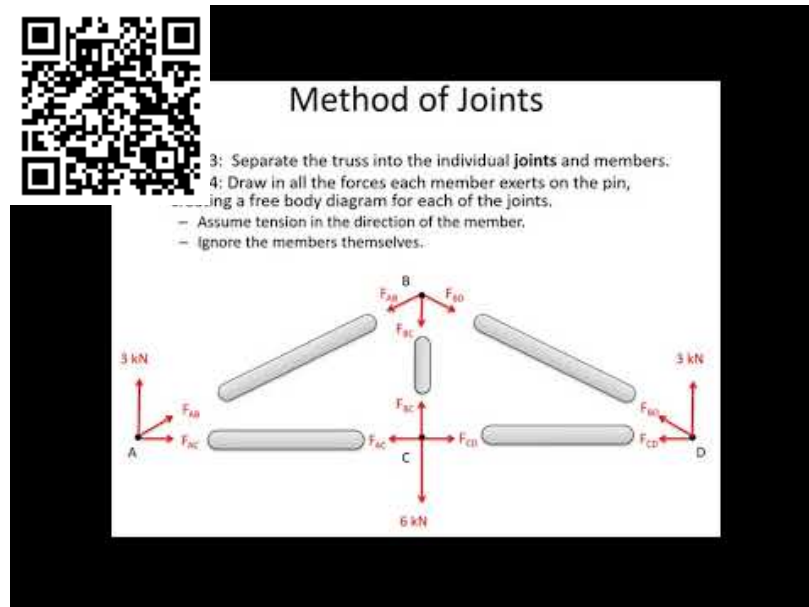
$$\sum F_x = 0; \sum F_y = 0 \quad (5.4.2)$$

- In space trusses, the sum of the forces in the x direction will be zero, the sum of the forces in the y direction will be zero, and the sum of forces in the z direction will be zero for each of the joints.

$$\sum \vec{F} = 0 \quad (5.4.3)$$

$$\sum F_x = 0; \sum F_y = 0; \sum F_z = 0 \quad (5.4.4)$$

- Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/B8SEG7xPI-o>. [Video](#) [Transcript](#)

? Example 5.4.1

Find the force acting in each of the members in the truss bridge shown below. Remember to specify if each member is in tension or compression.

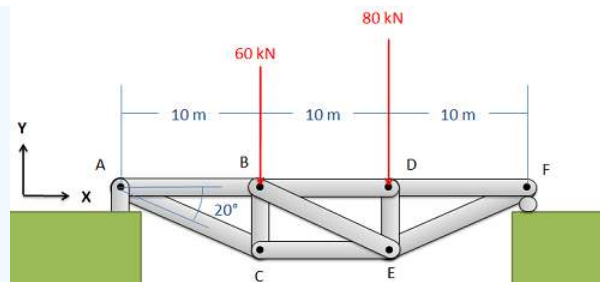
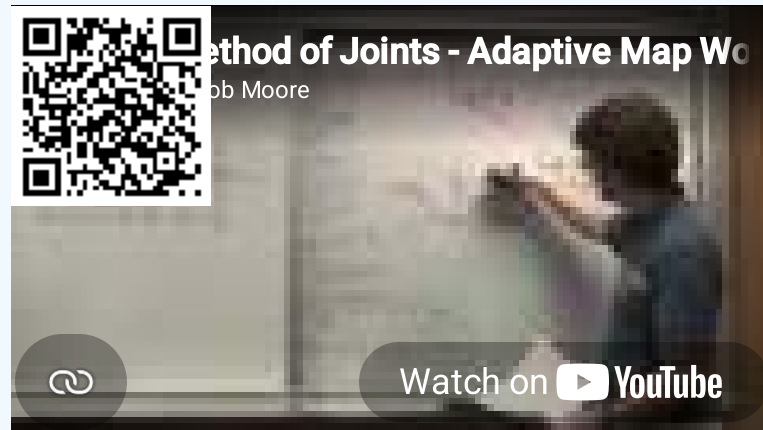


Figure 5.4.4: problem diagram for Example 5.4.1. A truss bridge represented as a 2D plane truss, with a standard-orientation xy -coordinate system.

Solution



Video 5.4.2: Worked solution to example problem 5.4.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/vowewkEdTzw>. [Video Transcript 2](#)

? Example 5.4.2

Find the force acting in each of the members of the truss shown below. Remember to specify if each member is in tension or compression.

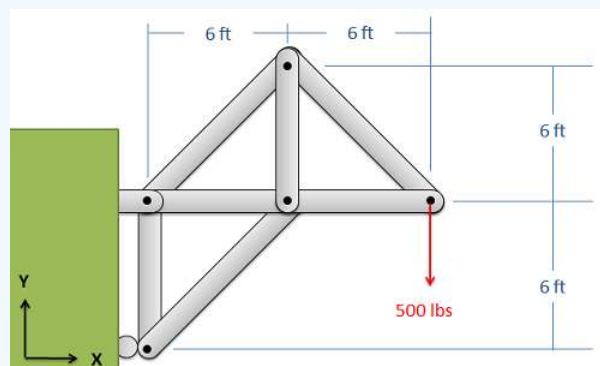


Figure 5.4.5: problem diagram for Example 5.4.2. A plane truss mounted on a wall, with a standard-orientation xy -coordinate system.

Solution



Video 5.4.3: Worked solution to example problem 5.4.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/IxnClZ-ppjM>. [Video Transcript 3](#)

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5.5: Method of Sections

The **method of sections** is a process used to solve for the unknown forces acting on members of a **truss**. The method involves breaking the truss down into individual sections and analyzing each section as a separate rigid body. The method of sections is usually the fastest and easiest way to determine the unknown forces acting in a specific member of the truss.

Using the Method of Sections:

The process used in the method of sections is outlined below.

1. In the beginning, it is usually useful to label the members in your truss. This will help you keep everything organized and consistent in later analysis. In this book, the members will be labeled with letters.

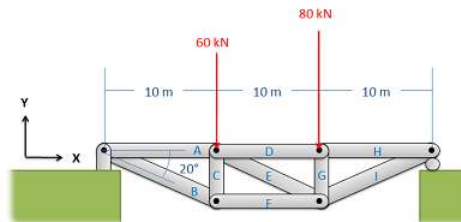


Figure 5.5.1: The first step in the method of sections is to label each member.

2. Treating the entire truss structure as a rigid body, draw a free body diagram, write out the equilibrium equations, and solve for the external reacting forces acting on the truss structure. This analysis should not differ from the analysis of a single rigid body.

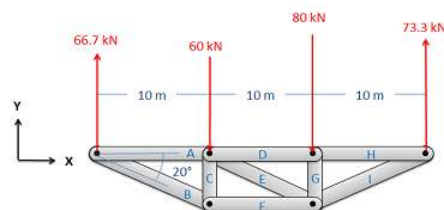


Figure 5.5.2: Treat the entire truss as a rigid body and solve for the reaction forces supporting the truss structure.

3. Next, you will imagine cutting your truss into two separate sections. The cut should travel through the member that you are trying to solve for the forces in, and should cut through as few members as possible. The cut does not need to be a straight line.

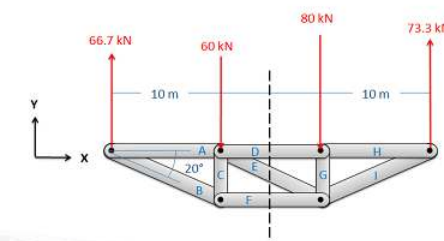


Figure 5.5.3: Next you will imagine cutting the truss into two parts. If you want to find the forces in a specific member, be sure to cut through that member. It also makes things easier if you cut through as few members as possible.

4. Next, you will draw a free body diagram for either one or both sections that you created. Be sure to include all the forces acting on each section.
 - Any external reaction or load forces that may be acting at the section.
 - An internal force in each member that was cut when splitting the truss into sections. Remember that for a two-force member, the force will be acting along the line between the two connection points on the member. We will also need to guess if it will be a tensile or a compressive force. An incorrect guess now, though, will simply lead to a negative solution later on. A common strategy then is to assume all forces are tensile; then later in the solution any positive forces will be tensile forces and any negative forces will be compressive forces.

- Label each force in the diagram. Include any known magnitudes and directions and provide variable names for each unknown.

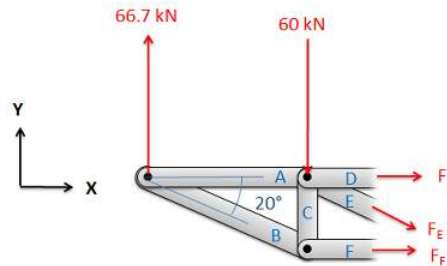


Figure 5.5.4: Next, draw a free body diagram of one or both halves of the truss. Add the known forces, as well as unknown tensile forces for each member that you cut.

- Write out the equilibrium equations for each section you drew a free body diagram of. These will be extended bodies, so you will need to write out the force and the moment equations.
 - For 2D problems you will have three possible equations for each section: two force equations and one moment equation.

$$\sum \vec{F} = 0 \quad \sum \vec{M} = 0 \quad (5.5.1)$$

$$\sum F_x = 0; \sum F_y = 0; \sum M_z = 0 \quad (5.5.2)$$

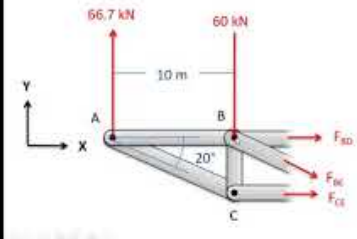
- Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If you assumed that all forces were tensile earlier, remember that negative answers indicate compressive forces in the members.



Method of Sections

Step 5: Write out the equations of equilibrium (forces in the x, forces in the y, and moments)

Step 6: Solve the equations.



$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum M = 0$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/XcRn776w22Q>. [Video Transcript](#)

? Example 5.5.1

Find the forces acting on members BD and CE. Be sure to indicate if the forces are tensile or compressive.

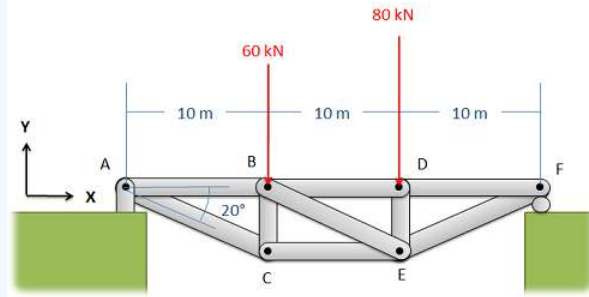
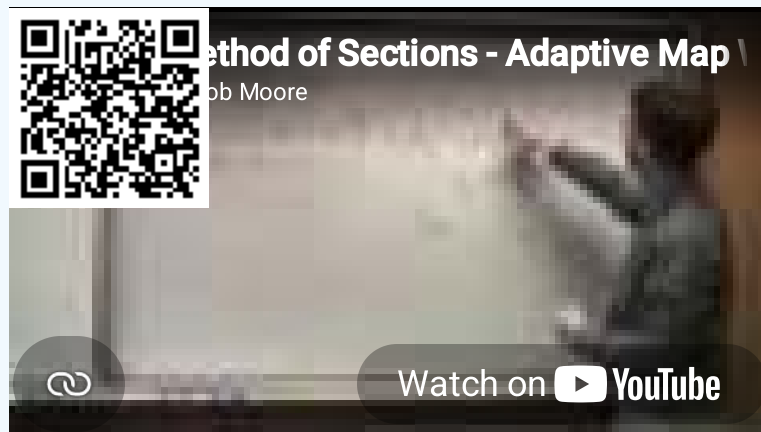


Figure 5.5.5: problem diagram for Example 5.5.1. A two-dimensional representation of a truss bridge, with a standard-orientation xy -coordinate system.

Solution



Video 5.5.2: Worked solution to example problem 5.5.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/9xxmHpLB1uU>. [Video Transcript 2](#)

? Example 5.5.2

Find the forces acting on members AC, BC, and BD of the truss. Be sure to indicate if the forces are tensile or compressive.

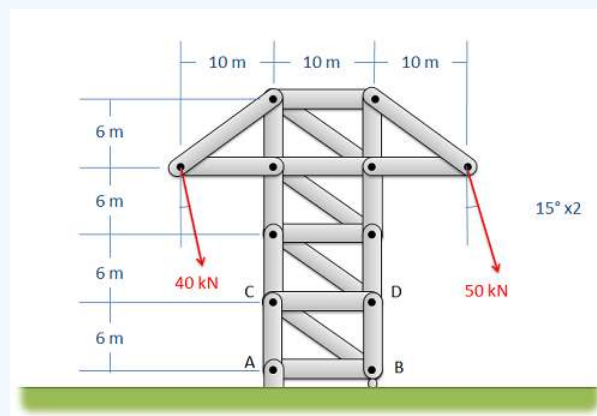


Figure 5.5.6: Problem diagram for Example 5.5.2. A two-dimensional representation of a tower composed of trusses, arranged in a tall rectangle made of rectangular subunits with a trapezoidal top.

Solution



Video 5.5.3: Worked solution to example problem 5.5.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/Kp9U4d2qbvE>. [Video Transcript 3](#)

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5.6: Frames and Machines

A **frame** or a **machine** is an engineering structure that contains at least one member that is not a two-force member.

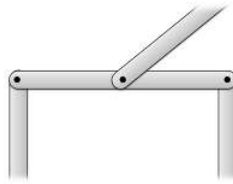


Figure 5.6.1: This horizontal beam is connected to other members (where normal forces would exist) at more than two locations. This beam is therefore not a two-force member.

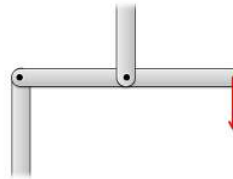


Figure 5.6.2: This beam has two connection points, but a force is acting on a third point. Therefore the beam has forces acting on it at more than two locations and it is not a two-force member.

A frame is a rigid structure, while a machine is not rigid. This means that no part can move relative to the other parts in a frame, while parts can move relative to one another in a machine. Though there is a difference in vocabulary in describing frames and machines, they are grouped together here because we use the same process to analyze both of these structures.



Figure 5.6.3: This stool contains non-two-force members (the legs) and no part can move relative to the other parts (it is rigid). Therefore it is a frame. Image by Besceh31 CC-BY-SA 2.5.



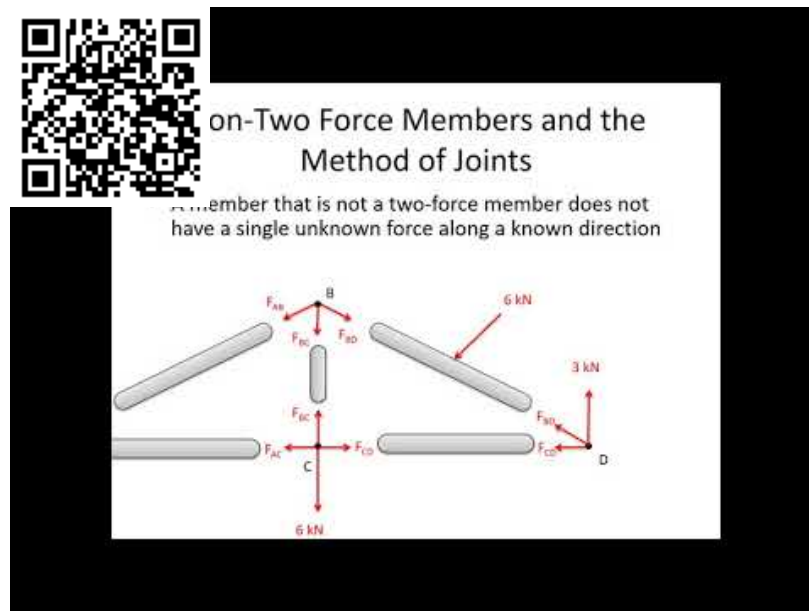
Figure 5.6.4: This pair of locking pliers contains non-two-force members and has parts that can move relative to one another (it is not rigid). Therefore it is a machine. Image by Duk CC-BY-SA 3.0.

Analyzing Frames and Machines:

When we talk about analyzing frames or machines, we are usually looking to identify both the external forces acting on the structure and the internal forces acting between members within the structure.

The method we use to analyze frames and machines (no special name here) centers around the process of breaking the structure down into individual components and analyzing each component as a rigid body. Where the components are connected, Newton's Third Law states that each body will exert an equal and opposite force on the other body. Each component will be analyzed as an

independent rigid body leading to equilibrium equations for each component, but because of Newton's Third Law, some unknowns may show up acting on two bodies.



Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: <https://youtu.be/QY-zWXzL7aI>. [Video Transcript](#)

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5.7: Analysis of Frames and Machines

The process used to analyze frames and machines involves breaking the structure down into individual components in order to solve for the forces acting on each component. **Sometimes** the structure as a whole can be analyzed as a rigid body, and each component can **always** be analyzed as a rigid body.

The Process for Analyzing Frames and Machines:

1. In the beginning it is usually useful to label the members in your structure. This will help you keep everything organized and consistent in later analysis. In this book, we will label everything by assigning letters to each of the joints.

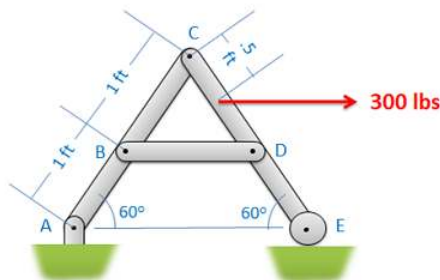


Figure 5.7.1: The first step in the analysis of frames and machines is to label the members.

2. Next you will need to determine if we can analyze the entire structure as a rigid body. In order to do this, the structure needs to be independently rigid. This means that it would be rigid even if we separated it from its supports. If the structure is independently rigid (no machines, and only some frames, will be independently rigid), then analyze the structure as a single rigid body to determine the reaction forces acting on the structure. If the structure is not independently rigid, skip this step.

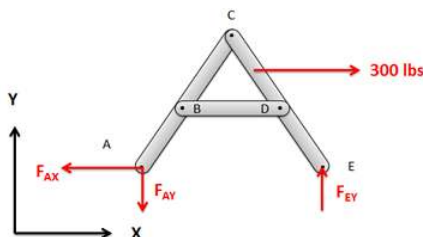


Figure 5.7.2: If, and only if, the structure is independently rigid, you should analyze the whole structure as a single rigid body to solve for the reaction forces.

3. Next you will draw a free body diagram for each of the components in the structure. You will need to include all forces acting on each member:
 - First, add any external reaction or load forces that may be acting on the components.
 - Second, identify any **two-force members** in the structure. At their connection points, they will cause a force with an unknown magnitude but a **known direction** (the forces will act along the line between the two connection points on the member).
 - Next, add in the reaction forces (and possibly moments) at the connection points between non-two-force members. For forces with an unknown magnitude and direction (such as in pin joints), the forces are often drawn in as having unknown x and y components (x , y and z for 3D truss problems).
 - Remember that the forces at each of the connection points will be a **Newton's Third Law pair**. This means that if one member exerts some force on some other member, then the second member will exert an equal and opposite force back on the first. When we draw out our unknown forces at the connection points, we must make sure that the forces acting on each member are **opposite in direction**.

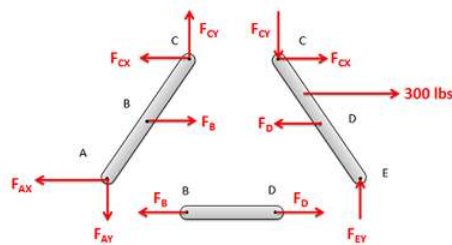


Figure \(\backslash\)
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{3}\):

Separate the structure into individual components and draw a free body diagram of each component. It is important to remember that the forces at each connection point are a Newton's Third Law pair.

4. Write out the equilibrium equations for each component you drew a free body diagram of. These will be extended bodies, so you will need to write out the force and the moment equations.
 - For 2D problems you will have three possible equations for each section: two force equations and one moment equation.

$$\sum \vec{F} = 0 \quad \sum \vec{M} = 0 \quad (5.7.1)$$

$$\sum F_x = 0; \sum F_y = 0; \sum M_z = 0 \quad (5.7.2)$$

5. Finally, solve the equilibrium equations for the unknowns. You can do this algebraically, solving for one variable at a time, or you can use matrix equations to solve for everything at once. If any force turns out to be negative, that indicates that the force actually travels in the opposite direction from what is indicated in your initial free body diagram.

Analysis of Frames and Machines

Step 4: Write out the equilibrium equations for **each component**.

— You may have more than three unknowns for any one component, and that's fine as long as the total number of unknowns is less than or equal to the total number of equations you generate (three equations per component).

$$\sum F_x = 0$$

$$\sum F_y = 0$$

$$\sum M_A = 0$$

Video lecture covering this section, delivered by Dr. Jacob Moore. YouTube source: https://youtu.be/6DK6IOU_TmU. [Video Transcript](#)

? Example 5.7.1

Find all the forces acting on each of the members in the structure below.

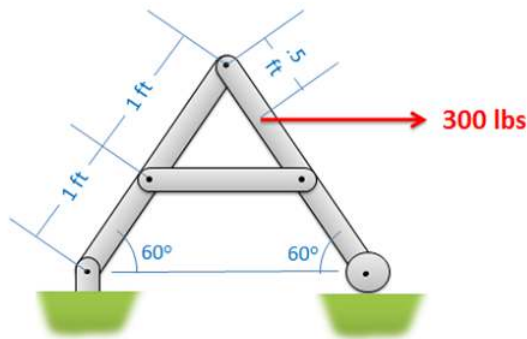


Figure 5.7.4: problem diagram for Example 5.7.1. A symmetrical A-shaped structure experiences an external force applied at one point.

Solution



Video 5.7.2: Worked solution to example problem 5.7.1, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/ix2BuRGMGBs>. [Video Transcript 2](#)

? Example 5.7.2

Find all the forces acting on each of the members in the structure below.

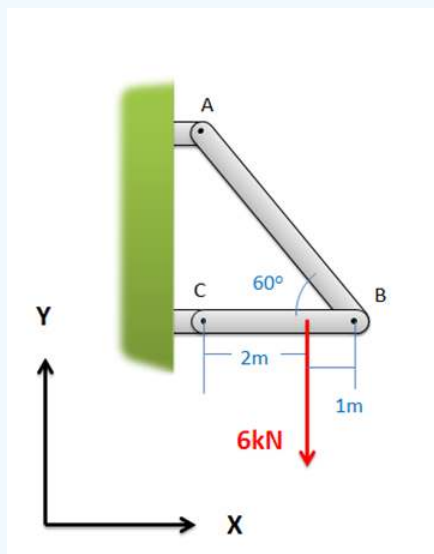


Figure 5.7.5: problem diagram for Example 5.7.2. A two-member structure is attached to a wall with pin joints and experiences an external force applied at one point.

Solution



Video 5.7.3: Worked solution to example problem 5.7.2, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/TJjsz5Yt3Y0>. [Video Transcript 3](#)

? Example 5.7.3

If two 150-Newton forces are exerted on the handles of the bolt cutter shown below, determine the reaction forces (F_{R1} and F_{R2}) exerted on the blades of the bolt cutter (this will be equal to the cutting forces exerted by the bolt cutters).

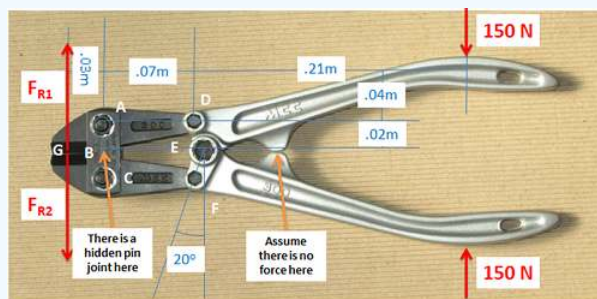


Figure 5.7.6: problem diagram for Example 5.7.3. Top-down view of a bolt cutter lying on a table facing left, divided into members by labeled points.

Solution



Video 5.7.4: Worked solution to example problem 5.7.3, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/uLsfSMzc5eQ>. [Video Transcript 4](#)

? Example 5.7.4

A 100-lb force is exerted on one side of a TV tray as shown below. Assuming there are no friction forces at the base, determine all forces acting on each of the three parts of the TV tray.

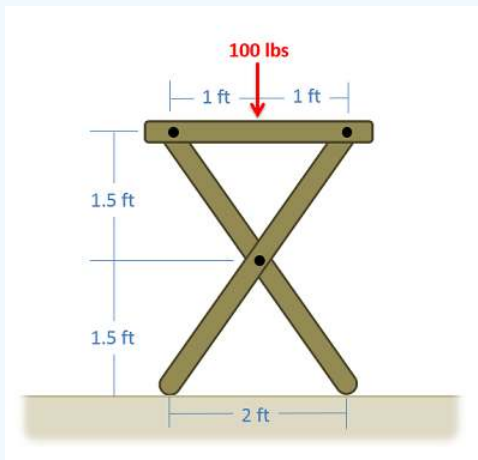


Figure 5.7.7: problem diagram for Example 5.7.4. Side view of a rectangular TV tray on two legs arranged in an X-shape, with a downwards force applied on the tray.

Solution



WP 004 - Analysis of Frames and

Jacob Moore



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Video 5.7.5: Worked solution to example problem 5.7.4, provided by Dr. Jacob Moore. YouTube source: <https://youtu.be/qi7WNDSb43k>. [Video Transcript 5](#)

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5.8: Chapter 5 Homework Problems

? Exercise 5.8.1

Use the method of joints to solve for the forces in each member of the lifting gantry truss shown below.

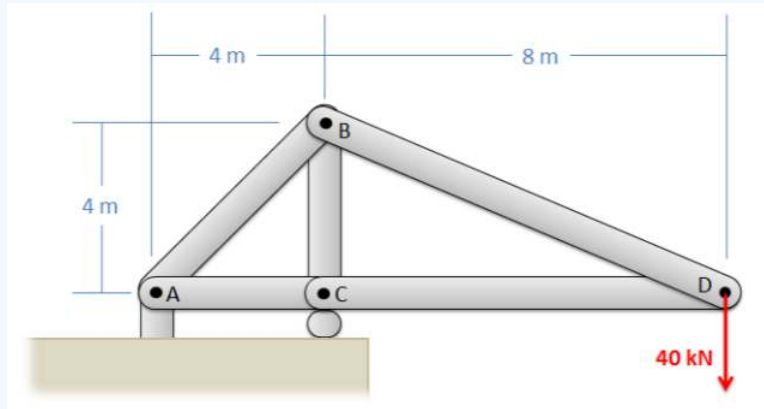


Figure 5.8.1: problem diagram for Exercise 5.8.1. A two-dimensional representation of a lifting gantry truss, which experiences a downwards force of 40 kN at one end.

Solution

$$F_{AB} = 113.14 \text{ kN T}, F_{AC} = 80 \text{ kN C}, F_{BC} = 120 \text{ kN C}$$

$$F_{BD} = 89.44 \text{ kN T}, F_{CD} = 80 \text{ kN C}$$

? Exercise 5.8.2

The truss shown below is supported by two cables at A and E, and supports two lighting rigs at D and F, as shown by the loads. Use the method of joints to determine the forces in each of the members.

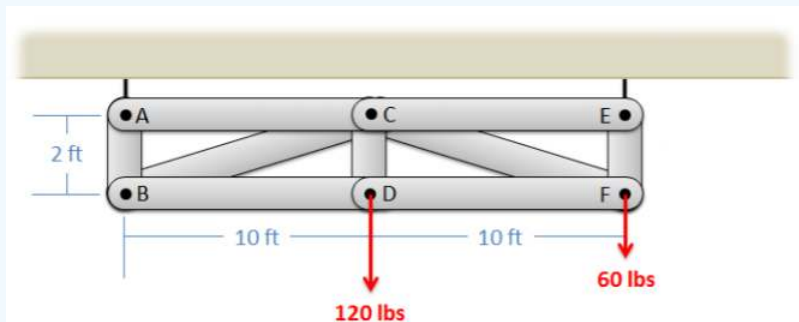


Figure 5.8.2: problem diagram for Exercise 5.8.2. A two-dimensional representation of a lighting support truss, which experiences the weights of lighting rigs hung at two points.

Solution

$$F_{AB} = 60 \text{ lbs T}, F_{AC} = 0, F_{BC} = 305.94 \text{ lbs C}$$

$$F_{BD} = 300 \text{ lbs T}, F_{CD} = 120 \text{ lbs T}, F_{CE} = 0$$

$$F_{CF} = 305.94 \text{ lbs C}, F_{DF} = 300 \text{ lbs T}, F_{EF} = 120 \text{ lbs T}$$

? Exercise 5.8.3

The truss shown below is supported by a pin joint at A, a cable at D, and is supporting a 600 N load at point C. Use the method of joints to determine the forces in each of the members. Assume the mass of the beams are negligible.

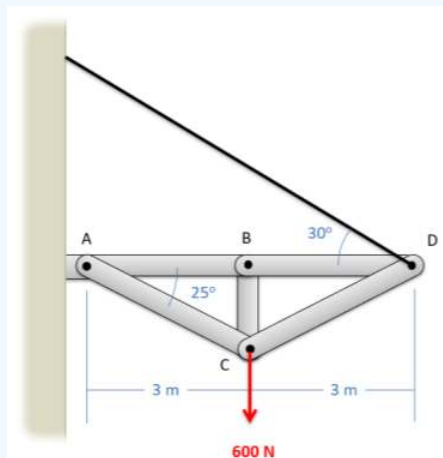


Figure 5.8.3: problem diagram for Exercise 5.8.3. A two-dimensional representation of a truss connected to the wall at one end, supported by a cable at the other, and experiencing a downwards force at one point.

Solution

$$F_{AB} = 1162.97 \text{ N C}, F_{AC} = 709.86 \text{ N T}, F_{BC} = 0$$

$$F_{BD} = 1162.97 \text{ N C}, F_{CD} = 709.86 \text{ N T}$$

? Exercise 5.8.4

Use the method of sections to solve for the forces acting on members CE, CF, and DF of the gantry truss shown below.

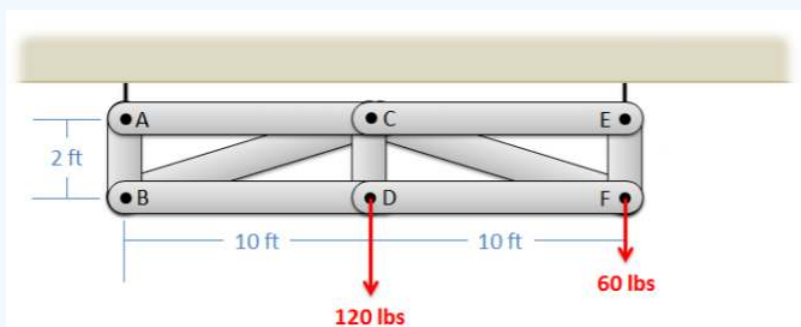


Figure 5.8.4: problem diagram for Exercise 5.8.4. A two-dimensional representation of a gantry truss, which experiences downwards forces applied at two points.

Solution

$$F_{CE} = 0, F_{CF} = 306.2 \text{ lbs C}, F_{DF} = 300.2 \text{ lbs T}$$

? Exercise 5.8.5

You are asked to compare two crane truss designs as shown below. Find the forces in members AB, BC, and CD for Design 1 and find forces AB, AD, and CD for Design 2. What member is subjected to the highest loads in either case?

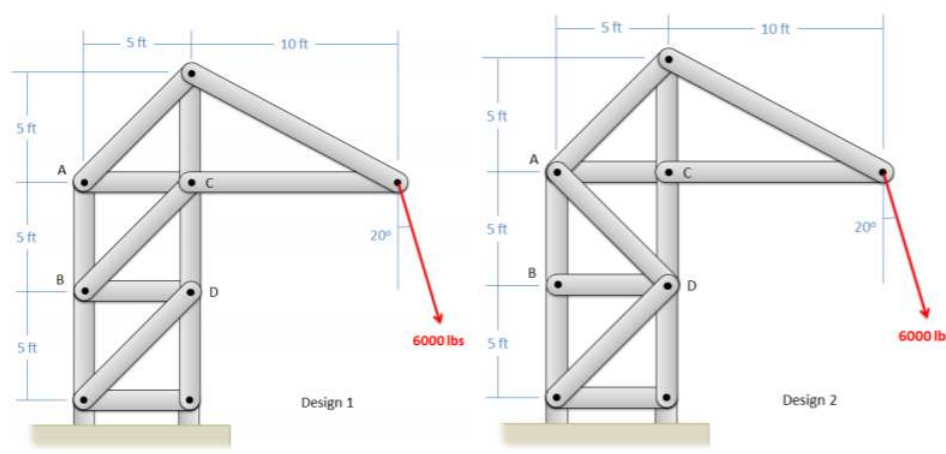


Figure 5.8.5: problem diagram for Exercise 5.8.5. Two versions of a crane truss design that differ only in the orientation of their support beams.

Solution

Design 1: $F_{AB} = 11,276 \text{ lbs T}$, $F_{BC} = 2902 \text{ lbs T}$, $F_{CD} = 18,967 \text{ lbs C}$

Design 2: $F_{AB} = 13,322 \text{ lbs T}$, $F_{AD} = 2902 \text{ lbs C}$, $F_{CD} = 16,914 \text{ lbs C}$

The largest forces are in member CD for both designs.

? Exercise 5.8.6

The K truss shown below supports three loads. Assume only vertical reaction forces at the supports. Use the method of sections to determine the forces in members AB and FG. (Hint: you will need to cut through more than three members, but you can use your moment equations strategically to solve for exactly what you need).

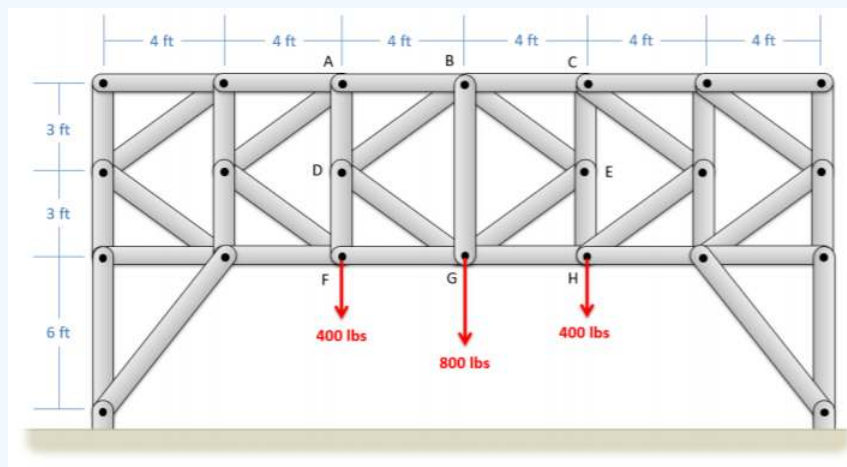


Figure 5.8.6: problem diagram for Exercise 5.8.6. A two-dimensional representation of a K-truss which experiences downwards forces applied at three points.

Solution

$F_{AB} = 1066.67 \text{ lbs C}$, $F_{FG} = 1066.67 \text{ lbs T}$

? Exercise 5.8.7

The truss shown below is supported by a pin support at A and a roller support at B. Use the hybrid method of sections and joints to determine the forces in members CE, CF, and CD.

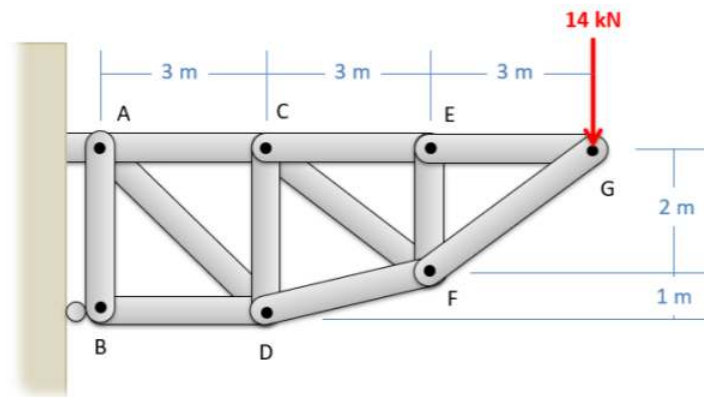


Figure 5.8.7: problem diagram for Exercise 5.8.7. A two-dimensional representation of a truss attached to a wall, experiencing a downwards force at one point.

Solution

$$F_{CE} = 21 \text{ kN T}, F_{CF} = 8.41 \text{ kN T}, F_{CD} = 4.67 \text{ kN C}$$

? Exercise 5.8.8

The shelf shown below is used to support a 50-lb weight. Determine the forces on members ACD and BC in the structure. Draw those forces on diagrams of each member.

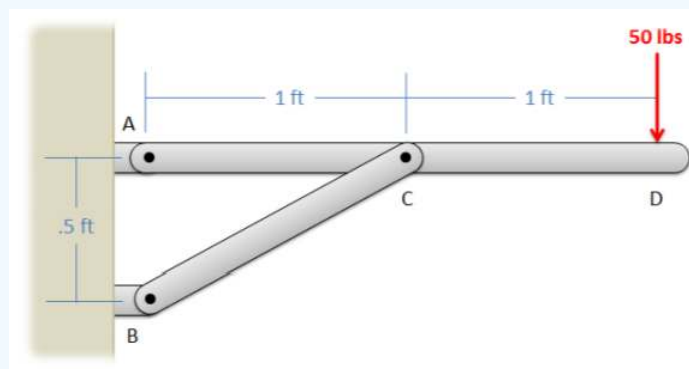


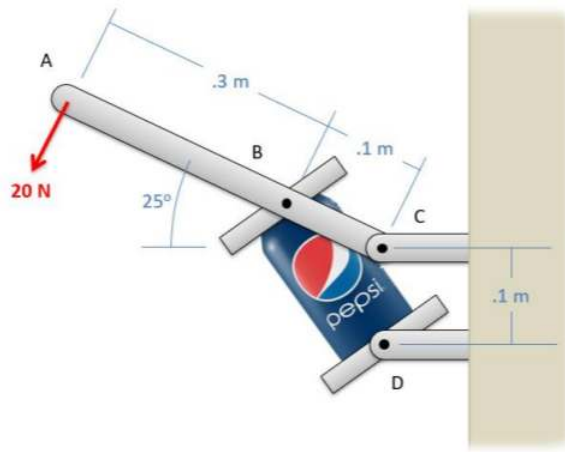
Figure 5.8.8: problem diagram for Example 5.8.8. Side view of a wall-mounted horizontal shelf supporting a weight at its free end.

Solution

$$F_{BC} = 223.6 \text{ lbs (Compression)}, F_{Ax} = -200 \text{ lbs}, F_{Ay} = -50 \text{ lbs}$$

? Exercise 5.8.9

A 20 N force is applied to a can-crushing mechanism as shown below. If the distance between points C and D is 0.1 meters, what are the forces being applied to the can at points B and D? (Hint: treat the can as a two-force member)



Solution

? Exercise 5.8.10

The diagram shows a mechanical system. A vertical wall on the left has points A, B, and D. A horizontal beam is pivoted at B. A spring is attached to the wall at A and to the beam at C. The distance from B to A is 10 cm. The distance from B to C is 20 cm. The beam is horizontal. A vertical force of 3300 N is applied upwards at point F, which is 20 cm to the right of point E. The distance from B to E is 40 cm. The distance from E to F is 10 cm. The beam is pivoted at B. The spring is attached to the wall at A and to the beam at C. The distance from B to A is 10 cm. The distance from B to C is 20 cm. The beam is horizontal. A vertical force of 3300 N is applied upwards at point F, which is 20 cm to the right of point E. The distance from B to E is 40 cm. The distance from E to F is 10 cm.

Solution

? Exercise 5.8.11

The chair shown below is subjected to forces at A and B by a person sitting in the chair. Assuming that normal forces exist at F and G, and that friction forces only act at point G (not at F), determine all the forces acting on each of the three members in the chair. Draw these forces acting on each part of the chair on a diagram.

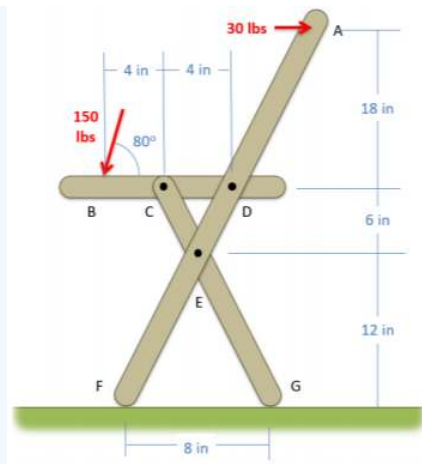


Figure 5.8.11: problem diagram for Exercise 5.8.11. Side view of a folding chair with a square seat, facing to the left, that experiences forces at two points from a person sitting in the chair.

Solution

$$F_F = 108.3 \text{ lbs}, F_{G_x} = -3.95 \text{ lbs}, F_{G_y} = 39.5 \text{ lbs}$$

$$F_{C_x} = \pm 16.89 \text{ lbs}, F_{C_y} = \pm 295.4 \text{ lbs}$$

$$F_{D_x} = \pm 142.9 \text{ lbs}, F_{D_y} = \pm 147.7 \text{ lbs}$$

$$F_{E_x} = \pm 112.9 \text{ lbs}, F_{E_y} = \pm 256.0 \text{ lbs}$$

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